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Second order backward SDE with random terminal time *

Yiqing Lin† Zhenjie Ren‡ Nizar Touzi§ Junjian Yang¶

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Abstract

Backward stochastic differential equations extend the martingale representation theorem to the nonlinear setting. This can be seen as path-dependent counterpart of the extension from the heat equation to fully nonlinear parabolic equations in the Markov setting. This paper extends such a nonlinear representation to the context where the random variable of interest is measurable with respect to the information at a finite stopping time. We provide a complete wellposedness theory which covers the semilinear case (backward SDE), the semilinear case with obstacle (reflected backward SDE), and the fully nonlinear case (second order backward SDE).

MSC2010. 60H10, 60H30

Keywords. Backward SDE, second order backward SDE, quasi-sure stochastic analysis, random horizon

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, supporting a $d-$dimensional Brownian motion W. The martingale representation theorem states that any integrable $\mathcal{F}_\tau-$measurable random variable ξ, for some $\mathbb{F}-$stopping time τ, can be represented as $\xi = \mathbb{E}\xi + (Z \cdot W)_\tau + N_\tau$, for some square integrable $\mathbb{F}-$predictable process Z, and some martingale N with $N_0 = 0$ and $[N, W] = 0$. In particular when $\mathbb{F}$ is the (augmented) canonical filtration of the Brownian motion, $N = 0$. This result can be seen as the path-dependent counterpart of the heat equation. Indeed, a standard density argument reduces to the case $\xi = g(W_{t_0}, \ldots, W_{t_n})$ for an arbitrary partition $0 = t_0 < \ldots < t_n = T$ of $[0, T]$, where the representation follows from a backward resolution of the heat equation $\partial_t v + \frac{1}{2}\Delta v = 0$ on each time interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, and the Z process is identified to the space gradient of the solution.

As a first extension of the martingale representation theorem, the seminal work of Pardoux & Peng [PP90] introduced the theory of backward stochastic differential equations in finite horizon, extended further to the random horizon setting by Darling & Pardoux [DP97]. In words, this theory provides a representation of an $\mathcal{F}_\tau-$measurable random variable ξ with appropriate

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integrability as $\xi = Y_\tau$ with $Y_{t\wedge\tau} = Y_0 - \int_0^{t\wedge\tau} f_s(Y_s, Z_s)ds + (Z \cdot W)_{t\wedge\tau} + N_{t\wedge\tau}$, $t \geq 0$, where f is a given random field. In the Markov setting where $\xi = g(W_T)$ and $f_t(\omega, y, z) = f(t, W_t(\omega), y, z)$, $t \geq 0$, it turns out that $Y_t(\omega) = v(t, W_t(\omega))$ for some deterministic function $v : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$, which is easily seen to correspond to the semilinear heat equation $\partial_t v + \frac{1}{2}\Delta v + f(., v, Dv) = 0$, by the fact that the Z process again identifies the space gradient of v.

As our interest in this paper is on the random horizon setting, we refer the interested reader to the related works by Briand & Hu [BH98], Briand and Carmona [BC00], Royer [Roy04], Bahlali, Elouaflin & N'zi [BEN04], Popier [Pop07], Briand and Confortola [BC08]. We also mention the related work of Hamadène, Lepeltier & Wu [HLW99] which considers the infinite horizon.

Our main interest in this paper is on the extension to the fully nonlinear second order parabolic equations, as initiated in the finite horizon setting by Soner, Touzi & Zhang [STZ12], and further developed by Possamaï, Tan & Zhou [PTZ17], see also the first attempt by Cheridito, Soner, Touzi & Victoir [CSTV07], and the closely connected BSDEs in a nonlinear expectation framework of Hu, Ji, Peng & Song [HJPS14a, HJPS14b] (called GBSDEs). This extension is performed on the canonical space of continuous paths with canonical process denoted by X. The key idea is to reduce the fully nonlinear representation to a semilinear representation which is required to hold simultaneously under an appropriate family $\mathcal{P}$ of singular semimartingale measures on the canonical space. Namely, an $\mathcal{F}_T-$ random variable ξ with appropriate integrability is represented as

$$\xi = Y_T, \text{ where } Y_t = Y_0 - \int_0^t F_s(Y_s, Z_s, \hat{\sigma}_s)ds + (Z \bullet X)_t + U_t^{\mathbb{P}},\ t \geq 0,\ \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}.$$

Here, $\hat{\sigma}_s^2 ds = d\langle X\rangle_s$, and $U^{\mathbb{P}}$ is a supermartingale with $U_0^{\mathbb{P}} = 0$, $[U^{\mathbb{P}}, X] = 0$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}$ satisfying the minimality condition $\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[U_T^{\mathbb{P}}] = 0$. Loosely speaking, in the Markov setting where $Y_t(\omega) = v(t, X_t(\omega))$ for some deterministic function v, the last representation implies that v is a supersolution of a semilinear parabolic PDE parameterized by the diffusion coefficient $-\partial_t v - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}} D^2 v] - F(t, x, v, Dv, \sigma) \geq 0$, and the minimality condition induces the fully nonlinear parabolic PDE $-\partial_t v - \sup_\sigma\{\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\mathrm{T}} D^2 v] + F(t, x, v, Dv, \sigma)\} = 0$.

Our main contribution is to extend the finite horizon fully nonlinear representation of [STZ12] and [PTZ17] to the context of a random horizon defined by a finite $\mathbb{F}-$stopping time. In view of the formulation of second order backward SDEs as backward SDEs holding simultaneously under a non-dominated family of singular measures, we review –and in fact complement– the corresponding theory of backward SDEs, and we develop the theory of reflected backward SDEs, which is missing in the literature, and which plays a crucial role in the well-posedness of second order backward SDEs.

Finally, we emphasize that backward SDEs and their second order extension provide a Sobolev-type of wellposedness as uniqueness holds within an appropriate integrability class of the solution Y and the corresponding "space gradient" Z. Also, our extension to the random horizon setting allows in particular to cover the elliptic fully nonlinear second order PDEs with convex dependence on the Hessian component.

The paper is organized as follows. Section 2 sets the notations used throughout the paper. Our main results are contained in Section 3, with proofs reported in the remaining sections. Namely, Section 4 contains the proofs related to backward SDEs and the corresponding reflected version, while Sections 5 and 6 focus on the uniqueness and the existence, respectively, for the second order backward SDEs.

2 Preliminaries

2.1 Canonical space

Fix $d \in \mathbb{N}_+$, $m \in \mathbb{N}_+$, $m \geq d$. Let

$$\Omega \;=\; \Big\{\overline{\omega} := (\omega^1, \omega^2) : \omega^1 \in \mathcal{C}\big([0,\infty); \mathbb{R}^d\big),\ \omega^2 \in \mathcal{C}\big([0,\infty),\ \mathbb{R}^m\big)\,;\ \overline{\omega}_0 = \mathbf{0}\Big\},$$

be the space of continuous paths starting from the origin equipped with the distance defined by $\|\overline{\omega} - \overline{\omega}'\|_\infty := \sum_{n\geq 0} 2^{-n}\big(\sup_{0\leq t\leq n} \|\overline{\omega}_t - \overline{\omega}'_t\| \wedge 1\big)$. Define the canonical process $\overline{X} := (X, W)$ by

$$X_t(\overline{\omega}) := \omega^1_t \quad \text{and} \quad W_t(\overline{\omega}) = \omega^2_t,\ t \geq 0.$$

Let $\mathcal{M}_1$ be the collection of all probability measures on $(\Omega, \mathcal{F})$, equipped with the topology of weak convergence. Denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the raw filtration generated by the canonical process $\overline{X}$. Denote by $\mathbb{F}^+ := (\mathcal{F}^+_t)_{t\geq 0}$ the right limit of $(\mathcal{F}_t)_{t\geq 0}$. For each $\mathbb{P} \in \mathcal{M}_1$, we denote by $\mathbb{F}^{+,\mathbb{P}}$ the augmented filtration of $\mathbb{F}^+$ under $\mathbb{P}$. The filtration $\mathbb{F}^{+,\mathbb{P}}$ is the coarsest filtration satisfying the usual conditions. We denote by $\mathbb{F}^U := \big(\mathcal{F}^U_t\big)_{t\geq 0}$ and $\mathbb{F}^{+,U} := \big(\mathcal{F}^{+,U}_t\big)_{t\geq 0}$ the (right-continuous) universal completed filtration defined by

$$\mathcal{F}^U_t := \cap_{\mathbb{P}\in\mathcal{M}_1} \mathcal{F}^{\mathbb{P}}_t \quad \text{and} \quad \mathcal{F}^{+,U}_t := \cap_{\mathbb{P}\in\mathcal{M}_1} \mathcal{F}^{+,\mathbb{P}}_t.$$

Clearly, $\mathbb{F}^{+,U}$ is right-continuous. Simialrly, for $\mathcal{P} \subseteq \mathcal{M}_1$, we introduce $\mathbb{F}^{\mathcal{P}} := \big(\mathcal{F}^{\mathcal{P}}_t\big)_{t\geq 0}$ and $\mathbb{F}^{+,\mathcal{P}} := \big(\mathcal{F}^{+,\mathcal{P}}_t\big)_{t\geq 0}$, where

$$\mathcal{F}^{\mathcal{P}}_t := \cap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}^{\mathbb{P}}_t \quad \text{and} \quad \mathcal{F}^{+,\mathcal{P}}_t := \cap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}^{+,\mathbb{P}}_t.$$

For any family $\Pi \subset \mathcal{M}_1$, we say that a property holds $\Pi-$quasi-surely, abbreviated as $\Pi-$q.s., if it holds $\mathbb{P}-$a.s. for all $\mathbb{P} \in \Pi$.

We denote by $\mathcal{P}_{loc} \subseteq \mathcal{M}_1$ the collection of probability measures such that for each $\mathbb{P} \in \mathcal{P}_{loc}$,
• X is a continuous $\mathbb{P}$-local martingale whose quadratic variation is absolutely continuous in t with respect to the Lebesgue measure;
• W is an m-dimensional $\mathbb{P}$-Browinian motion such that $\langle X, W\rangle$ is absolutely continuous in t with respect to the Lebesgue measure.

Due to the continuity of X, that X is an $\mathbb{F}$-local martingale under $\mathbb{P}$ implies that X is an $\mathbb{F}^{+,\mathbb{P}}$-local martingale. Similarly, W is an $\mathbb{F}^{+,\mathbb{P}}$-Brownian motion under $\mathbb{P}$.

As in [Kar95], we can define a pathwise version of a $(d+m)\times(d+m)$-matrix-valued process $\langle\overline{X}\rangle$. The constructed process is $\mathbb{F}$-progressively measurable and coincides with the quadratic variation of $\overline{X}$ under all $\mathbb{P} \in \mathcal{P}_{loc}$. In particular, the $d\times d$-matrix-valued and $d\times m$-matrix-valued processes $\langle X, X\rangle$ and $\langle X, W\rangle$ are defined pathwisely, and we may introduce the corresponding $\mathbb{F}$-progressively measurable density processes

$$\widehat{a}_t := \limsup_{n\to\infty} n\big(\langle X, X\rangle_t - \langle X, X\rangle_{t-\frac{1}{n}}\big),\ \text{and}\ \widehat{\sigma}_t := \limsup_{n\to\infty} n\big(\langle X, W\rangle_t - \langle X, W\rangle_{t-\frac{1}{n}}\big),\ t > 0,$$

so that $\langle X, X\rangle_t = \int_0^t \widehat{a}_s ds$ and $\langle X, W\rangle_t = \int_0^t \widehat{\sigma}_s ds$, $t \geq 0$, $\mathbb{P}-$a.s., for all $\mathbb{P} \in \mathcal{P}_{loc}$.

Remark 2.1. For later use, we observe that, as $\widehat{a} \in \mathbb{S}^{\geq 0}_d$, the set of $d \times d$ nonnegative-definite symmetric matrices, we may define a measurable[1] generalized inverse $\widehat{a}^{-1}$.

[1]Any matrix $S \in \mathbb{S}^{\geq 0}_d$ has a decomposition $S = Q^T_S \Lambda_S Q_S$ for some orthogonal matrix Q_S, and a diagonal matrix Λ_S, with Borel-measurable maps $S \mapsto Q_S$ and $S \mapsto \Lambda_S$, as this decomposition can be obtained by e.g. the Rayleigh quotient iteration. This implies the Borel measurability of the generalized inverse map $S \in \mathbb{S}^{\geq 0}_d \longmapsto S^{-1} := Q^T \Lambda^{-1} Q \in \mathbb{S}^{\geq 0}$, where Λ^{-1} is the diagonal element defined by $\Lambda^{-1}_{ii} := \Lambda_{ii}\mathbf{1}_{\{\Lambda_{ii}\neq 0\}}$, $i = 1, \ldots, d$.

Throughout this paper, we shall work with the the following subset of $\mathcal{P}_{loc}$:

$$\mathcal{P}_b \quad := \quad \big\{\mathbb{P} \in \mathcal{P}_{loc} : \widehat{\sigma} \text{ is bounded and } \widehat{a}_t = \widehat{\sigma}_t \widehat{\sigma}_t^{\mathbf{T}},\ dt \otimes \mathbb{P}(d\omega)\text{-a.e.}\big\}.$$

Lemma 2.2. *$\mathcal{P}_b \in \mathcal{B}(\mathcal{M}_1)$, and we have $X_t = \int_0^t \widehat{\sigma}_s dW_s$, $t \geq 0$, $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}_b$.*

Proof. The measurability of $\mathcal{P}_b$ follows from Nutz & von Handel [NvH13, Lemma 4.5]. We consider the extended space $\widetilde{\Omega} := \Omega \times \Omega'$, where $\Omega' := \mathcal{C}([0,\infty); \mathbb{R}^{d+m})$ equipped with the filtration $(\mathcal{F}'_t)_{t\geq 0}$ generated by the canonical process. Denote by $\mathbb{P}'_0$ the Wiener measure on Ω'. Set $\widetilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}'_t$, $\widetilde{\mathbb{F}} := \mathbb{F} \otimes \mathbb{F}'$ and $\widetilde{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}'_0$. Extend $\widehat{a}$, $\widehat{\sigma}$, X and W from Ω to $\widetilde{\Omega}$ in the obvious way, and denote these extensions by $\widetilde{a}$, $\widetilde{\sigma}$, $\widetilde{X}$ and $\widetilde{W}$. Note that

$$\begin{pmatrix} \widetilde{a}_t & \widetilde{\sigma}_t \\ \widetilde{\sigma}_t^{\mathbf{T}} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \widetilde{\sigma}_t & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widetilde{\sigma}_t^{\mathbf{T}} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad dt \otimes \widetilde{\mathbb{P}}(d\omega)\text{-a.s.}$$

By [SV97, Theorem 4.5.2], there is a $d+m$-dimensional Brownian motion $\widetilde{B}$ on $\big(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}}\big)$, such that

$$d\begin{pmatrix} \widetilde{X}_t \\ \widetilde{W}_t \end{pmatrix} = \begin{pmatrix} \widetilde{\sigma}_t & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} d\widetilde{B}_t.$$

Obviously, we have $d\widetilde{W}_t = d\big(\widetilde{B}^1_t, \widetilde{B}^2_t, \ldots, \widetilde{B}^m_t\big)^{\mathbf{T}}$. Then, $\widetilde{X}_t = \int_0^t (\widetilde{\sigma}_s, \mathbf{0})\, d\widetilde{B}_s = \int_0^t \widetilde{\sigma}_s d\widetilde{W}_s$, $t \geq 0$, $\widetilde{\mathbb{P}}$-a.s. which implies the desired result. □

2.2 Spaces and norms

Let $p > 1$ and $\alpha \in \mathbb{R}$.

(i) *One-measure integrability classes:* for a probability measure $\mathbb{P} \in \mathcal{M}_1$, let τ be an $\mathbb{F}^{+,\mathbb{P}}$-stopping time. We denote:

- $\mathbb{L}^p_{\alpha,\tau}(\mathbb{P})$ is the space of $\mathbb{R}$-valued and $\mathcal{F}^{+,\mathbb{P}}_\tau$-measurable random variables ξ, such that

$$\|\xi\|^p_{\mathbb{L}^p_{\alpha,\tau}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\big[\big|e^{\alpha\tau}\xi\big|^p\big] < \infty.$$

- $\mathbb{D}^p_{\alpha,\tau}(\mathbb{P})$ is the space of $\mathbb{R}$-valued, $\mathbb{F}^{+,\mathbb{P}}$-adapted processes Y with càdlàg paths, such that

$$\|Y\|^p_{\mathbb{D}^p_{\alpha,\tau}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq\tau}\big|e^{\alpha t}Y_t\big|^p\Big] < \infty.$$

- $\mathbb{H}^p_{\alpha,\tau}(\mathbb{P})$ is the space of $\mathbb{R}^d$-valued, $\mathbb{F}^{+,\mathbb{P}}$-progressively measurable processes Z such that

$$\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\alpha t}\widehat{\sigma}_t^{\mathbf{T}} Z_t\big|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbb{N}^p_{\alpha,\tau}(\mathbb{P})$ is the space of $\mathbb{R}$-valued, $\mathbb{F}^{+,\mathbb{P}}$-adapted martingales N such that

$$\|N\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau e^{2\alpha t} d[N]_t\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbb{I}^p_{\alpha,\tau}(\mathbb{P})$ is the set of scalar $\mathbb{F}^{+,\mathbb{P}}$-predictable processes K with càdlàg nondecreasing paths, s.t.

$$\|K\|^p_{\mathbb{K}^p_{\alpha,\tau}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau e^{\alpha t} dK_t\Big)^p\Big] < \infty.$$

• $\mathbb{U}^p_{\alpha,\tau}(\mathbb{P})$ is the set of càdlàg $\mathbb{F}$-supermartingales U, with Doob-Meyer decomposition $U = N - K$ into the difference of a martingale and a predictable non-decreasing process, such that

$$\|U\|^p_{\mathbb{U}^p_{\alpha,\tau}(\mathbb{P})} := \|N\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{P})} + \|K\|^p_{\mathbb{I}^p_{\alpha,\tau}(\mathbb{P})} < \infty.$$

(ii) *Integrability classes under dominated nonlinear expectation:* For $\mathbb{P} \in \mathcal{P}_b$, denote by $\mathcal{Q}_L(\mathbb{P})$ the set of all probability measures $\mathbb{Q}^\lambda$ such that

$$\mathrm{D}_t^{\mathbb{Q}^\lambda|\mathbb{P}} := \frac{d\mathbb{Q}^\lambda}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\Big(\int_0^t \lambda_s \cdot dW_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds\Big), \quad t \geq 0,$$

for some $\mathbb{F}^{+,\mathbb{P}}$-progressively measurable process $\lambda = (\lambda)_{t\geq 0}$ uniformly bounded by L, which is a fixed Lipschitz constant throughout this paper, see Assumption 3.1. By Girsanov's Theorem,

$$W^\lambda_{\cdot\wedge t} := W_{\cdot\wedge t} - \int_0^{\cdot\wedge t} \lambda_s ds \quad \text{is an } \big(\mathbb{F}, \mathbb{Q}^\lambda\big)\text{−Brownian motion on } [0,t],$$

for all $t > 0$. For $\mathbb{P} \in \mathcal{P}_b$, we denote

$$\mathcal{E}^{\mathbb{P}}[\cdot] \quad := \quad \sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})} \mathbb{E}^{\mathbb{Q}}[\cdot],$$

and we introduce the subspace $\mathcal{L}^p_{\alpha,\tau}(\mathbb{P}) \subset \cap_{\mathbb{Q}\in\mathcal{Q}_L} \mathbb{L}^p_{\alpha,\tau}(\mathbb{P})$ of r.v. ξ such that

$$\sup_{\mathbb{Q}\in\mathcal{Q}_L} \|\xi\|_{\mathbb{L}^p_{\alpha,\tau}} \;=\; \mathcal{E}^{\mathbb{P}}\big[|e^{\alpha\tau}\xi|^p\big] \;<\; \infty.$$

We define similarly the subspaces $\mathcal{D}^p_{\alpha,\tau}(\mathbb{P})$, $\mathcal{H}^p_{\alpha,\tau}(\mathbb{P})$, $\mathcal{N}^p_{\alpha,\tau}(\mathbb{P})$, and the subsets $\mathcal{I}^p_{\alpha,\tau}(\mathbb{P})$, $\mathcal{U}^p_{\alpha,\tau}(\mathbb{P})$.

(iii) *Integrability classes under non-dominated nonlinear expectation:* Let $\mathcal{P} \subseteq \mathcal{P}_b$ be a subset of probability measures, and denote

$$\mathcal{E}^{\mathcal{P}}[\cdot] \quad := \quad \sup_{\mathbb{P}\in\mathcal{P}} \mathcal{E}^{\mathbb{P}}[\cdot].$$

Let $\mathbb{G} := \{\mathcal{G}_t\}_{t\geq 0}$ be a filtration with $\mathcal{G}_t \supseteq \mathcal{F}_t$ for all $t \geq 0$, so that τ is also a $\mathbb{G}$-stopping time. We define the subspace $\mathcal{L}^p_{\alpha,\tau}(\mathcal{P},\mathbb{G})$ as the collection of all $\mathcal{G}_\tau$-measurable $\mathbb{R}$-valued random variables ξ, such that

$$\|\xi\|^p_{\mathcal{L}^p_{\alpha,\tau}(\mathcal{P})} := \mathcal{E}^{\mathcal{P}}\big[\big|e^{\alpha\tau}\xi\big|^p\big] < \infty.$$

We define similarly the subspaces $\mathcal{D}^p_{\alpha,\tau}(\mathcal{P},\mathbb{G})$ and $\mathcal{H}^p_{\alpha,\tau}(\mathcal{P},\mathbb{G})$ by replacing $\mathbb{F}^{+,\mathbb{P}}$ with $\mathbb{G}$.

3 Main results

3.1 Random horizon backward SDE

For a probability measure $\mathbb{P} \in \mathcal{P}_b$, a finite $\mathbb{F}$-stopping time τ, an $\mathcal{F}^{+,\mathbb{P}}_\tau$-measurable r.v. ξ, and a *generator* $F : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d \longrightarrow \mathbb{R} \cup \{\infty\}$, $\mathrm{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{S}^d)$-measurable [2], we set

$$f_t(\omega, y, z) \quad := \quad F_t\big(\omega, y, z, \widehat{\sigma}_t(\omega)\big), \; (t, \omega, y, z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d,$$

[2] By Prog we denote the σ-algebra generated by progressively measurable processes. Consequently, for every fixed $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the process $\big(F_t(y, z, \widehat{\sigma}_t)\big)_{t\geq 0}$ is progressively measurable.

and we consider the following backward stochastic differential equation (BSDE):

$$Y_{t\wedge\tau} \;=\; \xi + \int_{t\wedge\tau}^{\tau} f_s(Y_s, Z_s)ds - \int_{t\wedge\tau}^{\tau} (Z_s \cdot dX_s + dN_s), \quad \mathbb{P}-\text{a.s.} \tag{3.1}$$

Here, Y is a càdlàg adapted scalar process, Z is a predictable $\mathbb{R}^d$-valued process, and N a càdlàg $\mathbb{R}$-valued martingale with $N_0 = 0$ orthogonal to X, i.e., $[X, N] = 0$. We recall from Lemma 2.2 that $dX_s = \widehat{\sigma}_s dW_s$, $\mathbb{P}-$a.s.

By freezing the pair (y, z) to 0, we set $f_t^0 := f_t(0, 0)$.

Assumption 3.1. *The generator satisfies the following conditions.*

(i) F Lipschitz: *there is a constant $L \geq 0$, such that for all (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$, $\sigma \in \mathbb{S}_d$,*

$$\big|F_t(y_1, z_1, \sigma) - F_t(y_2, z_2, \sigma)\big| \;\leq\; L\big(|y_1 - y_2| + \big|\sigma^{\mathbf{T}}(z_1 - z_2)\big|\big), \quad dt \otimes d\mathbb{P} - a.e..$$

(ii) F Monotone: *there is a constant $\mu \in \mathbb{R}$, such that for all $z \in \mathbb{R}^d$, $(y_1, y_2) \in \mathbb{R}^2$, $\sigma \in \mathbb{S}_d$,*

$$(y_1 - y_2)\big(F_t(y_1, z, \sigma) - F_t(y_2, z, \sigma)\big) \;\leq\; -\mu|y_1 - y_2|^2, \quad dt \otimes d\mathbb{P} - a.e.$$

Assumption 3.2. *τ is a finite stopping time, ξ is $\mathcal{F}_\tau-$measurable, and*

$$\|\xi\|_{\mathcal{L}^q_{\rho,\tau}(\mathbb{P})} < \infty, \text{ and } \overline{f}^{\mathbb{P}}_{\rho,q,\tau} := \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\rho t} f_t^0\big|^2 ds\Big)^{\frac{q}{2}}\Big]^{\frac{1}{q}} < \infty, \quad \text{for some} \quad \rho > -\mu,\ q > 1.$$

Theorem 3.3 (Existence and uniqueness)**.** *Under Assumptions 3.1 and 3.2, the backward SDE (3.1) has a unique*[3] *solution $(Y, Z, N) \in \mathcal{D}^p_{\eta,\tau}(\mathbb{P}) \times \mathcal{H}^p_{\eta,\tau}(\mathbb{P}) \times \mathcal{N}^p_{\eta,\tau}(\mathbb{P})$, for all $p \in (1, q)$ and $\eta \in [-\mu, \rho)$, with*

$$\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathbb{P})} + \|Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} + \|N\|^p_{\mathcal{N}^p_{\eta,\tau}(\mathbb{P})} \;\leq\; Const\Big(\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathbb{P})} + \big(\overline{f}^{\mathbb{P}}_{\rho,q,\tau}\big)^p\Big). \tag{3.2}$$

Except for the estimate (3.2), whose proof is postponed in Section 4.5, the wellposedness part of the last result is a special case of Theorem 3.7 below, with obstacle $S \equiv -\infty$.

We emphasize that Darling & Pardoux [DP97] requires a similar integrability condition as Assumption 3.2 with $\bar{\rho} := \rho + L^2/2$ instead of ρ and $\mathbb{E}^{\mathbb{P}}$ instead of $\mathcal{E}^{\mathbb{P}}$. The following example illustrates the relevance of our assumption in the simple case of a linear generator.

Example 3.4. Let $\mathbb{P} := \mathbb{P}_0$, be the Wiener measure on Ω, so that X is a $\mathbb{P}_0-$Brownian motion. Let $\tau := H_1$, where $H_x := \inf\{t > 0 \,:\, X_t \geq x\}$, $\xi := |X_{1\wedge\tau}|$, and $f_t(\omega, y, z) := -\mu y + Lz$ for some constants $0 < \mu < 1 \leq L$. Notice that $f^0 = 0$, and $\xi \in \mathcal{L}^2_{0,\tau}(\mathbb{P}_0)$ directly verification:

$$\mathcal{E}^{\mathbb{P}_0}\big[|\xi|^2\big] \leq \sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}_0)} \mathbb{E}^{\mathbb{P}_0}\big[\mathrm{D}_1^{\mathbb{Q}|\mathbb{P}_0}|\xi|^2\big] \leq \sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}_0)} \mathbb{E}^{\mathbb{P}_0}\big[\big(\mathrm{D}_1^{\mathbb{Q}|\mathbb{P}_0}\big)^2\big]^{\frac{1}{2}} \mathbb{E}^{\mathbb{P}_0}\big[|\xi|^4\big]^{\frac{1}{2}} < \infty.$$

We next show that Darling & Pardoux's condition is not satisfied. To see this, observe that the event set $A := \big\{\omega \in \Omega \,:\, \sup_{0\leq t\leq 1} X_t < 1,\ X_1 \in \big[\frac{1}{2}, \frac{3}{4}\big]\big\}$ satisfies $\mathbb{P}_0[A] > 0$, and therefore

$$\begin{aligned}\mathbb{E}^{\mathbb{P}_0}\big[e^{2L^2\tau}|\xi|^2\big] \geq \frac{1}{4}\mathbb{E}^{\mathbb{P}_0}\big[e^{2L^2\tau}\mathbf{1}_A\big] &\geq \frac{1}{4}\mathbb{E}^{\mathbb{P}_0}\Big[\mathbf{1}_A \mathbb{E}^{\mathbb{P}_0}\big[e^{2L^2 H_{1-X_1}}\big|X_1\big]\Big] \\ &\geq \frac{1}{4}\mathbb{E}^{\mathbb{P}_0}\Big[\mathbf{1}_A \mathbb{E}^{\mathbb{P}_0}\big[e^{2L^2 H_{1/4}}\big]\Big] = \infty.\end{aligned}$$

□

[3]The solution is unique modulo the norms of the corresponding spaces.

We also have the following comparison and stability results, which are direct consequences of Theorem 3.8 below, obtained by setting the obstacle to $-\infty$ therein, together with the estimate (3.2) in Theorem 3.3.

Theorem 3.5. *Let (f,ξ), (f',ξ') be two sets of parameters satisfying the conditions of Theorem 3.3 with some stopping time τ, and the corresponding solutions (Y,Z,N), (Y',Z',N').*

(i) Stability. *Denoting $\delta\xi := \xi - \xi'$, $\delta Y := Y - Y'$, $\delta Z := Z - Z'$, $\delta U := U - U'$ and $\delta f = f - f'$, we have for all $1 < p < p' < q$ and $-\mu < \eta < \eta' < \rho$:*

$$\begin{aligned} \|\delta Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathbb{P})} &\leq C_{p,p',\eta}\Big\{\|\delta\xi\|^p_{\mathcal{L}^{p'}_{\eta,\tau}(\mathbb{P})} + \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\eta t}\delta f_t(Y_t,Z_t)\big|dt\Big)^{p'}\Big]^{\frac{p}{p'}}\Big\}, \\ \|\delta Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} + \|\delta N\|^p_{\mathcal{N}^p_{\eta,\tau}(\mathbb{P})} &\leq C_{p,\eta,\eta'}\Big\{\|\delta Y\|^p_{\mathcal{D}^p_{\eta',\tau}(\mathbb{P})} + \mathcal{E}^{\mathbb{P}}\Big[\Big(\int_0^\tau \big|e^{\eta t}\delta f_t(Y_t,Z_t)\big|dt\Big)^{p}\Big]\Big\}. \end{aligned}$$

(ii) Comparison. *Assume $\xi \leq \xi'$, $\mathbb{P}$-a.s., and $f(y,z) \leq f'(y,z)$ for all $(y,z) \in \mathbb{R}\times\mathbb{R}^d$, $dt\otimes\mathbb{P}-$a.e. Then, $Y_{\tau_0} \leq Y'_{\tau_0}$, $\mathbb{P}$-a.s. for all stopping time $\tau_0 \leq \tau$, $\mathbb{P}-$a.s.*

Remark 3.6. Following [EPQ97] we say that (Y,Z) is a supersolution (resp. subsolution) of the BSDE with parameters (f,ξ) if the martingale N in (3.1) is replaced by a supermartingale (resp. submartingale). A direct examination of the proof of the last comparison result reveals that the conclusion is unchanged if (Y,Z) is a subsolution of BSDE(f,ξ), and (Y',Z') is a supersolution of BSDE(f',ξ').

3.2 Random horizon reflected backward SDE

We now consider an obstacle defined by $(S_t)_{t\geq 0}$, and we search for a representation similar to (3.1) with the additional requirement that $Y \geq S$. This is achieved at the price of pushing up the solution Y by substracting a supermartingale U with minimal action. We then consider the following reflected backward stochastic differential equation (RBSDE):

$$\begin{aligned} Y_{\cdot\wedge\tau} &= \xi + \int_{\cdot\wedge\tau}^\tau f_s(Y_s,Z_s)ds - (Z_s\cdot dX_s + dU_s),\ Y \geq S,\ \mathbb{P}-\text{a.s.} \\ &\text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{t\wedge\tau} 1\wedge\big((Y_{r-}-S_{r-})\big)dU_r\Big] = 0,\ \text{for all } t\geq 0, \end{aligned} \tag{3.3}$$

where $U_{\wedge t}$ is a càdlàg $\mathbb{P}-$supermartingale, for all $t \geq 0$, starting from $U_0 = 0$, orthogonal to X, i.e. $[X,U] = 0$. The last minimality requirement is the so-called Skorokhod condition.[4]

Theorem 3.7 (Existence and uniqueness)**.** *Let Assumptions 3.1 and 3.2 hold true, and let S be a càdlàg $\mathbb{F}^{+,\mathbb{P}}-$adapted process with $\|S^+\|_{\mathcal{D}^q_{\rho,\tau}(\mathbb{P})} < \infty$. Then, the reflected backward SDE* (3.3) *has a unique solution $(Y,Z,U) \in \mathcal{D}^p_{\eta,\tau}(\mathbb{P}) \times \mathcal{H}^p_{\eta,\tau}(\mathbb{P}) \times \mathcal{U}^p_{\eta,\tau}(\mathbb{P})$, for all $p \in (1,q)$ and $\eta \in [-\mu,\rho)$.*

The existence part of this result is proved in Section 4.4. The uniqueness is a consequence of claim (i) of the following stability and comparison results.

Theorem 3.8. *Let (f,ξ,S) and (f',ξ',S') be two sets of parameters satisfying the conditions of Theorem 3.7, with corresponding solutions (Y,Z,U) and (Y',Z',U').*

[4]This condition indeed coincides the standard Skorokhod condition in the literature. Indeed, by using the corresponding Doob-Meyer decomposition $U = N - K$ into a martingale N and a nondecreasing process K, and recalling that $Y \geq S$, it follows that $0 = \mathbb{E}^{\mathbb{P}}\big[\int_0^{\tau\wedge t}\big(1\wedge(Y_{r-}-S_{r-})\big)dU_r\big] = \mathbb{E}^{\mathbb{P}}\big[-\int_0^{\tau\wedge t}\big(1\wedge(Y_{r-}-S_{r-})\big)dK_r\big]$ is equivalent to $\int_0^\tau (Y_{r-}-S_{r-})dK_r = 0$, $\mathbb{P}-$a.s. by the arbitrariness of $t \geq 0$.

(i) Comparison. *Assume $\xi \le \xi'$, $\mathbb{P}$-a.s., $f(y,z) \le f'(y,z)$ for all $(y,z) \in \mathbb{R}\times\mathbb{R}^d$, and $S \le S'$, $dt\otimes\mathbb{P}$-a.e. Then, $Y_{\tau_0} \le Y'_{\tau_0}$, $\mathbb{P}$-a.s., for all stopping time $\tau_0 \le \tau$, $\mathbb{P}$-a.s.*

(ii) Stability. *Let $S = S'$, and denote $\delta\xi := \xi - \xi'$, $\delta Y := Y - Y'$, $\delta Z := Z - Z'$, $\delta U := U - U'$ and $\delta f = f - f'$. Then, for all $1 < p < p' < q$ and $-\mu \le \eta < \eta' < \rho$, we have:*

$$\begin{aligned}&\|\delta Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathbb{P})} + \|\delta Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} + \|\delta U\|^p_{\mathcal{N}^p_{\eta,\tau}(\mathbb{P})}\\ &\quad\le C_{p,p',\eta,\eta'}\Big\{\Delta_\xi + \Delta_f + \Big(\Delta_\xi^{\frac12} + \Delta_f^{\frac12}\Big)\Big(\big(\overline{f}^{\mathbb{P}}_{\eta',p,\tau}\big)^{\frac p2} + \big(\overline{f'}^{\mathbb{P}}_{\eta',p,\tau}\big)^{\frac p2} + \|Y\|^{\frac p2}_{\mathcal{D}^p_{\eta',\tau}(\mathbb{P})} + \|Y'\|^{\frac p2}_{\mathcal{D}^p_{\eta',\tau}(\mathbb{P})}\Big)\Big\}.\end{aligned}$$

where $\Delta_\xi := \big\|\delta\xi\big\|^p_{\mathcal{L}^{p'}_{\eta',\tau}(\mathbb{P})}$ and $\Delta_f := \mathcal{E}^{\mathbb{P}}\big[\big(\int_0^\tau e^{\eta' s}\big|\delta f_s(Y_s,Z_s)\big|ds\big)^{p'}\big]^{\frac{p}{p'}}$.

Moreover, $\delta\overline{U} := \int_0^\cdot e^{\eta s} d\delta U_s$ satisfies

$$\big\|\delta\overline{U}\big\|^p_{\mathcal{D}^p_{0,\tau}} \quad\le\quad C_{p,L,\eta,\eta'}\Big(\|\delta Y\|^p_{\mathcal{D}^p_{\eta',\tau}(\mathbb{P})} + \|\delta Z\|^p_{\mathcal{H}^p_{\eta',\tau}(\mathbb{P})} + \Delta_f\Big).$$

The proof of **(ii)** is reported in Section 4.3, while **(i)** is proved at the end of Section 4.4.

Notice that the stability result is incomplete as the differences δY, δZ and δU are controlled by the norms of Y and Y'. However, in contrast with the estimate (3.2) in the backward SDE context, we have unfortunately failed to derive a similar control of (Y,Z,U) by the ingredients ξ and f^0 in the present context of random horizon reflected backward SDE .

3.3 Random horizon second order backward SDE

Following Soner, Touzi & Zhang [STZ12], we introduce second order backward SDE as a family of backward SDEs defined on the supports of a convenient family of singular probability measures. For this reason, we introduce the subset of $\mathcal{P}_b$:

$$\mathcal{P}_0 \;:=\; \big\{\mathbb{P}\in\mathcal{P}_b : f^0_t(\omega) < \infty,\ \text{for Leb}\otimes\mathbb{P}\text{-a.e. } (t,\omega)\in\mathbb{R}_+\times\Omega\big\}, \tag{3.4}$$

where we recall that $f^0_t(\omega) = F_t\big(\omega,0,0,\widehat{\sigma}_t(\omega)\big)$. We also define for all finite stopping times τ_0:

$$\mathcal{P}_{\mathbb{P}}(\tau_0) := \big\{\mathbb{P}'\in\mathcal{P}_0 :\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{\tau_0}\big\},\quad\text{and}\quad \mathcal{P}^+_{\mathbb{P}}(\tau_0) := \cup_{h>0}\mathcal{P}_{\mathbb{P}}\big(\tau_0+h\big). \tag{3.5}$$

The second order backward SDE (2BSDE, hereafter) is defined by

$$Y_{t\wedge\tau} \;=\; \xi + \int_{t\wedge\tau}^\tau F_s(Y_s,Z_s,\widehat{\sigma}_s)ds - \int_{t\wedge\tau}^\tau (Z_s\cdot dX_s + dU_s),\ \mathcal{P}_0-\text{q.s.} \tag{3.6}$$

for some supermartingale U together with a convenient minimality condition.

Definition 3.9. *Let $p>1$ and $\eta\in\mathbb{R}$. A process $(Y,Z)\in\mathcal{D}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{+,\mathcal{P}_0}\big)\times\mathcal{H}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{\mathcal{P}_0}\big)$ is said to be a solution of the 2BSDE (3.6), if for all $\mathbb{P}\in\mathcal{P}_0$, the process*

$$U^{\mathbb{P}}_{t\wedge\tau} := Y_{t\wedge\tau} - \xi - \int_{t\wedge\tau}^\tau F_s(Y_s,Z_s,\widehat{\sigma}_s)ds + \int_{t\wedge\tau}^\tau Z_s\cdot dX_s,\quad t\ge 0,\quad \mathbb{P}-a.s.$$

is a càdlàg $\mathbb{P}-$local supermartingale starting from $U^{\mathbb{P}}_0 = 0$, orthogonal to X, i.e. $[X,U^{\mathbb{P}}] = 0$, $\mathbb{P}-$a.s. and satisfying the minimality condition

$$U^{\mathbb{P}}_{s\wedge\tau} \quad=\quad \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(s\wedge\tau)}\ \mathbb{E}^{\mathbb{P}'}\big[U^{\mathbb{P}'}_{t\wedge\tau}\big|\mathcal{F}^{+,\mathbb{P}'}_{s\wedge\tau}\big],\ \mathbb{P}-a.s.\ \textit{for all } 0\le s\le t.$$

Remark 3.10. Notice that the last definition relaxes slightly (3.6) by allowing for a dependence of U on the underlying probability measure. This dependence is due to the fact that the stochastic integral $Z \bullet X := \int_0^{\cdot} Z_s \cdot dX_s$ is defined $\mathbb{P}-$a.s. under all $\mathbb{P} \in \mathcal{P}_0$, and should rather be denoted $(Z \bullet X)^{\mathbb{P}}$ in order to emphasize the $\mathbb{P}-$dependence.

By Theorem 2.2 in Nutz [Nut12], the family $\{(Z \bullet X)^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_0}$ can be aggregated as a medial limit $(Z \bullet X)$ under the acceptance of Zermelo-Fraenkel set theory with axiom of choice together with the continuum hypothesis into our framework. In this case, $(Z \bullet X)$ can be chosen as an $\mathbb{F}^{+,\mathcal{P}_0}$-adapted process, and the family $\{U^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_0}$ can be aggregated into the resulting medial limit U, i.e., $U = U^{\mathbb{P}}$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_0$. □

The following assumption requires the following additional notations:

$$\xi^{t,\omega}(\omega') := \xi(\omega \otimes_t \omega'),\ f_s^{0,t,\omega}(\omega') := F_{t+s}\big(\omega \otimes_t \omega', 0, 0, \widehat{\sigma}_s(\omega')\big),\ \bar{\tau}^{t,\omega} := \tau^{t,\omega} - t,$$

which involve the paths concatenation operator $(\omega \otimes_t \omega')_s := \mathbf{1}_{\{s\le t\}}\omega_s + \mathbf{1}_{\{s>t\}}(\omega_t + \omega'_{s-t})$, and

$$\mathcal{P}(t,\omega) := \big\{\mathbb{P} \in \mathcal{P}_b :\ f_s^{0,t,\omega}(\omega') < \infty,\ \text{for Leb}\otimes\mathbb{P} - \text{a.e. } (s,\omega') \in \mathbb{R}_+ \times \Omega\big\},$$

so that $\mathcal{P}_0 = \mathcal{P}(0, \mathbf{0})$.

Assumption 3.11. (i) *τ is a stopping time with $\lim_{n\to\infty} \mathcal{E}^{\mathcal{P}_0}\big[\mathbf{1}_{\{\tau\ge n\}}\big] = 0$, ξ is $\mathcal{F}_\tau-$measurable, and there are constants $\rho > -\mu$, and $q > 1$ such that*

$$\big\|\xi\big\|_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}_0)} < \infty, \quad \text{and} \quad \overline{F}^0_{\rho,q,\tau} := \mathcal{E}^{\mathcal{P}_0}\Big[\Big(\int_0^\tau \big|e^{\rho t} f_t^0\big|^2 dt\Big)^{\frac{q}{2}}\Big]^{\frac{1}{q}} < \infty.$$

(ii) *Furthermore, the following dynamic version of* (i) *holds for all $(t,\omega) \in [\![0,\tau]\!]$:*

$$\big\|\xi^{t,\omega}\big\|_{\mathcal{L}^q_{\rho,\bar{\tau}^{t,\omega}}(\mathcal{P}(t,\omega))} < \infty, \quad \text{and} \quad \overline{F}^{0,t,\omega}_{\rho,q} := \mathcal{E}^{\mathcal{P}(t,\omega)}\Big[\Big(\int_0^{\bar{\tau}^{t,\omega}} \big|e^{\rho s} f_s^{0,t,\omega}\big|^2 ds\Big)^{\frac{q}{2}}\Big]^{\frac{1}{q}} < \infty.$$

Theorem 3.12. *Under Assumptions 3.1 and 3.11* (i)*, the 2BSDE* (3.6) *has at most one solution $(Y,Z) \in \mathcal{D}^p_{\eta,\tau}\big(\mathcal{P}_0, \mathbb{F}^{+,\mathcal{P}_0}\big) \times \mathcal{H}^p_{\eta,\tau}\big(\mathcal{P}_0, \mathbb{F}^{\mathcal{P}_0}\big)$, for all $p \in (1,q)$ and $\eta \in [-\mu,\rho)$, with*

$$\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathcal{P}_0)} + \|Z\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathcal{P}_0)} \quad \le \quad C_{p,q,\eta,\rho}\big(\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}_0)} + \big(\overline{F}^0_{\rho,q,\tau}\big)^p\big). \tag{3.7}$$

Under the additional Assumption 3.11 (ii)*, such a solution (Y,Z) for the 2BSDE* (3.6) *exists. If $\mathcal{P}_0$ is saturated*[5]*, then $U^{\mathbb{P}}$ is a non-increasing process for all $\mathbb{P} \in \mathcal{P}_0$.*

Similar to Soner, Touzi & Zhang [STZ12], the following comparison result for second order backward SDEs is a by-product of our construction; the proof is provided in Proposition 5.2.

Proposition 3.13. *Let (Y,Z) and (Y',Z') be solutions of 2BSDEs with parameters (F,ξ) and (F',ξ'), respectively, which satisfy Assumptions 3.1 and 3.11. Suppose further that $\xi \le \xi'$ and $F_t\big(y,z,\widehat{\sigma}_t\big) \le F'_t\big(y,z,\widehat{\sigma}_t\big)$ for all $(y,z) \in \mathbb{R}\times\mathbb{R}^d$, $dt \otimes \mathcal{P}_0$-q.s. Then, we have $Y \le Y'$, $dt \otimes \mathcal{P}_0$-q.s. on $[\![0,\tau]\!]$.*

[5]We say that the family $\mathcal{P}_0$ is saturated if, for all $\mathbb{P} \in \mathcal{P}_0$, we have $\mathbb{Q} \in \mathcal{P}_0$ for every probability measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega,\mathcal{F})$ such that X is $\mathbb{Q}-$local martingale. The assertion follows by the same argument as in [PTZ17, Theorem 5.1].

4 Wellposedness of random horizon reflected BSDEs

Throughout this section, we fix a probability measure $\mathbb{P} \in \mathcal{P}_b$, and we omit the dependence on $\mathbb{P}$ in all of our notations. We also observe that $\mathcal{Q}_L := \mathcal{Q}_L(\mathbb{P})$ is stable under concatenation.

For all $\mathbb{Q}^\lambda \in \mathcal{Q}_L$, it follows from Girsanov's Theorem that

• U remains a $\mathbb{Q}^\lambda-$supermartingale, with the same Doob-Meyer decomposition as under $\mathbb{P}$,

• and $W^\lambda := W - \int_0^\cdot \lambda_s ds$ is a $\mathbb{Q}^\lambda-$Brownian motion, $X^\lambda := X - \int_0^\cdot \widehat{\sigma}_s \lambda_s ds$ is a $\mathbb{Q}^\lambda-$local martingale, and and may we rewrite the RBSDE as

$$dY_t = -f_t^\lambda(Y_t, Z_t)dt + Z_t \cdot dX_t^\lambda + dU_t, \quad \text{where} \quad f_t^\lambda(y,z) := f_t(y,z) - \widehat{\sigma}_t^{\mathbf{T}} z \cdot \lambda_t$$

satisfies the Assumption 3.1 with Lipschitz coefficient $2L$.

4.1 Some useful inequalities

We first state a Doob-type inequality. For simplicity, we write $\mathcal{E}[\cdot] := \mathcal{E}^{\mathbb{P}}[\cdot]$.

Lemma 4.1. *Let $(M_t)_{0\le t\le \tau}$ be a uniformly integrable martingale under some $\widehat{\mathbb{Q}} \in \mathcal{Q}_L$. Then,*

$$\mathcal{E}\Big[\sup_{0\le t\le\tau} |M_t|^p\Big] \quad \le \quad \frac{q}{q-p}\big(\mathcal{E}\big[|M_\tau|^q\big]\big)^{\frac{p}{q}}, \text{ for all } 0<p<q.$$

Proof. Let $x>0$ and $T_x := \tau \wedge \inf\{t \ge 0,\, |M_t| > x\}$, with the convention $\inf \emptyset = \infty$. From the definition of concatenation and the optional sampling theorem, we obtain for all $\mathbb{Q} \in \mathcal{Q}_L$:

$$\mathbb{E}^{\mathbb{Q}}\big[|M_{T_x}|^q\big] = \mathbb{E}^{\mathbb{Q}}\Big[\big|\mathbb{E}^{\widehat{\mathbb{Q}}}\big[M_\tau\big|\mathcal{F}_{T_x}\big]\big|^q\Big] \le \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\widehat{\mathbb{Q}}}\big[|M_\tau|^q\big|\mathcal{F}_{T_x}\big]\Big] = \mathbb{E}^{\mathbb{Q}\otimes_{T_x}\widehat{\mathbb{Q}}}\big[|M_\tau|^q\big] \le \mathcal{E}\big[|M_\tau|^q\big] =: c,$$

as $\mathbb{Q} \otimes_{T_x} \widehat{\mathbb{Q}} \in \mathcal{Q}_L$. Then, denoting $M_* := \sup_{0\le t\le\tau}|M_t|$, we see that

$$x^q \mathbb{Q}\left[M_* > x\right] = x^q \mathbb{Q}[T_x < \tau] \le \mathbb{E}^{\mathbb{Q}}\big[|M_{T_x}|^q \mathbf{1}_{\{T_x<\tau\}}\big] \le \mathbb{E}^{\mathbb{Q}}\big[|M_{T_x}|^q\big] \le c.$$

and we deduce that

$$\mathbb{E}^{\mathbb{Q}}\big[M_*^p\big] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty \mathbf{1}_{\{M_*>x\}} p x^{p-1} dx\Big] = \int_0^\infty \mathbb{Q}\left[M_* > x\right] p x^{p-1} dx \le \int_0^\infty [1 \wedge (cx^{-q})] p x^{p-1} dx = \frac{q c^{\frac{p}{q}}}{q-p}.$$

The required inequality follows from the arbitrariness of $\mathbb{Q} \in \mathcal{Q}_L$. □

The following result is well-known, we report its proof for completeness as we could not find a reference for it. We shall denote $\mathrm{sgn}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$, for all $x \in \mathbb{R}$.

Proposition 4.2. *For any semimartingale X, we have $|X_t| - |X_0| \ge \int_0^t \mathrm{sgn}(X_{s-})dX_s$, $t \ge 0$.*

Proof. Consider a decreasing sequence of C^2, symmetric convex functions φ_n on $\mathbb{R}$, such that $\varphi_n(x) = |x|$ on $(-\frac{1}{n^2}, \frac{1}{n^2})^c$, and $\varphi_n'(x)$ increases to 1 for $x>0$ and $\varphi_n'(x)$ decreases to -1 for $x<0$, i.e., $\varphi_n'(x)$ converges to $\mathrm{sgn}(x)$. By Itô's formula and convexity of φ_n, we obtain that

$$\varphi_n(X_t) - \varphi_n(X_0) = \int_0^t \varphi_n'(X_{s-})dX_s + \frac{1}{2}\int_0^t \varphi_n''(X_{s-})d[X^c]_s + \sum_{0<s\le t}\big\{\Delta\varphi_n(X_s) - \varphi_n'(X_{s-})\Delta X_s\big\}$$

By convexity of φ_n, this implies that $\varphi_n(X_t) - \varphi_n(X_0) \ge \int_0^t \varphi_n'(X_{s-})dX_s$. The required inequality follows by sending $n \to \infty$ in the above inequality and by applying the dominated convergence theorem for stochastic integrals (see, e.g., [Pro05, Section IV, Theorem 32]). □

4.2 A priori estimates

Proposition 4.3. *Under the conditions of Theorem 3.7, let* $(Y,Z,U)\in\mathcal{D}^p_{\beta,\tau}\times\mathcal{H}^p_{\beta,\tau}\times\mathcal{U}^p_{\beta,\tau}$ *be a solution of RBSDE* (3.3)*. For each* $p\in(1,q)$ *and* $-\mu\le\alpha<\beta<\rho$*, there exists a constant* $C_{p,L,\alpha,\beta}$ *such that*

$$\|Z\|^p_{\mathcal{H}^p_{\alpha,\tau}}+\|U\|^p_{\mathcal{U}^p_{\alpha,\tau}}\ \le\ C_{p,L,\alpha,\beta}\Big(\big(\overline{f}^{\mathbb{P}}_{\beta,p,\tau}\big)^p+\|Y\|^p_{\mathcal{D}^p_{\beta,\tau}}\Big).$$

Proof. Let $U=N-K$ be the Doob-Meyer decomposition of the supermartingale U.

1. We first prove that

$$\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|N\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\le C_p\Big(\big\|Z\bullet X^\lambda+U\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|K\|^p_{\mathbb{K}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\Big),\tag{4.1}$$

$$\widetilde{c}_p\Big(\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|U\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\Big)\le\big\|Z\bullet X^\lambda+U\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\le\widetilde{C}_p\Big(\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|U\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\Big).\tag{4.2}$$

We only prove (4.1), the second claim follows by similar arguments.

As $[X^\lambda,N]=\widehat{\sigma}\bullet[W^\lambda,N]=0$, we obtain that

$$\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|N\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\le c_p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}\big(d\big[Z\bullet X^\lambda\big]_s+d[N]_s\big)\Big)^{\frac{p}{2}}\Big]=c_p\big\|Z\bullet X^\lambda+N\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}.$$

We continue by estimating the right hand side term:

$$\begin{aligned}\big\|Z\bullet X^\lambda+N\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}&\le 2^{\frac{p}{2}}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}d\big[Z\bullet X^\lambda+U\big]_s+\int_0^\tau e^{2\alpha s}d[K]_s\Big)^{\frac{p}{2}}\Big]\\&\le 2^p\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}d\big[Z\bullet X^\lambda+U\big]_s\Big)^{\frac{p}{2}}\Big]+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}d[K]_s\Big)^{\frac{p}{2}}\Big]\Big)\\&\le 2^p\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}d\big[Z\bullet X^\lambda+U\big]_s\Big)^{\frac{p}{2}}\Big]+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{\alpha s}dK_s\Big)^{p}\Big]\Big)\\&=2^p\big(\big\|Z\bullet X^\lambda+U\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|K\|^p_{\mathbb{K}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\big),\end{aligned}$$

where we used the estimate $\int_0^\tau e^{2\alpha s}d[K]_s=\sum_{0<s\le\tau}e^{2\alpha s}(\Delta K_s)^2\le\Big(\sum_{0<s\le\tau}e^{\alpha s}(\Delta K_s)\Big)^2\le\big(\int_0^\tau e^{\alpha s}dK_s\big)^2$, since K is non-decreasing.

2. Denote $U^\lambda:=Z\bullet X^\lambda+U=\sigma^{\mathbf{T}}Z\bullet W^\lambda+U$. By Itô's formula,

$$0\ \le\ Y_0^2=e^{2\alpha\tau}\xi^2+\int_0^\tau e^{2\alpha t}\big\{-2\alpha Y_t^2ds+2Y_{t-}\big(f_t^\lambda(Y_t,Z_t)dt-dU_t^\lambda\big)-\big|\widehat{\sigma}_t^{\mathbf{T}}Z_t\big|^2dt-d[U]_t\big\}.$$

It follows from Assumption 3.1 and Young's inequality that $2yf_t^\lambda(y,z)\le-2\mu y^2+2|y||f_t^0|+4L|y||\widehat{\sigma}_t^{\mathbf{T}}z|\le-2\mu y^2+|f_t^0|^2+\ell|y|^2+\frac{1}{2}\big|\widehat{\sigma}_t^{\mathbf{T}}z\big|^2$, with $\ell:=1+8L^2$. Then, as $\alpha+\mu\ge0$,

$$\begin{aligned}\int_0^\tau e^{2\alpha t}\Big(\frac{1}{2}\big|\widehat{\sigma}_t^{\mathbf{T}}Z_t\big|^2dt+d[U]_t\Big)\ &\le\ e^{2\alpha\tau}\xi^2+\int_0^\tau e^{2\alpha t}\big(\big|f_t^0\big|^2+\ell Y_t^2\big)dt-2\int_0^\tau e^{2\alpha t}Y_{t-}dU_t^\lambda\\&\le\int_0^\tau e^{2\alpha t}\big|f_t^0\big|^2dt+\Big(1+\frac{\ell}{2(\alpha'-\alpha)}\Big)\sup_{0\le t\le\tau}e^{2\alpha' t}Y_t^2-2\int_0^\tau e^{2\alpha t}Y_{t-}dU_t^\lambda,\end{aligned}$$

for an arbitrary $\alpha'\in(\alpha,\rho)$. This implies that

$$\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}+\|U\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}\le C_{p,\alpha,\alpha',L}\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau\big|e^{\alpha t}f_t^0\big|^2dt\Big)^{\frac{p}{2}}\Big]+\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+E^\lambda\Big),\tag{4.3}$$

where

$$
\begin{aligned}
E^\lambda &:= \mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big|\int_0^\tau e^{2\alpha t}Y_{t-}dU_t^\lambda\Big|^{\frac{p}{2}}\Big] \\
&\leq \overline{C}_p\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\leq t\leq\tau}\Big|\int_0^t e^{2\alpha s}Y_{s-}\big(Z_s\bullet dX_s^\lambda+dN_s\big)\Big|^{\frac{p}{2}}+\Big|\int_0^\tau e^{2\alpha s}Y_{s-}dK_s\Big|^{\frac{p}{2}}\Big]\Big) \\
&\leq C_p'\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{4\alpha s}Y_{s-}^2d\big[Z\bullet X^\lambda+N\big]_s\Big)^{\frac{p}{4}}\Big]+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big|\int_0^\tau e^{2\alpha s}Y_{s-}dK_s\Big|^{\frac{p}{2}}\Big]\Big),
\end{aligned}
$$

by the BDG inequality. Since K is non-decreasing, we applying Young's inequality with an arbitrary $\varepsilon>0$ to deduce

$$
\begin{aligned}
E^\lambda &\leq C_p'\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\leq s\leq\tau}|e^{\alpha' s}Y_s|^{\frac{p}{2}}\Big\{\Big(\int_0^\tau e^{2(2\alpha-\alpha')s}d\big[Z\bullet X^\lambda+N\big]_s\Big)^{\frac{p}{4}}+\Big(\int_0^\tau e^{(2\alpha-\alpha')s}dK_s\Big)^{\frac{p}{2}}\Big\}\Big] \\
&\leq \frac{(C_p')^2}{\varepsilon}\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\frac{\varepsilon}{2}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2(2\alpha-\alpha')s}d\big[Z\bullet X^\lambda+N\big]_s\Big)^{\frac{p}{2}}\Big]+\frac{\varepsilon}{2}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{(2\alpha-\alpha')s}dK_s\Big)^p\Big] \\
&\leq \frac{(C_p')^2}{\varepsilon}\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\frac{\varepsilon}{2}(C_p''+1)\|K\|^p_{\mathbb{K}^p_{2\alpha-\alpha',\tau}(\mathbb{Q}^\lambda)}+\frac{\varepsilon}{2}C_p''\,\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2(2\alpha-\alpha')s}d\big[U^\lambda\big]_s\Big)^{\frac{p}{2}}\Big],
\end{aligned}
$$

where the last inequality follows from (4.1). Plugging this estimate into (4.3), and using (4.2) together with the fact that $2\alpha-\alpha'<\alpha$, we obtain

$$
\begin{aligned}
&\Big(1-\frac{\overline{C}_pC_{p,\alpha,\alpha',L}C_p''}{2}\varepsilon\Big)\big\|U^\lambda\big\|_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)} \qquad (4.4)\\
&\leq \overline{C}_pC_{p,\alpha,\alpha',L}\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau\big|e^{\alpha s}f_s^0\big|^2ds\Big)^{\frac{p}{2}}\Big]+\Big(1+\frac{(C_p')^2}{\varepsilon}\Big)\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\frac{\varepsilon}{2}(C_p''+1)\|K\|^p_{\mathbb{K}^p_{2\alpha-\alpha',\tau}(\mathbb{Q}^\lambda)}\Big).
\end{aligned}
$$

3. We shall prove in Step 4 below that for $\delta<\delta'<\rho$:

$$
\|K\|^p_{\mathbb{K}^p_{\delta,\tau}(\mathbb{Q}^\lambda)} \leq \overline{C}^K_{p,\delta,\delta',L}\Big(\|Y\|^p_{\mathbb{D}^p_{\delta',\tau}(\mathbb{Q}^\lambda)}+\|Z\|^p_{\mathbb{H}^p_{\delta',\tau}(\mathbb{Q}^\lambda)}+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau\big|e^{\delta' s}f_s^0\big|^2ds\Big)^{\frac{p}{2}}\Big]\Big). \quad (4.5)
$$

Plugging this inequality with $\delta:=2\alpha-\alpha'$ and $\delta':=\alpha$ in (4.4), and using the left hand side inequality of (4.2), we see that we may choose $\varepsilon>0$ conveniently such that

$$
\|Z\|^p_{\mathbb{H}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)} \leq C^Z_{p,\alpha,\alpha',L}\Big(\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}|f_s^0|^2ds\Big)^{\frac{p}{2}}\Big]\Big), \quad (4.6)
$$

for some constant $C^Z_{p,\alpha,\alpha',L}>0$. Plugging this inequality into (4.5) with $(\delta,\delta'):=(\alpha,\alpha')$ induces the estimate

$$
\|K\|^p_{\mathbb{K}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)} \leq C^K_{p,\alpha,\alpha',L}\Big(\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha' s}\big|f_s^0\big|^2ds\Big)^{\frac{p}{2}}\Big]\Big), \quad (4.7)
$$

for some constant $C^K_{p,\alpha,\alpha',L}$. Combining with (4.4), and recalling that $2\alpha-\alpha'<\alpha$, in turn, this implies an estimate for $\big\|U^\lambda\big\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)}$ which can be plugged into (4.1) to provide:

$$
\|N\|^p_{\mathbb{N}^p_{\alpha,\tau}(\mathbb{Q}^\lambda)} \leq C^N_{p,\alpha,\alpha',L}\Big(\|Y\|^p_{\mathbb{D}^p_{\alpha',\tau}(\mathbb{Q}^\lambda)}+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha' s}\big|f_s^0\big|^2ds\Big)^{\frac{p}{2}}\Big]\Big). \quad (4.8)
$$

Since the constants in (4.6), (4.7) and (4.8) do not depent on $\mathbb{Q}\in\mathcal{Q}_L$, the proof of the proposition is completed by taking supremum over the family of measures $\mathbb{Q}\in\mathcal{Q}_L$.

4. We now prove (4.5). By Itô's formula, we have

$$e^{\delta t}Y_t + \int_0^t e^{\delta s}\big(f_s^\lambda(Y_s,Z_s) - \delta Y_s\big)ds \;=\; Y_0 + \int_0^t e^{\delta s}(Z_s\cdot dX_s^\lambda + dU_s).$$

As $(Z,N)\in\mathbb{H}^p_{\delta,\tau}(\mathbb{Q}^\lambda)\times\mathbb{N}^p_{\delta,\tau}(\mathbb{Q}^\lambda)$ and K is nondecreasing, the process $e^{\delta t}Y_t+\int_0^t e^{\delta s}\big(f_s^\lambda(Y_s,Z_s)-\delta Y_s\big)ds$, is a supermartingale under $\mathbb{Q}^\lambda$. By [BPTZ16, Lemma A.1] and Assumption 3.1, we obtain that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{\delta s}dK_s\Big)^p\Big] &\le C_p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\le u\le\tau}\Big(e^{\delta u}Y_u+\int_0^u e^{\delta s}\big(f_s^\lambda(Y_s,Z_s)-\delta Y_s\big)ds\Big)^p\Big] \qquad (4.9)\\ &\le C_{p,\delta,L}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\le u\le\tau}|e^{\delta u}Y_u|^p+\Big(\int_0^\tau e^{\delta s}\big|f_s^0\big|ds\Big)^p\\ &\qquad +\Big(\int_0^\tau e^{\delta s}|Y_s|ds\Big)^p+\Big(\int_0^\tau e^{\delta s}|\widehat{\sigma}_s^{\mathbf{T}}Z_s|ds\Big)^p\Big].\end{aligned}$$

Finally, for $\delta'\in(\delta,\rho)$, we observe that

$$\Big(\int_0^\tau e^{\delta s}|Y_s|ds\Big)^p \le \sup_{0\le s\le\tau}|e^{\delta' s}Y_s|^p\Big(\int_0^\tau e^{-(\delta'-\delta)s}ds\Big)^p \le \frac{\sup_{0\le s\le\tau}|e^{\delta' s}Y_s|^p}{(\delta'-\delta)^p}, \qquad (4.10)$$

and by the Cauchy-Schwarz inequality

$$\Big(\int_0^\tau e^{\delta s}\big|f_s^0\big|ds\Big)^p \le \Big(\int_0^\tau \big|e^{\delta' s}f_s^0\big|^2ds\Big)^{\frac{p}{2}}\Big(\int_0^\tau e^{-2(\delta'-\delta)s}ds\Big)^{\frac{p}{2}} \le \frac{\big(\int_0^\tau \big|e^{\delta' s}f_s^0\big|^2ds\big)^{\frac{p}{2}}}{(2\delta'-2\delta)^{\frac{p}{2}}}, \qquad (4.11)$$

and, similarly,

$$\Big(\int_0^\tau e^{\delta s}\big|\widehat{\sigma}_s^{\mathbf{T}}Z_s\big|ds\Big)^p \le \frac{\big(\int_0^\tau \big|e^{\delta' s}\widehat{\sigma}_s^{\mathbf{T}}Z_s\big|^2ds\big)^{\frac{p}{2}}}{(2\delta'-2\delta)^{\frac{p}{2}}}. \qquad (4.12)$$

The required inequality (4.5) follows from (4.9), (4.10), (4.11) and (4.12). □

4.3 Stability of reflected backward SDEs

Proof of Theorem 3.8 (ii). Clearly, the process $(\delta Y,\delta Z,\delta U)$ satisfies the following equation

$$\delta Y_{t\wedge\tau} \;=\; \delta\xi+\int_{t\wedge\tau}^\tau g_s(\delta Y_s,\delta Z_s)ds-\int_{t\wedge\tau}^\tau\big(\delta Z_s\cdot dX_s+d\delta U_s\big), \qquad (4.13)$$

where $g_s(\delta Y_s,\delta Z_s):=f_s'(Y_s+\delta Y_s,Z_s+\delta Z_s)-f_s(Y_s,Z_s)$.

1. In this step, we prove that, for some constant $C_{p,p'}$,

$$\mathcal{E}\Big[\sup_{0\le t\le\tau}e^{p\eta' t}|\delta Y_t|^p\Big] \le C_{p,p'}\mathcal{E}\Big[e^{p'\eta'\tau}|\delta\xi|^{p'}+\Big(\int_0^\tau e^{\eta' s}|\delta f_s(Y_s,Z_s)|ds\Big)^{p'}\Big]^{\frac{p}{p'}}. \qquad (4.14)$$

It follows from Proposition 4.2 that

$$\begin{aligned}e^{\eta'\tau}|\delta Y_\tau|-e^{\eta'(t\wedge\tau)}|\delta Y_{t\wedge\tau}| &\ge \int_{t\wedge\tau}^\tau e^{\eta' s}\big(\eta'|\delta Y_s|-\mathrm{sgn}(\delta Y_s)g_s(\delta Y_s,\delta Z_s)\big)ds \qquad (4.15)\\ &\quad+\int_{t\wedge\tau}^\tau e^{\eta' s}\big(\mathrm{sgn}(\delta Y_s)\delta Z_s\cdot dX_s+\mathrm{sgn}(\delta Y_{s-})d\delta U_s\big).\end{aligned}$$

As f and f' satisfy Assumption 3.1, we obtain that

$$\operatorname{sgn}(\delta Y_s) g_s(\delta Y_s, \delta Z_s) \leq |\delta f_s(Y_s, Z_s)| + L\big|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s\big| - \mu |\delta Y_s|.$$

Considering the Doob-Meyer decomposition $U = N - K$ and $U' = N' - K'$, and denoting δN and δK the corresponding differences, it follows from the Skorokhod condition that

$$\begin{aligned} \delta Y_{s-} d\delta K_s &= (Y'_{s-} - Y_{s-})(dK'_s - dK_s) \\ &= (Y'_{s-} - S_{s-})dK'_s - (Y'_{s-} - S_{s-})dK_s - (Y_{s-} - S_{s-})dK'_s + (Y_{s-} - S_{s-})dK_s \\ &= -(Y'_{s-} - S_{s-})dK_s - (Y_{s-} - S_{s-})dK'_s \ \leq \ 0, \end{aligned} \tag{4.16}$$

so that

$$\operatorname{sgn}(\delta Y_{s-}) d\delta K_s = \mathbf{1}_{\{\delta Y_{s-} \neq 0\}} \frac{\operatorname{sgn}(\delta Y_{s-})}{\delta Y_{s-}} \delta Y_{s-} d\delta K_s \leq 0.$$

Then, denoting $\widehat{\lambda}_s := L \operatorname{sgn}(\delta Y_s) \frac{\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s}{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s|} \mathbf{1}_{\{|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s| \neq 0\}}$ and $X^{\widehat{\lambda}} := X - \int_0^{\cdot} \widehat{\sigma}_s \widehat{\lambda}_s ds$, it follows from inequality (4.15) and $-\mu < \eta'$ that

$$\begin{aligned} e^{\eta'(t\wedge\tau)} |\delta Y_{t\wedge\tau}| \ &\leq \ e^{\eta'\tau} |\delta Y_\tau| + \int_{t\wedge\tau}^{\tau} e^{\eta' s} |\delta f_s(Y_s, Z_s)| ds \\ &\quad - \int_{t\wedge\tau}^{\tau} e^{\eta' s} \big(\operatorname{sgn}(\delta Y_s) \delta Z_s \cdot dX_s^{\widehat{\lambda}} + \operatorname{sgn}(\delta Y_{s-}) d\delta N_s \big). \end{aligned}$$

As $\delta Z \in \mathcal{H}^p_{\eta,\tau}(\mathbb{P})$ and $\delta N \in \mathcal{N}^p_{\eta,\tau}(\mathbb{P})$, we deduce from the BDG inequality that the last two terms are $\mathbb{Q}^{\widehat{\lambda}}-$uniformly integrable martingales. Then, with $\tau_n := n \wedge \tau$ and $n \geq t$:

$$\begin{aligned} e^{\eta'(t\wedge\tau)} |\delta Y_{t\wedge\tau}| \ &\leq \ \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\widehat{\lambda}}} \Big[e^{\eta'\tau_n} |\delta Y_{\tau_n}| + \int_{t\wedge\tau}^{\tau_n} e^{\eta' s} |\delta f_s(Y_s, Z_s)| ds \Big| \mathcal{F}^{+,\mathbb{P}}_{t\wedge\tau} \Big] \\ &\leq \ \mathbb{E}^{\mathbb{Q}^{\widehat{\lambda}}} \Big[e^{\eta'\tau} |\delta\xi| + \int_0^{\tau} e^{\eta' s} |\delta f_s(Y_s, Z_s)| ds \Big| \mathcal{F}^{+,\mathbb{P}}_{t\wedge\tau} \Big], \end{aligned}$$

by the dominated convergence theorem and monotone convergence theorem, as $e^{\eta' t} Y_t$ and $e^{\eta' t} Y'_t$ are uniformly integrable. By Lemma 4.1 , we deduce that for any $p' \in (p, q)$:

$$\mathcal{E}\Big[\sup_{0\leq t\leq\tau} \big| e^{\eta' t} \delta Y_t \big|^p \Big] \ \leq \ \frac{p'}{p' - p} \mathcal{E}\Big[\Big(e^{\eta'\tau} |\delta\xi| + \int_0^{\tau} e^{\eta' s} |\delta f_s(Y_s, Z_s)| ds \Big)^{p'} \Big]^{\frac{p}{p'}},$$

which induces the required inequality (4.14).

2. Let $-\mu \leq \eta < \eta'$. By Itô's formula, we have

$$\begin{aligned} e^{2\eta\tau} (\delta Y_\tau)^2 - (\delta Y_0)^2 = 2\eta \int_0^{\tau} e^{2\eta s} (\delta Y_s)^2 ds - 2 \int_0^{\tau} e^{2\eta s} \delta Y_s g_s(\delta Y_s, \delta Z_s) ds \\ + 2 \int_0^{\tau} e^{2\eta s} \delta Y_{s-} \delta Z_s \cdot dX_s + 2 \int_0^{\tau} e^{2\eta s} \delta Y_{s-} d\delta N_s \\ - 2 \int_0^{\tau} e^{2\eta s} \delta Y_{s-} d\delta K_s + \int_0^{\tau} e^{2\eta s} \Big(\big|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s\big|^2 ds + d[\delta U]_s \Big). \end{aligned}$$

Again Assumption 3.1 implies that $\delta Y_s g_s(\delta Y_s, \delta Z_s) \ \leq \ |\delta Y_s| |\delta f_s(Y_s, Z_s)| + L |\delta Y_s| \big|\widehat{\sigma}_s^{\mathbf{T}} \delta Z_s\big| -$

$\mu|\delta Y_s|^2$, and therefore, together with (4.16) and the fact that $\eta+\mu\geq 0$, we obtain that

$$\begin{aligned}\int_0^\tau e^{2\eta s}\big(\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2 ds+d[\delta U]_s\big) &\leq e^{2\eta\tau}(\delta Y_\tau)^2+2\int_0^\tau e^{2\eta s}|\delta Y_s|\big(|\delta f_s(Y_s,Z_s)|+L\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|\big)ds\\ &\quad -2\int_0^\tau e^{2\eta s}\delta Y_{s-}\big(\delta Z_s\cdot dX_s+d\delta N_s\big)\\ &\leq e^{2\eta\tau}(\delta Y_\tau)^2+2\int_0^\tau e^{2\eta s}|\delta Y_s|\big(|\delta f_s(Y_s,Z_s)|+2L\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|\big)ds\\ &\quad +2\sup_{0\leq u\leq\tau}\Big|\int_0^u e^{2\eta s}\delta Y_s\delta Z_s\cdot dX_s^\lambda+\int_0^u e^{2\eta s}\delta Y_{s-}d\delta N_s\Big|,\end{aligned}$$

where $\lambda=(\lambda_s)_{0\leq s\leq\tau}$ is an arbitrary process uniformly bounded by L. By Young's inequality and the fact that $\eta<\eta'$, we have

$$\begin{aligned}2\int_0^\tau e^{2\eta s}|\delta Y_s||\delta f_s(Y_s,Z_s)|ds &\leq 2\Big(\sup_{0\leq s\leq\tau}e^{\eta s}|\delta Y_s|\Big)\int_0^\tau e^{\eta s}|\delta f_s(Y_s,Z_s)|ds\\ &\leq \sup_{0\leq s\leq\tau}e^{2\eta s}|\delta Y_s|^2+\Big(\int_0^\tau e^{\eta s}|\delta f_s(Y_s,Z_s)|ds\Big)^2,\end{aligned}$$

and

$$\begin{aligned}2\int_0^\tau e^{2\eta s}|\delta Y_s|\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|ds &\leq \frac{1}{\varepsilon}\int_0^\tau e^{2\eta s}|\delta Y_s|^2ds+\varepsilon\int_0^\tau e^{2\eta s}\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2ds\\ &\leq \frac{1}{\varepsilon}\Big(\sup_{0\leq s\leq\tau}e^{2\eta' s}|\delta Y_s|^2\Big)\int_0^\tau e^{-2(\eta'-\eta)s}ds+\varepsilon\int_0^\tau e^{2\eta s}\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2ds\\ &\leq \frac{1}{2\varepsilon(\eta'-\eta)}\sup_{0\leq s\leq\tau}e^{2\eta' s}|\delta Y_s|^2+\varepsilon\int_0^\tau e^{2\eta s}\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2ds,\end{aligned}$$

for an arbitrary $\varepsilon>0$. Therefore, by choosing an $\varepsilon>0$ conveniently, we obtain

$$\begin{aligned}\big\|\delta Z\big\|^p_{\mathbb{H}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}+\big\|\delta U\big\|^p_{\mathbb{N}^p_{\eta,\tau}(\mathbb{Q}^\lambda)} &\leq C_{p,\eta,\eta'}\Big(\big\|\delta Y\big\|^p_{\mathbb{D}^p_{\eta',\tau}(\mathbb{Q}^\lambda)}+\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{\eta s}|\delta f_s(Y_s,Z_s)|ds\Big)^p\Big]\Big)\\ &\quad +C_{p,\eta,\eta'}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\leq u\leq\tau}\Big|\int_0^u e^{2\eta s}\big(\delta Y_s\delta Z_s\cdot dX_s^\lambda+\delta Y_{s-}d\delta N_s\big)\Big|^{\frac{p}{2}}\Big].\end{aligned}\tag{4.17}$$

for some constant $C_{p,\eta,\eta'}>0$. By the BDG inequality, Young's inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\leq u\leq\tau}\Big|\int_0^u e^{2\eta s}\delta Y_{s-}(\delta Z_s\cdot dX_s^\lambda+d\delta N_s)\Big|^{\frac{p}{2}}\Big]\\ \leq\ & d_p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big[\int_0^\cdot e^{2\eta s}\delta Y_{s-}(\delta Z_s\cdot dX_s^\lambda+d\delta N_s)\Big]_\tau^{\frac{p}{4}}\Big]\\ =\ & d_p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{4\eta s}|\delta Y_{s-}|^2\big(\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2ds+d[\delta N]_s\big)\Big)^{\frac{p}{4}}\Big]\\ \leq\ & d_p'\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{4\eta s}|\delta Y_s|^2\big|\widehat{\sigma}_s^{\mathbf{T}}\delta Z_s\big|^2ds\Big)^{\frac{p}{4}}+\Big(\int_0^\tau e^{4\eta s}|\delta Y_{s-}|^2d[\delta N]_s\Big)^{\frac{p}{4}}\Big]\\ \leq\ & \frac{d_p'}{2\varepsilon'}\big\|\delta Y\big\|^p_{\mathbb{D}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}+\frac{d_p'\varepsilon'}{2}\big\|\delta Z\big\|^p_{\mathbb{H}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}+d_p'\big\|\delta Y\big\|^{\frac{p}{2}}_{\mathbb{D}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}\Big(\big\|N\big\|^{\frac{p}{2}}_{\mathbb{N}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}+\big\|N'\big\|^{\frac{p}{2}}_{\mathbb{N}^p_{\eta,\tau}(\mathbb{Q}^\lambda)}\Big),\end{aligned}$$

for some $\varepsilon' > 0$, where we used $d[\delta N]_s \le 2(d[N]_s + d[N']_s)$. Plugging this estimate into (4.17), and by the arbitrariness of λ, we obtain

$$\begin{aligned}\|\delta Z\|^p_{\mathcal{H}^p_{\eta,\tau}} + \|\delta U\|^p_{\mathcal{N}^p_{\eta,\tau}} \le C'_{p,\eta,\eta'}\Big\{ &\|\delta Y\|^p_{\mathcal{D}^p_{\eta',\tau}} + \mathcal{E}\Big[\Big(\int_0^\tau e^{\eta s}|\delta f_s(Y_s,Z_s)|ds\Big)^p\Big] \\ &+ \|\delta Y\|^{\frac{p}{2}}_{\mathcal{D}^p_{\eta',\tau}}\Big(\|N\|^{\frac{p}{2}}_{\mathcal{N}^p_{\eta,\tau}} + \|N'\|^{\frac{p}{2}}_{\mathcal{N}^p_{\eta,\tau}}\Big)\Big\}.\end{aligned}$$

Together with (4.14) from Step 1, and Proposition 4.3, this induces the first estimate in Theorem 3.8 (ii).

3. It remains to verify the announced estimate on $\int_0^t e^{\alpha s} d\delta U_s$. Given the dynamics of δY in (4.13), it follows from a direct application of Itô's formula and the use of Assumption 3.1 that:

$$\begin{aligned}\sup_{0\le u\le\tau}\Big|\int_0^u e^{\alpha s} d\delta U_s\Big| \;\le\; & 2\sup_{0\le s\le\tau} e^{\alpha s}|\delta Y_s| + (|\alpha|+L)\int_0^\tau e^{\alpha s}|\delta Y_s|ds + 2L\int_0^\tau e^{\alpha s}\big|\widehat{\sigma}^{\mathbf{T}}_s \delta Z_s\big|ds \\ &+ \int_0^\tau e^{\alpha s}\big|\delta f_s(Y_s,Z_s)\big|ds + \sup_{0\le u\le\tau}\Big|\int_0^u e^{\alpha s}\delta Z_s\cdot dX^\lambda_s\Big|.\end{aligned}$$

By the BDG inequality and the Cauchy-Schwarz inequality, we obtain for $\beta\in(\alpha,\rho)$:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\le u\le\tau}\Big|\int_0^u e^{\alpha s} d\delta U_s\Big|^p\Big] \\ &\le C_p\Big(\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\le s\le\tau}\big|e^{\alpha s}\delta Y_s\big|^p\Big] + (|\alpha|+L)^p\Big(\int_0^\infty e^{-(\beta-\alpha)s}ds\Big)^p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\sup_{0\le s\le\tau} e^{p\beta s}|\delta Y_s|^p\Big] \\ &\qquad + (2L)^p\Big(\int_0^\infty e^{-2(\beta-\alpha)s}ds\Big)^{\frac{p}{2}}\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\beta s}\big|\widehat{\sigma}^{\mathbf{T}}_s\delta Z_s\big|^2 ds\Big)^{\frac{p}{2}}\Big] \\ &\qquad + \mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{\alpha s}\big|\delta f_s(Y_s,Z_s)\big|ds\Big)^p\Big] + d_p\mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{2\alpha s}\big|\widehat{\sigma}^{\mathbf{T}}_s\delta Z_s\big|^2 ds\Big)^{\frac{p}{2}}\Big]\Big) \\ &\le C_{p,\alpha,\beta,L}\Big(\|\delta Y\|^p_{\mathbb{D}^p_{\beta,\tau}(\mathbb{Q}^\lambda)} + \|\delta Z\|^p_{\mathbb{H}^p_{\beta,\tau}(\mathbb{Q}^\lambda)} + \mathbb{E}^{\mathbb{Q}^\lambda}\Big[\Big(\int_0^\tau e^{\alpha s}\big|\delta f_s(Y_s,Z_s)\big|ds\Big)^p\Big]\Big),\end{aligned}$$

for some constant $C_{p,\alpha,\beta,L}$. □

4.4 Wellposedness of reflected backward SDEs

We start from the so-called Snell envelope defined by the dynamic optimal stopping problem:

$$\widehat{y}_{t\wedge\tau} := \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,\tau}}\mathbb{E}^{\mathbb{P}}\Big[\widehat{\xi}\mathbf{1}_{\{\theta\ge\tau\}} + \widehat{S}_\theta\mathbf{1}_{\{\theta<\tau\}}\Big|\mathcal{F}^{+,\mathbb{P}}_{t\wedge\tau}\Big], \quad \text{with} \quad \widehat{\xi} := e^{-\mu\tau}\xi,\ \widehat{S}_t := e^{-\mu t}S_t,$$

where $\mathcal{T}_{t,\tau}$ denotes the set of all $\mathbb{F}^{+,\mathbb{P}}-$stopping times θ with $t\wedge\tau\le\theta\le\tau$. Following the proof of [EKP+97, Proposition 5.1] and the theory of optimal stopping, see e.g., [El 81], we deduce that there exists an $X-$integrable process $\widehat{z}$, such that:

$$\begin{aligned}\widehat{y}_{t\wedge\tau} = \widehat{\xi} - \int_{t\wedge\tau}^\tau \widehat{z}_s\cdot dX_s - \int_{t\wedge\tau}^\tau d\widehat{u}_s, \quad \widehat{y}_t\ge\widehat{S}_t, \quad t\ge 0,\ \mathbb{P}-\text{a.s.} \\ \text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\tau\wedge t}\big(1\wedge(\widehat{y}_{t-}-\widehat{S}_{t-})\big)d\widehat{u}_t\Big] \;=\; 0, \text{ for all } t\ge 0,\end{aligned}$$

where $\widehat{u}$ is local supermartingale, starting from $\widehat{u}_0 = 0$, orthogonal to X, i.e., $[X,\widehat{u}] = 0$. In other words, $(\widehat{y},\widehat{z},\widehat{u})$ is a solution of the RBSDE with generator $f\equiv 0$ and obstacle $\widehat{S}$. Then, it follows by Itô's formula that the triple (y,z,u), defined by

$$y_t := e^{\mu t}\widehat{y}_t, \quad z_t := e^{\mu t}\widehat{z}_t, \quad u_t := \int_0^t e^{\mu s}d\widehat{u}_s, \quad t\ge 0,$$

is a solution of the following RBSDE

$$y_{t\wedge\tau} = \xi - \mu\int_{t\wedge\tau}^{\tau} y_s ds - \int_{t\wedge\tau}^{\tau} z_s\cdot dX_s - \int_{t\wedge\tau}^{\tau} du_s,\ y_t \geq S_t,\ t\geq 0,\ \mathbb{P}-\text{a.s.}$$
$$\text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{\tau\wedge t}\big(1\wedge(y_{t-}-S_{t-})\big)du_t\Big] \ = \ 0,\ \text{for all } t\geq 0,$$

where u is local supermartingale, starting from $u_0 = 0$, orthogonal to X, i.e., $[X,u]=0$.

Lemma 4.4. *For all $\alpha\in[-\mu,\rho)$ and $p\in(1,q)$, we have*

$$\|y\|^p_{\mathcal{D}^p_{\rho,\tau}} + \|z\|^p_{\mathcal{H}^p_{\alpha,\tau}} \ \leq\ C_{p,q,L,\alpha,\rho}\Big(\|\xi\|^q_{\mathcal{L}^q_{\rho,\tau}} + \|S^+\|^q_{\mathcal{L}^q_{\rho,\tau}}\Big)^{\frac{p}{q}}.$$

Proof. By the definition of $\widehat{y}$, we have

$$\mathbb{E}^{\mathbb{P}}\big[e^{-\mu\tau}\xi\big|\mathcal{F}_t^{+,\mathbb{P}}\big] \ \leq\ \widehat{y}_t \ \leq\ \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,\tau}}\mathbb{E}^{\mathbb{P}}\big[e^{-\mu\tau}|\xi| + e^{-\mu\theta}S^+_\theta\big|\mathcal{F}_t^{+,\mathbb{P}}\big].$$

Then, for $\alpha\in[-\mu,\rho]$,

$$\begin{aligned}
-\mathbb{E}^{\mathbb{P}}\big[e^{\alpha\tau}|\xi|\big|\mathcal{F}_t^{+,\mathbb{P}}\big] \ &\leq\ -e^{(\alpha+\mu)t}\mathbb{E}^{\mathbb{P}}\big[e^{-\mu\tau}|\xi|\big|\mathcal{F}_t^{+,\mathbb{P}}\big] \ \leq\ e^{(\alpha+\mu)t}\widehat{y}_t = e^{\alpha t}y_t\\
&\leq\ \operatorname*{ess\,sup}_{\theta\in\mathcal{T}_{t,\tau}}\mathbb{E}^{\mathbb{P}}\big[e^{\alpha\tau}|\xi| + e^{\alpha\theta}S^+_\theta\big|\mathcal{F}_t^{+,\mathbb{P}}\big] \ \leq\ \mathbb{E}^{\mathbb{P}}\Big[e^{\alpha\tau}|\xi| + \sup_{0\leq s\leq\tau}e^{\alpha s}S^+_s\Big|\mathcal{F}_t^{+,\mathbb{P}}\Big],
\end{aligned}$$

and therefore $e^{\alpha t}|y_t| \leq \mathbb{E}^{\mathbb{P}}\big[e^{\alpha\tau}|\xi| + \sup_{0\leq s\leq\tau}e^{\alpha s}S^+_s\big|\mathcal{F}_t^{+,\mathbb{P}}\big]$. By Lemma 4.1, this implies that

$$\begin{aligned}
\mathcal{E}\Big[\sup_{0\leq s\leq\tau}\big|e^{\alpha s}y_s\big|^p\Big] \ &\leq\ C_p\mathcal{E}\Big[\sup_{0\leq s\leq\tau}\mathbb{E}^{\mathbb{P}}\Big[|e^{\alpha\tau}\xi|^p + \sup_{0\leq u\leq\tau}\big(e^{\alpha u}S^+_u\big)^p\Big|\mathcal{F}_s^{+,\mathbb{P}}\Big]\Big]\\
&\leq\ C_{p,p'}\Big(\|\xi\|^{p'}_{\mathcal{L}^{p'}_{\alpha,\tau}} + \|S^+\|^{p'}_{\mathcal{D}^{p'}_{\alpha,\tau}}\Big)^{\frac{p}{p'}},
\end{aligned}$$

for all $1<p<p'$. By our assumption on ξ and S^+, we see that we need to restrict to $p'<q$ in order to ensure that the last bound is finite. Moreover, by Proposition 4.3, we have for some $\alpha'>\alpha$,

$$\mathcal{E}\Big[\Big(\int_0^\tau e^{2\alpha t}\big|\widehat{\sigma}_t^{\mathbf{T}}z_t\big|^2dt\Big)^{\frac{p}{2}}\Big] \leq C_{p,\alpha,\alpha',L}\mathcal{E}\Big[\sup_{0\leq t\leq\tau}\big|e^{\alpha' t}y_t\big|^p\Big] \leq C_{p,p',\alpha,\alpha',L}\mathcal{E}\Big[\big|e^{\alpha'\tau}\xi\big|^{p'} + \sup_{0\leq t\leq\tau}\big(e^{\alpha' t}S^+_t\big)^{p'}\Big]^{\frac{p}{p'}}.$$

By our assumption on ξ and S^+, we see that we need to restrict α to the interval $[-\mu,\rho)$ in order to ensure that the last bound is finite. □

Now, we construct a sequence of approximating solutions to the RBSDE, using the finite horizon RBSDE result in [BPTZ16] and on the optimal stopping problem above.

Let $\tau_n := \tau\wedge n$, and (Y^n, Z^n, U^n) be the solution to the following RBSDE

$$Y^n_{t\wedge\tau} = y_{\tau_n} + \int_{t\wedge\tau_n}^{\tau_n} f_s(Y^n_s, Z^n_s)ds - \int_{t\wedge\tau_n}^{\tau_n}\big(Z^n_s\cdot dX_s + dU^n_s\big),\ Y^n_{t\wedge\tau_n}\geq S_{t\wedge\tau_n},\ t\geq 0,\quad \mathbb{P}-\text{a.s.}$$
$$\text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{t\wedge\tau_n}\big(1\wedge(Y^n_{t-}-S_{t-})\big)dU^n_t\Big] = 0,\ \text{for all } t\geq 0.$$

We extend the definition of Y^n to $[\![0,\tau]\!]$ by

$$Y^n_t = y_t,\quad Z^n_t = z_t,\quad U^n_t = u_t,\quad \tau_n\leq t\leq\tau,$$

so that (Y^n, Z^n, U^n) is a solution of the RBSDE with parameters (f^n, ξ, S):

$$Y^n_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f^n_s(Y^n_s, Z^n_s)ds - \int_{t\wedge\tau}^{\tau} \big(Z^n_s \cdot dX_s + dU^n_s\big),\ Y^n_{t\wedge\tau} \geq S_{t\wedge\tau},\ t \geq 0,\ \mathbb{P}-\text{a.s.} \tag{4.18}$$

$$\text{and } \mathbb{E}^{\mathbb{P}}\Big[\int_0^{t\wedge\tau} \big(1 \wedge (Y^n_{t-} - S_{t-})\big) dU^n_t\Big] = 0,\ \text{for all } t \geq 0, \tag{4.19}$$

where

$$f^n_t(y,z) := f_s(y,z)\mathbf{1}_{\{s\leq n\}} - \mu y \mathbf{1}_{\{s>n\}}, \quad t \geq 0, \quad (y,z) \in \mathbb{R}\times\mathbb{R}^d.$$

The following result justifies the existence statement in Theorem 3.7.

Proposition 4.5. *For all $\eta \in [-\mu, \rho)$ and $p \in (1, q)$, the sequence $\{(Y^n, Z^n, U^n)\}_{n\in\mathbb{N}}$ converges in $\mathcal{D}^p_{\eta,\tau} \times \mathcal{H}^p_{\eta,\tau} \times \mathcal{U}^p_{\eta,\tau}$ to some (Y, Z, U), which is a solution of the random horizon RBSDE with the parameters (f, ξ, S).*

Proof. **1.** We first show that $\{(Y^n, Z^n, U^n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{D}^p_{\eta,\tau} \times \mathcal{H}^p_{\eta,\tau} \times \mathcal{U}^p_{\eta,\tau}$, which induces the convergence of (Y^n, Z^n, U^n) towards some (Y, Z, U) in $\mathcal{D}^p_{\eta,\tau} \times \mathcal{H}^p_{\eta,\tau} \times \mathcal{U}^p_{\eta,\tau}$.

By the stability result of Theorem 3.8 (ii), we have the following estimate for the differences $(\delta Y, \delta Z, \delta U) := (Y^n - Y^m, Z^n - Z^m, U^n - U^m)$, $n > m$,

$$\begin{aligned}&\|\delta Y\|^p_{\mathcal{D}^p_{\eta,\tau}} + \|\delta Z\|^p_{\mathcal{H}^p_{\eta,\tau}} + \|\delta U\|^p_{\mathcal{N}^p_{\eta,\tau}}\\ &\qquad \leq C\Big\{\Delta_f + \Delta_f^{\frac{1}{2}}\Big(2\big(\overline{f}^{\mathbb{P}}_{\eta',p,\tau}\big)^{\frac{p}{2}} + \|Y^m\|^{\frac{p}{2}}_{\mathcal{D}^p_{\eta',\tau}} + \|Y^n\|^{\frac{p}{2}}_{\mathcal{D}^p_{\eta',\tau}}\Big)\Big\},\end{aligned} \tag{4.20}$$

where, by the Lipschitz property of f in Assumption 3.1,

$$\begin{aligned}\Delta_f^{\frac{p'}{p}} &= \mathcal{E}\Big[\Big(\int_{\tau_m}^{\tau_n} e^{\eta' s}\big|\delta f_s(Y^m_s, Z^m_s)\big|ds\Big)^{p'}\Big]\\ &\leq C_{p',\eta',L}\Big(\mathcal{E}\Big[\Big(\int_{\tau_m}^{\tau_n} e^{\eta' s}\big|f^0_s\big|ds\Big)^{p'}\Big] + \mathcal{E}\Big[\Big(\int_{\tau_m}^{\tau_n} e^{\eta' s}|y_s|ds\Big)^{p'}\Big] + \mathcal{E}\Big[\Big(\int_{\tau_m}^{\tau_n} e^{\eta' s}\big|\widehat{\sigma}^{\mathbf{T}}_s z_s\big|ds\Big)^{p'}\Big]\Big).\end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$\mathcal{E}\Big[\Big(\int_{\tau\wedge m}^{\tau_n} e^{\eta' s}\big|f^0_s\big|ds\Big)^{p'}\Big] \leq \Big(\frac{e^{-2(\rho-\eta')m}}{2(\rho-\eta')}\Big)^{\frac{p'}{2}} \mathcal{E}\Big[\Big(\int_0^{\tau} e^{2\rho s}\big|f^0_s\big|^2 ds\Big)^{\frac{p'}{2}}\Big] \leq \Big(\frac{e^{-2(\rho-\eta')m}}{2(\rho-\eta')}\Big)^{\frac{p'}{2}} \big(\overline{f}^{\mathbb{P}}_{\rho,q,\tau}\big)^{p'}.$$

Similarly, for $\eta < \eta' < \eta'' < \rho$, we obtain that

$$\mathcal{E}\Big[\Big(\int_{\tau_m}^{\tau_n} e^{\eta' s}\big|\widehat{\sigma}^{\mathbf{T}}_s z_s\big|ds\Big)^{p'}\Big] \leq \left(\frac{e^{-2(\eta''-\eta')m}}{2(\eta''-\eta')}\right)^{\frac{p'}{2}} \|\delta Z\|^{p'}_{\mathcal{H}^{p'}_{\eta'',\tau}} \leq C\Big(\frac{e^{-2(\eta''-\eta')m}}{2(\eta''-\eta')}\Big)^{\frac{p'}{2}}\big(\|\xi\|^{p'}_{\mathcal{L}^q_{\rho,\tau}} + \|S^+\|^{p'}_{\mathcal{D}^q_{\rho,\tau}}\big),$$

and

$$\mathcal{E}\Big[\Big(\int_{\tau\wedge m}^{\tau_n} e^{\eta' s}|y_s|ds\Big)^{p'}\Big] \leq \Big(\frac{e^{-(\rho-\eta')m}}{\rho-\eta'}\Big)^{p'} \|y\|^{p'}_{\mathcal{D}^{p'}_{\rho,\tau}} \leq C\Big(\frac{e^{-(\rho-\eta')m}}{\rho-\eta'}\Big)^{p'} \big(\|\xi\|^{p'}_{\mathcal{L}^q_{\rho,\tau}} + \|S^+\|^{p'}_{\mathcal{D}^q_{\rho,\tau}}\big).$$

The last three estimates show that $\Delta_f \longrightarrow 0$ as $m, n \to \infty$, so that the required Cauchy property would follow from (4.20) once we establish that $\big\|Y^n\big\|_{\mathcal{D}^p_{\eta',\tau}}$ is bounded uniformly in n. To see

this, notice that $\|Y^n\|_{\mathcal{D}^p_{\eta',\tau}} \leq \|y\|_{\mathcal{D}^p_{\eta',\tau}} + \|Y^n - y\|_{\mathcal{D}^p_{\eta',\tau}}$, where $\|y\|_{\mathcal{D}^p_{\eta',\tau}}$ is finite by Lemma 4.4, and thus it reduces our task to controlling $\|Y^n - y\|_{\mathcal{D}^p_{\eta',\tau}}$. To do this, we use (4.14) to obtain

$$\begin{aligned}\mathcal{E}\Big[\sup_{0\leq s\leq\tau} e^{p\eta' s}|Y^n_s - y_s|^p\Big] &\leq C_{p,p'}\mathcal{E}\Big[\Big(\int_0^n e^{\eta' s}|f_s(y_s,z_s) - \mu y_s|ds\Big)^{p'}\Big]^{\frac{p}{p'}} \\ &\leq C_{p,p',\mu,L}\mathcal{E}\Big[\Big(\int_0^\tau e^{\eta' s}|f^0_s|ds\Big)^{p'} + \Big(\int_0^\tau e^{\eta' s}|y_s|ds\Big)^{p'} + \Big(\int_0^\tau e^{\eta' s}|\widehat{\sigma}^{\mathbf{T}}_s z_s|ds\Big)^{p'}\Big]^{\frac{p}{p'}},\end{aligned}$$

and we argue as above to verify that the last bound is finite, using the integrability condition on f^0 in Assumption 3.1, together with Lemma 4.4.

2. We next prove that the limit process U is a càdlàg supermartingale with $[U,X]=0$. Theorem 3.8 (ii) also implies that $\mathcal{E}\big[\sup_{0\leq t\leq\tau}|\overline{U}^n_t - \overline{U}^m_t|^p\big] \longrightarrow 0$, where $\overline{U}^n := \int_0^\cdot e^{\eta s}dU^n_s$. Then, there exists a limit process $\overline{U}\in\mathcal{D}^p_{0,\tau}(\mathbb{P})$. As $\overline{U}^n$ is a càdlàg $\mathbb{Q}-$uniformly integrable supermartingale for all $\mathbb{Q}\in\mathcal{Q}_L$, we may deduce that its limit $\overline{U}$ is also a càdlàg $\mathbb{Q}-$uniformly integrable supermartingale for all $\mathbb{Q}\in\mathcal{Q}_L$. Define $U_t := \int_0^t e^{-\eta s}d\overline{U}_s$, $t\geq 0$. Clearly, $U\in\mathcal{D}^p_{\eta,\tau}(\mathbb{P})$. As the integrand $e^{-\eta s}$ is positive, the process U is a supermartingale. By Kunita-Watanabe inequality for semimartingales, we obtain

$$\begin{aligned}\int_0^\tau e^{2\eta s}\big|d[U,X]_s\big| &\leq \int_0^\tau e^{2\eta s}\big|d[U-U^n,X]_s\big| + \int_0^\tau e^{2\eta s}\big|d[U^n,X]_s\big| \\ &\leq \Big(\int_0^\tau e^{2\eta s}d[U-U^n]_s\Big)^{\frac12}\Big(\int_0^\tau e^{2\eta s}d[X]_s\Big)^{\frac12}.\end{aligned}$$

Theorem 3.8 (ii) also states that the right-hand side converges a.s. to 0, at least along a subsequence, which implies that $[U,X]=0$.

3. Clearly, $Y\geq S$, $\mathbb{P}-$a.s. In this step, we prove that the limit supermartingale U satisfies the Skorokhod condition. To do this, denote $\varphi^n := 1\wedge(Y^n - S)$, $\varphi := 1\wedge(Y-S)$, and let us show that the convergence of (Y^n,U^n) to (Y,U) implies that

$$a_n := \mathbb{E}\Big[\int_0^{\tau\wedge t}\varphi^n_{r-}dU^n_r\Big] - \mathbb{E}\Big[\int_0^{\tau\wedge t}\varphi_{r-}dU_r\Big] \longrightarrow 0, \text{ as } n\to\infty, \text{ for all } t\geq 0.$$

For $\varepsilon>0$, let $\tau^\varepsilon_0 = 0$, $\tau^\varepsilon_{i+1} := \inf\big\{r>\tau^\varepsilon_i : |\varphi_r - \varphi_{\tau^\varepsilon_i}|\geq\varepsilon\big\}$, and $\varphi^\varepsilon := \sum_{i\geq 0}\varphi_{\tau^\varepsilon_i}\mathbf{1}_{[\tau^\varepsilon_i,\tau^\varepsilon_{i+1})}$, so that $|\varphi-\varphi^\varepsilon|\leq\varepsilon$. We first decompose

$$a_n \leq \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi^n-\varphi)_{r-}dU^n_r\Big]\Big| + \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi-\varphi^\varepsilon)_{r-}d(U^n-U)_r\Big]\Big| + \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}\varphi^\varepsilon_{r-}d(U^n-U)_r\Big]\Big|.$$

Since φ^ε is piecewise constant, bounded by 1, and $U^n\to U$ in $\mathcal{D}^p_{\eta,\tau}$, we get

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\tau\wedge t}\varphi^\varepsilon_{s-}d(U^n_s - U_s)\Big] = 0. \tag{4.21}$$

For the second term, we have

$$\begin{aligned}0 &\leq \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi_{s-}-\varphi^\varepsilon_{s-})d(U^n_s-U_s)\Big]\Big| \\ &\leq \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi_{s-}-\varphi^\varepsilon_{s-})d(N^n_s-N_s)\Big]\Big| + \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi_{s-}-\varphi^\varepsilon_{s-})d(K^n_s-K_s)\Big]\Big| \\ &= \varepsilon\mathbb{E}\Big[K_{\tau\wedge t} + K^n_{\tau\wedge t}\Big].\end{aligned}$$

By (4.7) and $|f^{n,0}| \le |f^0|$ we obtain that

$$\begin{aligned}\mathbb{E}[K^n_{\tau\wedge t}] &\le C\Big(\|Y^n\|_{\mathcal{D}^p_{\alpha,\tau}} + \mathbb{E}\Big[\Big(\int_0^{\tau\wedge t} e^{2\rho s}|f^{n,0}_s|^2 ds\Big)^{\frac{q}{2}}\Big]\Big)\\ &\le C\Big(\|Y^1\|_{\mathcal{D}^p_{\alpha,\tau}} + \|Y\|_{\mathcal{D}^p_{\alpha,\tau}} + \mathbb{E}\Big[\Big(\int_0^{\tau\wedge t} e^{2\rho s}|f^0_s|^2 ds\Big)^{\frac{q}{2}}\Big]\Big) < \infty.\end{aligned}$$

Hence, we may control the second term by choosing ε arbitrarily small. For the first term, we have

$$\begin{aligned}0 &\le \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi^n_{s-} - \varphi_{s-})dU^n_s\Big]\Big| = \Big|\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi^n_{s-} - \varphi_{s-})dK^n_s\Big]\Big|\\ &\le \mathbb{E}\Big[\Big(\sup_{0\le s\le\tau\wedge t}|Y^n_s - Y_s|\wedge 1\Big)K^n_{\tau\wedge t}\Big] \le \mathbb{E}\Big[\sup_{0\le s\le\tau\wedge t}|Y^n_s - Y_s|^{p^*}\wedge 1\Big]^{\frac{1}{p^*}}\mathbb{E}\big[(K^n_{\tau\wedge t})^p\big]^{\frac{1}{p}}.\end{aligned}$$

Again we may show that $\mathbb{E}\big[(K^n_{\tau\wedge t})^p\big]$ is bounded by a constant, independent of $n \in \mathbb{N}$. As $Y^n \to Y$ in $\mathcal{D}^p_{\eta,\tau}$, we have

$$\sup_{0\le s\le\tau\wedge t}|Y^n_s - Y_s|^{p^*} \to 0, \quad a.s.$$

By dominated convergence, we have

$$\lim_{n\to\infty}\mathbb{E}\Big[\sup_{0\le s\le\tau\wedge t}|Y^n_s - Y_s|^{p^*}\wedge 1\Big] = 0.$$

Hence, we have

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\tau\wedge t}(\varphi^n_{s-} - \varphi_{s-})dU^n_s\Big] = 0.$$

All together, we have

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\tau\wedge t}\varphi^n_{s-}dU^n_s - \int_0^{\tau\wedge t}\varphi_{s-}dU_s\Big] = 0,$$

and the assertion follows.

4. We finally verify that (Y, Z, U) satisfies the differential part of the RBSDE. The following verification is reported for the convenience of the reader, and reproduces exactly the line of argument in [DP97, Section 5.2, Step 3]. For any $\alpha \in \mathbb{R}$ and $t \ge 0$, we have by Itô's formula and (4.18) that

$$\begin{aligned}e^{\alpha(t\wedge\tau)}Y^n_{t\wedge\tau} &= e^{\alpha\tau}\xi + \int_{t\wedge\tau}^{\tau} e^{\alpha s}\big\{\big(f^n_s(Y^n_s, Z^n_s) - \alpha Y^n_s\big)ds - \big(Z^n_s\cdot dX_s + dU^n_s\big)\big\}\\ &= e^{\alpha\tau}\xi + \int_{t\wedge\tau}^{\tau} e^{\alpha s}\big\{\big(f_s(Y^n_s, Z^n_s) - \alpha Y^n_s\big)ds - \big(Z^n_s\cdot dX_s + dU^n_s\big)\big\}\\ &\quad - \int_{t\vee\tau_n}^{\tau} e^{\alpha s}\big(f_s(Y^n_s, Z^n_s) + \mu Y^n_s\big)ds.\end{aligned}$$

We choose $\alpha < \eta$. Then, it is easily seen that $e^{\alpha(t\wedge\tau)}Y^n_{t\wedge\tau} \longrightarrow e^{\alpha(t\wedge\tau)}Y_{t\wedge\tau}$, and for all $t \ge 0$, $\int_{t\wedge\tau}^{\tau} e^{\alpha s}dU^n_s \to \int_{t\wedge\tau}^{\tau} e^{\alpha s}dU_s$, in $\mathbb{L}^p$. By the BDG inequality, it also follows that $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Z^n_s \cdot dX_s \longrightarrow \int_{t\wedge\tau}^{\tau} e^{\alpha s}Z_s\cdot dX_s$, in $\mathbb{L}^p$, for all $t \ge 0$. Moreover, we have $\int_{t\wedge\tau}^{\tau} e^{\alpha s}Y^n_s ds \longrightarrow \int_{t\wedge\tau}^{\tau} e^{\alpha s}Y_s ds$, in $\mathbb{L}^p$ for all $t \ge 0$, due to the following estimate

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^{\tau} e^{\alpha s}|Y^n_s - Y_s|ds\Big)^p\Big] &\le \mathbb{E}\Big[\sup_{0\le s\le\tau} e^{p\eta s}|Y^n_s - Y_s|^p\Big(\int_0^{\tau} e^{-(\eta-\alpha)s}ds\Big)^p\Big]\\ &\le \frac{1}{(\eta-\alpha)^p}\mathbb{E}\Big[\sup_{0\le s\le\tau} e^{p\eta s}|Y^n_s - Y_s|^p\Big] \longrightarrow 0, \quad \text{as } n\to\infty.\end{aligned}$$

From a similar argument, we also have $\int_{\tau_n}^{\tau} e^{\alpha s}\big(f_s(Y_s^n, Z_s^n) + \mu Y_s^n\big)ds \longrightarrow 0$, in $\mathbb{L}^p$, and by Lipschitz continuity of f we see that $\int_{t\wedge\tau}^{\tau} e^{\alpha s}\big|f_s(Y_s^n, Z_s^n) - f_s(Y_s, Z_s)\big|ds \longrightarrow 0$, in $\mathbb{L}^p$ for all $t \geq 0$. Therefore, we have proved that

$$e^{\alpha(t\wedge\tau)}Y_{t\wedge\tau} \;=\; e^{\alpha\tau}\xi + \int_{t\wedge\tau}^{\tau} e^{\alpha s}\big\{\big(f_s(Y_s, Z_s) - \alpha Y_s\big)ds - \big(Z_s \cdot dX_s + dU_s\big)\big\},$$

thus completing the proof by Itô's formula. □

We now prove the comparison result. In particular, this justifies the uniqueness statement in Theorem 3.7.

Proof of Theorem 3.8 (i). Denote by $\big\{\big(Y^n, Z^n, U^n\big)\big\}_{n\in\mathbb{N}}$ and $\big\{\big(Y'^n, Z'^n, U'^n\big)\big\}_{n\in\mathbb{N}}$ the approximating sequence of (Y, Z, U) and (Y', Z', U'), using the triples (y, z, u) and (y', z', u'), respectively, as in the last proof. Since $\xi \leq \xi'$ and $S \leq S'$, we have $y_{\tau_n} \leq y'_{\tau_n}$. By standard comparison argument of BSDEs, see e.g. [RS12, Proposition 3.2], this in turns implies that $Y_{\tau_0}^n \leq Y'^n_{\tau_0}$ for all stopping time $\tau_0 \leq \tau$. The required result follows by sending $n \to \infty$. □

4.5 Special case: backward SDE

Proof of Theorem 3.3. By setting $S = -\infty$, the existence and uniqueness results follow from Theorem 3.7. In particular, the Skorokhod condition implies in the present setting that $U = N$ is a $\mathbb{P}-$martingale orthogonal to X. It remains to verify the estimates (3.2).

Let $\eta \geq -\mu$, and observe that Assumption 3.1 implies that $\mathrm{sgn}(y)f_s(y, z) \leq -\mu|y| + L\big|\widehat{\sigma}_s^{\mathbf{T}} z\big| + \big|f_s^0\big| \leq \eta|y| + L\big|\widehat{\sigma}_s^{\mathbf{T}} z\big| + \big|f_s^0\big|$. Then, by Itô's formula, together with Proposition 4.2, we have

$$\begin{aligned}
&e^{\eta(n\wedge\tau)}|Y_{n\wedge\tau}| - e^{\eta(t\wedge\tau)}|Y_{t\wedge\tau}| \\
&\quad\geq \int_{t\wedge\tau}^{n\wedge\tau} e^{\eta s}\Big(\eta|Y_s|ds - \mathrm{sgn}(Y_{s-})\big(f_s(Y_s, Z_s)ds - Z_s \cdot dX_s - dN_s\big)\Big) \\
&\quad\geq \int_{t\wedge\tau}^{n\wedge\tau} e^{\eta s}\big(- L\big|\widehat{\sigma}_s^{\mathbf{T}} Z_s\big|ds - |f_s^0|ds + \mathrm{sgn}(Y_s)Z_s \cdot dX_s + \mathrm{sgn}(Y_{s-})dN_s\big).
\end{aligned}$$

Introduce $\widehat{\lambda}_s := L\,\mathrm{sgn}(Y_s)\frac{\widehat{\sigma}_s^{\mathbf{T}} Z_s}{|\widehat{\sigma}_s^{\mathbf{T}} Z_s|}\mathbf{1}_{\{|\widehat{\sigma}_s^{\mathbf{T}} Z_s| \neq 0\}}$, and recall that N remains a martingale under $\mathbb{Q}^{\widehat{\lambda}}$ by the orthogonality $[X, N] = 0$. Then, taking conditional expectation under $\mathbb{Q}^{\widehat{\lambda}}$, we obtain

$$\begin{aligned}
e^{\eta(t\wedge\tau)}|Y_{t\wedge\tau}| \;&\leq\; \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\widehat{\lambda}}}\Big[e^{\eta(n\wedge\tau)}|Y_{n\wedge\tau}| + \int_0^{n\wedge\tau} e^{\eta s}\big|f_s^0\big|ds\;\Big|\mathcal{F}_{t\wedge\tau}^{+,\mathbb{P}}\Big] \\
&=\; \mathbb{E}^{\mathbb{Q}^{\widehat{\lambda}}}\Big[e^{\eta\tau}|\xi| + \int_0^{\tau} e^{\eta s}\big|f_s^0\big|ds\;\Big|\mathcal{F}_{t\wedge\tau}^{+,\mathbb{P}}\Big],
\end{aligned}$$

by the uniform integrability of the process $\{e^{\eta s}Y_s\}_{s\geq 0}$. By Lemma 4.1, this provides

$$\big\|Y\big\|^p_{\mathcal{D}^p_{\eta,\tau}} \leq \frac{p'}{p'-p}\mathcal{E}\Big[\Big(e^{\eta\tau}|\xi| + \int_0^{\tau} e^{\eta s}\big|f_s^0\big|ds\Big)^{p'}\Big]^{\frac{p}{p'}} \leq C_{p,p',\eta,\eta'}\Big\{\big\|\xi\big\|^p_{\mathcal{L}^{p'}_{\eta',\tau}} + \big(\overline{f}^0_{\eta',p',\tau}\big)^p\Big\}, \quad (4.22)$$

for all $p' \in (p, q)$ and $-\mu \leq \eta < \eta' \leq \rho$ with some constant $C_{p,p',\eta,\eta'}$. Next we can follow the lines of the proof of Proposition 4.3 and Theorem 3.5 (i) to show that

$$\big\|Z\big\|^p_{\mathcal{H}^p_{\eta'',\tau}(\mathbb{P})} + \big\|N\big\|^p_{\mathcal{N}^p_{\eta'',\tau}(\mathbb{P})} \;\leq\; C_{p,L,\eta,\eta''}\big(\big\|Y\big\|^p_{\mathcal{H}^p_{\eta,\tau}(\mathbb{P})} + \big(\overline{f}^0_{\eta,p,\tau}\big)^p\big),$$

for $\eta'' < \eta$. Combined with (4.22), this induces the required estimate. □

For later use, we need a version of the stability result allowing for different horizons. This requires to extend the generator and the solution of the BSDE beyond the terminal time by:

$$f_{t\vee\tau}(y,z)=0,\quad Y_{t\vee\tau}=\xi,\quad Z_{t\vee\tau}=0,\quad N_{t\vee\tau}=0,\quad \text{for all } t\geq 0.$$

Proposition 4.6. *Suppose (f,ξ,τ) and (f',ξ',τ') safisfy Assumptions 3.1 with the same parameters L and μ. Let $\delta Y=Y-Y'$, $\delta Z=Z-Z'$, $\delta N=N-N'$, $\delta f=f-f'$ and $\delta\xi=\xi-\xi'$. Then, for all stopping time $\tau_0\leq\tau\wedge\tau'$, and all $\eta\in[-\mu,\rho)$, $1<p<p'<q$, we have*

$$\big|e^{\eta\tau_0}\delta Y_{\tau_0}\big| \;\leq\; \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}_L}\mathbb{E}^{\mathbb{Q}}\Big[\big|e^{\eta\tau}\xi-e^{\eta\tau'}\xi'\big|+\int_{\tau_0}^{\tau\vee\tau'}e^{\eta s}\big|\delta f_s(Y_s,Z_s)\big|ds\Big|\mathcal{F}^{+,\mathbb{P}}_{\tau_0}\Big].$$

Proof. By Proposition 4.2 and the Lipschitz and monotonicity conditions of Assumption 3.1,

$$\begin{aligned}\big|e^{\eta\tau_0}\delta Y_{\tau_0}\big| \;\leq\; &\big|e^{\eta\tau}\xi-e^{\eta\tau'}\xi'\big|+\int_{\tau_0}^{\tau\vee\tau'}e^{\eta s}\big|\delta f_s(Y_s,Z_s)\big|ds+e^{\eta s}\operatorname{sgn}(\delta Y_s)\delta Z_s\cdot\big(dX_s-\widehat{\sigma}_s\widehat{\lambda}_s ds\big)\\ &-\int_{\tau_0}^{\tau\vee\tau'}e^{\eta s}\operatorname{sgn}(\delta Y_{s-})d\delta N_s,\end{aligned}$$

with $\widehat{\lambda}_s:=L\operatorname{sgn}(\delta Y_s)\frac{\widehat{\sigma}_s\delta Z_s}{|\widehat{\sigma}_s\delta Z_s|}\mathbf{1}_{\{|\widehat{\sigma}_s\delta Z_s|\neq 0\}}$. Taking conditional expectation under $\mathbb{Q}^{\widehat{\lambda}}\in\mathcal{Q}_L$ induces the required inequality. □

5 Second order backward SDE: representation and uniqueness

We shall use the additional notation:

$$\mathcal{E}^{\mathbb{P},+}_t[\cdot]:=\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t)}\operatorname*{ess\,sup}^{\mathbb{P}'}_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}')}\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}^+_t],\quad\text{for all}\quad t\geq 0,\ \mathbb{P}\in\mathcal{P}_0.$$

Remark 5.1. It follows from Assumption 3.11 and Doob's inequality that for any $q'<q$

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{0\leq t\leq\tau}\mathcal{E}^{\mathbb{P},+}_t\big[(e^{\rho\tau}|\xi|)^{q'}\big]\Big]+\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{0\leq t\leq\tau}\mathcal{E}^{\mathbb{P},+}_t\Big[\Big(\int_0^{\tau}\big|e^{\rho s}f^0_s\big|^2ds\Big)^{\frac{q'}{2}}\Big]\Big]<\infty,$$

We also note that $\mathcal{E}^{\mathbb{P},+}_t\big[\mathbf{1}_{\{\tau\geq n\}}\big]$ is a $\mathbb{P}$-supermartingale. Then, by Doob's martingale inequality, we have $\mathbb{E}^{\mathbb{P}}\Big[\mathcal{E}^{\mathbb{P},+}_t\big[\mathbf{1}_{\{\tau\geq n\}}\big]\Big]\leq C\mathcal{E}^{\mathcal{P}_0}\big[\mathbf{1}_{\{\tau\geq n\}}\big]\longrightarrow 0$, so that $\mathbb{E}^{\mathbb{P}}\Big[\lim_{n\to\infty}\mathcal{E}^{\mathbb{P},+}_t\big[\mathbf{1}_{\{\tau\geq n\}}\big]\Big]=0$, by dominated convergence theorem, and therefore

$$\lim_{n\to\infty}\mathcal{E}^{\mathbb{P},+}_t\big[\mathbf{1}_{\{\tau\geq n\}}\big]\;=\;0,\quad\mathbb{P}\text{-a.s.}$$

□

Similar to Soner, Touzi & Zhang [STZ12], the uniqueness follows from the representation of the Y component of the 2BDSE (3.6) by means of the family of backward SDEs. For all $\mathbb{P}\in\mathcal{P}_0$, we denote by $\mathcal{Y}^{\mathbb{P}}[\xi_0,\tau_0]$ the $Y-$component of the solution of the backward SDE:

$$\mathcal{Y}^{\mathbb{P}}_{t\wedge\tau}\;=\;\xi_0+\int_{t\wedge\tau_0}^{\tau_0}F_s\big(\mathcal{Y}^{\mathbb{P}}_s,\mathcal{Z}^{\mathbb{P}}_s,\widehat{\sigma}_s\big)ds-\int_{t\wedge\tau_0}^{\tau_0}\big(\mathcal{Z}^{\mathbb{P}}_s\cdot dX_s+d\mathcal{N}^{\mathbb{P}}_s\big),\quad t\geq 0,\quad\mathbb{P}\text{-a.s.}\quad(5.1)$$

where ξ_0 is an $\mathcal{F}_{\tau_0}-$measurable r.v. for some stopping time $\tau_0\leq\tau$. Under our conditions on (F,ξ), the wellposedness of these BSDEs for $\xi_0\in\mathcal{L}^p_{\eta,\tau_0}(\mathbb{P})$ follows from Theorem 3.3. Remark that in the sequel we always consider the version of $\mathcal{Y}^{\mathbb{P}}$ such that $\mathcal{Y}^{\mathbb{P}}_{t\wedge\tau}\in\mathcal{F}^+_{t\wedge\tau}$ by the result of Lemma 6.3.

The following statement provides a representation for the 2BSDE, and justifies the comparison (and uniqueness) result of Proposition 3.13.

Proposition 5.2. *Let Assumptions 3.1 and 3.11 hold true, and let* $(Y,Z)\in\mathcal{D}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{+,\mathcal{P}_0}\big)\times\mathcal{H}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{\mathcal{P}_0}\big)$ *be a solution of the 2BSDE* (3.6)*, for some* $p\in(1,q)$ *and* $\eta\in[-\mu,\rho)$*. Then,*

$$Y_{t_1\wedge\tau} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)} \mathcal{Y}^{\mathbb{P}'}_{t_1\wedge\tau}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big] \tag{5.2}$$

$$= \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)} \mathcal{Y}^{\mathbb{P}'}_{t_1\wedge\tau}[\xi,\tau],\quad \mathbb{P}-a.s.\ for\ all\ \mathbb{P}\in\mathcal{P}_0,\ 0\le t_1\le t_2. \tag{5.3}$$

In particular, the 2BSDE has at most one solution in $\mathcal{D}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{+,\mathcal{P}_0}\big)\times\mathcal{H}^p_{\eta,\tau}\big(\mathcal{P}_0,\mathbb{F}^{\mathcal{P}_0}\big)$*, satisfying the estimate* (3.7)*, and the comparison result of Proposition 3.13 holds true.*

Proof. The uniqueness of Y is an immediate consequence of (5.3), and implies the uniqueness of Z, $\widehat{a}_t dt\otimes\mathcal{P}_0$-q.s. by the fact that $\langle Y,X\rangle_t=\langle\int_0^\cdot Z_s\cdot X_s,X\rangle_t=\int_0^t\widehat{a}_sZ_sds$, $\mathbb{P}-$a.s. This representation also implies the comparison result as an immediate consequence of the corresponding comparison result of the BSDEs $\mathcal{Y}^{\mathbb{P}}[\xi,\tau]$.

1. We first prove (5.2). Fix some arbitrary $\mathbb{P}\in\mathcal{P}_0$ and $\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)$. By Definition 3.9 of the solution of the 2BSDE (3.6), we see that Y is a supersolution of the BSDE on $[\![t_1\wedge\tau,t_2\wedge\tau]\!]$ under $\mathbb{P}'$ with terminal condition $Y_{t_2\wedge\tau}$ at time $t_2\wedge\tau$. By the comparison result of Theorem 3.5 (ii), see also Remerk 3.6, this implies that $Y_{t_1\wedge\tau}\ge\mathcal{Y}^{\mathbb{P}'}_{t_1\wedge\tau}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big]$, $\mathbb{P}'$-a.s. As $\mathcal{Y}^{\mathbb{P}'}_{t_1}$ is $\mathcal{F}^+_{t_1\wedge\tau}$-measurable and Y_{t_1} is $\mathcal{F}^{+,\mathcal{P}_0}_{t_1\wedge\tau}$-measurable, the inequality also holds $\mathbb{P}$-a.s., by definition of $\mathcal{P}^+_{\mathbb{P}}(t_1)$ and the fact that measures extend uniquely to the completed σ-algebras. Then, by arbitrariness of $\mathbb{P}'$,

$$Y_{t_1\wedge\tau} \ge \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)} \mathcal{Y}^{\mathbb{P}'}_{t_1\wedge\tau}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big],\quad \mathbb{P}\text{-a.s. for all } \mathbb{P}\in\mathcal{P}_0.$$

We next prove the reverse inequality. Denote $\delta Y:=Y-\mathcal{Y}^{\mathbb{P}'}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big]$, $\delta Z:=Z-\mathcal{Z}^{\mathbb{P}'}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big]$ and $\delta U:=U^{\mathbb{P}'}-\mathcal{N}^{\mathbb{P}'}\big[Y_{t_2\wedge\tau},t_2\wedge\tau\big]$. Recall that $U^{\mathbb{P}'}$ is a $\mathbb{P}'$-supermartingale with decomposition $U^{\mathbb{P}'}=N^{\mathbb{P}'}-K^{\mathbb{P}'}$. For $\alpha\in[-\mu,\eta]$, it follows by Itô's formula, together with the Lipschitz property of F in Assumption 3.1 that there exist two bounded processes $a^{\mathbb{P}'}$ and $b^{\mathbb{P}'}$, uniformly bounded by the Lipschitz constant L of F, such that

$$e^{\alpha(t_1\wedge\tau)}\delta Y_{t_1\wedge\tau} = \int_{t_1\wedge\tau}^{t_2\wedge\tau} e^{\alpha s}\big(a^{\mathbb{P}'}_s\delta Y_s+b^{\mathbb{P}'}_s\cdot\widehat{\sigma}^{\mathbf{T}}_s\delta Z_s\big)ds-\int_{t_1\wedge\tau}^{t_2\wedge\tau} e^{\alpha s}\big(\widehat{\sigma}^{\mathbf{T}}_s\delta Z_s\cdot dW_s+d\delta U_s\big),$$

which implies that

$$e^{\alpha(t_1\wedge\tau)}\delta Y_{t_1\wedge\tau}=-\mathbb{E}^{\mathbb{P}'}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau}\Gamma^{\mathbb{P}'}_s e^{\alpha s}d\delta U^{\mathbb{P}'}_s\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]=\mathbb{E}^{\mathbb{P}'}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau}\Gamma^{\mathbb{P}'}_s e^{\alpha s}dK^{\mathbb{P}'}_s\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big],$$

with $\Gamma^{\mathbb{P}'}_s:=e^{\int_{t_1\wedge\tau}^s(a^{\mathbb{P}'}_u-\frac{1}{2}|b^{\mathbb{P}'}_u|^2)du+\int_{t_1\wedge\tau}^s b^{\mathbb{P}'}_u\cdot dW_u}$. As $a^{\mathbb{P}'}_u,b^{\mathbb{P}'}_u$ are uniformly bounded by L, it follows from the Doob maximal inequality that

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}'}\Big[\Big(\sup_{t_1\wedge\tau\le s\le t_2\wedge\tau}\Gamma^{\mathbb{P}'}_s\Big)^{\frac{p+1}{p-1}}\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]\\ \le\ & e^{L(t_2-t_1)}C'_p\mathbb{E}^{\mathbb{P}'}\Big[e^{-\int_{t_1\wedge\tau}^{t_2\wedge\tau}\frac{p+1}{2(p-1)}|b^{\mathbb{P}'}_u|^2du+\frac{p+1}{p-1}\int_{t_1\wedge\tau}^{t_2\wedge\tau}b^{\mathbb{P}'}_u\cdot dW_u}\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]\ <\ C_p\ <\ \infty,\end{aligned}$$

where C_p is a constant independent of $\mathbb{P}'$. Then, it follows from the Hölder inequality that

$$\begin{aligned} e^{-|\alpha|t_1}\delta Y_{t_1\wedge\tau} &\leq \mathbb{E}^{\mathbb{P}'}\Big[\Big(\sup_{t_1\wedge\tau\leq s\leq t_2\wedge\tau}\Gamma^{\mathbb{P}'}_s\Big)^{\frac{p+1}{p-1}}\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]^{\frac{p-1}{p+1}}\mathbb{E}^{\mathbb{P}'}\Big[\Big(\int_{t_1\wedge\tau}^{t_2\wedge\tau}e^{\alpha s}dK^{\mathbb{P}'}_s\Big)^{\frac{p+1}{2}}\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]^{\frac{2}{p+1}} \\ &\leq C\left(C^{\mathbb{P},p,\alpha}_{t_1}\right)^{\frac{1}{p+1}}\mathbb{E}^{\mathbb{P}'}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau}e^{\alpha s}dK^{\mathbb{P}'}_s\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]^{\frac{1}{p+1}} \\ &\leq C_p^{\frac{p+1}{p-1}}\left(C^{\mathbb{P},p,\alpha}_{t_1}\right)^{\frac{1}{p+1}}e^{(\alpha t_1)\vee(\alpha t_2)}\mathbb{E}^{\mathbb{P}'}\Big[\int_{t_1\wedge\tau}^{t_2\wedge\tau}dK^{\mathbb{P}'}_s\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big]^{\frac{1}{p+1}}, \end{aligned}$$

where

$$C^{\mathbb{P},p,\alpha}_{t_1} := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)}\mathbb{E}^{\mathbb{P}'}\Big[\Big(\int_{t_1\wedge\tau}^{t_2\wedge\tau}e^{\alpha s}dK^{\mathbb{P}'}_s\Big)^p\Big|\mathcal{F}^+_{t_1\wedge\tau}\Big].$$

As it follows from the minimality condition in Definition 3.9 that

$$K^{\mathbb{P}}_{t_1\wedge\tau} = \operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)}\mathbb{E}^{\mathbb{P}'}\big[K^{\mathbb{P}'}_{t_2\wedge\tau}\big|\mathcal{F}^+_{t_1\wedge\tau}\big],$$

and $C^{\mathbb{P},p,\alpha}_{t_1}<\infty$ (see (6.18)), we obtain that

$$Y_{t_1\wedge\tau} - \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t_1\wedge\tau)}\mathcal{Y}^{\mathbb{P}'}_{t_1\wedge\tau}\leq 0,\quad \mathbb{P}\text{-a.s.}$$

thus providing the required equality.

2. Given (5.2), we now show (5.3) by proving that

$$\lim_{n\to\infty}\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t\wedge\tau)}\big|\delta\mathcal{Y}^{\mathbb{P}',n}_{t\wedge\tau}\big| = 0,\quad \mathbb{P}\text{-a.s. where}\quad \delta\mathcal{Y}^{\mathbb{P}',n} := \mathcal{Y}^{\mathbb{P}'}[\xi,\tau]-\mathcal{Y}^{\mathbb{P}'}[Y_{n\wedge\tau},n\wedge\tau]. \tag{5.4}$$

By the stability result of Proposition 4.6, we have

$$\big|e^{\eta(t\wedge\tau)}\delta\mathcal{Y}^{\mathbb{P}',n}_{t\wedge\tau}\big| \leq \operatorname*{ess\,sup}^{\mathbb{P}'}_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}')}\mathbb{E}^{\mathbb{Q}}\Big[\big|e^{\eta\tau}\xi - e^{\eta(n\wedge\tau)}Y_{n\wedge\tau}\big| + \int_{n\wedge\tau}^{\tau}e^{\eta s}\big|F_s\big(\mathcal{Y}^{\mathbb{P}'}_s[\xi,\tau],\mathcal{Z}^{\mathbb{P}'}_s[\xi,\tau],\widehat{\sigma}_s\big)\big|ds\Big|\mathcal{F}^+_{t\wedge\tau}\Big].$$

Notice that $\big|e^{\eta\tau}\xi-e^{\eta(n\wedge\tau)}Y_{n\wedge\tau}\big| = \mathbf{1}_{\{\tau\geq n\}}\big|e^{\eta\tau}\xi-e^{\eta(n\wedge\tau)}Y_{n\wedge\tau}\big|\leq 2\mathbf{1}_{\{\tau\geq n\}}e^{\eta\tau}\sup_{0\leq s\leq\tau}|Y_s|$. Then, it follows from the Hölder's inequality that for some $p'<p$,

$$\mathcal{E}^{\mathbb{P},+}_{t\wedge\tau}\big[\big|e^{\eta\tau}\xi - e^{\eta(n\wedge\tau)}Y_{n\wedge\tau}\big|^p\big|\mathcal{F}^+_{t\wedge\tau}\big] \leq 2\mathcal{E}^{\mathbb{P},+}_{t\wedge\tau}\Big[\sup_{0\leq s\leq\tau}e^{p'\eta s}|Y_s|^{p'}\Big]^{\frac{p}{p'}}\mathcal{E}^{\mathbb{P},+}_{t\wedge\tau}\big[\mathbf{1}_{\{\tau\geq n\}}\big]^{\frac{p'-p}{p'}}\longrightarrow 0,$$

as $n\to\infty$, due to the fact that $Y\in\mathcal{D}^p_{\tau,\eta}(\mathcal{P}_0)$ and $\mathcal{E}^{\mathbb{P},+}_{t\wedge\tau}\big[\mathbf{1}_{\{\tau\geq n\}}\big]\longrightarrow 0$ by Remark 5.1. This leads to

$$\limsup_{n\to\infty}\big|e^{\eta(t\wedge\tau)}\delta\mathcal{Y}^{\mathbb{P}',n}_{t\wedge\tau}\big| \leq \limsup_{n\to\infty}\operatorname*{ess\,sup}^{\mathbb{P}'}_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}')}\mathbb{E}^{\mathbb{Q}}\Big[\int_{n\wedge\tau}^{\tau}e^{\eta s}\big|F_s\big(\mathcal{Y}^{\mathbb{P}'}_s[\xi,\tau],\mathcal{Z}^{\mathbb{P}'}_s[\xi,\tau],\widehat{\sigma}_s\big)\big|ds\Big|\mathcal{F}^+_{t\wedge\tau}\Big]. \tag{5.5}$$

We next write $\mathcal{Y}^{\mathbb{P}'}_s := \mathcal{Y}^{\mathbb{P}'}_s[\xi,\tau]$, $\mathcal{Z}^{\mathbb{P}'}_s := \mathcal{Z}^{\mathbb{P}'}_s[\xi,\tau]$, and estimate that

$$\begin{aligned} \int_{n\wedge\tau}^{\tau}e^{\eta s}\big|F_s\big(\mathcal{Y}^{\mathbb{P}'}_s,\mathcal{Z}^{\mathbb{P}'}_s,\widehat{\sigma}_s\big)\big|ds &\leq \int_{n\wedge\tau}^{\tau}e^{\eta s}\big|f^0_s\big|ds + L\int_{n\wedge\tau}^{\tau}e^{\eta s}\big|\mathcal{Y}^{\mathbb{P}'}_s\big|ds + L\int_{n\wedge\tau}^{\tau}e^{\eta s}\big|\widehat{\sigma}^{\mathbf{T}}_s\mathcal{Z}^{\mathbb{P}'}_s\big|ds \\ &\leq \Big(\frac{e^{-2(\eta'-\eta)n}}{2(\eta'-\eta)}\Big)^{\frac{1}{2}}\Big[\Big(\int_0^{\tau}e^{2\eta' s}\big|f^0_s\big|^2ds\Big)^{\frac{1}{2}} + L\Big(\int_0^{\tau}e^{2\eta' s}\big|\widehat{\sigma}^{\mathbf{T}}_s\mathcal{Z}^{\mathbb{P}'}_s\big|^2ds\Big)^{\frac{1}{2}}\Big] \\ &\quad + L\Big(\frac{e^{-(\eta'-\eta)n}}{\eta'-\eta}\Big)\sup_{0\leq s\leq\tau}e^{\eta' s}\big|\mathcal{Y}^{\mathbb{P}'}_s\big|. \end{aligned}$$

By the integrability condition on f^0 in Assumption 3.1, and the fact that $(\mathcal{Y}^{\mathbb{P}'}, \mathcal{Z}^{\mathbb{P}'}) \in \mathcal{D}^p_{\eta',\tau}(\mathbb{P}') \times \mathcal{H}^p_{\eta',\tau}(\mathbb{P}')$ by the wellposedness result of backward SDEs in Theorem 3.3, this implies that $\mathbb{P}-$a.s. $\mathcal{E}^{\mathbb{P},+}_{t\wedge\tau}\big[\big(\int_{n\wedge\tau}^{\tau} e^{\eta s}|F_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s, \widehat{\sigma}_s)\big|ds\big)^p\big] \longrightarrow 0$, and therefore $\big|e^{\eta(t\wedge\tau)}\delta\mathcal{Y}^{\mathbb{P}',n}_{t\wedge\tau}\big| \longrightarrow 0$, by (5.5).

3. We finally verify the estimate (3.7). By the representation (5.3) proved in the previous step, and following the proof of Proposition 6.8, we may show that

$$\mathcal{E}^{\mathcal{P}_0}\Big[\sup_{0\le t\le\tau} e^{p\eta t}|Y_t|^p\Big] \;\le\; C_p\mathcal{E}^{\mathcal{P}_0}\Big[\sup_{0\le t\le\tau} \mathcal{E}^{\mathbb{P},+}_t\Big[\big|e^{\eta\tau}\xi\big|^p + \Big(\int_0^\tau e^{\eta s}\big|f^0_s\big|ds\Big)^p\Big]\Big].$$

By Remark 5.1 we obtain that $\|Y\|^p_{\mathcal{D}^p_{\eta,\tau}(\mathcal{P}_0)} \le C_p\big(\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}_0)} + (\overline{F}^0_{\rho,q,\tau})^p\big)$. As, for each $\mathbb{P}\in\mathcal{P}_0$, $\big(Y, Z, U^{\mathbb{P}}\big)$ is a solution of the RBSDE (6.17), the required estimate for the Z component follows from Proposition 4.3. □

6 Second order backward SDE: existence

In view of the representation (5.3) in Proposition 5.2, we follow the methodology of Soner, Touzi & Zhang [STZ12, STZ13] by defining the dynamic version of this representation (which requires the additional notations of the next section), and proving that the induced process defines a solution of the 2BSDE. In order to bypass the strong regularity conditions of [STZ12, STZ13], we adapt the approach of Possamaï, Tan & Zhou [PTZ17] to ensure measurability of the process of interest.

6.1 Shifted space

We recall the concatenation map of two paths ω, ω' at the junction time t defined by $(\omega\otimes_t\omega')_s := \omega_s\mathbf{1}_{[0,t)}(s) + (\omega_t + \omega'_{s-t})\mathbf{1}_{[t,\infty)}(s)$, $s\ge 0$, and we define the $(t,\omega)-$shifted random variable

$$\xi^{t,\omega}(\omega') := \xi(\omega\otimes_t\omega'), \quad \text{for all} \quad \omega'\in\Omega.$$

By a standard monotone class argument, we see that $\xi^{t,\omega}$ is $\mathcal{F}_s$ whenever ξ is $\mathcal{F}_{t+s}$-measurable. In particular, for an $\mathbb{F}$-stopping time τ, $t\le\tau$, then $\theta := \tau^{t,\omega} - t$ is still an $\mathbb{F}$-stopping time. Similarly, for any $\mathbb{F}$-progressively measurable process Y, the shifted process

$$Y^{t,\omega}_s(\omega') := Y_{t+s}(\omega\otimes_t\omega'), \quad s\ge 0,$$

is also $\mathbb{F}$-progressively measurable. The above notations are naturally extended to $(\tau,\omega)-$shifting for any finite $\mathbb{F}$-stopping time τ.

Lemma 6.1. *The mapping $(\omega, t, \omega')\in\Omega\times\mathbb{R}_+\times\Omega \longmapsto \omega\otimes_t\omega'\in\Omega$ is continuous. In particular, if ξ is $\mathcal{F}_\infty$-measurable function, then $\xi^{\cdot,\cdot}(\cdot)$ is $\mathcal{F}_\infty\otimes\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty$-measurable.*

Proof. We directly estimate that

$$\|\omega\otimes_t\omega' - \overline{\omega}\otimes_{\overline{t}}\overline{\omega}'\|_\infty \le \|\omega-\overline{\omega}\|_\infty + \|\omega'-\overline{\omega}'\|_\infty + \sup_{s\le|t-\overline{t}|}\Big(\|\omega_{s+\cdot}-\omega\|_\infty + \|\omega'_{s+\cdot}-\omega'\|_\infty\Big).$$

□

For every probability measure $\mathbb{P}$ on Ω and $\mathbb{F}$-stopping time τ, there exists a family of regular conditional probability distribution (for short r.c.p.d.) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$, see Theorem 1.3.4 in [SV97].[6] The r.c.p.d. $\mathbb{P}^\tau_\omega$ induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ on $(\Omega,\mathcal{F})$ such that

$$\mathbb{P}^{\tau,\omega}(A) := \mathbb{P}^\tau_\omega(\omega\otimes_\tau A), \quad A\in\mathcal{F}, \quad \text{where } \omega\otimes_\tau A := \{\omega\otimes_\tau \omega' : \omega'\in A\}.$$

It is clear that $\mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]$, for every $\mathcal{F}$-measurable random variable ξ.

6.2 Backward SDEs on the shifted spaces

For all $\mathbb{P}\in\mathcal{P}(t,\omega)$, we introduce a family of random horizon BSDEs

$$\mathcal{Y}^{t,\omega,\mathbb{P}}_{s\wedge\theta} = \xi^{t,\omega} + \int_{s\wedge\theta}^{\theta} F^{t,\omega}_r(\mathcal{Y}^{t,\omega,\mathbb{P}}_r, \mathcal{Z}^{t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr - \mathcal{Z}^{t,\omega,\mathbb{P}}_r\cdot dX_r - d\mathcal{N}^{t,\omega,\mathbb{P}}_r,\ s\geq 0,\ \mathbb{P}\text{-a.s.} \quad (6.1)$$

By Theorem 3.3, this BSDE admits a unique solution. Define the value function

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau], \quad \text{with} \quad \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau] := \mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}^{t,\omega,\mathbb{P}}_0\right]. \quad (6.2)$$

In this section, we will prove the following measurability result, which is important for the discussion of the dynamic programming.

Proposition 6.2. *Under Assumptions 3.1, the mapping* $(t,\omega,\mathbb{P})\mapsto \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$ *is* $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\tau\otimes\mathcal{B}(\mathcal{M}_1)$*-measurable.*

We will first review in Section 6.2.1 the finite horizon argument of [PTZ17], and we next adapt it to our random horizon setting in Section 6.2.2.

6.2.1 Measurability - finite horizon

Let $\tau = T$, where T is a finite deterministic time. For the convenience of the reader we repeat the argument in [PTZ17] in order to prove the finite horizon version of Proposition 6.2. For each $\mathbb{P}\in\mathcal{P}_b$, we consider the following shifted BSDE

$$\mathcal{Y}^{t,\omega,\mathbb{P}}_s = \xi^{t,\omega} + \int_s^{T-t} F^{t,\omega}_r\big(\mathcal{Y}^{t,\omega,\mathbb{P}}_r, \mathcal{Z}^{t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r\big)dr - \mathcal{Z}^{t,\omega,\mathbb{P}}_r\cdot dX_r - d\mathcal{N}^{t,\omega,\mathbb{P}}_r, s\in[0,T-t],\ \mathbb{P}\text{-a.s.} \quad (6.3)$$

Lemma 6.3. *Let* $\tau = T$ *be a deterministic time. Then, there exists a version of* $\mathcal{Y}^{t,\omega,\mathbb{P}}$ *such that the mapping* $(t,\omega,s,\omega',\mathbb{P})\in[0,\infty)\times\Omega\times[0,\infty)\times\Omega\times\mathcal{P}_b\mapsto\mathcal{Y}^{t,\omega,\mathbb{P}}_s(\omega')\in\mathbb{R}$ *is* $\mathcal{B}\big([0,\infty)\big)\times\mathcal{F}_\infty\times\mathcal{B}\big([0,\infty)\big)\times\mathcal{F}_\infty\times\mathcal{B}(\mathcal{M}_1)$*-measurable.*

Proof. We shall exploit the construction of the solution of the BSDE (6.3) by the Picard iteration, thus proving that for each step of the iteration, the induced process $\mathcal{Y}^{n,t,\omega,\mathbb{P}}$ satisfies the required measurability.

[6] By definition, an r.c.p.d. satisfies:
- For every $\omega\in\Omega$, $\mathbb{P}^\tau_\omega$ is a probability measure on $(\Omega,\mathcal{F})$;
- For every $A\in\mathcal{F}$, the mapping $\omega\longmapsto\mathbb{P}^\tau_\omega(A)$ is $\mathcal{F}_\tau$-measurable;
- The family $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ is a version of $\mathbb{P}|_{\mathcal{F}_\tau}$, i.e. $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}^\tau_\omega}[\xi]$, $\mathbb{P}-$a.s. for all $\xi\in\mathbb{L}^1(\mathbb{P})$.
- For every $\omega\in\Omega$, $\mathbb{P}^\tau_\omega(\Omega^\omega_\tau) = 1$, where $\Omega^\omega_\tau := \{\overline{\omega}\in\Omega : \overline{\omega}_s = \omega_s,\ 0\leq s\leq\tau(\omega)\}$.

1. We start from the first step of the Picard iteration. Take the initial value $\mathcal{Y}^{0,t,\omega,\mathbb{P}} \equiv 0$ and $\mathcal{Z}^{0,t,\omega,\mathbb{P}} \equiv 0$. Define for all $t \le T$

$$\begin{aligned}\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s &:= \mathbb{E}^{\mathbb{P}}\Big[\xi^{t,\omega} + \int_s^{T-t} F^{t,\omega}_r(\mathcal{Y}^{0,t,\omega,\mathbb{P}}_r, \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr\Big|\mathcal{F}^+_s\Big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[\xi^{t,\omega} + \int_0^{T-t} F^{t,\omega}_r(\mathcal{Y}^{0,t,\omega,\mathbb{P}}_r, \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr\Big|\mathcal{F}^+_s\Big] - \int_0^s F^{t,\omega}_r(\mathcal{Y}^{0,t,\omega,\mathbb{P}}_r, \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr, \quad (6.4)\end{aligned}$$

for $s \in [0, T-t]$. We extend the definition so that $\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s := \xi^{t,\omega}$ on $\{s > T-t\} \cap \{t \le T\}$ and $\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s \equiv \xi(\omega_{T\wedge\cdot})$ for $t > T$. By Lemma 6.1, the mapping $\xi^{\cdot,\cdot}(\cdot) : \Omega \times [0,T] \times \Omega \longrightarrow \mathbb{R}$ is $\mathcal{F}_\infty \otimes \mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty$-measurable. Similarly, the mapping

$$(t, \omega, r, \omega', \mathbb{P}) \mapsto F^{t,\omega}_r\big(\omega', \mathcal{Y}^{0,t,\omega,\mathbb{P}}_r(\omega'), \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r(\omega'), \widehat{\sigma}_r(\omega')\big)$$

is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}(\mathcal{P}_b)$-measurable, and by the Fubini theorem,

$$(t, \omega, \omega', \mathbb{P}) \longmapsto \mathbf{1}_{\{t \le T\}} \int_0^{T-t} F^{t,\omega}_r\big(\omega', \mathcal{Y}^{0,t,\omega,\mathbb{P}}_r(\omega'), \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r(\omega'), \widehat{\sigma}_r(\omega')\big)dr$$

is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{F}_\infty \otimes \mathcal{B}(\mathcal{P}_b)$-measurable. It follows from Lemma 3.1 in [NN14] that there exists a version, still noted by $\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}$, such that the mapping $(t, \omega, \omega', \mathbb{P}) \mapsto \overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s(\omega')$ is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{F}^+_s \otimes \mathcal{B}(\mathcal{P}_b)$-measurable for each s.

2. The function $\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s$ we just constructed is not necessarily $\mathbb{P}$-a.s. càdlàg in s. We next construct a version $\mathcal{Y}^{1,t,\omega,\mathbb{P}}$ (i.e., $\mathcal{Y}^{1,t,\omega,\mathbb{P}}_s = \overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_s$, $\mathbb{P}$-a.s. for all s) which is measurable and $\mathbb{P}$-a.s. càdlàg in s. Let $t^n_i := i2^{-n}(T-t)$, and set for $s \ge 0$:

$$\mathcal{Y}^{1,t,\omega,\mathbb{P}}_s := \limsup_{m\to\infty} \mathcal{Y}^{1,m,t,\omega,\mathbb{P}}_s \quad \text{where} \quad \mathcal{Y}^{1,m,t,\omega,\mathbb{P}}_s := \sum_{i=1}^{2^m} \overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}_{t^m_i} \mathbf{1}_{[t^m_{i-1}, t^m_i)}(s) + \xi^{t,\omega}\mathbf{1}_{[T-t,\infty)}(s).$$

Clearly, $(t, \omega, s, \omega', \mathbb{P}) \mapsto \mathcal{Y}^{1,m,t,\omega,\mathbb{P}}_s(\omega')$ is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}(\mathcal{P}_b)$-measurable, and so is $(t, \omega, s, \omega', \mathbb{P}) \mapsto \mathcal{Y}^{1,t,\omega,\mathbb{P}}_s(\omega')$. Since the filtration $\mathbb{F}^{+,\mathbb{P}}$ satisfies the usual conditions and the conditional expectation in (6.4) is an $\mathbb{F}^{+,\mathbb{P}}$-martingale, one can prove by a standard argument (see e.g. [KS12, Proposition I 3.14]) that $\mathcal{Y}^{1,t,\omega,\mathbb{P}}$ is a $\mathbb{P}$-a.s. càdlàg version of $\overline{\mathcal{Y}}^{1,t,\omega,\mathbb{P}}$.

3. Recall the inverse of a nonnegative-definite matrix in Footnote 1. Define

$$\mathcal{Z}^{1,t,\omega,\mathbb{P}}_s := \widehat{a}^{-1}_s \limsup_{n\to\infty} n\left(\langle \mathcal{Y}^{1,t,\omega,\mathbb{P}}, X\rangle_s - \langle \mathcal{Y}^{1,t,\omega,\mathbb{P}}, X\rangle_{(s-1/n)\vee 0}\right), \quad (6.5)$$

where the lim sup is componentwise. Clearly, the mapping $(t, \omega, s, \omega', \mathbb{P}) \mapsto \mathcal{Z}^{1,t,\omega,\mathbb{P}}_s(\omega')$ is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}(\mathcal{P}_b)$-measurable. Since $\mathcal{Y}^{1,t,\omega,\mathbb{P}}$ is càdlàg, by the uniqueness of the martingale representation (see e.g. [JS03, Lemma III 4.24]), there exists an $\mathcal{F}^{+,\mathbb{P}}$-martingale $\mathcal{N}^{1,t,\omega,\mathbb{P}}$ orthogonal to X under $\mathbb{P}$, such that for $t \le T$ and $s \in [0, T-t]$,

$$\mathcal{Y}^{1,t,\omega,\mathbb{P}}_s = \xi^{t,\omega} + \int_s^{T-t} F^{t,\omega}_r(\mathcal{Y}^{0,t,\omega,\mathbb{P}}_r, \mathcal{Z}^{0,t,\omega,\mathbb{P}}_r, \widehat{\sigma}_r)dr - \mathcal{Z}^{1,t,\omega,\mathbb{P}}_r \cdot dX_r - d\mathcal{N}^{1,t,\omega,\mathbb{P}}_r, \ \mathbb{P}\text{-a.s.} \quad (6.6)$$

4. By replacing $\big(\mathcal{Y}^{0,t,\omega,\mathbb{P}}, \mathcal{Z}^{0,t,\omega,\mathbb{P}}\big)$ in Steps 1 - 3 by $\big(\mathcal{Y}^{n,t,\omega,\mathbb{P}}, \mathcal{Z}^{n,t,\omega,\mathbb{P}}\big)$, for an arbitrary $n \ge 1$, we may define $\big(\mathcal{Y}^{n+1,t,\omega,\mathbb{P}}, \mathcal{Z}^{n+1,t,\omega,\mathbb{P}}, \mathcal{N}^{n+1,t,\omega,\mathbb{P}}\big)$ such that the mappings

$$(t, \omega, s, \omega', \mathbb{P}) \mapsto \big(\mathcal{Y}^{n+1,t,\omega,\mathbb{P}}(\omega'), \mathcal{Z}^{n+1,t,\omega,\mathbb{P}}(\omega')\big)$$

are $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{P}_b)$-measurable. By the contracting feature of the Picard iteration, see e.g. El Karoui, Peng & Quenez [EPQ97], we have

$$\big\|\mathcal{Y}^{n,t,\omega,\mathbb{P}}-\mathcal{Y}^{t,\omega,\mathbb{P}}\big\|_{\mathbb{D}^2_{T-t,\alpha}(\mathbb{P})}\longrightarrow 0,\ \text{as } n\to\infty.$$

As before, we extend the definition so that $\mathcal{Y}^{t,\omega,\mathbb{P}}_s:=\xi^{t,\omega}$ on $\{s>T-t\}\cap\{t\le T\}$ and $\mathcal{Y}^{t,\omega,\mathbb{P}}_s\equiv\xi(\omega_{T\wedge\cdot})$ for $t>T$. Then it follows from [NN14, Lemma 3.2] that there exists an increasing sequence $\{n^{\mathbb{P}}_k\}_{k\in\mathbb{N}}\subseteq\mathbb{N}$ such that $\mathbb{P}\longmapsto n^{\mathbb{P}}_k$ is measurable for each k and

$$\lim_{k\to\infty}\sup_{0\le s\le T-t}\big|\mathcal{Y}^{n^{\mathbb{P}}_k,t,\omega,\mathbb{P}}_s-\mathcal{Y}^{t,\omega,\mathbb{P}}_s\big|\ =\ 0,\ \mathbb{P}-\text{a.s.}$$

Besides, there exist $\mathcal{Z}^{t,\omega,\mathbb{P}}\in\mathbb{H}^2_{T-t,\alpha}$ and $\mathcal{N}^{t,\omega,\mathbb{P}}\in\mathbb{N}^2_{T-t,\alpha}$ as limits of the Picard sequence under each $(t,\omega,\mathbb{P})\in[0,T]\times\Omega\times\mathcal{P}_b$. We conclude that $\big(\mathcal{Y}^{t,\omega,\mathbb{P}},\mathcal{Z}^{t,\omega,\mathbb{P}},\mathcal{N}^{t,\omega,\mathbb{P}}\big)$ is a solution to the BSDE (6.3), and that $(t,\omega,s,\omega',\mathbb{P})\mapsto\mathcal{Y}^{t,\omega,\mathbb{P}}_s(\omega')$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{P}_b)$-measurable. As $\mathcal{P}_b\in\mathcal{B}(\mathcal{M}_1)$, the assertion follows. □

Remark 6.4. In the finite horizon case, Proposition 6.2 is a direct corollary of Lemma 6.3.

6.2.2 Measurability - random horizon

Let us return to our construction of the solution of the random horizon BSDE by means of a sequence of finite horizon BSDEs on $[0,\tau_n]$, $n\ge1$, where $\tau_n:=n\wedge\tau$. For all $(t,\omega)\in[\![0,\tau]\!]$ and $\mathbb{P}\in\mathcal{P}_b$, consider the approximating sequence $\big(\mathcal{Y}^{n,t,\omega,\mathbb{P}},\mathcal{Z}^{n,t,\omega,\mathbb{P}},\mathcal{N}^{n,t,\omega,\mathbb{P}}\big)$ defined by:

$$\mathcal{Y}^{n,t,\omega,\mathbb{P}}_s=\xi^{n,t,\omega}+\int_s^{n-t}f^{t,\omega}_s\big(\mathcal{Y}^{n,t,\omega,\mathbb{P}}_s,\mathcal{Z}^{n,t,\omega,\mathbb{P}}_s\big)ds-\mathcal{Z}^{n,t,\omega,\mathbb{P}}_s dX_s-d\mathcal{N}^{n,t,\omega,\mathbb{P}}_s,\ s\le n-t,\quad(6.7)$$

$\mathbb{P}^{t,\omega}-$a.s., where $\bar\tau^{n,\omega\otimes_t\bar X}:=(\tau^{n,\omega\otimes_t\bar X}-n)^+$, recall the notation $\bar X$ from Section 2.1,

$$\xi^{n,t,\omega}:=\mathbb{E}^{\mathbb{P}^{n,\omega\otimes_t\bar X}}\Big[e^{-\mu\bar\tau^{n,\omega\otimes_t\bar X}}\xi^{n,\omega\otimes_t\overline{X}}\Big],\ \text{and}\ f^{t,\omega}_s(y,z):=F^{t,\omega}_s(y,z,\widehat\sigma^{t,\omega}_s)\mathbf{1}_{\{s\le(\tau^{t,\omega}-t)^+\}}$$

satisfies Assumption 3.1. Then $\big(\mathcal{Y}^{n,t,\omega,\mathbb{P}},\mathcal{Z}^{n,t,\omega,\mathbb{P}},\mathcal{N}^{n,t,\omega,\mathbb{P}}\big)$ is a well-defined in $\mathcal{D}^p_{\eta,\tau}(\mathbb{P})\times\mathcal{H}^p_{\eta,\tau}(\mathbb{P})\times\mathcal{N}^p_{\eta,\tau}(\mathbb{P})$ for all $p\in(1,q)$ and $\eta\in[-\mu,\rho)$.

Proof of Proposition 6.2. As $\big(\mathcal{Y}^{n,t,\omega,\mathbb{P}},\mathcal{Z}^{n,t,\omega,\mathbb{P}},\mathcal{N}^{n,t,\omega,\mathbb{P}}\big)$ is defined by the finite horizon BSDE, we may apply the results of previous subsection, thus obtaining a version of $\mathcal{Y}^{n,t,\omega,\mathbb{P}}$ such that $(t,\omega,s,\omega',\mathbb{P})\longmapsto\mathcal{Y}^{n,t,\omega,\mathbb{P}}_s(\omega')$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{P}_b)$-measurable. This in turn implies that the mapping $(t,\omega,\mathbb{P})\longmapsto\overline{\mathbb{Y}}^{n,t,\omega,\mathbb{P}}:=\mathbb{E}^{\mathbb{P}}\big[\mathcal{Y}^{n,t,\omega,\mathbb{P}}_0\big]$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{P}_b)$-measurable.

By Proposition 4.5 (with $S=-\infty$), it follows that $\lim_{n\to\infty}\overline{\mathbb{Y}}^{n,t,\omega,\mathbb{P}}=\mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$. Then, the mapping $(t,\omega,\mathbb{P})\mapsto\mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{P}_b)$-measurable. As $\mathcal{P}_b\in\mathcal{B}(\mathcal{M}_1)$, the mapping $(t,\omega,\mathbb{P})\mapsto\mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}_\infty\otimes\mathcal{B}(\mathcal{M}_1)$-measurable. □

6.3 Dynamic programming principle

The goal of this section is to prove that the dynamic value process V satisfies the dynamic programming principle. We first focus on the underlying BSDEs for which the dynamic programming principle reduces to the following tower property, where we denote by $\mathcal{Y}[\xi_0,\tau_0]$ the Y component of the solution of the BSDE with the terminal time τ_0 and value ξ_0.

Lemma 6.5. *Let Assumptions 3.1 and 3.2 hold true. Then, for all stopping time $\tau_0 \le \tau$, and $\mathbb{P} \in \mathcal{P}_b$:*

(i) $\mathbb{E}^{\mathbb{P}}\big[\mathcal{Y}^{\mathbb{P}}_{\tau_0}\big|\mathcal{F}_{\tau_0}\big](\omega) = \mathbb{Y}^{\tau_0,\omega,\mathbb{P}^{\tau_0,\omega}}[\xi,\tau]$, *for* $\mathbb{P}-$*a.e.* $\omega \in \Omega$.

(ii) $\mathcal{Y}^{\mathbb{P}}_{t\wedge\tau_0}[\xi,\tau] = \mathcal{Y}^{\mathbb{P}}_{t\wedge\tau_0}\big[\mathcal{Y}^{\mathbb{P}}_{\tau_0}[\xi,\tau],\tau_0\big] = \mathcal{Y}^{\mathbb{P}}_{t\wedge\tau_0}\big[\mathbb{E}^{\mathbb{P}}\big[\mathcal{Y}^{\mathbb{P}}_{\tau_0}[\xi,\tau]\big|\mathcal{F}_{\tau_0}\big],\tau_0\big]$, *for all* $t \ge 0$.

The proof is omitted as (i) is a direct consequence of the uniqueness of the solution to BSDE, and (ii) is similar to [PTZ17, Lemma 2.7]. In order to apply the classic measurable selection results, we need the following properties of the probability families $\{\mathcal{P}(t,\omega)\}_{(t,\omega)\in[\![0,\tau]\!]}$.

Lemma 6.6. *The graph $[\![\mathcal{P}]\!] := \{(t,\omega,\mathbb{P}) : \mathbb{P} \in \mathcal{P}(t,\omega)\}$, is Borel-measurable in $\mathbb{R}_+ \times \Omega \times \mathcal{M}_1$. Moreover for all $(t,\omega) \in [\![0,\tau]\!]$ and all stopping time τ_0 valued in $[t,\tau]$, denoting $\vec{\tau}^{t,\omega}_0 := \tau^{t,\omega}_0 - t$, we have:*

(i) $\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$, *and for all* $\mathbb{P} \in \mathcal{P}(t,\omega)$, *the r.c.p.d.* $\mathbb{P}^{\vec{\tau}^{t,\omega}_0,\omega'} \in \mathcal{P}(\tau_0,\omega\otimes_t\omega')$, *for* $\mathbb{P}$*-a.e.* $\omega' \in \Omega$.

(ii) *For any $\mathcal{F}_{\vec{\tau}^{t,\omega}_0}$-measurable kernel $\nu : \Omega \to \mathcal{M}_1$ with $\nu(\omega') \in \mathcal{P}(\tau_0,\omega\otimes_t\omega')$ for $\mathbb{P}$-a.e. $\omega' \in \Omega$, the map $\overline{\mathbb{P}} := \mathbb{P} \otimes_{\vec{\tau}^{t,\omega}_0} \nu$ defined by*

$$\overline{\mathbb{P}}(A) = \iint (\mathbf{1}_A)^{\vec{\tau}^{t,\omega}_0,\omega'}(\omega'')\nu(d\omega'';\omega')\mathbb{P}(d\omega'), \quad A \in \mathcal{F},$$

is a probability measure in $\mathcal{P}(t,\omega)$.

Proof. This follows from [NvH13, Theorem 4.3], see also Remark **??**. □

Theorem 6.7 (Dynamic programming for V)**.** *Let Assumption 3.1 hold true. The mapping $\omega \longmapsto V_{\tau_0}(\omega)$ is $\mathcal{F}^U_{\tau_0}$-measurable. Moreover, for $(t,\omega) \in [\![0,\tau]\!]$, and a stopping time τ_0 be an $\mathbb{F}$-stopping time with $t\wedge\tau \le \tau_0 \le \tau$, we have, denoting $\vec{\tau}^{t,\omega}_0 := \tau^{t,\omega}_0 - t$,*

$$\mathcal{E}^{\mathcal{P}(t,\omega)}\big[\big|e^{\eta\vec{\tau}^{t,\omega}_0}(V_{\tau_0})^{t,\omega}\big|^p\big] < \infty, \ \sup_{\tau_0\le\tau}\mathcal{E}^{\mathcal{P}_0}\big[\big|e^{\eta\tau_0}(V_{\tau_0})\big|^p\big] < \infty, \ \textit{for all } p \in (1,q), \ \eta \in [-\mu,\rho), \tag{6.8}$$

$$V_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{Y}^{t,\omega,\mathbb{P}}\big[V_{\tau_0},\tau_0\big], \tag{6.9}$$

$$\textit{and } V_t = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}(t)} \mathbb{E}^{\mathbb{P}'}\Big[\mathcal{Y}^{\mathbb{P}'}_t\big[V_{\tau_0},\tau_0\big]\Big|\mathcal{F}_t\Big], \ \mathbb{P}-\textit{a.s. for all } \mathbb{P} \in \mathcal{P}_0. \tag{6.10}$$

Proof. Without loss of generality, we assume in the proof that $(t,\omega) = (0,\mathbf{0})$.

1. It follows from Proposition 6.2 that $(t,\omega,\mathbb{P}) \mapsto \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$ is $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}_\infty \otimes \mathcal{B}(\mathcal{M}_1)$-measurable, and from Lemma 6.6 that $[\![\mathcal{P}]\!]$ is analytic. By [BS96, Theorem 7.47], we know that the mapping $(t,\omega) \mapsto V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{Y}^{t,\omega,\mathbb{P}}[\xi,\tau]$ is upper semi-analytic and thus universally measurable, i.e., $\mathcal{B}\big([0,\infty)\big) \otimes \mathcal{F}^U_\infty$-measurable. Finally, note that $V_t(\omega) = V_t(\omega_{t\wedge\cdot})$. So, it follows from Galmarino's test that V_{τ_0} is $\mathcal{F}^U_{\tau_0\wedge\tau}$-measurable.

2. We next pove (6.8). By the measurable selection theorem (see, e.g., [BS96, Proposition 7.50]), for each $\varepsilon > 0$, there exists an $\mathcal{F}^U_{\tau_0}$-measurable kernel $\nu^\varepsilon : \omega \mapsto \nu^\varepsilon(\omega) \in \mathcal{P}\big(\tau_0(\omega),\omega\big)$, such that for all $\omega \in \Omega$

$$e^{\eta\tau_0(\omega)}V_{\tau_0}(\omega) \ \le \ e^{\eta\tau_0(\omega)}\mathbb{Y}^{\tau_0,\omega,\nu^\varepsilon(\omega)}[\xi,\tau] + \varepsilon. \tag{6.11}$$

By Lemma 6.5, we have $\mathbb{Y}^{\tau_0,\omega,\nu^\varepsilon(\omega)}[\xi,\tau] = \mathbb{E}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big[\mathcal{Y}_{\tau_0}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big|\mathcal{F}_{\tau_0}\big](\omega)$, $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{P}_0$. Therefore, for $\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\big[\big|e^{\eta\tau_0}V_{\tau_0}\big|^p\big] &\leq \mathbb{E}^{\mathbb{Q}}\Big[\big|\mathbb{E}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big[e^{\eta\tau_0}\mathcal{Y}_{\tau_0}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big|\mathcal{F}_{\tau_0}\big]+\varepsilon\big|^p\Big]\\
&= C_p\Big(\mathbb{E}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big[\mathrm{D}_{\tau_0}^{\mathbb{Q}|\mathbb{P}}\big|e^{\eta\tau_0}\mathcal{Y}_{\tau_0}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big|^p\big]+\varepsilon^p\Big)\\
&\leq C_p\Big(\sup_{\mathbb{Q}'\in\mathcal{Q}_L(\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon)}\mathbb{E}^{\mathbb{Q}'}\Big[\big|e^{\eta\tau_0}\mathcal{Y}_{\tau_0}^{\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon}\big|^p\Big]+\varepsilon^p\Big).
\end{aligned}$$

Then, by the estimate (3.2), we obtain

$$\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})}\mathbb{E}^{\mathbb{Q}}\big[\big|e^{\eta\tau_0}V_{\tau_0}\big|^p\big] \leq C_{p,q}\Big(\|\xi\|^p_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}_0)}+\big(\overline{F}^0_{\rho,q,\tau}\big)^p\Big)+C_p\varepsilon^p,$$

which induce the required estimate by sending $\varepsilon\to 0$.

3. To prove (6.9), we start by observing that, by the tower property in Lemma 6.5, we have

$$V_0=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\big[\mathcal{Y}_0^{\mathbb{P}}[\xi,\tau]\big]=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\mathbb{P}}\big[\mathcal{Y}_{\tau_0}^{\mathbb{P}}[\xi,\tau],\tau_0\big]\Big]=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\big[\mathcal{Y}_{\tau_0}^{\mathbb{P}}[\xi,\tau]\big|\mathcal{F}_{\tau_0}\big],\tau_0\Big]\Big].$$

Note that, for all $\mathbb{P}\in\mathcal{P}_0$, we have by Lemma 6.5 that for $\mathbb{P}-$a.e. ω,

$$V_{\tau_0}(\omega)=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{Y}^{\tau_0,\omega,\mathbb{P}}[\xi,\tau]=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\tau_0,\omega,\mathbb{P}}[\xi,\tau]\Big]\geq\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_{\tau_0}^{\mathbb{P}}[\xi,\tau]\Big|\mathcal{F}_{\tau_0}\Big](\omega).$$

By the comparison result of Theorem 3.5 (ii) for the BSDE (6.3), we deduce that

$$V_0 \leq \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\mathbb{P}}\big[V_{\tau_0},\tau_0\big]\Big] = \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{Y}^{0,\mathbf{0},\mathbb{P}}\left[V_{\tau_0},\tau_0\right]. \tag{6.12}$$

To prove the reverse inequality, we use again the measurable selection theorem to deduce the existence of an $\mathcal{F}^U_{\tau_0}$-measurable kernel $\nu^\varepsilon:\omega\mapsto\nu^\varepsilon(\omega)\in\mathcal{P}(\tau_0(\omega),\omega)$ such that (6.11) holds true for $\eta\in[-\mu,\rho)$. Define the concatenated probability $\overline{\mathbb{P}}:=\mathbb{P}\otimes_{\tau_0}\nu^\varepsilon$ and note that $\overline{\mathbb{P}}|_{\mathcal{F}_{\tau_0}}=\mathbb{P}|_{\mathcal{F}_{\tau_0}}$. Then, by the stability result of Theorem 3.5 (i) and Lemma 6.5, we have

$$V_0\geq\mathbb{E}^{\overline{\mathbb{P}}}\big[\mathcal{Y}_0^{\overline{\mathbb{P}}}[\xi,\tau]\big]=\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\mathbb{P}}\Big[\mathbb{E}^{\overline{\mathbb{P}}^{\tau_0,\cdot}}\big[\mathcal{Y}_0^{\tau_0,\cdot,\overline{\mathbb{P}}^{\tau_0,\cdot}}[\xi,\tau]\big],\tau_0\Big]\Big]=\mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_0^{\mathbb{P}}\Big[\mathbb{E}^{\nu^\varepsilon(\cdot)}\big[\mathcal{Y}_0^{\tau_0,\cdot,\nu^\varepsilon(\cdot)}[\xi,\tau]\big],\tau_0\Big]\Big].$$

By (6.11), the right hand side is larger than $\mathbb{E}^{\mathbb{P}}\left[\mathcal{Y}_0^{\mathbb{P}}\left[V_{\tau_0},\tau_0\right]\right]-C\varepsilon$ for some $C>0$ independent of ε. Therefore, $V_0\geq\mathbb{Y}^{0,\mathbf{0},\mathbb{P}}\left[V_{\tau_0},\tau_0\right]-C\varepsilon$, and we obtain by sending $\varepsilon\to 0$ that

$$V_0 \geq \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{Y}^{0,\mathbf{0},\mathbb{P}}\left[V_{\tau_0},\tau_0\right].$$

4. We finally prove (6.10). Due to the previous step, we know

$$V_t(\omega) \geq \mathbb{Y}^{t,\omega,\mathbb{P}'}\big[V_{\tau_0},\tau_0\big],\quad\text{for all}\quad\mathbb{P}'\in\mathcal{P}(t,\omega).$$

Now fix a probability measure $\mathbb{P}\in\mathcal{P}_0$. It follows from Lemma 6.6 (i) that for all $\widetilde{\mathbb{P}}\in\mathcal{P}_{\mathbb{P}}(t)\subseteq\mathcal{P}_0$ we have $\widetilde{\mathbb{P}}^{t,\omega}\in\mathcal{P}(t,\omega)$. So $V_t(\omega)\geq\mathbb{Y}^{t,\omega,\widetilde{\mathbb{P}}^{t,\omega}}\big[V_{\tau_0},\tau_0\big]$. By Lemma 6.5, this provides

$$V_t \geq \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathcal{Y}_t^{\widetilde{\mathbb{P}}}[V_{\tau_0},\tau_0]\Big|\mathcal{F}_t\Big],\quad\mathbb{P}-\text{a.s., and therefore}\quad V_t \geq \operatorname*{ess\,sup}^{\mathbb{P}}_{\widetilde{\mathbb{P}}\in\mathcal{P}_{\mathbb{P}}(t)}\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathcal{Y}_t^{\widetilde{\mathbb{P}}}[V_{\tau_0},\tau_0]\Big|\mathcal{F}_t\Big],\ \mathbb{P}\text{-a.s.}$$

To prove the reverse inequality, we apply the measurable selection theorem on the optimization problem (6.9), to find an $\mathcal{F}_t^U$-measurable kernel $\nu^\varepsilon : \omega \mapsto \nu^\varepsilon(\omega) \in \mathcal{P}(t,\omega)$ such that $V_t(\omega) \leq \mathbb{Y}^{t,\omega,\nu(\omega)}[V_{\tau_0},\tau_0] + \varepsilon$. By Lemma 6.6, $\mathbb{P}^\varepsilon := \mathbb{P} \otimes_t \nu^\varepsilon \in \mathcal{P}_0$, and thus $\mathbb{P}^\varepsilon \in \mathcal{P}_{\mathbb{P}}(t)$. Together with Lemma 6.5, this provides

$$V_t \;\leq\; \mathbb{E}^{\mathbb{P}^\varepsilon}\Big[\mathcal{Y}_t^{\mathbb{P}^\varepsilon}[V_{\tau_0},\tau_0]\Big|\mathcal{F}_t\Big] + \varepsilon \;\leq\; \operatorname*{ess\,sup}^{\mathbb{P}}_{\widetilde{\mathbb{P}}\in\mathcal{P}_{\mathbb{P}}(t)} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathcal{Y}_t^{\widetilde{\mathbb{P}}}[V_{\tau_0},\tau_0]\Big|\mathcal{F}_t\Big] + \varepsilon.$$

The required inequality now follows by sending $\varepsilon \to 0$. □

6.4 A càdlàg version of the value function

By [PTZ17, Lemma 3.2], the right limit

$$V_t^+(\omega) \;:=\; \lim_{r\in\mathbf{Q},r\downarrow t} V_r(\omega)$$

exists $\mathcal{P}_0$-q.s. and the process V^+ is càdlàg $\mathcal{P}_0$-q.s. with $V_{\tau_0}^+ \in \mathbb{L}^p_{\eta,\tau}(\mathbb{Q})$ for all $\mathbb{Q} \in \cup_{\mathbb{P}\in\mathcal{P}_0}\mathcal{Q}_L(\mathbb{P})$, $\eta \in [-\mu,\rho)$, $p \in (1,q)$, and all stopping times $\tau_0 \leq \tau$.

Proposition 6.8 (Dynamic programming for V^+)**.** *Under Assumption 3.1, $V^+ \in \mathcal{D}^p_{\eta,\tau}(\mathcal{P}_0)$ for any $\eta \in [-\mu,\rho)$, $p \in (1,q)$, and for all $\mathbb{F}^+$-stopping times $0 \leq \tau_0 \leq \tau_1 \leq \tau$, and $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_{\tau_0}^+ \;=\; \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathcal{Y}_{\tau_0}^{\mathbb{P}'}\big[V_{\tau_1}^+,\tau_1\big], \qquad \mathbb{P}-\text{a.s.}$$

Proof. **1.** For an $\mathbb{F}^+-$stopping time $\bar\tau \leq \tau$, we introduce the approximating sequence of stopping times $\bar\tau^n := \frac{\lfloor 2^n\bar\tau\rfloor+1}{2^n}$, and we now verify that

$$V_{\bar\tau}^+ \in \mathcal{L}^p_{\eta,\bar\tau}(\mathcal{P}_0) \text{ and } \lim_{n\to\infty} \mathcal{E}^{\mathbb{P}}\big[\big|e^{\eta\bar\tau}V_{\bar\tau}^+ - e^{\eta\bar\tau^n}V_{\bar\tau^n}\big|^p\big] \;=\; 0, \quad \text{for all} \quad \mathbb{P}\in\mathcal{P}_0.$$

Indeed, for all $\mathbb{P} \in \mathcal{P}_0$, and $\mathbb{Q} \in \mathcal{Q}_L(\mathbb{P})$:

$$\mathbb{E}^{\mathbb{Q}}\big[\big(e^{\eta\bar\tau}V_{\bar\tau}^+\big)^p\big] \;=\; \lim_{n\to\infty}\mathbb{E}^{\mathbb{Q}}\big[\big|e^{\eta\bar\tau^n}V_{\bar\tau^n}\big|^p\big] \;\leq\; \sup_{\tau'\leq\tau}\mathcal{E}^{\mathcal{P}_0}\big[\big|e^{\eta\tau'}V_{\tau'}\big|^p\big] \;=:\; v_p \;<\; \infty,$$

by (6.8) in Theorem 6.7, implying that $\mathcal{E}^{\mathcal{P}_0}\big[\big|e^{\eta\bar\tau}V_{\bar\tau}^+\big|^p\big] \leq v_p$. Then $\delta_n := \big|e^{\eta\bar\tau}V_{\bar\tau}^+ - e^{\eta\bar\tau^n}V_{\bar\tau^n}\big|$ satisfies for an arbitrary $m \geq 1$:

$$\mathcal{E}^{\mathbb{P}}\big[\delta_n^p\big] \leq \mathcal{E}^{\mathbb{P}}\big[\delta_n^p\mathbf{1}_{\{\tau\geq m\}}\big] + \mathcal{E}^{\mathbb{P}}\big[\delta_n^p\mathbf{1}_{\{\tau<m\}}\big] \leq 2v_{p'}^{\frac{p}{p'}}\mathcal{E}^{\mathbb{P}}\big[\mathbf{1}_{\{\tau\geq m\}}\big]^{1-\frac{p}{p'}} + C_m\big(\mathbb{E}^{\mathbb{P}}\big[\delta_n^{p'}\big]\big)^{\frac{p}{p'}},$$

which implies the required convergence.

2. We now prove that $V_{\tau_0}^+ \geq \mathcal{Y}_{\tau_0}^{\mathbb{P}}\big[V_{\tau_1}^+,\tau_1\big]$, $\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_0$, where the right hand is well defined by the integrability of V^+ obtained in the previous step. Recall from Theorem 6.7 that

$$V_{\tau_0^m} \;\geq\; \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}_{\tau_0^m}^{\mathbb{P}}\big[V_{\tau_1^n},\tau_1^n\big]\Big|\mathcal{F}_{\tau_0^m}\Big], \quad \mathbb{P}\text{-a.s.},$$

where τ_0^m and τ_1^n are defined from τ_0 and τ_1 as in the previous step. By the stability result of BSDEs in Proposition 4.6, and the result of Step 1 of the present proof, we have

$$\lim_{n\to\infty}\Big\|\mathcal{Y}_{\tau_0^m}^{\mathbb{P}}\big[V_{\tau_1^n},\tau_1^n\big] - \mathcal{Y}_{\tau_0^m}^{\mathbb{P}}\big[V_{\tau_1}^+,\tau_1\big]\Big\|_{\mathbb{L}^p_{\eta,\tau_0^m}(\mathbb{P})} \leq \lim_{n\to\infty}\Big\|\mathcal{Y}_{\tau_0^m}^{\mathbb{P}}\big[V_{\tau_1^n},\tau_1^n\big] - \mathcal{Y}_{\tau_0^m}^{\mathbb{P}}\big[V_{\tau_1}^+,\tau_1\big]\Big\|_{\mathcal{L}^p_{\eta,\tau_0^m}(\mathbb{P})} = 0.$$

Then, $V_{\tau_0^m} \geq \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}^{\mathbb{P}}_{\tau_0^m}[V_{\tau_1^n}, \tau_1^n]\Big|\mathcal{F}_{\tau_0^m}\Big] = \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}^{\mathbb{P}}_{\tau_0^m}[V^+_{\tau_1}, \tau_1]\Big|\mathcal{F}_{\tau_0^m}\Big]$, $\mathbb{P}-$a.s., and therefore

$$V^+_{\tau_0} \;=\; \lim_{m\to\infty} V_{\tau_0^m} \;\geq\; \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}^{\mathbb{P}}_{\tau_0^m}[V^+_{\tau_1}, \tau_1]\Big|\mathcal{F}_{\tau_0^m}\Big] \;=\; \mathbb{E}^{\mathbb{P}}\Big[\mathcal{Y}^{\mathbb{P}}_{\tau_0}[V^+_{\tau_1}, \tau_1]\Big|\mathcal{F}_{\tau_0}\Big],$$

where the last equality is due to $\mathcal{Y}^{\mathbb{P}}\big[V^+_{\tau_1}, \tau_1\big] \in \mathcal{D}^p_{\eta,\tau_1}(\mathbb{P})$.

3. We next prove the reverse inequality. By the comparison result together with the last step of the present proof, we have

$$\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathcal{Y}^{\mathbb{P}'}_{\tau_0}\big[V^+_{\tau_1}, \tau_1\big] \;\geq\; \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathcal{Y}^{\mathbb{P}'}_{\tau_0}\big[\mathcal{Y}^{\mathbb{P}'}_{\tau_1}[\xi,\tau], \tau_1\big] \;=\; \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathcal{Y}^{\mathbb{P}'}_{\tau_0}[\xi,\tau]. \tag{6.13}$$

So it remains to prove that

$$V^+_{\tau_0} \;\leq\; \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathcal{Y}^{\mathbb{P}'}_{\tau_0}[\xi,\tau]. \tag{6.14}$$

In the remainder of Step 3, we omit the parameter $[\xi,\tau]$ without causing confusion. For any $\eta \in [-\mu, \rho)$, we obtain by the dominated convergence theorem together with the estimate (6.8) of Theorem 6.7 that

$$\begin{aligned}
e^{\eta\tau_0} V^+_{\tau_0} \;=\; \lim_{n\to\infty} \mathbb{E}\big[e^{\eta\tau_0^n} V_{\tau_0^n}\big|\mathcal{F}^+_{\tau_0}\big] \;&=\; \lim_{n\to\infty} \mathbb{E}\Big[e^{\eta\tau_0^n} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_{\mathbb{P}}(\tau_0^n)} \mathbb{E}^{\mathbb{P}'}\big[\mathcal{Y}^{\mathbb{P}'}_{\tau_0^n}\big|\mathcal{F}_{\tau_0^n}\big]\Big|\mathcal{F}^+_{\tau_0}\Big] \\
&\leq\; \lim_{n\to\infty} \mathbb{E}\Big[e^{\eta\tau_0^n} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathbb{E}^{\mathbb{P}'}\big[\mathcal{Y}^{\mathbb{P}'}_{\tau_0^n}\big|\mathcal{F}_{\tau_0^n}\big]\Big|\mathcal{F}^+_{\tau_0}\Big] \\
&=\; \lim_{n\to\infty} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \mathbb{E}^{\mathbb{P}'}\Big[e^{\eta\tau_0^n} \mathcal{Y}^{\mathbb{P}'}_{\tau_0^n}\Big|\mathcal{F}^+_{\tau_0}\Big] \\
&=\; \lim_{n\to\infty} \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(\tau_0)} \Big\{e^{\eta\tau_0}\mathcal{Y}^{\mathbb{P}'}_{\tau_0} + \mathbb{E}^{\mathbb{P}'}\Big[\int_{\tau_0}^{\tau_0^n} e^{\eta s}\big(f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s) + \eta\mathcal{Y}^{\mathbb{P}'}_s\big)ds\Big|\mathcal{F}^+_{\tau_0}\Big]\Big\}.
\end{aligned} \tag{6.15}$$

By the Lipschitz property of F in Assumption 3.1, we estimate that

$$\mathbb{E}^{\mathbb{P}'}\Big[\int_{\tau_0}^{\tau_0^n} e^{\eta s}\big(f_s(\mathcal{Y}^{\mathbb{P}'}_s, \mathcal{Z}^{\mathbb{P}'}_s) + \eta\mathcal{Y}^{\mathbb{P}'}_s\big)ds\Big|\mathcal{F}^+_{\tau_0}\Big] \;\leq\; C2^{-n}\Big(\|\xi\|_{\mathcal{L}^q_{\rho,\tau}(\mathcal{P}_0)} + \big(\overline{F}^0_{\rho,q,\tau}\big)\Big),$$

which provides (6.14) in view of (6.15).

4. It remains to prove that V^+ inherits the integrability property of V. By Proposition 6.8,

$$e^{p\eta t}\big|V^+_t\big|^p \;=\; e^{p\eta t}\Big|\operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t)} \mathcal{Y}^{\mathbb{P}'}_t[\xi,\tau]\Big|^p = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t)} e^{p\eta t}\big|\mathcal{Y}^{\mathbb{P}'}_t[\xi,\tau]\big|^p.$$

As in the proof of Theorem 3.3, we may find for each $\mathbb{P}'$ a measure $\mathbb{Q}^{\mathbb{P}'} \in \mathcal{Q}_L$, such that $e^{\eta t}\big|\mathcal{Y}^{\mathbb{P}'}_t[\xi,\tau]\big| \leq \mathbb{E}^{\mathbb{Q}^{\mathbb{P}'}}\big[e^{\eta\tau}|\xi| + \int_0^\tau e^{\eta s}\big|f^0_s\big|ds\big|\mathcal{F}^+_t\big]$. Then,

$$e^{p\eta t}\big|V^+_t\big|^p \;\leq\; C_p \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(t)} \operatorname*{ess\,sup}^{\mathbb{P}'}_{\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P}')} \mathbb{E}^{\mathbb{Q}}\Big[e^{p\eta\tau}|\xi|^p + \Big(\int_0^\tau e^{\eta s}|f^0_s|ds\Big)^p\Big|\mathcal{F}^+_t\Big].$$

and therefore,

$$\mathcal{E}^{\mathcal{P}_0}\Big[\sup_{0\leq t\leq\tau} e^{p\eta t}\big|V^+_t\big|^p\Big] \leq C_p\mathcal{E}^{\mathcal{P}_0}\Big[\sup_{0\leq t\leq\tau} \mathcal{E}^{\mathbb{P},+}_t\Big[e^{p\eta\tau}|\xi|^p + \Big(\int_0^\tau e^{\eta s}\big|f^0_s\big|ds\Big)^p\Big]\Big] < \infty,$$

which induces the required result by Remark 5.1. □

6.5 Proof of Theorem 3.12: existence

Proof. **1.** We first prove the existence of a process Z and a family $(U^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ such that for all $p\in(1,q)$ and $\eta\in[-\mu,\rho)$,

$$\begin{aligned}(Z,U^{\mathbb{P}})\in\mathcal{H}^p_{\eta,\tau}\big(\mathbb{P},\mathbb{F}^{+,\mathbb{P}}\big)\times\mathcal{U}^p_{\eta,\tau}\big(\mathbb{P},\mathbb{F}^{+,\mathbb{P}}\big),\ U^{\mathbb{P}}\text{ is càdlàg }\mathbb{P}\text{-supermartingale},\ [U^{\mathbb{P}},X]=0,\\ \text{and } V^+_{t\wedge\tau}=\xi+\int_{t\wedge\tau}^{\tau}F_s\big(V^+_s,Z^{\mathbb{P}}_s,\widehat{\sigma}_s\big)ds-\int_{t\wedge\tau}^{\tau}\big(Z^{\mathbb{P}}_s\cdot dX_s+dU^{\mathbb{P}}_s\big),\ t\geq 0,\ \mathbb{P}\text{-a.s.},\end{aligned}\tag{6.16}$$

Fix $\mathbb{P}\in\mathcal{P}_0$. As for any $p<p'<q$, $V^+\in\mathcal{D}^{p'}_{\eta,\tau}(\mathcal{P}_0)$, by Proposition 6.8, it follows from Theorem 3.7 that there exists an unique solution $(Y^{\mathbb{P}},Z^{\mathbb{P}},U^{\mathbb{P}})\in\mathcal{D}^p_{\eta,\tau}(\mathbb{P})\times\mathcal{H}^p_{\eta,\tau}(\mathbb{P})\times\mathcal{U}^p_{\eta,\tau}(\mathbb{P})$ to the RBSDE:

$$\begin{aligned}Y^{\mathbb{P}}_{\cdot\wedge\tau}\ &=\ \xi+\int_{\cdot\wedge\tau}^{\tau}f_s(Y^{\mathbb{P}}_s,Z^{\mathbb{P}}_s)ds-(Z^{\mathbb{P}}_s\cdot dX_s+dU^{\mathbb{P}}_s),\ Y^{\mathbb{P}}\geq V^+,\ \mathbb{P}-\text{a.s.}\\ &\quad\text{and }\mathbb{E}^{\mathbb{P}}\Big[\int_0^{t\wedge\tau}\big(1\wedge(Y^{\mathbb{P}}_{r-}-V^+_{r-})\big)dU_r\Big]=0,\ \text{for all } t\geq 0.\end{aligned}\tag{6.17}$$

Following the same argument as in [STZ12], see also [PTZ17, Lemma 3.6], we now verify that $Y^{\mathbb{P}}=V^+$, $\mathbb{P}$-a.s. Indeed, assume to the contrary that $2\varepsilon:=Y^{\mathbb{P}}_0-V^+_0>0$ (without loss of generality), so that $\tau_\varepsilon:=\inf\big\{t>0:e^{\eta_t}Y^{\mathbb{P}}_t\leq e^{\eta_t}V^+_t+\varepsilon\big\}>0$, $\mathbb{P}-$a.s. Notice that $\tau_\varepsilon\leq\tau$, as the two processes are equal to ξ at time τ. From the Skorokhod condition, it follows that $U^{\mathbb{P}}$ is a martingale on $[0,\tau_\varepsilon]$, thus reducing the RBSDE to a BSDE on this time interval. Denoting as usual by $\mathcal{Y}^{\mathbb{P}}\big[V^+_{\tau_\varepsilon},\tau_\varepsilon\big]$, we obtain by standard BSDE techniques that, for some probability measure $\mathbb{Q}\in\mathcal{Q}_L(\mathbb{P})$,

$$Y^{\mathbb{P}}_0\ \leq\ \mathcal{Y}^{\mathbb{P}}_0\big[V^+_{\tau_\varepsilon},\tau_\varepsilon\big]+\mathbb{E}^{\mathbb{Q}}\big[e^{\eta\tau_\varepsilon}\big(Y^{\mathbb{P}}_{\tau_\varepsilon}-V^+_{\tau_\varepsilon}\big)\big]\ \leq\ \mathcal{Y}^{\mathbb{P}}_0\big[V^+_{\tau_\varepsilon},\tau_\varepsilon\big]+\varepsilon\ \leq\ V^+_0+\varepsilon,$$

where the last inequality follows from the crucial dynamic programming principle of Proposition 6.8. By the definition of ε, the last inequality cannot happen.

Consequently $Y^{\mathbb{P}}=V^+$. In particular, V^+ is a càdlàg semimartingale which would satisfy (6.16) once we prove that the family $\{Z^{\mathbb{P}}\}_{\mathbb{P}\in\mathcal{P}_0}$ may be aggregated. By Karandikar [Kar95], the quadratic covariation process $\langle V^+,X\rangle$ may be defined on $\mathbb{R}_+\times\Omega$. Moreover, $\langle V^+,X\rangle$ is $\mathcal{P}_0$-q.s. continuous and hence is $\mathbb{F}^{+,\mathcal{P}_0}$-predictable, or equivalently $\mathbb{F}^{\mathcal{P}_0}$-predictable. Similar to the proof of [Nut15, Theorem 2.4], we can define a universal $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z by $Z_tdt:=\widehat{a}_t^{-1}d\langle V^+,X\rangle_t$, and by comparing to the corresponding covariation under each $\mathbb{P}\in\mathcal{P}_0$, we see that $Z=Z^{\mathbb{P}}$, $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{P}_0$. This completes the proof of (6.16).

2. It remains to prove that the family of supermartingales $\big\{U^{\mathbb{P}}\big\}_{\mathbb{P}\in\mathcal{P}_0}$ satisfies the minimality condition. Let $0\leq s\leq t$, $\mathbb{P}\in\mathcal{P}_0$, $\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(s\wedge\tau)$, and denote by $\big(\mathcal{Y}^{\mathbb{P}'},\mathcal{Z}^{\mathbb{P}'},\mathcal{N}^{\mathbb{P}'}\big)$ the solution of the BSDE with parameters (F,ξ). Define $\delta Y:=V^+-\mathcal{Y}^{\mathbb{P}'}$, $\delta Z:=Z-\mathcal{Z}^{\mathbb{P}'}$ and $\delta U:=U^{\mathbb{P}'}-\mathcal{N}^{\mathbb{P}'}$. By Itô's formula, we have for $\alpha\in[-\mu,\rho)$,

$$\begin{aligned}e^{\alpha(s\wedge\tau)}\delta Y_{s\wedge\tau}\ &=\ \int_{s\wedge\tau}^{\tau}e^{\alpha(r\wedge\tau)}\big(F_r\big(V^+_r,Z_r,\widehat{\sigma}_r\big)-F_r\big(\mathcal{Y}^{\mathbb{P}'}_r,\mathcal{Z}^{\mathbb{P}'}_r,\widehat{\sigma}_r\big)-\alpha\delta Y_r\big)dr-e^{\alpha(s\wedge\tau)}\big(\delta Z_r\cdot dX_r+\delta U_r\big)\\ &=\ \int_{s\wedge\tau}^{\tau}\big(a^{\mathbb{P}'}_r\delta Y_r+b^{\mathbb{P}'}_r\cdot\widehat{\sigma}^{\mathbf{T}}_r\delta Z_r\big)dr-\big(\widehat{\sigma}^{\mathbf{T}}_r\delta Z_r\cdot dW_r+d\delta U_r\big),\end{aligned}$$

for some bounded processes $a^{\mathbb{P}'}$ and $b^{\mathbb{P}'}$, by Assumption 3.1. This provides that

$$\Gamma^{\mathbb{P}'}_{t\wedge\tau}e^{\alpha(t\wedge\tau)}\delta Y_{t\wedge\tau}-\Gamma^{\mathbb{P}'}_{s\wedge\tau}e^{\alpha(s\wedge\tau)}\delta Y_{s\wedge\tau}\ =\ \int_{s\wedge\tau}^{t\wedge\tau}\Gamma^{\mathbb{P}'}_re^{\alpha r}\big\{\big(\delta Y_rb^{\mathbb{P}'}_r+\widehat{\sigma}^{\mathbf{T}}_r\delta Z_r\big)\cdot dW_r\big)+d\delta U_r\big\}.$$

where $\Gamma_r^{\mathbb{P}'} := e^{\int_{s\wedge\tau}^r (a_u^{\mathbb{P}'} - \frac{1}{2}|b_u^{\mathbb{P}'}|^2)du + \int_{s\wedge\tau}^r b_u^{\mathbb{P}'}\cdot dW_u}$. Recall that $\delta Y \geq 0$, and $U^{\mathbb{P}'}$ is a $\mathbb{P}'-$supermartingale with decomposition $U^{\mathbb{P}'} = N^{\mathbb{P}'} - K^{\mathbb{P}'}$, for some $\mathbb{P}'-$martingale $N^{\mathbb{P}'}$ and nondecreasing process $K^{\mathbb{P}'}$. Then, taking conditional expectation $\mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}[.] := \mathbb{E}^{\mathbb{P}'}\big[.\big|\mathcal{F}^+_{s\wedge\tau}\big]$, we obtain

$$e^{|\alpha|t}\delta Y_{s\wedge\tau} \;\geq\; \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\int_{s\wedge\tau}^{t\wedge\tau} \Gamma_r^{\mathbb{P}'}\, dK_r^{\mathbb{P}'}\Big] \;\geq\; \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\gamma^{\mathbb{P}'}_{s,t}\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}'}\Big], \quad \text{with} \quad \gamma^{\mathbb{P}'}_{s,t} := \inf_{s\wedge\tau\leq r\leq t\wedge\tau} \Gamma_r^{\mathbb{P}'},$$

and we then obtain by the Cauchy-Schwarz and the Hölder inequality:

$$\begin{aligned} 0 \leq \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\int_{s\wedge\tau}^{t\wedge\tau} -d\delta U_r\Big] = \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}'}\Big] &\leq \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\gamma^{\mathbb{P}'}_{s,t}\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}'}\Big]^{\frac{1}{2}} \big(C^{\mathbb{P},p}_{s,t}\big)^{\frac{1}{2p}} \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\big(\gamma^{\mathbb{P}'}_{s,t}\big)^{-\widetilde{p}}\Big]^{\frac{1}{2\widetilde{p}}} \\ &\leq Ce^{\frac{1}{2}|\alpha|t}\big(C^{\mathbb{P},p}_{s,t}\big)^{\frac{1}{2p}}\,\big(\delta Y_{s\wedge\tau}\big)^{\frac{1}{2}}, \end{aligned}$$

where $p \in (1,q)$, $p^{-1} + \widetilde{p}^{-1} = 1$, and

$$C^{\mathbb{P},p}_{s,t} := \operatorname*{ess\,sup}^{\mathbb{P}'}_{\mathbb{P}'\in\mathcal{P}^+_{\mathbb{P}}(s\wedge\tau)} \mathbb{E}^{\mathbb{P}'}_{s\wedge\tau}\Big[\Big(\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}'}\Big)^p\Big]. \tag{6.18}$$

Now, the minimality condition in Definition 3.9 follows immediately from Proposition 6.8, provided that $C^{\mathbb{P},p}_{s,t} < \infty$, $\mathbb{P}-$a.s. which we now prove.

The family $\big\{\mathbb{E}^{\mathbb{P}'}\big[\big|\int_{s\wedge\tau}^{t\wedge\tau} dK_s^{\mathbb{P}'}\big|^p\big|\mathcal{F}^+_{s\wedge\tau}\big],\, \mathbb{P}' \in \mathcal{P}^+_{\mathbb{P}}(t\wedge\tau)\big\}$ is directed upward.[7] Then, it follows from [Nev75, Proposition V-1-1] that the ess sup in (6.18) is attained as an increasing limit along some sequence $\{\mathbb{P}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}^+_{\mathbb{P}}(s\wedge\tau)$. By the monotone convergence theorem, we see that

$$\mathbb{E}^{\mathbb{P}}\big[C^{\mathbb{P},p}_{s,t}\big] \;=\; \lim_{n\to\infty} \uparrow \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}_n}_{s\wedge\tau}\Big[\Big(\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}_n}\Big)^p\Big]\Big] \;\leq\; C\,\|U\|^p_{\mathcal{U}^p_{\eta,\tau}} \;<\; \infty,$$

by Proposition 4.3 together with the fact that $\|V^+\|_{\mathcal{D}^{p'}_{\eta,\tau}(\mathcal{P}_0)} < \infty$ by the wellposedness of the RBSDE. Hence, $C^{\mathbb{P},p}_{s,t} < \infty$, $\mathbb{P}$-a.s. □

References

[BC00] Philippe Briand and René Carmona. BSDEs with polynomial growth generators. *J. Appl. Math. Stochastic Anal.*, 13(3):207–238, 2000.

[BC08] Philippe Briand and Fulvia Confortola. Quadratic BSDEs with random terminal time and elliptic PDEs in infinite dimension. *Electron. J. Probab.*, 13:no. 54, 1529–1561, 2008.

[BEN04] K. Bahlali, A. Elouaflin, and M. N'zi. Backward stochastic differential equations with stochastic monotone coefficients. *J. Appl. Math. Stoch. Anal.*, 2004(4):317–335, 2004.

[BH98] P. Briand and Y. Hu. Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs. *Journal of Functional Analysis*, 155(2):455–494, 1998.

[7] This follows from the same argument as in [STZ12, Theorem 4.3]. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}^+_{\mathbb{P}}(s\wedge\tau)$, denote $\kappa^{\mathbb{P}_i} := \mathbb{E}^{\mathbb{P}_i}_{s\wedge\tau}\big[\big|\int_{s\wedge\tau}^{t\wedge\tau} dK_r^{\mathbb{P}'}\big|^p\big]$, and $A := \{\kappa^{\mathbb{P}_1} > \kappa^{\mathbb{P}_2}\}$, and define $E \in \mathcal{F} \longmapsto \mathbb{P}_3(E) := \mathbb{P}_1(A\cap E) + \mathbb{P}_2(A^c\cap E)$; clearly, $\mathbb{P}_3 \in \mathcal{P}^+_{\mathbb{P}}(t\wedge\tau)$, and $\kappa^{\mathbb{P}_3} = \kappa^{\mathbb{P}_1} \vee \kappa^{\mathbb{P}_2}$.

[BPTZ16] B. Bouchard, D. Possamai, X. Tan, and C. Zhou. A unified approach to a priori estimates for supersolutions of BSDEs in general filtrations. *to appear in the Annales de l'Institut Henri Poincaré*, 2016.

[BS96] D. Bertsekas and S. Shreve. *Stochastic Optimal Control: The Discrete-Time Case.* Belmont: Athena Scientific, 1996.

[CSTV07] Patrick Cheridito, H. Mete Soner, Nizar Touzi, and Nicolas Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm. Pure Appl. Math.*, 60(7):1081–1110, 2007.

[DP97] R. W. R. Darling and E. Pardoux. Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *The Annals of Probability*, 25(3):1135–1159, 1997.

[EKP+97] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez. Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.*, 25(2):702–737, 1997.

[El 81] N. El Karoui. Les aspects probabilistes du contrôle stochastique. In *Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979)*, volume 876 of *Lecture Notes in Math.*, pages 73–238. Springer, Berlin-New York, 1981.

[EPQ97] N. El Karoui, S. Peng, and M.-C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.

[HJPS14a] M. Hu, S. Ji, S. Peng, and Y. Song. Backward stochastic differential equations driven by G-Brownian motion. *Stochastic Processes and their Applications*, 124:759–784, 2014.

[HJPS14b] Mingshang Hu, Shaolin Ji, Shige Peng, and Yongsheng Song. Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. *Stochastic Process. Appl.*, 124(2):1170–1195, 2014.

[HLW99] S. Hamadène, J.-P. Lepeltier, and Zhen Wu. Infinite horizon reflected backward stochastic differential equations and applications in mixed control and game problems. *Probab. Math. Statist.*, 19(2, Acta Univ. Wratislav. No. 2198):211–234, 1999.

[JS03] J. Jacob and A.N. Shiryaev. Limit Theorems for Stochastic Processes, 2003.

[Kar95] R. L. Karandikar. On pathwise stochastic integration. *Stochastic Processes and their applications*, 57(1):11–18, 1995.

[KS12] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[Nev75] J. Neveu. *Discrete-parameter martingales.* North-Holland Publishing Company, Amsterdam-Oxford; American Elsevier Publishing Company, Inc., New York, revised edition, 1975. Translated from the French by T. P. Speed, North-Holland Mathematical Library, Vol. 10.

[NN14] A. Neufeld and M. Nutz. Measurability of semimartingale characteristics with respect to the probability law. *Stochastic Processes and their Applications*, 124(11):3819–3845, 2014.

[Nut12] M. Nutz. Pathwise construction of stochastic integrals. *Electron. Commun. Probab*, 17(24):1–7, 2012.

[Nut15] Marcel Nutz. Robust superhedging with jumps and diffusion. *Stochastic Processes and their Applications*, 125(12):4543–4555, 2015.

[NvH13] M. Nutz and R. van Handel. Constructing sublinear expectations on path space. *Stochastic Processes and their Applications*, 123(8):3100–3121, 2013.

[Pop07] A. Popier. Backward stochastic differential equations with random stopping time and singular final condition. *Ann. Probab.*, 35(3):1071–1117, 2007.

[PP90] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[Pro05] Philip E. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag Berlin Heidelberg, 2005.

[PTZ17] D. Possamai, X. Tan, and C. Zhou. Stochastic control for a class of nonlinear kernels and applications. *to appear in the Annals of Probability*, 2017.

[Roy04] Manuela Royer. BSDEs with a random terminal time driven by a monotone generator and their links with PDEs. *Stoch. Stoch. Rep.*, 76(4):281–307, 2004.

[RS12] A. Rozkosz and L. Słomiński. $\mathbb{L}^p$ solutions of reflected BSDEs under monotonicity condition. *Stochastic Processes and their Applications*, 122(12):3875–3900, 2012.

[STZ12] H.M. Soner, N. Touzi, and J. Zhang. Wellposedness of second order backward SDEs. *Probability Theory and Related Fields*, 153(1-2):149–190, 2012.

[STZ13] H.M. Soner, N. Touzi, and J. Zhang. Dual formulation of second order target problems. *The Annals of Applied Probability*, 23(1):308–347, 2013.

[SV97] D. W. Stroock and S.R. S. Varadhan. *Multidimensional diffusion processes*. Springer, 1997.