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High order asymptotic expansion for Wiener functionals *

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Abstract

By combining the Malliavin calculus with Fourier techniques, we develop a high-order asymptotic expansion theory for a sequence of vector-valued random variables. Our asymptotic expansion formulas give the development of the characteristic functional and of the local density of the random vectors up to an arbitrary order. We analyzed in details an example related to the wave equation with space-time white noise which also provides interesting facts on the correlation structure of the solution to this equation.

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1 Introduction

The asymptotic expansion of probability distributions for random variables and vectors represents a fundamental topic in probability theory and mathematical statistics. This theory has been widely applied to several fields, including the efficiency of estimators, hypothesis testing, information criterion for model selection, prediction theory, bootstrap methods and resampling plans, and information geometry. There exists now huge literature on asymptotic expansion. We refer, among many others, to [2] and [1] for the case of sequences of i.i.d. random variables and applications, to [3] for sequences of weakly dependent variables, to [9], [25], [26], [7], [27], [16], [28], [15] and [14] for asymptotic expansion of martingales and of classical diffusions, and to [12] or [20] for general sequences of random variables. The reader may consult the monographs [2], [4] and [8] for complete expositions on these topics.

Generally speaking, the asymptotic expansion theory aims at finding the expansion of the density functions for a sequence of random variables that converges in law to a target distributions (usually, the Gaussian distribution, but other target distributions, such as mixed normal, are possible). The theory usually provides the leading term and second order term in the asymptotic development of the density function. For estimators and test statistics, while the leading term is used for confidence limits and testing, the higher order terms provide a more accurate inference. Some higher order-type asymptotic expansions for particular sequences of diffusion-type can be found in [22], [24] with applications to statistics, and in [23], [6], [21], [17], [18], [19] with applications to finance.

Our purpose is to provide a general method to obtain the asymptotic expansions of density functions up to an arbitrary order, i.e. to find the further terms that appear in the asymptotic behavior of the density. Our approach combines the so-called Fourier approach and the recent Stein-Malliavin theory (see [10]) and applies to general sequences of random vectors. Our main finding is that the asymptotic expansion up to any order of the family of densities of a vector-valued random sequence $(F_N)_{N \geq 1}$ is completely characterized by the expectation of the so-called *Gamma factors* associated to the sequence F_N , or equivalently, by the joint cumulants of the components of this sequence. These Gamma factors (defined in Section 2.1) are defined in terms of the Malliavin operators of F_N . Consequently, the knowledge of the Taylor expansions of the cumulants, together with some regularity in the Malliavin sense of F_N , gives the higher order asymptotic expansion of the density. We also mention that, in contrast with the classical assumptions in the martingale case (see e.g. [26], [28]) or in the Malliavin calculus case (see [20]), the joint convergence in distribution of F_N together with its "bracket" (which is the usual martingale bracket when we deal with sequences of martingales and it is defined in terms of the Malliavin

derivative in the non-martingale case) is not assumed in our work.

As mentioned above, our strategy is based on the Stein-Malliavin calculus combined with the so-called Fourier approach. We start by analyzing the behavior of the (truncated) characteristic function of the sequence $(F_N)_{N \geq 1}$ via an interpolation method and the Malliavin-type integration by parts. We notice the appearance of the Gamma factors in the principal part of asymptotic expansion of the characteristic function. The Fourier inversion, together with some regularity of the distribution expressed in term of the Malliavin calculus, allows to develop asymptotically the sequence of (local) densities of F_N . Some regular ordering of the cumulants is assumed and this is checked in examples. Usually, the second order term in the asymptotic expansion comes from leading term in the expansion of third cumulant, while the third order term is due to the second and the fourth cumulant. A general formula is obtained.

As an example, we analyze the behavior of the spatial quadratic variation for the solution to the wave equation driven by a space-time white noise. We treat both the one dimensional case (i.e we fix the time t and we study the quadratic variation in space for the solution), as well as a two-dimensional case (i.e. we consider a the two-dimensional random sequence whose components are the spatial quadratic variations of the solution at two different times). In both cases, based on a sharp analysis of the correlation structure of the solution, we are able to find the asymptotic expansion up to at least the third order term. Let us emphasize that, besides being a toy example to apply our asymptotic expansion theory, this last part of our work shows some interesting facts related to the solution to the wave equation driven by a space-time white noise. We obtain the precise correlation of the increments of the solution, at fixed time and when the time is moving and it appears that the dependence structure of these increments depends in a non-trivial way on the spatial and temporal lags.

We organized our paper as follows. Section 2 presents, after the definition and some basic properties of the Gamma-factors, the asymptotic expansion up to an arbitrary order of the (truncated) characteristic function of a sequence of vector-valued random variables $(F_N)_{N \geq 1}$. This expansion depends on the Gamma-factors (or equivalently, the cumulants) of the vector F_N . Based on this expansion, we obtain in Section 3, by inverting in Fourier sense the principal part of the characteristic function, the approximate density for our sequence. This will approximate the local (truncated) density of F_N . The asymptotic expansion is further explicited in Section 4 where we show that if the cumulants of F_N admit a specific Taylor expansion, then a more precise expansion of the local density can be derived. In Section 5 we treat in details a concrete example related to the solution to the wave equation driven by an additive space-time white noise. The last section is the Appendix which contains the basic tools of the Malliavin calculus.

2 Expansion of the characteristic functional

In this section, we analyze the asymptotic behavior of a general sequence of random vectors, by using an interpolation method and the Malliavin integration by parts. We will distinguish a principal part of the characteristic function, written in terms of the Gamma factors and a negligible part. These two parts are then estimated separately.

2.1 The Gamma factors

Let $(W(h), h \in H)$ be an isonormal process on a standard probability space (Ω, \mathcal{F}, P) . For the definition of the Malliavin operators with respect to W , see Section 6. The pseudo-inverse L^{-1} of L is defined by $L^{-1}F = \sum_{q=1}^{\infty} q^{-1} J_q F$ for $F = \sum_{q=0}^{\infty} J_q F \in L^2(\Omega)$, where J_q is the orthogonal projection to the q -th chaos. For $p \in \mathbb{N}$ and $F \in \mathbb{D}_{1,p}$, we define the *Gamma-factors* $\Gamma^{(p)}(F)$ in a recursive way, see e.g. [10].

$$\begin{aligned}\Gamma^{(1)}(F) &= F \\ \Gamma^{(2)}(F) &= \langle DF, D(-L)^{-1}F \rangle_H, \\ &\dots \\ \Gamma^{(p)}(F) &= \langle DF, D(-L)^{-1}\Gamma^{(p-1)}(F) \rangle_H.\end{aligned}$$

These variables are well defined by Lemma 1 below, if F is regular enough in the sense of the Malliavin calculus.

We have the following formula that links the Gamma-factors and the cumulants: for every $m \geq 1$

$$k_m(F) = (m-1)! \mathbf{E} [\Gamma^{(m)}(F)]. \quad (1)$$

Recall that the m th cumulant of a random variable $F \in L^m(\Omega)$ is given by

$$k_m(F) = (-i)^m \frac{\partial}{\partial t^m} \ln \mathbf{E} [e^{itF}] \Big|_{t=0}.$$

We also introduce *the multidimensional Gamma factors* of a random vector $F = (F^{(1)}, \dots, F^{(d)}) \in \mathbb{D}_{1,p}(\mathbb{R}^d)$ are defined in the following way. For $i = 1, \dots, d$,

$$\Gamma_i^{(1)}(F) = F^{(i)}$$

and for $i_1, i_2 = 1, \dots, d$,

$$\Gamma_{i_1, i_2}^{(2)}(F) = \langle DF^{(i_2)}, D(-L)^{-1}F^{(i_1)} \rangle_H$$

while for $i_1, \dots, i_p = 1, \dots, d$

$$\Gamma_{i_1, \dots, i_p}^{(p)}(F) = \langle DF^{(i_p)}, D(-L)^{-1}\Gamma_{i_1, \dots, i_{p-1}}^{(p-1)}(F) \rangle_H. \quad (2)$$

The multidimensional Gamma factors $\Gamma_{i_1, \dots, i_p}^{(p)}(F)$ are also related to the joint cumulants of the random vector F . Recall that if $m = (m_1, \dots, m_d) \in \mathbb{N}^d$, then the m th cumulant of the random vector $F = (F^{(1)}, \dots, F^{(d)})$ is

$$k_m(F) = k_{(m_1, \dots, m_d)}(F^{(1)}, \dots, F^{(d)}) = (-\mathbf{i})^{|m|} \frac{\partial^{|m|}}{\partial t^m} \log \mathbf{E} [e^{\mathbf{i}(t, F)}] |_{t=0}$$

where $|m| = m_1 + \dots + m_d$. See [10] for the precise link between the multidimensional Gamma factors and the cumulants.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

Lemma 1. (a) *Let $\ell \in \mathbb{Z}_+$ and $r > 1$. Then $D(-L)^{-1}F \in \mathbb{D}_{\ell+1, r}(H)$ if $F \in \mathbb{D}_{\ell, r}$, and there exists a constant $C_{\ell, r}$ such that*

$$\|D(-L)^{-1}F\|_{\ell+1, r} \leq C_{\ell, r} \|F\|_{\ell, r}$$

for all $F \in \mathbb{D}_{\ell, r}$.

(b) *Let $\ell \in \mathbb{Z}_+$, $r > 1$ and $p \in \{2, 3, \dots\}$. Then $\Gamma_{i_1, \dots, i_p}^{(p)}(F) \in \mathbb{D}_{\ell, r}$ if $F = (F^{(1)}, \dots, F^{(d)}) \in \mathbb{D}_{\ell+1, pr}(\mathbb{R}^d)$, and there exists a constant $C_{\ell, r, p}$ such that*

$$\|\Gamma_{i_1, \dots, i_p}^{(p)}(F)\|_{\ell, r} \leq C_{\ell, r, p} \|F\|_{\ell+1, pr}^p \quad (3)$$

for all $F \in \mathbb{D}_{\ell+1, pr}$ and $i_1, \dots, i_p \in \{1, \dots, d\}$. In particular, $\Gamma_{i_1, \dots, i_p}^{(p)}(F) \in \mathbb{D}_{\ell, \infty-}$ if $F \in \mathbb{D}_{\ell+1, \infty-} = \bigcap_{r>1} \mathbb{D}_{\ell+1, r}$.

Proof. Let $r > 1$. Since $D(-L)^{-1}F = (I - L)^{-1}DF$ for $F \in \mathcal{P}$, the set of polynomial functionals, we have

$$\begin{aligned} \|D(-L)^{-1}F\|_{\ell+1, r} &\lesssim \|(I - L)^{(\ell+1)/2} D(-L)^{-1}F\|_r = \|(I - L)^{(\ell-1)/2} DF\|_r \\ &\lesssim \|DF\|_{\ell-1, r} \lesssim \|F\|_{\ell, r} \end{aligned}$$

for all $F \in \mathcal{P}$ uniformly. Therefore, $D(-L)^{-1}$ is extended as a continuous linear operator from $\mathbb{D}_{\ell, r}$ to $\mathbb{D}_{\ell+1, r}(\mathcal{H})$. Thus we obtained (a).

Suppose that $F = (F^{(1)}, \dots, F^{(d)}) \in \mathbb{D}_{\ell+1, pr}(\mathbb{R}^d)$. The property (b) follows from (a) by induction. Indeed, since $\Gamma_{i_1, \dots, i_{k+1}}^{(k+1)}(F) = \langle DF^{(i_{k+1})}, D(-L)^{-1}\Gamma_{(i_1, \dots, i_k)}^{(k)} \rangle_H$, if (3) holds for k ($\leq p-1$) in place of p , then

$$\begin{aligned} \|\Gamma_{i_1, \dots, i_{k+1}}^{(k+1)}(F)\|_{\ell, r} &\lesssim \|F^{(i_{k+1})}\|_{\ell+1, (k+1)r} \|D(-L)^{-1}\Gamma_{(i_1, \dots, i_k)}^{(k)}(F)\|_{\ell, k^{-1}(k+1)r} \\ &\lesssim \|F^{(i_{k+1})}\|_{\ell+1, (k+1)r} \|\Gamma_{(i_1, \dots, i_k)}^{(k)}(F)\|_{\ell-1, k^{-1}(k+1)r} \\ &\lesssim \|F^{(i_{k+1})}\|_{\ell+1, (k+1)r} \|F\|_{\ell, (k+1)r}^k \\ &\lesssim \|F\|_{\ell+1, (k+1)r}^{k+1}. \end{aligned}$$

Thus, (3) holds for $p = k + 1$. The inequality (3) is trivial when $p = 1$, which completes the proof. \blacksquare

2.2 Interpolation

Let $d \geq 1$. Consider a sequence of centered random variables $(F_N)_{N \geq 1}$ in \mathbb{R}^d of the form

$$F_N = \left(F_N^{(1)}, \dots, F_N^{(d)} \right).$$

Let $C = (C_{i,j})_{i,j=1}^d$ be a deterministic $d \times d$ positive definite symmetric matrix. For every $N \geq 1$, we introduce a truncation functional Ψ_N which is a smooth random variable. A more precise form of this functional will be chosen later in Section 2. We consider the following condition for the truncation functional Ψ_N : if p is a positive integer

[Ψ] For each $N \in \mathbb{N}$, $\Psi_N : \Omega \rightarrow [0, 1]$ and $\Psi_N \in \mathbb{D}_{1,p+1}$.

Let us define the interpolation functional

$$e(\theta, \boldsymbol{\lambda}, F_N) = \exp \left(i\theta \langle \boldsymbol{\lambda}, F_N \rangle - \frac{1}{2}(1 - \theta^2) \boldsymbol{\lambda}^T C \boldsymbol{\lambda} \right) \quad (4)$$

for every $\theta \in [0, 1]$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$ and $\theta \in [0, 1]$ and let us consider the truncated interpolation

$$\varphi_N^\Psi(\theta, \boldsymbol{\lambda}) = \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)]. \quad (5)$$

Notice that $\varphi_N^\Psi(1, \boldsymbol{\lambda}) = \mathbf{E}[\Psi_N e^{i\langle \boldsymbol{\lambda}, F_N \rangle}]$ represents the "truncated" characteristic function of F_N , while $\varphi_N^\Psi(0, \boldsymbol{\lambda}) = \mathbf{E}[\Psi_N] e^{-\boldsymbol{\lambda}^T C \boldsymbol{\lambda}/2}$ is the "truncated" characteristic function of the limit in law of F_N .

The first step is to get the expansion of the derivative with respect to the variable θ of the characteristic functional.

Lemma 2. *Suppose that $F_N \in \mathbb{D}_{1,p+1}(\mathbb{R}^d)$ for any $N \geq 1$ and that [Ψ] holds. Then*

the functional $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ is well defined for $(\theta, \boldsymbol{\lambda}) \in [0, 1] \times \mathbb{R}^d$, and it holds that

$$\begin{aligned}
\frac{\partial}{\partial \theta} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) &= \mathbf{i}(\mathbf{i}\theta)^p \sum_{i_1, \dots, i_{p+1}=1}^d \lambda_{i_1} \dots \lambda_{i_{p+1}} \mathbf{E} \left[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right] \\
&+ \mathbf{i}(\mathbf{i}\theta)^{p-1} \sum_{i_1, \dots, i_p=1}^d \lambda_{i_1} \dots \lambda_{i_p} \mathbf{E} [\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] \mathbf{E} [\Gamma_{i_1, \dots, i_p}^{(p)}(F_N)] \\
&+ \mathbf{i}(\mathbf{i}\theta)^{p-2} \sum_{i_1, \dots, i_{p-1}=1}^d \lambda_{i_1} \dots \lambda_{i_{p-1}} \mathbf{E} [\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] \mathbf{E} [\Gamma_{i_1, \dots, i_{p-1}}^{(p-1)}(F_N)] \\
&+ \dots \\
&+ \mathbf{i}(\mathbf{i}\theta)^2 \sum_{i_1, i_2, i_3=1}^d \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \mathbf{E} [\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] \mathbf{E} [\Gamma_{i_1, i_2, i_3}^{(3)}(F_N)] \\
&+ \mathbf{i}(\mathbf{i}\theta) \sum_{i_1, i_2=1}^d \lambda_{i_1} \lambda_{i_2} \mathbf{E} [\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] (\mathbf{E} [\Gamma_{i_1, i_2}^{(2)}(F_N)] - C_{i_1, i_2}) \quad (6) \\
&+ \sum_{j=1}^p R_{j,N}(\theta, \boldsymbol{\lambda})
\end{aligned}$$

for $(\theta, \boldsymbol{\lambda}) \in [0, 1] \times \mathbb{R}^d$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)$, where

$$R_{j,N}(\theta, \boldsymbol{\lambda}) = \mathbf{i}(\mathbf{i}\theta)^{j-1} \sum_{i_1, \dots, i_j=1}^d \lambda_{i_1} \dots \lambda_{i_j} \mathbf{E} \left[e(\theta, \boldsymbol{\lambda}, F_N) \langle D\Psi_N, D(-L)^{-1} \Gamma_{i_1, \dots, i_j}^{(j)}(F_N) \rangle_H \right] \quad (7)$$

for $j = 1, 2, \dots, p$. In (6), $\Gamma_{i_1, i_2}^{(2)}(F_N)$ can be replaced by the symmetrized version $\Gamma_{i_1, i_2}^{(2sym)}(F_N) = 2^{-1}(\Gamma_{i_1, i_2}^{(2)}(F_N) + \Gamma_{i_2, i_1}^{(2)}(F_N))$.

Proof. The Malliavin derivative of the random variable $e(\theta, \boldsymbol{\lambda}, F_N)$ given by (4) can be calculated as follows

$$De(\theta, \boldsymbol{\lambda}, F_N) = \mathbf{i}\theta e(\theta, \boldsymbol{\lambda}, F_N) \sum_{k=1}^d \lambda_k DF_N^{(k)}. \quad (8)$$

We differentiate $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ given by (5) with respect to θ , and we use the identity (99) as $F_N^{(j)} = \delta D(-L)^{-1} F_N^{(j)}$ for every $j = 1, \dots, d$ and the duality relationship and (8) to

obtain

$$\begin{aligned}
\frac{\partial}{\partial \theta} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) &= \sum_{j=1}^d \mathbf{i} \lambda_j \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) F_N^{(j)}] + \sum_{j=1}^d \theta \lambda_j \lambda_k \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) C_{j,k}] \\
&= -\theta \sum_{j,k=1}^d \lambda_j \lambda_k \mathbf{E} \left[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \left(\langle DF_N^{(k)}, D(-L)^{-1} F_N^{(j)} \rangle_H - C_{j,k} \right) \right] \\
&\quad + \mathbf{i} \sum_{j=1}^d \lambda_j \mathbf{E} \left[e(\theta, \boldsymbol{\lambda}, F_N) \langle D\Psi_N, D(-L)^{-1} F_N^{(j)} \rangle_H \right] \\
&= \mathbf{i}(\mathbf{i}\theta) \sum_{i_1, i_2=1}^d \lambda_{i_1} \lambda_{i_2} \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] (\Gamma_{i_1, i_2}^{(2)}(F_N) - C_{i_1, i_2}) \\
&\quad + R_{1,N}(\theta, \boldsymbol{\lambda}) \tag{9}
\end{aligned}$$

with

$$R_{1,N}(\theta, \boldsymbol{\lambda}) = \mathbf{i} \sum_{j=1}^d \lambda_j \mathbf{E} \left[e(\theta, \boldsymbol{\lambda}, F_N) \langle D\Psi_N, D(-L)^{-1} F_N^{(j)} \rangle_H \right].$$

The formula (9) has been also obtain in [20] and it allows to obtain the second order terms in the asymptotic expansion of the sequence $(F_N)_{N \geq 1}$. In order to get the higher order term in this asymptotic expansion when $p \geq 2$, we refine the above formula (9). By (99), we write

$$\begin{aligned}
\Gamma_{i_1, i_2}^{(2)}(F_N) - C_{i_1, i_2} &= \Gamma_{i_1, i_2}^{(2)}(F_N) - E[\Gamma_{i_1, i_2}^{(2)}(F_N)] + E[\Gamma_{i_1, i_2}^{(2)}(F_N)] - C_{i_1, i_2} \\
&= \delta D(-L)^{-1} \Gamma_{i_1, i_2}(F_N) + E[\Gamma_{i_1, i_2}^{(2)}(F_N)] - C_{i_1, i_2} \tag{10}
\end{aligned}$$

for every $i_1, i_2 \in \{1, \dots, d\}$, and we apply the duality and (8) once again to obtain

$$\begin{aligned}
&\mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \delta D(-L)^{-1} \Gamma_{i_1, i_2}^{(2)}(F_N)] \\
&= \mathbf{i}\theta \mathbf{E} \left[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \sum_{i_3=1}^d \lambda_{i_3} \langle DF_N^{(i_3)}, D(-L)^{-1} \Gamma_{i_1, i_2}^{(2)}(F_N) \rangle_H \right] \\
&\quad + \mathbf{E} \left[e(\theta, \boldsymbol{\lambda}, F_N) \langle D\Psi_N, D(-L)^{-1} \Gamma_{i_1, i_2}^{(2)}(F_N) \rangle_H \right]. \tag{11}
\end{aligned}$$

From (9), (10) and (11), we obtain

$$\begin{aligned} \frac{\partial}{\partial \theta} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) &= \mathbf{i}(\mathbf{i}\theta)^2 \sum_{i_1, i_2, i_3=1}^d \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \Gamma_{i_1, i_2, i_3}^{(3)}(F_N)] \\ &\quad + \mathbf{i}(\mathbf{i}\theta) \sum_{i_1, i_2=1}^d \lambda_{i_1} \lambda_{i_2} \mathbf{E}[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N)] (\mathbf{E}[\Gamma_{i_1, i_2}^{(2)}(F_N)] - C_{i_1, i_2}) \\ &\quad + R_{1, N}(\theta, \boldsymbol{\lambda}) + R_{2, N}(\theta, \boldsymbol{\lambda}) \end{aligned}$$

for every $\theta \in [0, 1]$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$, where

$$R_{2, N}(\theta, \boldsymbol{\lambda}) = \mathbf{i}(\mathbf{i}\theta) \sum_{i_1, i_2=1}^d \lambda_{i_1} \lambda_{i_2} \mathbf{E} \left[e(\theta, \boldsymbol{\lambda}, F_N) \langle D\Psi_N, D(-L)^{-1} \Gamma_{i_1, i_2}^{(2)}(F_N) \rangle_H \right].$$

By iterating this procedure, we obtain the desired expansion of $\frac{\partial}{\partial \theta} \varphi_N^\Psi(\theta, \boldsymbol{\lambda})$. ■

2.3 Approximation to the characteristic functional and estimate of the error

The result in Lemma 2 shows that the behavior of the characteristic functional $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ is closely related to the behavior of the Gamma factors (or equivalently, the cumulants) of F_N . Let us define, for $(\theta, \boldsymbol{\lambda}) \in [0, 1] \times \mathbb{R}^d$, the following quantity, which will be called in the sequel *the principal part*

$$\begin{aligned} P_N(\theta, \boldsymbol{\lambda}) &= \mathbf{i}(\mathbf{i}\theta)^p \sum_{i_1, \dots, i_{p+1}=1}^d \lambda_{i_1} \dots \lambda_{i_{p+1}} \mathbf{E}[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)] \\ &\quad + \mathbf{i}(\mathbf{i}\theta)^{p-1} \sum_{i_1, \dots, i_p=1}^d \lambda_{i_1} \dots \lambda_{i_p} \mathbf{E}[\Gamma_{i_1, \dots, i_p}^{(p)}(F_N)] \\ &\quad + \mathbf{i}(\mathbf{i}\theta)^{p-2} \sum_{i_1, \dots, i_{p-1}=1}^d \lambda_{i_1} \dots \lambda_{i_{p-1}} \mathbf{E}[\Gamma_{i_1, \dots, i_{p-1}}^{(p-1)}(F_N)] \\ &\quad + \dots \\ &\quad + \mathbf{i}(\mathbf{i}\theta)^2 \sum_{i_1, i_2, i_3=1}^d \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(F_N)] \\ &\quad + \mathbf{i}(\mathbf{i}\theta) \sum_{i_1, i_2=1}^d \lambda_{i_1} \lambda_{i_2} (\mathbf{E}[\Gamma_{i_1, i_2}^{(2)}(F_N)] - C_{i_1, i_2}) \end{aligned} \tag{12}$$

and let

$$\begin{aligned} R_N(\theta, \boldsymbol{\lambda}) &= i(i\theta)^p \sum_{i_1, \dots, i_{p+1}=1}^d \lambda_{i_1} \dots \lambda_{i_{p+1}} \mathbf{E} \left[\Psi_N e(\theta, \boldsymbol{\lambda}, F_N) \left(\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) - \mathbf{E} \left[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right] \right) \right] \\ &\quad + \sum_{j=1}^p R_{j,N}(\theta, \boldsymbol{\lambda}). \end{aligned}$$

The truncated characteristic function can be written in terms of the principal part $P_N(\theta, \boldsymbol{\lambda})$ and of the residual term $R_N(\theta, \boldsymbol{\lambda})$.

Lemma 3. *Suppose that $F_N \in \mathbb{D}_{1,p+1}(\mathbb{R}^d)$ and that $[\Psi]$ holds. Then, for any positive number c , the functional $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ admits the expression*

$$\begin{aligned} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) &= \varphi_N^\Psi(0, \boldsymbol{\lambda}) \exp \left(\int_0^\theta P_N(\theta_1, \boldsymbol{\lambda}) d\theta_1 \right) \\ &\quad + \int_0^\theta \exp \left(\int_{\theta_1}^\theta P_N(\theta_2, \boldsymbol{\lambda}) d\theta_2 \right) R_N(\theta_1, \boldsymbol{\lambda}) d\theta_1 \end{aligned} \quad (13)$$

for $(\theta, \boldsymbol{\lambda}) \in [0, 1] \times \mathbb{R}^d$.

Proof. By Lemma 2, we have

$$\frac{\partial}{\partial \theta} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) = R_N(\theta, \boldsymbol{\lambda}) + \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) P_N(\theta, \boldsymbol{\lambda}). \quad (14)$$

Solve (14) to obtain (13). ■

In order to obtain the high-order asymptotic expansion, we need to assume several conditions. The first assumptions stated below concern the Malliavin regularity of F_N and of its associated Gamma factors. We fix a positive number \mathfrak{q} , that will be the order of the asymptotic expansion. Given a positive integer ℓ , we consider the following conditions.

[A1] (i) $F_N \in \mathbb{D}_{\ell+1, \infty}$ and

$$\sup_{N \in \mathbb{N}} \|F_N\|_{\ell+1, r} < \infty \quad (15)$$

for every $r > 1$.

(ii) For some positive constant \mathfrak{a} and some $d \times d$ non-singular matrix C_1 satisfying $C = 2^{-1}(C_1 + C_1^T)$, it holds that

$$\|\Gamma^{(2)}(F_N) - C_1\|_{\ell, r} = O(N^{-\mathfrak{a}})$$

as $N \rightarrow \infty$ for every $r > 1$.

Let $\ell_1 \in \{1, \dots, \ell\}$. Consider also the condition

[A2] (i) For some $r > 1$,

$$\left\| \Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) - \mathbf{E}[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)] \right\|_{\ell_1, r} = O(N^{-q}) \quad (16)$$

as $n \rightarrow \infty$ for $i_1, \dots, i_{p+1} \in \{1, \dots, d\}$.

(ii) For some $r > 1$,

$$\left\| \Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) - \mathbf{E}[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)] \right\|_r = o(N^{-q}) \quad (17)$$

as $n \rightarrow \infty$ for $i_1, \dots, i_{p+1} \in \{1, \dots, d\}$.

Obviously, a sufficient condition for [A2] is

[A2[#]] For some $r > 1$,

$$\left\| \Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) - \mathbf{E}[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)] \right\|_{\ell_1, r} = o(N^{-q})$$

as $n \rightarrow \infty$ for $i_1, \dots, i_{p+1} \in \{1, \dots, d\}$.

In what follows, we will work with Ψ_N defined by $\Psi_N = \psi(\Xi_N)$ for a function $\psi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\psi(x) = 1$ for $x \in [-1/2, 1/2]$ and $\psi(x) = 0$ for $x \in (-1, 1)^c$, and a functional Ξ_N given by

$$\Xi_N = K_* N^{2a'} |\Gamma^{(2)}(F_N) - C_1|^2, \quad (18)$$

where K_* is a positive number and $\Gamma^{(2)}(F_N) = (\Gamma_{i,j}^{(2)}(F_N))_{i,j=1}^d$. By choosing a sufficiently large K_* , we may assume, due to (18), that there exist positive constants c_1 and c_2 such that

$$c_1 \leq |\det \Gamma^{(2)}(F_N)| \leq c_2 \quad (19)$$

for all $N \in \mathbb{N}$ and a.s. $\omega \in \Omega$ whenever $|\Xi_N| < 1$.

Let $\Lambda_N = \{\boldsymbol{\lambda} \in \mathbb{R}^d; |\boldsymbol{\lambda}| \leq N^\xi\}$, where ξ is a positive constant. We will put a condition ([A3]) concerning ξ later.

Lemma 4. *Suppose that [A1] is fulfilled.*

(a) Suppose that [A2] (i) is satisfied. Then, for any $\alpha \in \mathbb{Z}_+^d$, there exists a constant C_α such that

$$\sup_{N \in \mathbb{N}} \sup_{\theta \in [0,1]} 1_{\{\boldsymbol{\lambda} \in \Lambda_N\}} N^q \left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N(\theta, \boldsymbol{\lambda}) \right| \leq C_\alpha (1 + |\boldsymbol{\lambda}|)^{-\ell_1 + p + 1} \quad (\boldsymbol{\lambda} \in \mathbb{R}^d) \quad (20)$$

(b) Suppose that [A2] (ii) is satisfied. Then

$$\sup_{\theta \in [0,1]} \left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N(\theta, \boldsymbol{\lambda}) \right| = o(N^{-q})$$

as $n \rightarrow \infty$ for every $\boldsymbol{\lambda} \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}_+^d$.

Remark 1. (i) Under the assumption that $\sup_{N \in \mathbb{N}} \|F_N\|_{\ell_{+1}, (p+1)r} < \infty$, we have

$$\sup_{N \in \mathbb{N}} \|\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)\|_{\ell, r} < \infty$$

thanks to Lemma 1. In this situation, Condition (16) is requesting the order N^{-q} to the norm. (ii) It is possible to show a similar result under the condition that

$$\|\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)\|_{\ell_1, r} = o(N^{-q})$$

instead of [A2] though we will not pursue it here. (iii) Logically, ℓ in Lemma 4 can be ℓ_1 .

Proof of Lemma 4. Let $\tilde{\Gamma}_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) = \Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) - \mathbf{E}[\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N)]$. For $\alpha \in \mathbb{Z}_+^d$, we have

$$\begin{aligned} \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N(\theta, \boldsymbol{\lambda}) &= \mathbf{i}(\mathbf{i}\theta)^p \sum_{i_1, \dots, i_{p+1}=1}^d \binom{\alpha}{\gamma} E_{\gamma, N}(\theta, \boldsymbol{\lambda}) \frac{\partial^{\alpha-\gamma}}{\partial \boldsymbol{\lambda}^{\alpha-\gamma}} (\lambda_{i_1} \dots \lambda_{i_{p+1}}) \\ &\quad + \sum_{j=1}^p \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} R_{j, N}(\theta, \boldsymbol{\lambda}) \end{aligned} \quad (21)$$

where

$$E_{\gamma, N}(\theta, \boldsymbol{\lambda}) = \mathbf{E} \left[\Psi_N \frac{\partial^\gamma}{\partial \boldsymbol{\lambda}^\gamma} e(\theta, \boldsymbol{\lambda}, F_N) \tilde{\Gamma}_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right].$$

By definition,

$$\frac{\partial^\gamma}{\partial \boldsymbol{\lambda}^\gamma} e(\theta, \boldsymbol{\lambda}, F_N) = p_\gamma(\theta F_N, (1 - \theta^2) C \boldsymbol{\lambda}, (1 - \theta^2) C) e(\theta, \boldsymbol{\lambda}, F_N) \quad (22)$$

for a polynomial p_γ .

By Lemma 1 and [A1] (i), $\Gamma_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \in \mathbb{D}_{\ell, r_1}$ for any $r_1 > 1$. We apply the IBP formula (see Lemma 2 in [20]) to the interpolation functional $e(\theta, \boldsymbol{\lambda}, F_N)$ that is taking advantage of (19) and based on the chain rule

$$\langle Df(F_N), D(-L)^{-1}F_N^{(i_1)} \rangle_H = \sum_{i_2=1}^d \frac{\partial f}{\partial x_{i_2}}(F_N) \Gamma_{i_1, i_2}^{(2)}(F_N)$$

for $f \in C^1(\mathbb{R}^d)$ of at most polynomial growth, in order to obtain

$$\begin{aligned} & (\mathbf{i}\theta\boldsymbol{\lambda})^\beta \mathbf{E} \left[\Psi_N e^{\mathbf{i}\theta\langle \boldsymbol{\lambda}, F_N \rangle} p_\gamma(\theta F_N, (1-\theta^2)C\boldsymbol{\lambda}, (1-\theta^2)C) \tilde{\Gamma}_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right] \\ &= \mathbf{E} \left[e^{\mathbf{i}\theta\langle \boldsymbol{\lambda}, F_N \rangle} \Phi_\beta^{F_N} \left(\Psi_N p_\gamma(\theta F_N, (1-\theta^2)C\boldsymbol{\lambda}, (1-\theta^2)C) \tilde{\Gamma}_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right) \right] \end{aligned}$$

for $\boldsymbol{\lambda} \in \Lambda_N$ and $\beta \in \mathbb{Z}_+^d$ with the sum of the components $|\beta| = \ell_1$. Here $\Phi_\beta^{F_N}$ is a linear functional and we see that

$$\sup_{\substack{N \in \mathbb{N} \\ \theta \in [0, 1] \\ \boldsymbol{\lambda} \in \mathbb{R}^d}} \left\| \Phi_\beta^{F_N} \left(\Psi_N p_\gamma(\theta F_N, (1-\theta^2)C\boldsymbol{\lambda}, (1-\theta^2)C) N^q \tilde{\Gamma}_{i_1, \dots, i_{p+1}}^{(p+1)}(F_N) \right) e^{-2^{-1}(1-\theta^2)\boldsymbol{\lambda}^T C \boldsymbol{\lambda}} \right\|_1 < \infty. \quad (23)$$

Here we used (15), (16) for which r is strictly larger than 1, and the estimate

$$|(1-\theta^2)C\boldsymbol{\lambda}| \leq |C^{1/2}|(1 + (1-\theta^2)\boldsymbol{\lambda}^T C \boldsymbol{\lambda}).$$

For each $\boldsymbol{\lambda} = (\lambda_i) \in \check{\Lambda}_N$, we choose a component i that attains $\max\{|\lambda^j|; j = 1, \dots, d\}$ and use the above estimate for $\beta = (i, i, \dots, i)$. Then we obtain

$$\sup_{n \in \mathbb{N}, \theta \in [1/2, 1], \boldsymbol{\lambda} \in \mathbb{R}^d} \left\{ N^q |\boldsymbol{\lambda}^{\ell_1}| |E_{\gamma, N}(\theta, \boldsymbol{\lambda})| \right\} < \infty. \quad (24)$$

Moreover, there exists a positive constant c_0 such that

$$\sup_{N \in \mathbb{N}, \theta \in [0, 1/2]} |e(\theta, \boldsymbol{\lambda}, F_N)| \leq c_0^{-1} \exp(-c_0 |\boldsymbol{\lambda}|^2) \quad (25)$$

for all $\boldsymbol{\lambda} \in \mathbb{R}^d$. From (24) and (25), we obtain

$$\begin{aligned} & \sup_{N \in \mathbb{N}} \sup_{\theta \in [0, 1]} \left| E_{\gamma, N}(\theta, \boldsymbol{\lambda}) \frac{\partial^{\alpha-\gamma}}{\partial \boldsymbol{\lambda}^{\alpha-\gamma}} (\lambda_{i_1} \dots \lambda_{i_{p+1}}) \right| \\ & \leq K_0 (1 + |\boldsymbol{\lambda}|)^{-\ell_1 + p+1} \end{aligned} \quad (26)$$

for all $\lambda \in \mathbb{R}^d$, where K_0 is a constant depending on $\alpha - \gamma$ but independent of $\lambda \in \mathbb{R}^d$.

Since $\|D\Psi_N\|_2 = O(N^{-m})$ for every $m > 0$, for $\Lambda_N = \{|\boldsymbol{\lambda}| \leq N^\xi\}$ and for every $k > 0$ and $s > 0$, we see

$$\overline{\lim}_{N \rightarrow \infty} \sup_{\boldsymbol{\lambda} \in \Lambda_N} \left(N^s |\boldsymbol{\lambda}|^k \left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} R_{j,N}(\theta, \boldsymbol{\lambda}) \right| \right) \lesssim \overline{\lim}_{N \rightarrow \infty} N^{s+(k+j)\xi-m} = 0 \quad (27)$$

if we choose a sufficiently large m . Now, from (26) and (27), we obtain Inequality (20).

The property (b) is rather easy to prove since it only requires $\boldsymbol{\lambda}$ -wise estimate of an explicit expression of $\frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N(\theta, \boldsymbol{\lambda})$ by using [A2] (ii). \blacksquare

Estimate of $\partial^\alpha / \partial \boldsymbol{\lambda}^\alpha \varphi_{F_N}^\Psi(\theta, \boldsymbol{\lambda})$ will be necessary in the proof of Proposition 1 down below. The only possibility is to repeatedly apply the IBP formula. The estimate outside of Λ_N cannot gain improvement by the decay of cumulants, and it is the worst one, in other words, it uses the highest order of repetition of the IBP formula among other terms.

Lemma 5. *Suppose that [A1] (i) is fulfilled. Then, for every $\alpha \in \mathbb{Z}_+^d$, there exists a constant C_α such that*

$$\left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \varphi_N^\Psi(\theta, \boldsymbol{\lambda}) \right| \leq C_\alpha (1 + |\boldsymbol{\lambda}|)^{-\ell} \quad (28)$$

for all $\theta \in [0, 1]$, $\boldsymbol{\lambda} \in \mathbb{R}^d$ and $N \in \mathbb{N}$, with ℓ from condition [A1].

Proof. This lemma can be proved by estimation quite similar to that for $E_{\gamma,N}(\theta, \boldsymbol{\lambda})$ in the proof of Lemma 4. We can repeat the IBP formula (i.e. Lemma 2 in [20]) ℓ -times in this case for $\theta \in [1/2, 1]$. Estimation for $\theta \in [0, 1/2)$ is also similar. \blacksquare

Recall the expression (13) of the characteristic function $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ in Lemma 3. Based on the estimated in Lemma 4, we can show that the last summand in the right-hand side can be neglected. Consequently, the dominant part of $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$ will come from the first term in right-hand side of (13), which involves the exponential of the principal part. We will introduce a new random variable, in which we keep the first terms in the Taylor expansion of the exponential function. For $k \in \mathbb{N}$, let

$$\mathbf{P}_N^*(\boldsymbol{\lambda}) = \sum_{j=0}^k \frac{1}{j!} \left(\int_0^1 \mathbf{P}_N(\theta, \boldsymbol{\lambda}) d\theta \right)^j$$

and

$$\mathbf{R}_N^*(\boldsymbol{\lambda}) = \varphi_N^\Psi(0, \boldsymbol{\lambda}) \sum_{j=k+1}^{\infty} \frac{1}{j!} \left(\int_0^1 \mathbf{P}_N(\theta, \boldsymbol{\lambda}) d\theta \right)^j + \int_0^1 \exp \left(\int_{\theta_1}^1 \mathbf{P}_N(\theta_2, \boldsymbol{\lambda}) d\theta_2 \right) \mathbf{R}_N(\theta_1, \boldsymbol{\lambda}) d\theta_1.$$

Then, if one has the expansion (13), then

$$\varphi_N^\Psi(1, \boldsymbol{\lambda}) = \varphi_N^\Psi(0, \boldsymbol{\lambda})\mathbf{P}_N^*(\boldsymbol{\lambda}) + \mathbf{R}_N^*(\boldsymbol{\lambda}). \quad (29)$$

At this point, let us make a brief summary to comment on the role of the parameters that appear in our work. Recall that d is the dimension of the random vector, $p+1$ is the maximum order of the cumulant that appear in the decomposition of the characteristic functional $\varphi_N^\Psi(\theta, \boldsymbol{\lambda})$, k comes from the Taylor expansion of the exponential function while ℓ, ℓ_1, q appear in assumptions [A1] – [A2]. N^{-q} will be the size of the error term. Next, we need to assume some relation between all these parameters. Condition (30) will be needed to evaluate the residual terms.

[A3] The numbers $\mathbf{q}_0 \in (0, \infty)$, $\xi \in (0, \infty)$, $\ell \in \mathbb{N}$ and $\ell_1 \in \mathbb{N}$ satisfy

$$\mathbf{q}_0(\mathbf{k} + 1) > \mathbf{q}, \quad \xi(\ell - d) > \mathbf{q}, \quad \ell_1 > p + 1 + d$$

and

$$N^{\mathbf{q}_0 + 2\xi} |\mathbf{E}[\Gamma^{(2sym)}(F_N)] - C| + \sum_{j=3}^{p+1} N^{\mathbf{q}_0 + j\xi} |\mathbf{E}[\Gamma^{(j)}(F_N)]| = O(1) \quad (30)$$

as $N \rightarrow \infty$. The expectation of a matrix is understood componentwise and $|\cdot|$ denotes the Euclidean norm.

Remark 2. (a) Condition (30) imposes a restriction on ξ from above. Besides, p will be asked to be large according to the value of \mathbf{q} in order to satisfy [A2]. In this sense, p is a function of \mathbf{q} : $p = p(\mathbf{q})$.

(b) If we take $\ell_1 = \ell$, then Condition [A3] requires

$$\ell > d + \max \left\{ \frac{\mathbf{q}}{\xi}, p + 1 \right\}$$

to ℓ . If $d = 1$, $p = 1$, then $\ell \geq 4$ at least.

(c) Increasing \mathbf{k} causes only increase of complexity of the formula and does not require F_N to pay more, once we found p for [A2] and \mathbf{q}_0 for (30).

(d) The index \mathbf{q}_0 depends on the largest order of terms among the terms appearing in (30). Often the term for $j = 3$ dominates other terms.

- (e) In most common situations, the order p of the controllable cumulants is the essential parameter. It determines the possible order \mathbf{q} of the asymptotic expansion. The parameters ℓ , \mathbf{k} , ξ and \mathbf{q}_0 are somewhat technical but automatically determined by p and the dimension d . See Section 4.2, that treats a regularly ordered expansion.

Lemma 6. (a) *Suppose that Conditions [A1], [A2] (i) and [A3] are satisfied. Then there exists a constant K such that*

$$N^{\mathbf{q}} 1_{\{\boldsymbol{\lambda} \in \Lambda_N\}} \left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N^*(\boldsymbol{\lambda}) \right| \leq K(1 + |\boldsymbol{\lambda}|)^{-\ell_1 + p + 1}$$

for all $(\boldsymbol{\lambda}, N) \in \mathbb{R}^d \times \mathbb{N}$.

(b) *Suppose that Conditions [A1], [A2] (ii) and [A3] are satisfied. Then*

$$\frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{R}_N^*(\boldsymbol{\lambda}) = o(N^{-\mathbf{q}})$$

as $n \rightarrow \infty$ for every $\boldsymbol{\lambda} \in \mathbb{R}^d$.

Proof. It follows from [A3] that

$$\sup_{\theta \in [0,1], \boldsymbol{\lambda} \in \mathbb{R}^d} 1_{\{\boldsymbol{\lambda} \in \Lambda_N\}} \left| \frac{\partial^\alpha}{\partial \boldsymbol{\lambda}^\alpha} \mathbf{P}_N(\theta, \boldsymbol{\lambda}) \right| = O(N^{-\mathbf{q}_0}) \quad (31)$$

as $N \rightarrow \infty$ for every $\alpha \in \mathbb{Z}_+^d$. Therefore \mathbf{P}_N undertakes $\boldsymbol{\lambda}$'s by itself.

The property (a) follows from Lemma 4 (a), and the representations of $\mathbf{R}_N^*(\boldsymbol{\lambda})$ by $\mathbf{R}_N(\theta, \boldsymbol{\lambda})$ and $\mathbf{P}_N(\theta, \boldsymbol{\lambda})$ and $\mathbf{q}_0(\mathbf{k} + 1) > \mathbf{q}$. Similarly, the property (b) can be proved by using Lemma 4 (b). \blacksquare

3 Asymptotic expansion of the expectation and its error bound

The approximate density of the sequence $(F_N)_{N \geq 1}$ is defined as the Fourier inverse of the dominant part of the truncated characteristic function of F_N as follows.

$$f_{N,p,\mathbf{k}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \boldsymbol{\lambda}, x \rangle} \varphi_{N,p,\mathbf{k}}(\boldsymbol{\lambda}) d\boldsymbol{\lambda}, \quad x \in \mathbb{R}^d \quad (32)$$

where $\varphi_{N,p,k}$ is given by

$$\varphi_{N,p,k}(\boldsymbol{\lambda}) = e^{-\frac{1}{2}\boldsymbol{\lambda}^T C \boldsymbol{\lambda}} \mathbf{P}_N^*(\boldsymbol{\lambda}). \quad (33)$$

We recall that \mathbf{P}_N^* depends on p and \mathbf{k} . Obviously by definition, $\varphi_{N,p,k}(\boldsymbol{\lambda})$ may include terms of higher-order than \mathbf{q} . It is necessary to find an expansion of each term in the expression (12) of $\mathbf{P}_N(\theta, \boldsymbol{\lambda})$ if one wants to extract the principal part from the expansion. The principal part depends on the structure of the model in question, and really we will specify it in the later sections.

Let

$$\varphi_N^\Psi(\boldsymbol{\lambda}) = \varphi_N^\Psi(1, \boldsymbol{\lambda}) \equiv \mathbf{E}[\Psi_N e^{i\langle \boldsymbol{\lambda}, F_N \rangle}]. \quad (34)$$

The local (or truncated) density p_N^Ψ of the random variable F_N is defined as the inverse Fourier transform of the truncated characteristic function:

$$p_N^\Psi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \boldsymbol{\lambda}, x \rangle} \varphi_N^\Psi(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$$

for $x \in \mathbb{R}^d$. The local density is well-defined since obviously the truncated characteristic function $\varphi_{F_N}^\Psi$ is integrable over \mathbb{R}^d under [A1] by Lemma 5 if $\ell > d$.

We estimate below the difference between the approximate density and the local density.

Proposition 1. *Suppose that Conditions [A1], [A2] and [A3] are satisfied. Then*

$$\sup_{x \in \mathbb{R}^d} \left(|x|^m |p_N^\Psi(x) - f_{N,p,k}(x)| \right) = o(N^{-q}) \quad (35)$$

as $N \rightarrow \infty$ for every $m > 0$.

Proof. Let

$$\begin{aligned} \tilde{\mathbf{R}}_N(\boldsymbol{\lambda}) &= \{\varphi_N^\Psi(0, \boldsymbol{\lambda}) - e^{-\frac{1}{2}\boldsymbol{\lambda}^T C \boldsymbol{\lambda}}\} \mathbf{P}_N^*(\boldsymbol{\lambda}) \\ &= \{\mathbf{E}(\Psi_N) - 1\} e^{-\frac{1}{2}\boldsymbol{\lambda}^T C \boldsymbol{\lambda}} \mathbf{P}_N^*(\boldsymbol{\lambda}). \end{aligned} \quad (36)$$

Since

$$\varphi_{N,p,k}(\boldsymbol{\lambda}) = \varphi_N^\Psi(0, \boldsymbol{\lambda}) \mathbf{P}_N^*(\boldsymbol{\lambda}) - \tilde{\mathbf{R}}_N(\boldsymbol{\lambda}),$$

we have

$$\varphi_N^\Psi(\boldsymbol{\lambda}) - \varphi_{N,p,k}(\boldsymbol{\lambda}) = \mathbf{R}_N^*(\boldsymbol{\lambda}) + \tilde{\mathbf{R}}_N(\boldsymbol{\lambda}) \quad (37)$$

from (29).

By the integrability ensured by Lemmas 5 and 6 and by (36), we obtain

$$\begin{aligned}
\left| x^\alpha \left(p_N^\Psi(x) - f_{N,p,k}(x) \right) \right| &= \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{-i\langle \lambda, x \rangle} \left(\frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_N^\Psi(\lambda) - \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_{N,p,k}(\lambda) \right) d\lambda \right| \\
&\leq \left| \int_{\Lambda_N} e^{-i\langle \lambda, x \rangle} \left(\frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_N^\Psi(\lambda) - \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_{N,p,k}(\lambda) \right) d\lambda \right| \\
&\quad + \int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_N^\Psi(\lambda) \right| d\lambda + \int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_{N,p,k}(\lambda) \right| d\lambda \\
&\leq \int_{\Lambda_N} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} R_N^*(\lambda) \right| d\lambda + \int_{\Lambda_N} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \tilde{R}_N(\lambda) \right| d\lambda \\
&\quad + \int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_N^\Psi(\lambda) \right| d\lambda + \int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_{N,p,k}(\lambda) \right| d\lambda
\end{aligned}$$

for every $\alpha \in \mathbb{Z}_+^d$. Thus, we obtain (35) by the following estimates.

$$\lim_{N \rightarrow \infty} N^q \int_{\Lambda_N} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} R_N^*(\lambda) \right| d\lambda = 0$$

by the dominated convergence theorem with Lemma 6 since $\ell_1 > p + 1 + d$ by [A3]. Since (31) holds and $\mathbf{E}(\Psi_N) - 1 = O(N^{-L})$ as $N \rightarrow \infty$ for any $L > 0$, the representation (36) gives

$$\lim_{N \rightarrow \infty} N^L \int_{\Lambda_N} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \tilde{R}_N(\lambda) \right| d\lambda = 0$$

for every $L > 0$. By Lemma 5,

$$\begin{aligned}
\int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_N^\Psi(\lambda) \right| d\lambda &\leq C_\alpha \int_{\Lambda_N^c} (1 + |\lambda|)^{-\ell} d\lambda \\
&\lesssim \int_{r > N^\xi} r^{-\ell+d-1} dr \\
&= O(N^{-(\ell-d)\xi}) = o(N^{-q})
\end{aligned}$$

as $N \rightarrow \infty$ since $(\ell - d)\xi > q$ by [A3]. Moreover

$$\int_{\Lambda_N^c} \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \varphi_{N,p,k}(\lambda) \right| d\lambda = O(N^{-L})$$

as $N \rightarrow \infty$ by the Gaussian factor with the assistance of (30). ■

For $a, b > 0$, we denote by $\mathcal{E}(a, b)$ the set of measurable functions g on \mathbb{R}^d satisfying $|g(x)| \leq a(1 + |x|)^b$.

Theorem 1. *Suppose that Conditions [A1], [A2] and [A3] are satisfied. Then, for*

$$\sup_{g \in \mathcal{E}(a, b)} \left| \mathbf{E}[g(F_N)] - \int_{\mathbb{R}^d} g(x) f_{N, p, k}(x) dx \right| = o(N^{-q})$$

as $N \rightarrow \infty$ for every $a, b > 0$.

Proof. Since the Fourier transform φ_N^Ψ of the measure $\nu(dx) := \mathbf{E}[\Psi_N | F_N = x] P^{F_N}(dx)$ is integrable, ν has a continuous density p_N^Ψ . Applying Proposition 1, we have

$$\begin{aligned} & \sup_{g \in \mathcal{E}(a, b)} \left| \mathbf{E}[\Psi_N g(F_N)] - \int_{\mathbb{R}^d} g(x) f_{N, p, k}(x) dx \right| \\ &= \sup_{g \in \mathcal{E}(a, b)} \left| \int_{\mathbb{R}^d} g(x) p_N^\Psi(x) dx - \int_{\mathbb{R}^d} g(x) f_{N, p, k}(x) dx \right| \\ &\leq \sup_{g \in \mathcal{E}(a, b)} \int_{\mathbb{R}^d} |g(x)| |p_N^\Psi(x) - f_{N, p, k}(x)| dx \\ &\leq \sup_{x \in \mathbb{R}^d} \left((1 + |x|)^{d+1+b} |p_N^\Psi(x) - f_{N, p, k}(x)| \right) \int_{\mathbb{R}^d} a(1 + |y|)^{-d-1} dy \\ &= o(N^{-q}) \end{aligned}$$

as $N \rightarrow \infty$ for every $a, b > 0$. This completes the proof because

$$\sup_{g \in \mathcal{E}(a, b)} \left| \mathbf{E}[g(F_N)] - \mathbf{E}[\Psi_N g(F_N)] \right| \leq \|1 - \Psi_N\|_2 \sup_{N' \in \mathbb{N}} \|a(1 + |F_{N'}|)^b\|_2 = O(N^{-L})$$

as $N \rightarrow \infty$ for every $L > 0$. ■

4 Reduced formulas

If the cumulants $\mathbf{E}[\Gamma^{(p)}(F_N)]$, $p \geq 2$ admit a specific Taylor decomposition, we obtain a more explicite asymptotic expansion for the sequence $(F_N)_{N \geq 1}$.

4.1 Principal part of $f_{N,p,k}$

The asymptotic expansion formula is given by (32) and (33). However, it involves terms that is higher than N^{-q} in general. If the coefficients in $\mathbf{P}_N(\theta, \boldsymbol{\lambda})$ admit a specific expansion, then we can extract the principal part of $f_{N,p,k}$.

Let $\mathbb{I} = \{1, \dots, d\}$. For simplicity of notation, we will denote by I_j a generic element (i_1, \dots, i_j) of \mathbb{I}^j . The summation \sum_{I_j} stands for $\sum_{(i_1, \dots, i_j) \in \mathbb{I}^j}$. For $j \in \{2, \dots, p+1\}$, suppose that a nonnegative integer $k(I_j)$ is given for each $I_j \in \mathbb{I}^j$.

The below assumption gives the concrete Taylor expansion of the cumulants of F_N in terms of power of N .

[B] For each $j \in \{2, \dots, p+1\}$ and $I_j \in \mathbb{I}^j$, if $k(I_j) \geq 1$, then for $k \in \{1, \dots, k(I_j)\}$, there exist sequences of real numbers $(c(I_j, k))_{k=1, \dots, k(I_j)}$ and $(\gamma(I_j, k))_{k=1, \dots, k(I_j)}$ such that the following conditions hold.

- (i) $0 < \gamma(I_j, 1) < \dots < \gamma(I_j, k(I_j)) \leq q$ (when $k(I_j) \geq 1$).
- (ii) For $I_2 \in \mathbb{I}^2$,

$$\mathbf{E}[\Gamma_{I_2}^{(2sym)}(F_N)] - C_{I_2} = \sum_{k=1}^{k(I_2)} c(I_2, k) N^{-\gamma(I_2, k)} + o(N^{-q}),$$

where the sum $\sum_{k=1}^{k(I_2)}$ reads 0 when $k(I_2) = 0$.

- (iii) For $j \in \{3, \dots, p+1\}$ and $I_j \in \mathbb{I}^j$,

$$\mathbf{E}[\Gamma_{I_j}^{(j)}(F_N)] = \sum_{k=1}^{k(I_j)} c(I_j, k) N^{-\gamma(I_j, k)} + o(N^{-q}),$$

where the sum $\sum_{k=1}^{k(I_j)}$ reads 0 when $k(I_j) = 0$.

- (iv) The numbers $q_0 \in (0, \infty)$, $\xi \in (0, \infty)$, $\ell \in \mathbb{N}$ and $\ell_1 \in \mathbb{N}$ satisfy

$$q_0(k+1) > q, \quad \xi(\ell - d) > q, \quad \ell \geq \ell_1 > p+1+d \quad (38)$$

and

$$q_0 \leq \min \left\{ \gamma(I_j, 1) - j\xi; I_j \in \mathbb{I}^j, j = 2, \dots, p+1 \right\} \quad (39)$$

with $\gamma(I_j, 1) = \infty$ when $k(I_j) = 0$.

Assumption [B](ii) – (iii) indicates that the random vector $(F_N)_{N \geq 1}$ converges in distribution to a centered Gaussian vector with covariance matrix C .

Remark 3. In many cases, $\gamma(I_j, k)$ is a multiple of a constant such as $1/2$. However, it is not always true. For example, the asymptotic expansion formula for $F_N = S_N^{(1)} + S_N^{(2)}$ has two scales $N^{-1/2}$ and $[N^\pi]^{-1/2}$ and their mixtures when $S_N^{(1)} = N^{-1/2} \sum_{j=1}^N ((\xi_j^{(1)})^2 - 1)$ and $S_N^{(2)} = [N^\pi]^{-1/2} \sum_{j=1}^{[N^\pi]} ((\xi_j^{(2)})^2 - 1)$, where $\{\xi_j^{(1)}, \xi_j^{(2)}; j \in \mathbb{N}\}$ are independent standard Gaussian random variables.

We write $\lambda_{I_m} = \lambda_{i_1} \cdots \lambda_{i_m}$ for $\lambda = (\lambda_1, \dots, \lambda_d)$ and $I_m = (i_1, \dots, i_m)$. Under [B], $\int_0^1 P_N(\theta, \lambda) d\theta$ is given by

$$\int_0^1 P_N(\theta, \lambda) d\theta = \sum_{m=2}^{p+1} \sum_{I_m \in \mathbb{I}^m} \sum_{k=1}^{k(I_m)} \lambda_{I_m} \left\{ \frac{\mathbf{i}^m c(I_m, k)}{m} N^{-\gamma(I_m, k)} + o(N^{-q}) \right\}$$

Therefore, $P_N^*(\lambda)$ is expressed as

$$\begin{aligned} P_N^*(\lambda) &= 1 + \sum_{j=1}^k \sum_{m_1=2}^{p+1} \cdots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \cdots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{k(I_{m_1}^{(1)})} \cdots \sum_{k_j=1}^{k(I_{m_j}^{(j)})} \lambda_{I_{m_1}^{(1)}} \cdots \lambda_{I_{m_j}^{(j)}} \\ &\times \left\{ \frac{\mathbf{i}^{m_1 + \cdots + m_j}}{j! m_1 \cdots m_j} c(I_{m_1}^{(1)}, k_1) \cdots c(I_{m_j}^{(j)}, k_j) \mathbf{1}_{\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j) \leq q\}} \right. \\ &\quad \left. \times N^{-\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j)\}} + \epsilon_N(j; m_1, \dots, m_j; I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)}; k_1, \dots, k_j) \right\}, \end{aligned}$$

where $\epsilon_N(j; m_1, \dots, m_j; I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)}; k_1, \dots, k_j) = o(N^{-q})$ independent of λ .

Given the Taylor expansion of the cumulants from Condition [B] and the expression of the principal part $P_N(\theta, \lambda)$ in (12), we will replace $P_N^*(\lambda)$ by a new functional written in terms of powers of N . Define $\tilde{P}_N(\lambda)$ by

$$\begin{aligned} \tilde{P}_N(\lambda) &= 1 + \sum_{j=1}^k \sum_{m_1=2}^{p+1} \cdots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \cdots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{k(I_{m_1}^{(1)})} \cdots \sum_{k_j=1}^{k(I_{m_j}^{(j)})} \lambda_{I_{m_1}^{(1)}} \cdots \lambda_{I_{m_j}^{(j)}} \\ &\times \frac{\mathbf{i}^{m_1 + \cdots + m_j}}{j! m_1 \cdots m_j} c(I_{m_1}^{(1)}, k_1) \cdots c(I_{m_j}^{(j)}, k_j) \mathbf{1}_{\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j) \leq q\}} \\ &\times N^{-\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j)\}}. \end{aligned} \tag{40}$$

Let

$$\tilde{f}_{N,p,k}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \boldsymbol{\lambda}, x \rangle} \tilde{\varphi}_{N,p,k}(\boldsymbol{\lambda}) d\boldsymbol{\lambda} \quad (x \in \mathbb{R}^d)$$

where $\tilde{\varphi}_{N,p,k}$ is given by

$$\tilde{\varphi}_{N,p,k}(\boldsymbol{\lambda}) = e^{-\frac{1}{2}\boldsymbol{\lambda}^T C \boldsymbol{\lambda}} \tilde{\mathbf{P}}_N(\boldsymbol{\lambda}).$$

Then

$$\tilde{f}_{N,p,k}(x) = \tilde{\mathbf{P}}_N(i\partial_x)\phi(x; 0, C), \quad (41)$$

where

$$\phi(x; 0, C) = (2\pi)^{-d/2} (\det C)^{-1/2} e^{-\frac{1}{2}x^T C^{-1}x}.$$

Define the α -th Hermite polynomial $H_\alpha(x; C)$ by

$$H_\alpha(x; C) = e^{x^T C^{-1}x/2} (-\partial_x)^\alpha e^{-x^T C^{-1}x/2} \quad (x \in \mathbb{R}^d)$$

for $\alpha \in \mathbb{Z}_+^d$. Define the multi-index $\alpha(I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)}) \in \mathbb{Z}_+^d$ by

$$\boldsymbol{\lambda}_{I_{m_1}^{(1)}} \cdots \boldsymbol{\lambda}_{I_{m_j}^{(j)}} = \boldsymbol{\lambda}^{\alpha(I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)})} \quad (\boldsymbol{\lambda} \in \mathbb{R}^d).$$

That is, the i -th component of $\alpha(I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)})$ is the number of i 's appearing in the sequence $I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)}$. Then the density function $\tilde{f}_{N,p,k}$ is expressed as

$$\begin{aligned} \tilde{f}_{N,p,k}(x) &= \phi(x; 0, C) \\ &+ \sum_{j=1}^k \sum_{m_1=2}^{p+1} \cdots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \cdots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{k(I_{m_1}^{(1)})} \cdots \sum_{k_j=1}^{k(I_{m_j}^{(j)})} \left\{ \frac{1}{j! m_1 \cdots m_j} \right. \\ &\times c(I_{m_1}^{(1)}, k_1) \cdots c(I_{m_j}^{(j)}, k_j) H_{\alpha(I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)})}(x; C) \phi(x; 0, C) \\ &\left. \times \mathbf{1}_{\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j) \leq \mathbf{q}\}} N^{-\{\gamma(I_{m_1}^{(1)}, k_1) + \cdots + \gamma(I_{m_j}^{(j)}, k_j)\}} \right\}. \quad (42) \end{aligned}$$

The following theorem validates $\tilde{f}_{N,p,k}$ as a reduced asymptotic expansion formula.

Theorem 2. *Suppose that Conditions [A1], [A2] and [B] are fulfilled. Then*

$$\sup_{g \in \mathcal{E}(a,b)} \left| \mathbf{E}[g(F_N)] - \int_{\mathbb{R}^d} g(x) \tilde{f}_{N,p,k}(x) dx \right| = o(N^{-q}) \quad (43)$$

as $N \rightarrow \infty$ for every $a, b > 0$.

Proof. We have

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \left(|x^\alpha| |f_{N,p,k}(x) - \tilde{f}_{N,p,k}(x)| \right) \\ &= o(N^{-q}) \sup_{x \in \mathbb{R}^d} \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} ((i\partial_\lambda)^\alpha e^{-i\langle \lambda, x \rangle}) \left\{ e^{-\frac{1}{2}\lambda^T C \lambda} \sum_{j=1}^k \sum_{m_1=2}^{p+1} \dots \right. \right. \\ & \quad \left. \left. \dots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \dots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{k(I_{m_1}^{(1)})} \dots \sum_{k_j=1}^{k(I_{m_j}^{(j)})} \lambda_{I_{m_1}^{(1)}} \dots \lambda_{I_{m_j}^{(j)}} \right\} d\lambda \right| \\ &= o(N^{-q}) \end{aligned}$$

as $N \rightarrow \infty$ for every $\alpha \in \mathbb{Z}_+^d$. Here the last equality follows from integration-by-parts and that $\epsilon_N = o(N^{-q})$. Therefore,

$$\sup_{g \in \mathcal{E}(a,b)} \left| \int_{\mathbb{R}^d} g(x) f_{N,p,k}(x) dx - \int_{\mathbb{R}^d} g(x) \tilde{f}_{N,p,k}(x) dx \right| = o(N^{-q})$$

for any $a, b > 0$. Now Theorem 2 follows from Theorem 1. ■

4.2 Regular ordering

In this section, we will consider the situation where the exponents $\gamma(I_j, k)$ are multiples of some positive number γ . Suppose that $p \geq 2$. We consider the following situation.

[C] (i) For each $I_2 \in \mathbb{I}^2$,

$$\mathbf{E}[\Gamma_{I_2}^{(2sym)}(F_N)] - C_{I_2} = \sum_{k=1}^{p-1} c(I_2, k) N^{-k\gamma} + o(N^{-(p-1)\gamma})$$

as $N \rightarrow \infty$ for some constants $c(I_2, k)$ ($k = 1, \dots, p-1$).

(ii) For each $j \in \{3, \dots, p+1\}$ and $I_j \in \mathbb{I}^j$,

$$\mathbf{E}[\Gamma_{I_j}^{(j)}(F_N)] = \sum_{k=1}^{p-j+2} c(I_j, k) N^{-(j-3+k)\gamma} + o(N^{-(p-1)\gamma})$$

as $N \rightarrow \infty$ for some constants $c(I_j, k)$ ($k = 1, \dots, p-j+2$).

Given an integer $p \geq 2$ and the dimension d of F_N , we suppose that the positive integers ℓ and ℓ_1 satisfy

$$\ell > 3(p-1) + d, \quad \ell \geq \ell_1 > p+1 + d. \quad (44)$$

For example, the inequalities (44) hold if $\ell = \ell_1 > 3(p-1) + d$, when $p \geq 2$.

Suppose that Condition [C] and (44) are fulfilled and that a positive number ξ and an integer k satisfy

$$\xi \in \left(\frac{p-1}{\ell-d} \gamma, \frac{1}{3} \gamma \right)$$

and

$$k > \frac{(p-2)\gamma + 3\xi}{\gamma - 3\xi} = \frac{(p-1)\gamma}{\gamma - 3\xi} - 1. \quad (45)$$

Such numbers ξ and k exist under (44). Let $\mathbf{q} = (p-1)\gamma$ and let $\mathbf{q}_0 = \gamma - 3\xi (> 0)$. Let

$$\gamma(I_2, k) = k\gamma \quad (k = 1, \dots, p-1)$$

and

$$\gamma(I_j, k) = (j-3+k)\gamma \quad (k = 1, \dots, p-j+2)$$

for $j = 3, \dots, p+1$. Then the inequalities in (38) and (39) are met, and hence Condition [B] holds for $k(I_2) = p-1$ and $k(I_j) = p-j+2$ for $j = 3, \dots, p+1$.

We can write $\gamma(I_j, k) = ((j-3)_+ + k)\gamma$ for $j \in \{2, \dots, p+1\}$, $x_+ = \max\{x, 0\}$. Then the symbol $\tilde{\mathbf{P}}_N(\boldsymbol{\lambda})$ takes the form of

$$\begin{aligned} \tilde{\mathbf{P}}_N(\boldsymbol{\lambda}) &= 1 + \sum_{j=1}^{p-1} \sum_{m_1=2}^{p+1} \cdots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \cdots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{p-1-(m_1-3)_+} \cdots \sum_{k_j=1}^{p-1-(m_j-3)_+} \lambda_{I_{m_1}^{(1)}} \cdots \lambda_{I_{m_j}^{(j)}} \\ &\times \frac{\mathbf{i}^{m_1+\cdots+m_j}}{j! m_1 \cdots m_j} c(I_{m_1}^{(1)}, k_1) \cdots c(I_{m_j}^{(j)}, k_j) \mathbf{1}_{\{\sum_{i=1}^j (m_i-3)_+ + \sum_{i=1}^j k_i \leq p-1\}} \\ &\times N^{-\{\sum_{i=1}^j (m_i-3)_+ + \sum_{i=1}^j k_i\} \gamma}. \end{aligned} \quad (46)$$

We remark that the first summation on the right-hand side of (46) has become $\sum_{j=1}^{p-1}$ though it was originally $\sum_{j=1}^k$, by the following reason. The condition (45) entails $k \geq p - 1$, however, for $j \geq p$, the summands vanish due to the indicator function. Therefore only the terms for j up to $p - 1$ can contribute. According to (46), the density $\tilde{f}_{N,p,k}$ has the expression

$$\begin{aligned} \tilde{f}_{N,p,k}(x) &= \phi(x; 0, C) \\ &+ \sum_{j=1}^{p-1} \sum_{m_1=2}^{p+1} \cdots \sum_{m_j=2}^{p+1} \sum_{I_{m_1}^{(1)} \in \mathbb{I}^{m_1}} \cdots \sum_{I_{m_j}^{(j)} \in \mathbb{I}^{m_j}} \sum_{k_1=1}^{p-1-(m_1-3)_+} \cdots \sum_{k_j=1}^{p-1-(m_j-3)_+} \left\{ \frac{1}{j! m_1 \cdots m_j} \right. \\ &\quad \times c(I_{m_1}^{(1)}, k_1) \cdots c(I_{m_j}^{(j)}, k_j) H_{\alpha(I_{m_1}^{(1)}, \dots, I_{m_j}^{(j)})}(x; C) \phi(x; 0, C) \\ &\quad \left. \times 1_{\{\sum_{i=1}^j (m_i-3)_+ + \sum_{i=1}^j k_i \leq p-1\}} N^{-\{\sum_{i=1}^j (m_i-3)_+ + \sum_{i=1}^j k_i\} \gamma} \right\}. \end{aligned} \quad (47)$$

Applying Theorem 2, we obtain an asymptotic expansion formula when the gamma factors have a regularly ordered expansion.

Theorem 3. *Assume [A1] and [A2] for some pair (ℓ, ℓ_1) of integers satisfying (44) for given integers $p \geq 2$ and the dimension d of F_N . Moreover assume [C]. Then*

$$\sup_{g \in \mathcal{E}(a,b)} \left| \mathbf{E}[g(F_N)] - \int_{\mathbb{R}^d} g(x) \tilde{f}_{N,p,k}(x) dx \right| = o(N^{-(p-1)\gamma}) \quad (48)$$

as $N \rightarrow \infty$ for every $a, b > 0$ for $\tilde{f}_{N,p,k}$ of (47).

In the rest of this section, we will state several special cases of the asymptotic expansion in terms of the symbol $\tilde{\mathbf{P}}_N(\boldsymbol{\lambda})$ of (46). When $p = 2$, Formula (46) gives

$$\tilde{\mathbf{P}}_N(\boldsymbol{\lambda}) = 1 + \left\{ \sum_{I_2^{(1)} \in \mathbb{I}^2} \frac{1}{2} \mathbf{i}^2 \boldsymbol{\lambda}_{I_2^{(1)}} c(I_2^{(1)}, 1) + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 1) \right\} N^{-\gamma}. \quad (49)$$

When $p = 3$, Formula (46) gives

$$\begin{aligned}
\tilde{\mathcal{P}}_N(\boldsymbol{\lambda}) &= 1 + \left\{ \sum_{I_2^{(1)} \in \mathbb{I}^2} \frac{1}{2} \mathbf{i}^2 \boldsymbol{\lambda}_{I_2^{(1)}} c(I_2^{(1)}, 1) + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 1) \right\} N^{-\gamma} \\
&+ \left\{ \sum_{I_2^{(1)} \in \mathbb{I}^2} \frac{1}{2} \mathbf{i}^2 \boldsymbol{\lambda}_{I_2^{(1)}} c(I_2^{(1)}, 2) + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 2) \right. \\
&\quad + \sum_{I_4^{(1)} \in \mathbb{I}^4} \frac{1}{4} \mathbf{i}^4 \boldsymbol{\lambda}_{I_4^{(1)}} c(I_4^{(1)}, 1) \\
&\quad + \sum_{I_2^{(1)} \in \mathbb{I}^2} \sum_{I_2^{(2)} \in \mathbb{I}^2} \frac{1}{8} \mathbf{i}^4 \boldsymbol{\lambda}_{I_2^{(1)}} \boldsymbol{\lambda}_{I_2^{(2)}} c(I_2^{(1)}, 1) c(I_2^{(2)}, 1) \\
&\quad + \sum_{I_2^{(1)} \in \mathbb{I}^2} \sum_{I_3^{(2)} \in \mathbb{I}^3} \frac{1}{6} \mathbf{i}^5 \boldsymbol{\lambda}_{I_2^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} c(I_2^{(1)}, 1) c(I_3^{(2)}, 1) \\
&\quad \left. + \sum_{I_3^{(1)} \in \mathbb{I}^2} \sum_{I_3^{(2)} \in \mathbb{I}^3} \frac{1}{18} \mathbf{i}^6 \boldsymbol{\lambda}_{I_3^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} c(I_3^{(1)}, 1) c(I_3^{(2)}, 1) \right\} N^{-2\gamma}. \quad (50)
\end{aligned}$$

In particular, when $p = 2$, Formula (49) is reduced to

$$\tilde{\mathcal{P}}_N(\boldsymbol{\lambda}) = 1 + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 1) N^{-\gamma}$$

if $c(I_2, 1) = 0$ for all $I_2 \in \mathbb{I}^2$. When $p = 3$, Formula (50) is reduced to

$$\begin{aligned}
\tilde{\mathcal{P}}_N(\boldsymbol{\lambda}) &= 1 + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 1) N^{-\gamma} \\
&+ \left\{ \sum_{I_2^{(1)} \in \mathbb{I}^2} \frac{1}{2} \mathbf{i}^2 \boldsymbol{\lambda}_{I_2^{(1)}} c(I_2^{(1)}, 2) + \sum_{I_4^{(1)} \in \mathbb{I}^4} \frac{1}{4} \mathbf{i}^4 \boldsymbol{\lambda}_{I_4^{(1)}} c(I_4^{(1)}, 1) \right. \\
&\quad \left. + \sum_{I_3^{(1)} \in \mathbb{I}^2} \sum_{I_3^{(2)} \in \mathbb{I}^3} \frac{1}{18} \mathbf{i}^6 \boldsymbol{\lambda}_{I_3^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} c(I_3^{(1)}, 1) c(I_3^{(2)}, 1) \right\} N^{-2\gamma}
\end{aligned}$$

if $c(I_2, 1) = 0$ for all $I_2 \in \mathbb{I}^2$ and if $c(I_3, 2) = 0$ for all $I_3 \in \mathbb{I}^3$. If additionally $c(I_2, 3) = 0$ for all $I_2 \in \mathbb{I}^2$ and if $c(I_4, 1) = 0$ for all $I_4 \in \mathbb{I}^4$, then for $p = 4$, we obtain

$$\begin{aligned}
\tilde{\mathbb{P}}_N(\boldsymbol{\lambda}) &= 1 + \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 1) N^{-\gamma} \\
&+ \left\{ \sum_{I_2^{(1)} \in \mathbb{I}^2} \frac{1}{2} \mathbf{i}^2 \boldsymbol{\lambda}_{I_2^{(1)}} c(I_2^{(1)}, 2) + \sum_{I_4^{(1)} \in \mathbb{I}^4} \frac{1}{4} \mathbf{i}^4 \boldsymbol{\lambda}_{I_4^{(1)}} c(I_4^{(1)}, 1) \right. \\
&\quad \left. + \sum_{I_3^{(1)} \in \mathbb{I}^3} \sum_{I_3^{(2)} \in \mathbb{I}^3} \frac{1}{18} \mathbf{i}^6 \boldsymbol{\lambda}_{I_3^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} c(I_3^{(1)}, 1) c(I_3^{(2)}, 1) \right\} N^{-2\gamma} \\
&+ \left\{ \sum_{I_3^{(1)} \in \mathbb{I}^3} \frac{1}{3} \mathbf{i}^3 \boldsymbol{\lambda}_{I_3^{(1)}} c(I_3^{(1)}, 3) + \sum_{I_5^{(1)} \in \mathbb{I}^5} \frac{1}{5} \mathbf{i}^5 \boldsymbol{\lambda}_{I_5^{(1)}} c(I_5^{(1)}, 1) \right. \\
&\quad + \sum_{I_2^{(1)} \in \mathbb{I}^2} \sum_{I_3^{(2)} \in \mathbb{I}^3} \frac{1}{6} \mathbf{i}^5 \boldsymbol{\lambda}_{I_2^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} c(I_2^{(1)}, 2) c(I_3^{(2)}, 1) \\
&\quad + \sum_{I_3^{(1)} \in \mathbb{I}^3} \sum_{I_4^{(2)} \in \mathbb{I}^4} \frac{1}{12} \mathbf{i}^7 \boldsymbol{\lambda}_{I_3^{(1)}} \boldsymbol{\lambda}_{I_4^{(2)}} c(I_3^{(1)}, 1) c(I_4^{(2)}, 2) \\
&\quad \left. + \sum_{I_3^{(1)} \in \mathbb{I}^3} \sum_{I_3^{(2)} \in \mathbb{I}^3} \sum_{I_3^{(3)} \in \mathbb{I}^3} \frac{1}{162} \mathbf{i}^9 \boldsymbol{\lambda}_{I_3^{(1)}} \boldsymbol{\lambda}_{I_3^{(2)}} \boldsymbol{\lambda}_{I_3^{(3)}} c(I_3^{(1)}, 1) c(I_3^{(2)}, 1) c(I_3^{(3)}, 1) \right\} N^{-3\gamma}.
\end{aligned}$$

In the last situation above (which is the case of the example studied below), the first order term comes from the leading term in the Taylor expansion of the third cumulant, the second order term comes in the sum of the leading terms in the expansion of the second and fourth cumulant, the high order terms being a mixture of terms in the Taylor expansion of all the cumulants.

5 Application to the wave equation

In order to illustrate our theoretical results, we will consider an example of a random sequence related to the solution to the wave equation driven by a space-time white noise. More precisely, we will analyze the asymptotic behavior of the quadratic variation in space of this solution. We will first analyzed the asymptotic expansion

of the spatial quadratic variations at a fixed time and then we will study the two-dimensional random vector whose components are the spatial quadratic variations at different times. We show that the assumptions considered in the previous sections are satisfied in this case.

Let us start by recalling some basic facts concerning the stochastic wave equation and its solution.

5.1 The wave equation with space-time white noise

Our object of study is the solution to the following stochastic partial differential equation in dimension 1

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t, x) = \Delta u(t, x) + \dot{W}(t, x), & t > 0, x \in \mathbb{R} \\ u(0, x) = 0, & x \in \mathbb{R} \\ \frac{\partial u}{\partial t}(0, x) = 0, & x \in \mathbb{R}. \end{cases} \quad (51)$$

We denote by Δ the Laplacian on \mathbb{R} and by $W = \{W_t(A); t \geq 0, A \in \mathcal{B}_b(\mathbb{R})\}$ a real valued centered Gaussian field, over a given complete filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F})_{t \geq 0}, \mathbb{P})$ with covariance:

$$\mathbf{E} \left[W_t(A) W_s(B) \right] = (t \wedge s) \lambda(A \cap B), \text{ for every } A, B \in \mathfrak{B}_d(\mathbb{R}) \quad (52)$$

where λ is the one-dimensional Lebesgue measure and $\mathfrak{B}_d(\mathbb{R})$ is the set of the Borel-subsets of \mathbb{R} with finite Lebesgue measure. This is usually called "the space-time white noise".

The mild solution to (51) is a square-integrable process $u = \{u(t, x); t \geq 0, x \in \mathbb{R}\}$ which is defined by:

$$u(t, x) = \int_0^t \int_{\mathbb{R}} G_1(t-s, x-y) W(ds, dy) \quad (53)$$

where the Green kernel G_1 is defined by

$$G_1(t, x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}}, \quad t > 0, x \in \mathbb{R}. \quad (54)$$

5.2 Computing the covariance and the correlation

Fix $t_1, t_2 > 0$ and $x, y \in \mathbb{R}$. We need a sharp evaluation of the correlation structure of the Gaussian process (53). We first calculate the quantity $\mathbf{E}[u(t_1, x)u(t_2, x)]$ at different times $t_1 \neq t_2$ and when $t_1 = t_2$.

Lemma 7. Let u be given by (53). For every $t_1, t_2 > 0$ and $x, y \in \mathbb{R}$, we have

$$\mathbf{E}[u(t_1, x)u(t_2, y)] = \frac{1}{16} \mathbf{1}_{\{|t_1 - t_2| \leq |y - x| < t_1 + t_2\}} (t_1 + t_2 - |x - y|)^2 + \frac{1}{4} \mathbf{1}_{\{|t_1 - t_2| > |y - x|\}} (t_1 \wedge t_2)^2. \quad (55)$$

In particular, for $t_1 = t_2 = t > 0$,

$$\mathbf{E}[u(t, x)u(t, y)] = \frac{1}{16} \mathbf{1}_{|x - y| < 2t} (2t - |x - y|)^2. \quad (56)$$

Proof. By the isometry of the Wiener integral and from (54), we obtain

$$\begin{aligned} \mathbf{E}[u(t_1, x)u(t_2, y)] &= \frac{1}{4} \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}} dz G_1(t_1 - u, x - z) G_1(t_2 - u, y - z) \\ &= \frac{1}{4} \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}} dz \mathbf{1}_{\{|x - z| \leq t_1 - s\}} \mathbf{1}_{\{|y - z| \leq t_2 - s\}} \\ &= \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1 + t_2 - |y - x|)} ds \int_{\mathbb{R}} dz \mathbf{1}_{\{|x - z| \leq t_1 - s\}} \mathbf{1}_{\{|y - z| \leq t_2 - s\}} \\ &\quad + \frac{1}{4} \int_0^{t_1 \wedge t_2} ds \int_{\mathbb{R}} dz \mathbf{1}_{\{2s > t_1 + t_2 - |y - x|\}} \mathbf{1}_{\{|x - z| \leq t_1 - s\}} \mathbf{1}_{\{|y - z| \leq t_2 - s\}} \\ &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1 + t_2 - |y - x|)} ds \int_{\mathbb{R}} dz \mathbf{1}_{\{|x - z| \leq t_1 - s\}} \mathbf{1}_{\{|y - z| \leq t_2 - s\}} \\ &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1 + t_2 - |y - x|)} ds \left(\int_{(x - t_1 + s) \vee (y - t_2 + s)}^{(x + t_1 - s) \vee (y + t_2 - s)} dz \right)_+. \end{aligned}$$

In order to find the integration domain for the integral dz , we will consider several situations. Assume $x \geq y$.

If $t_1 \geq t_2$ and $x - y \geq t_1 - t_2$ then

$$y + t_2 - s \leq x + t_1 - s \text{ and } x - t_1 + s \geq y - t_2 + s.$$

In this case,

$$\begin{aligned} \mathbf{E}[u(t_1, x)u(t_2, y)] &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1 + t_2 - |y - x|)} ds \left(\int_{x - t_1 + s}^{y + t_2 - s} dz \right)_+ \\ &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1 + t_2 - |y - x|)} ds (t_1 + t_2 - (x - y) - 2s) \\ &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{4} \int_0^{\frac{1}{2}(t_1 + t_2 - |y - x|)} ds (t_1 + t_2 - (x - y) - 2s) \\ &= \mathbf{1}_{\{t_1 + t_2 > |y - x|\}} \frac{1}{16} (t_1 + t_2 - (x - y))^2. \end{aligned}$$

If $t_1 \geq t_2$ and $x - y < t_1 - t_2$ then

$$y + t_2 - s \leq x + t_1 - s \text{ and } x - t_1 + s \leq y - t_2 + s.$$

Then

$$\begin{aligned} \mathbf{E}[u(t_1, x)u(t_2, y)] &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1+t_2-|y-x|)} ds \int_{y-t_2+s}^{y+t_2-s} dz \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1+t_2-|y-x|)} ds (2t_2 - 2s) \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_2} ds (2t_2 - 2s) = 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} t_2^2. \end{aligned}$$

If $t_1 \leq t_2$ and $x - y \leq t_2 - t_1$, then

$$x + t_1 - s \leq y + t_2 - s \text{ and } x - t_1 + s \geq y - t_2 + s.$$

So

$$\begin{aligned} \mathbf{E}[u(t_1, x)u(t_2, y)] &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1+t_2-|y-x|)} ds \int_{x-t_1+s}^{x+t_1-s} dz \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1+t_2-|y-x|)} ds (2t_1 - 2s) \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1} ds (2t_1 - 2s) = 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} t_1^2. \end{aligned}$$

If $t_1 \leq t_2$ and $x - y \geq t_2 - t_1$, then

$$y + t_2 - s \leq x + t_1 - s \text{ and } x - t_1 + s \geq y - t_2 + s$$

Consequently

$$\begin{aligned} \mathbf{E}[u(t_1, x)u(t_2, y)] &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{t_1 \wedge t_2 \wedge \frac{1}{2}(t_1+t_2-|y-x|)} ds \left(\int_{x-t_1+s}^{y+t_2-s} dz \right)_+ \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{4} \int_0^{\frac{1}{2}(t_1+t_2-|y-x|)} ds (t_1 + t_2 - (x - y) - 2s) \\ &= 1_{\{t_1+t_2>|y-x|\}} \frac{1}{16} (t_1 + t_2 - (x - y))^2. \end{aligned}$$

By the above four estimates, we showed (55) when $x \geq y$. By symmetry, it is also valid for $x < y$. ■

Notice that the formula (56) has been obtained in [5] in the case $t_1 = t_2$.

Now, we compute the correlation between the increments of the solution to the wave equation over small spatial intervals. Denote, for $i = 0, \dots, N - 1$ and $t \geq 0$

$$A_i = \left[\frac{i}{N}, \frac{i+1}{N} \right] \text{ and } u(t, A_i) = u\left(t, \frac{i+1}{N}\right) - u\left(t, \frac{i}{N}\right). \quad (57)$$

Lemma 8. *Let u be given by (53).*

(a) *Suppose that $t_1 \neq t_2$ and $t_1, t_2 > 0$ with $t_1 + t_2 > 1$. Then*

$$\begin{aligned} \mathbf{E}[u(t_1, A_i)u(t_2, A_j)] &= -\frac{1}{8N^2} \mathbf{1}_{|t_1-t_2| \leq \frac{|i-j|-1}{N}} + f_{t_1, t_2, N}^{(1)}(|i-j|) \mathbf{1}_{\frac{|i-j|-1}{N} < |t_1-t_2| \leq \frac{|i-j|}{N}} \\ &\quad + f_{t_1, t_2, N}^{(2)}(|i-j|) \mathbf{1}_{\frac{|i-j|}{N} < |t_1-t_2| \leq \frac{|i-j|+1}{N}} \end{aligned} \quad (58)$$

for $(i, j, N) \in \{0, \dots, N - 1\}^2 \times \mathbb{N}$ satisfying $i \neq j$, where we used the notation, for $1 \leq k \leq N$

$$f_{t_1, t_2, N}^{(1)}(k) = 2 \times \frac{1}{16} \left(t_1 + t_2 - \frac{k}{N} \right)^2 - \frac{1}{16} \left(t_1 + t_2 - \frac{k+1}{N} \right)^2 - \frac{1}{4} (t_1 \wedge t_2)^2 \quad (59)$$

and

$$f_{t_1, t_2, N}^{(2)}(k) = \frac{1}{4} (t_1 \wedge t_2)^2 - \frac{1}{16} \left(t_1 + t_2 - \frac{k+1}{N} \right)^2. \quad (60)$$

(b) *Suppose that $t_1 \neq t_2$ and $t_1, t_2 > 0$ with $t_1 + t_2 > 1$. Then*

$$\mathbf{E}[u(t_1, A_i)u(t_2, A_i)] = 0 \quad (61)$$

for any $i \in \{0, \dots, N - 1\}$ and any $N \in \mathbb{N}$ satisfying $N > |t_1 - t_2|^{-1}$.

(c) *Suppose that $t > \frac{1}{2}$. Then*

$$\mathbf{E}[u(t, A_i)^2] = \frac{1}{4N} 2t - \frac{1}{8N^2} = \frac{1}{4N} \left(2t - \frac{1}{2N} \right) \quad (62)$$

for any $N \in \mathbb{N}$ and any $i \in \{0, 1, \dots, N - 1\}$.

(d) *Suppose that $t > \frac{1}{2}$. Then*

$$\mathbf{E}[u(t, A_i)u(t, A_j)] = -\frac{1}{8N^2}. \quad (63)$$

for any $i, j \in \{0, 1, \dots, N - 1\}$ satisfying $i \neq j$.

Proof. Suppose that $t_1, t_2 > 0$ and $t_1 + t_2 > 1$. We have from (55), for every $i, j = 0, \dots, N-1$

$$\begin{aligned}
& \mathbf{E}[u(t_1, A_i)u(t_2, A_j)] \\
&= 2 \times \frac{1}{16} \left(t_1 + t_2 - \frac{|i-j|}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{|i-j|}{N}} + 2 \times \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{|i-j|}{N}} \\
&\quad - \frac{1}{16} \left(t_1 + t_2 - \frac{|i-j-1|}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{|i-j-1|}{N}} - \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{|i-j-1|}{N}} \\
&\quad - \frac{1}{16} \left(t_1 + t_2 - \frac{|i-j+1|}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{|i-j+1|}{N}} - \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{|i-j+1|}{N}}. \quad (64)
\end{aligned}$$

First we will show (61). Assume that $i = j$. Take N large enough such that $|t_1 - t_2| > \frac{1}{N}$.

$$\begin{aligned}
\mathbf{E}[u(t_1, A_i)u(t_2, A_i)] &= 2 \times \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > 0} \\
&\quad - 2 \times \frac{1}{16} \left(t_1 + t_2 - \frac{1}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{1}{N}} - 2 \times \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{1}{N}} \\
&= 0.
\end{aligned}$$

Next, we will verify (58). Let us assume $i > j$. In this case we have

$$\begin{aligned}
\mathbf{E}[u(t_1, A_i)u(t_2, A_j)] &= 2 \times \frac{1}{16} \left(t_1 + t_2 - \frac{i-j}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{i-j}{N}} + 2 \times \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{i-j}{N}} \\
&\quad - \frac{1}{16} \left(t_1 + t_2 - \frac{i-j-1}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{i-j-1}{N}} - \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{i-j-1}{N}} \\
&\quad - \frac{1}{16} \left(t_1 + t_2 - \frac{i-j+1}{N} \right)^2 \mathbf{1}_{|t_1-t_2| \leq \frac{i-j+1}{N}} - \frac{1}{4} (t_1 \wedge t_2)^2 \mathbf{1}_{|t_1-t_2| > \frac{i-j+1}{N}}. \quad (65)
\end{aligned}$$

If $|t_1 - t_2| \leq \frac{i-j-1}{N}$, the above expression gives

$$\begin{aligned}
\mathbf{E}[u(t_1, A_i)u(t_2, A_j)] &= \frac{1}{16} \left[2 \left(t_1 + t_2 - \frac{i-j}{N} \right)^2 \right. \\
&\quad \left. - \left(t_1 + t_2 - \frac{i-j-1}{N} \right)^2 - \left(t_1 + t_2 - \frac{i-j+1}{N} \right)^2 \right] \\
&= -\frac{1}{8N} (2(i-j) - (i-j-1) - (i-j+1)) \\
&\quad + \frac{1}{16N^2} (2(i-j)^2 - (i-j-1)^2 - (i-j+1)^2) \\
&= -\frac{1}{8N} (2(i-j) - (i-j-1) - (i-j+1)) - \frac{1}{8N^2} = -\frac{1}{8N^2}.
\end{aligned}$$

If $\frac{i-j-1}{N} < |t_1 - t_2| \leq \frac{i-j}{N}$ or $\frac{i-j}{N} < |t_1 - t_2| \leq \frac{i-j+1}{N}$, the conclusion is obtained directly from (65) and finally, if $|t_1 - t_2| > \frac{i-j+1}{N}$, then

$$\mathbf{E}[u(t_1, A_i)u(t_2, A_j)] = 2 \times \frac{1}{4}(t_1 \wedge t_2)^2 - \frac{1}{4}(t_1 \wedge t_2)^2 - \frac{1}{4}(t_1 \wedge t_2)^2 = 0.$$

We consider the case (62). For $t > \frac{1}{2}$, from (56),

$$\begin{aligned}
\mathbf{E}[u(t, A_i)^2] &= 2 \times \frac{1}{16}(2t)^2 - 2 \times \frac{1}{16} \left(2t - \frac{1}{N} \right)^2 \\
&= \frac{1}{4N} \left(2t - \frac{1}{2N} \right).
\end{aligned}$$

Regarding (63), for $i \neq j$, by (64), we have

$$\begin{aligned}
\mathbf{E}[u(t, A_i)u(t, A_j)] &= \frac{1}{16} \left[2 \left(2t - \frac{|i-j|}{N} \right)^2 \right. \\
&\quad \left. - \left(2t - \frac{|i-j-1|}{N} \right)^2 - \left(2t - \frac{|i-j+1|}{N} \right)^2 \right] \\
&= -\frac{t}{4N} (2|i-j| - |i-j-1| - |i-j+1|) \\
&\quad + \frac{1}{16N^2} (2|i-j|^2 - |i-j-1|^2 - |i-j+1|^2) \\
&= -\frac{t}{4N} (2|i-j| - |i-j-1| - |i-j+1|) - \frac{1}{8N^2} \\
&= -\frac{1}{8N^2}.
\end{aligned}$$

Note that

$$2|i-j| - |i-j-1| - |i-j+1| = -2 \text{ if } i = j$$

and

$$2|i-j| - |i-j-1| - |i-j+1| = 0 \text{ if } i \neq j.$$

The conclusion is obtained. ■

As an immediate consequence of Lemma 8, we have

Corollary 1. *If $t_1 \neq t_2$ are fixed in $(0, \infty)$ with $|t_1 - t_2| \geq 1$, then $u(t_1, A_i)$ and $u(t_2, A_j)$ are independent Gaussian random variables for every $i, j \in \{0, 1, \dots, N-1\}$.*

5.3 The quadratic variation

Fix the time $t > 1/2$. We denote $x_j = \frac{j}{N}$ for $j = 0, \dots, N$ for every $N \in \mathbb{N}$. x_j depends on n . The centered renormalized quadratic variation statistic over the unit interval $[0, 1]$ can be defined in the following way:

$$V_{N,t} = \sum_{j=0}^{N-1} \left[\frac{(u(t, x_{j+1}) - u(t, x_j))^2}{\mathbf{E}[(u(t, x_{j+1}) - u(t, x_j))^2]} - 1 \right] = \sum_{j=0}^{N-1} \left[\frac{u(t, A_j)^2}{\mathbf{E}[|u(t, A_j)|^2]} - 1 \right]. \quad (66)$$

Starting with this paragraph, we will denote by $I_q = I_q^W$ the multiple integral of order $q \geq 1$ with respect to the Gaussian noise W (given by (52)) and by $D = D^W$ the Malliavin derivative with respect to W . In this case, we will write

$$u(t, x_{i+1}) - u(t, x_i) = I_1(g_{t,i}) \text{ with } g_{t,i}(u, y) = G_1(t-u, x_{i+1}-y) - G_1(t-u, x_i-y) \quad (67)$$

where G_1 from (54).

Using the product formula for multiple Wiener-integrals (100), we can rewrite:

$$\begin{aligned} V_{N,t} &= \sum_{j=0}^{N-1} \left[\frac{I_1^2(g_{t,i})}{\mathbf{E} \left[\left(u(t, x_{j+1}) - u(t, x_j) \right)^2 \right]} - 1 \right] \\ &= \sum_{j=0}^{N-1} \left[\frac{I_2(g_{t,i}^{\otimes 2}) + \mathbf{E} \left[\left(u(t, x_{j+1}) - u(t, x_j) \right)^2 \right]}{\mathbf{E} \left[\left(u(t, x_{j+1}) - u(t, x_j) \right)^2 \right]} - 1 \right] \\ &= \sum_{j=0}^{N-1} \frac{I_2(g_{t,i}^{\otimes 2})}{\mathbf{E} \left[\left(u(t, x_{j+1}) - u(t, x_j) \right)^2 \right]} = I_2(f_N) \end{aligned} \quad (68)$$

with

$$f_N = \sum_{j=0}^{N-1} \frac{g_{t,i}^{\otimes 2}}{\mathbf{E}[|u(t, A_j)|^2]}.$$

We will use the symbolic notation, for $i, j = 0, \dots, N$,

$$\langle A_i, A_j \rangle := \langle g_{t,i}, g_{t,j} \rangle_{L^2([0, \infty) \times \mathbb{R}, dt dx)} = \mathbf{E}[u(t, A_i)u(t, A_j)].$$

Let us first estimate the L^2 -mean of the random variable $V_{N,t}$ as $N \rightarrow \infty$.

Lemma 9. *Suppose that $t > 1/2$. Let $v_{N,t}^2 = \mathbf{E}[V_{N,t}^2]$. Then*

$$\frac{1}{2N} v_{N,t}^2 = 1 + \frac{1}{2}(N-1) \frac{\left(-\frac{1}{8N^2}\right)^2}{\left(\frac{1}{4N}\left(2t - \frac{1}{2N}\right)\right)^2} = 1 + O\left(\frac{1}{N}\right). \quad (69)$$

Proof. Notice that

$$\begin{aligned} v_{N,t}^2 &= 2 \sum_{j,k=0}^{N-1} \frac{\langle A_j, A_k \rangle^2}{\mathbf{E}[|u(t, A_j)|^2] \mathbf{E}[|u(t, A_k)|^2]} \\ &= 2N + 2 \sum_{j,k=0; j \neq k}^{N-1} \frac{\langle A_j, A_k \rangle^2}{\mathbf{E}[|u(t, A_j)|^2] \mathbf{E}[|u(t, A_k)|^2]} = 2N + 2 \sum_{j,k=0; j \neq k}^{N-1} \frac{(-8N^2)^{-2}}{\left((4N)^{-1}\left(2t - \frac{1}{2N}\right)\right)^2} \\ &= 2N + N(N-1) \frac{(-8N^2)^{-2}}{\left((4N)^{-1}\left(2t - \frac{1}{2N}\right)\right)^2} \end{aligned}$$

where we used (62), (63). ■

We define

$$F_{N,t} = \frac{V_{N,t}}{\sqrt{2N}}. \quad (70)$$

We will show below that the sequence $(F_N)_{N \geq 1}$ satisfies the conditions [A1], [A2] and [C].

5.4 The Gamma-factors

We need to compute the Gamma-factors $\Gamma^{(p)}(F_{N,t})$ for any $p \geq 1$ at fixed time $t > 1/2$ with $\Gamma^{(p)}$ defined in (2). We have

$$\Gamma^{(1)}(F_{N,t}) = F_{N,t}$$

and

$$\begin{aligned}
\Gamma^{(2)}(F_{N,t}) &= \langle DF_{N,t}, D(-L)^{-1}F_{N,t} \rangle \\
&= \left\langle \frac{2}{(2N)^{\frac{1}{2}}} \sum_{j_1=0}^{N-1} \frac{I_1(A_{j_1})I_{A_1}}{\mathbf{E}[|u(t, A_{j_1})|^2]}, \frac{1}{(2N)^{\frac{1}{2}}} \sum_{j_2=0}^{N-1} \frac{I_1(A_{j_2})A_{j_2}}{\mathbf{E}[|u(t, A_{j_2})|^2]} \right\rangle \\
&= \frac{2}{2N} \sum_{j_1, j_2=0}^{N-1} \frac{I_1(A_{j_1})I_1(A_{j_2})\langle A_{j_1}, A_{j_2} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]} \\
&= \frac{2}{2N} \sum_{j_1, j_2=0}^{N-1} \frac{I_2(A_{j_1} \otimes A_{j_2})\langle A_{j_1}, A_{j_2} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]} + \frac{2}{2N} \sum_{j_1, j_2=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle^2}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]} \\
&= \frac{2}{2N} \sum_{j_1, j_2=0}^{N-1} \frac{I_2(A_{j_1} \otimes A_{j_2})\langle A_{j_1}, A_{j_2} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]} + \mathbf{E}[\Gamma^{(2)}(F_{N,t})]. \tag{71}
\end{aligned}$$

Next,

$$\begin{aligned}
\Gamma^{(3)}(F_{N,t}) &= \langle DF_{N,t}, D(-L)^{-1}\Gamma^{(2)}(F_{N,t}) \rangle \\
&= \frac{4}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{I_1(A_{j_1})I_1(A_{j_2})\langle A_{j_1}, A_{j_3} \rangle \langle A_{j_2}, A_{j_3} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]\mathbf{E}[|u(t, A_{j_3})|^2]} \\
&= \frac{4}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{I_2(A_{j_1} \otimes A_{j_2})\langle A_{j_1}, A_{j_3} \rangle \langle A_{j_2}, A_{j_3} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]\mathbf{E}[|u(t, A_{j_3})|^2]} \\
&\quad + \frac{4}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle \langle A_{j_1}, A_{j_3} \rangle \langle A_{j_2}, A_{j_3} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]\mathbf{E}[|u(t, A_{j_3})|^2]} \\
&= \frac{4}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{I_2(A_{j_1} \otimes A_{j_2})\langle A_{j_1}, A_{j_3} \rangle \langle A_{j_2}, A_{j_3} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2]\mathbf{E}[|u(t, A_{j_2})|^2]\mathbf{E}[|u(t, A_{j_3})|^2]} + \mathbf{E}\Gamma^{(3)}(F_{N,t}).
\end{aligned}$$

In the same way, for any $p \geq 2$,

$$\begin{aligned}
\Gamma^{(p)}(F_{N,t}) &= \langle DF_{N,t}, D(-L)^{-1}\Gamma_{p-1}(F_{N,t}) \rangle \\
&= \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{I_2(A_{j_1} \tilde{\otimes} A_{j_2}) \langle A_{j_2}, A_{j_3} \rangle \langle A_{j_3}, A_{j_4} \rangle \dots \langle A_{j_{p-1}}, A_{j_p} \rangle \langle A_{j_p}, A_{j_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_p})|^2]} \\
&\quad + \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle \langle A_{j_2}, A_{j_3} \rangle \langle A_{j_3}, A_{j_4} \rangle \dots \langle A_{j_{p-1}}, A_{j_p} \rangle \langle A_{j_p}, A_{j_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_p})|^2]} \\
&= \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{I_2(A_{j_1} \tilde{\otimes} A_{j_2}) \langle A_{j_2}, A_{j_3} \rangle \langle A_{j_3}, A_{j_4} \rangle \dots \langle A_{j_{p-1}}, A_{j_p} \rangle \langle A_{j_p}, A_{j_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_p})|^2]} \\
&\quad + \mathbf{E}\Gamma^{(p)}(F_{N,t}). \tag{72}
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{E}[\Gamma^{(p)}(F_{N,t})] \\
&= \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle \langle A_{j_2}, A_{j_3} \rangle \langle A_{j_3}, A_{j_4} \rangle \dots \langle A_{j_{p-1}}, A_{j_p} \rangle \langle A_{j_p}, A_{j_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_p})|^2]}. \tag{73}
\end{aligned}$$

Then, using (62) and (63) in Lemma 8

$$\begin{aligned}
&\mathbf{E}[\Gamma^{(p)}(F_{N,t})] \\
&= \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \left[N + a_{p,2}N(N-1) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^2 + a_{p,3}N(N-1)(N-2) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^3 \right. \\
&\quad \left. + \dots + a_{p,p}N(N-1) \dots (N-p+1) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^p \right] \tag{74}
\end{aligned}$$

where $a_{p,2}, \dots, a_{p,p}$ are combinatorial constants. In order to get the explicit asymptotic expansion of the $\mathbf{E}[\Gamma^{(p)}(F_{N,t})]$ (which is necessary in order to check [C]), we will need to evaluate the Taylor expansion of $\left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^k$ for every $k \geq 1$ integer. We can write

$$\left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^k = (-1)^k \left(\frac{1}{4Nt} \right)^k \left(\frac{1}{1 - \frac{1}{4Nt}} \right)^k.$$

Using

$$\left(\frac{1}{1-x} \right)^k = \frac{1}{(k-1)!} \left(\frac{1}{1-x} \right)^{(k-1)} = \frac{1}{(k-1)!} \sum_{n=k-1}^{\infty} \frac{n!}{(n-k+1)!} x^{n-k+1} = \sum_{n=0}^{\infty} C_{n+k-1}^{k-1} x^n,$$

where $C_b^a = \binom{b}{a}$, we get

$$\left(\frac{\binom{-\frac{1}{8N^2}}{\frac{1}{4N}(2t - \frac{1}{2N})}}{\binom{1}{4Nt}} \right)^k = (-1)^k \binom{1}{4Nt}^k \sum_{n=0}^{\infty} C_{n+k-1}^{k-1} \left(\frac{1}{4Nt} \right)^n. \quad (75)$$

Let us now check assumptions [A1], [A2] and [C] in order to apply Theorem 3.

5.5 Checking condition [C]

We will show that condition [C] is satisfied for $C = 1$ and $\gamma = \frac{1}{2}$. Let us first look to assumption [C](i). Since $d = 1$, we have $\mathbb{1} = \{1\}$ and the only non-empty subset of $\mathbb{1}^2$ is $\{(1, 1)\}$. Let $\Gamma^{(2)} = \Gamma_{I_2}^{(2)}$. In order to check [C](i), we need to find the asymptotic behavior of $\mathbf{E}[\Gamma^{(2)}(F_{N,t})] - 1$. From (73)

$$\begin{aligned} \mathbf{E}[\Gamma^{(2)}(F_{N,t})] - 1 &= \frac{2}{2N} \sum_{j_1, j_2=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle^2}{\mathbf{E}[|u(t, A_{j_1})|^2] \mathbf{E}[|u(t, A_{j_2})|^2]} - 1 \\ &= \frac{1}{N} N(N-1) \left(\frac{\binom{-\frac{1}{8N^2}}{\frac{1}{4N}(2t - \frac{1}{2N})}}{\binom{1}{4Nt}} \right)^2 \end{aligned}$$

and from (75), for every $q \geq 3$

$$\begin{aligned} &\mathbf{E}[\Gamma^{(2)}(F_{N,t})] - 1 \\ &= \frac{1}{N} N(N-1) (-1)^2 \frac{1}{(4Nt)^2} \sum_{n=0}^{\infty} C_{n+1}^1 \left(\frac{1}{4Nt} \right)^n \\ &= (N-1) \frac{1}{(4Nt)^2} \left(1 + 2 \frac{1}{4Nt} + 3 \frac{1}{(4Nt)^2} + \dots + (q+1) \frac{1}{(4Nt)^q} + o(N^{-q}) \right) \\ &= \frac{1}{(4t)^2} \frac{1}{N} + \left(2 \frac{1}{(4t)^3} - \frac{1}{(4t)^2} \right) \frac{1}{N^2} + \dots + \left(q \frac{1}{(4t)^{q+1}} - (q-1) \frac{1}{(4t)^q} \right) \frac{1}{N^q} + o(N^{-(q+1)}). \end{aligned} \quad (76)$$

Therefore condition [C](i) holds by taking $q = \frac{p-1}{2}$ (so $q+1 = \frac{p+1}{2}$) with

$$c(I_2, k) = 0 \text{ if } k \text{ is odd and } c(I_2, 2k) = k \frac{1}{(4t)^{k+1}} - (k-1) \frac{1}{(4t)^k}, \quad k \geq 1.$$

Next, we show [C](ii). We will obtain the asymptotic expansion of $\mathbf{E}[\Gamma^{(j)}(F_{N,t})]$ for every $j \geq 3$ integer. By using (74),

$$\begin{aligned} & \mathbf{E}[\Gamma^{(j)}(F_{N,t})] \\ &= \frac{2^{j-1}}{(2N)^{\frac{j}{2}}} \left[N + a_{j,2}N(N-1) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^2 + a_{j,3}N(N-1)(N-2) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^3 \right. \\ & \quad \left. + \dots + a_{j,j}N(N-1)\dots(N-j+1) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t - \frac{1}{2N}))} \right)^j \right] \end{aligned} \quad (77)$$

where $a_{j,2}, \dots, a_{j,j}$ are combinatorial constants. Therefore, via (75)

$$\begin{aligned} \mathbf{E}[\Gamma^{(j)}(F_{N,t})] &= \frac{2^{j-1}}{(2N)^{\frac{j}{2}}} \left[N + a_{j,2}(-1)^2 \frac{1}{(4t)^2} \frac{N!}{(N-2)!N^2} \sum_{n=0}^{\infty} C_{n+1}^1 \left(\frac{1}{4Nt} \right)^n \right. \\ & \quad + a_{j,3}(-1)^3 \frac{1}{(4t)^3} \frac{N!}{(N-3)!N^3} \sum_{n=0}^{\infty} C_{n+2}^2 \left(\frac{1}{4Nt} \right)^n + \dots + \\ & \quad \left. + a_{j,j}(-1)^j \frac{1}{(4t)^j} \frac{N!}{(N-j)!N^j} \sum_{n=0}^{\infty} C_{n+j}^j \left(\frac{1}{4Nt} \right)^n \right]. \end{aligned}$$

Since

$$\frac{N!}{(N-k)!N^k} = \left(1 - \frac{1}{N}\right) \dots \left(1 - \frac{k-1}{N}\right)$$

for $k \geq 2$, we obtain

$$\begin{aligned} \mathbf{E}[\Gamma^{(j)}(F_{N,t})] &= c(I_j, 1) \frac{1}{N^{\frac{j}{2}-1}} + c(I_j, 3) \frac{1}{N^{\frac{j}{2}}} + c(I_j, 5) \frac{1}{N^{\frac{j}{2}+1}} \\ & \quad + \dots + c(I_j, p-j+2) \frac{1}{N^{\frac{p-1}{2}}} + o\left(\frac{1}{N^{\frac{p-1}{2}}}\right) \end{aligned} \quad (78)$$

for $j \in \{3, \dots, p+1\}$. All the coefficients above can be written explicitly. In particular $c(I_j, 2k) = 0$ for every $k \geq 1$ integer while

$$c(I_j, 1) = 2^{\frac{j}{2}-1}, c(I_j, 3) = 2^{\frac{j}{2}-1} \left(a_{j,2}(-1)^2 \frac{1}{(4t)^2} + a_{j,3}(-1)^3 \frac{1}{(4t)^3} + \dots + a_{j,j}(-1)^j \frac{1}{(4t)^j} \right)$$

and

$$c(I_j, 5) = 2^{\frac{j}{2}-1} \sum_{k=2}^j a_{j,k}(-1)^k \frac{1}{(4t)^k} \left(\frac{k+1}{4t} - \frac{(k-1)k}{2} \right).$$

5.6 Checking conditions [A1] and [A2]

Condition [A1](i) is clearly verified for every l, r because $F_{N,t}$ is an element of the second Wiener chaos with $\mathbf{E}[F_N^2] \rightarrow 1$ as $N \rightarrow \infty$. Condition [A1](ii) can be checked similarly to [A2](i).

Let $\mathfrak{q} = (p-1)/2$. Due to Lemma 1 (b) and by the hypercontractivity property of multiple stochastic integrals (101), it suffices to check that for every $p \geq 3$

$$\|\Gamma^{(p+1)}(F_{N,t}) - \mathbf{E}[\Gamma^{(p+1)}(F_{N,t})]\|_2 = o(N^{-\frac{p-1}{2}}).$$

Let $\tilde{\Gamma}^{(p+1)}(F_{N,t}) = \Gamma^{(p+1)}(F_{N,t}) - \mathbf{E}[\Gamma^{(p+1)}(F_{N,t})]$. Then by (72),

$$\begin{aligned} & \mathbf{E} \left[\left(\tilde{\Gamma}^{(p+1)}(F_{N,t}) \right)^2 \right] \left(\frac{2^p}{(2N)^{\frac{p+1}{2}}} \right)^2 \\ & \times 2 \sum_{\substack{j_1, \dots, j_{p+1}, \\ k_1, \dots, k_{p+1} = 0}}^{N-1} \frac{\langle A_{j_1} \tilde{\otimes} A_{j_2}, A_{k_1} \tilde{\otimes} A_{k_2} \rangle \langle A_{j_2}, A_{j_3} \rangle \dots \langle A_{j_{p+1}}, A_{j_1} \rangle \langle A_{k_2}, A_{k_3} \rangle \dots \langle A_{k_{p+1}}, A_{k_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_{p+1}})|^2] \mathbf{E}[|u(t, A_{k_1})|^2] \dots \mathbf{E}[|u(t, A_{k_{p+1}})|^2]} \\ & = \left(\frac{2^{p+1}}{(2N)^{\frac{p+1}{2}}} \right)^2 2 \sum_{\substack{j_1, \dots, j_{p+1}, \\ k_1, \dots, k_{p+1} = 0}}^{N-1} \frac{\langle A_{j_1}, A_{k_1} \rangle \langle A_{j_2}, A_{k_2} \rangle \langle A_{j_2}, A_{j_3} \rangle \dots \langle A_{j_{p+1}}, A_{j_1} \rangle \langle A_{k_2}, A_{k_3} \rangle \dots \langle A_{k_{p+1}}, A_{k_1} \rangle}{\mathbf{E}[|u(t, A_{j_1})|^2] \dots \mathbf{E}[|u(t, A_{j_{p+1}})|^2] \mathbf{E}[|u(t, A_{k_1})|^2] \dots \mathbf{E}[|u(t, A_{k_{p+1}})|^2]} \end{aligned} \quad (79)$$

where we used the symmetry of the sums above and the formula

$$\langle f \tilde{\otimes} g, f_1, \tilde{\otimes} g_1 \rangle = \frac{1}{2} (\langle f, f_1 \rangle \langle g, g_1 \rangle + \langle f, g_1 \rangle \langle g, f_1 \rangle).$$

Consequently, by (62) and (63), we get

$$\begin{aligned} & \mathbf{E} \left[\left(\tilde{\Gamma}^{(p)}(F_{N,t}) \right)^2 \right] \\ & = 2 \left(\frac{2^p}{(2N)^{\frac{p+1}{2}}} \right)^2 (N + O(1)) = 2^p N^{-(p+1)} (N + O(1)) = 2^p N^{-p} + o(N^{-p}). \end{aligned} \quad (80)$$

The above estimate is true for every $p \geq 3$ and it clearly implies [A2].

To summarize the behavior of the cumulants, we have the situation described in Table 1 with $\gamma = \frac{1}{2}$.

Table 1: The asymptotic behavior of $\mathbf{E} [\Gamma^{(j)}]$

Summand	First order term	Second term	Third term	Fourth term
$\mathbf{E} [\Gamma^{(3)}(F_N)]$	$\frac{1}{N^\gamma}$	$\frac{1}{N^{3\gamma}}$	$\frac{1}{N^{5\gamma}}$	$\frac{1}{N^{7\gamma}}$
$\mathbf{E} [\Gamma^{(4)}(F_N)] + (\mathbf{E} [\Gamma^{(2)}(F_N)] - 1)$	$\frac{1}{N^{2\gamma}}$	$\frac{1}{N^{4\gamma}}$	$\frac{1}{N^{6\gamma}}$	$\frac{1}{N^{8\gamma}}$
$\mathbf{E} [\Gamma^{(5)}(F_N)]$	$\frac{1}{N^{3\gamma}}$	$\frac{1}{N^{5\gamma}}$	$\frac{1}{N^{7\gamma}}$	$\frac{1}{N^{9\gamma}}$
$\mathbf{E} [\Gamma^{(6)}(F_N)]$	$\frac{1}{N^{4\gamma}}$	$\frac{1}{N^{6\gamma}}$	$\frac{1}{N^{8\gamma}}$	$\frac{1}{N^{10\gamma}}$
$\mathbf{E} [\Gamma^{(7)}(F_N)]$	$\frac{1}{N^{5\gamma}}$	$\frac{1}{N^{7\gamma}}$	$\frac{1}{N^{9\gamma}}$	$\frac{1}{N^{11\gamma}}$
\dots				
$\mathbf{E} [\Gamma^{(p)}(F_N)]$	$\frac{1}{N^{(p-2)\gamma}}$	$\frac{1}{N^{p\gamma}}$	$\frac{1}{N^{(p+2)\gamma}}$	$\frac{1}{N^{(p+4)\gamma}}$
$\mathbf{E} [\Gamma^{(p+1)}(F_N)]$	$\frac{1}{N^{(p-1)\gamma}}$	$\frac{1}{N^{(p+1)\gamma}}$	$\frac{1}{N^{(p+3)\gamma}}$	$\frac{1}{N^{(p+5)\gamma}}$

5.7 The multidimensional case

Consider the sequence

$$\mathbf{F}_{N,t} = (F_{N,t_1}, F_{N,t_2}) \quad (81)$$

with $F_{N,t}$ given by (70) and $t_1 \neq t_2$ and $t_1, t_2 > 1/2$. Using relations (58) and (61) we can prove that this two-dimensional sequence satisfies [A1], [A2] and [C].

Recall that we denoted by $I_q = I_q^W$ the multiple integral of order $q \geq 1$ with respect to the Gaussian field W (whose covariance is given by (52)) and by $D = D^W$ the Malliavin derivative with respect to W . In this case, the sequence (70) can be written

$$F_{N,t_i} = \frac{1}{\sqrt{2N}} \sum_{j=0}^{N-1} \frac{I_2(g_{t_i,j}^{\otimes 2})}{\mathbf{E}[|u(t, A_j)|^2]}$$

for $i = 1, 2$, where we used the notation $u(t, A_j)$ from (57) and $g_{t,i}$ from (67). Let us first compute the Gamma factors (2) of the vector (81). Since, for $a = 1, 2$

$$DF_{N,t_a} = \frac{2}{\sqrt{2N}} \sum_{j=0}^{N-1} \frac{I_1(g_{t_a,j})g_{t_a,j}}{\mathbf{E}[|u(t_a, A_j)|^2]} \text{ and } D(-L)^{-1}F_{N,t_a} = \frac{1}{\sqrt{2N}} \sum_{j=0}^{N-1} \frac{I_1(g_{t_a,j})g_{t_a,j}}{\mathbf{E}[|u(t_a, A_j)|^2]}$$

and by using the symbolic notation, for $a, b = 1, 2$ and $j_1, j_2 = 0, 1, \dots, N-1$

$$\langle A_{j_1}, A_{j_2} \rangle_{\mathcal{H}_{t_a, t_b}} = \mathbf{E}[u(t_a, A_{j_1})u(t_b, A_{j_2})] = \langle g_{t_a, j_1}, g_{t_b, j_2} \rangle.$$

we get, for every $i_1, i_2 = 1, 2$

$$\begin{aligned}
\Gamma_{i_1, i_2}^{(2)}(\mathbf{F}_{N,t}) &= \langle DF_{N,t_{i_1}}, D(-L)^{-1}F_{N,t_{i_2}} \rangle \\
&= \frac{1}{N} \sum_{j_1, j_2=0}^{N-1} \frac{I_1(g_{t_{i_1}, j_1}) I_1(g_{t_{i_2}, j_2}) \langle A_{j_1}, A_{j_2} \rangle \mathcal{H}_{t_{i_1}, t_{i_2}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \mathbf{E}[|u(t_{i_2}, A_{j_2})|^2]} \\
&= \frac{1}{N} \sum_{j_1, j_2=0}^{N-1} \frac{I_2((g_{t_{i_1}, j_1}) \tilde{\otimes} (g_{t_{i_2}, j_2})) \langle A_{j_1}, A_{j_2} \rangle \mathcal{H}_{t_{i_1}, t_{i_2}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \mathbf{E}[|u(t_{i_2}, A_{j_2})|^2]} \\
&\quad + \frac{1}{N} \sum_{j_1, j_2=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle^2 \mathcal{H}_{t_{i_1}, t_{i_2}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \mathbf{E}[|u(t_{i_2}, A_{j_2})|^2]} \\
&= \frac{1}{N} \sum_{j_1, j_2=0}^{N-1} \frac{I_2((g_{t_{i_1}, j_1}) \tilde{\otimes} (g_{t_{i_2}, j_2})) \langle A_{j_1}, A_{j_2} \rangle \mathcal{H}_{t_{i_1}, t_{i_2}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \mathbf{E}[|u(t_{i_2}, A_{j_2})|^2]} \\
&\quad + \mathbf{E}[\Gamma_{(i_1, i_2)}^{(2)}(\mathbf{F}_{N,t})].
\end{aligned}$$

In the same way, we get for $p \geq 2$

$$\begin{aligned}
\Gamma_{i_1, \dots, i_p}^{(p)}(\mathbf{F}_{N,t}) &= \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{I_2((g_{t_{i_1}, x, j_1}) \tilde{\otimes} (g_{t_{i_2}, x, j_2})) \langle A_{j_2}, A_{j_3} \rangle \mathcal{H}_{t_{i_2}, t_{i_3}} \dots \langle A_{j_p}, A_{j_1} \rangle \mathcal{H}_{t_{i_p}, t_{i_1}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \dots \mathbf{E}[|u(t_{i_p}, A_{j_p})|^2]} \\
&\quad + \mathbf{E}[\Gamma_{i_1, \dots, i_p}^{(p)}(\mathbf{F}_{N,t})]
\end{aligned}$$

where

$$\mathbf{E}[\Gamma_{i_1, \dots, i_p}^{(p)}(\mathbf{F}_{N,t})] = \frac{2^{p-1}}{(2N)^{\frac{p}{2}}} \sum_{j_1, \dots, j_p=0}^{N-1} \frac{\langle A_{j_1}, A_{j_2} \rangle \mathcal{H}_{t_{i_1}, t_{i_2}} \langle A_{j_2}, A_{j_3} \rangle \mathcal{H}_{t_{i_2}, t_{i_3}} \dots \langle A_{j_p}, A_{j_1} \rangle \mathcal{H}_{t_{i_p}, t_{i_1}}}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \dots \mathbf{E}[|u(t_{i_p}, A_{j_p})|^2]}. \tag{82}$$

5.8 Checking conditions [A1], [A2] and [C]

We will check our main assumptions in the particular case $p = 2$ with $\gamma = \frac{1}{2}$ and C a symmetric matrix in dimension two. Let us first determine the limit covariance matrix C .

Denote by $\lceil x \rceil$ the minimum integer that is not less than x . Let $\eta_N = \lceil \tau N \rceil - N\tau$ for $\tau = |t_1 - t_2|$.

Lemma 10. *Let $t_1 \neq t_2$ and $t_1, t_2 > 1/2$.*

1. Assume $|t_1 - t_2| \geq 1$. Then

$$\mathbf{E}[\Gamma_{i,j}^{(2)}(\mathbf{F}_{N,t})] = 1_{\{i=j\}}(1 + O(N^{-1}))$$

as $N \rightarrow \infty$ for $i, j = 1, 2$.

2. Assume $|t_1 - t_2| < 1$ (and hence $t_1 + t_2 > 1$). Then, as $N \rightarrow \infty$,

$$\mathbf{E}[\Gamma_{i,i}^{(2)}(\mathbf{F}_{N,t})] = 1 + O(N^{-1})$$

for $i = 1, 2$, and

$$\mathbf{E}[\Gamma_{i,j}^{(2)}(\mathbf{F}_{N,t})] = \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{2t_1 t_2} (2\eta_N^2 - 2\eta_N + 1) + O(N^{-1}) \quad (83)$$

for $(i, j) = (1, 2), (2, 1)$.

Proof. In both cases, we have

$$\mathbf{E}[\Gamma_{i,i}^{(2)}(\mathbf{F}_{N,t})] = 1 + O(N^{-1})$$

for $i = 1, 2$ by Lemma 9.

We will investigate the asymptotic behavior of $\mathbf{E}[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t})]$. In Case 1, by Corollary 1, we obtain

$$\mathbf{E}[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t})] = \mathbf{E}[F_{N,t_1} F_{N,t_2}] = 0.$$

In Case 2, using (58) and (61), we can write

$$\begin{aligned} \mathbf{E}[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t})] &= \frac{1}{N} \sum_{j_1, j_2=0}^{N-1} \frac{(\mathbf{E}[u(t_1, A_{j_1})u(t_2, A_{j_2})])^2}{\mathbf{E}[|u(t_1, A_{j_1})|^2] \mathbf{E}[|u(t_2, A_{j_2})|^2]} \\ &= \frac{1}{N} \left(\frac{1}{4N} (2t_1 - \frac{1}{2N}) \right)^{-1} \left(\frac{1}{4N} (2t_2 - \frac{1}{2N}) \right)^{-1} \\ &\quad \times \sum_{j_1, j_2=0, j_1 \neq j_2}^{N-1} \left[\frac{-1}{8N^2} 1_{|t_1 - t_2| \leq \frac{|j_1 - j_2| - 1}{N}} + f_{t_1, t_2, N}^{(1)}(|j_1 - j_2|) 1_{\frac{|j_1 - j_2| - 1}{N} < |t_1 - t_2| \leq \frac{|j_1 - j_2|}{N}} \right. \\ &\quad \left. + f_{t_1, t_2, N}^{(2)}(|j_1 - j_2|) 1_{\frac{|j_1 - j_2|}{N} < |t_1 - t_2| \leq \frac{|j_1 - j_2| + 1}{N}} \right]^2 \end{aligned} \quad (84)$$

with $f^{(1)}, f^{(2)}$ from (59) and (60) respectively.

To get the limit of the sequence $u_N := \mathbf{E}[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t})]$, it suffices to get the limit of $u_N |t_1 - t_2|^{-1}$. We can write, by (84), with $N_{1,2} = N |t_1 - t_2|^{-1}$

$$\begin{aligned}
u_{N^{|t_1-t_2|^{-1}}}} &= \frac{1}{N_{1,2}} \left(\frac{1}{4N_{1,2}} (2t_1 - \frac{1}{2N_{1,2}}) \right)^{-1} \left(\frac{1}{4N_{1,2}} (2t_2 - \frac{1}{2N_{1,2}}) \right)^{-1} \\
&\quad \times (u_N^{(1)} + u_N^{(2)} + u_N^{(3)})
\end{aligned} \tag{85}$$

with

$$u_N^{(1)} = \left(-\frac{1}{8N^2} \right)^2 \sum_{j_1, j_2=0, j_1 \neq j_2}^{[N_{1,2}]-1} 1_{|j_1-j_2| \geq N+1},$$

$$u_N^{(2)} = \sum_{j_1, j_2=0, j_1 \neq j_2}^{[N_{1,2}]-1} \left(f_{t_1, t_2, N_{1,2}}^{(1)}(|j_1 - j_2|) \right)^2 1_{|j_1-j_2|=N}, \quad u_N^{(3)} = \sum_{j_1, j_2=0, j_1 \neq j_2}^{[N_{1,2}]-1} \left(f_{t_1, t_2, N_{1,2}}^{(2)}(|j_1 - j_2|) \right)^2 1_{|j_1-j_2|=N-1}.$$

Notice that

$$\begin{aligned}
f_{t_1, t_2, N_{1,2}}^{(1)}(N) &= f_{t_1, t_2, N_{1,2}}^{(2)}(N-1) \\
&= \frac{1}{16} (t_1 + t_2 - |t_1 - t_2|)^2 - \frac{1}{16} \left(t_1 + t_2 - |t_1 - t_2| - |t_1 - t_2| \frac{1}{N} \right)^2 \\
&= \frac{1}{8} |t_1 - t_2| (t_1 \wedge t_2) \frac{1}{N} + c_2 \frac{1}{N^2} + c_3 \frac{1}{N^3} + \dots
\end{aligned} \tag{86}$$

Coming back to (85), we deduce that

$$\begin{aligned}
\mathbf{E} \left[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t}) \right] &\sim \frac{1}{N_{1,2}} \frac{4N_{1,2}^2}{t_1 t_2} \left(\frac{1}{8} (t_1 \wedge t_2) \right)^2 \frac{1}{N_{1,2}^2} \sum_{j_1, j_2=0, j_1 \neq j_2}^{[N_{1,2}]-1} [1_{|j_1-j_2|=N+1} + 1_{|j_1-j_2|=N-1}] \\
&\sim \frac{4}{64} \frac{(t_1 \wedge t_2)^2}{t_1 t_2} 4 (|t_1 - t_2|^{-1} - 1) := C_{1,2} = C_{2,1}.
\end{aligned} \tag{87}$$

Denote $u_N = \mathbf{E}[\Gamma_{1,2}^{(2)}(\mathbf{F}_{N,t})]$. Let $\tau = |t_1 - t_2|$. Then

$$u_N = \frac{1}{N} \left(\frac{1}{4N} (2t_1 - \frac{1}{2N}) \right)^{-1} \left(\frac{1}{4N} (2t_2 - \frac{1}{2N}) \right)^{-1} (u_N^{(1)} + u_N^{(2)} + u_N^{(3)}) \tag{88}$$

with

$$\begin{aligned}
u_N^{(1)} &= \left(-\frac{1}{8N^2} \right)^2 \sum_{j_1, j_2=0, j_1 \neq j_2}^{N-1} 1_{\{|j_1-j_2| \geq \tau N+1\}}, \\
u_N^{(2)} &= \sum_{j_1, j_2=0, j_1 \neq j_2}^{N-1} \left(f_{t_1, t_2, N}^{(1)}(|j_1 - j_2|) \right)^2 1_{\{\tau N \leq |j_1-j_2| < \tau N+1\}}
\end{aligned}$$

and

$$u_N^{(3)} = \sum_{j_1, j_2=0, j_1 \neq j_2}^{N-1} \left(f_{t_1, t_2, N}^{(2)}(|j_1 - j_2|) \right)^2 \mathbf{1}_{\{\tau N - 1 \leq |j_1 - j_2| < \tau N\}}.$$

The contribution of $u_N^{(1)}$ to u_N is asymptotically negligible since

$$u_N^{(1)} = O(N^{-2}).$$

Moreover,

$$u_N^{(2)} = 2(N - \lceil \tau N \rceil) \times \left\{ \frac{2}{16} \left(t_1 + t_2 - \frac{\lceil \tau N \rceil}{N} \right)^2 - \frac{1}{16} \left(t_1 + t_2 - \frac{\lceil \tau N \rceil + 1}{N} \right)^2 - \frac{1}{4} (t_1 \wedge t_2)^2 \right\}^2$$

and

$$u_N^{(3)} = 2(N - \lceil \tau N \rceil + 1) \times \left\{ \frac{1}{4} (t_1 \wedge t_2)^2 - \frac{1}{16} \left(t_1 + t_2 - \frac{\lceil \tau N \rceil}{N} \right)^2 \right\}^2.$$

Since

$$t_1 + t_2 - \frac{\lceil \tau N \rceil}{N} = t_1 + t_2 - \tau - \frac{\eta_N}{N} = 2(t_1 \wedge t_2) - \frac{\eta_N}{N}$$

we can express $u_N^{(2)}, u_N^{(3)}$ as

$$u_N^{(2)} = 2(1 - \tau)N^{-1} \times \left\{ \frac{1}{4} (t_1 \wedge t_2)(1 - \eta_N) + O(N^{-1}) \right\}^2 + O(n^{-2})$$

and

$$u_N^{(3)} = 2(1 - \tau)N^{-1} \times \left\{ \frac{1}{4} (t_1 \wedge t_2)\eta_N + O(N^{-1}) \right\}^2 + O(n^{-2}).$$

Consequently,

$$u_N = \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{2t_1 t_2} \left\{ (1 - \eta_N)^2 + \eta_N^2 \right\} + O(N^{-1}).$$

■

When $|t_1 - t_2| < 1$, the sequence u_N does not converge. We need to consider the limit of u_N along a subsequence of N such that $\eta_N \rightarrow a \in [0, 1]$. More precisely, u_N does not converge, however,

$$u_N = \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{2t_1 t_2} (2a^2 - 2a + 1) + o(N^{-1/2}) \quad (89)$$

along any subsequence such that

$$\eta_N = a + o(N^{-1/2}) \quad (90)$$

as $N \rightarrow \infty$ for some $a \in [0, 1]$.

Thus, in Case 2, we can consider approximation to the distribution of $\mathbf{F}_{N,t}$ by the asymptotic expansion along the subsequence.

Define the matrix $C = (C_{i,j})_{i,j=1,2}$ by $C_{i,j} = 1_{\{i=j\}}$ in the case $|t_1 - t_2| \geq 1$, and by $C_{1,1} = C_{2,2} = 1$ and

$$C_{1,2} = C_{2,1} = \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{2t_1 t_2} (2a^2 - 2a + 1)$$

in the case $|t_1 - t_2| < 1$.

We now check [A1], [A2] and [C] for $p = 2$ and $\gamma = \frac{1}{2}$. In what follows, we will consider the full sequence $(N)_{N \in \mathbb{N}}$ when $|t_1 - t_2| \geq 1$, but only consider a subsequence of $(N)_{N \in \mathbb{N}}$ satisfying (90) when $|t_1 - t_2| < 1$. Since $\mathbb{l} = \{1, 2\}$, we need to evaluate

$$\mathbf{E}[\Gamma_{I_2}^{(2)}(\mathbf{F}_{N,t})] \text{ and } \mathbf{E}[\Gamma^{(3)}(\mathbf{F}_{N,t})]$$

for every I_2 in \mathbb{l}^2 .

By Lemma 10, we have the property [C](i), that is,

$$\mathbf{E}[\Gamma_{I_2}^{(2)}(\mathbf{F}_{N,t})] = C_{I_2} + o(N^{-1/2})$$

for $I_2 \in \mathbb{l}^2$ ($i = 1, 2$) as $N \rightarrow \infty$ (but along the subsequence when $|t_1 - t_2| < 1$).

In order to check [C](ii), we need to estimate $\mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(\mathbf{F}_{N,t})]$ with $(i_1, i_2, i_3) \in \{1, 2\}^3$.

If $i_1 = i_2 = i_3$ (and they are 1 or 2), then we can follow the lines from the one-dimensional case, see relation (77). We will have, for $i = 1, 2$,

$$\begin{aligned}
\mathbf{E}[\Gamma_{i,i,i}^{(3)}(\mathbf{F}_{N,t})] &= \mathbf{E}[\Gamma^{(3)}(F_{N,t_i})] \\
&= \frac{4}{(2N)^{\frac{3}{2}}} \left[N + 3N(N-1) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t_i - \frac{1}{2N}))} \right)^2 \right. \\
&\quad \left. + N(N-1)(N-2) \left(\frac{(-\frac{1}{8N^2})}{(\frac{1}{4N}(2t_i - \frac{1}{2N}))} \right)^3 \right]
\end{aligned}$$

and by (75) we obtain

$$\begin{aligned}
\mathbf{E}[\Gamma_{i,i,i}^{(3)}(\mathbf{F}_{N,t})] &= \frac{4}{(2N)^{\frac{3}{2}}} \left[N + 3(-1)^2 \frac{1}{(2N(2t_i))^2} \frac{N!}{(N-2)!} \sum_{n=0}^{\infty} C_{n+1}^1 \left(\frac{1}{2N(2t_i)} \right)^n \right. \\
&\quad \left. + (-1)^3 \frac{1}{(2N(2t_i))^3} \frac{N!}{(N-3)!} \sum_{n=0}^{\infty} C_{n+2}^2 \left(\frac{1}{2N(2t_i)} \right)^n \right].
\end{aligned}$$

We thus obtain the estimate (78) where in the expression of the coefficients we replace t by t_i .

If $i_1, i_2, i_3 \in \{1, 2\}$ are not all equal, then we have a different behaviors of the quantity $\mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(\mathbf{F}_{N,t})]$. We can assume $i_1 = i_2 = 1$ and $i_3 = 2$ since the other

cases can be treated analogously. In this situation, from (58)-(63)

$$\begin{aligned}
& \mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(\mathbf{F}_{N,t})] \\
= & \frac{4}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{\mathbf{E}[u(t_1, A_{j_1})u(t_1, A_{j_2})] \mathbf{E}[u(t_1, A_{j_2})u(t_2, A_{j_3})] \mathbf{E}[u(t_2, A_{j_3})u(t_1, A_{j_1})]}{\mathbf{E}[|u(t_1, A_{j_1})|^2] \mathbf{E}[|u(t_1, A_{j_2})|^2] \mathbf{E}[|u(t_2, A_{j_3})|^2]} \\
= & \frac{4}{(2N)^{\frac{3}{2}}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-2} \left(\frac{1}{4N} \left(2t_2 - \frac{1}{2N} \right) \right)^{-1} \\
& \times \sum_{j_1, j_2, j_3=0}^{N-1} \mathbf{E}[u(t_1, A_{j_1})u(t_1, A_{j_2})] \mathbf{E}[u(t_1, A_{j_2})u(t_2, A_{j_3})] \mathbf{E}[u(t_2, A_{j_3})u(t_1, A_{j_1})] \\
= & \frac{4}{(2N)^{\frac{3}{2}}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-2} \left(\frac{1}{4N} \left(2t_2 - \frac{1}{2N} \right) \right)^{-1} \\
& \times \sum_{j_1, j_2, j_3=0; j_2 \neq j_3 \neq j_1}^{N-1} \mathbf{E}[u(t_1, A_{j_1})u(t_1, A_{j_2})] \mathbf{E}[u(t_1, A_{j_2})u(t_2, A_{j_3})] \mathbf{E}[u(t_2, A_{j_3})u(t_1, A_{j_1})] \\
= & \frac{4}{(2N)^{\frac{3}{2}}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-2} \left(\frac{1}{4N} \left(2t_2 - \frac{1}{2N} \right) \right)^{-1} \\
& \times \left[\sum_{j_1, j_2=0; j_1 \neq j_2}^{N-1} \mathbf{E}[|u(t_1, A_{j_1})|^2] (\mathbf{E}[u(t_1, A_{j_1})u(t_2, A_{j_2})])^2 \right. \\
& \left. + \sum_{j_1, j_2, j_3=0; j_2 \neq j_3 \neq j_1 \neq j_2}^{N-1} \mathbf{E}[u(t_1, A_{j_1})u(t_1, A_{j_2})] \mathbf{E}[u(t_1, A_{j_2})u(t_2, A_{j_3})] \mathbf{E}[u(t_2, A_{j_3})u(t_1, A_{j_1})] \right] \\
=: & \frac{4}{(2N)^{\frac{3}{2}}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-2} \left(\frac{1}{4N} \left(2t_2 - \frac{1}{2N} \right) \right)^{-1} (v_N^{(1)} + v_N^{(2)}).
\end{aligned}$$

In Case $|t_1 - t_2| \geq 1$, we see $\mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(\mathbf{F}_{N,t})] = 0$ by Corollary 1. In Case $|t_1 - t_2| < 1$, we are only considering the subsequence of $(N)_{N \in \mathbb{N}}$. By Lemma 8 (d) and (a), we have

$$\begin{aligned}
|v_N^{(2)}| & \lesssim O(N^{-2}) \times \left| \sum_{j_1, j_2, j_3=0; j_2 \neq j_3 \neq j_1 \neq j_2}^{N-1} \mathbf{E}[u(t_1, A_{j_2})u(t_2, A_{j_3})] \mathbf{E}[u(t_2, A_{j_3})u(t_1, A_{j_1})] \right| \\
& = O(N^{-2}) \times \left(O(N^2 \times N^{-2} \times N^{-2}) + O(N \times N^{-1} \times N^{-1}) \right) = O(N^{-3})
\end{aligned}$$

since the last sum is essentially one-dimensional. Therefore, $v_N^{(2)}$ has no essential

contribution in the limit. As for $v_N^{(1)}$, by Lemma 8 (c),

$$\begin{aligned} v_N^{(1)} &= \sum_{j_1, j_2=0; j_1 \neq j_2}^{N-1} \mathbf{E}[|u(t_1, A_{j_1})|^2] (\mathbf{E}[u(t_1, A_{j_1})u(t_2, A_{j_2})])^2 \\ &= \left(\frac{t_1}{2N} + O(N^{-2}) \right) \sum_{j_1, j_2=0; j_1 \neq j_2}^{N-1} (\mathbf{E}[u(t_1, A_{j_1})u(t_2, A_{j_2})])^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{4}{(2N)^{\frac{3}{2}}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-2} \left(\frac{1}{4N} \left(2t_2 - \frac{1}{2N} \right) \right)^{-1} v_N^{(1)} \\ &= \frac{4}{2^{\frac{3}{2}} N^{1/2}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-1} \left(\frac{t_1}{2N} + O(N^{-2}) \right) u_N. \end{aligned}$$

From (89), we obtain, with a from (90),

$$\begin{aligned} \mathbf{E}[\Gamma_{1.1.2}^{(3)}(\mathbf{F}_{N,t})] &= \frac{4}{2^{\frac{3}{2}} N^{1/2}} \left(\frac{1}{4N} \left(2t_1 - \frac{1}{2N} \right) \right)^{-1} \left(\frac{t_1}{2N} + O(N^{-2}) \right) \\ &\quad \times \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{2t_1 t_2} (2a^2 - 2a + 1) + o(N^{-1/2}) \\ &= \frac{(1 - |t_1 - t_2|)(t_1 \wedge t_2)^2}{\sqrt{2N} t_1 t_2} (2a^2 - 2a + 1) + o(N^{-1/2}) \end{aligned}$$

Consequently, we obtained

$$\mathbf{E}[\Gamma_{i_1, i_2, i_3}^{(3)}(\mathbf{F}_{N,t})] = D_1 N^{-1/2} + o(N^{-1/2}) \quad (91)$$

and this verifies $[C](ii)$.

Similarly to the one-dimensional case, we can check $[A1] - [A2]$. Recall that

$$\tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) = \frac{2^2}{(2N)^{\frac{3}{2}}} \sum_{j_1, j_2, j_3=0}^{N-1} \frac{I_2((g_{t_{i_1}, x, j_1}) \tilde{\otimes} (g_{t_{i_2}, x, j_2})) \langle A_{j_2}, A_{j_3} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_3}}} \langle A_{j_3}, A_{j_1} \rangle_{\mathcal{H}_{t_{i_3}, t_{i_1}}})}{\mathbf{E}[|u(t_{i_1}, A_{j_1})|^2] \mathbf{E}[|u(t_{i_2}, A_{j_2})|^2] \mathbf{E}[|u(t_{i_3}, A_{j_3})|^2]}$$

for

$$\tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) = \Gamma_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) - \mathbf{E}[\Gamma_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t})].$$

Therefore,

$$\begin{aligned}
& \mathbf{E} \left[\left| \tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) \right|^2 \right] \\
&= \left(\frac{2^2}{(2N)^{\frac{3}{2}}} \right)^2 \sum_{\substack{j_1, j_2, j_3=0 \\ j'_1, j'_2, j'_3=0}}^{N-1} \left\{ \frac{\langle A_{j_1}, A_{j'_1} \rangle_{\mathcal{H}_{t_{i_1}, t_{i_1}}} \langle A_{j_2}, A_{j'_2} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_2}}} \langle A_{j_2}, A_{j_3} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_3}}}}{\mathbf{E} \left[|u(t_{i_1}, A_{j_1})|^2 \right] \mathbf{E} \left[|u(t_{i_2}, A_{j_2})|^2 \right] \mathbf{E} \left[|u(t_{i_3}, A_{j_3})|^2 \right]} \right. \\
&\quad \times \left. \frac{\langle A_{j_3}, A_{j_1} \rangle_{\mathcal{H}_{t_{i_3}, t_{i_1}}} \langle A_{j'_2}, A_{j'_3} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_3}}} \langle A_{j'_3}, A_{j'_1} \rangle_{\mathcal{H}_{t_{i_3}, t_{i_1}}}}{\mathbf{E} \left[|u(t_{i_1}, A_{j'_1})|^2 \right] \mathbf{E} \left[|u(t_{i_2}, A_{j'_2})|^2 \right] \mathbf{E} \left[|u(t_{i_3}, A_{j'_3})|^2 \right]} \right\} \\
&+ \left(\frac{2^2}{(2N)^{\frac{3}{2}}} \right)^2 \sum_{\substack{j_1, j_2, j_3=0 \\ j'_1, j'_2, j'_3=0}}^{N-1} \left\{ \frac{\langle A_{j_1}, A_{j'_2} \rangle_{\mathcal{H}_{t_{i_1}, t_{i_2}}} \langle A_{j_2}, A_{j'_1} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_1}}} \langle A_{j_2}, A_{j_3} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_3}}}}{\mathbf{E} \left[|u(t_{i_1}, A_{j_1})|^2 \right] \mathbf{E} \left[|u(t_{i_2}, A_{j_2})|^2 \right] \mathbf{E} \left[|u(t_{i_3}, A_{j_3})|^2 \right]} \right. \\
&\quad \times \left. \frac{\langle A_{j_3}, A_{j_1} \rangle_{\mathcal{H}_{t_{i_3}, t_{i_1}}} \langle A_{j'_2}, A_{j'_3} \rangle_{\mathcal{H}_{t_{i_2}, t_{i_3}}} \langle A_{j'_3}, A_{j'_1} \rangle_{\mathcal{H}_{t_{i_3}, t_{i_1}}}}{\mathbf{E} \left[|u(t_{i_1}, A_{j'_1})|^2 \right] \mathbf{E} \left[|u(t_{i_2}, A_{j'_2})|^2 \right] \mathbf{E} \left[|u(t_{i_3}, A_{j'_3})|^2 \right]} \right\}
\end{aligned}$$

and hence

$$\begin{aligned}
& \mathbf{E} \left[\left| \tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) \right|^2 \right] \\
&\lesssim \sum^* N^3 \sum_{j_1, \dots, j_6=0}^{N-1} \left\{ \left| \langle A_{j_1}, A_{j_2} \rangle_{\mathcal{H}_{t_{i'_1}, t_{i'_2}}} \right| \left| \langle A_{j_2}, A_{j_3} \rangle_{\mathcal{H}_{t_{i'_2}, t_{i'_3}}} \right| \left| \langle A_{j_3}, A_{j_4} \rangle_{\mathcal{H}_{t_{i'_3}, t_{i'_4}}} \right| \right. \\
&\quad \times \left. \left| \langle A_{j_4}, A_{j_5} \rangle_{\mathcal{H}_{t_{i'_4}, t_{i'_5}}} \right| \left| \langle A_{j_5}, A_{j_6} \rangle_{\mathcal{H}_{t_{i'_5}, t_{i'_6}}} \right| \left| \langle A_{j_6}, A_{j_1} \rangle_{\mathcal{H}_{t_{i'_6}, t_{i'_1}}} \right| \right\}
\end{aligned}$$

where \sum^* is the sum for all permutations (i'_1, \dots, i'_6) of $(i_1, i_1, i_1, i_1, i_2, i_2, i_2, i_2, i_3, i_3, i_3, i_3)$. By Lemma 8 (a), (c), (d), we conclude that there exists a constant K such that

$$\sup_{i=0, \dots, N-1} \sum_{j=0}^{N-1} \left| \langle A_i, A_j \rangle_{\mathcal{H}_{t_a, t_b}} \right| \leq KN^{-1}$$

for any $a, b \in \{1, 2\}$ and $n \in \mathbb{N}$. By the Schwarz inequality and Lemma 8 (c), we have

$$\left| \langle A_{j_1}, A_{j_2} \rangle_{\mathcal{H}_{t_{i_1}, t_{i_2}}} \right| \leq \frac{\max\{t_1, t_2\}}{2N} \quad (j_1, j_2 = 0, \dots, N-1; i_1, i_2 = 1, 2).$$

Then, by Lemma 11 below with $\Lambda = \{1, 2\}^2$, we obtain

$$\left\| \tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t}) \right\|_2 = O(N^{-1}),$$

in particular, [A2](ii). Since $\tilde{\Gamma}_{i_1, \dots, i_3}^{(3)}(\mathbf{F}_{N,t})$ is in the second chaos, we obtain [A2](i) for any $\ell_1 \in \mathbb{N}$.

Verification of [A1] can be done in a similar way. The components of $\mathbf{F}_{N,t}$ and $\Gamma^{(2)}(\mathbf{F}_{N,t}) - C$ are in the second chaos. First we can show $\sup_{N \in \mathbb{N}} \|\mathbf{F}_{N,t}\|_2 < \infty$ and $\|\Gamma^{(2)}(\mathbf{F}_{N,t}) - C\|_2 = O(N^{-1/2})$, and next use hypercontractivity to obtain L^r -estimates for any $r > 2$. Condition [A1] is verified if we follow the same procedure after applying the Malliavin operator $[(\ell + 1)/2]$ -times to these variables.

In conclusion, the asymptotic expansion for the multi-variate $\mathbf{F}_{N,t}$ is valid for $p = 2$ and $\gamma = 1/2$ as $N \rightarrow \infty$ when $|t_1 - t_2| \geq 1$, and so along a subsequence satisfying (90) when $|t_1 - t_2| < 1$. With more tedious computation, our approach can be extended to any $p \geq 2$.

In the above discussion, we used the following lemma.

Lemma 11. *Let Λ and \mathbb{J} be finite sets. Let $a^\lambda(j_1, j_2) \in \mathbb{R}$ for $\lambda \in \Lambda$, $j_1, j_2 \in \mathbb{J}$. Let $\Delta \in \mathbb{R}$. Suppose that there exists a constant K such that*

$$\max_{\lambda \in \Lambda} \max_{j_1 \in \mathbb{J}} \sum_{j_2 \in \mathbb{J}} |a^\lambda(j_1, j_2)| \leq K\Delta.$$

Let $k \in \mathbb{N}$ satisfying $k \geq 2$. Let

$$S(j_1; \lambda_1, \dots, \lambda_k) = \sum_{j_2, \dots, j_k \in \mathbb{J}_N} a^{\lambda_1}(j_1, j_2) a^{\lambda_2}(j_2, j_3) \cdots a^{\lambda_{k-1}}(j_{k-1}, j_k) a^{\lambda_k}(j_k, j_1).$$

Let

$$T(\lambda_1, \dots, \lambda_k) = \sum_{j_1, j_2, \dots, j_k \in \mathbb{J}_N} a^{\lambda_1}(j_1, j_2) a^{\lambda_2}(j_2, j_3) \cdots a^{\lambda_{k-1}}(j_{k-1}, j_k) a^{\lambda_k}(j_k, j_1).$$

Let

$$\Delta' = \max_{\lambda \in \Lambda} \max_{j_1, j_2 \in \mathbb{J}} |a^\lambda(j_1, j_2)|.$$

Then

$$(a) \quad \max_{\lambda_1, \dots, \lambda_k \in \Lambda} \max_{j_1 \in \mathbb{J}} |S(j_1; \lambda_1, \dots, \lambda_k)| \leq K^{k-1} \Delta^{k-1} \Delta'.$$

$$(b) \quad \max_{\lambda_1, \dots, \lambda_k \in \Lambda} |T(\lambda_1, \dots, \lambda_k)| \leq K^{k-1} \Delta^{k-1} \Delta' \#\mathbb{J}.$$

Proof. We may assume that $a^\lambda(j_1, j_2) \geq 0$. The property (b) follows from (a). We will show (a). Let

$$\mathfrak{S}^{(p-1)}(j_1) = \max_{\lambda_1, \dots, \lambda_p \in \Lambda} \max_{j'_1 \in \mathbb{J}} \sum_{j_2, \dots, j_p \in \mathbb{J}} a^{\lambda_1}(j'_1, j_2) a^{\lambda_2}(j_2, j_3) \cdots a^{\lambda_{p-1}}(j_{p-1}, j_p) a^{\lambda_p}(j_p, j_1)$$

for $p = 2, 3, \dots$. Then

$$\begin{aligned}
& \mathfrak{S}^{(p-1)}(j_1) \\
&= \max_{\lambda_1, \dots, \lambda_p \in \Lambda} \max_{j'_1 \in \mathbb{J}} \sum_{j_2 \in \mathbb{J}} a^{\lambda_1}(j'_1, j_2) \sum_{j_3, \dots, j_p \in \mathbb{J}} a^{\lambda_2}(j_2, j_3) \cdots a^{\lambda_{p-1}}(j_{p-1}, j_p) a^{\lambda_p}(j_p, j_1) \\
&\leq \max_{\lambda_1 \in \Lambda} \max_{j'_1 \in \mathbb{J}} \sum_{j_2 \in \mathbb{J}} a^{\lambda_1}(j'_1, j_2) \\
&\quad \times \max_{\lambda_2, \dots, \lambda_p \in \Lambda} \max_{j'_2 \in \mathbb{J}} \sum_{j_3, \dots, j_k \in \mathbb{J}} a^{\lambda_2}(j'_2, j_3) \cdots a^{\lambda_{p-1}}(j_{p-1}, j_p) a^{\lambda_p}(j_p, j_1),
\end{aligned}$$

therefore

$$\mathfrak{S}_N^{(p-1)}(j_1) \leq K \Delta \mathfrak{S}^{(p-2)}(j_1) \quad (92)$$

for all $j_1 \in \mathbb{J}_N$, $N \in \mathbb{N}$ and $p \in \{3, 4, \dots\}$. By inductively applying (92), we obtain

$$\mathfrak{S}^{(k-1)}(j_1) \leq (K \Delta)^{k-2} \mathfrak{S}^{(1)}(j_1). \quad (93)$$

Moreover,

$$\begin{aligned}
\mathfrak{S}^{(1)}(j_1) &= \max_{\lambda_1, \lambda_2 \in \Lambda} \max_{j'_1 \in \mathbb{J}} \sum_{j_2 \in \mathbb{J}_N} a^{\lambda_1}(j'_1, j_2) a^{\lambda_2}(j_2, j_1) \\
&\leq \max_{\lambda_1, \lambda_2 \in \Lambda} \max_{j'_1 \in \mathbb{J}} \sum_{j_2 \in \mathbb{J}} a^{\lambda_1}(j'_1, j_2) \max_{j'_2 \in \mathbb{J}} a^{\lambda_2}(j'_2, j_1) \\
&\leq K \Delta \max_{\lambda_2 \in \Lambda} \max_{j'_2 \in \mathbb{J}} a^{\lambda_2}(j'_2, j_1) \\
&\leq K \Delta \Delta'. \quad (94)
\end{aligned}$$

From (93) and (94), we conclude

$$\max_{j_1 \in \mathbb{J}} \mathfrak{S}^{(k-1)}(j_1) \leq (K \Delta)^{k-1} \Delta'.$$

Since $S(j_1; i_1, \dots, i_k) \leq \mathfrak{S}^{(k-1)}(j_1)$ by definition, we obtain the result. ■

6 Elements from Malliavin calculus

In this section, we recall the basics of the Malliavin calculus. For complete presentations, we refer to [11] or [10]. Consider H a real separable Hilbert space and

$(W(h), h \in H)$ an isonormal Gaussian process on a probability space (Ω, \mathcal{A}, P) , which is a centered Gaussian family of random variables such that $\mathbf{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_H$.

We denote by D the Malliavin derivative operator that acts on smooth functions \mathcal{S} of the form $F = g(W(h_1), \dots, W(h_n))$ (g is a smooth function with compact support and $h_i \in H$)

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(W(h_1), \dots, W(h_n))h_i.$$

By iteration, we can also define $D^k F$, the k th iterated Malliavin derivative. Let $\mathbb{D}_{k,p}$ (for any natural number k and for any real number $p \geq 1$) be the closure of \mathcal{S} with respect to the norm

$$\|F\|_{k,p}^p := E[|F|^p] + \sum_{i=1}^k E[\|D^i F\|_{H^{\otimes i}}^p].$$

The adjoint of D is denoted by δ and is called the divergence (or Skorohod) integral. Its domain ($Dom(\delta)$) coincides with the class of stochastic processes $u \in L^2(\Omega \times T)$ such that

$$|\mathbf{E}[\langle DF, u \rangle]| \leq c\|F\|_2$$

for all $F \in \mathbb{D}_{1,2}$ and $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relationship

$$\mathbf{E}[(F\delta(u))] = \mathbf{E}[\langle DF, u \rangle_H]. \quad (95)$$

The chain rule for the Malliavin derivative (see Proposition 1.2.4 in [11]) will be used several times. If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function having bounded derivative and $F \in \mathbb{D}_{1,2}$, then $\varphi(F) \in \mathbb{D}_{1,2}$ and

$$D\varphi(F) = \varphi'(F)DF. \quad (96)$$

Denote by I_n the multiple stochastic integral with respect to B (see [11]). This mapping I_n is actually an isometry between the Hilbert space $H^{\odot n}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{H^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $h_n(W(h))$ where $h \in H$, $\|h\|_H = 1$ and h_n is the Hermite polynomial of degree $n \in \mathbb{N}$

$$h_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \left(\exp\left(-\frac{x^2}{2}\right) \right), \quad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as follows: for m, n positive integers,

$$\begin{aligned} \mathbf{E}[I_n(f)I_m(g)] &= n!\langle \tilde{f}, \tilde{g} \rangle_{H^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}[I_n(f)I_m(g)] &= 0 \quad \text{if } m \neq n. \end{aligned} \quad (97)$$

It also holds that $I_n(f) = I_n(\tilde{f})$ where \tilde{f} denotes the symmetrization of f .

We recall that any square integrable random variable which is measurable with respect to the σ -algebra generated by W can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad (98)$$

where $f_n \in H^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = - \sum_{n \geq 0} n I_n(f_n)$$

if F is given by (98) and it is such that $\sum_{n=1}^{\infty} n^2 n! \|f_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty$. Notice that

$$LF = L(F - EF) \text{ and } L^{-1}F = L^{-1}(F - EF).$$

It holds that

$$\delta D(-L)^{-1}F = F - EF. \quad (99)$$

We recall the product formula for multiple integrals. It is well-known that for $f \in H^{\odot n}$ and $g \in H^{\odot m}$

$$I_n(f)I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{m+n-2r}(f \otimes_r g) \quad (100)$$

where $f \otimes_r g$ means the r -contraction of f and g .

Another important property of finite sums of multiple integrals is the hypercontractivity. Namely, if $F = \sum_{k=0}^n I_k(f_k)$ with $f_k \in H^{\otimes k}$ then

$$\mathbf{E}[|F|^p] \leq C_p (\mathbf{E}[F^2])^{\frac{p}{2}}. \quad (101)$$

for every $p \geq 2$.

We can also define associated Sobolev spaces and Malliavin derivatives for vector-valued random variables. Let V be an Hilbert space. Let \mathcal{S}_V denote the set

$$\mathcal{S}_V = \left\{ \sum_{i=1}^n F_i h_i \mid F_1, \dots, F_n \in \mathcal{S}, h_1, \dots, h_n \in V, n \geq 1 \right\}.$$

Then for $u = \sum_{i=1}^n F_i h_i$ we can define $Du := \sum_{i=1}^n DF_i \otimes h_i$ and consider the norm

$$\|u\|_{k,p,V} = \left(\mathbf{E}[\|u\|_V^2] + \sum_{i=1}^k \mathbf{E}[\|D^i u\|_{H^{\otimes i} \otimes V}^p] \right)^{\frac{1}{p}}.$$

Now we can, just as for the space \mathcal{S} , consider the closure of \mathcal{S}_V with respect to this norm and call it $\mathbb{D}_{k,p}(V)$.

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