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ON STABILITY RADII IN DELAY PARAMETERS

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Abstract: This paper focuses on the robust stability analysis of a class of linear systems including multiple delays subjected to constant or time-varying perturbations. The approach considered makes use of appropriate stability radius concepts (dynamic, static) and relies on a feedback interconnection interpretation of the uncertain system. Various computable bounds on stability radii are obtained that exploit the structure of the problem. Systems including perturbations on both system matrices and delays are also dealt with.

Keywords: delays, uncertainty, stability radius, feedback interconnection.

1. INTRODUCTION

It is well-known that the characterization of stability regions in the parameter space for time-delay systems is a difficult problem (see, e.g. [Diekmann et al. (1995)], and the references therein). Furthermore, the characterization in the delay parameter space for linear systems with multiple delays is NP-hard as proven by [Toker and Özbay (1996)].

Stability radii are well known in the context of matrix distance problems, see Hinrichsen and Pritchard (2005) and the references therein. Recently, they have been used to assess or optimize robustness of stability of linear time-delay systems subjected to structured uncertainty on the corresponding system matrices in Michiels et al. (2006); Michiels and Roose (2003). In this context stability radii correspond to the size of the smallest perturbations that render the system unstable.

The aim of this paper is to adopt the concept of stability radii to linear systems including multiple delays subject to constant or time-varying perturbations on the delay parameters, and to derive computable expressions. The case of constant perturbations for a class of quasi-polynomials including two delays was addressed in a geometrical setting in [Gu et al. (2007)], where the authors introduced the notion of delay deviation. The idea can be resumed in computing the distance between the “nominal” point (in
the delay parameter space) and the “closest” crossing curve for which there exists at least one characteristic root on the imaginary axis. Such a delay deviation characterization is nothing else than a characterization of a stability radius in the delay parameter space.

The approach considered in this paper is quite distinct to the one mentioned above. First, we will introduce two appropriate notions of stability radii: static and dynamic, in order to characterize constant and time-varying perturbations on the nominal system’s parameters. These stability radii are scalar robustness measures based on an a priori chosen weighting of the perturbations of delays and -as we shall discuss at the end- system matrices. Secondly, we will employ a feedback interconnection interpretation of the uncertain system in order to derive estimates for the defined stability radii. Note that a similar point of view was taken in (Gu et al., 2003, Chapter 3); Kao and Lin coln (2004); Fridman and Gil’ (2006); Fridman and Shaked (2006); Shustin and Fridman (2007) (L2 gain analysis applied to systems with time-varying delay perturbations), Huang and Zhou (2000) (µ analysis applied to systems with constant delay perturbations) and Michiels et al. (2006); Wagenknecht et al. (2007) (pseudospectra and stability radii for nonlinear eigenvalue problems), and some of the references therein. In the present paper the robust stability characterization in these references are combined and extended in a unifying framework, and reformulated in terms of appropriately defined stability radii. Finally, in Kao and Rantzer (2007) and the references therein the IQC approach is applied to deal with time-varying delays, leading to easy-to-check stability conditions expressed as LMIs, under the assumption that the corresponding delay-free system is asymptotically stable. Such an assumption will not be made in this paper.

Although in the case of uncertainty on the delays the feedback interconnection point of view and related tools from robust control will typically lead to expressions for lower bounds on stability radii (corresponding to sufficient yet not necessary robust stability conditions), as we shall see, they offer several advantages. Explicit computable expressions for bounds are namely obtained that impose no limitations on the number of delays and the dimension of the problem. Also time-varying perturbations and combined perturbation on delays and system matrices (matrix valued perturbations) can be easily dealt with, as we shall demonstrate. Finally the interconnection framework is appropriate for solving associated synthesis problems. The latter issue will however not be further addressed in this paper.

The paper adopts a step-by-step approach, by imposing more conditions on the perturbations and exploiting this information accordingly. More precisely, first time-varying perturbations are considered in a L2 analysis framework. Next it is shown how the derived explicit bounds on the stability can be improved for the special case of constant perturbations, where besides the inherent increase of the stability radii (due to the restriction of the perturbations), the structure of the interconnection can be better exploited by using frequency domain techniques. Finally, implicit expressions are given which rely on exploiting all structure of the problem and leave conservatism only in the fact that phase information is not fully exploited in the feedback loop (inherent to the adopted approach). For reasons of simplicity and clarity of the presentation the cases of uncertainty on delays only and of uncertainty on both delays and system matrices are treated separately. The notations are standard.

2. UNCERTAINTY ON THE DELAYS

2.1 Concept

We address the uncertain system

\[ \begin{align*}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i - w_i \delta \tau_i(t)), \\
\gamma(t) &= \phi(\theta), \quad -\eta \leq \theta \leq 0, \quad \phi \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n),
\end{align*} \]

where \( x(t) \in \mathbb{R}^n, \gamma > 0, A_i \in \mathbb{R}^{n \times n}, \text{ and } \tau_i \geq 0. \)

The uncertainty on the delays is modeled by uniformly bounded scalar functions \( t \in [0, \infty] \rightarrow \delta \tau_i(t) \) and scalar weights \( w_i > 0, \) which are such that

\[ w_i \delta \tau_i(t) \geq -\tau_i, \forall t \geq 0, \quad i = 1, \ldots, m. \]

We assume that the zero solution of the corresponding unperturbed system

\[ \begin{align*}
\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \\
x(\theta) &= \phi(\theta), \quad -\eta \leq \theta \leq 0,
\end{align*} \]

is asymptotically stable.

The dynamic stability radius \( r_{\delta \tau}^d \) of the system (2) w.r.t. the delays is defined as:

\[ r_{\delta \tau}^d := \sup \{ \gamma \geq 0 : \text{the zero solution of (1) is asymptotically stable for all functions } \delta \tau(t) \text{ satisfying } \sup_{t \geq 0} |\delta \tau_i(t)| \leq \gamma, \quad i = 1, \ldots, m \}. \]

Note that, although \( r_{\delta \tau}^d \) explicitly depends on the weights \( w_i, \) this dependence is suppressed in the notation, for reasons of simplicity.

Similarly, if the uncertainty on the delay is assumed time-invariant then the static stability radius w.r.t. the delays is defined as:

\[ r_{\delta \tau}^s := \sup \{ \gamma \geq 0 : \text{the zero solution of (1) is asymptotically stable for all constant } \delta \tau \text{ with } |\delta \tau_i| \leq \gamma, \quad i = 1, \ldots, m \}. \]

In the next paragraph several lower bounds on the above stability radii are derived. Such lower bounds correspond to robust stability conditions.
2.3 Time-varying perturbations

We factorize
\[ A_i = B_i C_i, \quad B_i \in \mathbb{R}^{n_i \times n}, \quad C_i \in \mathbb{R}^{n_i \times n}, \quad i = 1, \ldots, m, \]
where all \( B_i \) have full column rank, all \( C_i \) have full row rank, and we let \( n = \sum_{i=1}^{m} n_i \).

For \( u \in \mathcal{L}_2(0, \infty, \mathbb{R}^n) \), let \( y = G u \) be defined by
\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i) + \sum_{i=1}^{m} [B_1 \cdots B_m] u(t), \quad x(0) = 0, \quad 0 \leq t \leq \infty,
\]
\[
y(t) = [w_1 C_1^T \ldots w_m C_m^T]^T \dot{x}(t).
\]

Clearly \( y \in \mathcal{L}_2(0, \infty, \mathbb{R}^n) \). By the asymptotic stability of the unperturbed system and Parseval’s theorem the \( \mathcal{L}_2 \)-induced norm of \( G \) satisfies
\[
\|G\|_{\mathcal{L}_2} = \|G(j\omega)\|_{\mathcal{H}_\infty},
\]
where
\[
G(j\omega) = j\omega \begin{bmatrix} w_1 C_1 \\ \vdots \\ w_m C_m \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} B_1 \cdots B_m \end{bmatrix}.
\]

Next, we let
\[
S_i^\nu : \mathcal{L}_2([0, \infty), \mathbb{R}^n) \rightarrow \mathcal{L}_2([0, \infty), \mathbb{R}^n),
\]
\[
(S_i^\nu \xi)(t) = \begin{cases} \frac{1}{w_i} \int_{t-\tau_i}^{t} \xi(s) ds, & t \geq 0, \\ \xi(t), & t < 0, \end{cases}
\]
where \( \nu \in \mathbb{N}, \quad i \in \{1, \ldots, m\} \), and \( \xi \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n) \) satisfies
\[
\begin{cases} \xi(t), & t \geq 0, \\ \xi(0), & t < 0. \end{cases}
\]

By defining
\[
\mathcal{D} : \mathcal{L}_2([0, \infty), \mathbb{R}^n) \rightarrow \mathcal{L}_2([0, \infty), \mathbb{R}^n),
\]
\[
(D \xi)(t) = \text{diag}(\xi_1^T(t), \ldots, \xi_m^T(t)),
\]
where \( \xi(t) = [\xi_1^T(t) \ldots \xi_m^T(t)]^T \), with \( \xi_i(t) \in \mathbb{R}^{n_i}, \quad i = 1, \ldots, m \), we can interpret the system (1) as a feedback interconnection of \( G \) and \( \mathcal{D} \).

Remark 1. If some of the matrices \( A_i, \quad i = 1, \ldots, m \), have low rank, then \( G \) and \( \mathcal{D} \) have \( n < nm \) inputs and outputs, which is due to the factorization (5).

2.2 Feedback interconnection point of view

As a first step we characterize the induced \( \mathcal{L}_2 \) gain of \( \mathcal{D} \). We need the following result from Shustin and Fridman (2007, 2006):

**Lemma 1.** Assume that \( |\delta \tau_i(t)| \leq \mu_i \) for all \( t \geq 0 \). Then the induced \( \mathcal{L}_2 \) norm of \( S_i^\nu \) is bounded by \( \sqrt{\frac{1}{\gamma}} \mu_i \).

**Lemma 2.** Assume that \( |\delta \tau_i(t)| \leq \mu_i \) for all \( t \geq 0 \) and \( 1 \leq i \leq m \). Then
\[
\|D\|_{\mathcal{L}_2} \leq \sqrt{\frac{1}{\gamma} \mu_i}. \tag{9}
\]

**Proof.** Expression (9) follows from
\[
\|D\|_{\mathcal{L}_2} = \max_{1 \leq i \leq m} \|S_i^\nu\|_{\mathcal{L}_2}
\]
and Lemma 1.

By combining the above lemmas we arrive at the main result of this section:

**Proposition 1.** We have the following estimate:
\[
r_\tau^D \geq \frac{1}{\sqrt{\gamma}} \left( \|G(j\omega)\|_{\mathcal{H}_\infty} \right)^{-1}. \tag{10}
\]

**Proof.** From the small gain theorem we have that
\[
\|G\|_{\mathcal{L}_2} \|D\|_{\mathcal{L}_2} = \|G(j\omega)\|_{\mathcal{H}_\infty} \|D\|_{\mathcal{L}_2} < 1, \tag{11}
\]
then the feedback interconnection of \( G \) and \( D \) is \( \mathcal{L}_2 \)-stable, which induces the asymptotic stability of the zero solution of (1). Under the assumptions of Lemma 2 the condition (11) is fulfilled if
\[
\sqrt{\frac{1}{\gamma}} \|G(j\omega)\|_{\mathcal{H}_\infty} \|\mu\|_{\infty} < 1. \tag{12}
\]
The assertion of the proposition follows.

Proposition 1 can be strengthened by an appropriate scaling in the feedback loop. More precisely, with the set \( T \) defined as
\[
T = \{ \text{diag}(T_1, \ldots, T_m) : \quad T_i \in \mathbb{C}^{n_i \times n_i},
\]
\[
\det T_i \neq 0, \quad i = 1, \ldots, m \},
\]
we get:

**Proposition 2.** We have the following estimate:
\[
r_\tau^D \geq \frac{1}{\sqrt{\gamma} \mu_i} \left( \min_{T \in T} \|TG(j\omega)T^{-1}\|_{\mathcal{H}_\infty} \right)^{-1}. \tag{14}
\]

**Remark 2.** The optimization problem
\[
\min_{T \in T} \|TG(j\omega)T^{-1}\|_{\mathcal{H}_\infty}
\]
can be reformulated as
\[
\min_{\gamma, U_{\gamma}} \gamma,
\]
such that
\[
\gamma > 0, \quad U \in T, \quad U = U^* > 0,
\]
\[
G(j\omega)^* U G(j\omega) - \gamma^{-2} U < 0, \quad \forall \omega \geq 0, \tag{15}
\]
where \( T \) can be computed from \( U = T^* T \). Hence, a good scaling matrix \( T \) in (14) can for instance be obtained by relaxing (15) to values of \( \omega \) on a frequency grid and solving the resulting convex optimization problem.
Remark 3. If the delay perturbations are such that the functions
\[ t \mapsto t - \tau_i - w_i \delta \tau_i(t), \quad i = 1, \ldots, m, \]
are non-decreasing, then the factor $\sqrt{t/4}$ in (10) and (14) can be replaced with 1. This follows from the fact that in such case $\|S_i^r\|_{C_2} \leq \mu_i$ if $|\delta \tau_i(t)| \leq \mu_i$ for all $t \geq 0$, see Fridman and Shaked (2006).

For improvements of Lemma 1 for the case where the delays are differential functions with a given upper bound on their derivatives, we refer to Shustin and Fridman (2006).

2.4 Time-invariant perturbations

We reconsider the estimates for the stability radii under the additional assumption of constant delay perturbations. Then improvements can be made by decoupling the signals in the frequency domain, and by further exploiting the structure in the problem under consideration.

Let the entire functions $s_i$ be defined as
\[ s_i(\lambda) = \begin{cases} e^{-\lambda \tau_i} \frac{1 - e^{-\lambda(w_i \delta \tau_i)}}{w_i \lambda}, & \lambda \neq 0, \\ 1, & \lambda = 0 \end{cases}, \quad i = 1, \ldots, m. \]
As they satisfy
\[ |s_i(j \omega)| \leq \frac{1 - e^{-j \omega(w_i \delta \tau_i)}}{w_i \omega}, \quad \forall \omega \geq 0, \]
we obtain
\[ \|S_i^r\|_{C_2} = \|s_i(j \omega)I\|_{H_\infty} \leq \|s_i\|_{H_\infty} \leq \delta \tau_i. \]  \hspace{1cm} (16)
This result can also be derived in the time-domain, see [Fridman and Shaked (2006)]. Denote with
\[ D(\lambda) := \text{diag}(s_1(\lambda)I_{n_1}, \ldots, s_m(\lambda)I_{n_m}) \]
transfer function associated with the operator $D$, defined in (8). From (16) it follows that
\[ \|D(j \omega)\|_{H_\infty} \leq \|s_i\|_{H_\infty} \leq \delta \tau_i. \]
The characteristic equation of (1) can be written on the imaginary axis as
\[ \det \left( j \omega I - A_0 - \sum_{i=1}^m A_i e^{-j \omega \tau_i} \right) \cdot \det \left( I - G(j \omega)D(j \omega) \right) = 0, \]  \hspace{1cm} (18)
where the first factor is nonzero for all $\omega \geq 0$ because the unperturbed system is assumed to be asymptotically stable. The perturbed system is asymptotically stable if the perturbations cannot shift characteristic roots to the imaginary axis, that is, if (18) has no solutions. Based on this observation we have the following result, which makes use of structured singular values (see the appendix for a short introduction):

Proposition 3. Define the uncertainty set
\[ \Delta := \{ \text{diag}(d_1 I_{n_1}, \ldots, d_m I_{n_m}) : d_i \in \mathbb{C}, 1 \leq i \leq m \}. \]  \hspace{1cm} (19)
Then
\[ r^*_f \geq \left( \sup_{\omega \geq 0} \mu(\Delta G(j \omega)) \right)^{-1}, \]  \hspace{1cm} (20)
where $\mu(\cdot)$ is the structured singular value w.r.t. (19).

Proof. From (18) and the fact that $D(j \omega) \in \Delta$ for all $\omega \geq 0$, a sufficient stability condition is given by
\[ \|D(j \omega)\|_2 < \frac{1}{\mu(\Delta G(j \omega))}, \quad \forall \omega \geq 0. \]
This condition is satisfied if
\[ \|s_i\|_{H_\infty} < (\mu(\Delta G(j \omega)))^{-1}, \quad \forall \omega \geq 0, \]
which leads to the statement of the proposition. \hfill \square

Because the exact computation of the structured singular value of a complex $n \times n$ matrix $M$ w.r.t. the uncertainty structure (19) is a hard problem if $m$ is large [Toker and Özbay (1995)], the available numerical algorithms typically compute lower and upper bounds, see the appendix. We have for instance
\[ \mu(\Delta(M)) \leq \min_{T \in T} \|s_i\|_{H_\infty} \leq \|s_i\|_{H_\infty}, \]  \hspace{1cm} (21)
where $T$ is given by (13) and $\| \cdot \|_2$. The computation of the upper bound in (21) can be formulated as a convex optimization problem, using the arguments spelled out in Remark 2.

From Proposition 3 and the estimate (21) we obtain:
\[ r^*_f \geq \left( \sup_{\omega \geq 0} \min_{T \in T} \|T^{-1}G(j \omega)T\|_2 \right)^{-1}. \]  \hspace{1cm} (22)
It is instructive to compare expressions (22) and (14), the latter corresponding to:
\[ r^d_f \geq \left( \sqrt{T/4} \min_{T \in T} \|T^{-1}G(j \omega)T\|_2 \right)^{-1}. \]

Besides the factor $\sqrt{T/4}$ (due to the better estimate of $\|S_i^r\|_{C_2}$ in the time-invariant case), the outer and inner optimization have been inter-changed, that is, the scaling has become frequency dependent in (22).

Further improvements of the estimate (20) can be made by using the smallest possible upper bound on $|s_i(j \omega)|$ instead of (16). The price to be paid is that the expression for the stability radius is no longer explicit. The following result slightly generalizes Theorem 3 of Huang and Zhou (2000):
Proposition 4. Let \( s : \mathbb{R}_+ \to \mathbb{R}_+ \),
\[
   \omega \mapsto s(\omega) := \begin{cases} 
   \sin(\omega), & \omega \leq \frac{\pi}{2} \\
   1, & \omega > \frac{\pi}{2} 
\end{cases}
\] (23)

Define \( F : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \),
\[
   \alpha \to F(\alpha) := \sup_{\omega \geq 0} \mu_\Delta \left( \begin{bmatrix}
   2s \left( \frac{w_1 \omega}{2} \right) C_1 \\
   \vdots \\
   2s \left( \frac{w_m \omega}{2} \right) C_m 
\end{bmatrix} \right) \cdot
\]
\[
   \cdot \left( j\omega I - A_0 - \sum_{i=1}^m A_i e^{-j\omega \tau_i} \right)^{-1} \cdot
\]
\[
   \cdot [B_1 \ldots B_m],
\] (24)

where \( \mu_\Delta(\cdot) \) is the structured singular value w.r.t. the uncertainty set (19).

If \( F(\alpha) < 1 \), then \( r_{\tau}^* \geq \alpha \). Consequently,
\[
   r_{\tau}^* \geq \sup \{ \alpha > 0 : F(\alpha) < 1 \}.
\] (25)

Proof. The proof is based on an additional scaling within the feedback loop. Equation (18) is equivalent with
\[
   \det(I - \Lambda^{-1}(\omega; \alpha) G(j\omega) D(j\omega) \Lambda(\omega; \alpha)) = 0,
\]
where
\[
   \Lambda(\omega; \alpha) = \text{diag} \left( \frac{j\omega w_1}{2s \left( \frac{w_1 \omega}{2} \right)}, \ldots, \frac{j\omega w_m}{2s \left( \frac{w_1 \omega}{2} \right)} \right).
\]

By construction, we have
\[
   F(\alpha) = \sup_{\omega \geq 0} \mu_\Delta(\Lambda^{-1}(\omega; \alpha) G(j\omega)).
\]

Furthermore, the structure of \( D(j\omega) \) is not affected by the post-multiplication with \( \Lambda(\omega; \alpha) \). Hence, under the assumption \( F(\alpha) < 1 \), the system is stable if
\[
   \|\Lambda(\omega; \alpha) D(j\omega)\|_2 < 1, \forall \omega \geq 0
\]
\[
   \Longleftrightarrow \frac{1 - e^{-j\omega w_i \tau_i}}{2s \left( \frac{w_i \omega}{2} \right)} < 1, \forall \omega \geq 0, i = 1, \ldots, m
\]
\[
   \Longleftrightarrow \sin \left( \frac{w_i \omega \tau_i}{2s \left( \frac{w_i \omega}{2} \right)} \right) < 1, \forall \omega \geq 0, i = 1, \ldots, m
\]
\[
   \Longleftrightarrow \delta \tau_i < \alpha, i = 1, \ldots, m.
\]

The assertion of the proposition follows. \( \square \)

Remark 4. Since for all \( \omega \geq 1 \) and \( i = 1, \ldots, m \), we have
\[
   \sup_{|\delta \tau_i| < \alpha} \left| \frac{\sin \left( \frac{w_i \delta \tau_i \omega}{2} \right)}{s \left( \frac{w_i \omega}{2} \right)} \right| = 1, \forall \omega \geq 0,
\]
a further improvement of the estimate (25) can only be achieved by exploiting phase information in the feedback loop, which is not possible with the adopted \( \mu \) approach.

3.UNCERTAINTY IN COEFFICIENT MATRICES AND DELAYS

We consider the uncertain system
\[
   \dot{x}(t) = (A_0 + D_0 \delta A_0(t) E_0) x(t) + \sum_{i=1}^m (A_i + D_i \delta A_i(t) E_i) (x(t - \tau_i - w_i \delta \tau_i(t))),
\] (26)

under appropriate initial conditions. The uncertainty is expressed by the piece-wise continuous functions
\[
   \delta A_i \in L_\infty([0, \infty), \mathbb{R}^{n_i \times n_i}), \quad i = 0, \ldots, m,
\]
\[
   \delta \tau_i \in L_\infty([0, \infty), [-\tau_i, \infty)), \quad i = 1, \ldots, m,
\] (27)

while \( D_i \in \mathbb{R}^{n \times n_i} \) and \( E_i \in \mathbb{R}^{n \times n_i} \) are weight matrices, and \( w_i > 0 \) are scalar weights.

The dynamic stability radius of the unperturbed system (2) w.r.t. the combined uncertainty in (26) is defined as
\[
   r_{\epsilon}^C := \sup \{ \gamma \geq 0 : \text{the zero solution of (26) is asymptotically stable for all functions } \delta A_i(t) \text{ and } \delta \tau(t) \text{ with } \sup_{t \geq 0} \| \delta A_i(t) \|_2 \leq \gamma, i = 0, \ldots, m \}
\]
and \( \sup_{t \geq 0} \| \delta \tau(t) \| \leq \gamma, i = 1, \ldots, m \}. \) (28)

The corresponding static stability radius \( r_{\epsilon}^C \) is defined in a similar way, by assuming time-invariant perturbations.

From an analysis point of view the main difference w.r.t. case discussed in the previous section is the nonlinear dependence of the right-hand side of (26) on the uncertainty, in particular, on the products of \( \delta A_i \) and \( x(t - \tau_i - w_i \delta \tau_i(t)) \). This problem can be overcome by introducing additional input and outputs [Fridman and Shaked (2006)]. First, let \( B_i, C_i, \tilde{A}_i, \tilde{D}_i, \tilde{E}_i \) be such that
\[
   A_i + D_i \delta A_i(t) E_i = B_i \left( \tilde{A}_i + \tilde{D}_i \delta A_i(t) \tilde{E}_i \right) C_i,
\]
\[
   i = 1, \ldots, m,
\]
where each \( C_i \in \mathbb{R}^{n_i \times n} \) has full row rank. A trivial choice is given by
\[
   B_i = C_i = I, \quad \tilde{A}_i = A_i, \quad \tilde{D}_i = D_i,
\]
\[
   \tilde{E}_i = E_i, \quad i = 1, \ldots, m,
\]
yet it is beneficial if a decomposition can be chosen where \( \text{rank}(C_i) = \tilde{n}_i < n \) (as this leads to smaller block sizes in the uncertainty structure). Next, we write (26) as the feedback interconnection of the system.
Proposition 5. We have the following estimate:

\[ r_c \geq \left( \|G(j\omega; \sqrt{7/4})\|_{\mathcal{H}_\infty} \right)^{-1}. \]

This proposition can again be strengthened by an appropriate scaling in the feedback loop. With the set \( T \) defined as

\[ T = \{ \text{diag}(t_0I_{n_0}, \ldots, t_mI_{n_m}, T_1, \ldots, T_m) : \quad t_i > 0, \ i = 0, \ldots, m, \ T_i \in \mathbb{C}^{n_i \times n_i}, \ \det T_i \neq 0, \ i = 1, \ldots, m \}, \]

we obtain:

\[ r_c^d \geq \left( \min_{T \in T} \|T^{-1}G(j\omega; \sqrt{7/4})T\|_{\mathcal{H}_\infty} \right)^{-1}. \]

3.2 Time-invariant perturbations

Taking into account the structure of the feedback path (30) and the estimate (17), we arrive at:

**Proposition 6.** Define the uncertainty set

\[ \Delta := \{ \text{diag}(\Delta_0, \ldots, \Delta_m, d_1I_{\tilde{n}_1}, \ldots, d_mI_{\tilde{n}_m}) : \quad \Delta_i \in \mathbb{C}^{n_i \times n_i}, d_j \in \mathbb{C}, \quad i = 0, \ldots, m, \ 1 \leq j \leq m \}. \]

Then

\[ r_c \geq \left( \sup_{\omega \geq 0} \min_{T \in T} \|T^{-1}G(j\omega; 1)T\|_{2} \right)^{-1}, \]

where \( T \) is given by (32).

An improvement of estimate (17) finally leads to:

**Proposition 7.** Let the function \( s : \mathbb{R}_+ \to \mathbb{R}_+ \) be given by (23). Define \( F : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+ \),

\[ \alpha \mapsto F(\alpha) := \sup_{\omega \geq 0} \mu_{\Delta}(G(j\omega; \alpha)), \]

where

\[ G_2(\lambda; \alpha) = \begin{bmatrix} \alpha E_0 \\ e^{-\lambda \tau_1} \alpha \tilde{E}_1 C_1 \\ \vdots \\ e^{-\lambda \tau_m \alpha \tilde{E}_m C_m} - 2s \left( \frac{\omega_m \alpha \omega}{2} \right) C_m \\ \lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda \tau_i} \end{bmatrix}^{-1}. \]

If \( F(\alpha) \leq 1 \), then \( r_c^d \geq \alpha \). Consequently,

\[ r_c^d \geq \sup \{ \alpha > 0 : F(\alpha) < 1 \}. \]
4. CONCLUDING REMARKS

Stability radii of uncertain time-delay were defined and lower bounds were derived using a feedback interconnection point of view. Both constant and time-varying perturbations were considered on the delays, as well as delays and system matrices.

If information on the delays’ variation and/or derivatives is available, then the derived estimates for the dynamic stability radii may be further improved, as we indicated in Remark 3.

Alternative estimates for the stability radii can be obtained by rewriting the uncertain system as a feedback interconnection, where the operator $D_2$ or $D_3$,

$$D_2(\eta)(t) = [(S^m_1 \eta)(t) \ldots (S^m_m \eta)(t)]^T,$$

$$D_3(\zeta)(t) = \sum_{k=1}^n (S^m_k \zeta_k)(t),$$

appears in the feedback loop instead of $D$. For particular cases, e.g. scalar systems with multiple time-varying delays, this may lead to improved bounds. This is currently under investigation.

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REFERENCES


Appendix A. THE STRUCTURED SINGULAR VALUE

We introduce the concept of structured singular values of matrices and outline the main principles behind the standard computational schemes. A more elaborate introduction can be found in the review paper [Packard and Doyle (1993)], Chapter 11 of [Zhou et al. (1995)] and Chapter 4 of [Hinrichsen and Pritchard (2005)].

Let $G \in \mathbb{C}^{N \times M}$ and denote its singular values in decreasing order with $\sigma_1(G) \geq \sigma_2(G) \geq \ldots$. A classical result from linear algebra and robust control theory, which lays the basis for the celebrated small gain theorem, relates the largest singular value of $G$ to the solutions of the equation

$$\det(I + G\Delta) = 0$$

in the following way:

$$\sigma_1(G) = \begin{cases} 0, & \text{if } \det(I + G\Delta) \neq 0, \forall \Delta \in \mathbb{C}^{M \times N}, \\ m_u^{-1}, & \text{otherwise}, \end{cases}$$

where

$$m_u := \min \left\{ \sigma_1(\Delta) : \Delta \in \mathbb{C}^{M \times N} \text{ and } \det(I + G\Delta) = 0 \right\}. \quad (A.2)$$

We refer to $\Delta$ as the ‘uncertainty’, as in a robust control framework, (A.1) typically originates from a feedback interconnection of a nominal transfer function and an uncertainty block.

Next we reconsider the solutions of equation (A.1), where $\Delta$ is restricted to having a particular structure by imposing $\Delta \in \Delta$, with $\Delta$ a closed subset of $\mathbb{C}^{M \times N}$. In analogy with (A.2) one defines the \textit{structured singular value} of the matrix $G$ with respect to the uncertainty set $\Delta$ as

$$\mu(\Delta) := \begin{cases} 0, & \text{if } \det(I + G\Delta) \neq 0, \forall \Delta \in \Delta, \\ m_s^{-1}, & \text{otherwise}, \end{cases}$$

where

$$m_s := \min \left\{ \sigma_1(\Delta) : \Delta \in \Delta \text{ and } \det(I + G\Delta) = 0 \right\}. \quad (A.3)$$

It directly follows from the definition that

$$\mu(\Delta) \leq \sigma_1(G). \quad (A.4)$$

Furthermore, if $\mathcal{C} \Delta = \Delta$, then

$$\mu(\Delta) = \max_{\Delta \in \Delta, \sigma_1(\Delta)=1} \text{r}_\sigma(G\Delta), \quad (A.5)$$

with $r_\sigma(\cdot)$ the spectral radius.

In what follows we introduce ourselves for simplicity to an uncertainty set $\Delta$ of the form

$$\Delta := \left\{ \text{diag}(\Delta_0, \ldots, \Delta_f, d_0I_{m_0}, \ldots, d_sI_{m_s}) : \right.$$  
$$\left. \Delta_i \in \mathbb{C}^{k_i \times l_i}, d_j \in \mathbb{C}, \right.$$  
$$0 \leq i \leq f, 0 \leq j \leq s \right\}, \quad (A.6)$$

where $\text{diag}(\cdot)$ represents a block diagonal matrix, $\sum_{i=0}^{f} k_i + \sum_{i=0}^{s} l_i = M$ and $\sum_{i=0}^{f} m_i + \sum_{i=0}^{s} m_i = N$. Such a set satisfies $\mathbb{C} \Delta = \Delta$. Furthermore, based on a slight generalization of [Packard and Doyle, 1993, Lemma 6.3)] to non-square block diagonal perturbations, the search space of the optimization in the right hand side of (A.5) can be restricted. This results in

$$\mu(\Delta)(G) = \max_{U \in \mathcal{U}} r_\sigma(GU),$$

where $\mathcal{U} \subseteq \Delta$ is defined as

$$\mathcal{U} := \left\{ \text{diag}(U_0, \ldots, U_f, u_0I_{m_0}, \ldots, u_sI_{m_s}) : \right.$$  
$$\left. U_i \in \mathbb{C}^{k_i \times l_i}, u_j \in \mathbb{C}, \sigma_k(U_i) = 1, \right.$$  
$$1 \leq k \leq \min(k_i, l_i), |u_j| = 1, \right.$$  
$$0 \leq i \leq f, 0 \leq j \leq s \right\}.$$  

Note that the elements of $\mathcal{U}$ are \textit{unitary} matrices if the uncertainty structure only involves square blocks, that is, $k_i = l_i, i = 1, \ldots, f$.

Next, the following invariance property can easily be checked:

$$\mu(\Delta)(G) = \min_{(D_1, D_2) \in \mathcal{D}} \sigma_1(D_2GD_1^{-1}), \quad (A.8)$$

where

$$\mathcal{D} := \left\{ (D_1, D_2) : \right.$$  
$$\left. D_1 = \text{diag}(a_0I_{l_0}, \ldots, a_fI_{l_f}, D_0, \ldots, D_s), \right.$$  
$$D_2 = \text{diag}(a_0I_{l_0}, \ldots, a_fI_{l_f}, D_0, \ldots, D_s) : \right.$$  
$$a_i > 0, D_i \in \mathbb{C}^{m_i \times m_i}, D_i^* = D_i > 0 \right\}. \quad (A.7)$$

From (A.7) and the combination of (A.8) and (A.4) we finally obtain

$$\max_{U \in \mathcal{U}} r_\sigma(GU) = \mu(\Delta)(G) \leq \min_{(D_1, D_2) \in \mathcal{D}} \sigma_1(D_2GD_1^{-1}).$$

(A.9)

Therefore, optimization procedures are typically used to compute estimates for $\mu(\Delta)(G)$. The function $U \in \mathcal{U} \rightarrow r_\sigma(GU)$ may have several local maxima and, for this, a local search for a maximum is not guaranteed to lead to $\mu(\Delta)(G)$, but to lower bounds. An appropriate formulation of the optimality condition enables algorithms which resemble power algorithms for computing eigenvalues and singular values, see Ref. [Packard et al. (1988)] for an example. Although the convergence of such algorithms to $\mu(\Delta)(G)$ is not guaranteed either and they may converge to values corresponding to lower bounds on $\mu(\Delta)(G)$, they have proven their effectiveness in practise. The computation of the upper-bound in (A.9) can be recasted into a standard convex optimization problem. However, in general $\mu(\Delta)(G)$ is not equal to the upper-bound. An exception to this holds if the number of blocks in the matrices belonging to the uncertainty set $\Delta$ satisfies $f + 2s \leq 3$ and all the blocks are square, $k_i = l_i, i = 0, \ldots, f$. 