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KEYS AND DEMAZURE CRYSTALS FOR KAC-MOODY ALGEBRAS

NICOLAS JACON AND CÉDRIC LECOUCVEY

ABSTRACT. The Key map is an important tool in the determination of the Demazure crystals associated to Kac-Moody algebras. In finite type A, it can be computed in the tableau realization of crystals by a simple combinatorial procedure due to Lascoux and Schützenberger. We show that this procedure is a part of a more general construction holding in the Kac-Moody case that we illustrate in finite types and affine type A. In affine type A, we introduce higher level generalizations of core partitions which notably give interesting analogues of the Young lattice and are expected to parametrize distinguished elements of certain remarkable blocks for Ariki-Koike algebras.

1. INTRODUCTION

Kac-Moody algebras are infinite-dimensional analogues of semisimple Lie algebras. Their classification is based on the notion of Cartan datum, a generalization of the finite root systems. In particular, a Kac-Moody algebra \mathfrak{g} admits an enveloping algebra $U(\mathfrak{g})$, a Weyl group W , a weight lattice P and a cone P_+ of dominant weights. To each dominant weight λ is associated a highest weight $U(\mathfrak{g})$ -module $V(\lambda)$. The works of Kashiwara, Lusztig and Littelmann during the 90's have shown the existence of a fundamental object associated to $V(\lambda)$: the crystal $B(\lambda)$. It is an oriented graph whose combinatorics encodes many informations on $V(\lambda)$. In particular, it is endowed with a weight function with values in P whose generating series over $B(\lambda)$ coincides with the character of $V(\lambda)$ (see [21] and the references therein). The graph $B(\lambda)$ admits a unique source vertex b_λ (its highest weight vertex) and there is a simple action of the Weyl group W on $B(\lambda)$. Also for λ, μ in P_+ , the crystal $B(\lambda) \otimes B(\mu) = \{b \otimes b' \mid b \in B(\lambda), b' \in B(\mu)\}$ of the tensor product $V(\lambda) \otimes V(\mu)$ can be easily computed from $B(\lambda)$ and $B(\mu)$. In particular, $b_\lambda \otimes b_\mu$ is of highest weight $\lambda + \mu$ in $B(\lambda) \otimes B(\mu)$. The crystals with highest weight vertices $b_\lambda \otimes b_\mu$ and $b_\mu \otimes b_\lambda$ in $B(\lambda) \otimes B(\mu)$ and $B(\mu) \otimes B(\lambda)$ are then isomorphic. The corresponding isomorphism can be regarded as the restriction of more general isomorphisms between $B(\lambda) \otimes B(\mu)$ and $B(\mu) \otimes B(\lambda)$ called combinatorial R -matrices.

The Demazure modules $V(\lambda)_w$ are $U^+(\mathfrak{g})$ -submodules of $V(\lambda)$ defined for any $w \in W$. Quite remarkably, each such Demazure module $V(\lambda)$ also admits a crystal $B_w(\lambda)$ which is a subgraph of $B(\lambda)$. It has been proved by Littelmann that the generating series of the weight function over $B_w(\lambda)$ gives the Demazure character of $B_w(\lambda)$. Given the crystal $B(\lambda)$, it is a natural question to ask whether a vertex b in $B(\lambda)$ belongs to a Demazure crystal $B_w(\lambda)$. This problem may be solved by using a combinatorial procedure which involves the computation of a certain map called the right Key map. This map associates to each vertex b of $B(\lambda)$ an element $K^R(b)$ in the orbit $O(\lambda)$ of b_λ under the action of W . The Key map can be computed in any realization of the abstract crystal $B(\lambda)$ but has a great combinatorial complexity. Observe also that the algebra $U^+(\mathfrak{g})$ admits a crystal $B(\infty)$ with Demazure crystals $B_w(\infty)$ and associated Key maps.

In finite type A (i.e. for the Lie algebras \mathfrak{sl}_n), the dominant weights λ can be regarded as partitions and each crystal $B(\lambda)$ has a simple realization in terms of semistandard tableaux of shape λ . In [24], Lascoux and Schützenberger defined a simple procedure associating to such a tableau a “Key” tableau defined as a semistandard tableau such that each column of height h is included in any column of height $h' \geq h$. They then showed that these Key tableaux permit to compute the Demazure characters. By using the Littelmann path model and the dilatation of crystals introduced by Kashiwara, one can then prove that the Key map defined in [24] coincides with the previous general definition in the crystal realization of $B(\lambda)$ by semistandard tableaux.

The goal of this paper is to give a general reduction procedure to compute the Key map for any Kac-Moody algebra. Our strategy is to show that the approach of Lascoux and Schützenberger can be generalized to any crystal $B(\lambda)$ associated to any Kac-Moody algebra. More precisely, we explain how the Key map K^R can be computed for any weight λ , recursively on λ essentially by reduction to the case of the fundamental weights. In this perspective, the Demazure crystals can be characterized by the Key map for the fundamental weights,

the previous restrictions of combinatorial R -matrices and the description of the strong Bruhat order on cosets of W . In particular, in finite type A , the Key map for a fundamental weight is the identity, the combinatorial R -matrices can be computed on tableaux by the Jeu de Taquin procedure and the strong Bruhat order is easy to describe. Thus one recovers the results of [24]. For the classical types and for type G_2 , there are analogue simple tableaux models and we then illustrate our general procedure by giving natural extension of Lascoux-Schützenberger’s construction. They might also be adapted to the remaining exceptional cases based on the “tableaux” existing model for crystals (see [5]). This suggests that recent results by Brubaker and al [4], Masson [34] and Proctor [35] for type A might have generalizations in finite types. Note that we were informed during the redaction of this paper that Santos [36] also simultaneously got the description of the Key in type C . His approach, based on the symplectic plactic monoid, is nevertheless distinct from ours. It is also worth mentioning that the Key map can be computed as the last direction for paths corresponding to Lenart alcove model [29] and there exist crystal isomorphisms [30] between this model and the tableaux model of Kashiwara and Nakashima. We next focus on the affine type $A_{e-1}^{(1)}$ for which there also exists an interesting crystal model using multipartitions and related to the modular representation theory of Ariki-Koike algebras (some generalizations of the Hecke algebras). When $\lambda = \omega_i$ is a fundamental weight, $O(\omega_i)$ is parametrized by particular partitions called e -cores and the Key map can be computed thanks to a combinatorial procedure introduced in [1]. Also the strong Bruhat order on $O(\omega_i)$ corresponds to the inclusion of the Young diagrams of the e -cores. Thus, we can apply the previous reduction. Along the way, we introduce higher level generalizations of the core partitions which give interesting analogues of the Young lattice and which parametrize distinguished elements of certain remarkable blocks for Ariki-Koike algebras. Let us conclude by mentioning there are also quite simple combinatorial models for the highest weight crystals in any affine type (see for example [13]). It would be interesting to have a combinatorial description of the key maps and the R -matrices for the fundamental weights in this setting.

The paper is organized as follows. Section 2 is a recollection of basics facts on crystals and Demazure characters. In Section 3, we present the previous recursive procedure to compute the Key map. We explain how it can be used for Demazure crystals associated to finite types in Section 4. The affine type A case is studied in Section 5 where we introduce the notion of (e, \mathbf{s}) -core as a natural labelling of the orbit of the empty multipartition in Uglov’s and Kleshchev realizations of crystals. We also describe the Key map on Kleshchev multipartitions. Finally, we explain how our results on the the Demazure subcrystals in $B(\lambda)$ can be used to characterize the Demazure subcrystals in $B(\infty)$.

2. BACKGROUND ON KEYS AND DEMAZURE CRYSTALS

2.1. Crystals for integrable modules over Kac-Moody algebras.

2.1.1. *Background on root systems and Kac-Moody algebras.* Let I be a finite set and $A = (a_{i,j})_{(i,j) \in I^2}$ be a generalized Cartan matrix of rank r . This means that the entries of the matrix satisfy the following conditions

- (1) $a_{i,j} \in \mathbb{Z}$ for $i, j \in I^2$,
- (2) $a_{i,i} = 2$ for $i \in I^2$,
- (3) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$ for $i, j \in I^2$.

We will also assume that A is indecomposable: given subsets I and J of $\{1, \dots, n\}$, there exists $(i, j) \in I^2$ such that $a_{i,j} \neq 0$. We refer to [18] for the classification of indecomposable generalized Cartan matrices. Recall there exist only three kinds of such matrices: when all the principal minors of A are positive, A is of finite type and corresponds to the Cartan matrix of a simple Lie algebra over \mathbb{C} ; when all the proper principal minors of A are positive and $\det(A) = 0$ the matrix A is said of affine type; otherwise A is of indefinite type. For technical reasons, from now on, we will restrict ourselves to symmetrizable generalized Cartan matrices i.e. we will assume there exists a diagonal matrix D with entries in $\mathbb{Z}_{>0}$ such that DA is symmetric.

The root and weight lattices associated to a generalized symmetrizable Cartan matrix are defined by mimicking the construction for the Cartan matrices of finite type. Let P^\vee be a free abelian group of rank $2|I| - r$ with \mathbb{Z} -basis $\{h_i \mid i \in I\} \cup \{d_1, \dots, d_{|I|-r}\}$. Set $\mathfrak{h} := P^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{h}_{\mathbb{R}} := P^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. The weight lattice P is then defined by

$$P := \{\gamma \in \mathfrak{h}^* \mid \gamma(P^\vee) \subset \mathbb{Z}\}.$$

Set $\Pi^\vee := \{h_i \mid i \in I\}$. One can then choose a set $\Pi := \{\alpha_i \mid i \in I\}$ of linearly independent vectors in $P \subset \mathfrak{h}^*$ such that $\alpha_i(h_j) = a_{i,j}$ for $i, j \in I^2$ and $\alpha_i(d_j) \in \{0, 1\}$ for $i \in \{1, \dots, |I| - r\}$. The elements of Π are the simple

roots. The free abelian group $Q := \bigoplus_{i=1}^{|I|} \mathbb{Z}\alpha_i$ is the root lattice. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called a generalized Cartan datum associated to the matrix A . Let $P_+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for any } i \in I\}$ be the set of dominant weights. For any $i \in I$, the fundamental weight $\omega_i \in P$ is such that $\omega_i(h_j) = \delta_{i,j}$ for $j \in I$ and $\omega_i(d_j) = 0$ for $j \in \{1, \dots, |I| - r\}$.

For any $i \in I$, we define the simple reflection s_i on \mathfrak{h}^* by

$$(1) \quad s_i(\gamma) = \gamma - h_i(\gamma)\alpha_i \text{ for any } \gamma \in P.$$

The Weyl group W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections s_i . This is a Coxeter group acting on the weight lattice P and we refer the reader to [2] for a complete exposition. In particular, all the reduced decompositions of a fixed $w \in W$ have the same length $\ell(w)$. In the sequel we shall need the following characterizations of the strong Bruhat order \trianglelefteq and the weak Bruhat order \leq on W . Given u and v in W , we have

- $u \leq v$ if and only if every reduced decomposition of v admits a subword that is a reduced decomposition of u .
- $u \trianglelefteq v$ if and only if there are reduced decompositions of u and v such that u is a suffix of v .

Of course, if $u \leq v$, we have $u \trianglelefteq v$ but the converse is not true in general. For any dominant weight λ , write W_λ for the stabilizer of λ under the action of W . Every $w \in W$ then admits a unique decomposition on the form $w = p_\lambda(w)v$ with $v \in W_\lambda$ and $p_\lambda(w) \in W$ of minimal length. Let us denote by W^λ the image of W by the projection map p_λ . By setting $J_\lambda = \{i \in I \mid s_i(\lambda) = \lambda\}$, we get that u belongs to W^λ if and only if none of its reduced decompositions ends with a generator s_i such that $i \in J_\lambda$ (alternatively all its reduced expressions ends with a generator $s_i, i \notin J_\lambda$). Finally recall that for any w and w' in W , we have

$$(2) \quad w \leq w' \implies p_\lambda(w) \trianglelefteq p_\lambda(w').$$

We have in fact the more precise lemma (which follows from Theorem 2.6.1 in [2])

Lemma 2.1. *Assume λ, μ are dominant weights and $(w, w') \in W^{\lambda+\mu} \times W^{\lambda+\mu}$. Then*

$$w \trianglelefteq w' \iff \begin{cases} p_\lambda(w) \trianglelefteq p_\lambda(w'), \\ p_\mu(w) \trianglelefteq p_\mu(w'). \end{cases}$$

Let \mathfrak{g} be the symmetrizable Kac-Moody algebra associated to the generalized Cartan matrix A . We yet refer to [18] for a detailed definition of \mathfrak{g} and write as usual R its root system and P its weight lattice. The algebra \mathfrak{g} admits a presentation by relations on its Chevalley type generators $e_i, f_i, i \in I$ and $h \in P^*$. There exists a relevant semisimple category \mathcal{O}_{int} of integrable \mathfrak{g} -modules whose simple are parametrized by the dominant weights in P_+ . To each $\lambda \in P_+$ corresponds a unique (up to isomorphism) irreducible highest weight integrable \mathfrak{g} -module $V(\lambda)$ of highest weight λ . The irreducible module $V(\lambda)$ decomposes into weight spaces $V(\lambda) = \bigoplus_{\gamma \in P} V(\lambda)_\gamma$; and each weight space $V(\lambda)_\gamma$ is finite-dimensional. Consider the ring algebra $\mathbb{Z}[P]$ with basis the formal exponentials $e^\beta, \beta \in P$. We have an action of W on $\mathbb{Z}[P]$ defined by $w \cdot e^\beta = e^{w(\beta)}$. Set $\mathbb{Z}^W[P] = \{X \in \mathbb{Z}[P] \mid w(X) = X\}$. The character s_λ of $V(\lambda)$ is the element of $\mathbb{Z}[P]$ defined by $s_\lambda := \sum_{\gamma \in P} K_{\lambda, \gamma} e^\gamma$ where $K_{\lambda, \gamma} := \dim(V(\lambda)_\gamma)$. It belongs in fact to $\mathbb{Z}^W[P]$ because $K_{\lambda, \gamma} = K_{\lambda, w(\gamma)}$ for any $w \in W$. We have the Weyl-Kac character formula: for any $\lambda \in P_+$,

$$(3) \quad s_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})^{m_\alpha}}$$

where m_α is the multiplicity of the roots α (equal to 1 in the finite case).

The quantum group $U_q(\mathfrak{g})$ is also defined from the same generalized Cartan matrix A . It also admits a presentation by generators and relations which can be regarded as q -deformation of that of \mathfrak{g} (see [21]). Roughly speaking, one obtains the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} as the limit of $U_q(\mathfrak{g})$ when q tends to 1. This implies that the representation theory of $U_q(\mathfrak{g})$ is essentially similar to that of $U(\mathfrak{g})$ and thus also to that of \mathfrak{g} . Therefore, for simplicity and since we do not need to distinguish the different module structures in the sequel, we will use the same notation for the category of integrable modules of $\mathfrak{g}, U(\mathfrak{g})$ and $U_q(\mathfrak{g})$. In particular, for each dominant weight λ , there exists a unique $U_q(\mathfrak{g})$ -module in \mathcal{O}_{int} also denoted by $V(\lambda)$.

2.1.2. *Crystals of integrable modules.* To each dominant weight λ corresponds a crystal graph $B(\lambda)$ which can be regarded as the combinatorial skeleton of the simple module $V(\lambda)$. Its structure can be defined from the notion of canonical bases as introduced by Lusztig [33] and subsequently studied by Kashiwara under the name of global bases (see [21] and [22]). It also has a purely combinatorial definition in terms of Littelmann's path model (see [31]). The crystal $B(\lambda)$ is a graph whose set of vertices is endowed with a weight function $\text{wt} : B(\lambda) \rightarrow P$ and with the structure of a colored and oriented graph given by the action of the crystal operators \tilde{f}_i and \tilde{e}_i with $i \in I$. More precisely, we have an oriented arrow $b \xrightarrow{i} b'$ between two vertices b and b' in $B(\lambda)$ if and only if $b' = \tilde{f}_i(b)$ or equivalently $b = \tilde{e}_i(b')$. We have $\tilde{f}_i(b) = 0$ (resp. $\tilde{e}_i(b) = 0$) when no arrow i starts from b (resp. ends at b). There is a unique vertex b_λ in $B(\lambda)$ such that $\tilde{e}_i(b_\lambda) = 0$ for any $i \in I$ called the highest weight vertex of $B(\lambda)$ and we have $\text{wt}(b_\lambda) = \lambda$. Thus, for any $b \in B(\lambda)$, there is a path $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(b_\lambda)$ from b_λ to b . The weight function wt is such that

$$\text{wt}(b) = \lambda - \sum_{k=1}^r \alpha_{i_k}.$$

For any $i \in I$, the crystal $B(\lambda)$ decomposes into i -chains. For any vertex $b \in B(\lambda)$, set $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k(b) \neq 0\}$ and $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k(b) \neq 0\}$. We have

$$\text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i \text{ and } s_\lambda = \sum_{b \in B(\lambda)} e^{\text{wt}(b)}.$$

The Weyl group W acts on the vertices of $B(\lambda)$: the action of the simple reflection s_i on $B(\lambda)$ sends each vertex b on the unique vertex b' in the i -chain of b such that $\varphi_i(b') = \varepsilon_i(b)$ and $\varepsilon_i(b') = \varphi_i(b)$ for any $i \in I$. This simply means that b and b' correspond by the reflection with respect to the center of the i -chain containing b . We shall write

$$O(\lambda) = \{w \cdot b_\lambda = b_{w\lambda} \mid w \in W\}$$

for the orbit of the highest weight vertex of $B(\lambda)$. Observe $b_{w\lambda}$ is then the unique vertex in $B(\lambda)$ of weight $w\lambda$.

More generally, the crystal B_M of any module M in \mathcal{O}_{int} is the disjoint union of the crystals associated to the irreducible modules appearing in its decomposition. In particular, the multiplicity of the irreducible module $V(\lambda)$ in M corresponds to the number of copies of the crystal $B(\lambda)$ in B_M . Consider M and N two modules in \mathcal{O}_{int} with crystals B_M and B_N , respectively. The crystal associated to $M \otimes N$ is the crystal $B_M \otimes B_N$ whose set of vertices is the direct product of the sets of vertices of B_M and B_N and whose crystal structure is given by the following rules¹

$$(4) \quad \tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i(v) & \text{if } \varepsilon_i(u) \leq \varphi_i(v) \\ \tilde{e}_i(u) \otimes v & \text{if } \varepsilon_i(u) > \varphi_i(v) \end{cases} \quad \text{and} \quad \tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i(u) \otimes v & \text{if } \varphi_i(v) \leq \varepsilon_i(u) \\ u \otimes \tilde{f}_i(v) & \text{if } \varphi_i(v) > \varepsilon_i(u) \end{cases}.$$

A crystal $B(\infty)$ for the positive part $U_q^+(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ is also available by the results of Lusztig [33] and Kashiwara [21]. This crystal $B(\infty)$ admits a unique source vertex b_\emptyset . Moreover, for any $\lambda \in P_+$, there exists a unique embedding of crystals $\pi_\lambda : B(\lambda) \hookrightarrow B(\infty)$ so that for any path $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(b_\lambda)$ in $B(\lambda)$, we have $\pi_\lambda(b) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(b_\emptyset)$ in $B(\infty)$.

2.1.3. *Cosets of the Weyl group and crystals.* There is a one-to-one correspondence between W^λ and $O(\lambda)$ which associates to each $w \in W^\lambda$ the vertex $b_{w\lambda}$. Also in $O(\lambda)$, the vertices $b_{w\lambda}$ and $b_{s_i w\lambda}$ are such that

$$(5) \quad \begin{cases} b_{s_i w\lambda} = \tilde{f}_i^{\varphi_i(b_{w\lambda})} b_{w\lambda} & \text{with } \varepsilon_i(b_{w\lambda}) = 0 \text{ if } \ell(s_i w) = \ell(w) + 1, \\ b_{s_i w\lambda} = \tilde{e}_i^{\varepsilon_i(b_{w\lambda})} b_{w\lambda} & \text{with } \varphi_i(b_{w\lambda}) = 0 \text{ if } \ell(s_i w) = \ell(w) - 1. \end{cases}$$

In particular, for any element $b_{w\lambda} \in O(\lambda)$ and any $i \in I$, we have either $\varepsilon_i(b_{w\lambda}) = 0$, or $\varphi_i(b_{w\lambda}) = 0$ ². Now if $w \in W^\lambda$ with reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$, we will have

$$b_{w\lambda} = \tilde{f}_{i_\ell}^{a_\ell} \cdots \tilde{f}_{i_1}^{a_1}(b_\lambda) = s_{i_\ell} \cdots s_{i_1} \cdot b_\lambda$$

with $a_1 = \varphi_{i_1}(b_\lambda)$ and $a_k = \varphi_{i_k}(\tilde{f}_{i_{k-1}}^{a_{k-1}} \cdots \tilde{f}_{i_1}^{a_1} b_\lambda)$ for $k = 2, \dots, \ell$. The converse is true which permits to identify the reduced expressions of $w = s_{i_\ell} \cdots s_{i_1} \in W^\lambda$ with the directed paths in $O(\lambda)$ from b_λ to $b_{w\lambda}$. In some sense, the crystal $B(\lambda)$ can be regarded as an automaton which associates to any $w \in W$, its projection $p_\lambda(w)$ on W^λ .

¹Observe our convention here is not the same as in [21] and [22].

²Nevertheless, the condition $\varepsilon_i(b_{w\lambda}) = 0$ or $\varphi_i(b_{w\lambda}) = 0$ does not characterize the elements of $\mathcal{O}(\lambda)$.

One can also observe that $O(\lambda)$ has the structure of the Hasse diagram on W^λ by putting arrows $b_{w\lambda} \dashrightarrow b_{s_i w\lambda}$ when $\ell(s_i w) = \ell(w) + 1$.

2.1.4. *Dilatation of crystals.* Consider a positive integer m and λ a dominant weight. There exists a unique embedding of crystals $\psi_m : B(\lambda) \hookrightarrow B(m\lambda)$ such that for any vertex $b \in B(\lambda)$ and any path $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k}(b_\lambda)$ in $B(\lambda)$, we have

$$\psi_m(b) = \tilde{f}_{i_1}^m \cdots \tilde{f}_{i_k}^m(b_{m\lambda}).$$

Since the vertex $b_\lambda^{\otimes m}$ is of highest weight $m\lambda$ in $B(\lambda)^{\otimes m}$, one gets a particular realization $B(b_\lambda^{\otimes m})$ of $B(m\lambda)$ in $B(\lambda)^{\otimes m}$ with highest weight vertex $b_\lambda^{\otimes m}$. This thus gives a canonical embedding

$$(6) \quad K_m : \begin{cases} B(b_\lambda) \hookrightarrow B(b_\lambda^{\otimes m}) \subset B(b_\lambda)^{\otimes m} \\ b \longmapsto b_1 \otimes \cdots \otimes b_m \end{cases}$$

Consider λ and μ two dominant weights. Write b_λ and b_μ for the highest weight vertices of $B(\lambda)$ and $B(\mu)$. Then $B(b_\lambda \otimes b_\mu)$ is a realization of the abstract crystal $B(\lambda + \mu)$ and we can define the m -dilatation $K_m : B(b_\lambda \otimes b_\mu) \hookrightarrow B(b_{\lambda+\mu}^{\otimes m})$ as in (6). The next lemma shows there is another natural m -dilatation of $B(b_\lambda \otimes b_\mu)$ (see for example Corollary 2.1.3 in [27] for a proof).

Lemma 2.2. *The map*

$$K'_m : \begin{cases} B(b_\lambda \otimes b_\mu) \hookrightarrow B(b_\lambda^{\otimes m} \otimes b_\mu^{\otimes m}) \\ b_1 \otimes b_2 \longmapsto K'_m(b_1) \otimes K'_m(b_2) \end{cases}$$

is a m -dilatation of $B(b_\lambda \otimes b_\mu)$, that is for any $i \in I$ we have

$$K'_m(\tilde{f}_i(b_1 \otimes b_2)) = \tilde{f}_i^m K'_m(b_1 \otimes b_2) \text{ and } K'_m(\tilde{e}_i(b_1 \otimes b_2)) = \tilde{e}_i^m K'_m(b_1 \otimes b_2).$$

Theorem 2.3. (see [22])

- (1) For any $w \in W$, we have $K_m(b_{w\lambda}) = b_{w\lambda}^{\otimes m}$.
- (2) Consider $b \in B(\lambda)$. When m has sufficiently many factors, there exist elements w_1, \dots, w_m in W such that $K_m(b) = b_{w_1\lambda} \otimes \cdots \otimes b_{w_m\lambda}$. Moreover, in this case
 - (a) up to repetition, the elements $b_{w_1\lambda}$ and $b_{w_m\lambda}$ in $K_m(b)$ do not then depend on m ,
 - (b) the sequence $(w_1\lambda, \dots, w_m\lambda)$ in $K_m(b)$ does not depend on the realization of the crystal $B(\lambda)$ and we have $w_1 \triangleleft \cdots \triangleleft w_m$.

From Assertion 2 of the previous theorem, we can define the left and right Keys of an element in $B(\lambda)$.

Definition 2.4. Let $b \in B(\lambda)$, then the left Key $K^L(b)$ of b and the right Key $K^R(b)$ of b are defined as follows:

$$K^L(b) = b_{w_1\lambda} \text{ and } K^R(b) = b_{w_m\lambda}.$$

Remark 2.5.

- (1) By Assertion 4 of the theorem, the sequence $(w_1\lambda, \dots, w_m\lambda)$ does not depend on the realization of the crystal $B(\lambda)$. Nevertheless, the components $b_{w_k\lambda}$ do and thus also the left and right Keys.
- (2) Assume $B_1(\lambda)$ and $B_2(\lambda)$ are two realizations of the crystal $B(\lambda)$ and $\phi : B_1(\lambda) \rightarrow B_2(\lambda)$ the associated crystal isomorphisms. Let $K_m^{(1)}$ and $K_m^{(2)}$ be the crystal embedding defined from $B_1(\lambda)$ and $B(\lambda)$ as in (6). Since ϕ is a crystal isomorphism and $K_m^{(1)}, K_m^{(2)}$ are both crystal embedding we have $K_m^{(2)} \circ \phi = \phi^{\otimes m} \circ K_m^{(1)}$ where $\phi^{\otimes m}$ is defined on $B_1(\lambda)^{\otimes m}$ by applying ϕ to each factors. In particular, for any $b \in B_1(\lambda)$ we have

$$(7) \quad K^L \cdot \phi(b) = \phi \cdot K^L(b) \text{ and } K^R \cdot \phi(b) = \phi \cdot K^R(b).$$

Following Kashiwara, let us now define for any $\mu \in W \cdot \lambda$, the set

$$\overline{B}_\mu(\lambda) = \{b \in B(\lambda) \mid K^R(b) = b_\mu\}$$

We then have $B(\lambda) = \bigsqcup_{\mu \in W \cdot \lambda} \overline{B}_\mu(\lambda)$.

2.2. Crystals of Demazure modules. Let λ be a dominant weight and consider $w \in W$. Then, there exists (up to a constant) a unique highest weight vector $v_{w\lambda}$ in $V(\lambda)$. The Demazure module associated to $v_{w\lambda}$ is the $U_q^+(\mathfrak{g})$ -module defined by

$$D_w(\lambda) := U_q^+(\mathfrak{g}) \cdot v_{w\lambda}.$$

Demazure [6] introduced the character s_λ^w of $D_w(\lambda)$ and shows that it can be computed by applying to e^λ a sequence of divided difference operators given by any decomposition of w . More precisely, define for any $i \in I$ the operator D_i on $\mathbb{Z}[P]$ by

$$D_i(X) = \frac{X - e^{-\alpha_i}(s_i \cdot X)}{1 - e^{-\alpha_i}}.$$

Consider a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ of w . Then, Demazure proved that $D_w = D_{i_1} \cdots D_{i_\ell}$ depends only on w and not on the reduced decomposition considered. Then $s_\lambda^w = D_w(e^\lambda) \in \mathbb{Z}[P]$ is the Demazure character. Later Kashiwara [20] and Littelmann [31] defined a relevant notion of crystals for the Demazure modules. To do this, consider for any $w \in W$, the set

$$\overline{B}_{w\lambda}(\lambda) = \{b \in B(\lambda) \mid K^R(b) = b_{w\lambda}\}.$$

By definition we have $\overline{B}_{w\lambda}(\lambda) = \overline{B}_{w'\lambda}(\lambda)$ when w and w' belong to the same right coset of $W_\lambda \backslash W$. We also get $B(\lambda) = \bigsqcup_{w\lambda \in W_\lambda} \overline{B}_{w\lambda}(\lambda)$.

Definition 2.6. *The Demazure crystal $B_w(\lambda)$ is defined by*

$$(8) \quad B_w(\lambda) = \bigsqcup_{w' \leq w} \overline{B}_{w'\lambda}(\lambda).$$

By writing $w = uv$ with $u \in W^\lambda$ and $v \in W_\lambda$, we get $B_w(\lambda) = B_u(\lambda)$ from the characterization of the strong Bruhat order recalled in §2.1.1. Thus we can and shall assume that both w and w' belong to W^λ in (8). The following Theorem has been established by Kashiwara and Littelmann.

Theorem 2.7. *Assume λ is a dominant weight.*

- (1) *We have $s_\lambda^w = \sum_{b \in B_w(\lambda)} e^{\text{wt}(b)}$.*
- (2) *For any reduced decomposition $s_{i_1} \cdots s_{i_\ell}$ of w , we have $B_w(\lambda) := \{\tilde{f}_{i_1}^{k_1} \cdots \tilde{f}_{i_\ell}^{k_\ell}(b_\lambda) \mid (k_1, \dots, k_\ell) \in \mathbb{Z}_{\geq 0}^\ell\}$.*

It is also interesting to define

$$B_w(\infty) = \lim_{\lambda \rightarrow +\infty} B_w(\lambda) := \{\tilde{f}_{i_1}^{k_1} \cdots \tilde{f}_{i_\ell}^{k_\ell}(b_\emptyset) \mid (k_1, \dots, k_\ell) \in \mathbb{Z}_{\geq 0}^\ell\}.$$

Thus, from the above result, we deduce that to compute the Demazure crystal $B_w(\lambda)$, it suffices to

- compute the Key map K^R on $B(\lambda)$.
- compute the strong Bruhat order on W^λ , or alternatively on the vertices of $O(\lambda)$.

3. RECURSIVE COMPUTATIONS OF THE KEYS AND THE STRONG BRUHAT ORDER

We shall describe in this section procedures for computing the Keys using combinatorial R-matrices and the strong Bruhat order on the orbit of the highest weight vertex.

3.1. Keys and combinatorial R-matrices. Consider λ, μ two dominant weights. Then $B(\lambda) \otimes B(\mu)$ contains a unique connected component $B_{\lambda, \mu}(\lambda + \mu)$ isomorphic to the abstract crystal $B(\lambda + \mu)$ with highest weight vertex $b_{\lambda, \mu} = b_\lambda \otimes b_\mu$. Moreover, the crystal $B(\lambda) \otimes B(\mu)$ and $B(\mu) \otimes B(\lambda)$ are isomorphic. In general, there are fewer isomorphisms from $B(\lambda) \otimes B(\mu)$ to $B(\mu) \otimes B(\lambda)$ (see [19] for the description of such an isomorphism in the Kac-Moody case). Nevertheless, each such isomorphism sends $B_{\lambda, \mu}(\lambda + \mu)$ on $B_{\mu, \lambda}(\lambda + \mu)$ for there is only one connected component in $B(\lambda) \otimes B(\mu)$ and $B(\mu) \otimes B(\lambda)$ of highest weight $\lambda + \mu$ that we call *principal*. We shall write R the unique isomorphism from $B_{\lambda, \mu}(\lambda + \mu)$ to $B_{\mu, \lambda}(\lambda + \mu)$. Also recall that $O_{\lambda, \mu}(\lambda + \mu)$ is the orbit of $b_\lambda \otimes b_\mu$ in $B(\lambda) \otimes B(\mu)$ under the action of W .

Given two crystals B_1 and B_2 the flip F is the bijection from $B_1 \otimes B_2$ to $B_2 \otimes B_1$ defined by $F(u \otimes v) = v \otimes u$ for any $u \otimes v$ in $B_1 \otimes B_2$. This is not a crystal isomorphism in general.

The previous definitions of $B_{\lambda, \mu}(\mu + \lambda)$, $O_{\lambda, \mu}(\lambda + \mu)$ etc. extend naturally to the case where a sequence $\lambda^{(1)}, \dots, \lambda^{(m)}$ of dominant weights is considered (rather than just two dominant weights). Let us start with the easy following lemma.

Lemma 3.1. *Consider $w \cdot b_{\lambda^{(1)}} \otimes \cdots \otimes b_{\lambda^{(m)}} \in O_{\lambda^{(1)}, \dots, \lambda^{(m)}}(\lambda^{(1)} + \cdots + \lambda^{(m)})$. Then we have*

$$w \cdot b_{\lambda^{(1)}} \otimes \cdots \otimes b_{\lambda^{(m)}} = b_{w\lambda^{(1)}} \otimes \cdots \otimes b_{w\lambda^{(m)}}.$$

Moreover, for any $i \in I$ and any $k = 1, \dots, m$ we have

$$\begin{cases} \varepsilon_i(b_{w \cdot \lambda^{(k)}}) = 0 \text{ when } \ell(s_i w) = \ell(w) + 1, \\ \varphi_i(b_{w \cdot \lambda^{(k)}}) = 0 \text{ when } \ell(s_i w) = \ell(w) - 1. \end{cases}$$

Proof. This follows from (5), the definition of the action of the generators s_i and an easy induction on the length of w . \square

Now, consider $w \cdot b_{\lambda, \mu} \in O_{\lambda, \mu}(\lambda + \mu) \subset B(\lambda) \otimes B(\mu)$.

Lemma 3.2. *Assume $w \in W$. Then, we have*

$$w \cdot b_{\lambda, \mu} = b_{p_\lambda(w)\lambda} \otimes b_{p_\mu(w)\mu}.$$

Proof. We can assume that $w \in W^{\lambda+\mu}$ and set $w \cdot b_{\lambda, \mu} = b_1 \otimes b_2$. By Lemma 3.1, we know that $b_1 \in O_\lambda(b_\lambda)$ and $b_2 \in O_\mu(b_\mu)$. Thus we can set $w \cdot b_{\lambda, \mu} = b_{w_L \lambda} \otimes b_{w_R \lambda}$ with $(w_L, w_R) \in W^\lambda \times W^\mu$. Moreover, if we fix a reduced expression of $w = s_{i_1} \cdots s_{i_m} \in W^{\lambda+\mu}$, we get a directed path $\pi_{(\lambda, \mu)}^w$ in $O_{\lambda, \mu}(\lambda + \mu)$ from $b_{\lambda, \mu}$ to $b_1 \otimes b_2$ obtained by applying crystals operators $\tilde{f}_i, i \in I$. By using the tensor product rules (4) for these operators, this thus yields also a directed path $\pi_\lambda^{w_L}$ in $O_\lambda(b_\lambda)$ from b_λ to $b_{w_L \lambda}$ and a directed path $\pi_\mu^{w_R}$ in $O_\mu(b_\mu)$ from b_μ to $b_{w_R \mu}$. The equivalence between directed paths and elements in W^λ and W^μ (see §2.1.3) then imposes that we have $(w_L, w_R) = (p_\lambda(w), p_\mu(w))$. \square

Consider $b = u \otimes v$ in $B_{\lambda, \mu}(\lambda + \mu)$ and set

$$K^L(b) = u^L \otimes v^L, \quad K^R(b) = u^R \otimes v^R.$$

Lemma 3.3. *We have $u^L = K^L(u)$ and $v^R = K^R(v)$.*

Proof. For m with sufficiently many factors, we get by definition of $K^L(b)$ and $K^R(b)$

$$(9) \quad K_m(b) = K^L(b) \otimes \cdots \otimes K^R(b)$$

where K_m is the crystal embedding from $B(b_{\lambda, \mu})$ in $B(b_{\lambda, \mu}^{\otimes m})$ defined in (6). Now the crystals $B(b_{\lambda, \mu}^{\otimes m})$ and $B(b_\lambda^{\otimes m} \otimes b_\mu^m)$ are isomorphic for their highest weight vertices $b_{\lambda, \mu}^{\otimes m} = (b_\lambda \otimes b_\mu)^m$ and $b_\lambda^{\otimes m} \otimes b_\mu^m$ have the same highest weight $m(\lambda + \mu)$. The isomorphism I from $B(b_{\lambda, \mu}^{\otimes m})$ and $B(b_\lambda^{\otimes m} \otimes b_\mu^m)$ and its converse I^{-1} are obtained by composing R-matrices whose actions on the previous highest weight vertices reduce to the flip of components b_λ and b_μ . In particular I and I^{-1} fix the leftmost and rightmost components in the vertices of $B(b_{\lambda, \mu}^{\otimes m})$ and $B(b_\lambda^{\otimes m} \otimes b_\mu^m)$. Since m can be any integer with sufficiently many factors, one can choose such a integer m so that (9) holds and simultaneously

$$K'_m(b) = K_m(u) \otimes K_m(v) = K^L(u) \otimes \cdots \otimes K^R(u) \otimes K^L(u) \otimes \cdots \otimes K^R(u)$$

where $K'_m = I \circ K_m$ by Lemma 2.2. Since I^{-1} fixes the leftmost and rightmost components in $K'_m(b)$ we are done. \square

Now we can show that the action of W commutes with the flip F on the orbit $O_{\lambda, \mu}(\lambda + \mu)$.

Proposition 3.4. *For any $w \in W$ and any vertex $b \in O_{\lambda, \mu}(\lambda + \mu)$, we have*

$$w \circ F(b) = F \circ w(b).$$

Proof. Write $b = u(b_\lambda \otimes b_\mu)$. On the one hand, Lemma 3.2 gives $w(b) = wu(b_\lambda \otimes b_\mu) = b_{p_\lambda(wu)\lambda} \otimes b_{p_\mu(wu)\mu}$ and thus $F \circ w(b) = b_{p_\mu(wu)\mu} \otimes b_{p_\lambda(wu)\lambda}$. On the other hand, we get $w \circ F(b) = w \circ F(b_{p_\lambda(u)\lambda} \otimes b_{p_\mu(u)\mu}) = w(b_{p_\mu(u)\mu} \otimes b_{p_\lambda(u)\lambda}) = wu(b_\mu \otimes b_\lambda) = b_{p_\mu(wu)\mu} \otimes b_{p_\lambda(wu)\lambda}$. Therefore, we have $w \circ F(b) = F \circ w(b)$ as desired. \square

Corollary 3.5. *The maps R and F coincide on $O_{\lambda, \mu}(\lambda + \mu)$.*

Proof. Consider $b = w \cdot b_{\lambda, \mu}$ in $O_{\lambda, \mu}(\lambda + \mu)$. On the one hand side, we have

$$R(w \cdot b_{\lambda, \mu}) = w \cdot R(b_{\lambda, \mu}) = w \cdot b_{\mu, \lambda}$$

because R is a crystal isomorphism (and thus commutes with the action of W) and $R(b_{\lambda, \mu}) = b_{\mu, \lambda}$. On the other side we get

$$F(w \cdot b_{\lambda, \mu}) = w \cdot F(b_{\lambda, \mu}) = w \cdot b_{\mu, \lambda}$$

by using the previous proposition and the equality $F(b_{\lambda, \mu}) = b_{\mu, \lambda}$. \square

3.2. Reduction to smaller dominant weights. Denote by \prec the partial dominant order on P_+ such that $\mu \preceq \lambda$ if and only if $\lambda - \mu \in P_+$ and resume the notation of §3.1. Our aim is now to compute the left and right Keys of any vertex in $B_{\lambda, \mu}(\lambda + \mu)$ as a tensor product of Keys in $B(\lambda)$ and $B(\mu)$. For any $b = u \otimes v$ in $B_{\lambda, \mu}(\lambda + \mu)$, set $R(b) = \tilde{b} = \tilde{v} \otimes \tilde{u}$ in $B_{\mu, \lambda}(\lambda + \mu)$.

Theorem 3.6. *We have*

$$K^L(b) = K^L(u) \otimes K^L(\tilde{v}) \quad \text{and} \quad K^R(b) = K^R(\tilde{u}) \otimes K^R(v).$$

Proof. We prove the first equality, the arguments being similar for the second one. Write $K^L(b) = u^L \otimes v^L$. By Lemma 3.3, we first get that $u^L = K^L(u)$. We also have $K^L(R(b)) = R(K^L(b))$ because R is a crystal isomorphism as in (7). By Corollary 3.5 and once again, Lemma 3.3 we deduce the equality

$$K^L(R(b)) = K^L(\tilde{v} \otimes \tilde{u}) = K^L(\tilde{v}) \otimes \tilde{u}^L = R(K^L(b)) = R(K^L(u) \otimes v^L) = v^L \otimes K^L(u).$$

Thus, $v^L = K^L(\tilde{v})$ as desired. \square

Now consider $\mathcal{S} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ a sequence of dominant weights and write $\lambda = \lambda^{(1)} + \dots + \lambda^{(l)}$. Let $B_{\mathcal{S}}(\lambda)$ be the unique connected component in $\bigotimes_{k=1}^l B(\lambda^{(k)})$ of highest weight λ . Its highest weight vertex is $b_{\mathcal{S}} = b_{\lambda^{(1)}} \otimes \dots \otimes b_{\lambda^{(l)}}$. For any $k = 1, \dots, l$, denote by θ_k^L the unique crystal isomorphism from $B_{\mathcal{S}}(\lambda)$ to $B_{\mathcal{S}_L^{(k)}}(\lambda)$ where $\mathcal{S}_L^{(k)} = (\lambda^{(k)}, \lambda^{(1)}, \dots, \lambda^{(l)})$ (in particular $\mathcal{S}_L^{(1)} = \mathcal{S}$). Write similarly θ_k^R the unique crystal isomorphism from $B_{\mathcal{S}}(\lambda)$ to $B_{\mathcal{S}_R^{(k)}}(\lambda)$ where $\mathcal{S}_R^{(k)} = (\lambda^{(1)}, \dots, \lambda^{(l)}, \lambda^{(k)})$ (in particular $\mathcal{S}_R^{(l)} = \mathcal{S}$).

For any $b = b_1 \otimes \dots \otimes b_l$ in $B_{\mathcal{S}}(\lambda)$, set $\theta_k^L(b) = b_1^L(k) \otimes \dots \otimes b_l^L(k)$ and $\theta_k^R(b) = b_1^R(k) \otimes \dots \otimes b_l^R(k)$. In particular $b_1^L(k) \in B(\lambda^{(k)})$ and $b_l^R(k) \in B(\lambda^{(k)})$ for any $k = 1, \dots, l$. An easy induction yields the following corollary of Theorem 3.6.

Corollary 3.7. *For any $b \in B_{\mathcal{S}}(\lambda)$, we have*

$$K^L(b) = \bigotimes_{k=1}^l K^L(b_1^L(k)) \quad \text{and} \quad K^R(b) = \bigotimes_{k=1}^l K^R(b_l^R(k)).$$

Remark 3.8. *The previous corollary reduces the computation of the Keys for a dominant weight λ to that of R -matrices and Keys for dominant weights less than λ for the order \prec on P_+ . For finite types and for affine type A , we shall see that this gives an efficient procedure by decomposing λ on the basis of fundamental weights.*

3.3. Recursive computation of the strong Bruhat order. We resume the notation of the previous § of this section. In §2.1.3, we have also seen that the elements of $W^{\lambda+\mu}$ are matched with the vertices of $O_{\lambda, \mu}(\lambda + \mu)$.

Proposition 3.9. *Consider $w \cdot b_{\lambda, \mu}$ and $w' \cdot b_{\lambda, \mu}$ in $O_{\lambda, \mu}(\lambda + \mu) \subset B(\lambda) \otimes B(\mu)$ with w and w' in $W^{\lambda+\mu}$. Then*

$$w \cdot b_{\lambda, \mu} = b_{p_{\lambda}(w)\lambda} \otimes b_{p_{\mu}(w)\mu} \quad \text{and} \quad w' \cdot b_{\lambda, \mu} = b_{p_{\lambda}(w')\lambda} \otimes b_{p_{\mu}(w')\mu}.$$

Moreover

$$w \preceq w' \text{ if and only if } p_{\lambda}(w) \preceq p_{\lambda}(w') \text{ and } p_{\mu}(w) \preceq p_{\mu}(w').$$

Proof. The first statement of the proposition comes by applying Lemma 3.2 to w and w' and the second one is Lemma 2.1. \square

Now, consider $\mathcal{S} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ a sequence of dominant weights and write $O_{\mathcal{S}}(\lambda)$ for the orbit of $b_{\mathcal{S}}$ in $B_{\mathcal{S}}(\lambda)$.

Proposition 3.9 and an easy induction yields the following corollary which permits a recursive computation of the strong Bruhat order. It will be of particular interest in the following sections when the $\lambda^{(k)}$ are fundamental weights and the crystal $B(\lambda)$ has a convenient realization in terms of tableaux or abaci.

Corollary 3.10.

- (1) For any $b_1 \otimes \cdots \otimes b_l = w \cdot b_S \in O_S(\lambda)$ with $w \in W^{\lambda^{(1)} + \cdots + \lambda^{(l)}}$, there exists a unique l -tuple $(w_1, \dots, w_l) \in \prod_{k=1}^l W^{\lambda^{(k)}}$ such that $b_k = w_k \cdot b_{\lambda^{(k)}}$ for any $k = 1, \dots, l$.
- (2) We have $w_k = p_{\lambda^{(k)}}(w)$.
- (3) Given $w \cdot b_S \in O_S(\lambda)$ and $w' \cdot b_S \in O_S(\lambda)$ with $w, w' \in W^{\lambda^{(1)} + \cdots + \lambda^{(l)}}$, we have $w \trianglelefteq w'$ if and only if $w_k \trianglelefteq w'_k$ for any $k = 1, \dots, l$.

4. DETERMINATION OF THE DEMAZURE CRYSTALS BY KEYS IN FINITE TYPES

By Theorems 2.7 and 3.6, given any dominant weight λ expressed as a sum of fundamental weights, we can conveniently compute the Demazure crystals $B_w(\lambda)$ as soon as we have efficient procedures for

- computing the combinatorial R-matrix (or at least its restriction to the principal connected component) on tensor product of fundamental crystals (i.e. crystal with fundamental highest weights),
- computing the Key for fundamental crystals,
- computing the strong Bruhat order on W^λ .

4.1. The finite type A. We start by recalling the results of Lascoux and Schützenberger [24]. In type A_n , the crystal $B(\omega_i)$, $i = 1, \dots, n$ is conveniently realized as the set of columns of height i on $\{1 < \cdots < n < n + 1\}$. Then the dominant weight ω_i is minuscule which implies that $K^L(c) = K^R(c)$ for any column $C \in B(\omega_i)$. Also the combinatorial R-matrices can be computed by using the Jeu de Taquin procedure or the insertion scheme on semistandard tableaux. More generally given a sequence $\mathcal{S} = (\omega_{i_1}, \dots, \omega_{i_l})$ of dominant weights such that $i_1 \geq \cdots \geq i_l$, the vertices of the crystal $B_{\mathcal{S}}(\lambda)$ with $\lambda = \omega_{i_1} + \cdots + \omega_{i_l}$ defined in § 3.1 can be identified with the semistandard tableaux of shape λ (see [21] and the example below). The highest weight tableau is the tableau $T(\lambda)$ of shape λ with entries i in row i for any $i = 1, \dots, n$. The elements of the orbit $O_{\mathcal{S}}(\lambda)$ of $T(\lambda)$ are the semistandard tableaux $T = C_1 \otimes \cdots \otimes C_l$ of shape λ verifying the chain of inclusions $C_l \subset \cdots \subset C_2 \subset C_1$. Also for two such tableaux $T = C_1 \otimes \cdots \otimes C_l$ and $T' = C'_1 \otimes \cdots \otimes C'_l$ with $T = w \cdot T(\lambda)$ and $T' = w' \cdot T(\lambda)$ and $(w, w') \in (W^\lambda)^2$, we have $w \trianglelefteq w'$ if and only if $C_k C'_k$ is a semistandard tableau for any $k = 1, \dots, l$. This is a direct consequence of 3.10. Equivalently one gets that the Strong Bruhat order on W^λ is just the product of the strong Bruhat orders on the cosets W^{ω_i} , $i = 1, \dots, l$.

Example 4.1. Let us compute the Key of the tableau

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & 4 \\ \hline 4 & 5 & \\ \hline \end{array}$$

corresponding to $v = (3, 3, 2)$. By using the Jeu de Taquin procedure, we get for the associated generalized tableaux of shape $(3, 2, 3)$ and $(2, 3, 3)$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 4 & 4 \\ \hline 4 & & 5 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 4 & 3 & 4 \\ \hline & 4 & 5 \\ \hline \end{array}$$

which gives

$$T^L = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 3 & 3 & 4 \\ \hline 4 & 4 & \\ \hline \end{array} \quad \text{and} \quad T^R = \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 4 & 4 & 4 \\ \hline 5 & 5 & \\ \hline \end{array}.$$

4.2. Other finite types.

4.2.1. Classical types. Thanks to Corollary 3.7 the computation of the Keys in types B_n, C_n and D_n becomes very closed to that in type A_n . We shall describe it for type C_n and let to the reader its adaptation to types B_n and D_n . For a review on the combinatorics of crystals in classical types we refer to [28]. There exists a convenient notion of symplectic tableaux compatible with crystal basis theory [23]. In type C_n , the dominant weights can be identified with partitions exactly as in type A_n .

A tableau $T = C_1 \cdots C_l$ of type C_n and shape a partition λ is a filling of the Young diagram λ by letters of $\{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\}$ such that each column C_l is admissible and the split form of T is semistandard. A column C is admissible when it can be split in a pair (lC, rC) of columns contained no pair of letters (z, \bar{z})

with $z \in \{1, \dots, n\}$ by using the following procedure. Let $I = \{z_1 > \dots > z_r\}$ the set of unbarred letters z such that the pair (z, \bar{z}) occurs in C . The column C can be split when there exists (see the example below) a set $J = \{t_1 > \dots > t_r\} \subset \{1, \dots, n\}$ such that:

- t_1 is the greatest letter of $\{1, \dots, n\}$ satisfying: $t_1 < z_1, t_1 \notin C$ and $\bar{t}_1 \notin C$,
- for $i = 2, \dots, r$, t_i is the greatest letter of $\{1, \dots, n\}$ satisfying: $t_i < \min(t_{i-1}, z_i), t_i \notin C$ and $\bar{t}_i \notin C$.

In this case write:

rC for the column obtained by changing in C , \bar{z}_i into \bar{t}_i for each letter $z_i \in I$ and by reordering if necessary, lC for the column obtained by changing in C , z_i into t_i for each letter $z_i \in I$ and by reordering if necessary. Admissible columns with i boxes label the vertices of $B(\omega_i)$. Moreover for any C in $B(\omega_i)$, we have $C^L = lC$ and $C^R = rC$, that is the previous procedure give the left and right Keys of a column.

Now T is a tableau of type C_n when its split form $\text{spl}(T) = lC_1 rC_1 \dots lC_l rC_l$ is semistandard.

As in type A , one associates to the sequence $\mathcal{S} = (\omega_{i_1}, \dots, \omega_{i_l})$ of dominant weights such that $i_1 \geq \dots \geq i_l$ the crystal $B_{\mathcal{S}}(\lambda)$ with $\lambda = \omega_{i_1} + \dots + \omega_{i_l}$. Its vertices then coincide with the tableaux of type C_n and shape λ . The R -matrix $B_{(\omega_i, \omega_j)}(\omega_i + \omega_j) \rightarrow B_{(\omega_j, \omega_i)}(\omega_i + \omega_j)$ can be computed by using Sheats symplectic Jeu de Taquin (which does not coincide with the restriction of the usual Jeu de Taquin on symplectic tableaux) or the bumping procedure on symplectic tableaux. The vertices in $O_{\mathcal{S}}(\lambda)$ are the tableaux of type C_n of the form $T = C_1 \dots C_l$ where $C_l \subset \dots \subset C_1$ and no pair of letters (z, \bar{z}) in each column C_k . As in type A , for $T = w \cdot T(\lambda)$ and $T = w' \cdot T(\lambda)$ and $(w, w') \in (W^\lambda)^2$, we have $w \leq w'$ if and only if $C_k C'_k$ is a semistandard tableau for any $k = 1, \dots, l$.

Example 4.2. Let us assume $n = 4$ and compute the right Key of the tableau

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 4 & \bar{4} \\ \hline \bar{4} & \\ \hline \end{array} \quad \text{with } \text{spl}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 & \bar{4} & \bar{3} \\ \hline \bar{4} & \bar{3} & & \\ \hline \end{array}$$

We have

$$R_{(\omega_4, \omega_3)}(T) = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 & \bar{4} \\ \hline \bar{3} & \\ \hline \end{array} \quad \text{with } \text{spl}(R_{(\omega_4, \omega_3)}(T)) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 4 \\ \hline 3 & 3 & \bar{4} & \bar{3} \\ \hline \bar{3} & \bar{1} & & \\ \hline \end{array}$$

which gives

$$T^L = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \bar{4} & \\ \hline \end{array} \quad \text{and } T^R = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 4 & 4 \\ \hline \bar{3} & \bar{3} \\ \hline \bar{1} & \\ \hline \end{array}$$

4.2.2. Exceptional types. For exceptional types, the Key in fundamental crystals can yet be computed from the dilatation maps K_m defined in (6) with $m \leq 4$ (see [32]). There is also relevant notions of tableaux (see [5] and the references therein). Nevertheless, the combinatorial R -matrices for fundamental crystals, the orbit of the highest weight vertex in the crystals and the strong Bruhat order on this orbit become more complex to compute beyond type G_2 (for which the model remains simple and there is a bumping algorithm (see [28])).

5. DETERMINATION OF THE DEMAZURE CRYSTALS BY KEYS IN AFFINE TYPE A

In this section we assume $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$ is the affine Lie algebra of type $A_{e-1}^{(1)}$. A sequence $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ is called a multicharge. It defines the dominant weight $\Lambda_{\mathbf{s}} = \sum_{i=1}^l \omega_{s_i \bmod e}$ of level l where $\omega_0, \dots, \omega_{e-1}$ are the fundamental weights of $\widehat{\mathfrak{sl}}_e$.

5.1. The level 1. We now recall a convenient realization of the crystals $B(\omega_i), i = 0, \dots, e-1$ by abaci. Recall that a partition is a nonincreasing sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$ of nonnegative integers. One can assume this sequence is infinite by adding parts equal to zero. Each partition will be identified with its Young diagram. To each box (also called node) b of a partition λ one associates its content $c(b) = v - u$ where u and v are such that b belongs to the u -th row and the v -th column of λ , respectively. A partition is completely determined by its beta numbers. These are the contents of its extended rim obtained by adding one box to the right end of each row. The removable nodes of λ are the nodes located at the ends of its rows which yet yield a partition when

they are removed from λ . The addable nodes of λ are the nodes in its extended rim which yield a partition when they are added to λ .

Now fix $s \in \mathbb{Z}$. The symbol of λ , denoted by $S_s(\lambda)$ is the list of its beta numbers translated by s . Alternatively, one can consider the abacus $L_s(\lambda)$ which is obtained by decorating \mathbb{Z} with black and white beads such that the black beads corresponds to the integers in $S_s(\lambda)$. Since λ is assumed to have an infinite number of zero parts, both $S_s(\lambda)$ and $L_s(\lambda)$ are infinite. Nevertheless, only the nonzero parts of λ are relevant which is easy to make apparent when $S_s(\lambda)$ and $L_s(\lambda)$ are pictured (see the following example). An addable node in λ corresponds in $L_s(\lambda)$ to a black bead with a white bead at its right whereas a removable node corresponds to a black bead with a white bead at its left.

The set of symbols can be endowed with the structure of type $A_{e-1}^{(1)}$ -crystal. For any $i \in \{0, \dots, e-1\}$, the i -nodes of λ are those of content $x = i \bmod e$. Let w_i be the word on the alphabet $\{A, R\}$ obtained by reading from right to left the entries x of $S_s(\lambda)$ such that $x = i \bmod e$ or $x = i + 1 \bmod e$ corresponding to addable or removable i -nodes in λ . We shall say that w_i is the $\{A, R\}$ -word of λ . Delete recursively each factor RA until obtain a reduced word of the form $\tilde{w}_i = A^a R^r$. Then $f_i(S_s(\lambda))$ is obtained by changing in $S_s(\lambda)$ the rightmost entry x appearing in \tilde{w}_i into $x + 1$ if $a > 0$ and is zero otherwise. Is is easy to check that $S_s(\emptyset)$ is then a source vertex. In fact the connected component $B_s(\emptyset)$ of $S_s(\emptyset)$ is isomorphic to the $\widehat{\mathfrak{sl}}_e$ -crystal $B(\omega_{s \bmod e})$ (see for example [10] Chapter 6 and the references therein). Also one can prove that the vertices in $B_s(\emptyset)$ are the symbols $S_s(\lambda)$ corresponding to e -regular partitions, that is to partitions with no part repeated strictly more than $e - 1$ times. Alternatively, λ is e -regular if there is no sequence of e black beads in $L_s(\lambda)$.

Observe that when e tend to infinity, the previous construction yields the crystal $B_s^\infty(S_s(\emptyset))$ which is isomorphic to the \mathfrak{sl}_∞ -crystal with highest weight the s -th fundamental weight. Also, up to rotation, the symbol $S_s(\lambda)$ is nothing but the half-infinite column semistandard tableau on \mathbb{Z} which is the natural type A_∞ -extension of the finite columns used in §4.1.

Example 5.1. Consider the 3-regular partition $\lambda = (5, 3, 3, 2)$. Its beta numbers are easily deduced from its Young diagram

$$\lambda = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \begin{array}{l} 5 \\ 2 \\ 1 \\ -1 \\ -4 \\ -5 \end{array}$$

and we get

$$S_0(\lambda) = \cdots -5 \quad -4 \quad -1 \quad 1 \quad 2 \quad 5$$

The abacus $L_0(\lambda)$ is :



Recall that the hook length $h(b)$ of a node b in the partition λ (i.e. a box in its Young diagram) is the number of nodes located to the right or below b (weakly speaking, thus b contributes to $h(b)$). A partition λ is called a e -core if it does not contain a node with hook length e . There are alternative characterizations of the e -core (see for example [25]) of the partition λ .

Proposition 5.2. The following assertions are equivalent:

- (1) λ is a e -core,
- (2) λ does not contains any node with hook length e ,
- (3) for any $i = 0, \dots, e - 1$, w_i contains only nodes A or only nodes R ,
- (4) for any x in $S_e(\lambda)$, $x - e$ also belongs to $S_e(\lambda)$,
- (5) we have $L_t(\lambda) \subset L_{t+e}$ for any $t \in \mathbb{Z}$.

Given two partitions λ and μ , we write $\lambda \subseteq \mu$ when the Young diagram of λ is contained in that of μ . This defines the inclusion order on partitions.

Corollary 5.3. The orbit $O_s(\emptyset)$ of \emptyset in $B_s(\emptyset)$ under the action of the Weyl group W contains exactly the e -cores. Moreover, under this correspondence, the strong Bruhat order on W^{ω_s} coincides with the inclusion order on partitions.

In the following paragraph, we will see how generalize these two last results in highest level. Now let us recall a combinatorial procedure described in [1] yielding the right Key $K_s^R(\lambda)$ of $S_s(\lambda)$ in $B_s(\emptyset)$.³ First set $U(S(\lambda)) = \{x \in S(\lambda) \mid x - e \notin S(\lambda)\}$. Then $K_s^R(\lambda)$ can be computed by the following algorithm:

- (1) If $U(S(\lambda)) = \emptyset$, then $K_s^R(\lambda) = S_s(\lambda)$
- (2) Else let $p = \max\{x \in S(\lambda) \mid x - e \notin S(\lambda)\}$ and $q = \min\{x > p \mid x \notin S(\lambda), x - e \in S(\lambda), x \not\equiv p \pmod{e}\}$.
Replace $S(\lambda)$ by $S(\lambda) \setminus \{p\} \cup \{q\}$ and return to step 1.

Observe the algorithm is well-defined for $\{x > p \mid x \notin S(\lambda), x - e \in S(\lambda), x \not\equiv p \pmod{e}\}$ is not empty. Also it terminates since the cardinality of $U(S(\lambda))$ decreases after sufficiently iterations.

Example 5.4. Assume $e = 3$ and

$$S_0(\lambda) = \cdots \quad -5 \quad -4 \quad -1 \quad 1 \quad 2 \quad 5$$

Then we get $p = 1$ and $q = 8$. Only one iteration is needed and this gives

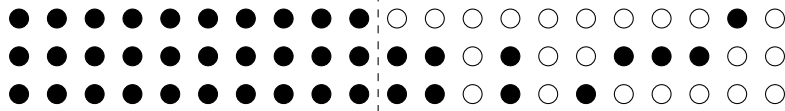
$$K_s^R(\lambda) = \cdots \quad -5 \quad -4 \quad -1 \quad 2 \quad 5 \quad 8$$

which is the symbol of the 3-core $\mu = (8, 6, 4, 2)$.

5.2. Higher level.

5.2.1. *Uglov realization.* Let $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ be an arbitrary multicharge. We now recall Uglov's realization of the crystal $B(\Lambda_{\mathbf{s}})$. Its is quite similar to the level 1 case except one has to consider l -partitions $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^l)$ (i.e. sequences of partitions of length l) instead of partitions. To the l -partition $\boldsymbol{\lambda}$ is associated its symbol which is the sequence $S_{\mathbf{s}}(\boldsymbol{\lambda}) = (S_{s_l}(\lambda^l), \dots, S_{s_1}(\lambda^1))$ of the the symbols associated to each pair (s_k, λ^k) . The abacus $L_{\mathbf{s}}(\boldsymbol{\lambda}) = (L_{s_l}(\lambda^l), \dots, L_{s_1}(\lambda^1))$ is defined similarly.

Example 5.5. The abacus of the 3-partition $(1.1, 2.2.2.1, 9)$ with $\mathbf{s} = (4, 6, 1)$ is



The set of symbols so obtained is also endowed with the structures of $\widehat{\mathfrak{sl}}_{\infty}$ and $\widehat{\mathfrak{sl}}_e$ -crystals of level l . Nevertheless the $\widehat{\mathfrak{sl}}_e$ -crystal structure is not a tensor product of level 1 crystals when $l > 1$. Thus, we cannot apply directly to the results of Section 3. We shall see in § 5.6 that there is another (closed) construction of level l $\widehat{\mathfrak{sl}}_e$ -crystals (called the Kleshchev realization) which is by definition a tensor product of level 1 affine crystals. We shall consider both in the sequel notably because Uglov's version is easier to connect to the combinatorics of non affine type A and the two versions are of common use in the literature.

First of all, to get the $\widehat{\mathfrak{sl}}_{\infty}$ -structure, consider $j \in \mathbb{Z}$ and W_j the word on the alphabet $\{A, R\}$ obtained by reading from right to left and successively in $L_{s_l}(\lambda^l), \dots, L_{s_1}(\lambda^1)$, the entries j or $j + 1$ corresponding to addable or removable nodes. Delete recursively each factor RA in W_j until get a reduced word of the form $\widetilde{W}_j = A^a R^r$. Then $\widetilde{f}_j(S_{\mathbf{s}}(\boldsymbol{\lambda}))$ is obtained by changing in $S_{\mathbf{s}}(\boldsymbol{\lambda})$ the rightmost j appearing in \widetilde{W}_j into $j + 1$ if $a > 0$ and is zero otherwise. It is easy to check that $S_{\mathbf{s}}(\emptyset)$ is then a source vertex of highest weight $\Lambda_{\mathbf{s}}^{\infty}$, thus its associated connected component $B^{\infty}(S_{\mathbf{s}}(\emptyset))$ is isomorphic to $B(\Lambda_{\mathbf{s}}^{\infty})$.

Now, to define the $\widehat{\mathfrak{sl}}_e$ -structure, consider $i \in \{0, \dots, e - 1\}$ and the word

$$(10) \quad w_i = \prod_{p=-\infty}^{+\infty} W_{i+pe}.$$

Define $\widetilde{w}_i = A^a R^r$ from w_i as previously by recursive deletion of the factors RA . The nodes surviving in \widetilde{w}_i are the normal i -nodes. Then $\widetilde{f}_i(S_{\mathbf{s}}(\boldsymbol{\lambda}))$ is obtained by changing in $S_{\mathbf{s}}(\boldsymbol{\lambda})$ the entry x appearing in \widetilde{w}_i corresponding to the rightmost (normal) node into $x + 1$ if $a > 0$ and is zero otherwise. The symbol $S_{\mathbf{s}}(\emptyset)$ becomes a source vertex of highest weight $\Lambda_{\mathbf{s}}$ and the associated connected component $B(S_{\mathbf{s}}(\emptyset))$ is isomorphic to $B(\Lambda_{\mathbf{s}})$. Observe that both crystal structures $B(S_{\mathbf{s}}(\emptyset))$ and $B^{\infty}(S_{\mathbf{s}}(\emptyset))$ are compatible: we have $B(S_{\mathbf{s}}(\emptyset)) \subset B^{\infty}(S_{\mathbf{s}}(\emptyset))$ (i.e.

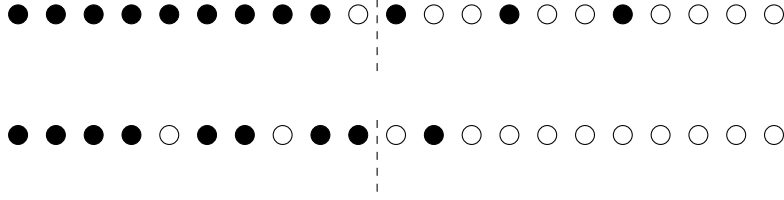
³There is a similar procedure for computing the left key $K_s^L(\lambda)$ also described in [1]. Thus our foregoing results can also be used to compute the left Key in arbitrary level.

an inclusion of the sets of vertices) and each arrow $S_s(\lambda) \xrightarrow{i} S_s(\mu)$ in $B(S_s(\emptyset))$ is an arrow $S_s(\lambda) \xrightarrow{j} S_s(\mu)$ in $B^\infty(S_s(\emptyset))$ with $j = i \bmod e$ where W_j is the factor of w_i modified in (10) when \tilde{f}_i is applied to $S_s(\lambda)$.

5.3. Orbit of the highest weight vertex. Let $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$ be an arbitrary multicharge and $e \in \mathbb{Z}^{>0}$. We now give a characterization of the l -partitions in the orbit $O(\mathbf{s}, e)$ of $S_s(\emptyset)$ modulo the action of the affine Weyl group similar to Corollary 5.3. To an l -partition λ , we attach its abacus (which depends on \mathbf{s}). Recall that $L_s(\lambda) = (L_{s_1}(\lambda^1), \dots, L_{s_l}(\lambda^l))$ is the abacus of λ . For two runners $L_s(\lambda)$ and $L_t(\mu)$ in one abacus, write $L_s(\lambda) \subset L_t(\mu)$ when for each black bead in the runner $L_s(\lambda)$, there is a black bead at the same position in the runner $L_t(\mu)$. Alternatively, let $S_s(\lambda)$ and $S_t(\mu)$ be the symbols corresponding to these two runners. We have $L_s(\lambda) \subset L_t(\mu)$ if and only if $S_s(\lambda)$ is contained in $S_t(\mu)$. Note that for all $k \in \mathbb{Z}$, we have $L_s(\lambda) \subset L_t(\mu)$ if and only if $L_{s+k}(\lambda) \subset L_{t+k}(\mu)$. By a slight abuse of notation, we shall say that x is in the abacus $L_s(\lambda)$ if and only if there is a black bead in position x in one of its runner (equivalently, x appear in a row of $S_s(\lambda)$).

For any partition μ and any $s \in \mathbb{Z}$, the abacus $L_{-s}(\mu^t)$ of the transpose partition μ^t is obtained from $L_s(\mu)$ by switching the black and white beads and performing a mirror image. Observe also that λ is an e -core if and only if λ^t is. This is easy to see on the abacus where it suffices to check that for all black bead in position x , there is a black bead in position $x - e$.

Example 5.6. Compare below the abaci of the partitions $(5, 3, 1)$ with charge 2 and (3.2.2.1.1) with charge -2 .



Assume $l > 1$ and $\mathbf{s} = (s_1, \dots, s_l)$. For any $1 \leq a < b < l$, let s'_a and s'_b be integers such that $s_a = s'_a + p_a e$, $s_b = s'_b + p_b e$ with $(p_a, p_b) \in \mathbb{Z}^2$ and $0 \leq s'_a - s'_b < e$. When $l = 1$, we set $s'_1 = s_1$ for completeness. Note that s'_a and s'_b are in fact defined modulo translation by the same multiple of e (one can see that that such a translation do not affect the definition below).

Definition 5.7. We say that the l -partition λ is a (e, \mathbf{s}) -core if it satisfies on of the following properties:

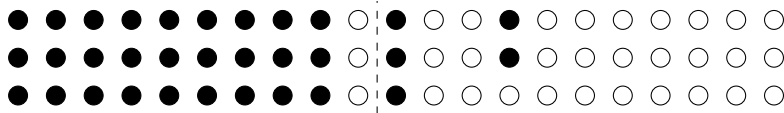
- (1) $l = 1$ and $L_{s'_1}(\lambda^1) \subset L_{s'_1+e}(\lambda^1)$
- (2) $l > 1$ and $L_{s'_a}(\lambda^a) \subset L_{s'_b}(\lambda^b) \subset L_{s'_a+e}(\lambda^a)$ for any $1 \leq a < b < l$.

We denote by $\mathfrak{L}(e, \mathbf{s})$ the set of all (e, \mathbf{s}) -cores.

Remark 5.8.

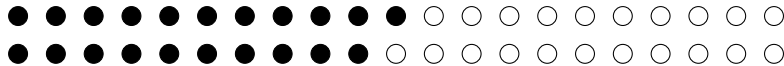
- (1) The condition $L_{s'_a}(\lambda^a) \subset L_{s'_a+e}(\lambda^b)$ means that for each x in $L_{s'_a}(\lambda^a)$, $x - e$ also belongs to $L_{s'_a}(\lambda^a)$ (since $x \in L_{s'_a+e}(\lambda^a)$). Thus, in the (e, \mathbf{s}) -core λ , each λ^a is a core.
- (2) When $l > 1$ and $\mathbf{s} = (s_1, \dots, s_l)$ is such that $0 \leq s_1 \leq \dots \leq s_l < e$, then λ is a (e, \mathbf{s}) -core if and only if for any $1 \leq a \leq l$, λ^a is a e -core and for any $1 \leq a < l - 1$, $L_{s_a}(\lambda^a) \subset L_{s_{a+1}}(\lambda^b) \subset L_{s_a+e}(\lambda^a)$.

Example 5.9. Assume $e = 3$ and $\mathbf{s} = (0, 1, 2)$, then the 3-partition $(1, 3, 1, 3, 1)$ is in $\mathfrak{L}(e, \mathbf{s})$.

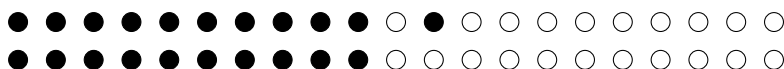


Example 5.10. Take $e = 3$ and $\mathbf{s} = (0, 1)$. The following are the 2-partitions of rank less than 3 in $\mathfrak{L}(a, \mathbf{s})$:

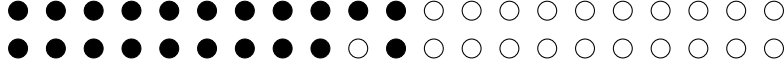
The empty bipartition (\emptyset, \emptyset) , with abacus:



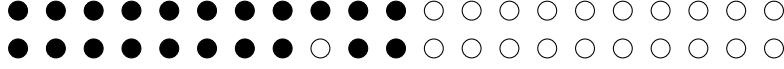
The bipartition $(\emptyset, 1)$, with abacus:



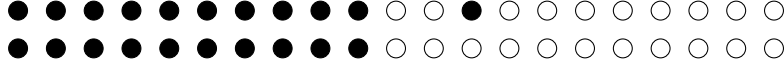
The bipartition $(1, \emptyset)$, with abacus:



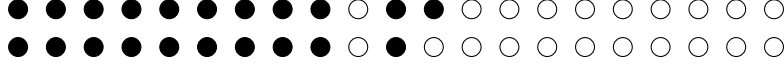
The bipartition $(1.1, \emptyset)$, with abacus:



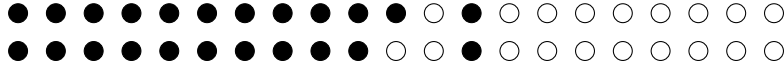
The bipartition $(\emptyset, 2)$, with abacus:



The bipartition $(1, 1.1)$, with abacus:



The bipartition $(2, 1)$, with abacus:



Given two multicharges \mathbf{s} and \mathbf{s}' in \mathbb{Z}^l , we have $\Lambda_{\mathbf{s}} = \Lambda_{\mathbf{s}'}$ if and only if $\mathbf{s} \bmod e$ and $\mathbf{s}' \bmod e$ coincide up to permutation of their components. Then the crystals $B(S_{\mathbf{s}}(\emptyset))$ and $B(S_{\mathbf{s}'}(\emptyset))$ are isomorphic. In [17], we establish that the associated isomorphism $\Phi_{\mathbf{s} \rightarrow \mathbf{s}'}$ can always be obtained by composing two types of elementary isomorphisms. The first one is denoted by $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e$ and corresponds to the permutation of s_k and s_{k+1} in \mathbf{s} . It is just the restriction to the affine crystals of the combinatorial R-matrix $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e$ (which can be computed by Jeu de Taquin operations). The second one is denoted $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e$ where $\tau \cdot \mathbf{s} = (s_2, \dots, s_l, s_1 + e)$ and sends the symbol $S_{\mathbf{s}}(\boldsymbol{\lambda})$ on $S_{\tau \cdot \mathbf{s}}(\boldsymbol{\mu})$ where $\boldsymbol{\mu} = (\lambda^2, \dots, \lambda^l, \lambda^1)$. Let us first prove our set $\mathfrak{L}(e, \mathbf{s})$ is stable by these isomorphisms.

Lemma 5.11. *Let $\mathbf{s} \in \mathbb{Z}^l$ and $k \in \{1, \dots, l-2\}$ be such that $s_k \leq s_{k+1}$ then*

(1) *We have $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\mathfrak{L}(e, \mathbf{s})) = \mathfrak{L}(e, \mathbf{s})$ and for any (e, \mathbf{s}) -core $\boldsymbol{\lambda}$, we get*

$$\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^1, \dots, \lambda^{k+1}, \lambda^k, \dots, \lambda^l).$$

(2) *We have $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\mathfrak{L}(e, \mathbf{s})) = \mathfrak{L}(e, \mathbf{s})$ and for any (e, \mathbf{s}) -core $\boldsymbol{\lambda}$, we get*

$$\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^2, \dots, \lambda^l, \lambda^1).$$

Proof. Let $k \in \{1, \dots, l-2\}$ be such that $s_k \leq s_{k+1}$. We first show that if $\boldsymbol{\lambda} \in \mathfrak{L}(e, \mathbf{s})$, we have $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^1, \dots, \lambda^{k+1}, \lambda^k, \dots, \lambda^l)$. Since the computation of $\Phi_{\mathbf{s} \rightarrow \sigma_k \mathbf{s}}^e$ reduces to Jeu de Taquin, the equality $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^1, \dots, \lambda^{k+1}, \lambda^k, \dots, \lambda^l)$ is equivalent to the condition $L_{s_k}(\lambda^k) \subset L_{s_{k+1}}(\lambda^{k+1})$. Let $(s'_k, s'_{k+1}) \in \mathbb{Z}^2$ and $(p_k, p_{k+1}) \in \mathbb{Z}^2$ be such that $s_k = s'_k + p_k e$ and $s_{k+1} = s'_{k+1} + p_{k+1} e$ and $0 \leq s'_{k+1} - s'_k < e$. By definition we have $L_{s'_k}(\lambda^k) \subset L_{s'_{k+1}}(\lambda^{k+1}) \subset L_{s'_k + e}(\lambda^k)$. As $s_k \leq s_{k+1}$, we must have $p_k \leq p_{k+1}$. By hypothesis, we have $L_{s'_k + p_k e}(\lambda^k) \subset L_{s'_{k+1} + p_k e}(\lambda^{k+1})$. This gives $L_{s'_k + p_k e}(\lambda^k) \subset L_{s'_{k+1} + p_k e}(\lambda^{k+1}) \subset L_{s'_{k+1} + p_{k+1} e}(\lambda^{k+1})$ because λ^{k+1} is an e -core (see Assertion 5 of Proposition 5.2). Thus $L_{s_k}(\lambda^k) \subset L_{s_{k+1}}(\lambda^{k+1})$ as desired. We now show that $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\boldsymbol{\lambda})$ is a $(e, (k, k+1) \cdot \mathbf{s})$ -core. So consider $(h_k, h_{k+1}) \in \mathbb{Z}^2$ such that $s_{k+1} = s''_{k+1} + h'_k e$ and $s_k = s''_{k+1} + h_{k+1} e$ where $0 \leq s''_{k+1} - s''_k < e$. Keeping the above notation, we have that $s''_k = s'_{k+1}$ and $s''_{k+1} = s'_k + e$ (up to a translation by a the same integer). On the one hand, we have $L_{s'_{k+1}}(\lambda^{k+1}) \subset L_{s'_k + e}(\lambda^k)$ and thus $L_{s''_k}(\lambda^{k+1}) \subset L_{s''_{k+1}}(\lambda^k)$. On the second hand, we have $L_{s'_k}(\lambda^k) \subset L_{s'_{k+1}}(\lambda^{k+1})$ and thus $L_{s''_{k+1} - e}(\lambda^k) \subset L_{s''_k}(\lambda^{k+1})$ or equivalently $L_{s''_{k+1}}(\lambda^k) \subset L_{s''_k + e}(\lambda^{k+1})$. Finally, we get the inclusion $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\mathfrak{L}(e, \mathbf{s})) \subset \mathfrak{L}(e, \mathbf{s})$ and conclude that $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e(\mathfrak{L}(e, \mathbf{s})) = \mathfrak{L}(e, \mathbf{s})$ because $\Phi_{\mathbf{s} \rightarrow (k, k+1) \cdot \mathbf{s}}^e$ is a crystal isomorphism. This proves our first assertion.

For the second one, we use that $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^2, \dots, \lambda^l, \lambda^1)$. We just need to show that this is a $(e, \tau \cdot \mathbf{s})$ -core. To do this, take $k \in \{2, \dots, l\}$. Let $(s'_1, s'_k) \in \mathbb{Z}^2$ and $(p_1, p_k) \in \mathbb{Z}^2$ be such that $s_1 = s'_1 + p_1 e$ and $s_k = s'_k + p_k e$

and $0 \leq s'_k - s'_1 < e$. Then by definition we have $L_{s'_1}(\lambda^1) \subset L_{s'_k}(\lambda^k) \subset L_{s'_1+e}(\lambda^1)$. Also $s_1 + e = s'_1 + e + p_1e$ and $s_k = s'_k + p_k e$ with $0 \leq (s'_1 + e) - s'_k < e$. Thus it suffices to see that $L_{s'_k}(\lambda^k) \subset L_{s'_1+e}(\lambda^1) \subset L_{s'_k+e}(\lambda^k)$ which is clear from the above property. Again, we obtain that $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\mathfrak{L}(e, \mathbf{s})) = \mathfrak{L}(e, \mathbf{s})$. \square

Lemma 5.12. *For any $\mathbf{s} \in \mathbb{Z}^l$ the empty l -partition is a (e, \mathbf{s}) -core.*

Proof. Assume that for $\mathbf{s} \in \mathbb{Z}^l$ and $a = 1, \dots, l-1$, $b > a$ we have $0 \leq s_b - s_a < e$, then it is clear that $L_{s_a}(\emptyset) \subset L_{s_b}(\emptyset) \subset L_{s_a+e}(\emptyset)$.

Now let us consider any $\mathbf{s} \in \mathbb{Z}^l$. Let $a = 1, \dots, l-1$ and let $b > a$. Write as in the definition, $s_a = s'_a + p_a e$ and $s_b = s'_b + p_b e$ with $(p_a, p_b) \in \mathbb{Z}^2$ such that $0 \leq s'_b - s'_a < e$. Then we have that $L_{s'_a}(\emptyset) \subset L_{s'_b}(\emptyset) \subset L_{s'_b+e}(\emptyset)$ by the previous case. The result follows. \square

Lemma 5.13. *Let $\mathbf{s} \in \mathbb{Z}^l$ and assume that $\boldsymbol{\lambda}$ is a (e, \mathbf{s}) -core. Then $\boldsymbol{\lambda}^t := ((\lambda^1)^t, \dots, (\lambda^l)^t)$ is a $(e, (-s_l, \dots, -s_1))$ -core.*

Proof. Assume that for any $a = 1, \dots, l-1$ and $b > a$ we have $0 \leq s_b - s_a < e$. Then for the multicharge $\mathbf{t} := (-s_l, \dots, -s_1)$ we also have for any $a = 1, \dots, l-1$ and $b > a$ the inequalities $0 \leq t_b - t_a < e$. Our result then follows from the interpretation of the transposition on abaci (see Example 5.6). \square

We can now describe the orbit $O(\mathbf{s}, e)$ in the Uglov realization of the crystal $B(\Lambda_{\mathbf{s}})$.

Proposition 5.14. *Let $\mathbf{s} \in \mathbb{Z}^l$. The l -partition $\boldsymbol{\lambda}$ yields a symbol in $O(\mathbf{s}, e)$ if and only if it is a (e, \mathbf{s}) -core. In particular, when $l > 1$ and $0 \leq s_1 \leq \dots \leq s_l < e$, the symbols in $O(\mathbf{s}, e)$ are exactly those such that $L_{s_a}(\lambda^a) \subset L_{s_{a+1}}(\lambda^b) \subset L_{s_a+e}(\lambda^a)$ for any $1 \leq a < l$.*

We shall first need the following lemma.

Lemma 5.15. *Let $\mathbf{s} \in \mathbb{Z}^l$ and $\boldsymbol{\lambda}$ be a (e, \mathbf{s}) -core. Assume that we have a removable j -node at the position x of the abacus $S_{\mathbf{s}}(\boldsymbol{\lambda})$ (thus $x \equiv j \pmod{e}$). Then there is no addable j -node in $\boldsymbol{\lambda}$.*

Proof. Assume first $l = 1$. Since we have a removable node at position x , x lies in the abacus $S_{\mathbf{s}}(\boldsymbol{\lambda})$ but not $x - 1$. As a consequence, for all $a \in \mathbb{Z}_{>0}$, $x - 1 + ae$ does not belong to $S_{\mathbf{s}}(\boldsymbol{\lambda})$ which thus has no addable node greater than x . In addition, for all $a \in \mathbb{Z}_{>0}$, the node $x - a.e$ is in the abacus and this implies that there is no addable node in the abacus.

Now, assume $l > 1$ and x is a removable j -node on k -th runner of $\boldsymbol{\lambda}$. When $k \neq 1$, we have $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\boldsymbol{\lambda}) = (\lambda^2, \dots, \lambda^l, \lambda^1)$ which is a $(e, \tau \cdot \mathbf{s})$ -core by Lemma 5.11. Therefore, x is a removable j -node in the $k-1$ -runner of $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\boldsymbol{\lambda})$. Clearly, $\Phi_{\mathbf{s} \rightarrow \tau \cdot \mathbf{s}}^e(\boldsymbol{\lambda})$ has no addable j -node if and only if this holds in $\boldsymbol{\lambda}$. By repeating this argument we can restrict the proof to the case $k = 1$.

First, by the same arguments as in the case $l = 1$, there is no addable j -node in $L_{s'_1}(\mu^1)$. The condition $L_{s'_1}(\mu^1) \subset L_{s'_b}(\lambda^b)$ for any $b = 2, \dots, l$ implies that the j -node x belongs to the b -th runner of $L_{\mathbf{s}}(\boldsymbol{\lambda})$. If it is removable, we get as in the case $l = 1$ that there is no addable node in $L_{s'_b}(\lambda^b)$ since λ^b is an e -core. If not, there is no addable node in $L_{s'_b}(\lambda^b)$ which is less than x because all the positions $x - a.e$ and $x - 1 - a.e$ with $a > 0$ are occupied. Also the condition $L_{s_b}(\lambda^b) \subset L_{s_1+e}(\lambda^1)$ implies that there is no node $x - 1 + e$ in $L_{s'_b}(\lambda^b)$. Otherwise $x - 1$ would belong to $L_{s'_1}(\lambda^1)$ and x could not be removable in $L_{s'_1}(\lambda^1)$. Thus, there is also no addable j -node greater than x in $L_{s'_b}(\lambda^b)$ for any $b = 2, \dots, l$. Finally we have showed there is no addable j -node in the runners $L_{s'_a}(\lambda^a)$, $a = 1, \dots, l$. Since $s_a = s'_a \pmod{e}$ for any $a = 1, \dots, l$, this is also true for the runners $L_{s_a}(\lambda^a)$, $a = 1, \dots, l$. \square

Proof of Proposition 5.14. Let us prove first the inclusion $\mathfrak{L}(e, \mathbf{s}) \subset O(\mathbf{s}, e)$. Consider $\boldsymbol{\lambda} \in \mathfrak{L}(e, \mathbf{s})$, we show that $\boldsymbol{\lambda}$ is in $O(\mathbf{s}, e)$ by induction on the rank n of $\boldsymbol{\lambda}$. For $n = 0$, the result is true by Lemma 5.12.

Assume that $n > 0$. Then $\boldsymbol{\lambda}$ is non empty and there exists a removable node for $\boldsymbol{\lambda}$. Let j be its residue. As $\boldsymbol{\lambda}$ has no addable j -node by Lemma 5.15, all the removable nodes with residue j are normal nodes. If we remove them, the resulting l -partition $\boldsymbol{\mu}$ is clearly in $\mathfrak{L}(e, \mathbf{s})$ and thus also in $O(\mathbf{s}, e)$ by induction. Then, adding to $\boldsymbol{\mu}$ all the normal addable j -nodes gives $\boldsymbol{\lambda}$ which is thus in $O(\mathbf{s}, e)$.

To prove the inclusion $\mathfrak{L}(e, \mathbf{s}) \supset O(\mathbf{s}, e)$, consider $\boldsymbol{\lambda}$ such that $S_{\mathbf{s}}(\boldsymbol{\lambda}) \in O(\mathbf{s}, e)$. By (5), there exists at least an integer $j \in \{1, \dots, e-1\}$ such that $\boldsymbol{\lambda}$ contains only removable j -nodes. Then $S_{\mathbf{s}}(\boldsymbol{\mu}) = s_j \cdot S_{\mathbf{s}}(\boldsymbol{\lambda}) \in O(\mathbf{s}, e)$ and by the induction hypothesis, $\boldsymbol{\mu} \in \mathfrak{L}(e, \mathbf{s})$. Then adding to $\boldsymbol{\mu}$ all its addable j -nodes gives the l -partition $\boldsymbol{\lambda}$ in $\mathfrak{L}(e, \mathbf{s})$. \square

Example 5.16. Let us resume Example 5.10. Denote by $\{s_0, s_1, s_2\}$ the simple reflections of the affine Weyl group \widehat{W}_3 . We have:

$$\begin{aligned} S_{(0,1)}(\emptyset, 1) &= s_1 S_{(0,1)}(\emptyset, \emptyset), \quad S_{(0,1)}(1, \emptyset) = s_0 S_{(0,1)}(\emptyset, \emptyset), \quad S_{(0,1)}(1.1, \emptyset) = s_2 \cdot s_0 S_{(0,1)}(\emptyset, \emptyset) \\ S_{(0,1)}(\emptyset, 2) &= s_2 s_1 S_{(0,1)}(\emptyset, \emptyset), \quad S_{(0,1)}(1, 1.1) = s_0 s_1 S_{(0,1)}(\emptyset, \emptyset), \quad S_{(0,1)}(2, 1) = s_1 s_0 S_{(0,1)}(\emptyset, \emptyset). \end{aligned}$$

5.4. More on (e, \mathbf{s}) -cores. We here point an interesting property of the set of (e, \mathbf{s}) -cores. Assume that \mathbf{s} is an arbitrary multicharge. Following [8], for any l -partition λ and for each pairs of integers $(i, j) \in \{0, \dots, e-1\} \times \{1, \dots, l\}$ set

$$b_{i,j}^{\mathbf{s}}(\lambda) := \max(\beta \in S_{s_j}(\lambda^j) | \beta \equiv i \pmod{e}).$$

Now let $\tilde{\mathbf{s}} = (\tilde{s}_1, \dots, \tilde{s}_l) \in \{0, 1, \dots, e-1\}^l$ be such that $\mathbf{s} \equiv \tilde{\mathbf{s}} \pmod{e}$.

Proposition 5.17. For any $\lambda \in O(\mathbf{s}, e)$ and any $i \in \{0, 1, \dots, e-1\}$ there exists an integer α_i such that $b_{i,j}^{\tilde{\mathbf{s}}}(\lambda) \in \{\gamma_i, \gamma_i + e\}$ for all $j \in \{1, \dots, l\}$.

Proof. Fix $i \in \{0, 1, \dots, e-1\}$. To prove the proposition, it suffices to show that for all $(j, k) \in \{1, \dots, l\}^2$, we have that $|b_{i,j}^{\tilde{\mathbf{s}}}(\lambda) - b_{i,k}^{\tilde{\mathbf{s}}}(\lambda)| \in \{0, e\}$. Assume first that $\tilde{s}_j \leq \tilde{s}_k$. In this case, since $\lambda \in O(\mathbf{s}, e)$, we have $L_{\tilde{s}_j}(\lambda^j) \subset L_{\tilde{s}_k}(\lambda^k) \subset L_{\tilde{s}_j+e}(\lambda^j)$ and this implies that $b_{i,k}^{\tilde{\mathbf{s}}}(\lambda) - b_{i,j}^{\tilde{\mathbf{s}}}(\lambda) \in \{0, e\}$. If we have now $\tilde{s}_j \geq \tilde{s}_k$, we thus have $\tilde{s}_j \leq \tilde{s}_k + e$ and we get $L_{\tilde{s}_j}(\lambda^j) \subset L_{\tilde{s}_k+e}(\lambda^k) \subset L_{\tilde{s}_j+e}(\lambda^j)$. This implies now that $b_{i,j}^{\tilde{\mathbf{s}}}(\lambda) - b_{i,k}^{\tilde{\mathbf{s}}}(\lambda) \in \{0, e\}$ as desired. \square

It follows from the previous proposition and results in [8, Th. 3.1] that the (e, \mathbf{s}) -cores parametrize distinguished elements of certain remarkable blocks for Ariki-Koike algebras which may be seen as analogues of simple blocks for Iwahori-Hecke algebras. These blocks have been defined by Fayers and are known as core blocks. This interesting fact together with its consequences will be developed elsewhere.

5.5. Strong Bruhat order on Keys. Consider $S_{\mathbf{s}}(\lambda) \in O(\mathbf{s}, e)$ and $S_{\mathbf{s}}(\mu) \in O(\mathbf{s}, e)$. Let u and v be the elements in $W^{\Lambda_{\mathbf{s}}}$ such that $S_{\mathbf{s}}(\lambda) = u \cdot S_{\mathbf{s}}(\emptyset)$ and $S_{\mathbf{s}}(\mu) = v \cdot S_{\mathbf{s}}(\emptyset)$, respectively. Recall we have written \subseteq for the inclusion order on partitions. Since the higher level Uglov $\widehat{\mathfrak{sl}}_e$ -crystal structure on the set of symbols is not a tensor product of level 1 $\widehat{\mathfrak{sl}}_e$ -crystals, we cannot directly use the results of Section 3. Nevertheless, we can get the following description of the strong Bruhat order on $O(\mathbf{s}, e)$.

Proposition 5.18. With the previous notation, we have $u \trianglelefteq v$ if and only if $\lambda^k \subseteq \mu^k$ for any $k = 1, \dots, l$.

Proof. Recall that each symbol $S_{\mathbf{s}}(\nu)$ can be regarded as a vertex of the type $A_{e-1}^{(1)}$ and A_{∞} crystals $B(S_{\mathbf{s}}(\emptyset))$ and $B^{\infty}(S_{\mathbf{s}}(\emptyset))$. Now consider the finite set $S_{\mathbf{s}}^N(\nu)$ of symbols of l -partitions ν with rank less or equal to a fixed integer N . Then, there exists an integer m such the action of any simple reflection $s_i \in \widehat{W}_e$ on $S_{\mathbf{s}}^N(\nu)$ coincide with that of the permutation $\sigma_i := \prod_{-m \leq k \leq m} (i + ke, i + 1 + ke) \in S_{[-m, m]} \subset W_{\infty}$ where $S_{[-m, m]}$ is the symmetric group on the integers between $-m$ and m . More generally, the action of $w \in \widehat{W}_e$ with minimal decomposition $w = s_{i_1} \cdots s_{i_a}$, on $S_{\mathbf{s}}^N(\nu)$ will coincide with that of $\tilde{w} = \sigma_{i_1} \cdots \sigma_{i_a} \in S_{[-m, m]}$ and $\sigma_{i_1} \cdots \sigma_{i_a}$ is also a minimal decomposition of \tilde{w} . Also by the definition of the strong Bruhat order we have $u \trianglelefteq v$ in \widehat{W}_e if and only if $\tilde{u} \trianglelefteq \tilde{v}$ in $S_{[-m, m]}$. Therefore, we are reduced to the finite type A for which the strong Bruhat order of level l is the product of l strong Bruhat orders of level 1 as observed in §4.1. \square

5.6. Kleshchev realization and computation of the Keys. As we have explained in Section 3, the computation of the Keys for an element in $B(\Lambda_{\mathbf{s}})$ can be reduced to the computation of the Keys for the crystals $B(\omega_{s_j})$ associated to the fundamental highest weights once the orbit of the highest weight vertex in $B(\Lambda_{\mathbf{s}})$ and the combinatorial R -matrices between fundamental crystals can be described. However, to do this the crystal $B(\Lambda_{\mathbf{s}})$ should be realized as a connected component in a tensor product of level 1 crystals. Unfortunately, this is not the case for the Uglov realization when e is finite. The realization relevant in order to use Corollary 3.7 is Kleshchev's one (see for example [10, §6.2.16]). The vertices of the associated crystal are called the Kleshchev multipartitions. They have, in principle, a non trivial inductive definition but an elementary characterization of them has been recently given in [15].

Fortunately, Uglov and Kleshchev realizations can be easily connected. In particular, one can deduce from the above results that the characterization of the multipartitions in the orbit of the empty multipartition are the same in Kleshchev and Uglov realizations.

Proposition 5.19. *Let $\mathbf{s} \in \mathbb{Z}^l$. In the Kleshchev realization of $B(\Lambda_{\mathbf{s}})$, a l -partition λ yields a symbol in the orbit of the empty l -partition if and only if it is a (e, \mathbf{s}) -core. In particular, when $l > 1$ and $0 \leq s_1 \leq \dots \leq s_l < e$, the symbols in this orbit are exactly those such that $L_{s_a}(\lambda^a) \subset L_{s_{a+1}}(\lambda^b) \subset L_{s_a+e}(\lambda^a)$ for any $1 \leq a < l$.⁴*

Proof. Let $n \in \mathbb{Z}_{>0}$. Take $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{Z}^l$ such that $t_j \equiv s_j \pmod{e}$ for all $j = 1, \dots, l$ and such that $t_j - t_1 \geq n + e$ for all $j = 2, \dots, l$. By [10, §6.2.16], the subcrystals containing the multipartitions of rank less than n in the Kleshchev realization for the multicharge \mathbf{s} and in the Uglov realization for the multicharge \mathbf{t} coincide. Thus we can conclude by using the fact that $\mathbf{t}' = \mathbf{s}'$. \square

Now the second crucial ingredient in our procedure for computing the key by reduction to the fundamental weights as prescribed by 3.7 is the combinatorial R -matrix associated to a pair of fundamental weights. It corresponds to a transposition $(i, i + 1)$ in the multicharge \mathbf{s} for the Kleshchev realization of crystals (the rank of the multipartitions being fixed). Since only the components i and $i + 1$ are affected by this R -matrix, we are reduced to the case where $\mathbf{s} = (s_1, s_2)$ and $i = 1$.

Let $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ be such that $0 \leq v_1 \leq v_2 < e$ and $v_j \equiv s_j \pmod{e}$ for $j = 1, 2$. Then the subcrystal containing the multipartitions of rank less than n in the Kleshchev realization for $\mathbf{s} = (s_1, s_2)$ coincides with that in the Uglov realization for the multicharge $\mathbf{v}^> := (v_1, v_2 + ke)$ where $k \in \mathbb{N}$ is such that $k \cdot e > n + e$. The desired R -matrix thus corresponds to a crystal isomorphism between the crystal associated to the multicharge $\mathbf{v}^>$ and the crystal associated to the multicharge $(v_2, v_1 + ke)$. This isomorphism can be computed on the bipartition (λ^1, λ^2) by composing the crystal isomorphisms described in §5.3 as follows.

- (1) First apply the crystal isomorphism $\Phi_{(v_1, v_2 + ke) \rightarrow (v_2 + ke, v_1 + e)}^e$ which exchanges the two components of the bipartition, that is exchanges the two rows in the symbols and next translates the bottom one by e .
- (2) Apply the crystal isomorphism $\Phi_{(v_2 + ke, v_1 + e) \rightarrow (v_1 + e, v_2 + k \cdot e)}^e$ which reduces to a ‘‘Jeu de taquin’’ switching the lengths of the two rows in the symbols.
- (3) Repeat the two previous steps $2k$ times to get the image of (λ^1, λ^2) in the crystal with multicharge $(v_1 + 2e, v_2 + k \cdot e)$.
- (4) Finally, apply one more isomorphism $\Phi_{(v_1 + 2ke, v_2 + ke) \rightarrow (v_2 + ke, v_1 + 2ke + e)}^e$ and use the fact that the isomorphism between the crystals with multicharge $(v_2 + ke, v_1 + 2ke + e)$ in the Uglov realization and (s_2, s_1) in the Kleshchev realization is trivial.

Remark 5.20. *The crystal isomorphism between the Uglov and Kleshchev realizations of $B(\Lambda_{\mathbf{s}})$ can also be obtained from the results in [17] although it is not easy to make explicit. By 2 of Remark 2.5, we so obtain a characterization of the Demazure crystals in the Uglov realization. Nevertheless, we can just use the Kleshchev realization in which is the orbit of the highest weight and the relevant combinatorial R -matrices are easy to describe.*

5.7. Generalization of the Young Lattice. When $e = \infty$, $l = 1$ and $\mathbf{s} = (0)$, the orbit $O(\mathbf{s}, e)$ coincides with the whole crystal $B(S_{\mathbf{s}}(\emptyset))$. By forgetting the labels i of the arrows in $B(S_{\mathbf{s}}(\emptyset))$, one then recovers the Young lattice \mathcal{Y} of partitions which is strongly connected with the combinatorics of Schur functions. This lattice admits an interesting generalization \mathcal{Y}_{e-1} where the ordinary partitions are replaced by the e -cores (or by the k bounded partitions with $k = e - 1$) connected this times with the combinatorics of k -Schur functions (see [26]). The graph \mathcal{Y}_{e-1} corresponds to the Hasse diagram of the orbit $O(\mathbf{s}, e)$ when $l = 1$ and $\mathbf{s} = (0)$ and we have an arrow $\lambda \rightarrow \mu$ between the two e -cores λ and μ when μ is obtained by adding all the possible addable i -nodes in λ corresponding to a fixed $i \in \{0, \dots, e - 1\}$. When $l > 1$, the notion of (e, \mathbf{s}) -core yields generalizations of the graph \mathcal{Y}_{e-1} whose structure is obtained similarly from the orbit $O(\mathbf{s}, e)$. It is a natural question to ask whether its combinatorial properties (for e finite or not) can also be encoded in the combinatorics of a distinguished basis in a polynomial algebra analogous to k -Schur functions in level 1.

6. DEMAZURE CRYSTALS IN $B(\infty)$

6.1. Link with the Demazure crystals in $B(\lambda)$. Consider \mathfrak{g} a Kac-Moody algebra and λ a dominant weight for \mathfrak{g} . We now explain how it is possible to characterize the elements of a Demazure crystal $B(\infty)_w$ from the

⁴Therefore, λ is a (e, \mathbf{s}) -core if and only if for any $1 \leq a \leq l$, λ^a is a e -core and for any $1 \leq a < l - 1$, $L_{s_a}(\lambda^a) \subset L_{s_{a+1}}(\lambda^b) \subset L_{s_a+e}(\lambda^a)$.

computation of keys in the crystals $B(\lambda)_w, \lambda \in P_+$. First recall that there exists a unique embedding

$$\pi_\lambda : \begin{cases} B(\lambda) \hookrightarrow B(\infty) \\ b \longmapsto \pi_\lambda(b) \end{cases}$$

such that for any path $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_\lambda$ we have $\pi_\lambda(b) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_0$ where b_0 is the highest weight vertex of $B(\infty)$. Also the crystal $B(\infty)$ is endowed with the Kashiwara involution $*$ and the crystal operators have starred versions $\tilde{f}_i^* = * \circ \tilde{f}_i \circ *$ and $\tilde{e}_i^* = * \circ \tilde{e}_i \circ *$. Thanks to the operators \tilde{e}_i^* , we get a simple characterization of the image of π_λ . Namely, we have

$$\text{Im } \pi_\lambda = \{u \in B(\infty) \mid \varepsilon^*(u) \preceq \lambda\}$$

where $\varepsilon^*(u) = \sum_{i \in I} \varepsilon_i^*(u) \omega_i$ and $\varepsilon^*(u) \preceq \lambda$ means that $\lambda - \varepsilon^*(u)$ is a dominant weight.

Given any w in the Weyl group W , we also have by Theorem 2.7

$$\pi_\lambda(B(\lambda)_w) = B(\infty)_w \cap \text{Im } \pi_\lambda.$$

From the previous properties, for deciding if a vertex u belongs to $B(\infty)_w$, it suffices to have a realization of $B(\lambda)$ and $B(\infty)$ satisfying the properties below.

- The embedding π_λ is easy to describe.
- The actions of both the ordinary and $*$ -crystal operators are explicit.
- For any $u \in \text{Im } \pi_\lambda$, one can compute the unique vertex $b \in B(\lambda)$ such that $\pi_\lambda(b) = u$.
- One can decide if a vertex b in $B(\lambda)$ belongs to $B(\lambda)_w$.

For deciding whether $u \in B(\infty)_w$ it then suffices to proceed as follows.

- (1) Compute $\lambda = \varepsilon^*(u)$, we get that $u \in \text{Im } \pi_\lambda$.
- (2) Determine $b \in B(\lambda)$ such that $\pi_\lambda(b) = u$.
- (3) Then, $u \in B(\infty)_w$ if and only if $b \in B(\lambda)_w$.

6.2. Finite, infinite and affine type A. In type A (finite, infinite and affine), vertices of $B(\infty)$ are parametrized by multisegments that we now define.

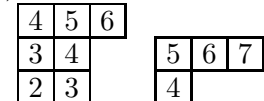
Definition 6.1. *A segment is a sequence of consecutive integers $[a, a+1, \dots, b]$. We denote it by $[a; b]$. A collection (or a formal sum) of segments is called a multisegment. The empty multisegment is denoted by \emptyset and we write \mathfrak{M} for the set of all multisegments.*

For $e \in \mathbb{Z}_{\geq 2}$, let us define \mathfrak{M}_e as the subset of \mathfrak{M} of the multisegments \mathfrak{m} in which each segment $[a, b]$ is such that $1 \leq a < b \leq e$. Also define $\mathfrak{M}_e^{\text{aff}}$ as the subset of aperiodic multisegments of \mathfrak{M} , that is the subset of multisegments \mathfrak{m} such that for each length l there exists at least an integer in $\{0, \dots, e-1\}$ for which \mathfrak{m} does not contain a segment $[b-l+1, b]$ of length l with $b = i \bmod e$. It is then known that in types A_{e-1}, A_∞ and $A_{e-1}^{(1)}$, the crystal $B(\infty)$ has a simple realization with \emptyset as highest weight vertex and in which the vertices are parametrized by the multisegments in $\mathfrak{M}_e, \mathfrak{M}$ and $\mathfrak{M}_e^{\text{aff}}$, respectively. Also we determined in [16] the corresponding embedding

$$\Pi_{\Lambda_{\mathbf{s}}} : B(S_{\mathbf{s}}(\emptyset)) \hookrightarrow B(\infty)$$

compatible with the Uglov realization of crystals for a multicharge $\mathbf{s} \in \mathbb{Z}^l$ such that $0 \leq s_1 \leq \dots \leq s_l < e$ (with $s_1 \geq 1$ in type A_e and $e = \infty$ in type A_∞). The embedding $\Pi_{\Lambda_{\mathbf{s}}}$ can be described as follows. Take $\boldsymbol{\lambda}$ an l -partition regarded as a sequence of l Young diagrams. Then associate to each row λ_i^k of $\boldsymbol{\lambda}$ the segment $[a; b]$ where $a = 1 - i + s_k$ and $b = \lambda_i^k - i + s_k$ are the contents of the leftmost and rightmost boxes in λ_i^k translated by s_k , respectively. The multisegment $\Pi_{\Lambda_{\mathbf{s}}}(\boldsymbol{\lambda})$ is then the formal sum of the segments associated to each non empty row of $\boldsymbol{\lambda}$.

Example 6.2. *Consider the multicharge $\mathbf{s} := (4, 5)$ and the bipartition (3.2.2, 3.1). Write the associated Young*



diagrams and fill each box in $\lambda^{(k)}, k \in \{1, 2\}$ with its content translated by s_k :

Then, we have

$$\Pi_{\Lambda_{\mathbf{s}}} = [4; 6] + [3; 4] + [2; 3] + [5; 7] + [4].$$

In [16], we also got the action of the $*$ -crystal operators and a procedure to compute the minimal symbol associated to a multisegment. Thus, by the previous arguments, we can use the results of Section 5 to decide whether a multisegment \mathbf{m} belongs to $B(\infty)$. This is direct for types A_{e-1} and A_∞ but in type $A_{e-1}^{(1)}$, one needs the characterization of the Demazure crystals in the Uglov realization (see Remark 5.20).

6.3. Multisegments associated to a (e, s) -core. Given a segment $\mathbf{m} \in \mathfrak{M}_e^{\text{aff}}$, we now give a direct procedure deciding whether $\mathbf{m} \in \Pi_{\Lambda_s}(O(\mathbf{s}, e))$ or not⁵, that is characterizing the image of the Key map for the Demazure crystals $B(\infty)_e$. To do this, It will be convenient to write our aperiodic multisegments by gathering segments with the same right end as follows:

$$\mathbf{m} = \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq r_j} [a_j^i, b_j]$$

where $m \in \mathbb{N}$ and where, for each $1 \leq j \leq m$, we have $r_j \in \mathbb{N}$. We can also assume that $b_1 \leq \dots \leq b_m$ and that for each $1 \leq j \leq m$, we have $a_j^1 \leq \dots \leq a_j^{r_j}$. Our algorithm (illustrated by the example below) decides if $\mathbf{m} \in \Pi_{\Lambda_s}(O(\mathbf{s}, \infty))$ and then construct recursively $m + 1$ sequences of segments $(L_1^j, \dots, L_l^j), j = 0, \dots, m$ starting from $(L_1^0, \dots, L_l^0) = (\emptyset, \dots, \emptyset)$.

- If $r_m > l$ then the algorithm stops. Otherwise set

$$L_l^1 = ([a_m^{r_m}, b_m]), \dots, L_{l-r_m+1}^1 = ([a_m^1, b_m]), L_{l-r_m}^1 = \emptyset, \dots, L_1^1 = \emptyset.$$

- More generally, assume we have the sequence $(L_1^{m-j}, \dots, L_l^{m-j})$ and consider the segments $[a_j^i, b_j]$ for $i = 1, \dots, r_j$. When $r_j > l$ the algorithm stops. Otherwise, set $L_{l+r_j-k}^{m-j+1} = L_{l+r_j-k}^{m-j}$ for $r_j < k \leq l$ and for each $1 \leq k \leq r_j$, $L_{l+r_j-k}^{m-j+1}$ is obtained by adding the segment $[a_j^k, b_j]$ at the beginning of the sequence $L_{l+r_j-k}^{m-j}$ if this sequence is empty or its first segment $[a, b]$ is such that $a = a_j^k + 1$ and $b > b_j$. If not, the algorithm stops.

At the end of the procedure either the algorithm stops before all the segments of \mathbf{m} have been considered and we then conclude $\mathbf{m} \notin \Pi_{\Lambda_s}(O(\mathbf{s}, e))$ or we get a sequence (L_1^m, \dots, L_l^m) of segments :

$$L_j^m := ([\alpha_1, \beta_1], \dots, [\alpha_{p_j}, \beta_{p_j}])$$

Then we consider the symbol

$$S_j = \alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_{p_j}$$

We have $\mathbf{m} \in \Pi_{\Lambda_s}(O(\mathbf{s}, e))$ if and only if for all $(i, j) \in \{1, \dots, l\}^2$ we have $s_i - s_j = p_i - p_j$. Moreover, it is easy to see that the symbol we have constructed is nothing but the symbol associated to $\Pi_{\Lambda_s}^{-1}(\mathbf{m})$.

More generally this algorithm shows when there exists $\mathbf{s} \in \mathbb{Z}^l$ such that $\mathbf{m} \in \Pi_{\Lambda_s}(O(\mathbf{s}, e))$. The proof of the the rightness of the algorithm is straightforward. The algorithm simply construct if possible the symbol of a multipartition which satisfies all the properties of being in $\Pi_{\Lambda_s}(O(\mathbf{s}, e))$.

Example 6.3. Assume $e = \infty$ and consider the multisegment

$$[2] + [3] + [2, 3] + [2, 3] + [4] + [3, 4] + [5, 6] + [6, 7] + [4, 7] + [7, 9] + [5, 9] + [3, 9]$$

We take $\mathbf{s} = (0, 2, 4)$.

- We start with the segments $[7, 9]$, $[5, 9]$ and $[3, 9]$ and we get $L_1^1 = ([3, 9])$, $L_2^1 = ([5, 9])$ and $L_3^1 = ([7, 9])$.
- We then take the segments $[6, 7]$ and $[4, 7]$ and we get $L_1^2 = ([3, 9])$, $L_2^2 = ([4, 7], [5, 9])$ and $L_3^2 = ([6, 7], [7, 9])$.
- We then take the segments $[5, 6]$ and we get $L_1^3 = ([3, 9])$, $L_2^3 = ([4, 7], [5, 9])$ and $L_3^3 = ([5, 6], [6, 7], [7, 9])$.
- We then take the segments $[4]$ and $[3, 4]$ and we get $L_1^4 = ([3, 9])$, $L_2^4 = ([3, 4], [4, 7], [5, 9])$ and $L_3^4 = ([4], [5, 6], [6, 7], [7, 9])$.
- We then take the segments $[3]$, $[2, 3]$ and $[2, 3]$ and we get $L_1^5 = ([2, 3], [3, 9])$, $L_2^5 = ([2, 3], [3, 4], [4, 7], [5, 9])$ and $L_3^5 = ([3], [4], [5, 6], [6, 7], [7, 9])$.
- We finally take the segment $[2]$ and we get $L_1^6 = ([2, 3], [3, 9])$, $L_2^6 = ([2, 3], [3, 4], [4, 7], [5, 9])$ and $L_3^6 = ([2], [3], [4], [5, 6], [6, 7], [7, 9])$.

We see that all the properties are satisfied and thus that $\mathbf{m} \in \Pi_{\Lambda_s}(O(\mathbf{s}, \infty))$, the associated 3-partition is

$$(7.2, 5.4.2.2, 3.2.2.1.1.1).$$

⁵It also hold in type A_{e-1} and A_∞ by using the relevant set of multisegments.

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