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Systoles and Lagrangians of random complex algebraic hypersurfaces

Damien Gayet

September 19, 2019

Abstract

Let $n \geq 1$ be an integer, $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth affine real hypersurface, not necessarily connected. We prove that there exists $c > 0$ and $d_0 \geq 1$, such that for any $d \geq d_0$, any smooth complex projective hypersurface Z in $\mathbb{C}P^n$ of degree d contains at least $c \dim H_*(Z, \mathbb{R})$ disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$, where Z is equipped with the restriction of the Fubini-Study symplectic form (Theorem 1.1). If moreover the connected components of $\mathcal{L}$ have non vanishing Euler characteristic, which implies that n is odd, the latter Lagrangian submanifolds form an independent family of $H_{n-1}(Z, \mathbb{R})$ (Corollary 1.2). We use a probabilistic argument for the proof (Theorem 1.19) inspired by a result by J.-Y. Welschinger and the author on random real algebraic geometry [11], together with quantitative Moser-type constructions (Theorem 3.4). For $n = 2$, the method provides a uniform positive lower bound for the probability that a projective complex curve in $\mathbb{C}P^2$ of given degree equipped with the restriction of the ambient metric has a systole of small size (Theorem 1.6), which is an analog to a similar bound for hyperbolic curves given by M. Mirzakhani [18]. Our results hold in the more general setting of vanishing loci of holomorphic sections of vector bundles of rank between 1 and n tensored by a large power of an ample line bundle over a projective complex n-manifold (Theorem 1.16).

Keywords: Systole, complex algebraic curve, complex projective hypersurface, Lagrangian submanifold, random geometry, Kähler geometry

Mathematics subject classification 2010: 60D05, 53D12, 32Q15

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1 Introduction

1.1 Disjoint Lagrangian submanifolds

On a compact orientable smooth real surface of genus $g > 1$, there exists $3g-3$ disjoint non-contractible closed curves such that two of them are not isotopic. A natural generalization of this phenomenon in a closed symplectic manifold (X, ω) is to estimate the possible number of disjoint Lagrangian submanifolds of given diffeomorphism type in X. The answer is easy for submanifolds which exist as compact smooth manifolds in in $\mathbb{R}^{2n}$, like the torus, since by the Darboux theorem they can be implemented at any scale in X, so there exists an infinite number of them. Moreover, when the submanifold $\mathcal{L}$ possesses a smooth non-vanishing closed one-form, which is the case for the n-torus, this form produces an infinite number of disjoint Lagrangian graphs in $T^*\mathcal{L}$, hence by Weinstein tubular neighbourhood there exists an infinite number of disjoint Lagrangian submanifolds close to $\mathcal{L}$, see Remark 1.11 below. If this is not the case or if the Euler characteristic of $\mathcal{L}$ is not zero, then it cannot be displaced by a perturbation as a disjoint submanifold, see § 1.4. Furthermore, the classes of a finite family of disjoint such Lagrangian submanifolds with non zero Euler characteristic form an independant family of the ambient homology group of degree half of the dimension of X, see Lemma 2.2.

The main result. In this paper, we are interested in smooth projective complex submanifolds equipped with the restriction of the ambient Fubini-Study Kähler form. They have the same diffeomorphic type, because they can be isotoped through smooth hypersurface. For this latter reason, the Moser trick and the fact that the symplectic form has entire periods shows that they are all also symplectomorphic, see Proposition 4.2. Moreover, they benefit an interesting homological property: for any degree d hypersurface $Z \subset \mathbb{C}P^n$,

$$\dim H_*(Z, \mathbb{R}) \underset{d\to\infty}{\sim} \dim H_{n-1}(Z, \mathbb{R}) \underset{d\to\infty}{\sim} d^n.$$

The first asymptotic is a consequence of the Lefschetz hyperplane theorem [13] and the second one can be estimated through the Euler class of the tangent space of Z through Euler characteristic and Chern classes. The main goal of this paper is to prove the following theorem:

Theorem 1.1 *Let $n \geq 1$ be an integer and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth real affine hypersurface, not necessarily connected. Then, there exists $c > 0$ such that for any d large enough, any complex hypersurface Z of degree d in $\mathbb{C}P^n$ contains at least $c \dim H_*(Z, \mathbb{R})$ pairwise disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.*

In fact, we prove this result in the more general setting of vanishing loci of holomorphic sections of vector bundles of rank between 1 and n tensored by a large power of an ample line bundle over a projective complex n-manifold, see Theorem 1.16.

Corollary 1.2 *Under the hypotheses of Theorem 1.1,*

1. *if for any component $\mathcal{L}_i$ of $\mathcal{L}$, $\chi(\mathcal{L}_i) \neq 0$, then the classes in $H_{n-1}(Z, \mathbb{R})$ generated by their Lagrangian copies in Z are linearly independent;*

2. *if $\mathcal{L}$ is simply connected, its Lagrangian copies are not close perturbations of each other.*

Remark 1.3 1. *Note that $\chi(\mathcal{L}) \neq 0$ implies that n is odd.*

2. *The real projective plane $\mathbb{R}P^2$ is a Lagrangian submanifold of $Z = \mathbb{C}P^2 \subset \mathbb{C}P^3$ but cannot be a hypersurface in $\mathbb{R}^3$ since any compact hypersurface of $\mathbb{R}^n$ is orientable.*

3. *If $Z = \mathbb{C}P^2 \subset \mathbb{C}P^3$, $H_2(Z, \mathbb{Z})$ is generated by the class of a complex line $[D]$. The integral of the Fubini-Study Kähler form ω_{FS} over D is positive since ω_{FS} is positive over complex submanifolds, that is $\langle \omega_{FS}, [D] \rangle > 0$. However $\langle \omega_{FS}, [\mathcal{L}] \rangle = 0$ if $\mathcal{L}$ is a Lagrangian submanifold, so that $H_2(Z, \mathbb{Z})$ cannot be generated by Lagrangian classes.*

In this paper, we prove Theorem 1.1, which is a special case of Theorem 1.16, through a probabilistic argument, see Theorem 1.14: if we choose at random such a projective hypersurface of given large degree, the probability that the conclusion of the theorem holds is positive. Since the hypersurfaces have the same symplectomorphism type, see Proposition 4.2, they all satisfy this property. In a parallel paper [9], we prove this theorem with a deterministic proof based on Donaldson-Auroux method [6] and [2].

Other results on disjoint Lagrangian submanifolds. As far as the author of the present work knows, essentially three types of results for disjoint Lagrangian submanifolds have been proved.

- The oldest one concerns Lagrangian spheres that naturally germ from singularities of hypersurfaces by Picard-Lefschetz theory. For instance, S. V. Chmutov [1, p. 419] proved that there exists a singular projective hypersurface of degree d with $c_n d^n + o(d^n)$ singular points, with $c_n \sim_n \sqrt{\frac{2}{\pi n}}$. When the polynomial defining this hypersurface is perturbed into a non-singular polynomial, the singularities give birth to disjoint Lagrangian spheres of the associated smooth hypersurface of the same degree.

- The second result is due to G. Mikhalkin and uses toric arguments:

  Theorem 1.4 *[17, Corollary 3.1] For any $n \geq 2$ and $d \geq 1$, a $2h^{n-1,0}$-dimensional subspace of $H_{n-1}(Z, \mathbb{R})$ has a basis represented by embedded Lagrangian tori and spheres, where Z is any smooth projective hypersurface of $\mathbb{C}P^n$.*

  Here, $h^{n,0}$ is the geometric genus of Z, that is the dimension of space of the global holomorphic n-form $H^{n,0}(X) \subset H^n(X, \mathbb{C})$. It grows like cd^n for some $c > 0$, as does the dimension of $H_{n-1}(Z, \mathbb{C})$ and $\chi(Z)$. For Lagrangian spheres, Theorem 1.4 is more precise than our Theorem 1.1, since with our method, for an even dimension $n \geq 3$, we cannot know if our Lagrangian spheres have non-trivial class in H_{n-1}, and since the constant c in our bound should be very small compared to the one given by Mikhalkin, as in Chmutov's theorem. Moreover, our theorem does not say anything new for tori. On the other hand, Theorem 1.1 asserts that *any* real affine hypersurface is produced as Lagrangian submanifolds in a large quantity in

the projective hypersurfaces of large enough degree, and in odd dimension, with a simple topological restriction, it generates a uniform proportion of the homology of the complex hypersurface. Moreover, our result extends to any projective manifold equipped with any ample line bundle, see Theorem 1.16.

- The third type of results concerns upper bounds for the number of disjoint Lagrangian submanifolds (not necessarily spheres), and uses Floer techniques, see for instance [21] for results in open manifolds and a survey for older results of this kind.

1.2 Random complex projective hypersurfaces

The smooth projective complex hypersurfaces of a given degree d, that is smooth vanishing loci in $\mathbb{C}P^n$ of complex homogeneous degree d polynomials, form a very natural family of compact Kähler manifolds. On the contrary to the real projective hypersurfaces, that is the vanishing loci in $\mathbb{R}P^n$ of real polynomials, for fixed d the complex hypersurfaces have the same diffeomorphism type. In particular for $n = 2$, the smooth complex hypersurfaces in $\mathbb{C}P^2$ of degree d are compact connected Riemann surfaces of genus

$$g_d := \frac{1}{2}(d-1)(d-2).$$

Moreover, as said before, for any n and d, when the complex hypersurfaces are equipped with the restriction of the ambiant Kähler form, they all have the same symplectomorphism type, see Proposition 4.2.

In [23], the authors inaugurated the study of random vanishing loci of complex polynomials in higher dimensions (and zero sets of random holomorphic sections, see section 1.5), studying in particular the statistics of the current of integration over these loci. In this paper, we will study the statistics of some metric and symplectic properties of these hypersurfaces equipped with the restriction of the Fubini-Study Kähler metric g_{FS} and form ω_{FS} on $\mathbb{C}P^n$. More precisely, we will be concerned with systoles and small Lagrangian submanifolds.

- **$n = 2$.** A source of inspiration and motivation for this paper in the case where $n = 2$ was genuinely probabilistic and provided by M. Mirzhakani's theorem on systoles of random hyperbolic curves [18], see Theorem 1.5. One of the two main goals of the present work is in fact to find an analog of it for random complex projective curves, see Theorem 1.6.

- **$n \geq 3$.** With the methods we use, it happens that in higher dimension the natural generalization of small non-contractible loops are small Lagrangian submanifolds of the random hypersurfaces. Our motivation was nevertheless deterministic. The probabilistic method is partly inspired by the work of J.-Y. Welschinger and the author on random *real* algebraic manifolds [11], were we proved that any compact affine real hypersurface $\mathcal{L}$ appears a lot of times as a component of a random large degree real projective hypersurface with a uniform probability, see Theorem 1.8. Note that these components are Lagrangian submanifolds of the complexified hypersurface. In particular, this implies that any complex hypersurface of large enough degree d contains at least $c\sqrt{d}^n$ Lagrangian submanifolds diffeomorphic to $\mathcal{L}$, where $c > 0$ does not depend on d, see Remark 1.9. In this paper, we prove an analogous complex and symplectic result analogous to Theorem 1.8: that any compact real hypersurface appears at least cd^n times as a small Lagrangian submanifold in a random complex projective hypersurface with a uniform positive probability, see Theorem 1.10.

We emphasize that this improvement from $\sqrt{d}^n$ to d^n has an interesting topological implication: when $\chi(\mathcal{L}) \neq 0$, it implies that these disjoint submanifolds form an independent family of homology classes of a cardinal comparable to the dimension of the whole homology of the complex hypersurface. As said before, the deterministic Theorem 1.6 is a direct consequence of the probabilistic Theorem 1.10.

- It can be suprising that probabilistic arguments can have deterministic consequences in this situation. The main explanation is given by Theorem 1.19 which shows for any sequence of smaller and smaller balls B of size $1/\sqrt{d}$ in $\mathbb{C}P^n$, the Lagrangian of desired diffeomorphism type appears in the intersection of B and the random hypersurface of degree d with a uniform positive probability. This uniform localization easily implies the global Theorem 1.10, which says that with uniform probability, a uniform proportion of a packing of $\mathbb{C}P^n$ with disjoint balls of size $1/\sqrt{d}$ contain the wanted Lagrangian. It happens that the order d^n of growth of $\dim H_*(Z, \mathbb{R})$ is the same order of the number of these packed small balls. This result itself implies immediatly the deterministic consequence.

- Finally, using the universality of peak sections on Kähler manifolds equipped with ample line bundles, or the asymptotic (in the degree d) universality of the Bergmann kernel, we will explain that analogous results can be proved in this general setting, see the probabilistic Theorem 1.14 and the deterministic Theorem 1.16.

Let us define the measure on the space of complex polynomials used in [23] and in this paper. Let

$$H_{d,n+1} := \mathbb{C}^d_{hom}[Z_0, \cdots, Z_n]$$

be the space of complex homogeneous polynomial in $n+1$ complex variables. Its dimension equals $\binom{n+d}{n}$. For $P \in H_{d,n+1}$, denote by $Z(P) \subset \mathbb{C}P^n$ its projective vanishing locus. For P outside a codimension 1 complex subvariety of $H_{d,n+1}$, $Z(P)$ is a smooth complex hypersurface. Since for transverse polynomials P, Q, $Z(P) = Z(Q)$ is equivalent to $P = \lambda Q$ for some $\lambda \in \mathbb{C}^*$, the space of degree d hypersurfaces has the dimension of $H_{d,n+1}$ minus one. For $n = 2$ this is $\frac{1}{2}d(d+3) \underset{g_d \to \infty}{\sim} g_d$. Note that for the hyperbolic curves, the complex moduli space has dimension $3g - 3$. There exists a natural Hermitian product on $H_{d,n+1}$ given by

$$\forall P, Q \in H_{d,n+1}, \langle P, Q \rangle = \int_{\mathbb{C}P^n} h_{FS}(P, Q) d\mathrm{vol}_{g_{FS}},$$

where

$$h_{FS}(P, Q)([Z]) = \frac{P(Z)\overline{Q(Z)}}{|Z|^{2d}}$$

and g_{FS} denotes the Fubini-Study metric on $\mathbb{C}P^n$. Recall that the latter is the quotient metric induced by the projection $\mathbb{C}^{n+1} \supset \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ and the standard round metric on the sphere. Then, the monomials

$$\left(\sqrt{\frac{(d+n)!}{i_0! \cdots i_n!}} Z_0^{i_0} \cdots Z_n^{i_n} \right)_{\sum_{k=0}^n i_k = d} \tag{1.1}$$

form an orthonormal basis of $\mathbb{C}^d_{hom}[Z_0, \cdots, Z_n]$, see the end of the proof of Lemma 4.6. This Hermitian product induces a Gaussian probability measure on $H_{d,n+1}$. In other terms, we choose

$$P = \sum_{i_0 + \cdots + i_n = d} a_{i_0 \cdots i_n} \sqrt{\frac{(d+n)!}{i_0! \cdots i_n!}} Z_0^{i_0} \cdots Z_n^{i_n} \tag{1.2}$$

Parameters of the surfaces of genus g	Hyperbolic surfaces	Planar algebraic curves
Dimension of the moduli space	$\underset{g\to\infty}{\sim} 3g$	$\underset{d\to\infty}{\sim} g$
Curvature	-1	$\in]-\infty, 2]$ [20]
Volume	$\underset{g\to\infty}{\sim} 4\pi g$	$\underset{g\to\infty}{\sim} 4\pi g$
Diameter	$\in]0, +\infty[$	$\in [c, Cg^{5/2}]$ [8]

Figure 1: Deterministic parameters of the two different models of real surfaces, the Weil-Petersson one with hyperbolic surfaces, and the Fubini-Study model with complex algebraic curves equipped with the induced rescaled induced metric $\sqrt{2\pi d} g_{FS}$ on $\mathbb{C}P^2$.

with i.i.d. Gaussian coefficients $a_I \in \mathbb{C}$ such that $\Re a_I \sim N(0,1)$ and $\Im a_I \sim N(0,1)$ and are independent. We denote by $\mathbb{P}_d$ the measure.

1.3 Systoles of random projective curves

Let (X, h) be a compact smooth real manifold equipped with a metric h. In [18], M. Mirzakhani studied probabilistic aspects of metric parameters of (X, h), when (X, h) is taken at random in $\mathcal{M}_g$, the moduli space of hyperbolic genus g compact Riemann surfaces. This moduli space is equipped with a natural symplectic form, the Weil-Petersson form, hence a volume form, for which $\mathcal{M}_g$ has a finite volume, and provides a natural probality measure $\mathbb{P}_{WP,g}$ on it, see [18]. Denote by

- $\ell_{sys}(X)$ the least length of the noncontractible loops in (X, h).

M. Mirzakhani proved the following theorem:

Theorem 1.5 *[18, Theorem 4.2] There exists $\varepsilon_0 > 0$ and $0 < c < C$ such that for any $\varepsilon \leq \varepsilon_0$ and every $g \geq 2$,*

$$c\varepsilon^2 \leq \mathbb{P}_{WP,g}\big[X \in \mathcal{M}_g \mid \ell_{sys}(X) < \varepsilon\big] \leq C\varepsilon^2.$$

We now introduce a partial analogous result for random projective curves of given degree, with an homological point of view. For any (X, h) as above, $\delta > 0$ and $c > 1$,

- denote by $N_{sys}(X, \delta, c)$ the maximal cardinal of an independent family of classes in $H_1(X, \mathbb{Z})$ such that any class in the family is represented by a circle of length between δ/c and $c\delta$.

Our first main result concerns the systoles of the random complex curves in $\mathbb{C}P^2$:

Theorem 1.6 *There exists a constant $c \geq 1$ and $d_0 \geq 1$ such that for every $0 < \varepsilon \leq 1$ and every $d \geq d_0$,*

$$\exp(-\frac{c}{\varepsilon^6}) < \mathbb{P}_d\Big[P \in H_{d,3} \mid N_{sys}\big(Z(P), \frac{\varepsilon}{\sqrt{d}}, c\big) \geq d^2 \exp\big(-\frac{c}{\varepsilon^6}\big)\Big],$$

where $Z(P)$ is equipped with $g_{FS|Z(P)}$. In particular,

$$\exp\big(-\frac{c}{\varepsilon^6}\big) < \mathbb{P}_d\Big[P \in H_{d,3} \mid \ell_{sys}\big(Z(P)\big) \leq \frac{\varepsilon}{\sqrt{d}}\Big].$$

Parameters of the surfaces of genus g	Hyperbolic surfaces Weil-Petersson measure	Planar algebraic curves Fubini-Study measure
Curvature	-1	$\mathbb{E}_d\big(K(x)\vert x\in C\big)\asymp -1$
Diameter	$\mathbb{P}_{WP,g}(\mathrm{Diam}\geq 40\log g)\underset{g\to\infty}{\to} 0$ [18]	?
Systole	$\mathbb{P}_{WP,g}(\ell_{sys}\leq\varepsilon)\asymp\varepsilon^2$ [18]	$\mathbb{P}_d(\ell_{sys}\leq\varepsilon)\geq\exp(-\frac{c}{\varepsilon^6})$ [this paper]

Figure 2: Statistics of some metric parameters. The complex algebraic curves equipped with the induced rescaled induced metric $\sqrt{2\pi d}g_{FS}$ on $\mathbb{C}P^2$.

Theorem 1.6 is a particular case of the more general Theorem 1.12, which holds for random complex curves in a projective complex manifold.

Remark 1.7 *1. Since* $\dim H_1(Z(P),\mathbb{R})=2g_d\sim_d d^2$*, the first assertion of this theorem proves that with uniform probability, there exists a basis of* $H_1(Z(P),\mathbb{R})$ *such that a uniform proportion of its members is represented by a loop of size less than* $\varepsilon/\sqrt{d}$*.*

2. If we want to compare the Fubini-Study model with the Weil-Petersson model, we would like that the volumes equal at given genus. This implies that the metric in the projective setting has to be rescaled with a $\sqrt{d}$ *factor. In this case the size estimates given by Theorem 1.6 become similar to the lower bound of Theorem 1.5. Note however that, although our bound is uniform in* d *or* g_d *as in [18], the dependence in* ε *is very bad compared to Mirzakhani's bound.*

3. In fact, for any $x\in\mathbb{C}P^n$*, with the same probability, a noncontractible loop lies in* $Z(P)\cap B(x,\varepsilon/\sqrt{d})$*, see Theorem 1.19.*

Other metric parameters. For the reader's convenience, we present some known results for other metric properties of the projective curves. Figure 1. and 2. compares deterministic and probabilistic observables for the Weil-Petersson and Fubini-Study models.

- **Volume.** By the Wirtinger theorem, any curve of degree d in $\mathbb{C}P^2$ (and any degree d hypersurface of $\mathbb{C}P^n$) has a volume equal to d, see [13]. By the Gauss-Bonnet theorem, for any hyperbolic curve of genus g, the volume equals $2\pi(2g-2)$. Hence, for $n=2$, for comparison with the Weil-Petersson model, we should rescale the metric g_{FS} on $\mathbb{C}P^2$ by $\sqrt{2\pi d}$, so that
$$\mathrm{Vol}_{\sqrt{2\pi d}g_{FS|Z(P)}}\big(Z(P)\big)=2\pi d^2\underset{d\to\infty}{\sim}4\pi g_d.$$
- **Curvature.** By a result by L. Ness [20, Corollary p. 60], the Gaussian curvature K of a degree d complex curve in $\mathbb{C}P^2$ equipped with the induced metric g_{FS} belongs to $]-\infty,2]$. Besides, by the Gauss-Bonnet theorem, the average on $Z(P)$ of K equals
$$K_{mean}=-2\pi\frac{2g_d-2}{d}\underset{g\to\infty}{\sim}-2\pi d.$$
We can prove moreover that $\forall x\in\mathbb{C}P^2,\ \mathbb{E}\big(K(x)\vert P(x)=0\big)\asymp -d$.

- **Diameter.** Since by the maximum principle there are no compact complex curves in $\mathbb{C}^2$, no algebraic complex curve in $\mathbb{C}P^2$ does exist in a ball, so that

$$\exists c > 0,\ \forall P \in \bigcup_{d\geq 1} H_{d,n+1},\ \mathrm{Diam}\big(Z(P), g_{FS|Z(P)}\big) \geq c. \tag{1.3}$$

  F. Bogomolov [5] has proved that the intrisic diameter of planar complex curves is not bounded when the degree grows to infinity. However S.-T. Feng and G. Schumacher [8] showed that for a given degree there exists an upper bound for the diameter given by:

$$\forall d \geq 1,\ \forall P \in H_{d,3},\ \mathrm{Diam}\big(Z(P), g_{FS|Z(P)}\big) \leq 32\pi g_d^2 + o(g_d^2).$$

  It should be possible, like in [18], to find a better probabilistic estimate for the diameter, and one can wonder if it is also logarithmic in d.

1.4 Small Lagrangian submanifolds of random hypersurfaces

Let (X^{2n}, ω) be a smooth symplectic manifold of dimension $2n$. Recall that ω is a closed non-degenerate two-form. A Lagrangian submanifold $\mathcal{L}$ of X is a n-dimensional submanifold such that $\omega_{|TL}$ vanishes. For instance, a real analytic hypersurface in $\mathbb{R}^n$ is a Lagrangian submanifold of its associated complex extension, which is a Kähler manifold for the restricton of the standard Kähler form in $\mathbb{C}^n$.

Universal real components. In [11], J.-Y. Welschinger and the author of the present paper studied random *real* projective hypersurfaces, that is the real loci of random elements of $\mathbb{R}H_{d,n+1}$, the space of real homogeneous polynomials in $(n+1)$ variables and of degree d. The measure was the complex Fubini-Study (1.2) restricted to $\mathbb{R}H_{d,n+1}$. In the litterature, this measure is often called the *Kostlan measure*. Let $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth real hypersurface. For any real homogeneous polymial P, let $Z_{\mathbb{R}}(P) := Z(P) \cap \mathbb{R}P^{n+1}$, and denote by

- $N_{\mathbb{R}}(\mathcal{L}, Z_{\mathbb{R}}(P))$ be the number of disjoint balls in $\mathbb{R}P^n$ such that for any such ball B, $B \cap Z_{\mathbb{R}}(P)$ contains a submanifold $\mathcal{L}'$ diffeomorphic to $\mathcal{L}$.

Theorem 1.8 *[11, Theorem 1.2] and [12, Theorem 2.1.1] Let $n \geq 1$ and $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth hypersurface, not necessarily connected. Then there exists $c > 0$ and d_0, such that for every $d \geq d_0$,*

$$c < \mathbb{P}_d\Big[P \in \mathbb{R}H_{d,n+1} \mid N_{\mathbb{R}}\big(\mathcal{L}, Z_{\mathbb{R}}(P)\big) > c\sqrt{d}^n\Big].$$

Remark 1.9 *1. Note that this theorem has a deterministic corollary, using the same argument given in this paper: any compact real affine hypersurface appears at least $c\sqrt{d}^n$ times as disjoint Lagrangian submanifolds in any complex projective hypersurface of high enough degree. Indeed, the real part of a complex hypersurface defined over the reals is Lagrangian for the restriction for the Fubini-Study Kähler form, and the complex projective hypersurface are all symplectomorphic.*

2. In [10], the author constructed real hypersurfaces with $c\sqrt{d}^n$ real spheres. The same proof, replacing a polynomial vanishing on a sphere by another polynomial gives the same corollary as the latter. Theorem 1.1 gives a cd^n lower bound, which is of the order of $\dim H_*(Z(P), \mathbb{R})$ when d grows to infinity.

3. In fact, Theorem 1.8 holds in the more general context of Kähler compact manifolds with holomorphic line bundles equipped with real structures, see [11].

Universal Lagrangian submanifolds. We turn now to a complex and Lagrangian analog of this theorem. As before, let $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth real hypersurface. For any compact symplectic manifold (Z, ω, h) equipped with a Riemannian metric h, any $\delta > 0$ and $c \geq 1$,

- denote by $N_{Lag}(\mathcal{L}, Z, \delta, c)$ the number of pairwise disjoint open sets containing a Lagrangian submanifold $\mathcal{L}'$ diffeomorphic to $\mathcal{L}$ and satisfying:

$$\frac{\delta}{c} \leq \mathrm{Diam}\big(\mathcal{L}', h_{|\mathcal{L}'}\big) \leq c\delta. \tag{1.4}$$

For polynomials, the following theorem is the main probabilistic result of this paper. It is a particular case of Theorem 1.14 below:

Theorem 1.10 *Let $n \geq 2$, $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth hypersurface, not necessarily connected. Then there exists $c \geq 1, D \geq 1, d_0 \geq 1$ such that for any $0 < \varepsilon \leq 1$ and $d \geq d_0$*

$$\exp(-\frac{c}{\varepsilon^D}) < \mathbb{P}_d \left[P \in H_{d,n+1} \mid N_{Lag}\Big(\mathcal{L}, Z(P), \frac{\varepsilon}{\sqrt{d}}, c\Big) > d^n \exp\big(-\frac{c}{\varepsilon^D}\big)\right],$$

where the metric and the symplectic form on $Z(P)$ are the ones induced by the Fubini-Study metric and symplectic form on $\mathbb{C}P^n$. Moreover, is $\mathcal{L}$ is real algebraic, that is if there exists $p \in \mathbb{R}[x_1, \cdots, x_n]$ such that $\mathcal{L} = Z_{\mathbb{R}}(p)$, then D can be chosen to be $D = 2 \deg p$.

Remark 1.11

1. *In fact, Theorems 1.1, 1.6 and 1.10 have a higher codimension generalization: instead of taking one unique random polynomial, one can choose $1 \leq r \leq n$ random independent polynomials $(P_1, \cdots, P_r)$ of the same degree, and look at their common vanishing locus $Z(P_1, \cdots, P_r) := \cap_{i=1}^r Z(P_i) \in \mathbb{C}P^n$, which is now almost surely of complex codimension r. Then, the same conclusions hold with the following changes: for complex curves (Theorem 1.6), we take $n \geq 2$ instead of $n = 2$, and choose $r = n - 1$. For Lagrangians (Theorem 1.10), we take $\mathcal{L} \subset \mathbb{R}^{n-r+1}$ instead of $\mathcal{L} \subset \mathbb{R}^n$. However, if $r \geq 2$, $\mathcal{L} \subset \mathbb{R}^{n-r+1}$ must satisfy a further necessary condition: its normal bundle must be trivial. These generalizations are direct consequences of Theorem 1.12 for the curves and Corollary 1.15 for the higher dimensions.*

2. *By the Weinstein theorem [27], a tubular neighborhood of a closed Lagrangian submanifold $\mathcal{L}$ is symplectomorphic to a tubular neighborhood of the zero section in $T^*\mathcal{L}$, so that the local Lagrangian deformations of $\mathcal{L}$ can be viewed in $T^*\mathcal{L}$ as graphs of closed 1-forms on $\mathcal{L}$. If the form is exact, then it has at least two zeros and the associated graph intersects $\mathcal{L}$. In particular, if $H^1(\mathcal{L}, \mathbb{R}) = 0$, $\mathcal{L}$ cannot locally be deformed as a disjoint Lagrangian submanifold. On the other hand, if $\mathcal{L}$ possesses a closed $1-$form which does not vanish, like the torus, then there exists an infinite number of Lagrangian submanifold of diffeomorphic to $\mathcal{L}$. See [7] for topological conditions on $\mathcal{L}$ which imply non-existence of such non-vanishing forms. This remark shows that in the case of spheres, the disjoint Lagrangian spheres produced by Theorem 1.1 are not small deformations of each others.*

3. *Theorem 1.10 is the consequence of the more precise Theorem 1.19, which asserts that for any sequence of balls centered on a fixed point x in $\mathbb{C}P^n$ and of size $1/\sqrt{d}$, with uniform probability $\mathcal{L}$ appears as a Lagrangian submanifold of the random vanishing locus.*

1.5 Random sections of a holomorphic vector bundle

There is at least two natural generalizations of Theorems 1.6 and 1.10: firstly, we can work in the setting of ample holomorphic line bundles over compact Kähler manifolds introduced by B. Schiffman and S. Zelditch in [23]. Secondly, we can study the statistics of the vanishing locus of *several* random polynomials or sections, as said in Remark 1.11. We present the fusion of the two generalization, as in [12]. Let $n \geq 1$ and X be a compact complex n-dimensional manifold equipped with an ample holomorphic line bundle $L \to X$, that is there exists a Hermitian metric h_L on L with curvature $-2i\pi\omega$, such that ω is Kähler. We denote by g_ω the associated Kähler metric. Note that by the Kodaira theorem, X can be holomorphically embedded in $\mathbb{C}P^N$ for N large enough. Let $1 \leq r \leq n$ be an integer and $E \to X$ be a holomorphic vector bundle of rank r equipped with a Hermitian metric h_E. For any *degree* $d \geq 1$, denote by $H^0(X, E \otimes L^d)$ the space of holomorphic sections of $E \otimes L^{\otimes d}$. By the HirzebruchRiemannRoch theorem,

$$\dim H^0(X, E \otimes L^{\otimes d}) \underset{d\to\infty}{\sim} rd^n \int_X \frac{\omega^n}{n!}.$$

Let $d\mathrm{vol}$ be any volume form on X, and define for any $d \geq 1$ the Hermitian product on $H^0(X, E \otimes L^d)$:

$$\forall s, t \in H^0(X, E \otimes L^d),\ \langle s, t\rangle := \int_X h_{E,L^d}(s, t) d\mathrm{vol}, \tag{1.5}$$

where $h^d_{E,L}$ is the Hermitian metric on $E \otimes L^{\otimes d}$ associated to h_E and h_L. Then we associate to this Hermitian product the Gaussian probability measure $d\mathbb{P}_d$ on $H^0(X, E \otimes L^d)$. In other term, for any $d \geq 1$, choosing an orthnormal basis $(S_i)_{i\in\{1,\cdots N_d\}}$ of $H^0(X, E \otimes L^d)$, where $N_d := \dim H^0(X, E \otimes L^d)$, a random section $s \in H^0(X, E \otimes L^d)$ writes

$$s = \sum_{i=1}^{N_d} a_i S_i,$$

where the complex coefficients $(\Re a_i)_i$ and $(\Im a_i)_i$ are i.i.d. and follow the same normal law $N(0, 1)$. In the sequel

- $Z(s)$ will denote the vanishing locus in X of $s \in H^0(X, E \otimes L^d)$,
- and the tuple $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\mathrm{vol}, (\mathbb{P}_d)_{d\geq 1})$ will be called an *ample probabilistic model*, and *ample model*, if no probability is involved.

By Bertini's theorem, almost surely $Z(s)$ is a compact smooth codimension r complex submanifold of X.

Standard example: the Fubini-Study random polynomial mappings. For $X = \mathbb{C}P^n$, $E = \mathbb{C}P^n \times \mathbb{C}^r$, h_E the standard metric on $\mathbb{C}^r$, $L = \mathcal{O}(1)$ the hyperplane bundle, $h_L = h_{FS}$ the Fubini-Study metric, then

$$H^0(\mathbb{C}P^n, E \otimes L^d) = \left(\mathbb{C}^d_{hom}[Z_0, \cdots, Z_n]\right)^r.$$

Moreover, the monomials given by (1.1) make this identification an isometry. In other terms, a random polynomial mapping for the standard stuctures is a r-uple of independent random polyomials in $H_{d,n+1}$ equipped with the Gaussian measure (1.2).

Random curves. When $r = n-1$, the vanishing locus of the sections of $H^0(X, E \otimes L^d)$ is generically a smooth compact complex curve. When $n = 2$ and $r = 1$, the adjunction formula shows that their genus equals

$$g_d = \frac{1}{2} d^2 \int_X \omega^2 - \frac{1}{2} d \int_X c_1(X) \wedge \omega + 1,$$

where $c_1(X)$ denotes the first Chern class of the surface X, see [13]. Theorem 1.6 has the following natural generalization:

Theorem 1.12 *Let $n \geq 2$ be an integer. Then, there exists a universal constant $c \geq 1$ such that the following holds. Let $(n, n-1, X, L, E, h_L, \omega, g_\omega, h_E, dvol, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Then, there exists $d_0 \geq 1$ such that for every $0 < \varepsilon \leq 1$ and every $d \geq d_0$,*

$$\exp(-\frac{c}{\varepsilon^6}) < \mathbb{P}_d\Big[s \in H^0(X, E \otimes L^d) \mid N_{sys}\big(Z(s), \frac{\varepsilon}{\sqrt{d}}, c\big) > d^n \, Vol_{g_\omega}(X) \exp(-\frac{c}{\varepsilon^6})\Big].$$

Here, the metric on $Z(s)$ is the restriction of the Kähler metric g_ω associated to ω.

Recall that N_{sys} is defined in § 1.3. Note that the volume involved in the Theorem 1.12 is the one associated to g_ω and not with the arbitrary volume form $dvol$ used for the definition of the scalar product (1.5).

Theorem 1.12 means that for any degree large enough, with uniform probability in d, there exists a basis $H_1(Z(s), \mathbb{R})$ such that a uniform proportion of its elements are represented by loops of size bounded by $\varepsilon/\sqrt{d}$.

Remark 1.13 *It is classical [14, Corollary 3.6] that any compact orientable Riemann surface embeds in $\mathbb{C}P^3$. However, a degree d curve in $\mathbb{C}P^3$, that is a holomorphic curve whose class in $H_2(\mathbb{C}P^3, \mathbb{Z})$ equals $d[D]$, where D is a line, can have different topologies, and it is not known which pairs of genus and degree do exist, see [14, IV, 6]. Finally, if E is of rank 2, our model of sections of $H^0(\mathbb{C}P^3, E \otimes L^d)$ only provides strict families of curves of the whole set of curves.*

Lagrangian submanifolds. We provide now a similar Kähler generalization of Theorem 1.10, that is for Lagrangian submanifolds. Let Σ be a complex submanifold in $\mathbb{B} \subset \mathbb{C}^n$, and $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$. For any symplectic manifold (Z, ω, h) equipped with a metric h and $\delta > 0, c > 1$,

- denote by $N(\Sigma, \mathcal{L}, Z, \delta, c)$ the maximal number of pairwise disjoint open sets $\Sigma' \subset Z$ such that Σ' contains a Lagrangian submanifold $\mathcal{L}'$ such that

$$(\mathcal{L}', \Sigma') \sim_{diff} (\mathcal{L}, \Sigma) \text{ and } \frac{\delta}{c} \leq \mathrm{Diam}_{\mathcal{L}'} \mathcal{L}' \leq c\delta.$$

Theorem 1.14 *Let $n \geq 2$, $1 \leq r \leq n-1$, $\Sigma \subset \mathbb{B} \subset \mathbb{C}^n$ be a complex algebraic smooth codimension r submanifold, let $\mathcal{L} \subset \Sigma$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$. Then there exist $c, D \geq 1$ such that the following holds. Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, dvol, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Then, there exists $d_0 \geq 1$ such that for every $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-\frac{c}{\varepsilon^D}) \quad < \quad \mathbb{P}_d\Big[s \in H^0(X, E \otimes L^{\otimes d}) \mid N\big(\Sigma, \mathcal{L}, Z(s), \frac{\varepsilon}{\sqrt{d}}, c\big) > d^n \, Vol_{g_\omega}(X) \exp(-\frac{c}{\varepsilon^D})\Big].$$

The following corollary proves that any compact smooth real affine codimension $(n-r)$ submanifold with trivial normal bundle appears a large number of times in the random complex codimension r submanifold, with a uniform probability:

Corollary 1.15 *Let $n \geq 2$, $1 \leq r \leq n-1$, and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth codimension r submanifold with trivial normal bundle. Then there exist $c, D \geq 1$ such that the following holds. Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, dvol, (\mathbb{P}_d)_{d\geq 1})$ be an ample probabilistic model. Then, there exists $d_0 \geq 1$ such that for every $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-\frac{c}{\varepsilon^D}) \quad < \quad \mathbb{P}_d\Big[s \in H^0(X, E \otimes L^{\otimes d}) \mid N_{Lag}\big(\mathcal{L}, Z(s), \frac{\varepsilon}{\sqrt{d}}, c\big) > d^n \, Vol_{g_\omega}(X) \exp(-\frac{c}{\varepsilon^D})\Big].$$

If $\mathcal{L}$ is algebraic, one can choose D as the double of the degree of $\mathcal{L}$.

Recall that N_{Lag} is defined in § 1.4. Note that when $r = 1$, that is if $\mathcal{L}$ is a hypersurface, the condition on its normal bundle is always satisfied. Corollary 1.15 implies the following generalization of the deterministic Theorem 1.1:

Theorem 1.16 *Let $n \geq 2$, $1 \leq r \leq n$ and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth $(n-r)$-submanifold with trivial normal bundle. Then, there exists $c > 0$ such that for any ample model $(n, r, X, L, E, h_L, \omega, g_\omega, h_E)$, for d large enough, the zero locus of any section $s \in H^0(X, E \otimes L^d)$ vanishing transversally contains at least $cd^n \, Vol_{g_\omega}(X)$ disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.*

Again, by the Lefschetz theorem and a computation with Chern classes, there exists $c > 0$ such that

$$\forall d \gg 1, \forall s \in H^0(X, E \otimes L^d), \ \dim H_*(Z(s), \mathbb{R}) \underset{d\to\infty}{\sim} \dim H_{n-r}(Z(s), \mathbb{R}) \underset{d\to\infty}{\sim} cd^n,$$

see [12, Corollary 3.5.2] for a proof of it with an explicit constant c.

Corollary 1.17 *Under the hypotheses of Theorem 1.16,*

1. *if for every connected component $\mathcal{L}_i$ of $\mathcal{L}$, $\chi(\mathcal{L}_i) \neq 0$, then the classes in $H_{n-r}(Z(s), \mathbb{R})$ generated by these disjoint submanifolds are linearly independent;*
2. *if the $\mathcal{L}_i's$ are simply connected, no Lagrangian copy of them can be isotoped to another one as disjoint Lagrangian submanifolds.*

1.6 Prescribed topology in a small ball

Theorem 1.14 is a consequence of the more precise Theorem 1.19 below. This theorem is partly inspired by the work of J.-Y. Welschinger and the author. For this reason, we recall it. In [12], it was proved the following:

Theorem 1.18 *[12, Proposition 2.4.2] Let $n \geq 2$ and $1 \leq r \leq n$. Then, for any real compact smooth $(n-r)$-submanifold $\mathcal{L} \subset \mathbb{R}^n$ with trivial normal bundle, for any d large enough and any $x \in \mathbb{R}P^n$, with uniform positive probability in d, the zero set $Z(P)$ of a random real polynomial $P \in \mathbb{R}^d_{hom}[X_0, \cdots, X_n]$ intersects $B(x, 1/\sqrt{d})$ along some components, ones of which are diffeomorphic to $\mathcal{L}$.*

This theorem was in fact proved in the general setting of random sections of holomorphic real vector bundles over a projective manifold, see [12]. We begin with an analogous version of Proposition 1.18 for smooth complex algebraic affine hypersurfaces $\Sigma \subset \mathbb{C}^n$

containing a Lagrangian submanifold $\mathcal{L}$. Note that the latter condition is not a constraint since every symplectic manifold contains a Lagrangian torus of any small size enough near every point. Note that contrary to the real case, an affine algebraic complex hypersurface is never compact, and is connected if and only it is the vanishing locus of an irreducible polynomial. Let $n \geq 2$ and $1 \leq r \leq n$ be integers, Σ be a complex submanifold in $\mathbb{B} \subset \mathbb{C}^n$, $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$, and $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\text{vol}, (\mathbb{P}_d)_{d\geq 1})$ be an ample probabilistic model, see § 1.5 for the definition. For any $x \in X$, $\delta > 0$ and $C > 1$, define:

- for any $s \in H^0(X, E \otimes L^d)$, $A\big(\Sigma, \mathcal{L}, Z(s), x, \delta, C\big)$ denotes the event that there exists a smooth topological ball $B \subset X$ containing x and a Lagrangian submanifold $\mathcal{L}'$ of $\big(Z(s) \cap B, \omega_{|Z(s)}\big)$, such that

$$(\mathcal{L}', Z(s) \cap B) \sim_{diff} (\mathcal{L}, \Sigma) \text{ and } \frac{\delta}{c} \leq \text{Diam}(\mathcal{L}') \leq c\delta.$$

Here, the diameter is computed with respect to the induced metric on $\mathcal{L}'$. The main theorem of this paper is the following:

Theorem 1.19 *Let $n \geq 2$, $1 \leq r \leq n-1$ be integers, $\Sigma \subset \overline{\mathbb{B}} \subset \mathbb{C}^n$ be a smooth complex algebraic $(n-r)$-submanifold, and $\mathcal{L} \subset \Sigma$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$. Then there exists $c \geq 1$, such that for any ample probabilistic model $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, dvol, (\mathbb{P}_d)_{d\geq 1})$, there exists $d_0 \geq 1$ such that for any $0 < \varepsilon \leq 1$ and for any $x \in X$,*

$$\forall d \geq d_0,\ \exp(-\frac{c}{\varepsilon^D}) \leq \mathbb{P}_d\Big[A\big(\Sigma, \mathcal{L}, Z(s), x, \frac{\varepsilon}{\sqrt{d}}, c\big)\Big].$$

This theorem implies quickly Theorem 1.14, see below. In fact, the same result holds for affine real hypersurfaces, not only Lagrangians, as in Corollary 1.15 and Theorem 1.18.

1.7 Ideas of the proof of the main theorems

We present the strategy of the proofs of Theorems 1.19, 1.12 and 1.14 for $r = 1$, $\varepsilon = 1$ and for polynomials. The proof holds on two main tools, namely the barrier method for proving uniform probability of some local topological event, together with a quantitative Moser-type construction to make this event symplectic and Lagrangian. The barrier method was used for instance in a real deterministic context in [10] to construct a lot of small spheres in the real part of holomorphic or symplectic Donaldson hypersurfaces. In probabilistic contexts similar to this present work, it was used for instance in [19] to produce small components of the vanishing locus of a random function with uniform probability, and in [11] to produce small components with prescribed diffeomorphism types. The proof of the main Theorem 1.19 is roughly the following:

- fix a point $x \in \mathbb{C}P^n$ and choose for any d a polynomial $Q_{x,d}$ vanishing along a hypersurface $Z(Q_{x,d})$ intersecting $B(x, 1/\sqrt{d})$ along a hypersurface diffeomorphic to Σ. Here, $1/\sqrt{d}$ of the natural scale for Fubini-Study or Kostlan measures. The easiest way to do this is to rescale for every d the same polynomial in an affine chart centered on x.

- Then, for small enough perturbations, the perturbed polynomial still vanishes in $B(x, 1/\sqrt{d})$ along a hypersurface isotopic to Σ. If the allowed pertubation can be quantified, typically when the two-point correlation function of the random function

converges locally to a universal random function after rescaling, one can prove that with a uniform positive probability, a random polynomial of degree d vanishes in the sequence of balls $B(x, 1/\sqrt{d})$ along a hypersurface diffeomorphic to Σ. In our case, we specialize this method in two different ways, depending on the dimension n of the ambient space.

- For $n = 2$ (Theorem 1.12), we choose $\Sigma \subset \mathbb{B} \subset \mathbb{C}^2$ to be a complex curve of degree 3, hence a torus without three small disks. Then a circle whose class in $H_1(\Sigma, \mathbb{R})$ is non-trival will still be non-trivial in $H_1(Z(P), \mathbb{R})$.

- For $n \geq 3$ (Theorem 1.14), In normal affine complex coordinates on the small ball $B(x, 1/\sqrt{d})$, the Fubini-Study form equals the standard form at x, so that the local implementation in $\mathbb{C}P^n$ of $\mathcal{L}$ is almost Lagrangian in $\big(Z(Q_{x,d}), \omega_{FS|TZ(Q_{x,d})}\big)$. Since the perturbation of $Q_{x,d}$ by a random polynomial is complex and not real, there is no natural way to follow $\mathcal{L}$ as a Lagrangian perturbation in the perturbed vanishing locus Σ'. The classical way to deform object of symplectic nature, like the Lagrangians, is the Moser method. We reprove it in our particular situation, but with a quantitative point of view (Theorem 3.4). Thanks to the latter the method keeps the perturbation of $\mathcal{L}$ inside the small ball, so that this small Lagrangian displacement happens with uniform probability. These points provide the idea of the proof of Theorem 1.19. Note that the quantiative Moser trick is needed only for dimensions $n \geq 3$ and not for our result on systoles.

- Then, Theorems 1.12 and 1.14 are direct consequences of Theorem 1.19: choosing in $\mathbb{C}P^n$ a maximal set of small disjoint balls, automatically with uniform probability, at least cd^n of these balls intersect $Z(P)$ along a component diffeomorphic to Σ and contain a Lagrangian copy of $\mathcal{L}$ with the good diameter.

Organization of the paper. In section 2, we we assume Theorem 1.19 and we give the proofs of its consequences presented above. In section 3, we give a quantitative version of the Moser trick. This part is deterministic. In section 4, we give the proof of Theorem 1.19.

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2 Direct proofs

In this section we assume Theorem 1.19 and we give the proofs of its consequences.

2.1 From local to global

Proof of Theorem 1.14. We follow the proof given in [12, §2.5]. Let $c \geq 1$ be given by Theorem 1.19, and let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\mathrm{vol}, (\mathbb{P}_d)_{d\geq 1})$ be an ample probabilistic model. Let $\Lambda_{\varepsilon,d}$ be a subset on X, maximal for the property that any pair of distinct points in $\Lambda_{\varepsilon,d}$ are distant from at least $2\varepsilon/\sqrt{d}$. Then, the union of the balls $B(x, 2\varepsilon/\sqrt{d})$ centered on the points of $\Lambda_{\varepsilon,d}$ cover X, and the balls $B(x, \varepsilon/\sqrt{d})$ are disjoint. Denote by

$N(\Lambda_{\varepsilon,d})$ the number of elements x of $\Lambda_{\varepsilon,d}$ where $A(\Sigma,\mathcal{L},Z(s),x,\varepsilon/\sqrt{d},c)$ happens. Then, by Theorem 1.19,

$$\begin{aligned}
|\Lambda_{\varepsilon,d}|\exp\big(-\frac{c}{\varepsilon^D}\big) &\leq \sum_{x\in\Lambda_{\varepsilon,d}} \mathbb{P}_d\big[A(\Sigma,\mathcal{L},Z(s),x,\varepsilon/\sqrt{d},c)\big] \\
&= \sum_{k=1}^{|\Lambda_{\varepsilon,d}|} k\mathbb{P}_d(N(\Lambda_{\varepsilon,d})=k) \\
&\leq \frac{1}{2}|\Lambda_{\varepsilon,d}|e^{-\frac{c}{\varepsilon^D}}\mathbb{P}_d\Big[N(\Lambda_{\varepsilon,d})\leq\frac{1}{2}|\Lambda_{\varepsilon,d}|e^{-\frac{c}{\varepsilon^D}}\Big] \\
&\quad +|\Lambda_{\varepsilon,d}|\mathbb{P}_d\Big[N(\Lambda_{\varepsilon,d})\geq\frac{1}{2}|\Lambda_{\varepsilon,d}|e^{-\frac{c}{\varepsilon^D}}\Big].
\end{aligned}$$

Consequently, $\mathbb{P}_d\big[N(\Lambda_{\varepsilon,d})\geq\frac{1}{2}|\Lambda_{\varepsilon,d}|e^{-\frac{c}{\varepsilon^D}}\big]\geq\frac{1}{2}\exp\big(-\frac{c}{\varepsilon^D}\big)$. Since

$$\mathrm{Vol}_{g_\omega}(X)\leq\sum_{x\in\Lambda_{\varepsilon,d}}\mathrm{Vol}_{g_\omega}(x,2\varepsilon/\sqrt{d})\underset{d\to\infty}{\sim}|\Lambda_{\varepsilon,d}|(\frac{2\varepsilon}{\sqrt{d}})^{2n}\mathrm{Vol}_{g_0}(\mathbb{B}),$$

there exists a universal $c_n>0$ and d_0 independent of $\varepsilon\leq 1$ but depending on the ample probabilistic model, such that $|\Lambda_{\varepsilon,d}|\geq c_n\mathrm{Vol}_{g_\omega}(X)d^n\varepsilon^{-2n}$ so that

$$\mathbb{P}_d\left[N_{Lag}\big(\Sigma,\mathcal{L},Z(s),x,\frac{\varepsilon}{\sqrt{d}},c\big)\geq c_nd^ne^{-\frac{c}{\varepsilon^D}}\mathrm{Vol}_{g_\omega}(X)\right]\geq\frac{1}{2}e^{-\frac{c}{\varepsilon^D}}.$$

We can now absorb c_n into the exponential, replacing c by smaller positive constant. □

Proof of Theorem 1.10. This is Theorem 1.14 in the standard case and for $r=1$. □

2.2 From probabilistic to deterministic

Proof of Theorem 1.16. Theorem 1.16 is a direct consequence of Corollary 1.15 and the fact that the zeros of holomorphic sections of given degree d have the same diffeomorphism and symplectomorphism type, when there are equipped with the restriction of the ambient Kähler form ω, see Proposition 4.2. □

Remark 2.1 *As said before for projective hypersurfaces, in a parallel paper [9], we prove the deterministic Theorem 1.16 using the deterministic Donaldson [6] and Auroux [2] methods. In the two types of proofs, we use peak sections and a lattice of mesh of order* $1\sqrt{d}$. *In both cases we prove that the Lagrangian submanifolds appear in a uniform proportion of disjoint balls centered on the vertices of the lattice. An advantage of the Donaldson method is that it can be used for Donaldson hypersurfaces in a symplectic compact manifold* (M,ω) *equipped with an almost complex structure* J. *These hypersurfaces are in fact codimension 2 symplectic submanifolds which are vanishing loci of almost holomorphic sections of high powers* $L^{\otimes d}$ *of a complex line bundle* L *over* M, *where* L *is equipped with a Hermitian metric of curvature* $-2i\pi\omega$. *In this general symplectic context, it is not clear which natural space of symplectic hypersurfaces can be used for probabilistic considerations. In [24], the authors replaced the holomorphic sections (which no longer exist in this general context) by the kernel of a certain elliptic operator acting on the bundle, which is the* $\bar\partial_L$ *operator if the almost complex stucture is integrable and the bundle is holomorphic. However, the vanishing locus of a section in this space is a priori not symplectic. The deterministic proof is not easier, since we also need the quantitative version of the Moser method given by Theorem 3.4.*

Proof of Theorem 1.1. This is Theorem 1.16 in the standard case and for $r=1$. □

2.3 Small non-contractible curves

We turn now to the proof of the Theorem 1.12 for the systoles of the random complex curves.

Proof of Theorem 1.12. Define

$$\forall (z_1, z_2) \in \mathbb{C}^2,\ p(z_1, z_2) = z_1^3 + z_2^3 - 1.$$

By the genus formula applied to the homogenization

$$P := Z_0^3 p\big(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}, \frac{Z_3}{Z_0}\big),$$

$Z(P) \subset \mathbb{C}P^2$ is a smooth torus, so that for $\rho > 0$ large enough,

$$\tilde{\Sigma} := \frac{1}{\rho}\big(Z(p) \cap \mathbb{B}(0, \rho)\big) \subset \mathbb{B} \subset \mathbb{C}^2$$

is an affine algebraic complex curve diffeomorphic to $\mathbb{T}^2 \setminus \cup_{i=1}^3 D_i$, where $(D_i)_{i=1}^3$ are three disjoint disc in $\mathbb{T}^2$. Embedding $\mathbb{C}^2$ into $\mathbb{C}^n$ turns $\tilde{\Sigma}$ into an affine algebraic complex curve Σ in $\mathbb{C}^n$. Let $\mathcal{L} \subset \Sigma$ be a smooth circle which is non-trivial in $H_1(\Sigma, \mathbb{Z})$, see Figure 3. Since $\mathcal{L}$ is a Lagrangian, by Theorem 1.14 there exists at least $d^n \mathrm{Vol}_{g_\omega}(X) \exp(-\frac{c}{\varepsilon^D})$ copies of $(\Sigma, \mathcal{L})$ in a random curve $Z(s)$ such that any compy γ_i of $\mathcal{L}$ has intrisic diameter of the order $\varepsilon/\sqrt{d}$, with a uniform probability given by the theorem. The classes in $H_1(Z(s), \mathbb{R})$ generated by the copies of γ form an independent family. Indeed, if $\sum_i \lambda_i [\gamma_i] = 0$, where $(\lambda_i)_i \in \mathbb{R}^N$ and the γ_i are the distinct copies of γ, then there exist codimension 0 surfaces with boundaries $\Sigma_1, \cdots \Sigma_{N'}$ in $Z(s)$ and $(\mu_j)_j \in \mathbb{R}^{N'}$ such that $\sum_i \lambda_i \gamma_i = \sum_j \mu_j \partial \Sigma_j$. This implies that $\partial \Sigma_j$ is a sum of distincts γ_j's. However, if γ_i is one component of the boundary of Σ_j, then the latter must contain the punctured torus $\tilde{\Sigma}$ which contains γ_i, which implies that γ_i bounds on the other side Σ_j, which is a contradiction. □

Proof of Theorem 1.6. Theorem 1.6 is a particular case of Theorem 1.12, choosing $n = 2$, $r = 1$, $X = \mathbb{C}P^2$, $E = \mathbb{C}P^2 \times \mathbb{C}$, h_E the Euclidean metric, $L = \mathcal{O}(1)$ the hyperplane bundle, h_L the Fubini-Study metric and ω the Fubini-Study Kähler form. □

2.4 From disjoint to homologically non-trivial

Proof of Corollary 1.17. The first assertion is a direct consequence of the classical Lemma 2.2 below, remembering that Lagrangian submanifold are totally real for any almost complex stucture tamed by the symplectic form ω. The second assertion was explained in Remark 1.11. □

Lemma 2.2 *Let $\mathcal{L} \subset (X, J)$ be any closed oriented smooth totally real dimension n submanifold in an almost complex manifold X of dimension $2n$. Then,*

$$[\mathcal{L}] \cdot [\mathcal{L}] = \chi(\mathcal{L}),$$

where $[\mathcal{L}] \in H_n(X, \mathbb{Z})$ and $\chi(\mathcal{L})$ denotes the Euler characteristic of $\mathcal{L}$. If $\mathcal{L}_1, \cdots, \mathcal{L}_N$ is a family of disjoint totally real submanifolds of X with nonvanishing Euler characteristic, then the family made of their classes $[\mathcal{L}_1], \cdots, [\mathcal{L}_N]$ in $H_n(X, \mathbb{R})$ is independent.

Proof. For a closed totally real $\mathcal{L} \subset X$, if h is any metric, then $JT\mathcal{L} \sim NL$, where $N\mathcal{L}$ is the normal bundle over $\mathcal{L}$. Then $\chi(\mathcal{L}) = \int_{\mathcal{L}} e(T\mathcal{L}) = \int_L e(N\mathcal{L})$ which equals $[\mathcal{L}| \cdot [\mathcal{L}]$. For the second assertion, if $\sum_{i=k}^m a_i[\mathcal{L}_k] = 0$ in $H_n(X, \mathbb{R})$, where $\mathcal{L}_1, \cdots, \mathcal{L}_m$ are pairwise disjoint totally real submanifolds, then for every j, intersecting by $[\mathcal{L}_j]$ gives $a_j[\mathcal{L}_j]\cdot[\mathcal{L}_j] = 0$ so that in our case, $a_j = 0$. $\square$

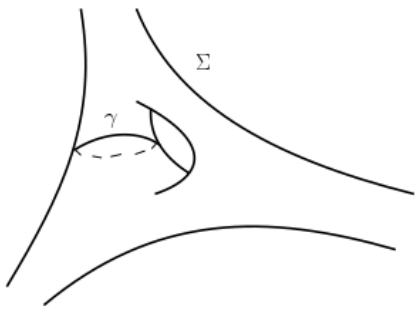

Figure 3: A degree 3 affine complex curve Σ in $\mathbb{C}^2$ with a non-trivial loop.

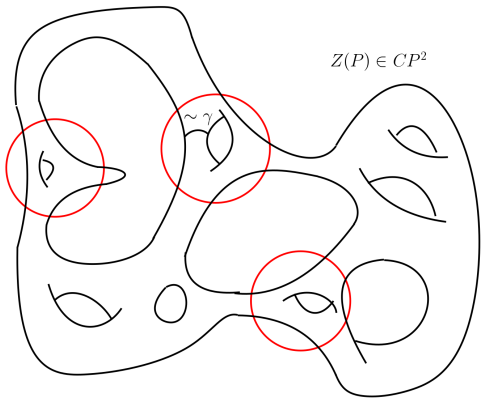

Figure 4: A (nonrealistic) degree 6 curve in $\mathbb{C}P^2$ and three small balls of size $1/\sqrt{d}$ containing the affine complex curve Σ and the non-trivial real curve γ of Figure 3.

2.5 From smooth to algebraic

For the proof of Corollary 1.15 we will use the classical theorem of H. Seifert:

Theorem 2.3 *[22] Let $n \geq 2$, $1 \leq r \leq n$ and $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth $(n-r)$-submanifold with trivial normal bundle. Then, there exists a real polynomial map $p := (p_1, \cdots, p_r) : \mathbb{R}^n \to \mathbb{R}^r$ and a diffeotopy of $\mathbb{R}^n$ sending $\mathcal{L}$ onto some connected components of $Z_{\mathbb{R}}(p)$. The diffeotopy can be chosen as C^1-close to the identity map as we want.*

It is not known which hypersurfaces are diffeotopic to algebraic ones, see [4, Remark 14.1.1].

Proof of Corollary 1.15. By Theorem 2.3, there exists a regular real polynomial map $p = (p_1, \cdots, p_r) : \mathbb{R}^n \to \mathbb{R}^p$ of maximal degree $d(p) := \max_i \deg p_i$ such that $Z_{\mathbb{R}}(p) := Z(p) \cap \mathbb{R}^n$ has a compact component $\mathcal{L}'$ or a set of components diffeomorphic to $\mathcal{L}$. If $\mathcal{L}$ is algebraic, we can choose p such that $Z_{\mathbb{R}}(p) = \mathcal{L}$. By a comprehensible abuse of notation, we keep the notation $\mathcal{L}$ for $\mathcal{L}$. After perturbation, we can assume that p, when considered as defined on $\mathbb{C}^n$, is regular, too. Then, $Z_{\mathbb{R}}(p)$ is a Lagrangian submanifold of its complex

vanishing locus $\Sigma := Z(p)$ equipped by the the restriction of the standard Kähler form ω_0. For a large enough $\rho > 0$, $\rho\mathbb{B}$ contains $\mathcal{L}$. We rescale the polynomial by $1/\rho$ and keep the notation p, so that $Z_{\mathbb{R}}(p) \cap \mathbb{B}$ contains $\mathcal{L}$. Then Corollary 1.15 is a consequence of Theorem 1.14 applied to the couple $(\Sigma \cap \mathbb{B}, \mathcal{L})$. $\square$

3 Quantitative deformations

In this section we introduce and prove deterministic lemmas and propositions which quantify how much a given specific geometrical situation can be perturbed keeping its specificity. The first part concerns the topology, the second part being Lagrangian.

3.1 Preserving the topology

The next proposition is a quantitative and deterministic version of the barrier method for functions. We need first a notation. For any linear mapping $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, where $1 \leq p \leq m$ are integers, define

$$T(A) := \inf_{|w|=1} |A^* w|, \tag{3.1}$$

where $|\cdot|$ denotes the standard Euclidean norm. In the sequel we will use the following simple properties: for any $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$,

- $T(A) > 0$ if and only if A is onto;
- $T(A) \leq \|A\|$, where $\|A\| := \sup_{|v|=1} |Av|$;
- $\|(AA^*)^{-1}\| \leq T(A)^{-2}$;
- if $p = 1$, then $T(A) = \|A\|$;
- for any $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, $T(A + B) \geq T(A) - \|B\|$.

The following proposition provides quantitative estimates for the perturbation of a vanishing locus on $2\mathbb{B}$. It differs from [11, Proposition 3.4] in two ways. First, it allows the vanishing locus to cross the boundary of the ball. Second, it explicits quantitatively the existence of a diffeomorphism sending the vanishing locus to its perturbation. We need indeed to understand how a Lagrangian submanifold of the locus can be moved into another Lagrangian submanifold of the perturbed locus. For this, we give quantitative estimates of the difference between the diffeomorphism and the identity. For $\eta > 0$ and $f, g : 2\mathbb{B} \to \mathbb{R}^p$ C^k mappings, define

$$\forall 0 \leq j \leq k-1,\ c_j(\eta, f, g) := \frac{1}{\eta^{2(j+1)}} \|f\|^{2j+1}_{C^{j+1}(2\mathbb{B})} \|g\|_{C^j(2\mathbb{B})}. \tag{3.2}$$

Note that c_j is an homogeneous function of degree 0; this will be crucial for probabilistic estimates, see (4.5) below.

Proposition 3.1 *Let $m \geq 1$, $1 \leq p \leq m$ and $k \geq 3$ be integers, $\eta > 0$, and $f, g : 2\overline{\mathbb{B}} \subset \mathbb{R}^m \to \mathbb{R}^p$ be two C^k maps, such that $\|g\|_{C^1(2\mathbb{B})} \leq \eta/8$, $c_0(\eta, f, g) \leq 1/8$ and*

$$\forall x \in 2\mathbb{B},\ |f(x)| < \eta \Rightarrow T\big(df(x)\big) > \eta.$$

1. *Then, there exists a* $1-$*parameter family* $(\phi_t)_{t\in[0,1]}$ *of diffeomorphisms with support in* $2\mathbb{B}$ *such that*

$$\forall t \in [0,1],\ \big(Z(f),\overline{\mathbb{B}}\big) \sim_{\phi_t} \big(Z(f+tg),\phi_t(\overline{\mathbb{B}})\big)$$

with $Z(f+tg)\cap \frac{1}{2}\mathbb{B} \subset \phi_t(Z(f)\cap\mathbb{B}) \subset Z(f+tg)\cap\frac{3}{2}\mathbb{B}$, $(x,t)\mapsto \phi_t(x)$ *is* C^{k-1} *and*

$$\forall t \in [0,1],\ \|\phi_t - Id\|_{C^0(2\mathbb{B})} \ \leq\ tc_0(\eta,f,g). \tag{3.3}$$

2. *Let* $j=1,2$ *and* $C>1$ *such that* $c_j(\eta,f,g)\leq C$. *Then there exists* C' *depending only on* C *such that*

$$\forall t \in [0,1],\ \|\phi_t - Id\|_{C^j(2\mathbb{B})} \ \leq\ C'tc_j(\eta,f,g). \tag{3.4}$$

In the proof of the main probabilistic Theorem 1.19 below, the different estimates for the various norms of $\phi_1 - \mathrm{Id}$ in Proposition 3.1 will be used in different ways:

- a small C^0 norm will imply that a Lagrangian submanifold of $Z(f)$ in $\frac{1}{2}\mathbb{B}$ will be send by ϕ_1 in $Z(f+g)$ into a submanifold of $\mathbb{B}$;
- a small C^1 norm implies that ϕ_1 is close to be symplectic, so that the image of the Lagrangian is close to be Lagrangian and can be perturbed into a genuine Lagrangian submanifold of $Z(f+g)$, see Theorem 3.4;
- the bound for the C^2 norm will be used to estimate the intrisic metric on the perturbation of $Z(f)$ on $Z(f+g)$, in order ton obtain the estimates for diameters.

Proof of Proposition 3.1. Define for any $t\in[0,1]$, $f_t := f+tg$. We first prove that

$$\forall (x,t)\in 2\mathbb{B}\times[0,1],\ |f_t(x)|<\eta/2 \Rightarrow T\big(df_t(x)\big)>\eta/2. \tag{3.5}$$

Indeed, $|f_t(x)|<\eta/2$ implies $|f(x)|<\eta$ since $|g(x)|<\eta/2$, so that $T(df(x))>\eta$ by hypothesis, and since $df_t = df + tdg$ and $\|dg(x)\|<\eta/2$, then $T(df_t(x))>\eta/2$. In particular, for all $t\in[0,1]$, $Z(f_t)$ is a C^{k-1} codimension p submanifold of $2\mathbb{B}$. For any $t\in[0,1]$, $\beta>0$, let $V_t(\beta) := \{x\in 2\mathbb{B}, |f_t(x)|\leq\beta\}$. Then, by hypothesis on g,

$$\forall t\in[0,1],\ Z(f_t)\subset V_0\big(\eta/8\big)\subset V_0\big(\eta/4\big)\subset V_t\big(\eta/2\big). \tag{3.6}$$

For all $(x,t)\in V_0\big(\eta/4\big)\times[0,1]$ define $X_t(x)\in\mathbb{R}^m$ to be the projection of the origin onto the $(m-p)$-plane

$$df_t(x)^{(-1)}(\{-\partial_t f_t(x)\})\subset\mathbb{R}^m,$$

which is well defined by (3.6) and (3.5). Note that $X(x,t)=\Phi(df_t(x),g(x))$ where Φ is defined in Lemma 4.5 of the annex. This Lemma 4.5 shows that Φ is a smooth mapping where the first variable is onto, so that X is C^{k-1} where it is defined. Let $\chi:\mathbb{R}\to[0,1]$ be a smooth cut-off function satisfying $\chi_{|(-\infty,1/4]}=1$ and $\chi_{|[1/2,1]}=0$, and define on $2\mathbb{B}$ the family of vector fields

$$\forall (x,t)\in 2\mathbb{B}\times[0,1],\ \tilde{X}_t(x) := \chi\Big(\frac{2}{\eta}|f(x)|\Big)\chi\Big(\frac{|x|-1}{2}\Big)X_t(x).$$

Then, $\tilde{X}_t$ is C^{k-1} in (t,x), for any $t\in[0,1]$ $\tilde{X}_t = X_t$ over $V_0\big(\eta/8\big)\cap\frac{3}{2}\mathbb{B}$, and $\tilde{X}_t=0$ on $\big(V_0(\eta/4)\big)^c$ and on $\partial(2\mathbb{B})$. Now define $(\phi_t)_{t\in[0,1]}$ the family of diffeomorphisms generated by $(\tilde{X}_t)_{t\in[0,1]}$ on $2\mathbb{B}$, that is

$$\forall (x,t)\in 2\mathbb{B}\times[0,1],\ \partial_t\phi_t(x) = \tilde{X}_t\big(\phi_t(x)\big),\ \phi_0=\mathrm{Id}.$$

Note that $(x,t) \mapsto \phi_t(x)$ is C^{k-1}. By construction, ϕ_t can be extend smoothly as the identity outside $2\mathbb{B}$. Since the C^0 norm of $\tilde{X}$ is bounded by the one of X, by Lemma 4.3 and Lemma 4.5,

$$\forall t \in [0,1],\ \|\phi_t - \mathrm{Id}\|_{C^0(2\mathbb{B})} \leq \frac{4t}{\eta^2}\|df\|_{C^0(2\mathbb{B})}\|g\|_{C^0(2\mathbb{B})} \leq 1/2$$

by hypothesis, so that $\forall t \in [0,1],\ \frac{1}{2}\mathbb{B} \subset \phi_t(\mathbb{B}) \subset \frac{3}{2}\mathbb{B}$. Moreover, for any (x,t) such that $\phi_t(x) \in V_0\big(\eta/8\big) \cap \frac{3}{2}\mathbb{B}$,

$$\partial_t\big(f_t(\phi_t(x))\big) = g(\phi_t(x)) + df_t(\phi_t(x)(X(\phi_t(x),t) = 0.$$

By an open-closed argument and the inclusions (3.6), this condition is satisfied if $x \in \mathbb{B}$. Consequently,

$$\forall t \in [0,1],\ Z(f_t) \cap \frac{1}{2}\mathbb{B} \subset \phi_t(Z(f) \cap \mathbb{B}) \subset Z(f_t) \cap \frac{3}{2}\mathbb{B}$$

and the first assertion 1. of the proposition is proved.

Now, since $\forall t \in [0,1],\ dX_t = d\Phi\big(df_t(x), g(x)\big)(d^2 f_t, dg(x))$, Lemma 4.5 gives

$$\begin{aligned}\max_{t\in[0,1]} \|dX_t\|_{C^0(2\mathbb{B})} &\leq \frac{16}{\eta^4}\|g\|_{C^0(2\mathbb{B})}\big(2\|df\|^2_{C^0} + \eta^2/4)\|d^2 f\|_{C^0}\big) + \frac{4}{\eta^2}\|df\|_{C^0}\|dg\|_{C^0} \\ &\leq Kc_1(\eta, f, g),\end{aligned}$$

where K is a universal constant. Moreover, $d\tilde{X}_t = \frac{\nabla|f|}{\eta}\chi' X + \chi dX$, so that

$$\|d\tilde{X}_t\|_{C^0(2\mathbb{B})} \leq K'\big(\frac{\|df\|_{C^0}}{\eta}c_0(\eta, f, g) + c_1(\eta, f, g)\big) \leq K'' c_1(\eta, f, g),$$

where K', K'' depend only on χ. Consequently, by Lemma 4.3, there exists C' depending only on C, such that

$$\|\phi_t - \mathrm{Id}\|_{C^1(2\mathbb{B})} \leq C' t c_1(\eta, f, g).$$

This proves assertion 2. for $j = 1$.

For $j = 2$ in assertion 2., we compute

$$d^2_x X_t(x,t) = d^2\Phi\big(df_t(x), g(x)\big)(d^2 f_t, dg)^2 + d\Phi\big(df_t(x), g(x)\big)(d^3 f_t, d^2 g(x)).$$

so that by Lemma 4.5,

$$\begin{aligned}\|d^2_x X_t\| &\leq 14\|df\|^3\eta^{-6}|g|\|d^2 f\|^2 + 6\eta^{-4}\|df\|^2\|d^2 f\|\|dg\| + \\ &\quad 3\eta^{-4}\|df\|^2|g|\|d^3 f\| + \eta^{-2}\|df\|\|d^2 g\| \\ &\leq 24c_2(\eta, f, g).\end{aligned}$$

A similar estimate for $d^2\tilde{X}$ and Lemma 4.3 imply

$$\forall t \in [0,1],\ \|\phi_t - \mathrm{Id}\|_{C^2(2\mathbb{B})} \leq tC'' c_2(\eta, f, g),$$

where C'' is a constant depending only on C. $\square$

3.2 Preserving the Lagrangianity

The main goal of this paragraph is to prove the technical Proposition 3.3 below. The latter asserts that, in a quantitative way,

- if some compact Lagrangian submanifold $\mathcal{L}$ lies inside a compact symplectic submanifold Σ of a symplectic manifold (M, ω),
- if Σ is perturbed into $\phi(\Sigma)$ by a diffeomorphism ϕ close to the identity,
- and if ω is exact and perturbed by a small 2-form $d\mu$,

then there exists a perturbation $\mathcal{L}' \subset \phi(\Sigma)$ of $\mathcal{L}$ which is Lagrangian for the restriction of the perturbed form. Since we think that this quantitative proposition has its own interest, we provide a general statement and a proof for symplectic manifolds. However, in this paper we will apply it in the simple case where the ambient manifold is the unit ball of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, see Theorem 3.4 below. Before the statement of Proposition 3.3, we need some definitions:

Definition 3.2 *Let (M, h) be a smooth Riemannian manifold, possibly with boundary.*

- *For any pair of continuous maps $f, g : M \to M$ define $d(f, g) := \sup_{x \in M} d(f(x), g(x))$, where d is the distance associated to h.*
- *For any $k \geq 0$ and any C^k vector field X on M, define $N_k(X, M) = \sup_{x \in M, 0 \leq p \leq k} \|\nabla^p X\|$ and similarly $N_k(\alpha, M)$ for any C^k form α on M. Here, ∇ denotes the Levi-Civita connection associated to h.*
- *For any submanifold $\mathcal{L} \subset M$, define $Diam_M(\mathcal{L}) := \max_{p,q \in \mathcal{L}} d(p, q)$.*
- *For any 2-form ω defined on a neighborhood of an open U set of a manifold M equipped with a metric h, let $S(\omega, U) := \inf_{x \in U, X \in T_x M, |X|=1} \sup_{Y \in T_x M, |Y|=1} |\omega(X, Y)|$.*

Note that:

- $\mathrm{Diam}_{\mathcal{L}}(\mathcal{L})$ is the intrisic diameter of $\mathcal{L}$. Note also that if $\mathcal{L}$ is a circle, then its length is bounded by its intrisic diameter.
- If U is relatively compact, then ω is symplectic over U if and only if $S(\omega, U) > 0$;
- if ω_0 denotes the standard symplectic form on $\mathbb{R}^{2n}$, that is $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, then $S(\omega_0, \mathbb{R}^{2n}) = 1$;
- for any pairs of 2-forms ω and ω',
$$S(\omega + \omega', U) \geq S(\omega, U) - \|\omega'\|; \tag{3.7}$$
- if X is a vector field on U, $i_X \omega = \lambda$ and ω is symplectic, then
$$\|X\|_{C^0(U)} \leq \frac{1}{S(\omega, U)} \|\lambda\|_{C^0(U)}; \tag{3.8}$$
- if $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic and vanishes transversally, then $S(\omega_{0|TZ(f)}, U) = 1$.

Proposition 3.3 *Let $1 \leq r \leq n$ be integers, (M, ω, h) a smooth symplectic $2n-$manifold equipped with a metric, $U \subset V \subset W \subset Y$ four relatively compact open sets such that $\overline{U} \subset V$, $\overline{V} \subset W$, $\overline{W} \subset Y$, and assume that there exists λ a smooth 1-form on $\overline{Y}$ such that $\omega_{|\overline{Y}} = d\lambda$. Let $\Sigma \subset \overline{Y}$ be a compact smooth codimension $2r$ submanifold, symplectic for $\omega_{|T\Sigma}$, $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(\Sigma \cap U, \omega_{|T\Sigma})$, $\phi : Y \to Y$ be a smooth diffeomorphism with support in Y, and μ be a smooth 1- form on $\overline{W}$ satisfying*

$$\begin{aligned} d(\phi, Id) &\leq dist(V, \partial W), \\ \|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} &\leq \frac{1}{2} S(\omega_{|T\Sigma}, W \cap \Sigma) dist(U, \partial V) \\ \text{and } \|\phi^*(d\lambda + d\mu) - d\lambda\|_{C^0(W)} &\leq \frac{1}{2} S(\omega_{|T\Sigma}, W \cap \Sigma). \end{aligned}$$

1. *Then, there exists $\mathcal{L}'$ a compact smooth Lagrangian submanifold of $\big(\phi(\Sigma) \cap W, (\omega + d\mu)_{|\phi(\Sigma)}\big)$, such that $\big(\mathcal{L}, \Sigma \cap V\big) \sim_\phi \big(\mathcal{L}', \phi(\Sigma \cap V)\big)$.*
2. *If furthermore $d(\phi, Id) \leq \frac{1}{8} Diam_M(\mathcal{L})$ and*
$$\|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} \leq \frac{1}{16} S(\omega_{|T\Sigma}, W \cap \Sigma) Diam_M(\mathcal{L}),$$
then $\frac{1}{2} Diam_M(\mathcal{L}) \leq Diam_{\mathcal{L}'}(\mathcal{L}')$.
3. *Let $C > 1$. If furthermore*
$$\max\Big(S(\omega_{|T\Sigma}, W \cap \Sigma)^{-1}, N_1\big(\phi^*(\lambda + \mu) - \lambda, W\big), N_1(\omega, W), \|d\phi\|_{C^0(W)}\Big) \leq C,$$
then there exists $C' > 0$ depending only on C, on the pair (V, W) and on the C^1 norm of h over W such that $Diam_{\mathcal{L}'}(\mathcal{L}') \leq C' Diam_{\mathcal{L}}(\mathcal{L})$.

In the proof of Theorem 1.19, where we prove that a given affine complex hypersurface Σ with a Lagrangian submanifold $\mathcal{L}$ appears with uniform probability in a sequence of small balls, we will need Proposition 3.3 applied to the concrete context of Proposition 3.1, where Σ is the vanishing locus of a holomorphic function f, ϕ is a diffeomorphism sending $Z(f)$ onto the perturbed submanifold $Z(f+g)$ and ω the Kähler form viewed in the chart on the standard ball. The next Theorem 3.4 below synthesizes these two propositions for this goal: it asserts, in a quantiative way, that if $\mathcal{L}$ is a compact Lagrangian of a vanishing locus $Z(f)$ which is symplectic for the restriction of the standard form inside the standard ball, as it is the case for the real part of a complex hypersurface defined by a real polynomial, and if g is a small perturbing function, then there exists a perturbation $\mathcal{L}'$ of $\mathcal{L}$ which is a Lagrangian submanifold of $Z(f+g)$ equipped with the restriction of a perturbation $\omega_0 + d\mu$ of the standard form.

Theorem 3.4 *Let $n \geq 1$ and $1 \leq r \leq n$ be integers, $\eta > 0$, and $f, g : 2\overline{\mathbb{B}} \subset \mathbb{R}^{2n} \to \mathbb{R}^{2r}$ be two smooth maps, such that $\|g\|_{C^1(2\mathbb{B})} \leq \eta/8$, and*

$$\forall x \in 2\mathbb{B},\ |f(x)| < \eta \Rightarrow T\big(df(x)\big) > \eta.$$

Let ω be a smooth symplectic form on $2\overline{\mathbb{B}}$ and μ be a smooth 1-form on $2\overline{\mathbb{B}}$ satisfying $\omega = \omega_0 + d\mu$, with

$$\max\big(c_0(\eta, f, g), c_1(\eta, f, g), \|\mu\|_{C^0(2\mathbb{B})}, \|d\mu\|_{C^0(2\mathbb{B})}\big) \leq \frac{1}{16}. \tag{3.9}$$

Let $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $\big(Z(f) \cap \frac{1}{2}\mathbb{B}, \omega_{0|Z(f)}\big)$. Then,

1. *there exists a smooth ball B satisfying $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$ and $\mathcal{L}'$ a compact smooth Lagrangian submanifold of $\big(Z(f+g)\cap B, \omega_{|Z(f+g)}\big)$, satisfying $(\mathcal{L}, Z(f)\cap\mathbb{B}) \sim_{diff} (\mathcal{L}', Z(f+g)\cap B)$.*

2. *If furthermore*

$$\max\big(c_0(\eta,f,g), c_1(\eta,f,g), \|\mu\|_{C^0(2\mathbb{B})}\big) \quad \leq \quad \frac{1}{16} Diam_{Z(f)}(\mathcal{L}), \tag{3.10}$$

then $\frac{1}{2}Diam_{Z(f)}(\mathcal{L}) \leq Diam_{\mathcal{L}'}(\mathcal{L}')$.

3. *Let $C > 1$ be such that, furthermore,*

$$\max\big(c_2(\eta,f,g), N_1(\mu, 2\mathbb{B})\big) \leq C. \tag{3.11}$$

Then there exists $C'' > 1$ depending only on C, such that $Diam_{\mathcal{L}'}(\mathcal{L}') \leq C'' Diam_{\mathcal{L}}(\mathcal{L})$.

The various estimates for the diameters concern the restriction of the standard metric g_0 on $\mathbb{R}^{2n}$. We postpone the proof of Proposition 3.3 and prove the theorem now, which is the consequence of the latter Proposition 3.3 and the former Proposition 3.1.

Proof of Theorem 3.4. By the two first assertions 1. and 2. of Proposition 3.1, there exists a family of diffeomorphisms $(\phi_t)_{t\in[0,1]} : 2\mathbb{B} \to 2\mathbb{B}$ with compact support and a universal constant $K' \geq 1$ such that, writing $\phi = \phi_1$,

$$d(\phi, \mathrm{Id}) = \|\phi - \mathrm{Id}\|_{C^0} \leq c_0(\eta,f,g) \leq \frac{1}{2} \text{ and } \|d\phi - \mathrm{Id}\|_{C^0} \leq K' c_1(\eta,f,g),$$

and $(Z(f), \mathbb{B}) \sim_\phi (Z(f+g), \phi(\mathbb{B})$ with

$$Z(f+g)\cap\mathbb{B} \subset \phi(Z(f)\cap\mathbb{B}) \subset Z(f+g)\cap\frac{3}{2}\mathbb{B}.$$

Let $\lambda_0 := \sum_{i=1}^n x_i dy_i$ be the standard Liouville form, which satisfies $d\lambda_0 = \omega_0$. Note that for any $x \in \mathbb{R}^{2n}$, $\|\lambda_0(x)\| \leq |x|$. Then, using that $S(\omega_{0|TZ(f)}, 2\mathbb{B}) = 1$,

$$\begin{aligned}
\|\phi^*(\lambda_0+\mu) - \lambda_0\|_{C^0(2\mathbb{B})} &\leq \|\phi - \mathrm{Id}\|_{C^0} + \|\lambda_0\|_{C^0}\|d\phi - \mathrm{Id}\|_{C^0} + \|d\phi\|_{C^0}\|\mu\|_{C^0} \\
&\leq c_0(\eta,f,g) + 2c_1(\eta,f,g) + (1+c_1)\|\mu\|_{C^0} \\
&\leq 5\max(c_0, c_1, \|\mu\|_{C^0}) \\
&\leq \frac{1}{2} S\big(\omega_{0|TZ(f)}, 2\mathbb{B}\cap Z(f)\big)\ \mathrm{dist}(\mathbb{B}, 2\mathbb{B})
\end{aligned} \tag{3.12}$$

by (3.9). Similarly,

$$\begin{aligned}
\|\phi^*(d\lambda_0 + d\mu) - d\lambda_0\|_{C^0(2\mathbb{B})} &\leq \|d\phi - \mathrm{Id}\|_{C^0}^2 + 2\|d\phi - \mathrm{Id}\|_{C^0} + \|d\phi\|_{C^0}^2\|d\mu\|_{C^0} \\
&\leq c_1^2 + 2c_1 + (1+c_1)^2\|d\mu\|_{C^0} \\
&\leq \frac{1}{2} S\big(\omega_{0|TZ(f)}, 2\mathbb{B}\cap Z(f)\big)
\end{aligned}$$

again by (3.9). By assertion 1. of Proposition 3.3 applied to $Y = 2\mathbb{B}$, $W = \frac{3}{2}\mathbb{B}$, $V = \mathbb{B}$, $U = \frac{1}{2}\mathbb{B}$, $\Sigma = Z(f)$, and $\omega = \omega_0$, there exists a Lagrangian submanifold $\mathcal{L}'$ of $\big(Z(f+g)\cap\phi(\mathbb{B}), (\omega_0 + d\mu)_{|TZ(f+g)}\big)$, such that

$$\big(\mathcal{L}, Z(f)\cap\mathbb{B}\big) \sim_\phi \big(\mathcal{L}', Z(f+g)\cap\phi(\mathbb{B})\big).$$

If $B := \phi(\mathbb{B})$, then $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$. Hence, the first assertion of the theorem is proved. If furthermore (3.10) is satisfied, using (3.12), the hypotheses of assertion 2. of Proposition 3.3 are satisfied, so that $\frac{1}{2}\mathrm{Diam}_{Z(f)}(\mathcal{L}) \leq \mathrm{Diam}(\mathcal{L}')$. This proves the second assertion. Now, if (3.11) is satisfied, by assertion 2. of Proposition 3.1, there exists C'' depending only on C such that $\|d^2\phi\|_{C^0(2\mathbb{B})} \leq C''$. This implies that there is a universal constant K''' and a constant C''' depending only on C such that

$$N_1\big[\phi^*(d\lambda_0 + d\mu) - d\lambda_0), 2\mathbb{B}\big] \leq K'''\big(\|d\phi\|_{C^0} N_1(d\mu, 2\mathbb{B}) + \|d^2\phi\|_{C^0}\|d\mu\|_{C^0}\big) \leq C''.$$

Consequently,

$$\max\Big(S\big(\omega_{0|TZ(f)}, \frac{3}{2}\mathbb{B} \cap Z(f)\big)^{-1}, N_1\big(\phi^*(d\lambda_0 + d\mu) - d\lambda_0, \frac{3}{2}\mathbb{B}\big), N_1(\omega_0, 2\mathbb{B})\Big) \leq C''',$$

where C''' depends only on C. We can now apply assertion 3. of Proposition 3.3: there exists a constant $C'''' > 0$ depending only on C such that $\mathrm{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C'\mathrm{Diam}_{\mathcal{L}}(\mathcal{L})$. □

The main steps for proving Proposition 3.3 are the following:

- Recall that in Proposition 3.3, a symplectic submanifold Σ is deformed by a diffeomorphism ϕ into Σ', and the ambient symplectic form ω is deformed into $\omega + d\mu$.
- The restriction of the perturbed form on Σ' can be viewed as $\Omega' = \phi^*(\omega + d\mu)_{|T\Sigma}$ on Σ. Proposition 3.5 constructs, in a general setting, an isotopy of local diffeomorphisms $(\psi_t)_t$ on Σ, such that ψ_1 is a symplectomorphism between Ω' and a given symplectic form Ω, which is $\Omega = \omega_{|T\Sigma}$ in our case, with an explicit control of $\psi_1 - \mathrm{Id}$ depending on $\Omega - \Omega'$ and its primitive.
- Corollary 3.6 applies this intrisic Proposition 3.5 to the relative situation of Proposition 3.3, and transfers the latter control to controls depending on $\phi - \mathrm{Id}$ and the perturbation of the ambient symplectic form.
- The proof of Proposition 3.3 consists in applying this corollary to the deformation of the Lagrangian submanifold.

Moser trick. The next Proposition is a quantitative version of the Moser's trick.

Proposition 3.5 *Let (Σ, Ω, H) be a smooth symplectic manifold, possibly with boundary, equipped with a metric H, and $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be three relatively compact open sets in Σ such that $\overline{\mathcal{U}} \subset \mathcal{V}$ and $\overline{\mathcal{V}} \subset \mathcal{W}$. Let ν be a smooth 1-form on $\overline{\mathcal{W}}$ satisfying*

$$\|\nu\|_{C^0(\mathcal{W})} \leq \frac{1}{2}S(\Omega, \mathcal{W}))\,dist(\mathcal{U}, \partial\mathcal{V}) \text{ and } \|d\nu\|_{C^0(\mathcal{W})} \leq \frac{1}{2}S(\Omega, \mathcal{W}).$$

1. *Then, there exists a smooth family of diffeomorphisms $(\psi_t)_{t\in[0,1]} : \mathcal{W} \to \mathcal{W}$ with support in $\mathcal{W}$ such that*

$$\forall t \in [0,1],\ \psi_t^*\big(\Omega + t d\nu\big) = \Omega \text{ on } \mathcal{U},\ \psi_t(\mathcal{U}) \subset \mathcal{V}, \text{ and } d(\psi_t, Id) \leq \frac{2t}{S(\Omega, \mathcal{W})}\|\nu\|_{C^0(\mathcal{W})}.$$

2. *Let $C > 1$ and assume that*

$$\max\big(S(\Omega, \mathcal{W})^{-1}, N_1(\nu), N_1(\Omega)\big) \leq C. \tag{3.13}$$

Then, there exists $C' > 0$ depending only on $(\mathcal{U}, \mathcal{V})$, on the C^1 norm of H on $\mathcal{W}$ and on C, such that $\|d\psi_t\|_{C^0(\mathcal{W})} \leq C'$.

Proof. For any $t \in [0,1]$, let $\Omega_t := \Omega + td\nu$. Then for every $t \in [0,1]$, by hypothesis and (3.7), $S(\Omega_t, \mathcal{W}) \geq \frac{1}{2}S(\Omega, \mathcal{W})$ which is positive since $\overline{\mathcal{W}}$ is compact, so that Ω_t is symplectic on $\overline{\mathcal{W}}$. We are looking for a 1-parameter family of diffeomorphisms $(\psi_t)_{t\in[0,1]}$ of $\mathcal{W}$ such that $\forall t \in [0,1], \psi_t^*(\Omega_t) = \Omega$ over $\psi_t(\mathcal{U})$. Derivating in time, and assuming that $(X_t)_{t\in[0,1]}$ is a vector field that generates $(\psi_t)_t$, we obtain $\partial_t\Omega_t + d(i_{X_t}\Omega_t) = 0$, or $d(\nu + i_{X_t}\Omega_t) = 0$. We now inverse now the procedure. Let $(X_t)_{t\in[0,1]}$ be a family of vector fields on $\mathcal{W}$ such that

$$\forall t \in [0,1],\ \forall x \in \mathcal{W},\ i_{X_t(x)}\Omega_t(x) = -\nu(x). \tag{3.14}$$

Since Ω_t is non-degenerate, X_t is uniquely defined, smooth and by (3.8),

$$\forall t \in [0,1],\ \|X_t\|_{C^0(\mathcal{W})} \leq \frac{2}{S(\Omega, \mathcal{W})}\|\nu\|_{C^0(\mathcal{W})}. \tag{3.15}$$

Let $\chi : \mathcal{W} \to [0,1]$ be a smooth cut-off function such that $\chi_{\overline{\mathcal{V}}} = 1$ and χ has support in $\mathcal{W}$. Let $(\psi_t)_{t\in[0,1]}$ be the 1-parameter family of diffeomorphisms associated to χX_t. By Lemma 4.4,

$$\forall t \in [0,1],\ d(\psi_t, \mathrm{Id}) \leq \frac{2t}{S(\Omega, \mathcal{W})}\|\nu\|_{C^0(\mathcal{W})}. \tag{3.16}$$

By hypothesis on $\|\nu\|$, this implies that $\forall t \in [0,1],\ \psi_t(\mathcal{U}) \subset \mathcal{V}$. Since $\chi = 1$ over $\mathcal{V}$, we obtain $\psi_1^*\Omega_1 = \Omega$ over $\psi_1(\mathcal{U}) \subset \mathcal{V}$.

We now assume that (3.13) is satisfied and want a bound for the derivative of ψ_1. Derivating equation (3.14) gives $\forall t \in [0,1],\ i_{X_t}\nabla\Omega_t + i_{\nabla X_t}\Omega_t = -\nabla\nu$ over $\mathcal{W}$, so that

$$\max_{t\in[0,1]} \|\nabla X_t\|_{C^0(\mathcal{W})} \leq \frac{2}{S(\Omega, \mathcal{W})}(\|\nabla\Omega_t\|_{C^0(\mathcal{W})}\|X_t\|_{C^0(\mathcal{W})} + \|\nabla\nu\|_{C^0(\mathcal{W})})$$

and $\max_{t\in[0,1]} \|\nabla(\chi X_t)\| \leq \max_{t\in[0,1]} \|\nabla X_t\|_{C^0} + \|d\chi\|\|X_t\|_{C^0}$. By Lemma 4.4 and (3.16), this implies that $\max_{t\in[0,1]} \|d\psi_t\|_{C^0} \leq C'$, where C' depends only on the derivative of the metric on $\mathcal{W}$, on C and χ, hence on $(\mathcal{V}, \mathcal{W})$. $\square$

In Corollary 3.6 below, we apply the latter proposition to the situation that is of interest for us: the construction of a symplectomorphism Ψ between a symplectic submanifold Σ in a ambient manifold (M, ω) and another submanifold $\phi(\Sigma)$ equipped the restriction of another symplectic structure $\omega + d\mu$ which is close to ω. Then, the proof of Proposition 3.3 will be a direct consequence of Corollary 3.6.

Corollary 3.6 *Under the hypotheses of Proposition 3.3,*

1. *there exists a smooth isotopy of embeddings* $(\Psi_t)_{t\in[0,1]} : \Sigma \cap W \to \phi(\Sigma)$ *satisfying*

$$\begin{aligned} \forall t \in [0,1],\ \Psi_t^*\big((\omega + td\mu)_{|T\phi(\Sigma)}\big) &= \omega_{|T\Sigma} \text{ on } U \cap \Sigma \text{ with } \Psi_t(U \cap \Sigma) \subset W \cap \phi(\Sigma), \\ \text{and } d(\Psi_1, Id_{|\Sigma\cap W}) &\leq \frac{2}{S(\omega_{|T\Sigma}, W \cap \Sigma)}\|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} + d(\phi, Id). \end{aligned}$$

2. *If furthermore the hypotheses of 3. of Proposition 3.3 are satisfied, there exists* $C' > 0$ *depending only on* U, V, Σ*, the* C^1 *norm of* h *on* W *and on* C*, such that* $\forall t \in [0,1], \|d\Psi_t\|_{C^0(\Sigma\cap W)} \leq C'$.

Proof. Let $j : \Sigma \to W$ be the natural injection, $\Omega := j^*\omega$ and $\nu := j^*\big(\phi^*(\lambda + \mu) - \lambda\big)$, so that $j^*\big(\phi^*(\omega + d\mu)\big) = \Omega + d\nu$. Choosing the metric H on Σ to be the induced one by the

ambient metric h, the various estimates for ν are bounded by the ones for $\phi^*(\lambda+\mu)-\lambda$, so that, using that the induced distance in Σ is larger than the one in M,

$$\|\nu\|_{C^0(W\cap\Sigma)} \leq \frac{1}{2}S(\omega_{|T\Sigma}, W\cap\Sigma)\ \mathrm{dist}_\Sigma(U\cap\Sigma, \partial V\cap\Sigma) \text{ and } \|d\nu\|_{C^0(W\cap\Sigma)} \leq \frac{1}{2}S(\omega_{|T\Sigma}, W\cap\Sigma).$$

By assertion 1 of Proposition 3.5 applied to (Σ,Ω), $\mathcal{W}=\Sigma\cap W$, $\mathcal{V}=\Sigma\cap V$, $\mathcal{U}=\Sigma\cap U$, $H=h_{|T\Sigma}$, and ν, there exists a 1-parameter family of diffeomorphisms: $\psi_t:\Sigma\cap W\to\Sigma\cap W$ with support in $W\cap\Sigma$ such that for any $t\in[0,1]$, $\psi_t(U\cap\Sigma)\subset V\cap\Sigma$,

$$d(\psi_t,\mathrm{Id}) \leq \frac{2t}{S(\omega_{|T\Sigma}, W\cap\Sigma)}\|\nu\|_{C^0(W\cap\Sigma)} \leq \mathrm{dist}(\mathcal{U},\partial V), \tag{3.17}$$

and $\psi_t^*\Big(j^*\big(\phi^*(\omega+d\mu)\big)\Big)=\omega_{|T\Sigma}$ on $U\cap\Sigma$. For any $t\in[0,1]$, let $\Psi_t:=\phi\circ\psi_t: W\cap\Sigma\to Y$. Then, since by hypothesis $d(\phi,\mathrm{Id})\leq\mathrm{dist}(V,\partial W)$, we have by (3.17) $\Psi_t(U\cap\Sigma)\subset W\cap\phi(\Sigma)$. Moreover,

$$\big(\Psi_t^*(\omega+d\mu)\big)_{|T\Sigma} = \Big(\psi_t^*\big(\phi^*(\omega+td\mu)\big)\Big)_{|T\Sigma} = \Omega \text{ on } U\cap\Sigma.$$

This proves the first assertion.

Now, assume that the hypotheses of 3. in Proposition 3.3 are satisfied. Then $N_1(\nu, W\cap\Sigma)\leq C$, so that by assertion 2 of Proposition 3.5, there exists $C'>0$ depending only on $(\mathcal{U},\mathcal{V})$, the C^1 norm of H and C, $\|d\psi_t\|_{C^0(W\cap\Sigma)}\leq C'$. Since $\|d\phi\|_{C^0(Y)}\leq C$, we have $\|d\Psi_t\|_{C^0(W\cap\Sigma)}\leq CC'$, hence the result after changing the definition of C'. □

We can now give the proof of Proposition 3.3, which demonstrates the stability of a Lagrangian submanifold in a symplectic submanifold when the latter and the symplectic form are perturbed.

Proof of Proposition 3.3. By Corollary 3.6 there exists a smooth diffeomorphism

$$\Psi:\Sigma\cap U\to\Psi(\Sigma\cap U)\subset\phi(\Sigma)\cap W$$

such that $\big(\Psi^*(\omega+d\mu)\big)_{|T\Sigma}=\omega_{|T\Sigma}$ on $U\cap\Sigma$. This implies that $\mathcal{L}':=\Psi(\mathcal{L})$ is a smooth compact Lagrangian submanifold in $\big(\phi(\Sigma)\cap W, (\omega+d\mu)_{|T\phi(\Sigma)}\big)$.

Assume now that the hypotheses of 2 in Proposition 3.3 are satisfied. Then by Corollary 3.6, $d(\Psi,\mathrm{Id})\leq\frac{1}{4}\mathrm{Diam}_M\mathcal{L}$. Let $p,q\in\mathcal{L}$ such that $\mathrm{Diam}_M(\mathcal{L})=d_M(p,q)$. Then

$$\mathrm{Diam}_{\mathcal{L}'}(\mathcal{L}')\geq d_{\mathcal{L}'}(\Psi(p),\Psi(q))\geq d_M(\Psi(p),\Psi(q))\geq d_M(p,q)-2d(\Psi,\mathrm{Id})\geq\frac{1}{2}\mathrm{Diam}_M(\mathcal{L}).$$

Assume now that the hypothesis of 3 in Proposition 3.3 is satisfied. Again by Corollary 3.6, there exists C' such that $\|d\Psi\|_{C^0(\Sigma\cap W)}\leq C'$. This implies $\mathrm{Diam}_{\mathcal{L}'}(\mathcal{L}')\leq C'\mathrm{Diam}_{\mathcal{L}}(\mathcal{L})$. Indeed, let $p',q'\in\mathcal{L}'$ and $\gamma:[a,b]\to\mathcal{L}$ such that γ is a shortest path in $\mathcal{L}$ between $p:=\Psi^{-1}(p')\in\mathcal{L}$ and $q:=\Psi^{-1}(q')\in\mathcal{L}$. Then,

$$d_{\mathcal{L}'}(p',q')\leq Length_{\mathcal{L}'}(\Psi(\gamma))=\int_a^b|d\Psi(\gamma)(\gamma'(t))|dt\leq C'\mathrm{Diam}(\mathcal{L}).$$

□

4 Proof of the main local theorem

4.1 The standard setting

Proof of Theorem 1.19. We adapt the barrier method of the real context in [11] to our complex algebraic situation, and will use the quantitative Moser method given by Theorem 3.4. For the reader's convenience, we begin by the proof in the case of the standard random polynomials. Then we sketch the proof for the general setting of random holomorphic sections.

Let $p \in \mathbb{C}[z_1, \cdots, Zz_n]$ regular such that $Z(p) \cap \mathbb{B} = \Sigma$. Since p is regular, there exists $\eta > 0$ such that $p : 2\mathbb{B} \to \mathbb{C}^r$ satisfies the transversality condition

$$\forall z \in 2\mathbb{B},\ |p(z)| < \eta \Rightarrow T\big(dp(z)\big) > \eta, \tag{4.1}$$

where T is defined by (3.1).

Since the probability measure is invariant under the symmetries of $\mathbb{C}P^n$, as well as the assertion of Theorem 1.19, it is enough to prove the theorem for $x = [1 : 0 : \cdots : 0]$. Let z be the local holomorphic affine coordinates:

$$z = (z_1, \cdots, z_n) := \Big(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0}\Big) \in \mathbb{C}^n$$

defined on $\mathbb{C}P^n \setminus \{Z_0 = 0\}$. Fix $\varepsilon > 0$ and let $p_{\varepsilon,d}(z) := p\Big(z\frac{\sqrt{d}}{\varepsilon}\Big)$. Note that $Z(p_{\varepsilon,d}) = \frac{\varepsilon}{\sqrt{d}}\Sigma$. Then for any $d \geq d(p)$, let

$$P_{\varepsilon,d}(Z) := Z_0^d p_{\varepsilon,d}\Big(\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0}\Big) \in \big(\mathbb{C}^d_{hom}[Z_0, \cdots, Z_n]\big)^r. \tag{4.2}$$

By construction, $Z(P_{\varepsilon,d}) \subset \mathbb{C}P^n$ intersects the affine coordinate ball $B(0, \varepsilon/\sqrt{d})$ around $[1 : 0 \cdots : 0]$ along a small homothetical copy of Σ and contains a copy of $\mathcal{L}$. Notice that $P_{\varepsilon,d}$ is singular, since $Z(P_{\varepsilon,d})$ contains the hyperplane $\{X_0 = 0\}$ with multiplicity $d - d_0$.

In order to apply the first item of Theorem 3.4, we must have a bound for the perturbation of ω_0 in ω. For this, in our affine coordinates, let $\lambda_{FS} = d^c \log(1 + |z|^2)$ and $\lambda_0 = d^c|z|^2$, that is

$$\lambda_{FS} = \frac{1}{2i}\frac{\sum_{i=1}^n z_i d\bar{z}_i - \bar{z}_i dz_i}{1 + |z|^2}.$$

By definition $\omega_{FS} = d\lambda_{FS}$ and $\omega_0 = d\lambda_0$, so that $\lambda_{FS} = \lambda_0 + O(\|z\|^3)$, and $d\lambda_{FS} = \omega_0 + O(\|z\|^2)$. Let ψ the linear map $\psi(z) = z\frac{\varepsilon}{\sqrt{d}}$. Then, there exists a universal constant $K > 0$ such that the pull-backs $\lambda = \frac{d}{\varepsilon^2}\psi^*\lambda_{FS}$ and $\omega = \frac{d}{\varepsilon^2}\psi^*\omega_{FS}$ satisfy

$$\forall d \geq d_{\mathcal{L}} := \frac{16K}{\min(1, \mathrm{Diam}_{Z(p)}\mathcal{L})},\ \|\lambda - \lambda_0\|_{C^0(2\mathbb{B})} + \|\omega - \omega_0\|_{C^0(2\mathbb{B})} \leq K\frac{\varepsilon^2}{d} \leq \frac{1}{16}\min(1, \mathrm{Diam}_{Z(p)}\mathcal{L}),$$

which is the bound needed in Theorem 3.4 for the perturbation form μ and its differential, see conditions (3.9) and (3.10).

Now let $H_P := P_{\varepsilon,d}^{\perp}$ be the orthogonal space to $P_{\varepsilon,d}$ in $\big(\mathbb{C}^d_{hom}[Z], \langle\ ,\rangle\big)$. We use a decomposition for our random polynomials adapted to $P_{\varepsilon,d}$ an H_P. Since the random polynomial can be written in any fixed orthormal basis, we can decompose our random polynomial P as

$$P = a\frac{P_{\varepsilon,d}}{\|P_{\varepsilon,d}\|_{L^2}} + R, \tag{4.3}$$

where a is a complex Gaussian variable and $R \in H_P$ is a Gaussian random polynomial for the induced law on H_P and independent of a. The L^2-norm of $P_{\varepsilon,d}$ is computed by Lemma 4.6 below. We want to prove that with uniform positive lower bound, R does not perturb too much the first term, so that P still vanishes on a hypersurface diffeomorphic to Σ. Hence, we need to know when the vanishing locus of a perturbation of a function gives a diffeotopic perturbation of the vanishing locus of the function. For this, for any $d \geq d(p)$, we apply Theorem 3.4 to

$$\forall z \in 2\mathbb{B},\ f(z) := aP_{\varepsilon,d}(1, z\frac{\varepsilon}{\sqrt{d}}) = ap(z) \text{ and } g(z) := \|P_{\varepsilon,d}\|_{L^2} R(1, z\frac{\varepsilon}{\sqrt{d}}).$$

By (4.1) we have

$$\forall a \in \mathbb{C}^*,\ \forall z \in 2\mathbb{B},\ |f(x)| < |a|\eta \Rightarrow T\big(df(x)\big) > |a|\eta.$$

We want now to give a uniform lower bound for the probability that the pair of random functions (f, g) on $2\mathbb{B}$ satisfies the various conditions of Theorem 3.4. In order to control the peturbation g, we decompose it as

$$g = p_1 + p_2 := \frac{1}{2}(g + f) + \frac{1}{2}(g - f).$$

Note that the law of $p_1 := g + f$ is the same of $r(z) := \|P_{\varepsilon,d}\|_{L^2} P(1, z\frac{\varepsilon}{\sqrt{d}})$, where P follows the Fubini-Study law. The same holds for $p_2 := g - f$. We use the trivial inequality

$$\mathbb{E}\sup_{2\mathbb{B}}|g| \leq \frac{1}{2}\big(\mathbb{E}\sup_{2\mathbb{B}}|p_1| + \mathbb{E}\sup_{2\mathbb{B}}|p_2|\big) \leq \mathbb{E}\sup_{2\mathbb{B}}|r|, \tag{4.4}$$

and similarly for the average of the derivative of g. Hence, it is enough to bound from above the norms of a random q.

By Markov inequality, by the independence between f and p_1, p_2, by the bound (4.4), by Remark 4.7 and by Lemma 4.8, there exists $K_P > 0$ depending only on P that for all $0 < \varepsilon \leq 1$, $d \geq d(p)$, $F > 0$, $0 < \alpha \leq 1$, and $\forall j \in \{0, 1, 2\}$,

$$\begin{aligned} \mathbb{P}_d\Big[\exists a \in \mathbb{C}^*, \|g\|_{C^1} \leq |a|\eta/8, c_j(|a|\eta, f, g) \leq \alpha\Big] &\geq \\ \mathbb{P}_d\left[F \leq |a|, \|g\|_{C^1} \leq F\eta/8, \|g\|_{C^j} \leq F\alpha\frac{\eta^{2j+2}}{\|p\|_{C^3}^{2j+1}}\right] &\geq \\ \frac{1}{\pi}\int_{F<|a|} e^{-|a|^2}|da|\Big(1 - \frac{K_P^2}{\varepsilon^{2d(p)}\alpha^2 F^2}\Big). \end{aligned} \tag{4.5}$$

Recall that c_j is defined by (3.2). For $j = 2$, let $F = F_\varepsilon := 2\frac{K_P}{\alpha\varepsilon^{d(p)}}$ and $\alpha = \frac{1}{16}\mathrm{Diam}_{\mathbb{R}^{2n}}\mathcal{L}$. Then, there exists a constant $C_P > 0$ depending only on P such that for all $d \geq d(p)$, $0 < \varepsilon \leq 1$, the probability (4.5) is bounded from above by $C_P \exp(-\frac{C_P}{\varepsilon^{2d(P)}\mathrm{Diam}^2_{\mathbb{R}^{2n}}\mathcal{L}})$. By assertions 1., 2. and 3. of Theorem 3.4, there exists C'' depending only on $\mathrm{Diam}_{\mathbb{R}^{2n}}\mathcal{L}$, such that for $d \geq \max\big(d(p), d_{\mathcal{L}}\big)$, with the same probability, there exists a topological ball B satisfying $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$, and $\mathcal{L}'$ a compact smooth Lagrangian submanifold of $\big(Z(f+g) \cap B, \omega_{|Z(f+g)}\big)$, satisfying

$$(\mathcal{L}, Z(f) \cap \mathbb{B}) \sim_{diff} (\mathcal{L}', Z(f+g) \cap B)$$

with $\frac{1}{2}\mathrm{Diam}_{Z(f)}(\mathcal{L}) \leq \mathrm{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq \mathcal{C}''\mathrm{Diam}_{\mathcal{L}}(\mathcal{L})$. Here, the metrics are the various restrictions of the standard metric g_0 on the ball. However, the push-forward of the metric g_ω on the unit ball by the coordinates $z\varepsilon/\sqrt{d}$ converges uniformly in $0 \leq \varepsilon \leq 1$ to g_0 when d grows to infinity. This implies the theorem. □

4.2 The general Kähler setting

The generalization of the proof of Theorem 1.19 to random holomorphic sections holds onto the concept of *peak sections*, as in [11] and [12]. This object was used by Tian [26] to give estimates for the Bergman kernel, and by S. K. Donaldson [6] to prove the existence of codimension 2 symplectic submanifolds. In a way, they were already used by Hörmander to solve the Levi problem for Stein manifolds [15, Theorem 5.1.6] . They are used in the parallel paper [9] for a deterministic proof of Corollary 1.16.

Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\mathrm{vol}, (\mathbb{P}_d)_{d\geq 1})$ be an ample probabilistic model A peak section of $L^{\otimes d}$ at $x \in X$ is a holomorphic section which norm decreases exponentially fast outside x, and almost vanishing at scale $\gg 1/\sqrt{d}$, like X_0^d in the standard projective case near the point $[1 : 0 \cdots 0]$. One of their crucial interest lies in the fact that a given peak section times the monomials (1.1) in normal holomorphic coordinates form asymptotically an orthonormal family, which make the general Kähler situation locally very similar to the standard projective one.

Proof of Theorem 1.14. Let $x \in X$, and e a local holomorphic trivialization of L near x such that $\|e\|_{h_L}$ is locally maximal at x, with $\|e(x)\|_{h_L} = 1$. Then there exists a uniform (in $x \in X$) constant $c > 0$ such that for any y in a fixed neighborhood of x,

$$\|e^{\otimes d}(y)\|_{h_L} \leq \exp(-cd\|x - y\|^2). \tag{4.6}$$

This is implied by the fact that the curvature of h_L is a Kähler form and the uniformity is implied by the compacity of X. Again, $X_0^d = e^d$ in the standard case. Let $(e_1, \cdots, e_r)$ be a local holomorphic trivialization of E, orthonormal at x. Then, $(e_1 \otimes e^d, \cdots, e_r \otimes e^d)$ is a local holomorphic trivialization of $E \otimes L^d$ whose coordinates are *called peak sections* for x. Now, let $(p_1, \cdots, p_r)$ be a polynomial map that defines the complex algebraic hypersurface Σ, and

$$s_{\varepsilon,d,p} := \Big(p_i\big(\cdot\,\frac{\sqrt{d}}{\varepsilon}\big)e_i\Big)_{1\leq i\leq r} \otimes e^d,$$

which is a section of $E \otimes L^d$ defined in a fixed neighborhood of x, and is the equivalent of $P_{\varepsilon,d}$, see (4.2) in the standard case. Now by the Hörmander L^2-estimates, see [15] or [25] for a bundle version of it, $s_{\varepsilon,d,p}$ can be perturbed in a global section $\sigma_{\varepsilon,d,p} \in H^0(X, E \otimes L^d)$. Moreover, this is a classical result in Hörmander theory that the C^1 error produced by the perturbation on $B(x, \frac{\log d}{\sqrt{d}})$ is bounded by $\exp(-cd)$, see [11, Lemma 3.5]. Here, the estimates (4.6) are crucial. By Lemma 3.1 this implies that $Z(s_{\varepsilon,d,p})$ is a complex $(n-r)$-submanifold which is an isotopic peturbation of $Z(p)$.

The rest of the proof is very similar to the standard case. We decompose the random section $s \in H^0(X, E \otimes L^d)$ as

$$s = a\frac{s_{\varepsilon,d,p}}{\|s_{\varepsilon,d,p}\|_{L^2}} + \rho,$$

where $\rho \in s_{\varepsilon,d,p}^{\perp}$ and $s_{\varepsilon,d,p}^{\perp}$ is equipped with the restriction of the Gaussian measure, and a follows a complex normal law $N_{\mathbb{C}}(0, 1)$. The L^2-norm of $s_{\varepsilon,d,p}$ has a similar equivalent as $\|P_{\varepsilon,d}\|_{L^2}$ given by Lemma 4.6. Then we look the situation on $B(x, \varepsilon/\sqrt{d})$ which becomes a fixed $\mathbb{B} \in \mathbb{C}^n$ after rescaling, and the sections are trivialized as functions with values in $\mathbb{C}^r$. Lemma 4.8 still holds for the trivialization q of the pertubation. In the proof of it, the only essential adaptation in the bundle case is the estimate (4.10), where the modulus of the function is compared on $B(0, \varepsilon/\sqrt{d})$ with the Fubini-Study norm of it. In the present situation, a similar comparison holds, since the norm of e^d varies only of a uniform positive multiplicative constant over $B(x, \varepsilon/\sqrt{d})$.

The Lagrangian part of the proof is the same, since the coordinates at a point $x \in X$ we can choose holomorphic coordinates z such that $\omega = z^*\omega_0$ at x, so that we can find a $1-$form λ on the chart so that $\lambda - z^*\lambda_0 = O(|z|)$, which is the two only thing we need. $\square$

Remark 4.1 *Instead of peak sections, we could have use the Bergman kernel, the Schwartz kernel for the projection onto the space of holomorphic sections, and the 2-point correlation function for our random model. This kernel converges at scale $1/\sqrt{d}$ to a universel kernel, the* Bargmann-Fock *kernel, see [3], which explains why the results on standard Fubini-Study random polynomials are similar to those for random holomorphic sections. This universality can be proved by peak sections, see [26]. The kernel approach has the virtue that parts of the proof can be adapted to other random models. However, we must not overestimate this interest for some reasons. Firstly, the fact that the zeros of the sections of given degree have the same topology and symplectomophism type is very dependent on holomorphicity, or at least asymptotic holomorphy in the Donaldson [6] and Auroux settings [2]. Secondly, the projective hypersurface inherit a natural symplectic form, which is rarely the case for other models. Thirdly, the barrier method is very adapted to explicit local sections, like peak sections. Fourthly, the fact that this model is particularly well suited for polynomials is not directly seen from the kernel and need some asymptotic computation. Lastly, let us notice that the Bergman kernel beween x and y is essentially represented by the value of the peak section associated to x evaluated at y.*

We finish this section with the proof that the smooth vanishing loci have all the same symplectomorphism type:

Proposition 4.2 *Let $1 \leq r \leq n$ be an integer and $E \to X$ be a holomorphic vector bundle of rank r, $L \to X$ be a holomorphic line bundle equipped with a metric h with positive curvature $-i\omega$. For any* degree *$d \geq 1$, denote by $H^0_{reg}(X, E \otimes L^d)$ the space of holomorphic sections of $E \otimes L^{\otimes d}$which vanish transversally. Then for any d large enough,*

$$\forall (s,t) \in H^0_{reg}(X, E \otimes L^d)^2, \big(Z(s), \omega_{|Z(s)}\big) \sim_{symp} \big(Z(t), \omega_{|Z(t)}\big).$$

Proof. First, by Bertini's theorem [13, p. 137], $H^0_{sing} := H^0_{reg}(X, E \otimes L^d) \backslash H^0_{reg}(X, E \otimes L^d)$ is of real codimension at least 2 in H^0. This implies that any pair $(s,t) \in H^0_{reg}$ are joined by a path of sections in H^0_{reg}. By Ehresmann theorem, this implies that $Z(s)$ is diffeomorphic to $Z(t)$. Now, for a continuous family of sections $(s_t)_{t\in[0,1]}$ in H^0_{reg}, since ω is a curvature of a line bundle, as its restriction to $Z(s_t)$, $[\omega_{|Z(s_t)}] \in H^2(Z(s_t), \mathbb{Z})$. Consequently, the pullback in $H^2(Z(s_0), \mathbb{Z})$ of $[\omega_{|Z(s_t)}]$ by the diffeomorphism $\psi_t : Z(s_0) \to Z(s_1)$ given by the former argument is constant. In other terms, $\psi_t^*[\omega_{|Z(s_t)}] = [\omega_{|Z(s_0)}]$. Then, the Moser argument (see [16, Theorem 3.17]) implies that the zero sets are in fact symplectomorphic. $\square$

4.3 Some simple lemmas

In this paragraph we give the proofs of elementary and technical lemmas that are used in the core of the proof of the quantitative Moser deformation Proposition 3.3.

Lemmas for the deformations

Lemma 4.3 *Let $m \geq 1$ be an integer, $(X_t)_{t\in[0,1]}$ be a C^2 family of vector field on $\mathbb{R}^n$ with compact support, and $(\phi_t)_{t\in[0,1]}$ be the associated flow. Then,*

1. $\forall t \in [0,1],\ \|\phi_t - Id\|_{C^0(\mathbb{R}^m)} \leq t \max_{t\in[0,1]} \|X_t\|_{C^0(\mathbb{R}^m)}$.
2. *Let* $0 \leq j \leq 2$ *and* $C > 0$ *be such that* $\max_{t\in[0,1]} N_j(X_t, \mathbb{R}^m) \leq C$. *Then, there exists* C' *depending only on* C *such that*

$$\forall t \in [0,1],\ \|\phi_t - Id\|_{C^j(\mathbb{R}^m)} \leq C' t \max_{t\in[0,1]} N_j(X_t).$$

Proof. First, it is classical that ϕ_t is C^k in (t,x). We have

$$\forall (x,t) \in \mathbb{R}^n \times [0,1],\ \phi_t(x) - x = \int_0^t X_s(\phi_s(x))ds,$$

which implies $\|\phi_t - \mathrm{Id}\|_{C^0(M)} \leq \max_{t\in[0,1]} \|X_t\|_{C^0(M)}$ and

$$d\phi_t - \mathrm{Id} = \int_0^t d_x X_s(\phi_s(x)) \circ d\phi_s ds.$$

Consequently, $\|d\phi_t - \mathrm{Id}\|_{C^0} \leq \max_t \|dX_t\|_{C^0}\big(t + \int_0^t \|d\phi_s - \mathrm{Id}\|ds\big)$. By Gronwall, this implies

$$\|d\phi_t - \mathrm{Id}\|_{C^0} \leq t \max_t \|dX_t\|_{C^0} \exp\big(\max_t \|dX_t\|_{C^0}\big) \leq te^C N_1(X). \tag{4.7}$$

Now, $d^2(\phi_t - Id) = d^2\phi_t = \int_0^t d_x^2 X_s(\phi_s) d\phi_s \otimes d\phi_s + d_x X_s \circ d^2\phi_s ds$. Together with estimate (4.7), this implies

$$\|d^2\phi_t\| \leq \max_t \|d^2 X_t\|_{C^0}(1 + Ce^C)^2 + \max_t \|dX_t\|_{C^0} \int_0^t \|d^2\phi_s\| ds$$

so that by Gronwall, $\|d^2\phi_t\| \leq \max_t \|d^2 X_t\|_{C^0}(1 + Ce^C)^2 \exp(C)$. □

Unfortunately, for manifolds we need a simplier version of the latter affine lemma.

Lemma 4.4 *Let* (M,h) *be a smooth Riemannian manifold,* $(X_t)_{t\in[0,1]}$ *be a* C^k *family of vector fields with compact support* N *and* $(\phi_t)_{t\in[0,1]}$ *the 1-parameter group of diffeomorphism generated by* $(X_t)_t$. *Then*

1. $\forall t \in [0,1],\ d(\phi_t, Id) \leq t \max_{t\in[0,1]} \|X_s\|_{C^0(M)}$.
2. *Let* $C > 0$ *be such that* $\max_{t\in[0,1]} N_1(X_t, \mathbb{R}^m) \leq C$. *Then, there exists* C' *depending only on* C *and the* C^1 *norm of the metric on* N, *such that* $\max_{t\in[0,1]} \|d\phi_t\|_{C^0(M)} \leq C'$.

Proof. Again, it is classical that ϕ_t is C^k in (t,x), and

$$\forall (x,t) \in M \times [0,1],\ d(\phi_t(x), x) \leq Length(\{\phi_t(x)\}_{t\in[0,1]}) \leq t \max_{s\in[0,1]} \|X_s\|_{C^0(M)}.$$

Let $x \in M$ in a local chart. If t is small enough, *In coordinates*, we have

$$\phi_t(x) - x = \int_0^t X(\phi_s(x), s)ds,$$

so that $d\phi_t - \mathrm{Id} = \int_0^t d_x X(\phi_s(x), s) \circ d\phi_s ds$. Then, there exists a constant C depending only on the compact support of X and the C^1 norm of the metric h in the coordinates, such that for any vector $Y \in \mathbb{R}^n$,

$$|d\phi_t(Y)|_{\phi_t(x)} \leq C|Y|_x\big(1 + \max_{s\in[0,t]} \|d_x X\|_{C^0(M)} \int_0^t \|d\phi_s\|_{\phi_s(x)} ds\big)$$

which implies $\|d\phi_t\|_{\phi_t(x)} \leq C\big(1+\max_{s\in[0,t]}\|d_xX\|_{C^0(M)}\|\int_0^t\|d\phi_s\|_{\phi_s(x)}ds\big)$ and by Gronwall

$$\|d\phi_t\|_{C^0} \leq C\exp(C\max_{[0,1]}\|d_xX\|),$$

so that there exists another constant C' depending on the chart, such that

$$\|d\phi_t\|_{C^0} \leq C\exp(C'\max_{[0,1]}N_1(X)\|).$$

Since we can cover the support of X by a finite number of charts, this implies the result. □

The following Lemma 4.5 was used for the proof of last assertion (3.3) of Proposition 3.1.

Lemma 4.5 *Let $m \geq 1$, $1 \leq p \leq m$ be two integers, and $\Phi : M(p,m)\times\mathbb{R}^m \to \mathbb{R}^m$, where for any $(A,Y)\in M(p,m)\times\mathbb{R}^m$, $\Phi(A,Y)$ denotes the orthogonal projection of the origin onto the space $\{X\in\mathbb{R}^m, AX = Y\}$. Then, for any $0\leq j\leq 2$ and for any $(A,Y)\in M(p,m)\times\mathbb{R}^m$ such that A is onto,*

$$\begin{aligned}
\Phi(A,Y) &\leq T^{-2}(A)\|A\||Y|,\\
\|d_A\Phi(A,Y)\| &\leq 3T^{-4}(A)\|A\|^2|Y| \text{ and } \|d_Y\Phi(A,Y)\| \leq T^{-2}(A)\|A\|,\\
\|d^2_{A^2}\Phi(A,Y)\| &\leq 14\|A\|^3T(A)^{-6}|Y| \text{ and } \|d^2_{AY}\Phi\| \leq 3T(A)^{-4}\|A\|^2,
\end{aligned}$$

where T has been defined by (3.1).

Proof. Write $A = (A_1,\cdots,A_p)^t$, where $A_k\in\mathbb{R}^m$ are column vectors. Since A is onto, $(\ker A)^\perp = \langle A_1,\cdots,A_p\rangle$ and there exist a unique $\lambda(A,Y) = (\lambda_1,\cdots,\lambda_p)\in\mathbb{R}^p$, such that $\Phi(A,Y) = \sum_{i=1}^p\lambda_iA_i \in A^{(-1)}(\{Y\})$, which means $AA^t\lambda = Y$, so that

$$\Phi(A,Y) = \big[K(A)Y, A\big],$$

where $[\lambda, A] := \sum_{i=1}^p\lambda_iA_i$ and $K(A) := (AA^t)^{-1}$. This implies that Φ is a smooth near (A,Y) for any A onto, and linear in Y. Since

$$K(A) \leq T(A)^{-2},$$

by Cauchy-Schwarz $|\Phi(A,Y)| \leq T(A)^{-2}\|A\||Y|$. Derivating gives

$$d\Phi(A,Y) = \big[dK(A)Y + KdY, A\big] + [KY, dA] \tag{4.8}$$

so that $\|d_Y\Phi\| \leq T(A)^{-2}\|A\|$. If $Q(A) := AA^t$, then $\|dQ(A)\| \leq 2\|A\|$ and

$$\forall B\in M_{p,m}(\mathbb{R}),\ dK(A)(B) = -KdQ(A)BK,$$

so that, using $T(A)\leq\|A\|$,

$$\begin{aligned}
\|dK(A)\| &\leq 2T(A)^{-4}\|A\|\\
\text{and } \|d_A\Phi\| &\leq 2T(A)^{-4}\|A\|^2|Y| + T(A)^{-2}|Y| \leq 3T(A)^{-4}\|A\|^2|Y|.
\end{aligned}$$

Now

$$d^2\Phi(A,Y) = [d^2K(A)Y, A] + \text{Sym}\Big([dK(A)Y, dA] + [K(A)dY, dA] + [dK(A)dY, A]\Big), \tag{4.9}$$

where Sym means that the bracket is symmetrized in the two vectors in $M_{p,m}(\mathbb{R})\times\mathbb{R}^m$ on which applies $d^2\Phi(A,Y)$. Since $d^2K(A) = \text{Sym}\ KdQKdQK - Kd^2QK$, we obtain

$$\begin{aligned}
\|d^2K(A)\| &\leq 8\|A\|^2T(A)^{-6} + 2T(A)^{-4} \leq 10\|A\|^2T(A)^{-6},\\
\|d^2_{A^2}\Phi(A,Y)\| &\leq 10\|A\|^3T(A)^{-6}|Y| + 4T(A)^{-4}\|A\||Y| \leq 14\|A\|^3T(A)^{-6}|Y|\\
\text{and } \|d^2_{AY}\Phi\| &\leq 3T(A)^{-4}\|A\|^2.
\end{aligned}$$

□

Lemmas for the barrier methods.

Lemma 4.6 *Let $0 < \varepsilon \leq 1$, $p \in (\mathbb{C}[z_1, \cdots, z_n])^r$ and $P_{\varepsilon,d} = Z_0^d p\big(\frac{\sqrt{d}}{\varepsilon} \cdot\big)$. Then, uniformly in ε,*

$$\|P_{\varepsilon,d}\|_{L^2} \underset{d\to\infty}{\sim} \frac{1}{\pi d^{n/2}} \|p\big(\frac{\cdot}{\varepsilon}\big)\|_{BF},$$

where $\|p\|_{BF}^2 := \frac{1}{\pi^n}\int_{\mathbb{C}^n} |p(y)|^2 e^{-|y|^2}|dy|$ defines the Bargmann-Fock norm of p.

Remark 4.7 *Note that for any p, there exists a constant $c > 0$ such that for any $0 < \varepsilon \leq 1$, and $d \geq 1$, $\|P_{\varepsilon,d}\|_{L^2} \leq \frac{c}{d^{\frac{n}{2}}\varepsilon^{\deg p}}$.*

Proof. We have, by definition of the Fubini-Study measure on $\mathbb{C}P^n$,

$$\|P_{\varepsilon,d}\|_{L^2}^2 = \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon,d}|^2 \frac{d\sigma}{2\pi},$$

where $d\sigma$ is the canonical measure on the sphere with volume 1 and the 2π factor corresponds to the volume of the fiber $U(1)$ of the quotient $S^{2n+1} \to \mathbb{C}P^n$. Since

$$\int_{\mathbb{C}^{n+1}} |P_{\varepsilon,d}|^2 e^{-\|Z\|^2}|dZ| = \int_0^\infty r^{2d+2n+1}e^{-r^2}dr \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon,d}|^2 d\sigma = (d+n)! \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon,d}|^2 d\sigma,$$

where $|dZ|$ denotes the Lebesgue measure on $\mathbb{C}^{n+1}$, we have

$$\|P_{\varepsilon,d}\|_{L^2}^2 = \frac{1}{2\pi(d+n)!}\int_{\mathbb{C}^{n+1}} |Z_0|^{2d}\Big|p\big(\frac{\sqrt{d}}{\varepsilon}\frac{Z'}{Z_0}\big)\Big|^2 e^{-\|Z\|^2}|dZ|,$$

where $Z' = (Z_1, \cdots, Z_n)$. We use the change of variable $(W_0, w) = (Z_0, \frac{Z'}{Z_0})$, then $(w_0, w) = (W_0\sqrt{1+|w|^2}, w)$, and finally $y = \sqrt{d}w$ so that

$$\begin{aligned}
\|P_{\varepsilon,d}\|_{L^2}^2 &= \frac{1}{2\pi(d+n)!\pi^{n+1}}\int_{\mathbb{C}^{n+1}} |W_0|^{2(d+n)}|p(\frac{\sqrt{d}}{\varepsilon}w)|^2 e^{-|W_0|^2(1+\|w\|^2)}|dW_0||dw| \\
&= \frac{1}{2\pi(d+n)!\pi^{n+1}}\int_{\mathbb{C}^{n+1}} |w_0|^{2(d+n)}e^{-|w_0|^2}|p(\frac{\sqrt{d}}{\varepsilon}w)|^2 \frac{1}{(1+\|w\|^2)^{d+n+1}}|dw_0||dw| \\
&= \frac{1}{d^n}\frac{1}{2\pi(d+n)!\pi^{n+1}}\int_{\mathbb{C}} |w_0|^{2(d+n)}e^{-|w_0|^2}|dw_0|\int_{\mathbb{C}^n} |p(\frac{y}{\varepsilon})|^2 \frac{1}{(1+\frac{1}{d}\|y\|^2)^{d+n+1}}|dy| \\
&\underset{d\to\infty}{\sim} \frac{1}{d^n}\frac{1}{\pi^{n+1}}\int_{\mathbb{C}^n} |p(\frac{y}{\varepsilon})|^2 e^{-|y|^2}|dy|
\end{aligned}$$

uniformly in $\varepsilon \leq 1$.

Note that for any $(i_0, \cdots, i_n) \in \mathbb{N}^{n+1}$ such that $\sum_k i_k = d$,

$$\begin{aligned}
\|Z_0^{i_0}\cdots Z_n^{i_n}\|_{L^2}^2 &= \frac{1}{2\pi(d+n)!}\int_{\mathbb{C}^{n+1}} \Pi_{k=0}^{n+1}|Z_k^{i_k}|^2 e^{-\|Z\|^2}|dZ| \\
&= \frac{1}{2\pi(d+n)!}\Pi_{k=0}^{n+1}\int_{\mathbb{C}} |z|^{2i_k}e^{-|z|^2}|dz| \\
&= \frac{1}{(d+n)!}\Pi_{k=0}^{n+1}\int_0^\infty r^{2i_k+1}e^{-r^2}dr = \frac{i_0!\cdots i_n!}{(d+n)!}
\end{aligned}$$

□

The next lemma was proved in a real and general Kähler situation in [11]. We give a proof of it in the polynomial setting, in order the article to be self-contained and simple.

Lemma 4.8 *Let $1 \leq r \leq n$ be integers, $\varepsilon > 0$, $R \in (H_{d,n+1})^r$ be a random polynomial mapping of maximal degree d and $q(z) = R(1, z\frac{\varepsilon}{\sqrt{d}})$, where $z = (z_1, \cdots, z_n)$. Then, there exists $C > 0$ depending only on n and r such that for any $d \geq 1$, any $0 < \varepsilon \leq 1$, any $0 \leq j \leq 2$, $\mathbb{E}\big(\sup_{2\mathbb{B}} |d^j q|^2\big) \leq C_n \frac{(d+n)!}{d!}$.*

Proof. Since q is holomorphic, we can use the mean value inequality for plurisubharmonic functions applied to $|q|^2$ (see [15]):

$$\forall z \in 2\mathbb{B},\ |q(z)|^2 \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{z+\mathbb{B}} |q|^2(u)du,$$

so that $\mathbb{E}\big(\sup_{2\mathbb{B}} |q|^2\big) \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{3\mathbb{B}} \mathbb{E}(|q(u)|^2)du$. We have by (1.2)

$$\forall z \in 2\mathbb{B},\ \mathbb{E}(|q(z)|^2) = \mathbb{E}\big(|R(1, z\frac{\varepsilon}{\sqrt{d}})|^2\big) = \mathbb{E}\big(\|R\|^2_{FS}(1, z\frac{\varepsilon}{\sqrt{d}})\big)\big(1 + \frac{|z|^2\varepsilon^2}{d}\big)^{2d}. \tag{4.10}$$

By definition of the measure, $\mathbb{E}\big(\|R\|^2_{FS}\big)$ is constant over $\mathbb{C}P^n$. Remembering that the coordinates of R are independent random polynomials, we obtain (see decomposition 1.2),

$$\mathbb{E}\big(\|R\|^2_{FS}\big)(1, z\frac{\varepsilon}{\sqrt{d}}) = \mathbb{E}\big(\|R\|^2_{FS}(0)\big) = r\mathbb{E}\Big(\frac{(d+n)!}{d!}|a_0|^2\Big) = r\frac{(d+n)!}{d!}.$$

Moreover $\forall d \geq 1$, $\forall z \in 2\mathbb{B}$, $\big(1 + \frac{|z|^2\varepsilon}{d}\big)^{2d} \leq e^{18\varepsilon^2}$, hence the first estimate of the Lemma.

For the second estimate, define for any holomorphic function $f = (f_1, \cdots, f_r) : \mathbb{C}^n \to \mathbb{C}^r$

$$\|df\|_2^2 := \sum_{i=1}^r \sum_{j=1}^n \Big|\frac{\partial f_i}{\partial z_j}\Big|^2.$$

Notice that $\|df\| \leq \|df\|_2$, where $\|df\|$ is the operator norm used in Proposition 3.1. We have, similarly to the first estimate since the complex derivatives of q are holomorphic,

$$\mathbb{E}\big(\sup_{2\mathbb{B}} \|dq\|_2^2\big) \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{3\mathbb{B}} \mathbb{E}\big(\|dq(u)\|_2^2\big)du.$$

with $\|dq(u)\|_2^2 = \frac{\varepsilon^2}{d}\|d_{Z'}R(1, u\frac{\varepsilon}{\sqrt{d}})\|_2^2$, where $Z' = (Z_1, \cdots, Z_n)$. As before

$$\mathbb{E}\big(\|d_{Z'}P(1, u\frac{\varepsilon}{\sqrt{d}})\|_2^2\big) \leq \mathbb{E}\big(\|d_{Z'}P\|^2_{FS}(1, 0)\big)e^{18\varepsilon^2}$$

with, using the linear part in Z' of the decomposition (1.2),

$$\mathbb{E}\big(\|d_{Z'}P(1, 0)\|^2_{FS}\big) = r\sum_{i=1}^n \mathbb{E}\Big(\frac{(d+n)!}{(d-1)!1!}|a_{0\cdots1\cdots0}|^2\Big) = rn\frac{(d+n)!}{(d-1)!}$$

which implies the second estimate of the Lemma. The last estimate is similar. □

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