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Nonlinear predictable representation and \mathbb{L}^1 -solutions of second-order backward SDEs

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Abstract

The theory of backward SDEs extends the predictable representation property of Brownian motion to the nonlinear framework, thus providing a path-dependent analog of fully nonlinear parabolic PDEs. In this paper, we consider backward SDEs, their reflected version, and their second-order extension, in the context where the final data and the generator satisfy \mathbb{L}^1 -type of integrability condition. Our main objective is to provide the corresponding existence and uniqueness results for general Lipschitz generators. The uniqueness holds in the so-called Doob class of processes, simultaneously under an appropriate class of measures. We emphasize that the previous literature only deals with backward SDEs, and requires either that the generator is separable in (y, z) , see Peng [Pen97], or strictly sublinear in the gradient variable z , see [BDH⁺03], or that the final data satisfies an $L \ln L$ -integrability condition, see [HT18]. We by-pass these conditions by defining \mathbb{L}^1 -integrability under the nonlinear expectation operator induced by the previously mentioned class of measures.

MSC 2010 Subject Classification: 60H10

Key words: Backward SDE, second-order backward SDE, nonlinear expectation, nondominated probability measures.

1 Introduction

Backward stochastic differential equations extend the martingale representation theorem to the nonlinear setting. It is well-known that the martingale representation theorem is the path-dependent counterpart of the heat equation. Similarly, it has been proved in the seminal paper of Pardoux and Peng [PP90] that backward SDEs provide a path-dependent substitute to semilinear PDEs. Finally, the path-dependent counterpart of parabolic fully nonlinear parabolic PDEs was obtained by Soner, Touzi & Zhang [STZ12] and later by Hu, Ji, Peng & Song [HJPS14a, HJPS14b]. The standard case of a Lipschitz nonlinearity (or generator), has been studied extensively in the literature, the solution is defined on an appropriate \mathbb{L}^p -space for some $p > 1$, and wellposedness is guaranteed whenever the final data and the generator are \mathbb{L}^p -integrable.

In this paper, our interest is on the limiting \mathbb{L}^1 -case. It is well-known that the martingale representation, which is first proved for square integrable random variables, holds also in \mathbb{L}^1 -

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by a density argument. This is closely related to the connexion with the conditional expectation operator.

The first attempt for an \mathbb{L}^1 -theory of backward SDEs is by Peng [Pen97] in the context of a separable nonlinearity $f_1(t, y) + f_2(t, z)$, Lipschitz in (y, z) , with $f_1(t, 0) = 0$, $f_2(t, 0) \geq 0$, and final data $\xi \geq 0$. The wellposedness result of this paper is specific to the scar case, and follows the lines of the extension of the expectation operator to \mathbb{L}^1 .

Afterwards, Briand, Delyon, Hu, Pardoux & Stoica [BDH⁺03] consider the case of multi-dimensional backward SDEs, and obtain a wellposedness result in \mathbb{L}^1 by using a truncation technique leading to a Cauchy sequence. This approach is extended by Rozkosz & Słomiński [RS12] and Klimsiak [Kli12] to the context of reflected backward SDEs. However, the main result of these papers requires the nonlinearity to be strictly sublinear in the gradient variable. In particular, this does not cover the linear case, whose unique solution is immediately obtained by a change of measure. More generally, the last restriction excludes the nonlinearities generated by stochastic control problems (with uncontrolled diffusion), which is a substantial field of application of backward SDEs, see El Karoui, Peng & Quenez [EPQ97] and Cvitanić, Possamaï & Touzi [CPT18].

We finally refer to the recent work by Hu and Tang [HT18] who provide an $L \ln L$ -integrability condition which guarantees the wellposedness in \mathbb{L}^1 of the backward SDE for a Lipschitz nonlinearity.

In this paper, we consider an alternative integrability class for the solution of the backward SDE by requiring an \mathbb{L}^1 -integrability under a nonlinear expectation induced by an appropriate family of probability measures. In the context of a Lipschitz nonlinearity, the first main result of this paper provides wellposedness of the backward SDE for a final condition and a nonlinearity satisfying a uniform integrability type of condition under the same nonlinear expectation. This result is obtained by appropriately adapting the arguments of [BDH⁺03]. Although all of our results are stated in the one-dimensional framework, we emphasize that the arguments used for the last wellposedness results are unchanged in the multi-dimensional context.

We also provide a similar wellposedness result for (scalar) reflected backward SDEs, under the same conditions as for the corresponding backward SDE, with an obstacle process whose positive value satisfies the same type of uniform integrability under nonlinear expectation. This improves the existence and uniqueness results of [RS12, Kli12].

Our third main result is the wellposedness of second order backward SDEs in \mathbb{L}^1 . Here again, the \mathbb{L}^1 -integrability is in the sense of a nonlinear expectation induced by a family of measure. In the present setting, unlike the case of backward SDEs and their reflected version, the family of measures is non-dominated as in Soner, Touzi & Zhang [STZ12] and Possamaï, Tan and Zhou [PTZ18].

The paper is organized as follows. Section 2 introduces the notations used throughout the paper. Our main results are contained in Section 3, with proofs postponed in the rest of the paper. Section 4 contains the proofs related to (reflected) backward SDEs, and Sections 5 and 6 focus on the uniqueness and the existence, respectively, for the second-order backward SDEs.

2 Preliminaries

2.1 Canonical space

For a given fixed maturity $T > 0$ and $d \in \mathbb{N}$, we denote by

$$\Omega := \left\{ \omega \in \mathcal{C}([0, T]; \mathbb{R}^d) : \omega_0 = \mathbf{0} \right\}$$

the canonical space equipped with the norm of uniform convergence $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$ and by X the canonical process. Let \mathcal{M}_1 be the collection of all probability measures on (Ω, \mathcal{F}) , equipped with the topology of weak convergence. Denote by $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ the raw filtration generated by the canonical process X . We denote by $\mathbb{F}^+ := (\mathcal{F}_t^+)_{0 \leq t \leq T}$ the right limit of \mathbb{F} . For each $\mathbb{P} \in \mathcal{M}_1$, we denote by $\mathbb{F}^{+, \mathbb{P}}$ the augmented filtration of \mathbb{F}^+ under \mathbb{P} . The filtration $\mathbb{F}^{+, \mathbb{P}}$ is the coarsest filtration satisfying the usual conditions. Moreover, for $\mathcal{P} \subseteq \mathcal{M}_1$, we introduce the universally completed filtration $\mathbb{F}^U := (\mathcal{F}_t^U)_{0 \leq t \leq T}$, $\mathbb{F}^{\mathcal{P}} := (\mathcal{F}_t^{\mathcal{P}})_{0 \leq t \leq T}$, and $\mathbb{F}^{+, \mathcal{P}} := (\mathcal{F}_t^{+, \mathcal{P}})_{0 \leq t \leq T}$, defined as follows

$$\mathcal{F}_t^U := \bigcap_{\mathbb{P} \in \mathcal{M}_1} \mathcal{F}_t^{\mathbb{P}}, \quad \mathcal{F}_t^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{F}_t^{\mathbb{P}}, \quad \mathcal{F}_t^{+, \mathcal{P}} := \mathcal{F}_{t+}^{\mathcal{P}}, \quad t \in [0, T), \quad \text{and} \quad \mathcal{F}_T^{+, \mathcal{P}} := \mathcal{F}_T^{\mathcal{P}}.$$

For any family $\mathcal{P} \subseteq \mathcal{M}_1$, we say that a property holds \mathcal{P} -quasi-surely, abbreviated as \mathcal{P} -q.s., if it holds \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}$.

Finally, for $0 \leq s \leq t \leq T$, we denote by $\mathcal{T}_{s,t}$ the collection of all $[s, t]$ -valued \mathbb{F} -stopping times.

2.2 Local martingale measures

We denote by $\mathcal{P}_{loc} \subseteq \mathcal{M}_1$ the collection of probability measures such that for each $\mathbb{P} \in \mathcal{P}_{loc}$ the canonical process X is a continuous \mathbb{P} -local martingale whose quadratic variation $\langle X \rangle$ is absolutely continuous in t with respect to the Lebesgue measure. Due to the continuity, X is an \mathbb{F} -local martingale under \mathbb{P} implies that X is an $\mathbb{F}^{+, \mathbb{P}}$ -local martingale.

As in [Kar95], we can define pathwisely a version of a $d \times d$ -matrix-valued process $\langle X \rangle$. The constructed process is \mathbb{F} -progressively measurable and coincides with the cross-variation of X under all $\mathbb{P} \in \mathcal{P}_{loc}$. We may introduce

$$\widehat{a}_t := \limsup_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{t-\varepsilon}}{\varepsilon}, \quad \text{so that} \quad \langle X \rangle_t = \int_0^t \widehat{a}_s ds, \quad t \in [0, T], \quad \mathbb{P} - a.s., \quad \text{for all } \mathbb{P} \in \mathcal{P}_{loc}.$$

Note that $\widehat{a}_t \in \mathbb{S}_{\geq 0}^d$ (the set of $d \times d$ symmetric nonnegative-definite matrices). Therefore, we may define a measurable square root $\widehat{\sigma}_t := \widehat{a}_t^{\frac{1}{2}}$. Define

$$\mathcal{P}_b := \{ \mathbb{P} \in \mathcal{P}_{loc} \mid \widehat{\sigma} \text{ is bounded, } dt \otimes \mathbb{P}(d\omega) - a.e. \}.$$

By [NvH13, Lemma 4.5], $\mathcal{P}_b \in \mathcal{B}(\mathcal{M}_1)$.

2.3 Spaces and norms

(i) *One-measure integrability classes:* For a probability measure $\mathbb{P} \in \mathcal{M}_1$ and $p > 0$, we denote:

- $\mathbb{L}^p(\mathbb{P})$ is the space of \mathbb{R} -valued and $\mathcal{F}_T^{+, \mathbb{P}}$ -measurable random variables ξ , such that

$$\|\xi\|_{\mathbb{L}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}} [|\xi|^p]^{1 \wedge \frac{1}{p}} < \infty.$$

- $\mathbb{S}^p(\mathbb{P})$ is the space of \mathbb{R} -valued, $\mathbb{F}^{+, \mathbb{P}}$ -adapted processes Y with càdlàg paths, such that

$$\|Y\|_{\mathbb{S}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^p \right]^{1 \wedge \frac{1}{p}} < \infty.$$

- $\mathbb{H}^p(\mathbb{P})$ is the space of \mathbb{R}^d -valued, $\mathbb{F}^{+, \mathbb{P}}$ -progressively measurable processes Z such that

$$\|Z\|_{\mathbb{H}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\widehat{\sigma}_s^{\mathbf{T}} Z_s|^2 ds \right)^{\frac{p}{2}} \right]^{1 \wedge \frac{1}{p}} < \infty.$$

- $\mathbb{N}^p(\mathbb{P})$ is the space of \mathbb{R} -valued, $\mathbb{F}^{+, \mathbb{P}}$ -adapted local martingales N such that

$$\|N\|_{\mathbb{N}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}} \left[[N]_T^{\frac{p}{2}} \right]^{1 \wedge \frac{1}{p}} < \infty.$$

- $\mathbb{I}^p(\mathbb{P})$ is the set of \mathbb{R} -valued, $\mathbb{F}^{+, \mathbb{P}}$ -predictable processes K of bounded variation with càdlàg nondecreasing paths, such that

$$\|K\|_{\mathbb{I}^p(\mathbb{P})} := \mathbb{E}^{\mathbb{P}} [K_T^p]^{1 \wedge \frac{1}{p}} < \infty.$$

The spaces above are Banach spaces for $p \geq 1$ and complete metric spaces if $p \in (0, 1)$. A process Y belongs to class $\mathbb{D}(\mathbb{P})$ if the family $\{Y_\tau, \tau \in \mathcal{T}_{0,T}\}$ is uniformly integrable under \mathbb{P} . Here, we denote by $\mathcal{T}_{0,T}$ the set of all $[0, T]$ -valued stopping times. We define the norm

$$\|Y\|_{\mathbb{D}(\mathbb{P})} := \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^{\mathbb{P}} [|Y_\tau|].$$

The space of progressive measurable càdlàg processes which belong to class $\mathbb{D}(\mathbb{P})$ is complete under this norm. See Theorem [DM82, VI Theorem 22, Page 83].

(ii) Integrability classes under dominated nonlinear expectation: Let us enlarge the canonical space to $\overline{\Omega} = \Omega \times \Omega$ and denote by (X, W) the coordinate process on $\overline{\Omega}$. Denote by $\overline{\mathbb{F}}$ the filtration generated by (X, W) . For each $\mathbb{P} \in \mathcal{P}_b$, we may construct a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$ such that $\overline{\mathbb{P}} \circ X^{-1} = \mathbb{P}$, W is a $\overline{\mathbb{P}}$ -Brownian motion and $dX_t = \widehat{\sigma}_t dW_t$, $\overline{\mathbb{P}}$ -a.s. By abuse of notation, we keep using \mathbb{P} to represent $\overline{\mathbb{P}}$ on Ω . Denote by $\mathcal{Q}_L(\mathbb{P})$ the set of all probability measures \mathbb{Q}^λ such that

$$\frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} \Big|_{\overline{\mathcal{F}}_t} = G_t^\lambda := \exp \left\{ \int_0^t \lambda_s \cdot dW_s - \frac{1}{2} \int_0^t |\lambda_s|^2 ds \right\}, \quad t \in [0, T],$$

for some $\mathbb{F}^{+, \mathbb{P}}$ -progressively measurable process $(\lambda_t)_{0 \leq t \leq T}$ bounded uniformly by L . It is straightforward to check that the set $\mathcal{Q}_L(\mathbb{P})$ is stable under concatenation, i.e., for $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{Q}_L(\mathbb{P})$, $\tau \in \mathcal{T}_{0,T}$, we have $\mathbb{Q}_1 \otimes_\tau \mathbb{Q}_2 \in \mathcal{Q}_L(\mathbb{P})$, where

$$\mathbb{Q}_1 \otimes_\tau \mathbb{Q}_2(A) := \mathbb{E}^{\mathbb{Q}_1} [\mathbb{E}^{\mathbb{Q}_2} [\mathbf{1}_A | \mathcal{F}_\tau]], \quad A \in \mathcal{F}_T.$$

It is clear from Girsanov's Theorem that under a measure $\mathbb{Q}^\lambda \in \mathcal{Q}_L(\mathbb{P})$, the process $W_t^\lambda := W_t - \int_0^t \lambda_s ds$ is a Brownian motion under \mathbb{Q}^λ . Thus, $X_t^\lambda := X_t - \int_0^t \widehat{\sigma}_t \lambda_t dt$ is a \mathbb{Q}^λ -martingale. Given a $\mathbb{P} \in \mathcal{P}_b$, we denote

$$\mathcal{E}^{\mathbb{P}}[X] := \sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}[X],$$

and introduce the space $\mathcal{L}^p(\mathbb{P}) \subseteq \bigcap_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{L}^p(\mathbb{Q})$ of random variables ξ such that

$$\|\xi\|_{\mathcal{L}^p(\mathbb{P})} := \mathcal{E}^{\mathbb{P}} [|\xi|^p]^{1 \wedge \frac{1}{p}} < \infty.$$

We define similarly the subspaces $\mathcal{S}^p(\mathbb{P})$, $\mathcal{H}^p(\mathbb{P})$, $\mathcal{N}^p(\mathbb{P})$, $\mathcal{K}^p(\mathbb{P})$ and the subsets $\mathcal{I}^p(\mathbb{P})$.

A process Y belongs to $\mathcal{D}(\mathbb{P})$ if Y is progressive measurable and càdlàg, and the family $\{Y_\tau, \tau \in \mathcal{T}_{0,T}\}$ is uniformly integrable under $\mathcal{Q}_L(\mathbb{P})$, i.e., $\lim_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}} [|Y_\tau| \mathbf{1}_{\{|Y_\tau| \geq N\}}] = 0$. We define the norm

$$\|Y\|_{\mathcal{D}(\mathbb{P})} := \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}} [|Y_\tau|].$$

Note that $\|Y\|_{\mathcal{D}(\mathbb{P})} < \infty$ does not imply $Y \in \mathcal{D}(\mathbb{P})$. However, the space $\mathcal{D}(\mathbb{P})$ is complete under this norm. See Theorem A.2.

(iii) *Integrability classes under non-dominated nonlinear expectation:* Let $\mathcal{P} \subseteq \mathcal{P}_0$ be a subset of probability measures, and denote

$$\mathcal{E}^{\mathcal{P}}[X] := \sup_{\mathbb{P} \in \mathcal{P}} \mathcal{E}^{\mathbb{P}}[X].$$

Let $\mathbb{G} := \{\mathcal{G}_t\}_{0 \leq t \leq T}$ be a filtration with $\mathcal{G}_t \supseteq \mathcal{F}_t$ for all $0 \leq t \leq T$. We define the subspace $\mathcal{L}^p(\mathcal{P}, \mathbb{G})$ as the collection of all \mathcal{G}_T -measurable \mathbb{R} -valued random variables ξ , such that

$$\|\xi\|_{\mathcal{L}^p(\mathcal{P})} := \mathcal{E}^{\mathcal{P}} [|\xi|^p]^{1/p} < \infty.$$

We define similarly the subspaces $\mathcal{S}^p(\mathcal{P}, \mathbb{G})$ and $\mathcal{H}^p(\mathcal{P}, \mathbb{G})$ by replacing $\mathbb{F}^{+, \mathbb{P}}$ with \mathbb{G} . Similarly, we denote by $\mathcal{D}(\mathcal{P}, \mathbb{G})$ the space of \mathbb{R} -valued, \mathbb{G} -adapted processes Y with càdlàg paths, such that $\lim_{N \rightarrow \infty} \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}} [|Y_\tau| \mathbf{1}_{\{|Y_\tau| \geq N\}}] = 0$.

3 Main results

Throughout this paper, we fix a finite time horizon $0 < T < \infty$. Let ξ be an $\mathcal{F}_T^{+, \mathcal{P}_b}$ -measurable random variable, and $F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$, a $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{S}^d)$ -measurable map,¹ called generator, and denote

$$f_t(\omega, y, z) := F_t(\omega, y, z, \widehat{\sigma}_t(\omega)), \quad (t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d.$$

By freezing the pair (y, z) to 0, we set $f_t^0 = f_t(0, 0)$.

Assumption 3.1. *The coefficient F is uniformly Lipschitz in (y, z) in the following sense: there exist constants $L_y, L_z \geq 0$, such that for all $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$ and $\sigma \in \mathbb{S}^d$,*

$$|F_s(y_1, z_1, \sigma) - F_s(y_2, z_2, \sigma)| \leq L_y |y_1 - y_2| + L_z |\sigma^{\mathbf{T}}(z_1 - z_2)|, \quad ds \otimes d\mathbb{P} - a.e.$$

Remark 3.2. Without loss of generality, we may assume that F is nonincreasing in y . Indeed, we may always reduce to this context by using the standard change of variable $(\widetilde{Y}_t, \widetilde{Z}_t) := e^{at}(Y_t, Z_t)$ for sufficiently large a .

3.1 \mathbb{L}^1 -solution of backward SDE

For a probability measure $\mathbb{P} \in \mathcal{P}_b$, consider the following backward stochastic differential equation (BSDE):

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - Z_s \cdot dX_s - dN_s, \quad t \in [0, T], \quad \mathbb{P} - a.s. \quad (3.1)$$

¹We denote by Prog the σ -algebra generated by progressively measurable processes.

Here, Y is a càdlàg process adapted \mathbb{R} -valued process, Z is a predictable \mathbb{R}^d -valued process, and N a càdlàg \mathbb{R} -valued local martingale with $N_0 = 0$ orthogonal to X , i.e., $[X, N] = 0$. Recall that $dX_s = \widehat{\sigma}_s dW_s$, \mathbb{P} -a.s. for some \mathbb{P} -Brownian motion W .

We shall use the Lipschitz constant L_z of Assumption 3.1 as the bound of the coefficients of the Girsanov transformations introduced in Section 2.3 (ii). In particular, we denote

$$\mathcal{E}^{\mathbb{P}}[X] := \sup_{\mathbb{Q} \in \mathcal{Q}_{L_z}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}[X].$$

Assumption 3.3. $\lim_{n \rightarrow \infty} \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right] = 0$.

Theorem 3.4. *Let Assumptions 3.1 and 3.3 hold true. Then, the BSDE (3.1) has a unique solution $(Y, Z, N) \in (\mathcal{S}^{\beta}(\mathbb{P}) \cap \mathcal{D}(\mathbb{P})) \times \mathcal{H}^{\beta}(\mathbb{P}) \times \mathcal{N}^{\beta}(\mathbb{P})$ for all $\beta \in (0, 1)$, with*

$$\|Y\|_{\mathcal{D}(\mathbb{P})} \leq \mathcal{E}^{\mathbb{P}} \left[|\xi| + \int_0^T |f_s^0| ds \right], \quad (3.2)$$

$$\|Y\|_{\mathcal{S}^{\beta}(\mathbb{P})} + \|Z\|_{\mathcal{H}^{\beta}(\mathbb{P})} + \|N\|_{\mathcal{N}^{\beta}(\mathbb{P})} \leq C_{\beta, L, T} \left(\mathcal{E}^{\mathbb{P}}[|\xi|]^{\beta} + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| ds \right]^{\beta} \right). \quad (3.3)$$

for some constant $C_{\beta, L, T}$.

We also have the following comparison and stability results, which are direct consequences of Theorem 3.7 and the estimates (3.2)-(3.3) of Theorem 3.4.

Theorem 3.5. *Let (f, ξ) and (f', ξ') satisfy the assumptions of Theorem 3.4, and (Y, Z, N) and (Y', Z', N') be the corresponding solutions.*

- (i) *Stability: Denoting $\delta Y := Y' - Y$, $\delta Z := Z' - Z$, $\delta N := N' - N$ and $\delta \xi := \xi' - \xi$, $\delta f_t(y, z) := f'_t(y, z) - f_t(y, z)$, we have for all $\beta \in (0, 1)$, and some constant $C_{\beta, L, T}$:*

$$\begin{aligned} \|\delta Y\|_{\mathcal{D}(\mathbb{P})} &\leq \mathcal{E}^{\mathbb{P}} \left[|\delta \xi| + \int_0^T |\delta f_s(Y_s, Z_s)| ds \right], \\ \|\delta Y\|_{\mathcal{S}^{\beta}(\mathbb{P})} + \|\delta Z\|_{\mathcal{H}^{\beta}(\mathbb{P})} + \|\delta N\|_{\mathcal{N}^{\beta}(\mathbb{P})} &\leq C_{\beta, L, T} \left(\mathcal{E}^{\mathbb{P}}[|\delta \xi|]^{\beta} + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |\delta f_s(Y_s, Z_s)| ds \right]^{\beta} \right). \end{aligned}$$

- (ii) *Comparison: Suppose that $\xi \leq \xi'$, \mathbb{P} -a.s., and $f(y, z) \leq f'(y, z)$, $dt \otimes \mathbb{P}$ -a.e., for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. Then, $Y_{\tau} \leq Y'_{\tau}$, \mathbb{P} -a.s., for all $\tau \in \mathcal{T}_{0, T}$.*

3.2 \mathbb{L}^1 -solution of reflected backward SDE

Consider the following reflected backward stochastic differential equation (RBSDE)

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - Z_s \cdot dX_s - dU_s, \quad t \in [0, T], \quad \mathbb{P} - a.s. \quad (3.4)$$

where Z is a predictable \mathbb{R}^d -valued process, U is a local supermartingale orthogonal to X , i.e., $[X, U] = 0$, starting from $U_0 = 0$, and Y is a scalar càdlàg process satisfying the following Skorokhod condition with càdlàg obstacle $(S_t)_{0 \leq t \leq T}$:

$$Y_t \geq S_t, \quad t \in [0, T], \quad \text{and} \quad \int_0^T (Y_{t-} - S_{t-}) dK_t = 0, \quad \mathbb{P} - a.s. \quad \text{where} \quad U = N - K$$

is the Doob-Meyer decomposition of U into a local martingale N and a nondecreasing process K starting from $N_0 = K_0 = 0$.

Our second wellposedness result is the following.

Theorem 3.6. *Let Assumptions 3.1 and 3.3 hold true. Assume that $S^+ \in \mathcal{D}(\mathbb{P})$. Then, the RBSDE (3.4) has a unique solution $(Y, Z, N, K) \in (\mathcal{S}^\beta(\mathbb{P}) \cap \mathcal{D}(\mathbb{P})) \times \mathcal{H}^\beta(\mathbb{P}) \times \mathcal{N}^\beta(\mathbb{P}) \times \mathcal{I}^\beta(\mathbb{P})$ for all $\beta \in (0, 1)$.*

We also have the following stability and comparison results.

Theorem 3.7. *Let (f, ξ, S) and (f', ξ', S') satisfy the assumptions of Theorem 3.6 with corresponding solutions (Y, Z, N, K) and (Y', Z', N', K') .*

(i) Stability: with $\delta Y := Y' - Y$, $\delta Z := Z' - Z$, $\delta U := U' - U$, $\delta \xi := \xi' - \xi$, $\delta f_t := f'_t - f_t$, we have

$$\|\delta Y\|_{\mathcal{D}(\mathbb{P})} \leq \mathcal{E}^\mathbb{P} \left[|\delta \xi| + \int_0^T |\delta f_s(\Theta_s)| ds \right], \quad \Theta_s := (Y_s, Z_s),$$

and for all $\beta \in (0, 1)$, there exists a constant $C = C_{\beta, L, T}$ such that

$$\|\delta Y\|_{\mathcal{S}^\beta(\mathbb{P})} + \|\delta Z\|_{\mathcal{H}^\beta(\mathbb{P})} + \|\delta U\|_{\mathcal{N}^\beta(\mathbb{P})} \leq C \left\{ \Delta_\xi^\beta + \Delta_f^\beta + (\Delta_\xi^{\frac{\beta}{2}} + \Delta_f^{\frac{\beta}{2}}) (C^Y + C^{Y'})^{\frac{1}{2}} \right\},$$

with $\Delta_\xi := \mathcal{E}^\mathbb{P} [|\delta \xi|]$, $\Delta_f := \mathcal{E}^\mathbb{P} \left[\int_0^T |\delta f_s(\Theta_s)| ds \right]$, $C^Y := \|Y\|_{\mathcal{S}^\beta(\mathbb{P})} + \|Y\|_{\mathcal{D}(\mathbb{P})}^\beta + \mathcal{E}^\mathbb{P} \left[\int_0^T |f_s^0| ds \right]^\beta$, and $C^{Y'}$ defined similarly.

(ii) Comparison: Suppose that $\xi \leq \xi'$, \mathbb{P} -a.s.; $f(y, z) \leq f'(y, z)$, $dt \otimes \mathbb{P}$ -a.e., for all $y, z \in \mathbb{R} \times \mathbb{R}^d$; and $S \leq S'$, $dt \otimes \mathbb{P}$ -a.e. Then, $Y_\tau \leq Y'_\tau$, for all $\tau \in \mathcal{T}_{0, T}$.

3.3 \mathbb{L}^1 -solution of second-order backward SDE

Following Soner, Touzi & Zhang [STZ12], we introduce second-order backward SDE as a family of backward SDEs defined on the supports of a convenient family of singular probability measures. We introduce the subset of \mathcal{P}_b :

$$\mathcal{P}_0 := \{ \mathbb{P} \in \mathcal{P}_b : f_t^0(\omega) < \infty, \text{ for } \text{Leb} \otimes \mathbb{P}\text{-a.e. } (t, \omega) \in [0, T] \times \Omega \}.$$

We also define for all stopping times $\tau \in \mathcal{T}_{0, T}$:

$$\mathcal{P}(\tau, \mathbb{P}) := \{ \mathbb{P}' \in \mathcal{P}_0 : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_\tau \} \quad \text{and} \quad \mathcal{P}_+(\tau, \mathbb{P}) := \bigcup_{h>0} \mathcal{P}((\tau + h) \wedge T, \mathbb{P}).$$

Our general 2BSDE takes the following form:

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - Z_s \cdot dX_s - dU_s, \quad \mathcal{P}_0 - q.s. \quad (3.5)$$

for some local supermartingale U satisfying with $[X, U] = 0$ and together with an appropriate minimality condition. A property is said to hold \mathcal{P}_0 -quasi surely, abbreviated as \mathcal{P}_0 -q.s., if it holds \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}_0$.

Definition 3.8. *For $\beta \in (0, 1)$, the process $(Y, Z) \in \mathcal{D}(\mathcal{P}_0, \mathbb{F}^{+, \mathcal{P}_0}) \times \mathcal{H}^\beta(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0})$ is a supersolution of the 2BSDE (3.5), if for all $\mathbb{P} \in \mathcal{P}_0$, the process*

$$U_t^\mathbb{P} := Y_t - Y_0 + \int_0^t F_s(Y_s, Z_s, \hat{\sigma}_s) ds - Z_s \cdot dX_s, \quad t \in [0, T], \quad \mathbb{P} - a.s.$$

is a \mathbb{P} -supermartingale, with $U_0^\mathbb{P} = 0$, $[X, U^\mathbb{P}] = 0$, \mathbb{P} -a.s. and corresponding Doob-Meyer decomposition $U^\mathbb{P} = N^\mathbb{P} - K^\mathbb{P}$ into a \mathbb{P} -local martingale $N^\mathbb{P} \in \mathcal{N}^\beta(\mathbb{P})$ and a \mathbb{P} -a.s. nondecreasing process $K^\mathbb{P} \in \mathcal{I}^\beta(\mathbb{P})$ starting from the origin $N_0^\mathbb{P} = K_0^\mathbb{P} = 0$.

The dependence of the supermartingale $U^{\mathbb{P}}$ on \mathbb{P} is inherited from the dependence of the stochastic integral $Z \bullet X := \int_0^\cdot Z_s \cdot dX_s$ on the underlying semimartingale measure \mathbb{P} .² Because of this the 2BSDE representation (3.5) should be rather written under each $\mathbb{P} \in \mathcal{P}_0$ as:

$$Y_t = \xi + \int_t^T F_s(Y_s, Z_s, \widehat{\sigma}_s) ds - Z_s \cdot dX_s - dN_s^{\mathbb{P}} + dK_s^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \quad (3.6)$$

We next introduce the notations of the shifted variables:

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'), \quad f_s^{0,t,\omega}(\omega') := F_{t+s}(\omega \otimes_t \omega', 0, 0, \widehat{\sigma}_s(\omega')),$$

which involve the paths concatenation operator $(\omega \otimes_t \omega')_s := \mathbf{1}_{\{s \leq t\}} \omega_s + \mathbf{1}_{\{s > t\}}(\omega_t + \omega'_{s-t})$. Define

$$\mathcal{P}(t, \omega) := \{ \mathbb{P} \in \mathcal{P}_b : f_s^{0,t,\omega}(\omega') < \infty, \text{ for Leb} \otimes \mathbb{P} - \text{a.e. } (s, \omega') \in \mathbb{R}_+ \times \Omega \},$$

so that $\mathcal{P}_0 = \mathcal{P}(0, \mathbf{0})$, in particular.

Assumption 3.9. *The terminal condition ξ and the generator F satisfy the integrability:*

$$\lim_{n \rightarrow \infty} \left\| \left| \xi^{t,\omega} \mathbf{1}_{\{|\xi^{t,\omega}| \geq n\}} + \int_0^{T-t} |f_s^{0,t,\omega}| \mathbf{1}_{\{|f_s^{0,t,\omega}| \geq n\}} ds \right\|_{\mathcal{L}^1(\mathcal{P}(t,\omega))} = 0 \quad \text{for all } (t, \omega) \in [0, T] \times \Omega.$$

For all $\mathbb{P} \in \mathcal{P}_0$, we denote by $(\mathcal{Y}^{\mathbb{P}}, \mathcal{Z}^{\mathbb{P}}, \mathcal{N}^{\mathbb{P}})$ the unique solution of the backward SDE (3.1). By (H1), there exist two random fields $a^{\mathbb{P}}(y, z)$ and $b^{\mathbb{P}}(y, z)$ bounded by L such that

$$f_s(y, z) - f_s(\mathcal{Y}_s^{\mathbb{P}}, \mathcal{Z}_s^{\mathbb{P}}) = a_s^{\mathbb{P}}(y - \mathcal{Y}_s^{\mathbb{P}}) + b_s^{\mathbb{P}} \cdot \widehat{\sigma}_s(z - \mathcal{Z}_s^{\mathbb{P}}).$$

We now introduce our notion of second order backward SDE by means of a minimality condition involving the last function $b^{\mathbb{P}}$.

Definition 3.10. *For $\beta \in (0, 1)$, the process $(Y, Z) \in \mathcal{D}(\mathcal{P}_0, \mathbb{F}^{+, \mathcal{P}_0}) \times \mathcal{H}^\beta(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0})$ is a solution to 2BSDE (3.5) if it is a supersolution in the sense of Definition 3.8, and it satisfies the minimality condition:*

$$K_\tau^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} \left[K_T^{\mathbb{P}'} \middle| \mathcal{F}_\tau^{+, \mathbb{P}'} \right], \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}_0, \quad \tau \in \mathcal{T}_{0,T}, \quad (3.7)$$

where $\mathbb{Q}_\tau^{\mathbb{P}'} \in \mathcal{Q}_{L_z}(\mathbb{P}')$ is defined by the density $\frac{d\mathbb{Q}_\tau^{\mathbb{P}'}}{d\mathbb{P}'} := \frac{G_T^{b^{\mathbb{P}'}}(Y, Z)}{G_\tau^{b^{\mathbb{P}'}}(Y, Z)}$.

Note that $\mathbb{Q}_\tau^{\mathbb{P}'}|_{\mathcal{F}_\tau^+} = \mathbb{P}'|_{\mathcal{F}_\tau^+} = \mathbb{P}|_{\mathcal{F}_\tau^+}$ and the process $W_s - \int_\tau^s b_s^{\mathbb{P}'} ds$ is a Brownian motion starting from W_τ .

Theorem 3.11. *Under Assumptions 3.1 and 3.9, the 2BSDE (3.5) has a unique solution $(Y, Z) \in \mathcal{D}(\mathcal{P}_0, \mathbb{F}^{+, \mathcal{P}_0}) \times \mathcal{H}^\beta(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0})$, for all $\beta \in (0, 1)$.*

Moreover, if \mathcal{P}_0 is saturated³, then $N^{\mathbb{P}} = 0$ for all $\mathbb{P} \in \mathcal{P}_0$.

Similar to Soner, Touzi & Zhang [STZ12], the following comparison result for second order backward SDEs is a by-product of our construction; the proof is provided in Theorem 5.1.

Proposition 3.12. *Let (Y, Z) and (Y', Z') be solutions of 2BSDEs with parameters (F, ξ) and (F', ξ') , respectively, which satisfy Assumptions 3.1 and 3.9. Suppose further that $\xi \leq \xi'$ and $F_t(y, z, \widehat{\sigma}_t) \leq F'_t(y, z, \widehat{\sigma}_t)$ for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $dt \otimes \mathcal{P}_0$ -q.s. Then, we have $Y \leq Y'$, $dt \otimes \mathcal{P}_0$ -q.s.*

²By Theorem 2.2 in Nutz [Nut12], the family $\{(Z \bullet X)^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_0}$ can be aggregated as a medial limit $(Z \bullet X)$ under the acceptance of Zermelo-Fraenkel set theory with axiom of choice together with the continuum hypothesis into our framework. In this case, $(Z \bullet X)$ can be chosen as an $\mathbb{F}^{+, \mathcal{P}_0}$ -adapted process, and the family $\{U^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_0}$ can be aggregated into the resulting medial limit U , i.e., $U = U^{\mathbb{P}}$, \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}_0$.

³We say that the family \mathcal{P}_0 is saturated if, for all $\mathbb{P} \in \mathcal{P}_0$, we have $\mathbb{Q} \in \mathcal{P}_0$ for every probability measure $\mathbb{Q} \sim \mathbb{P}$ on (Ω, \mathcal{F}) such that X is \mathbb{Q} -local martingale. The assertion follows by the same argument as in [PTZ18, Theorem 5.1].

4 Wellposedness of reflected BSDEs

Throughout this section, we fix a probability measure $\mathbb{P} \in \mathcal{P}_b$, and we omit the dependence on \mathbb{P} in all of our notations (e.g. $\mathcal{D}(\mathbb{P})$ denoted as \mathcal{D}). It is clear from Girsanov's Theorem that under a measure $\mathbb{Q}^\lambda \in \mathcal{Q}_{L_z}$, the process $W^\lambda := W - \int_0^\cdot \lambda_s ds$ is a Brownian motion under \mathbb{Q}^λ .

Remark 4.1. We note that under a measure $\mathbb{Q}^\lambda \in \mathcal{Q}_{L_z}$ defined as above, the RBSDE satisfies

$$dY_t = -(f_t(Y_t, Z_t) - \widehat{\sigma}_t^\mathbf{T} Z_t \cdot \lambda_t) dt + Z_t \cdot dX_t^\lambda + dN_t - dK_t,$$

where the process $X_t^\lambda := X_t - \int_0^t \widehat{\sigma}_s \lambda_s ds$ is a local martingale under \mathbb{Q}^λ , and the generator $f_t(y, z) - \widehat{\sigma}_t^\mathbf{T} z \cdot \lambda_t$ satisfies the Assumption 3.1 with Lipschitz coefficients L_y and $2L_z$.

4.1 Some useful inequalities

First of all, we provide an estimation of a running supremum process.

Lemma 4.2. *Let X be a nonnegative càdlàg process, and $X_t^* := \max_{s \leq t} X_s$. Then,*

$$\mathbb{E}[(X_T^*)^\beta] \leq \frac{1}{1-\beta} \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[X_\tau]^\beta, \quad \text{for all } \beta \in (0, 1).$$

Proof. For $x > 0$, let us define $\tau_x := \inf\{t > 0 \mid X_t \geq x\} \wedge T$. We have that $X_{\tau_x}^* = X_{\tau_x}$, and

$$\mathbb{P}[\tau_x < T] = \mathbb{P}[X_{\tau_x} \geq x] \leq \frac{\mathbb{E}[X_{\tau_x}]}{x} \leq \frac{c}{x},$$

with $c := \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[X_\tau]$, which implies that $\mathbb{P}[\tau_x < T] \leq \frac{c}{x} \wedge 1$. Then, for $\beta \in (0, 1)$

$$\begin{aligned} \mathbb{E}[(X_T^*)^\beta] &= \mathbb{E}\left[\int_0^\infty \mathbf{1}_{\{X_T^* > x\}} \beta x^{\beta-1} dx\right] = \int_0^\infty \mathbb{P}[X_T^* > x] \beta x^{\beta-1} dx \\ &= \int_0^\infty \mathbb{P}[\tau_x < T] \beta x^{\beta-1} dx \leq \int_0^\infty \left(\frac{c}{x} \wedge 1\right) \beta x^{\beta-1} dx = \frac{c^\beta}{1-\beta}. \end{aligned}$$

□

Lemma 4.3. *Let ζ be a nonnegative \mathcal{F}_T -measurable r.v. and Y a nonnegative process such that*

$$\sup_{\tau \in \mathcal{T}_{0,T}} \left\{ Y_\tau - \mathbb{E}^{\widehat{\mathbb{Q}}}[\zeta \mid \mathcal{F}_\tau^+, \mathbb{P}] \right\} \leq 0, \quad \text{for some } \widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}. \quad (4.1)$$

Then, $\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^\mathbb{P}[Y_\tau] \leq \mathcal{E}^\mathbb{P}[\zeta]$.

Proof. Fix $\mathbb{Q} \in \mathcal{Q}_{L_z}$ and $\tau \in \mathcal{T}_{0,T}$. Notice that $\mathbb{Q} \otimes_\tau \widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$ for all $\widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$. Then, it follows from (4.1) that

$$\mathbb{E}^\mathbb{Q}[|Y_\tau|] \leq \mathbb{E}^\mathbb{Q}\left[\mathbb{E}^{\widehat{\mathbb{Q}}}[\zeta \mid \mathcal{F}_\tau]\right] = \mathbb{E}^{\mathbb{Q} \otimes_\tau \widehat{\mathbb{Q}}}[\zeta] \leq \mathcal{E}^\mathbb{P}[\zeta].$$

The required inequality follows by taking supremum over all stopping times and $\mathbb{Q} \in \mathcal{Q}_{L_z}$. □

Now, we show a Doob-type inequality under the nonlinear expectation $\mathcal{E}^\mathbb{P}$, which turns out to be crucial for our analysis.

Lemma 4.4. *Let $(M_t)_{0 \leq t \leq T}$ be a nonnegative submartingale under some $\widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$. Then,*

$$\|M\|_{\mathcal{S}^\beta} \leq \frac{1}{1-\beta} \mathcal{E}^\mathbb{P}[M_T]^\beta \quad \text{for all } 0 < \beta < 1.$$

Proof. Let $x > 0$ and $\mathbb{Q} \in \mathcal{Q}_{L_z}$ be arbitrary. Define

$$\tau_x := \inf \{t \geq 0 \mid M_t > x\} \wedge T,$$

with the usual convention that $\inf \emptyset = \infty$. From the optional sampling theorem, Jensen's inequality and the definition of concatenation, we obtain that

$$\mathbb{E}^{\mathbb{Q}} [M_{\tau_x}] \leq \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\widehat{\mathbb{Q}}} [M_T \mid \mathcal{F}_{\tau_x}^{+, \mathbb{P}}] \right] = \mathbb{E}^{\mathbb{Q} \otimes_{\tau_x} \widehat{\mathbb{Q}}} [M_T],$$

for each $\mathbb{Q} \in \mathcal{Q}_{L_z}$. As $\mathbb{Q} \otimes_{\tau_x} \widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$, this provides that $\mathbb{E}^{\mathbb{Q}} [M_{\tau_x}] \leq \mathcal{E}^{\mathbb{P}} [M_T] =: c$.

Let us denote $M_* := \sup_{0 \leq t \leq T} M_t$. It follows that

$$x \mathbb{Q} [M_* > x] = x \mathbb{Q} [\tau_x < T] \leq \mathbb{E}^{\mathbb{Q}} [M_{\tau_x} \mathbf{1}_{\{\tau_x < T\}}] \leq \mathbb{E}^{\mathbb{Q}} [M_{\tau_x}] \leq c.$$

Then,

$$\mathbb{E}^{\mathbb{Q}} [M_*^\beta] = \int_0^\infty \mathbb{Q} [M_* > x] \beta x^{\beta-1} dx \leq \int_0^\infty \left(1 \wedge \frac{c}{x}\right) \beta x^{\beta-1} dx = \frac{c^\beta}{1-\beta}.$$

As $\mathbb{Q} \in \mathcal{Q}_{L_z}$ is arbitrary, the assertion follows. \square

4.2 A priori estimates for reflected backward SDEs

We will construct a solution of the RBSDE (3.4), using a sequence of \mathbb{L}^2 -solutions to the related RBSDEs. The following a priori estimation is crucial for the existence result.

Proposition 4.5. *Let (Y, Z, N, K) be a solution of RBSDE (3.4). For all $\beta \in (0, 1)$, there exists a constant $C_{\beta, L, T} > 0$ such that*

$$\|Z\|_{\mathcal{H}^\beta} + \|N\|_{\mathcal{N}^\beta} + \|K\|_{\mathcal{S}^\beta} \leq C_{\beta, L, T} \left(\|Y\|_{\mathcal{S}^\beta} + \|Y\|_{\mathcal{D}}^\beta + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| ds \right]^\beta \right).$$

Before proving this result, we establish some more general intermediate estimates.

Lemma 4.6. *Let $\beta \in (0, 1)$. For $(Y, Z, N, K) \in \mathbb{S}^\beta(\mathbb{Q}^\lambda) \times \mathbb{H}^\beta(\mathbb{Q}^\lambda) \times \mathbb{N}^\beta(\mathbb{Q}^\lambda) \times \mathbb{I}^\beta(\mathbb{Q}^\lambda)$, define*

$$M^\lambda := Z \cdot X^\lambda + N - K = \widehat{\sigma}^{\mathbf{T}} Z \cdot W^\lambda + N - K.$$

Then,

$$c_\beta \left(\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right) \leq \|M^\lambda + K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq C_\beta \left(\|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \right), \quad (4.2)$$

$$c_\beta \left(\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right) \leq \|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq C_\beta \left(\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right). \quad (4.3)$$

Proof. As $[X^\lambda, N] = \widehat{\sigma} \cdot [W^\lambda, N] = 0$, we obtain that

$$\begin{aligned} c_\beta \left(\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right) &\leq \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left([Z \cdot X^\lambda]_T + [N]_T \right)^{\frac{\beta}{2}} \right] \\ &= \mathbb{E}^{\mathbb{Q}^\lambda} \left[[Z \cdot X^\lambda + N]_T^{\frac{\beta}{2}} \right] = \|M^\lambda + K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq 2^{\frac{\beta}{2}} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left([M^\lambda]_T + [K]_T \right)^{\frac{\beta}{2}} \right] \\ &\leq C_\beta \left(\mathbb{E}^{\mathbb{Q}^\lambda} \left[[M^\lambda]_T^{\frac{\beta}{2}} \right] + \mathbb{E}^{\mathbb{Q}^\lambda} \left[[K]_T^{\frac{\beta}{2}} \right] \right) \leq C_\beta \left(\|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \right), \end{aligned}$$

where the last inequality is deduced from the fact that K is nondecreasing and of finite variation, together with the following simple calculation

$$[K]_T \leq [K]_T + 2 \int_0^T K_{s-} dK_s = K_T^2.$$

Again by $[X^\lambda, N] = 0$, we have $[M^\lambda] = [Z \cdot X^\lambda + N - K] = [Z \cdot X^\lambda] + [N - K]$. With the similar calculation as above, we obtain (4.3). \square

Proof of Proposition 4.5. Step 1. We first derive the following estimate of K :

$$\|K\|_{\mathcal{S}^\beta} \leq C_{\beta,L,T}^K \left(\|Y\|_{\mathcal{D}}^\beta + \mathcal{E}^\mathbb{P} \left[\int_0^T |f_s^0| ds \right]^\beta \right), \quad (4.4)$$

where $C_{\beta,L,T}^K$ is a positive constant depending on β, L_y, L_z and T . Indeed, it follows from (3.4) and Assumption 3.1 that

$$K_t \leq |Y_0| + |Y_t| + \int_0^t |f_s^0| ds + L_y \int_0^t |Y_s| ds + L_z \int_0^t |\widehat{\sigma}_s^\mathbf{T} Z_s| ds + \int_0^t Z_s \cdot dX_s + N_t.$$

Define

$$\lambda_s := L_z \frac{\widehat{\sigma}_s^\mathbf{T} Z_s}{|\widehat{\sigma}_s^\mathbf{T} Z_s|} \mathbf{1}_{\{|\widehat{\sigma}_s^\mathbf{T} Z_s| \neq 0\}} \quad \text{and} \quad \frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} := \exp \left\{ - \int_0^T \lambda_s \cdot dW_s - \frac{1}{2} \int_0^T |\lambda_s|^2 ds \right\}. \quad (4.5)$$

By a localization argument, we obtain

$$\mathbb{E}^{\mathbb{Q}^\lambda} [K_T] \leq (2 + L_y T) \|Y\|_{\mathbb{D}(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\int_0^T |f_s^0| ds \right] \leq (2 + L_y T) \|Y\|_{\mathcal{D}} + \mathcal{E}^\mathbb{P} \left[\int_0^T |f_s^0| ds \right] < \infty. \quad (4.6)$$

Now take any $\|\lambda'\|_\infty \leq L_z$. By the Girsanov transformation and the Hölder inequality, we have

$$\mathbb{E}^{\mathbb{Q}^{\lambda'}} [K_T^\beta] \leq \exp(3qL_z^2 T) \mathbb{E}^{\mathbb{Q}^\lambda} [K_T]^\beta, \quad \text{with} \quad \frac{1}{q} = 1 - \beta.$$

As λ' is arbitrary, together with (4.6) we obtain (4.4).

Step 2. We next estimate the stochastic integral $\int_0^T Y_{s-} dM_s^\lambda$. Using Burkholder-Davis-Gundy inequality and Young's inequality, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left| \int_0^T Y_{s-} dM_s^\lambda \right|^{\frac{\beta}{2}} \right] &\leq \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq u \leq T} \left| \int_0^u Y_{s-} (Z_s \cdot dX_s^\lambda + dN_s) \right|^{\frac{\beta}{2}} + \left| \int_0^T Y_{s-} dK_s \right|^{\frac{\beta}{2}} \right] \\ &\leq C_\beta \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T Y_{s-}^2 d[Z \cdot X^\lambda + N]_s \right)^{\frac{\beta}{4}} \right] + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq s \leq T} |Y_s|^{\frac{\beta}{2}} K_T^{\frac{\beta}{2}} \right] \\ &\leq C_\beta \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq s \leq T} |Y_s|^{\frac{\beta}{2}} [Z \cdot X^\lambda + N]_T^{\frac{\beta}{4}} \right] + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq s \leq T} |Y_s|^{\frac{\beta}{2}} K_T^{\frac{\beta}{2}} \right] \\ &\leq \frac{C_\beta + 1}{2\varepsilon} \|Y\|_{\mathcal{S}^\beta(\mathbb{Q}^\lambda)} + \frac{C_\beta \varepsilon}{2} \|Z \cdot X^\lambda + N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \frac{\varepsilon}{2} \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \\ &\leq \frac{C_\beta + 1}{2\varepsilon} \|Y\|_{\mathcal{S}^\beta(\mathbb{Q}^\lambda)} + \frac{C_\beta C'_\beta \varepsilon}{2} \|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \frac{(C_\beta C'_\beta + 1)\varepsilon}{2} \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)}, \end{aligned} \quad (4.7)$$

where the last inequality follows from (4.2) with parameter C'_β .

Step 3. We now show that

$$\|Z\|_{\mathcal{H}^\beta} + \|N - K\|_{\mathcal{N}^\beta} \leq C''_{L,T,\beta} \left(\|Y\|_{\mathcal{S}^\beta} + \|Y\|_{\mathcal{D}}^\beta + \mathcal{E}^\mathbb{P} \left[\int_0^T |f_s^0| ds \right]^\beta \right).$$

Applying Itô's formula on Y^2 , we obtain

$$\int_0^T |\widehat{\sigma}_s^\mathbf{T} Z_s|^2 ds + [N - K]_T = \xi^2 - Y_0^2 + \int_0^T 2Y_{s-} (f_s(Y_s, Z_s) - \widehat{\sigma}_s^\mathbf{T} Z_s \cdot \lambda_s) ds - 2Y_{s-} dM_s^\lambda,$$

where λ is defined as in (4.5). Hence, by Assumption 3.1 and Young's inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} Z_s|^2 ds + [N - K]_T \\ & \leq (3 + 2L_y T + 8L_z^2 T) \sup_{0 \leq s \leq T} Y_s^2 + \left(\int_0^T |f_s^0| ds \right)^2 + 2 \left| \int_0^T Y_{s-} dM_s^\lambda \right|. \end{aligned}$$

Together with (4.7) and (4.3), we have for $\beta \in (0, 1)$

$$\begin{aligned} & \|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \\ & \leq C_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left| \int_0^T Y_{s-} dM_s^\lambda \right|^{\frac{\beta}{2}} \right] \right) \\ & \leq C_{L,T,\beta} \left(\frac{C_\beta + 1}{2\varepsilon} + 1 \right) \|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + C_{L,T,\beta} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] \\ & \quad + \frac{C_{L,T,\beta} C_\beta C'_\beta \varepsilon}{2} C''_\beta \left(\|Z \cdot X^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right) + \frac{C_{L,T,\beta} (C_\beta C'_\beta + 1) \varepsilon}{2} \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)}. \end{aligned} \quad (4.8)$$

Choosing $\varepsilon = \frac{1}{C_{L,T,\beta} C_\beta C'_\beta C''_\beta}$, together with (4.4) we get

$$\begin{aligned} \|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} & \leq C'_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] + \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \right) \\ & \leq C''_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta} + \|Y\|_{\mathcal{D}}^\beta + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| ds \right]^\beta \right). \end{aligned}$$

Step 4. It remains to prove that:

$$\|N\|_{\mathcal{N}^\beta} \leq C_{L,T,\beta}^N \left(\|Y\|_{\mathbb{S}^\beta} + \|Y\|_{\mathcal{D}}^\beta + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| ds \right]^\beta \right). \quad (4.9)$$

By (4.3), (4.8) and (4.7), we get

$$\begin{aligned} \|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} & \leq C_\beta \left(\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N - K\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right) \\ & \leq C_\beta C_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left| \int_0^T Y_{s-} dM_s^\lambda \right|^{\frac{\beta}{2}} \right] \right) \\ & \leq C_\beta C_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] \right. \\ & \quad \left. + \frac{C'_\beta}{\varepsilon} \|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + C'_\beta \varepsilon \|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + C'_\beta \varepsilon \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \right). \end{aligned}$$

Choosing $\varepsilon = \frac{1}{2C_\beta C_{L,T,\beta} C'_\beta}$, we have

$$\|M^\lambda\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq C_{L,T,\beta}^{M^\lambda} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] + \|K\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)} \right). \quad (4.10)$$

Then, (4.9) follows by (4.2), (4.4) and (4.10). \square

4.3 Existence and Uniqueness

4.3.1 Square integrable obstacle

Theorem 4.7. *Let Assumption 3.1 hold true. Assume that $S^+ \in \mathcal{S}^2$, then Theorem 3.6 holds true.*

Proof. Existence: For each $n \in \mathbb{N}$, we denote $\xi^n := q_n(\xi)$ and $f_t^n(y, z) := f_t(y, z) - f_t^0 + q_n(f_t^0)$, where $q_n(x) := \frac{xn}{|x|^{\vee n}}$. As $S^+ \in \mathcal{S}^2$, by [BPTZ16, Theorem 3.1], RBSDE(f^n, ξ^n, S) has a unique solution $(Y^n, Z^n, N^n, K^n) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{N}^2 \times \mathbb{I}^2$, and Y^n belongs to class $\mathbb{D}(\mathbb{Q})$ for each $\mathbb{Q} \in \mathcal{Q}_{L_z}$.

Step 1: We are going to show that $\{Y^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{S}^β and \mathcal{D} . Let $m, n \in \mathbb{N}$ and $n \geq m$. Set $\delta Y := Y^n - Y^m$, $\delta Z := Z^n - Z^m$, $\delta N := N^n - N^m$ and $\delta K := K^n - K^m$. Clearly, the process $(\delta Y, \delta Z, \delta N, \delta K)$ satisfies the following equation

$$\delta Y_t = \delta \xi + \int_t^T g_s(\delta Y_s, \delta Z_s) ds - \delta Z_s \cdot dX_s - d\delta N_s + d\delta K_s, \quad (4.11)$$

where

$$g_s(\delta Y_s, \delta Z_s) := f_s^n(Y_s^m + \delta Y_s, Z_s^m + \delta Z_s) - f_s^m(Y_s^m, Z_s^m).$$

It follows by Proposition 4.2 in [LRTY18] that

$$\begin{aligned} |\delta Y_\tau| &\leq |\delta Y_T| - \int_\tau^T \text{sgn}(\delta Y_{s-}) d\delta Y_s \\ &= |\delta \xi| + \int_\tau^T \text{sgn}(\delta Y_s) \{g_s(\delta Y_s, \delta Z_s) ds - \delta Z_s \cdot dX_s - d\delta N_s + d\delta K_s\}. \end{aligned}$$

By Assumption 3.1 we obtain

$$\begin{aligned} \text{sgn}(\delta Y_s) g_s(\delta Y_s, \delta Z_s) &= \text{sgn}(\delta Y_s) (f_s^n(Y_s^m + \delta Y_s, Z_s^m + \delta Z_s) - f_s^n(Y_s^m, Z_s^m + \delta Z_s)) \\ &\quad + \text{sgn}(\delta Y_s) (f_s^n(Y_s^m, Z_s^m + \delta Z_s) - f_s^n(Y_s^m, Z_s^m)) \\ &\quad + \text{sgn}(\delta Y_s) (f_s^n(Y_s^m, Z_s^m) - f_s^m(Y_s^m, Z_s^m)) \\ &\leq L_z |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| + |\delta f_s(Y_s^m, Z_s^m)| \leq L_z |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| + |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}}. \end{aligned}$$

We note that by Skorokhod condition,

$$\begin{aligned} \delta Y_{s-} d\delta K_s &= (Y_{s-}^n - S_{s-}) dK_s^n - (Y_{s-}^n - S_{s-}) dK_s^m - (Y_{s-}^m - S_{s-}) dK_s^n + (Y_{s-}^m - S_{s-}) dK_s^m \\ &= -(Y_{s-}^n - S_{s-}) dK_s^m - (Y_{s-}^m - S_{s-}) dK_s^n \leq 0, \end{aligned} \quad (4.12)$$

and

$$\text{sgn}(\delta Y_{s-}) d\delta K_s = \frac{\text{sgn}(\delta Y_{s-})}{\delta Y_{s-}} \delta Y_{s-} d\delta K_s \leq 0. \quad (4.13)$$

Therefore, we obtain

$$\begin{aligned} |\delta Y_\tau| &\leq |\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_\tau^T (L_z |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| + |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}}) ds - \text{sgn}(\delta Y_{s-}) (d\delta N_s + \delta Z_s \cdot dX_s) \\ &= |\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_\tau^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds - \text{sgn}(\delta Y_{s-}) (d\delta N_s + \delta Z_s \cdot (dX_s - \widehat{\sigma}_s \widehat{\lambda}_s ds)), \end{aligned}$$

where $\widehat{\lambda} := L_z \operatorname{sgn}(\delta Y_s) \frac{\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s}{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|} \mathbf{1}_{\{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| \neq 0\}}$. As $\delta Z \in \mathcal{H}^2$ and $\delta N \in \mathcal{N}^2$, we deduce from Burkholder-Davis-Gundy inequality that the last two terms are uniformly integrable martingales under the measure $\widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$ such that $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} := G_T^{\widehat{\lambda}}$. Taking conditional expectation with respect to $\mathcal{F}_\tau^{+, \mathbb{P}}$ under the equivalent measure $\widehat{\mathbb{Q}}$, we obtain that

$$|\delta Y_\tau| \leq \mathbb{E}^{\widehat{\mathbb{Q}}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \middle| \mathcal{F}_\tau^{+, \mathbb{P}} \right].$$

We deduce immediately from Lemma 4.3 that

$$\|\delta Y\|_{\mathcal{D}} \leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right], \quad (4.14)$$

and from Lemma 4.4 that for any $\beta \in (0, 1)$,

$$\|\delta Y\|_{\mathcal{S}^\beta} \leq \frac{1}{1-\beta} \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right]^\beta. \quad (4.15)$$

This shows that $\{Y^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{D} and \mathcal{S}^β . By completeness of \mathcal{D} and \mathcal{S}^β , there exists a limit $Y \in \mathcal{D} \cap \mathcal{S}^\beta$.

Step 2: We prove that $\{Z^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H}^β . By Itô's formula, we have

$$\begin{aligned} (\delta Y_T)^2 - (\delta Y_0)^2 &= -2 \int_0^T \delta Y_{s-} g_s(\delta Y_s, \delta Z_s) ds + 2 \int_0^T \delta Y_{s-} \delta Z_s \cdot dX_s + 2 \int_0^T \delta Y_{s-} d\delta N_s \\ &\quad - 2 \int_0^T \delta Y_{s-} d\delta K_s + \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds + [\delta(N - K)]_T, \end{aligned}$$

and therefore by Assumption 3.1, Skorokhod condition (4.12) and Young's inequality

$$\begin{aligned} &\int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds + [\delta(N - K)]_T \\ &\leq \sup_{0 \leq s \leq T} (\delta Y_s)^2 + 2 \int_0^T \delta Y_s |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds + 2L_z \int_0^T |\delta Y_s| |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| ds \\ &\quad - 2 \int_0^T \delta Y_s \delta Z_s \cdot dX_s - \int_0^T \delta Y_{s-} d\delta N_s \\ &\leq \sup_{0 \leq s \leq T} (\delta Y_s)^2 + 2 \int_0^T |\delta Y_s| |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds + 4L_z \int_0^T |\delta Y_s| |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| ds \\ &\quad + 2 \sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right| \\ &\leq (2 + 8L_z^2 T) \sup_{0 \leq s \leq T} (\delta Y_s)^2 + \left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^2 + \frac{1}{2} \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds \\ &\quad + 2 \sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|, \end{aligned}$$

which implies that

$$\int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds \leq 2(2 + 8L_z^2 T) \sup_{0 \leq s \leq T} (\delta Y_s)^2 + 2 \left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^2$$

$$+ 4 \sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|. \quad (4.16)$$

Taking expectation we obtain

$$\begin{aligned} \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} &\leq C'_{\beta,L,T} \left(\|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right] \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|^{\frac{\beta}{2}} \right] \right). \end{aligned}$$

By Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^\lambda} \left[\sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|^{\frac{\beta}{2}} \right] \\ &\leq C'_{\beta} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |\delta Y_s|^2 |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds \right)^{\frac{\beta}{4}} + \left(\int_0^T |\delta Y_{s-}|^2 d[\delta N]_s \right)^{\frac{\beta}{4}} \right] \\ &\leq \frac{C'_{\beta}}{2\varepsilon} \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \frac{C'_{\beta}\varepsilon}{2} \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + C'_{\beta} \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \|\delta N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}}. \end{aligned}$$

Choosing $\varepsilon := \frac{1}{C'_{\beta,L,T} C'_{\beta}}$, we obtain by Jensen's inequality for $\beta \in (0, 1)$ that

$$\begin{aligned} \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} &\leq 2C'_{\beta,L,T} \left(1 + \frac{(C'_{\beta})^2 C'_{\beta,L,T}}{2} \right) \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + 2C'_{\beta,L,T} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right]^{\frac{1}{\beta}} \\ &\quad + 2C'_{\beta,L,T} C'_{\beta} \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \|\delta N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}}. \end{aligned} \quad (4.17)$$

It remains to show that the term $\|\delta N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}$ is bounded. Clearly, we have

$$\|\delta N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq 2^{\frac{\beta}{2}} \left(\|N^n\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} + \|N^m\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \right).$$

Hence, it is enough to show that $\|N^n\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}$ is bounded uniformly for all $n \in \mathbb{N}$ and $\mathbb{Q}^\lambda \in \mathcal{Q}_{L_z}$. By Proposition 4.5, we have

$$\|N^n\|_{\mathcal{N}^\beta} \leq C_{\beta,L,T} \left(\|Y^n\|_{\mathcal{S}^\beta} + \|Y^n\|_{\mathcal{D}}^\beta + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| ds \right]^\beta \right).$$

Since $\{Y^n\}_{n \in \mathbb{N}}$ converges to Y in \mathcal{D} and \mathcal{S}^β , we deduce that

$$\sup_{n \in \mathbb{N}} \|Y^n\|_{\mathcal{D}} < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|Y^n\|_{\mathcal{S}^\beta} < \infty.$$

Therefore, $\|N^n\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}$ are uniformly bounded. Further, it follows from (4.17) that

$$\|Z\|_{\mathcal{H}^\beta} \leq C_{\beta,L,T,Y}^Z \left(\|Y\|_{\mathcal{S}^\beta} + \|Y\|_{\mathcal{S}^\beta}^{\frac{1}{2}} + \mathcal{E}^{\mathbb{P}} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right] \right). \quad (4.18)$$

As the right-hand side converges to 0 for $m, n \rightarrow \infty$, $\{Z^n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}^\beta$ is a Cauchy sequence. By completeness of \mathcal{H}^β , there exists a limit $Z \in \mathcal{H}^\beta$.

Step 3: We next show that $\{U^n := N^n - K^n\}_n$ is a Cauchy sequence in \mathcal{S}^β . By (4.11), we have

$$\delta U_t = \delta Y_t - \delta Y_0 + \int_0^t (g_s(\delta Y_s, \delta Z_s) - \widehat{\sigma}_s^{\mathbf{T}} \delta Z_s \cdot \lambda_s) ds - \int_0^t \delta Z_s \cdot dX_s^\lambda,$$

and therefore, together with Assumption 3.1

$$\begin{aligned} & \sup_{0 \leq s \leq T} |\delta U_s| \\ & \leq 2 \sup_{0 \leq s \leq T} |\delta Y_s| + \int_0^T (L_y |\delta Y_s| + 2L_z |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| + |\delta f_s(Y_s^m, Z_s^m)|) ds + \sup_{0 \leq u \leq T} \left| \int_0^u \delta Z_s \cdot dX_s^\lambda \right| \\ & \leq (2 + TL_y) \sup_{0 \leq s \leq T} |\delta Y_s| + \int_0^T (2L_z |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| ds + |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}}) ds + \sup_{0 \leq u \leq T} \left| \int_0^u \delta Z_s \cdot dX_s^\lambda \right|. \end{aligned}$$

Applying Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequality, we obtain that for $\beta \in (0, 1)$

$$\begin{aligned} \|\delta U\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} & \leq (2 + TL_y)^\beta \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + ((2L_z)^\beta T^{\frac{\beta}{2}} + C'_\beta) \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} \\ & \quad + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right] \\ & \leq C_{\beta, L, T}^U \left(\|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right] \right), \end{aligned}$$

and hence

$$\|\delta U\|_{\mathcal{S}^\beta} \leq C_{\beta, L, T}^U \left(\|\delta Y\|_{\mathcal{S}^\beta} + \|Z\|_{\mathcal{H}^\beta} + \mathcal{E}^\mathbb{P} \left[\left(\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right)^\beta \right] \right). \quad (4.19)$$

Since the right-hand side converges to 0, we obtain $\lim_{m, n \rightarrow \infty} \mathcal{E}^\mathbb{P} [\sup_{0 \leq s \leq T} |U_s^m - U_s^n|^\beta] = 0$, and that by completeness of $\mathcal{S}^\beta(\mathbb{P})$ there exists a limit $U \in \mathcal{S}^\beta(\mathbb{P})$.

Step 4: The process U is a local supermartingale. To see this, we shall find a localizing sequence of stopping times $\{\tau_k\}_{k \in \mathbb{N}}$, such that up to τ_k , we have U^n converges to U in “ \mathbb{L}^1 -sense”. Indeed, for $t \in [0, T]$ and $\varepsilon > 0$, by Markov’s inequality and BDG inequality, we have

$$\mathbb{P} \left[\sup_{0 \leq u \leq t} \left| \int_0^u Z_s^n \cdot dX_s - \int_0^u Z_s \cdot dX_s \right| > \varepsilon \right] \leq \frac{1}{\varepsilon^\beta} \mathbb{E} \left[\sup_{0 \leq u \leq t} \left| \int_0^u (Z_s^n - Z_s) \cdot dX_s \right|^\beta \right] \leq \frac{C_\beta}{\varepsilon^\beta} \|Z^n - Z\|_{\mathbb{H}^\beta},$$

which implies that $\int_0^t Z_s^n \cdot dX_s$ converges to $\int_0^t Z_s \cdot dX_s$ in ucp. We may extract a subsequence such that $\int_0^T Z_s^n \cdot dX_s$ converges to $\int_0^T Z_s \cdot dX_s$ almost surely and $\sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \left| \int_0^u Z_s^n \cdot dX_s \right| < \infty$, a.s. By Assumption 3.1, we have

$$\begin{aligned} & \int_0^T |f_s^n(Y_s^n, Z_s^n) - f_s(Y_s, Z_s)| ds \\ & \leq \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds + L_y \int_0^T |Y_s^n - Y_s| ds + L_z \int_0^T |\widehat{\sigma}_s^{\mathbf{T}}(Z_s^n - Z_s)| ds \\ & \leq \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds + L_y T \sup_{0 \leq s \leq T} |Y_s^n - Y_s| + L_z T^{\frac{1}{2}} \left(\int_0^T |\widehat{\sigma}_s^{\mathbf{T}}(Z_s^n - Z_s)|^2 ds \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, and hence $\sup_{n \in \mathbb{N}} \sup_{0 \leq u \leq T} \int_0^u |f_s^n(Y_s^n, Z_s^n)| ds < \infty$, a.s. We now define

$$\tau_m := \left\{ t > 0 : \sup_{n \in \mathbb{N}} \int_0^t |f_s^n(Y_s^n, Z_s^n)| ds + \sup_{n \in \mathbb{N}} \left| \int_0^t Z_s^n \cdot dX_s \right| \geq m \right\}.$$

From above the sequence of stopping times $\{\tau_m\}_{m \in \mathbb{N}}$ converges almost surely to ∞ and the processes $\int_0^{t \wedge \tau_m} f_s^n(Y_s^n, Z_s^n) ds - \int_0^{t \wedge \tau_m} Z_s^n \cdot dX_s$ are uniformly bounded by m . Hence, the process

$$U_{t \wedge \tau_m}^n = N_{t \wedge \tau_m}^n - K_{t \wedge \tau_m}^n = Y_{t \wedge \tau_m}^n - Y_0^n + \int_0^{t \wedge \tau_m} f_s^n(Y_s^n, Z_s^n) ds - \int_0^{t \wedge \tau_m} Z_s^n \cdot dX_s$$

is of class $\mathcal{D}(\mathbb{P})$, and $U_{t \wedge \tau_m}^n$ converges to $U_{t \wedge \tau_m}$ in $\mathcal{L}^1(\mathbb{P})$ for each $t \in [0, T]$. This implies that U is a local supermartingale under each $\mathbb{Q} \in \mathcal{Q}_{L^z}$.

Step 5: We now show that the limiting process (Y, Z, N, K) solves the RBSDE (3.4). By a general version of Doob-Meyer decomposition, see e.g. [CE15, Theorem 9.2.7], the local supermartingale U uniquely decomposes as $U = N - K$, where N is a local martingale and K is a nondecreasing predictable process starting from zero. By Kunita-Watanabe inequality for semimartingale, we obtain that

$$[N, X]_T = [U, X]_T \leq [U - U^n, X]_T + [U^n, X]_T \leq [U - U^n]_T^{\frac{1}{2}} [X]_T^{\frac{1}{2}}.$$

The right-hand side converges a.s. to 0, at least along a subsequence. Therefore $[N, X] = 0$.

As $\int_0^u Z_s^n \cdot dX_s$ converges to $\int_0^u Z_s \cdot dX_s$ in ucp and the map $(y, z) \mapsto f_t(y, z)$ is continuous, taking a limit in ucp implies that (Y, Z, N, K) solves the correct RBSDE.

Step 6: We now Snell envelop approach to optimal stopping in order to derive the Skorokhod condition. By following the proof of [LX05, Proposition 3.1], we may show the following representation for each $n \in \mathbb{N}$

$$\begin{aligned} Y_{t \wedge \tau_m}^n &= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t \wedge \tau_m, T \wedge \tau_m}} \mathbb{E} \left[\int_{t \wedge \tau_m}^{\tau} f_s^n(Y_s^n, Z_s^n) ds + S_\tau \mathbf{1}_{\{\tau < T \wedge \tau_m\}} + Y_{T \wedge \tau_m}^n \mathbf{1}_{\{\tau = T \wedge \tau_m\}} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right] \\ &\leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t \wedge \tau_m, T \wedge \tau_m}} \mathbb{E} \left[\int_{t \wedge \tau_m}^{\tau} f_s(Y_s, Z_s) ds + S_\tau \mathbf{1}_{\{\tau < T \wedge \tau_m\}} + Y_{T \wedge \tau_m} \mathbf{1}_{\{\tau = T \wedge \tau_m\}} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right] \\ &\quad + \mathbb{E} \left[\int_0^{T \wedge \tau_m} |f_s^n(Y_s^n, Z_s^n) - f_s(Y_s, Z_s)| ds + |Y_{T \wedge \tau_m}^n - Y_{T \wedge \tau_m}| \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right]. \end{aligned}$$

It follows from $Y_{t \wedge \tau_m}^n \rightarrow Y_{t \wedge \tau_m}$, $Y_{T \wedge \tau_m}^n \rightarrow Y_{T \wedge \tau_m}$ and $\int_0^{T \wedge \tau_m} |f_s^n(Y_s^n, Z_s^n) - f_s(Y_s, Z_s)| ds \rightarrow 0$ in \mathbb{L}^1 that

$$Y_{t \wedge \tau_m} \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t \wedge \tau_m, T \wedge \tau_m}} \mathbb{E} \left[\int_{t \wedge \tau_m}^{\tau} f_s(Y_s, Z_s) ds + S_\tau \mathbf{1}_{\{\tau < T \wedge \tau_m\}} + Y_{T \wedge \tau_m} \mathbf{1}_{\{\tau = T \wedge \tau_m\}} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right].$$

On the other hand, it is clear that $Y_{t \wedge \tau_m} \geq S_{t \wedge \tau_m} \mathbf{1}_{\{t < T \wedge \tau_m\}} + Y_{T \wedge \tau_m} \mathbf{1}_{\{t = T \wedge \tau_m\}}$. Since $Y_{t \wedge \tau_m} + \int_0^{t \wedge \tau_m} f_s(Y_s, Z_s) ds$ is a supermartingale, we have

$$Y_{t \wedge \tau_m} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t \wedge \tau_m, T \wedge \tau_m}} \mathbb{E} \left[\int_{t \wedge \tau_m}^{\tau} f_s(Y_s, Z_s) ds + S_\tau \mathbf{1}_{\{\tau < T \wedge \tau_m\}} + Y_{T \wedge \tau_m} \mathbf{1}_{\{\tau = T \wedge \tau_m\}} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right]. \quad (4.20)$$

Define

$$\eta_t^m := \int_0^t f_s(Y_s, Z_s) ds + S_t \mathbf{1}_{\{t < T \wedge \tau_m\}} + Y_{T \wedge \tau_m} \mathbf{1}_{\{t = T \wedge \tau_m\}}$$

$$- \mathbb{E} \left[\int_0^{T \wedge \tau_m} f_s(Y_s, Z_s) ds + Y_{T \wedge \tau_m} \middle| \mathcal{F}_t^{+, \mathbb{P}} \right].$$

Clearly, $\eta_{T \wedge \tau_m}^m = 0$. Note that $\eta^m = (\eta_t^m)_{0 \leq t \leq T \wedge \tau_m}$ is of class $\mathbb{D}(\mathbb{P})$. Let J_t^m be the Snell envelope of η^m

$$J_t^m := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t \wedge \tau_m, T \wedge \tau_m}} \mathbb{E} \left[\eta_\tau^m \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right],$$

which is a càdlàg process of class $\mathbb{D}(\mathbb{P})$ and is the smallest supermartingale dominating the process η . Hence, by the Doob-Meyer decomposition, there exist a martingale M^m and a predictable nondecreasing process A^m such that $J_t^m = J_0^m + M_t^m - A_t^m$. By the definition of J^m and the representation (4.20), we obtain

$$J_t^m = Y_{t \wedge \tau_m} - \mathbb{E} \left[\int_{t \wedge \tau_m}^{T \wedge \tau_m} f_s(Y_s, Z_s) ds + Y_{T \wedge \tau_m} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right]. \quad (4.21)$$

We have that

$$J_t^m + \mathbb{E} \left[\int_0^{T \wedge \tau_m} f_s(Y_s, Z_s) ds + Y_{T \wedge \tau_m} \middle| \mathcal{F}_{t \wedge \tau_m}^{+, \mathbb{P}} \right] = Y_{t \wedge \tau_m} + \int_0^{t \wedge \tau_m} f_s(Y_s, Z_s) ds$$

is a supermartingale. Therefore, by the uniqueness of the Doob-Meyer decomposition, $A_t^m = K_{t \wedge \tau_m}$. Decompose A^m (and the same for K) in continuous part $A^{m,c}$ (K^c) and pure-jumps part $A^{m,d}$ (K^d). By [KQ12, Proposition B.11], see also [El 81, Proposition 2.34], we obtain

$$\int_0^{T \wedge \tau_m} (J_t^m - \eta_t^m) dA_t^{m,c} = 0 \quad \text{and} \quad \Delta A_t^{m,d} = \Delta A_t^{m,d} \mathbf{1}_{\{J_{t-}^m - \eta_{t-}^m\}}, \quad a.s. \text{ for } t \leq \tau_m.$$

By noticing that $J_t^m - \eta_t^m = Y_{t \wedge \tau_m} - S_{t \wedge \tau_m} \mathbf{1}_{\{t < T \wedge \tau_m\}} - Y_{T \wedge \tau_m} \mathbf{1}_{\{t = T \wedge \tau_m\}}$, we obtain that

$$\int_0^{T \wedge \tau_m} (Y_{t-} - S_{t-}) dK_t = \int_0^{T \wedge \tau_m} (J_{t-}^m - \eta_{t-}^m) dK_t = 0, \quad a.s.$$

Letting $m \rightarrow \infty$, the Skorokhod condition holds true for K .

Uniqueness: Let (Y, Z, N, K) and (Y', Z', N', K') be two solutions to RBSDE(f, ξ, S). Set $\delta Y = Y - Y'$, $\delta Z = Z - Z'$, $\delta N = N - N'$ and $\delta K = K - K'$. Using the similar computation as above, we have

$$|\delta Y_{\tau \wedge \tau_m}| \leq |\delta Y_{\tau_m}| - \int_{\tau \wedge \tau_m}^{\tau_m} \operatorname{sgn}(\delta Y_s) \delta Z_s \cdot (dX_s - \sigma_s \widehat{\lambda}_s ds) - \int_{\tau \wedge \tau_m}^{\tau_m} \operatorname{sgn}(\delta Y_{s-}) d\delta N_s,$$

where $\widehat{\lambda}_s := L \operatorname{sgn}(\delta Y_s) \frac{\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s}{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|} \mathbf{1}_{\{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| \neq 0\}}$ and

$$\tau_m := \inf \left\{ t \geq 0 : \int_0^t |Z_s|^2 + |Z'_s|^2 ds \geq m \right\} \wedge T \wedge \tau_m^N,$$

and $\{\tau_m^N\}_{m \in \mathbb{N}}$ denotes the localizing sequence of the local martingale δN . Taking the conditional expectation with respect to \mathcal{F}_τ under the equivalent measure $\widehat{\mathbb{Q}} \sim \mathbb{P}$, defined by $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = G_T^{\widehat{\lambda}}$, we obtain that

$$|\delta Y_{\tau \wedge \tau_m}| \leq \mathbb{E}^{\widehat{\mathbb{Q}}} [|\delta Y_{\tau_m}| | \mathcal{F}_\tau^{+, \mathbb{P}}].$$

Again, since δY belongs to $\mathbb{D}(\widehat{\mathbb{Q}})$, it follows that $\delta Y_{\tau_m} \rightarrow 0$ in $\mathbb{L}^1(\widehat{\mathbb{Q}})$, therefore $|\delta Y_\tau| = 0$. It follows by the section theorem that Y and Y' are indistinguishable. By (4.18) and (4.19), $(\delta Z, \delta N, \delta K) = (0, 0, 0)$. \square

4.3.2 General obstacles

Before proving the wellposedness result, we state the following comparison result for the general càdlàg solution and general filtration in the \mathbb{L}^2 -setting, which is a generalization of [RS12, Proposition 3.2]. The proof is omitted as it follows the same argument as in the classical one.

Proposition 4.8. *Let (f, ξ, S) and (f', ξ', S') be such that f and f' satisfy Assumption 3.1 and $\mathbb{E}[\int_0^T |f_s^0|^2 ds] < \infty$, $\mathbb{E}[\int_0^T |f_s'^0|^2 ds] < \infty$, $\xi, \xi' \in \mathbb{L}^2$, and $S, S' \in \mathbb{S}^2$, and let (Y, Z, N, K) and (Y', Z', N', K') be the corresponding solutions.*

Assume that $\xi \leq \xi'$, $S_t \leq S'_t$, and $(f_t - f'_t)(Y'_t, Z'_t) \leq 0$, \mathbb{P} -a.s., $t \in [0, T]$. Then $Y_\tau \leq Y'_\tau$, for all $\tau \in \mathcal{T}_{0,T}$.

Proposition 4.9. *Let (f, ξ, S) and (f', ξ', S') satisfy the assumptions of Theorem 4.7. Let (Y, Z, N, K) and (Y', Z', N', K') be solutions of corresponding RBSDEs. Suppose that $\xi \leq \xi'$, \mathbb{P} -a.s.; $f(y, z) \leq f'(y, z)$, $dt \otimes \mathbb{P}$ -a.e., for each $y, z \in \mathbb{R} \times \mathbb{R}^d$; and $S \leq S'$, $dt \otimes \mathbb{P}$ -a.e. Then, $Y_\tau \leq Y'_\tau$ for each $\tau \in \mathcal{T}_{0,T}$.*

Proof. Let (Y^n, Z^n, N^n, K^n) and (Y^m, Z^m, N^m, K^m) be the approximation sequences of the solutions of RBSDE with (f, ξ, S) and (f', ξ', S') , respectively. By the comparison result, Proposition 4.8, we have $Y_\tau^n \leq Y_\tau^m$, therefore $Y_\tau \leq Y'_\tau$ for each $\tau \in \mathcal{T}_{0,T}$. \square

Now we are ready to prove the main theorem.

Proof of Theorem 3.6. Define $S_t^n := S_t \wedge n$. Clearly, $S^n \geq S^m$ for $n \geq m$. By Theorem 4.7, RBSDE with (f, ξ, S^n) has a unique solution (Y^n, Z^n, N^n, K^n) . Define $(\delta Y, \delta Z, \delta N, \delta K) := (Y^n - Y^m, Z^n - Z^m, N^n - N^m, K^n - K^m)$. By Proposition 4.9 we have $\delta Y \geq 0$.

Step 1: We are going to show that $\{Y^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{D} and \mathcal{S}^β . Let $\sigma \in \mathcal{T}_{0,T}$ be arbitrary. Define

$$\begin{aligned} \tau_\sigma^\varepsilon &:= \inf\{t \geq \sigma : Y_t^n \leq S_t^n + \varepsilon\} \wedge T, \\ \tau_k &:= \inf\left\{t \geq 0 : \int_0^t (|\widehat{\sigma}_s^{\mathbf{T}} Z_s^m|^2 + |\widehat{\sigma}_s^{\mathbf{T}} Z_s^n|^2) ds \geq k\right\} \wedge \tau_k^N \wedge T, \end{aligned}$$

where $\{\tau_k^N\}_{k \in \mathbb{N}}$ is the localizing sequence for the local martingales N^m and N^n . It follows from the definition of τ_σ^ε that K^n is flat on $[\sigma, \tau_\sigma^\varepsilon]$, hence $\text{sgn}(\delta Y_{s-}) d\delta K_s \leq 0$ on $[\sigma, \tau_\sigma^\varepsilon]$. Again by Proposition 4.2 in [LRTY18] and Assumption 3.1, we obtain

$$\begin{aligned} |\delta Y_{\sigma \wedge \tau_k}| &\leq |\delta Y_{\tau_\sigma^\varepsilon \wedge \tau_k}| - \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_{s-}) d\delta Y_s \\ &= |\delta Y_{\tau^* \wedge \tau_k}| + \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_s) (f_s(Y_s^n, Z_s^n) - f_s(Y_s^m, Z_s^m)) ds \\ &\quad - \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_s) \delta Z_s \cdot dX_s - \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_{s-}) d\delta N_s + \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_{s-}) d\delta K_s \\ &\leq |\delta Y_{\tau_\sigma^\varepsilon \wedge \tau_k}| - \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_s) \delta Z_s \cdot (dX_s - \widehat{\sigma}_s \widehat{\lambda}_s ds) - \int_{\sigma \wedge \tau_k}^{\tau_\sigma^\varepsilon \wedge \tau_k} \text{sgn}(\delta Y_{s-}) d\delta N_s, \end{aligned}$$

where $\widehat{\lambda}_s := L_z \text{sgn}(\delta Y_s) \frac{\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s}{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|} \mathbf{1}_{\{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| \neq 0\}}$. Conditioning with respect to $\mathcal{F}_\sigma^{+, \mathbb{P}}$ under the equivalent measure $\widehat{\mathbb{Q}} \in \mathcal{Q}_{L_z}$ defined by $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = G_T^{\widehat{\lambda}}$, and then, as δY is of class $\mathcal{D}(\mathbb{P})$, letting $k \rightarrow \infty$, we deduce from the above inequality that

$$|\delta Y_\sigma| \leq \mathbb{E}^{\widehat{\mathbb{Q}}} \left[|\delta Y_{\tau_\sigma^\varepsilon}| \middle| \mathcal{F}_\sigma^{+, \mathbb{P}} \right] \leq \mathbb{E}^{\widehat{\mathbb{Q}}} \left[S_{\tau_\sigma^\varepsilon}^+ \mathbf{1}_{\{S_{\tau_\sigma^\varepsilon}^+ \geq m\}} \middle| \mathcal{F}_\sigma^{+, \mathbb{P}} \right] + \varepsilon,$$

where the last inequality follows from $0 \leq \delta Y_{\tau_\varepsilon} = S_{\tau_\varepsilon}^n + \varepsilon - Y_{\tau_\varepsilon}^m \leq S_{\tau_\varepsilon}^n + \varepsilon - S_{\tau_\varepsilon}^m \leq S_{\tau_\varepsilon}^+ \mathbf{1}_{\{S_{\tau_\varepsilon}^+ \geq m\}} + \varepsilon$.

Let $\mathbb{Q} \in \mathcal{Q}_{L_z}$ be arbitrary. We obtain that

$$\mathbb{E}^{\mathbb{Q}}[|\delta Y_\sigma|] \leq \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\widehat{\mathbb{Q}}} \left[S_{\tau_\varepsilon}^+ \mathbf{1}_{\{S_{\tau_\varepsilon}^+ \geq m\}} \middle| \mathcal{F}_\sigma^{+, \mathbb{P}} \right] \right] + \varepsilon \leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}} \left[S_\tau^+ \mathbf{1}_{\{S_\tau^+ \geq m\}} \right] + \varepsilon.$$

Together with Lemma 4.2, we obtain that for any $\beta \in (0, 1)$

$$\|\delta Y\|_{\mathcal{S}^\beta} \leq \frac{1}{1-\beta} \|\delta Y\|_{\mathcal{D}}^\beta \leq \frac{1}{1-\beta} \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}} \left[S_\tau^+ \mathbf{1}_{\{S_\tau^+ \geq m\}} \right]^\beta.$$

As the spaces \mathcal{D} and \mathcal{S}^β are complete, we may find a limit $Y \in \mathcal{D} \cap \mathcal{S}^\beta$.

Step 2: We will show that $\{Z^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H}^β . Similar to (4.16), we have

$$\begin{aligned} \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds &\leq 2(2 + 8L_z^2 T) \sup_{0 \leq s \leq T} (\delta Y_s)^2 + 4 \int_0^T \delta Y_{s-} d\delta K_s \\ &\quad + 4 \sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|. \end{aligned} \quad (4.22)$$

Comparing to (4.16), the extra term $\delta Y_{s-} d\delta K_s$ is due to the different obstacles $S^n \neq S^m$. Note that by Skorokhod condition and $Y^m \geq S^m$, $Y^n \geq S^n$

$$\begin{aligned} \int_0^t \delta Y_{s-} d\delta K_s &= \int_0^t (Y_{s-}^n - S_{s-}^n) dK_s^n + \int_0^t (S_{s-}^n - Y_{s-}^m) dK_s^n \\ &\quad - \int_0^t (Y_{s-}^n - S_{s-}^m) dK_s^m + \int_0^t (Y_{s-}^m - S_{s-}^m) dK_s^m \\ &= \int_0^t S_{s-}^n dK_s^n - \int_0^t Y_{s-}^m dK_s^n - \int_0^t Y_{s-}^n dK_s^m + \int_0^t S_{s-}^m dK_s^m \\ &\leq \int_0^t \delta S_{s-} dK_s^n - \int_0^t \delta S_{s-} dK_s^m \\ &\leq \sup_{0 \leq s \leq T} |\delta S_s| (K_T^m + K_T^n). \end{aligned}$$

Plugging this inequality in (4.22), we obtain

$$\begin{aligned} \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|^2 ds &\leq 2(2 + 8L_z^2 T) \sup_{0 \leq s \leq T} (\delta Y_s)^2 + 4 \sup_{0 \leq s \leq T} |\delta S_s| (K_T^m + K_T^n) \\ &\quad + 4 \sup_{0 \leq u \leq T} \left| \int_0^u \delta Y_s \delta Z_s \cdot dX_s^\lambda + \int_0^u \delta Y_{s-} d\delta N_s \right|. \end{aligned} \quad (4.23)$$

Taking expectation and using Young and Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} \|\delta Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} &\leq C_{\beta, L, T} \left(\|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \|\delta S\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \left(\|K^m\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} + \|K^n\|_{\mathbb{I}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \right) \right. \\ &\quad \left. + \|\delta Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \left(\|N^m\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} + \|N^n\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)}^{\frac{1}{2}} \right) \right). \end{aligned}$$

Since $0 \leq \delta S_t \leq S_t^+ \mathbf{1}_{\{S_t^+ \geq m\}}$ for $t \in [0, T]$, we have $0 \leq \sup_{0 \leq s \leq T} |\delta S_s| \leq \sup_{0 \leq s \leq T} S_s^+ \mathbf{1}_{\{S_s^+ \geq m\}}$ and by Lemma 4.2

$$\|\delta S\|_{\mathcal{S}^\beta(\mathbb{Q}^\lambda)} \leq \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\sup_{0 \leq s \leq T} S_s^+ \mathbf{1}_{\{S_s^+ \geq m\}} \right)^\beta \right] \leq \frac{1}{1-\beta} \sup_{\tau \in \mathcal{T}_{0,T}} \left(\mathbb{E}^{\mathbb{Q}^\lambda} \left[S_\tau^+ \mathbf{1}_{\{S_\tau^+ \geq m\}} \right] \right)^\beta.$$

As in the proof of Theorem 4.7, by the convergence of $\{Y^n\}_{n \in \mathbb{N}}$ and Proposition 4.5 we have $\sup_{n \in \mathbb{N}} \|N^n\|_{\mathcal{N}^\beta} < \infty$ and $\sup_{n \in \mathbb{N}} \|K^n\|_{\mathcal{K}^\beta} < \infty$. Therefore, we obtain

$$\|\delta Z\|_{\mathcal{H}^\beta} \leq C_{\beta,L,T,Y} \left(\|\delta Y\|_{\mathcal{S}^\beta} + \|\delta Y\|_{\mathcal{S}^\beta}^{\frac{1}{2}} + \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}} \left[S_\tau^+ \mathbf{1}_{\{S_\tau^+ \geq m\}} \right]^{\frac{\beta}{2}} \right).$$

As the right-hand side converges to 0, $\{Z^n\}_{n \in \mathbb{N}} \subseteq \mathcal{H}^\beta$ is a Cauchy sequence. Again by completeness, there exists a limit $Z \in \mathcal{H}^\beta$.

Step 3: Using the same argument as in Step 3 of the proof of Theorem 4.7, we show that $\{U^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{S}^β . Hence, there exists a limit $U \in \mathcal{S}^\beta$. We also show similarly that U is a local supermartingale, and can be uniquely decomposed as $N - K$, where N is a local martingale satisfying $[N, X] = 0$ and K is a nondecreasing predictable process starting from zero.

Clearly $Y \geq S$. In the same way, we show the Skorokhod condition and that (Y, Z, N, K) solves the correct RBSDE with (f, ξ, S) .

The uniqueness follows by the same argument as in the proof of Theorem 4.7. \square

Proof of Theorem 3.7 (ii). Let (Y^n, Z^n, N^n, K^n) and (Y^m, Z^m, N^m, K^m) be the approximation sequences of the solutions of RBSDEs with (f, ξ, S) and (f', ξ', S') , respectively. By the comparison result, Proposition 4.9, we have $Y_\tau^n \leq Y_\tau^m$, therefore $Y_\tau \leq Y_\tau'$ for each τ . \square

4.4 Stability of reflected BSDE

Proof of Theorem 3.7 (i). Obviously, the process $(\delta Y, \delta Z, \delta N, \delta K)$ satisfies

$$\delta Y_t = \delta \xi + \int_t^T g_s(\delta Y_s, \delta Z_s) ds - \delta Z_s \cdot dX_s - d\delta N_s + d\delta K_s, \quad \mathbb{P} - \text{a.s.}$$

where $g_s(\delta Y_s, \delta Z_s) := f'_s(Y_s + \delta Y_s, Z_s + \delta Z_s) - f_s(Y_s, Z_s)$. Define $\tau_m := \inf\{t \geq 0 \mid \int_0^t |Z_s|^2 + |Z'_s|^2 ds \geq m\} \wedge T \wedge \tau_m^N$, and $\{\tau_m^N\}_{m \in \mathbb{N}}$ denotes the localizing sequence of the local martingale δN . Following the same argument as in Step 1 of the proof of Theorem 4.7, we obtain that

$$|\delta Y_{\tau \wedge \tau_m}| \leq |\delta Y_{T \wedge \tau_m}| + \int_0^T |\delta f_s(Y_s, Z_s)| ds - \int_{\tau \wedge \tau_m}^{T \wedge \tau_m} \text{sgn}(\delta Y_{s-}) \{d\delta N_s - \delta Z_s \cdot (dX_s - \hat{\sigma}_s \hat{\lambda}_s ds)\},$$

where $\hat{\lambda}_s := L_z \text{sgn}(\delta Y_s) \frac{\hat{\sigma}_s^T \delta Z_s}{|\hat{\sigma}_s^T \delta Z_s|} \mathbf{1}_{\{|\hat{\sigma}_s^T \delta Z_s| \neq 0\}}$. Define $d\mathbb{Q}^{\hat{\lambda}} := G_T^{\hat{\lambda}} d\mathbb{P}$. Since $\delta Y \in \mathcal{D}$, it follows that $\delta Y_{\tau_m} \rightarrow \delta \xi$ in \mathcal{L}^1 , and therefore

$$|\delta Y_\tau| \leq \mathbb{E}^{\mathbb{Q}^{\hat{\lambda}}} \left[|\delta \xi| + \int_0^T |\delta f_s(Y_s, Z_s)| ds \middle| \mathcal{F}_\tau^{+, \mathbb{P}} \right].$$

We deduce immediately from Lemma 4.3 that

$$\|\delta Y\|_{\mathcal{D}} \leq \mathcal{E}^{\mathbb{P}} \left[|\delta \xi| + \int_0^T |\delta f_s(Y_s, Z_s)| ds \right], \quad (4.24)$$

and from Lemma 4.4 that for any $\beta \in (0, 1)$,

$$\|\delta Y\|_{S^\beta} \leq \frac{1}{1-\beta} \mathcal{E}^\mathbb{P} \left[|\delta\xi| + \int_0^T |\delta f_s(Y_s, Z_s)| ds \right]^\beta. \quad (4.25)$$

Further, following **Step 2** in Theorem 4.7, we obtain that

$$\begin{aligned} & \|\delta Z\|_{\mathcal{H}^\beta} + \|\delta(N - K)\|_{\mathcal{N}^\beta} \\ & \leq C_{\beta, L, T}^1 \left(\|\delta Y\|_{S^\beta} + \mathcal{E}^\mathbb{P} \left[\left(\int_0^T |\delta f_s(Y_s, Z_s)| ds \right)^\beta \right] + \|\delta Y\|_{S^\beta}^{\frac{1}{2}} (\|N\|_{\mathcal{N}^\beta} + \|N'\|_{\mathcal{N}^\beta})^{\frac{1}{2}} \right). \end{aligned}$$

The assertion follows from (4.25) and Proposition 4.5. \square

4.5 A priori estimates and stability of BSDE

For $S = -\infty$, we have the existence and uniqueness result of the BSDE in general filtration. As we have seen in Proposition 4.5, there is no a priori estimate for Y for reflected BSDE. However, for the BSDE (3.1) without reflection we may find a priori estimate for Y .

Proof of Theorem 3.4: estimates (3.2)-(3.3). Let $\tau_n := \inf\{t \geq 0 \mid \int_0^t |\hat{\sigma}_s^\mathbf{T} Z_s|^2 ds \geq n\} \wedge \tau_n^N \wedge T$, where $\{\tau_n^N\}_{n \in \mathbb{N}}$ denotes the localizing sequence for the local martingale N . Applying Tanaka's formula, by Assumption 3.1 and Remark 3.2, we obtain that

$$\begin{aligned} |Y_{\tau \wedge \tau_n}| & \leq |Y_{\tau_n}| - \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_{s-}) dY_s \\ & = |Y_{\tau_n}| + \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_s) f_s(Y_s, Z_s) ds - \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_s) Z_s \cdot dX_s - \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_{s-}) dN_s \\ & \leq |Y_{\tau_n}| + \int_0^T |f_s^0| ds - \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_s) \hat{\sigma}_s^\mathbf{T} Z_s \cdot dW_s^{\hat{\lambda}} - \int_{\tau \wedge \tau_n}^{\tau_n} \text{sgn}(Y_{s-}) dN_s, \end{aligned}$$

with $\hat{\lambda}_s := L_z \text{sgn}(Y_s) \frac{\hat{\sigma}_s^\mathbf{T} Z_s}{|\hat{\sigma}_s^\mathbf{T} Z_s|} \mathbf{1}_{\{|\hat{\sigma}_s^\mathbf{T} Z_s| \neq 0\}}$. Taking conditional expectation with respect to $\mathcal{F}_\tau^{+, \mathbb{P}}$ under the measure $\mathbb{Q}^{\hat{\lambda}}$ defined by $\frac{d\mathbb{Q}^{\hat{\lambda}}}{d\mathbb{P}} := G_T^{\hat{\lambda}}$, we obtain $|Y_{\tau \wedge \tau_n}| \leq \mathbb{E}^{\mathbb{Q}^{\hat{\lambda}}} \left[|Y_{\tau_n}| + \int_0^T |f_s^0| ds \mid \mathcal{F}_\tau^{+, \mathbb{P}} \right]$. As $Y \in \mathcal{D}(\mathbb{P})$, letting $n \rightarrow \infty$, we obtain that

$$|Y_\tau| \leq \mathbb{E}^{\mathbb{Q}^{\hat{\lambda}}} \left[|\xi| + \int_0^T |f_s^0| ds \mid \mathcal{F}_\tau^{+, \mathbb{P}} \right], \quad (4.26)$$

and (3.2) follows immediately from Lemma 4.3. By Lemma 4.4, (4.26) implies that

$$\|Y\|_{S^\beta} \leq \frac{1}{1-\beta} \mathcal{E}^\mathbb{P} \left[|\xi| + \int_0^T |f_s^0| ds \right]^\beta \quad \text{for all } \beta \in (0, 1). \quad (4.27)$$

Further, by applying Itô's formula on Y^2 , we see that

$$\begin{aligned} & \int_0^T |\hat{\sigma}_s^\mathbf{T} Z_s|^2 ds + [N]_T \\ & = Y_T^2 - Y_0^2 + 2 \int_0^T Y_s (f_s(Y_s, Z_s) - \hat{\sigma}_s^\mathbf{T} Z_s \cdot \lambda_s) ds - 2 \int_0^T Y_s Z_s \cdot dX_s^\lambda - 2 \int_0^T Y_{s-} dN_s \\ & \leq \sup_{0 \leq s \leq T} Y_s^2 + 2 \int_0^T |Y_s| (L_z |\hat{\sigma}_s^\mathbf{T} Z_s| + |f_s^0|) ds + 2 \sup_{0 \leq u \leq T} \left| \int_0^u Y_s Z_s \cdot dX_s^\lambda + 2 \int_0^u Y_{s-} dN_s \right| \end{aligned}$$

$$\leq (2 + 2L_z^2 T) \sup_{0 \leq s \leq T} |Y_s|^2 + \frac{1}{2} \int_0^T |\widehat{\sigma}_s^{\mathbf{T}} Z_s|^2 ds + \left(\int_0^T |f_s^0| ds \right)^2 + 2 \sup_{0 \leq u \leq T} \left| \int_0^u Y_s Z_s \cdot dX_s^\lambda + \int_0^u Y_{s-} dN_s \right|.$$

Finally, by Burkholder-Davis-Gundy inequality and Young's inequality, we obtain

$$\|Z\|_{\mathbb{H}^\beta(\mathbb{Q}^\lambda)} + \|N\|_{\mathbb{N}^\beta(\mathbb{Q}^\lambda)} \leq C''_{L,T,\beta} \left(\|Y\|_{\mathbb{S}^\beta(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda} \left[\left(\int_0^T |f_s^0| ds \right)^\beta \right] \right).$$

Taking supremum over all $\mathbb{Q}^\lambda \in \mathcal{Q}_{L_z}$, (3.3) follows from the above inequality and (4.27). \square

As in the proof of Theorem 4.7 and that of the estimates (3.2)-(3.3) of Theorem 3.4, we may estimate the difference of two solutions of two BSDEs. Let (Y^n, Z^n, N^n) be the solution of the approximating BSDE with (f^n, ξ^n) as in previous section. Define $(\delta Y, \delta Z, \delta N) := (Y - Y^n, Z - Z^n, N - N^n)$.

Proposition 4.10. *Under Assumptions 3.1 and 3.3, we have*

$$\|\delta Y\|_{\mathcal{D}} \leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right],$$

and $\|\delta Y\|_{\mathbb{S}^\beta} + \|\delta Z\|_{\mathcal{H}^\beta} + \|\delta N\|_{\mathcal{N}^\beta} \leq C_{\beta,L,T} \left(\mathcal{E}^{\mathbb{P}} [|\xi| \mathbf{1}_{\{|\xi| \geq n\}}]^\beta + \mathcal{E}^{\mathbb{P}} \left[\int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right]^\beta \right).$

Corollary 4.11. *For any $\delta > 0$ and $A \in \mathcal{F}_T^{+, \mathbb{P}}$ such that $\mathcal{E}^{\mathbb{P}}[\mathbf{1}_A] < \delta$ we have*

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|Y_\tau| \mathbf{1}_A] \leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right] + C_n \delta^{\frac{1}{2}},$$

where C_n is a constant dependent on n .

Proof. It is clear that for any $n \in \mathbb{N}$

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|Y_\tau| \mathbf{1}_A] &\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|\delta Y_\tau| \mathbf{1}_A] + \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|Y_\tau^n| \mathbf{1}_A] \\ &\leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right] + \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|Y_\tau^n|^2]^{\frac{1}{2}} \mathcal{E}^{\mathbb{P}}[\mathbf{1}_A]^{\frac{1}{2}} \\ &\leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq n\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq n\}} ds \right] + C_n \delta^{\frac{1}{2}}. \end{aligned}$$

The second inequality is due to Proposition 4.10, and the last inequality is due to the classical estimate on \mathbb{L}^2 solution of BSDE. \square

5 Second-order backward SDE: representation and uniqueness

We now prove the following representation theorem for the solution of the 2BSDE (3.5). Note that this representation implies the uniqueness of the process Y , and further that of the process Z as $d\langle Y, X \rangle = Zd\langle X \rangle$.

Theorem 5.1. *Let Assumption 3.9 hold true and (Y, Z) be a solution to the 2BSDE (3.5) satisfying the minimality condition (3.7). For each $\mathbb{P} \in \mathcal{P}_0$, let $(\mathcal{Y}^{\mathbb{P}}, \mathcal{Z}^{\mathbb{P}}, \mathcal{N}^{\mathbb{P}})$ be the solution of the corresponding BSDE (3.1). Then, for any $\mathbb{P} \in \mathcal{P}_0$ and $\tau \in \mathcal{T}_{0,T}$,*

$$Y_\tau = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_\tau^{\mathbb{P}'}, \quad \mathbb{P} - a.s. \quad (5.1)$$

In particular, the 2BSDE has at most one solution in $\mathcal{D}(\mathcal{P}_0, \mathbb{F}^{+, \mathcal{P}_0}) \times \mathcal{H}^\beta(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0})$ for all $\beta \in (0, 1)$ satisfying the minimality condition (3.7), and the comparison result of Proposition 3.12 holds true.

Proof. The uniqueness of Y is an immediate consequence of (5.1), and implies the uniqueness of Z , $\widehat{a}_t dt \otimes \mathcal{P}_0$ -q.s. by the fact that $\langle Y, X \rangle_t = \langle \int_0^t Z_s \cdot X_s, X \rangle_t = \int_0^t \widehat{a}_s Z_s ds$, \mathbb{P} -a.s. This representation also implies the comparison result as an immediate consequence of the corresponding comparison result of the BSDEs $\mathcal{Y}^{\mathbb{P}}$.

This proof of the representation is similar to the one in [STZ12]. The only difference is due to the different minimality condition (3.7). Let $\mathbb{P} \in \mathcal{P}_0$ and $\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})$ be arbitrary. Since (3.6) holds \mathbb{P}' -a.s., we can consider Y as a supersolution of the BSDE on $[\tau, T]$ under \mathbb{P}' . By comparison result, Proposition 3.7(ii), we obtain that $Y_\tau \geq \mathcal{Y}_\tau^{\mathbb{P}'}$, \mathbb{P}' -a.s. As $\mathcal{Y}_\tau^{\mathbb{P}'}$ is \mathcal{F}_τ^+ -measurable and Y_τ is $\mathcal{F}_\tau^{+, \mathcal{P}_0}$ -measurable, we deduce that the inequality also holds \mathbb{P} -a.s., by definition of $\mathcal{P}_+(\tau, \mathbb{P})$ and the fact that measures extend uniquely to the completed σ -algebras. Therefore,

$$Y_\tau \geq \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_\tau^{\mathbb{P}'}, \quad \mathbb{P} - a.s. \quad (5.2)$$

by arbitrariness of \mathbb{P}' .

We now show the reverse inequality. Define $\delta := Y - \mathcal{Y}^{\mathbb{P}'}$, $\delta Z := Z - \mathcal{Z}^{\mathbb{P}'}$ and $\delta N := N^{\mathbb{P}'} - \mathcal{N}^{\mathbb{P}'}$. By Assumption 3.9, there exist two bounded processes $a^{\mathbb{P}'}$ and $b^{\mathbb{P}'}$ such that

$$\begin{aligned} \delta Y_\tau &= \int_\tau^T \left(a_s^{\mathbb{P}'} \delta Y_s + b_s^{\mathbb{P}'} \cdot \widehat{\sigma}_s^{\mathbf{T}} \delta Z_s \right) ds - \int_\tau^T \widehat{\sigma}_s^{\mathbf{T}} \delta Z_s \cdot dW_s - \int_\tau^T d\delta N_s + \int_\tau^T dK_s^{\mathbb{P}'} \\ &= \int_\tau^T a_s^{\mathbb{P}'} \delta Y_s - \int_\tau^T \widehat{\sigma}_s^{\mathbf{T}} \delta Z_s \cdot (dW_s - b_s^{\mathbb{P}'} ds) - \int_\tau^T d\delta N_s + \int_\tau^T dK_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s. \end{aligned}$$

Under the measure $\mathbb{Q}_\tau^{\mathbb{P}'}$, the process $W_s^{\mathbb{Q}_\tau^{\mathbb{P}'}} := W_s - \int_\tau^s b_u^{\mathbb{P}'} du$ is a Brownian motion on $[\tau, T]$ beginning with W_τ . Applying Itô's formula with $\delta Y_s e^{\int_\tau^s a_u^{\mathbb{P}'} du}$,

$$\delta Y_\tau = - \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} \widehat{\sigma}_s^{\mathbf{T}} \delta Z_s \cdot dW_s^{\mathbb{Q}_\tau^{\mathbb{P}'}} - \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} d\delta N_s + \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} dK_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s.$$

Taking conditional expectation with respect to $\mathbb{Q}_\tau^{\mathbb{P}'}$ and localization procedure if necessary, we obtain that

$$\delta Y_\tau = \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} \left[\int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} dK_s^{\mathbb{P}'} \middle| \mathcal{F}_\tau^+ \right] \leq e^{L_y T} \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} [K_T^{\mathbb{P}'} - K_\tau^{\mathbb{P}'} | \mathcal{F}_\tau^+].$$

By minimality condition (3.7)

$$0 \leq Y_\tau - \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})}^{\mathbb{P}} \delta Y_\tau \leq e^{L_y T} \left(\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} [K_T^{\mathbb{P}'} | \mathcal{F}_\tau^+] - K_\tau^{\mathbb{P}} \right) = 0, \quad \mathbb{P} - a.s.$$

Together with (5.2) the assertion follows. \square

6 Second-order backward SDE: existence

To prove the existence, we first define a value function V by means of the solutions of BSDEs on shifted spaces, then we show that V satisfies the dynamic programming principle, and introduce the corresponding pathwise right limit V^+ . By combining the standard Doob-Meyer decomposition with our results on reflected BSDEs, we obtain that V^+ satisfies the required 2BSDE.

We shall use the following notations for on the shifted space for some \mathbb{F} -stopping time τ :

$$\omega \otimes_{\tau} \omega' := \omega \otimes_{\tau(\omega)} \omega', \quad \xi^{\tau, \omega} := \xi^{\tau(\omega), \omega}, \quad Y^{\tau, \omega} := Y^{\tau(\omega), \omega}, \quad \text{and } Y_s^{t, \omega}(\omega') := Y_{t+s}(\omega \otimes_t \omega'),$$

for all $0 \leq t \leq s \leq T$. In the context of the canonical process X , this reduces to

$$X_s^{t, \omega}(\omega') = X_{t+s}(\omega \otimes_t \omega') = (\omega \otimes_t \omega')_{t+s} = \omega_t + \omega'_s, \quad s \in [0, T-t].$$

6.1 Backward SDEs on the shifted spaces

For every $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}(t, \omega)$, we consider the following BSDE

$$\mathcal{Y}_s^{t, \omega, \mathbb{P}} = \xi^{t, \omega} + \int_s^{T-t} F_r^{t, \omega}(\mathcal{Y}_r^{t, \omega, \mathbb{P}}, \mathcal{Z}_r^{t, \omega, \mathbb{P}}, \widehat{\sigma}_r) dr - \mathcal{Z}_r^{t, \omega, \mathbb{P}} \cdot dX_r - d\mathcal{N}_r^{t, \omega, \mathbb{P}}, \quad \mathbb{P} - \text{a.e.} \quad (6.1)$$

with $s \in [0, T-t]$. By Theorem 3.4 we have a unique solution $(\mathcal{Y}^{t, \omega, \mathbb{P}}, \mathcal{Z}^{t, \omega, \mathbb{P}}, \mathcal{N}^{t, \omega, \mathbb{P}}) \in \mathcal{S}_{T-t}^{\beta}(\mathbb{P}) \times \mathcal{H}_{T-t}^{\beta}(\mathbb{P}) \times \mathcal{N}_{T-t}^{\beta}(\mathbb{P})$ and $\mathcal{Y}^{t, \omega, \mathbb{P}} \in \mathcal{D}_{T-t}^{\beta}(\mathbb{P})$.

In this section, we will prove the following measurability result, which is important for the dynamic programming.

Proposition 6.1. *Under Assumption 3.1, the mapping $(t, \omega, \mathbb{P}) \mapsto \mathbb{Y}^{t, \omega, \mathbb{P}}[\xi, T]$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathcal{M}_1)$ -measurable.*

Proof. Let ξ^n and f^n be defined as in Section 2. Following Step 1-4 in the proof of [LRTY18, Lemma 4.2], we may construct the solution $(\mathcal{Y}_s^{n, t, \omega, \mathbb{P}}, \mathcal{Z}_s^{n, t, \omega, \mathbb{P}}, \mathcal{N}_s^{n, t, \omega, \mathbb{P}})$ of the following BSDE

$$\mathcal{Y}_s^{n, t, \omega, \mathbb{P}} = \xi^{n, t, \omega} + \int_s^{T-t} F_r^{n, t, \omega}(\mathcal{Y}_r^{n, t, \omega, \mathbb{P}}, \mathcal{Z}_r^{n, t, \omega, \mathbb{P}}, \widehat{\sigma}_r) dr - \int_s^{T-t} \mathcal{Z}_r^{n, t, \omega, \mathbb{P}} \cdot dX_r - \int_s^{T-t} d\mathcal{N}_r^{n, t, \omega, \mathbb{P}}$$

in a measurable way, such that $(t, \omega, s, \omega', \mathbb{P}) \mapsto \mathcal{Y}_s^{n, t, \omega, \mathbb{P}}(\omega')$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathcal{P}_b)$ -measurable. By Proposition 4.10 and **(H2)** we have that

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq s \leq T-t} \left| \mathcal{Y}_s^{n, t, \omega, \mathbb{P}} - \mathcal{Y}_s^{t, \omega, \mathbb{P}} \right|^{\beta} \right] \longrightarrow 0,$$

where $\mathcal{Y}^{t, \omega, \mathbb{P}}$ is the solution associated with $(F^{t, \omega}, \xi^{t, \omega})$. Then, it follows from [NN14, Lemma 3.2] that there exists an increasing sequence $\{n_k^{\mathbb{P}}\}_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\mathbb{P} \mapsto n_k^{\mathbb{P}}$ is measurable for each $k \in \mathbb{N}$ and

$$\lim_{k \rightarrow \infty} \sup_{0 \leq s \leq T-t} \left| \mathcal{Y}_s^{n_k^{\mathbb{P}}, t, \omega, \mathbb{P}} - \mathcal{Y}_s^{t, \omega, \mathbb{P}} \right| = 0.$$

Therefore, $(t, \omega, s, \omega', \mathbb{P}) \mapsto \mathcal{Y}_s^{t, \omega, \mathbb{P}}(\omega')$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathcal{P}_b)$ -measurable, and the mapping

$$(t, \omega, \mathbb{P}) \mapsto \mathbb{Y}^{t, \omega, \mathbb{P}}[\xi, T] = \mathbb{E}^{\mathbb{P}} \left[\mathcal{Y}_0^{t, \omega, \mathbb{P}} \right]$$

is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathcal{P}_b)$ -measurable. Since $\mathcal{P}_b \in \mathcal{B}(\mathcal{M}_1)$, the mapping $(t, \omega, \mathbb{P}) \mapsto \mathbb{Y}^{t, \omega, \mathbb{P}}[\xi, T]$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(\mathcal{M}_1)$ -measurable. \square

Lemma 6.2. *Let Assumptions 3.1 and 3.9 hold true. Then, for all $\tau \in \mathcal{T}_{0,T}$ and $\mathbb{P} \in \mathcal{P}_b$:*

- (i) *BSDE (3.1) and shifted version (6.1): $\mathbb{E}^{\mathbb{P}} \left[\mathcal{Y}_{\sigma}^{\mathbb{P}} \middle| \mathcal{F}_{\sigma} \right] (\omega) = \mathbb{Y}^{\sigma, \omega, \mathbb{P}^{\sigma, \omega}}[\xi, T]$, for \mathbb{P} -a.e. $\omega \in \Omega$;*
- (ii) *Tower property of BSDE: $Y_t[\xi, T] = Y_t[Y_{\sigma}, \sigma] = Y_t \left[\mathbb{E} \left[Y_{\sigma}[\xi, T] \middle| \mathcal{F}_{\sigma} \right], \sigma \right]$.*

We omit the proof as the assertion (i) is a direct result of the uniqueness of the solution to BSDE and the assertion (ii) is similar to [PTZ18, Lemma 2.7].

6.2 Dynamic programming

We define the value function

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}(t, \omega)} \mathbb{Y}^{t, \omega, \mathbb{P}}[\xi, T], \quad \text{with} \quad \mathbb{Y}^{t, \omega, \mathbb{P}}[\xi, T] := \mathbb{E}^{\mathbb{P}} \left[\mathcal{Y}_0^{t, \omega, \mathbb{P}} \right].$$

Now, we show the dynamic programming result by the measurable selection theorem. We first prove the following class $\mathcal{D}(\mathcal{P})$ integrability result for the process V .

Lemma 6.3. *Let Assumption 3.9 hold true. Then, the mapping $\omega \mapsto V_{\tau}(\omega)$ is \mathcal{F}_{τ}^U -measurable for each $[0, T]$ -valued \mathbb{F} -stopping time τ . For any $(t, \omega) \in [0, T] \times \Omega$,*

$$\lim_{n \rightarrow \infty} \sup_{\tau \in \mathcal{T}_{0,T}} \sup_{\mathbb{P} \in \mathcal{P}(t, \omega)} \sup_{\mathbb{Q} \in \mathcal{Q}_{L_z}(\mathbb{P})} \mathbb{E}^{\mathbb{Q}} \left[|(V_{\tau})^{t, \omega}| \mathbf{1}_{\{|(V_{\tau})^{t, \omega}| \geq n\}} \right] = 0.$$

Proof. By the measurability result proved in Proposition 6.1 and the measurable selection theorem (see, e.g., [BS96, Proposition 7.50]), for each $\varepsilon > 0$, there exists an \mathcal{F}_{τ}^U -measurable kernel $\nu^{\varepsilon} : \omega \mapsto \nu^{\varepsilon}(\omega) \in \mathcal{P}(\tau(\omega), \omega)$, such that for all $\omega \in \Omega$

$$V_{\tau}(\omega) \leq \mathbb{Y}^{\tau, \omega, \nu^{\varepsilon}(\omega)}[\xi, T] + \varepsilon. \quad (6.2)$$

This implies that $\omega \mapsto V_{\tau}(\omega)$ is \mathcal{F}_{τ}^U -measurable. Further it follows from Lemma 6.2 (i) that

$$\mathbb{Y}^{\tau, \omega, \nu^{\varepsilon}(\omega)}[\xi, T] = \mathbb{E}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \left[\mathcal{Y}_{\tau}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \middle| \mathcal{F}_{\tau} \right] (\omega), \quad \mathbb{P}\text{-a.s. for each } \mathbb{P} \in \mathcal{P}_0. \quad (6.3)$$

Together with (6.2) we have for $\mathbb{Q} \in \mathcal{Q}_{L_z}(\mathbb{P})$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[|V_{\tau}|] &\leq \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \left[|\mathcal{Y}_{\tau}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \middle| \mathcal{F}_{\tau} \right] \right] + \varepsilon \leq \mathcal{E}^{\mathbb{P}} \left[|\mathcal{Y}_{\tau}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \right] + \varepsilon \\ &\leq \mathcal{E}^{\mathbb{P}} \left[|\xi| + \int_0^T |f_s^0| ds \right] + \varepsilon \leq \mathcal{E}^{\mathcal{P}_0} \left[|\xi| + \int_0^T |f_s^0| ds \right] + \varepsilon. \end{aligned}$$

The second last inequality is due to the estimate (3.2) on the BSDE solution. So we have

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}}[|V_{\tau}|] \leq \mathcal{E}^{\mathcal{P}} \left[|\xi| + \int_0^T |f_s^0| ds \right] + \varepsilon < \infty. \quad (6.4)$$

Further, fix any $\delta > 0$ and $A \in \mathcal{F}_{\tau}^U$ such that $\mathcal{E}^{\mathcal{P}_0}[\mathbf{1}_A] < \delta$. It follows again from (6.2) and (6.3) that for $\mathbb{Q} \in \mathcal{Q}_{L_z}(\mathbb{P})$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[|V_{\tau}| \mathbf{1}_A] &\leq \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \left[|\mathcal{Y}_{\tau}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \middle| \mathbf{1}_A \middle| \mathcal{F}_{\tau} \right] \right] + \varepsilon \leq \mathcal{E}^{\mathbb{P}} \left[|\mathcal{Y}_{\tau}^{\mathbb{P}^{\otimes_{\tau} \nu^{\varepsilon}}} \middle| \mathbf{1}_A \right] + \varepsilon \\ &\leq \mathcal{E}^{\mathbb{P}} \left[|\xi| \mathbf{1}_{\{|\xi| \geq m\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq m\}} ds \right] + C_m \delta^{\frac{1}{2}} + \varepsilon, \quad \text{for all } m \in \mathbb{N}. \end{aligned}$$

The last inequality is due to Corollary 4.11. Now let m be large enough such that

$$\mathcal{E}^{\mathcal{P}_0} \left[|\xi| \mathbf{1}_{\{|\xi| \geq m\}} + \int_0^T |f_s^0| \mathbf{1}_{\{|f_s^0| \geq m\}} ds \right] < \varepsilon$$

and δ be small enough such that $C_m \delta^{\frac{1}{2}} < \varepsilon$. Then we obtain $\mathbb{E}^{\mathbb{Q}}[|V_\tau| \mathbf{1}_A] < 3\varepsilon$. Further note that the choice of m and δ is independent from \mathbb{P} and τ , so we have

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}_0} [|V_\tau| \mathbf{1}_A] < 3\varepsilon.$$

Finally, since

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}_0} [\mathbf{1}_{\{|V_\tau| \geq n\}}] \leq \frac{1}{n} \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}_0} [|V_\tau|],$$

for n big enough, $\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}_0} [\mathbf{1}_{\{|V_\tau| \geq n\}}] \leq \delta$ and thus $\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathcal{P}_0} [|V_\tau| \mathbf{1}_{\{|V_\tau| \geq n\}}] < 3\varepsilon$. \square

Using the last integrability result, we now show the following results using the same argument as in [PTZ18] and [LRTY18].

Proposition 6.4. *Under Assumption 3.9, we have*

$$V_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{Y}^{t,\omega,\mathbb{P}} [V_\tau, \tau] \text{ for all } (t, \omega) \in [0, T] \times \Omega, \text{ and } \tau \in \mathcal{T}_{t,T}. \quad (6.5)$$

Moreover, we have for all $\mathbb{P} \in \mathcal{P}_0$ and $\tau \in \mathcal{T}_{0,T}$:

$$V_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} [\mathcal{Y}_t^{\mathbb{P}'} [V_\tau, \tau] | \mathcal{F}_t], \quad \mathbb{P}\text{-a.s.} \quad (6.6)$$

Proof. See [LRTY18, Theorem 6.7]. \square

Based on the previous result we can define the right limit of the value function, and the next result shows that V^+ is actually a semimartingale under any $\mathbb{P} \in \mathcal{P}_0$, and gives its decomposition.

Lemma 6.5. *Let Assumptions 3.1 and 3.9 hold true. The right limit*

$$V_t^+(\omega) := \lim_{r \in \mathbb{Q}, r \downarrow t} V_r(\omega) \quad (6.7)$$

exists \mathcal{P}_0 -q.s. and the process V^+ is càdlàg \mathcal{P}_0 -q.s. Also we have:

- (i) *The process $V^+ \in \mathcal{D}(\mathcal{P}_0)$.*
- (ii) *For any \mathbb{F}^+ -stopping times $0 \leq \tau_1 \leq \tau_2 \leq T$*

$$V_{\tau_1}^+ = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_+(\tau_1, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_{\tau_1}^{\mathbb{P}'} [V_{\tau_2}^+, \tau_2], \quad \mathbb{P}\text{-a.s.} \quad (6.8)$$

Further, for any $\mathbb{P} \in \mathcal{P}_0$ and $\beta \in (0, 1)$, there is $(Z^\mathbb{P}, M^\mathbb{P}, K^\mathbb{P}) \in \mathcal{H}^\beta(\mathbb{P}, \mathbb{F}^+, \mathbb{P}) \times \mathcal{N}^\beta(\mathbb{P}, \mathbb{F}^+, \mathbb{P}) \times \mathcal{I}^\beta(\mathbb{P}, \mathbb{F}^+, \mathbb{P})$, such that

$$V_t^+ = \xi + \int_t^T F_s(V_s^+, Z_s^\mathbb{P}, \hat{\sigma}_s) ds - \int_t^T Z_s^\mathbb{P} \cdot dX_s - \int_t^T dN_s^\mathbb{P} + \int_t^T dK_s^\mathbb{P}, \quad \mathbb{P}\text{-a.s.}$$

where $[N^\mathbb{P}, X] = 0$. Moreover, there is some $\mathbb{F}^{\mathcal{P}_0}$ -predictable process Z which aggregates the family $\{Z^\mathbb{P}\}_{\mathbb{P} \in \mathcal{P}_0}$.

Proof. See [LRTY18, Proposition 6.8] and the step 1 in the proof of the existence part of [LRTY18, Theorem 3.12]. \square

6.3 Existence through dynamic programming

Lemma 6.5 above provides us a candidate $(Y, Z) = (V^+, Z)$ of the solution of the 2BSDE (3.5). Then, it suffices to verify that the family $\{K^{\mathbb{P}}\}_{\mathbb{P} \in \mathcal{P}_0}$ satisfies the minimality condition (3.7).

Proof. Let $\mathbb{P} \in \mathcal{P}_0$, $\tau \in \mathcal{T}_{0,T}$ and $\mathbb{P}' \in \mathcal{P}_+(\tau, \mathbb{P})$ be arbitrary. Let $(\mathcal{Y}^{\mathbb{P}'}, \mathcal{Z}^{\mathbb{P}'}, \mathcal{N}^{\mathbb{P}'})$ be the solution of

$$\mathcal{Y}_t^{\mathbb{P}'} = \xi + \int_t^T F_s(\mathcal{Y}^{\mathbb{P}'}, \mathcal{Z}^{\mathbb{P}'}, \hat{\sigma}_s) ds - \int_t^T \mathcal{Z}_s^{\mathbb{P}'} \cdot dX_s - \int_t^T d\mathcal{N}_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s.$$

Define $\delta Y := V^+ - \mathcal{Y}^{\mathbb{P}'}$, $\delta Z := Z - \mathcal{Z}^{\mathbb{P}'}$ and $\delta N^{\mathbb{P}'} := N^{\mathbb{P}'} - \mathcal{N}^{\mathbb{P}'}$. Then,

$$\begin{aligned} \delta Y_\tau &= \int_\tau^T \left(F_s(V_s^+, Z_s, \hat{\sigma}_s) - F_s(\mathcal{Y}_s^{\mathbb{P}'}, \mathcal{Z}_s^{\mathbb{P}'}, \hat{\sigma}_s) \right) ds - \int_\tau^T \delta Z_s \cdot dX_s - \int_\tau^T d\delta N_s^{\mathbb{P}'} + \int_\tau^T dK_s^{\mathbb{P}'} \\ &= \int_\tau^T (a_s^{\mathbb{P}'} \delta Y_s + b_s^{\mathbb{P}'} \cdot \hat{\sigma}_s^{\mathbb{T}} \delta Z_s) ds - \int_\tau^T \hat{\sigma}_s^{\mathbb{T}} \delta Z_s \cdot dW_s - \int_\tau^T d\delta N_s^{\mathbb{P}'} + \int_\tau^T dK_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s. \end{aligned}$$

where $a^{\mathbb{P}'}$ and $b^{\mathbb{P}'}$ are two bounded processes bounded by L . Under the measure $\mathbb{Q}_\tau^{\mathbb{P}'}$, we have

$$\delta Y_\tau = \int_\tau^T a_s^{\mathbb{P}'} \delta Y_s ds - \int_\tau^T \hat{\sigma}_s^{\mathbb{T}} \delta Z_s \cdot dW_s^{\mathbb{Q}_\tau^{\mathbb{P}'}} - \int_\tau^T d\delta N_s^{\mathbb{P}'} + \int_\tau^T dK_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s.$$

We next get rid of the linear term in δY by introducing $\bar{\delta Y}_s := \delta Y_s e^{\int_\tau^s a_u^{\mathbb{P}'} du}$, $\tau \leq s \leq T$ so that

$$\delta Y_\tau = - \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} \hat{\sigma}_s^{\mathbb{T}} \delta Z_s \cdot dW_s^{\mathbb{Q}_\tau^{\mathbb{P}'}} - \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} d\delta N_s^{\mathbb{P}'} + \int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} dK_s^{\mathbb{P}'}, \quad \mathbb{P}' - a.s.$$

Taking conditional expectation with respect to $\mathbb{Q}_\tau^{\mathbb{P}'}$ and localization procedure if necessary, we obtain that

$$\delta Y_\tau = \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} \left[\int_\tau^T e^{\int_\tau^s a_u^{\mathbb{P}'} du} dK_s^{\mathbb{P}'} \middle| \mathcal{F}_\tau^+ \right] \geq e^{-L_y T} \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} [K_T^{\mathbb{P}'} - K_\tau^{\mathbb{P}'} | \mathcal{F}_\tau^+],$$

therefore,

$$0 \leq \mathbb{E}^{\mathbb{Q}_\tau^{\mathbb{P}'}} [K_T^{\mathbb{P}'} | \mathcal{F}_\tau^+] - K_\tau^{\mathbb{P}'} \leq e^{L_y T} \delta Y_\tau.$$

Then, the result follows immediately thanks to (6.8). \square

A Appendix

A.1 Uniform integrability under $\mathcal{Q}_L(\mathbb{P})$

Here, we show that the space of progressive measurable càdlàg processes which belong to class $\mathcal{D}(\mathbb{P})$ is complete under the norm

$$\|Y\|_{\mathcal{D}(\mathbb{P})} := \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}^{\mathbb{P}}[|Y_\tau|].$$

First of all, we proof an equivalent characterization of the concept of uniform integrability.

Proposition A.1. *A family $\{X_t\}_{t \in \mathbb{T}}$ of random variables, where \mathbb{T} is an arbitrary index set, is uniformly integrable under $\mathcal{Q}_L(\mathbb{P})$, i.e.,*

$$\lim_{N \rightarrow \infty} \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_{\{|X_t| \geq N\}}] = 0, \quad (\text{A.1})$$

if and only if the following two conditions are satisfied:

(a) $\sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t|] < \infty$,

(b) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for any $A \in \mathcal{F}$ with $\sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{Q}[A] < \delta$ we have

$$\sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_A] < \varepsilon.$$

Proof. Clearly, (A.1) implies (a). Next, let $A \in \mathcal{F}$ and write $A_t := \{|X_t| \geq N\}$. Then, we have

$$\begin{aligned} \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_A] &= \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t|(\mathbf{1}_{A \cap A_t} + \mathbf{1}_{A \setminus A_t})] \\ &\leq \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_{A_t}] + N \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[\mathbf{1}_{A_t}] \\ &\leq \sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_{A_t}] + N \sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{Q}[A]. \end{aligned}$$

Given $\varepsilon > 0$, by (A.1), we may find N such that $\sup_{t \in \mathbb{T}} \mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_{A_t}] < \frac{\varepsilon}{2}$. Therefore, (b) follows by setting $\delta = \frac{\varepsilon}{2N}$.

Conversely, suppose that (a) and (b) hold. Then, by Markov inequality, we obtain that

$$\sup_{t \in \mathbb{T}} \sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{Q}[|X_t| \geq N] \leq \sup_{t \in \mathbb{T}} \frac{1}{N} \mathcal{E}^{\mathbb{P}}[|X_t|] \leq \frac{M}{N},$$

where M is the bound indicated in (a). Hence, if $N \geq \frac{M}{\delta}$, then $\sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{Q}[A_t] < \delta$, for each $t \in \mathbb{T}$. By (b), we have for each $t \in \mathbb{T}$ that $\mathcal{E}^{\mathbb{P}}[|X_t| \mathbf{1}_{A_t}] < \varepsilon$. Thus, (A.1) follows. \square

Now, we show the completeness of $\mathcal{D}(\mathbb{P})$.

Theorem A.2. *The space $\mathcal{D}(\mathbb{P})$ is complete with respect to the norm $\|\cdot\|_{\mathcal{D}(\mathbb{P})}$.*

Proof. Let $\{X^n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{P})$ be a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{D}(\mathbb{P})}$. In particular, this is a Cauchy sequence with respect to $\|\cdot\|_{\mathbb{D}(\mathbb{P})}$. By [DM82, VI Theorem 22, Page 83], there exists a càdlàg process X such that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} (X_t - X_t^n) = 0, \quad \mathbb{P} - a.s.,$$

and $\|X\|_{\mathbb{D}(\mathbb{P})} < \infty$. Since $\mathbb{Q} \sim \mathbb{P}$ for each $\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})$, the above convergence holds also for each $\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})$. As $\{X^n\}_{n \in \mathbb{N}}$ is a Cauchy with respect to $\|\cdot\|_{\mathcal{D}(\mathbb{P})}$, for each $\varepsilon > 0$ there exists a $N \in \mathbb{N}$, such that $\|X - X^N\|_{\mathcal{D}(\mathbb{P})} < \varepsilon$, and by triangle inequality we obtain

$$\|X\|_{\mathcal{D}(\mathbb{P})} \leq \|X - X^N\|_{\mathcal{D}(\mathbb{P})} + \|X^N\|_{\mathcal{D}(\mathbb{P})} < \infty.$$

To show the uniform integrability, it suffices to show (b) in Proposition A.1. For each $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $\sup_{\tau \in \mathcal{T}_{0, T}} \mathcal{E}^{\mathbb{P}}[|X_\tau - X_\tau^N|] < \frac{\varepsilon}{2}$, and $\delta > 0$ such that $\sup_{\tau \in \mathcal{T}_{0, T}} \mathcal{E}^{\mathbb{P}}[|X_\tau^N| \mathbf{1}_A] < \frac{\varepsilon}{2}$, for each $\sup_{\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})} \mathbb{Q}[A] < \delta$. Therefore, by triangle inequality, we obtain that

$$\sup_{\tau \in \mathcal{T}_{0, T}} \mathcal{E}^{\mathbb{P}}[|X_\tau| \mathbf{1}_A] \leq \sup_{\tau \in \mathcal{T}_{0, T}} \mathcal{E}^{\mathbb{P}}[|X_\tau - X_\tau^N|] + \sup_{\tau \in \mathcal{T}_{0, T}} \mathcal{E}^{\mathbb{P}}[|X_\tau^N| \mathbf{1}_A] < \varepsilon,$$

and the assertion follows. \square

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