A Decision-Making Computational Methodology for a Class of Type-2 Fuzzy Intervals: An Interval-Based Approach

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ABSTRACT

This paper proposes an interval-based computational formulation of the Bellman-Zadeh decision-making approach when the handled information (goals and constraints) is represented by type-2 fuzzy intervals (FIs). Our method, which maintains the flexibility of interval arithmetic and interval reasoning as major objectives, consists of representing an FI by its profiles, which are considered gradual numbers. The developed reflection is based on interval relations to determine a generic formulation of the intersection operation between type-2 FIs, where a computational mechanism can be easily derived. This intersection area is considered an uncertain decision domain that is represented by lower type-1 FI situations and upper type-1 FI bounds that are considered extreme situations in adverse situations and favorable situations, respectively. In this framework, any FI between these FI bounds can be chosen by decision makers as an optimal solution according to a specified decision criterion. In this paper, a risk decision-making criterion is considered; however, other decision criteria can be employed in a similar manner. The proposed vision offers a convenient tool that enables decision makers to manage their judgment in the possible uncertain domain of a decision. The interest of the proposed approach is the extension of inter-interval relations to type-1 and type-2 FIs, where the Bellman-Zadeh decision-making problem using membership functions can be transformed into an interval arithmetic problem using the FI profiles.

Keywords: Type-1 and Type-2 Fuzzy Intervals, Bellman-Zadeh decision-making principle, Intersection operator, Imprecision-Uncertainty, Interval relations and Interval arithmetic, Risk decision-making problem.

I. INTRODUCTION

Due to the presence of imprecision and uncertainty in complex environments, decision makers are often unable to provide crisp numerical values to quantify their evaluations and/or judgements. Decision makers often use some degree of imprecision and/or uncertainty to formulate their subjective judgments. To address this situation, fuzzy subsets (often referred to as fuzzy sets (FSs)), which are a useful tool for handling the imprecision and uncertainty of decision makers, have been substantially exploited in decision-making problems [3][40][41][42].

The philosophy of fuzzy decision-making is based on the concept of FSs (also known as type-1 FSs). The underlying theory—the FS theory proposed by Zadeh [48]—provides a reasonable mathematical tool for explicitly representing imprecise (vague) information in the form of membership functions. Imprecision is primarily attributed to vague or even approximate characteristics (ill-defined limits) that are expressed in a linguistic form using a natural language. After a few years, Zadeh expressed his doubts about the ability of a type-1 FS to exhibit the uncertainty of word-based representations. By handling words via a type-1 FS based on membership degrees, the uncertainty of words is absent. Mendel [31] relies on Popper's falsification principle to express the following statement: a type-1 FS is certain and cannot properly represent a word that is uncertain by essence.

To better handle imprecision and uncertainty, the type-2 FS concept has been proposed. Historically, the type-2 FS concept was introduced by Zadeh as an extension of type-1 FS [49]. The issue of representing words in a natural language motivated Zadeh's initial proposition. Zadeh's basic idea can be summarized by the following sentence: "The same words have a different meaning for different people". Uncertainties are inherent in the handling of words in a natural language. Thus, the type-2 vision enables the integration of uncertainty into the answers of experts and the simultaneous consideration of their different opinions (as originally proposed by Zadeh). Based on the extension principle, algebraic structures of type-2 FSs were extensively investigated [36][37]. Over the past 30 years, type-2 fuzzy representation has advanced significantly due to the research of Mendel et al. [24][32][33][34]. In this framework, interval type-2 FSs are the most commonly employed FS of the higher order FS due to the high computational complexity of using general type-2 FSs. Interval type-2 FSs have been applied in many practical domains, especially in modeling, control and decision-making [32][33][40][41][42][45].

In the fuzzy literature, an FS is sometimes referred to as a "fuzzy number". In general, this denomination refers to an FS whose α-cuts are conventional intervals. Philosophically, and as discussed in [17], a fuzzy number does not generalize the concept of a real number but rather the concept of a real-valued interval. In this framework, a fuzzy number should inherit the properties of intervals and not those of real numbers, which explains why the appellation "fuzzy interval" (FI) is employed instead of "fuzzy number" throughout this study. An FI is a convex FS, where all α-cuts are intervals. An FI can be considered a stack of nested intervals defined by the α-cuts concept [5].

Bellman and Zadeh [3] originally proposed the concept of fuzzy decision-making based on a compromise between goals and constraints that are represented by type-1 FSs. This concept of optimization aims to determine an optimal solution, where both goals and constraints are represented by their membership functions. The fuzzy decision domain, which is denoted D and represented by its membership function μ_D on a referential X, is issued from the intersection operation between the fuzzy goals and the fuzzy constraints. In this context, on the referential set X, an optimal solution x_* of the decision-making problem must reflect the maximum fulfillment degree of the compromise between goals and constraints, i.e., x_* corresponds to the highest degree of belonging to D.

Generally, in type-1 decision-making techniques, the FS (FI) that represents the perception of the decision makers is assumed to be fixed, and the optimal decision-making solution is considered certain. This optimal solution can be altered if uncertainties are attached to the type-1 FIs. To address this uncertainty phenomenon, decision-making methodologies have been extended to type-2 FSs and type-2 FIs. This extension is not a new problem. Numerous useful and excellent methods for handling decision-making problems using type-2 representation, especially in multiple attribute group decision-making problems, have been published in the literature [11][12][40][41][42][43]. For instance, in [11][40], ranking values and arithmetic operations techniques are exploited. In [12][43], the TOPSIS method is used. In [13], arithmetic operations and fuzzy preference relations are proposed. In [47], a linguistic weighted average is exploited. This research domain has expanded, and it is now difficult to compose an exhaustive list of all the work that has been published in the literature. Regrettably, many excellent pieces of work are not mentioned in this paper.

Recently, an interesting type-2 risk decision-making methodology was proposed based on the Bellman-Zadeh principle [42]. Using this method, the type-2 goals and constraints are represented by type-2 membership functions. As explained in [42], if the decision-making methodologies that were previously mentioned are effective and useful, they do not index the decision process for the notion of risk. This paper aims to revisit this concept of decision-making, where an alternative computational approach is proposed. The philosophy of the proposed method is not limited to the risk framework and can be applied to other issues.

Generally, fuzzy computations (standard arithmetic operations, intersection, union, …) that are based on membership function formalism are implemented using the Zadeh extension principle. However, computations based on the Zadeh's extension principle are expensive due to the need to solve a nonlinear programming problem. To overcome this problem, approximation *via* α-cuts (and its hybridizations) is often employed [21][47][49]. Due to its simplicity and the availability of computational methods, fuzzy computation based on α-cuts is the most common approach for implementing fuzzy operations in different applications. However, the literature is unanimous regarding the fact that the α-cut approach is time consuming. In this framework, regardless of the method, computing operations on type-2 FIs remain computationally expensive due to the 3D nature of type-2 FIs. Although α-cuts were sound and useful in some situations, this method was computationally expensive and required significant preliminary computations. As stated in [22], the implementation of type-2 FIs operations sometimes requires the use of massively parallel processing units, such as graphical processing units (GPUs).

This paper proposes an alternative computational formulation of the Bellman-Zadeh decision-making method according to an interval-based vision, where the flexibility of interval arithmetic and interval relations is maintained as a major objective. Our work aims to replace the membership function formalism that is often employed in decision-making methods by an FI representation via the concept of gradual numbers. An interval arithmetic methodology, where a generic computational mechanism can be easily derived, is proposed to avoid the discretization procedure, which is necessary for implementing the α-cuts principle. In this vision, an FI is regarded as a pair of lower and upper gradual numbers (bounds), which are referred to as left and right profiles. Fortin *et al.* introduced the notion of gradual numbers, which provides a new outlook on FI and their manipulation [8][17][20]. This vision differs from existing methods in the literature and enables the extension of interval arithmetic and reasoning methods to FI and decision-making strategies. The proposed method has been applied in the decision-making context; however, many uses of its potentialities can be imagined in the frameworks of type-2 fuzzy control [10][29], type-2 fuzzy multicriteria decision-making and aggregation operators [14][46] in type-2 regression [2][23].

From a methodological point of view, when goals and constraints are represented by a type-2 FI, the originality of the proposed methodology exploits the interval relations to express the decision domain (intersection domain) as a type-2 FI defined by its lower and upper type-1 FI boundaries. The type-1 FI bounds can be interpreted as extreme situations in most adverse and favorable situations. They frame a domain that represents an uncertainty footprint of the decision. According to a specific criterion, a decision maker in this case can select any optimal solution within this domain. Thus, this methodology offers a convenient and flexible tool that enables decision makers to directly manage and adjust their decisions within the possible decision domain according to their specified decision criteria (e.g., a risk decision-making criterion). The proposed approach extends inter-interval relations to type-1 and type-2 FIs to propose an interval-based vision of the Bellman-Zadeh decision-making problem.

This paper is organized as follows. Section II provides some preliminaries about intervals and FIs. Semantics and interpretations of type-2 FIs are detailed in section III. In section IV, partial interval relations are introduced to provide general analytical expressions of the intersection operator for type-1 and type-2 FIs. The decision-making methodology for type-1 and type-2 FIs with associated application examples is detailed in section V. Concluding remarks are given in section VI.

II. PRELIMINARIES: INTERVALS AND FUZZY INTERVALS

For the sake of simplicity without the loss of generality, the FI in this paper is considered to be unimodal and piecewise linear. However, the proposed methodology remains adaptable regardless of the form of the FI. Generally, a conventional interval $\boldsymbol{a}$ (the interval is denoted in bold) can be expressed by two main representations. The first representation is the endpoints (*EP*) representation, where $\boldsymbol{a}$ is denoted by its endpoints, i.e., $\boldsymbol{a} = [a^-, a^+]$ with $a^- \leq a^+$. The second representation is the midpoint-radius (*MR*) representation. In this case, the interval $\boldsymbol{a}$ is denoted by $\boldsymbol{a} = (M_a, R_a)$; $R_a \geq 0$. The midpoint M_a and the radius R_a are defined by $M_a = (a^- + a^+)/2$ and $R_a = (a^+ - a^-)/2$. The *MR* representation facilitates the interpretation of the interval relations and arithmetic [6][25]. The relation between the *EP* and *MR* representations is simple, i.e., $a^- = M_a - R_a$ and $a^+ = M_a + R_a$.

II.1. Type-1 fuzzy intervals

A type-1 FS is constructed by generalizing the traditional notion of the characteristic function of a set. An information is a member of an FS, which is denoted A, with a certain degree of belonging to the interval $[0, w_A]$. If $w_A = 1$, then A becomes a normal FS. With the referential set X, the unimodal FS A with this reference is characterized by its membership function, which is denoted μ_A, such that

- μ_A is a continuous mapping from $X \to [0, w_A]$, and $\mu_A(x = k) = w_A$;
- its support, i.e., $\{x \mid \mu_A(x) > 0\}$, is the open interval (s^-, s^+), and its modal value is $x = k$; and
- $\mu_A(x)$ is nondecreasing for $x \in (-\infty, k]$ and nonincreasing for $x \in [k, +\infty)$.

Let us denote $\mu_A{-}$ and $\mu_A{+}$ as the restrictions of μ_A to $(s^-, k]$ and μ_A to $[k, s^+)$, respectively, i.e., $\mu_A{-}(x) = \mu_A(x)$ for $x \in (s^-, k]$ and $\mu_A{+}(x) = \mu_A(x)$ for $x \in [k, s^+)$. Let us also assume that these functions are injective ($\mu_A{-}$ is increasing and $\mu_A{+}$ is decreasing). Generally, when the bounds of a conventional interval $\boldsymbol{a}$ are flexible and characterize a gradual transition over the interval, they can be represented by gradual numbers [8][9][17][20]. A gradual number is a real-valued function that is parameterized by a degree of relevance λ. Similar to a conventional interval, an FI can be represented by the ordered pair of its two bounds, i.e., $a^-(\lambda)$ and $a^+(\lambda)$ (gradual numbers), which are referred to as left and right profiles. In the *EP* space, an FI is denoted by $\boldsymbol{a}(\lambda) = [a^-(\lambda), a^+(\lambda)]$, where $a^-(\lambda) \leq a^+(\lambda)$. The type-1 FS A is interpreted as a type-1 FI $\boldsymbol{a}(\lambda)$, where $a^-(\lambda)$ and $a^+(\lambda)$ are defined by the inverse functions $(\mu_A{-})^{-1}$ and $(\mu_A{+})^{-1}$, respectively:

$$a^-(\lambda) = \inf\{x \mid \mu_A(x) \geq \lambda\} = (\mu_A{-})^{-1}(\lambda); \text{ and } a^+(\lambda) = \sup\{x \mid \mu_A(x) \geq \lambda\} = (\mu_A{+})^{-1}(\lambda).$$

In this paper, $a^-(\lambda)$ and $a^+(\lambda)$ are assumed to be continuous, and $a^-(0)$ and $a^+(0)$ are defined. For instance, Fig. 1 shows a normalized type-1 triangular FS A and its representation as a type-1 triangular FI $\boldsymbol{a}(\lambda)$. For the sake of clarity, and as habitually applied in FS representations, a rotation of an angle of $\pi/2$ is shown in Fig. 1.b, which generates Fig. 2.a. An FI is represented by the coordinates $(\boldsymbol{a}(\lambda), \lambda)$ instead of $(\lambda, \boldsymbol{a}(\lambda))$.

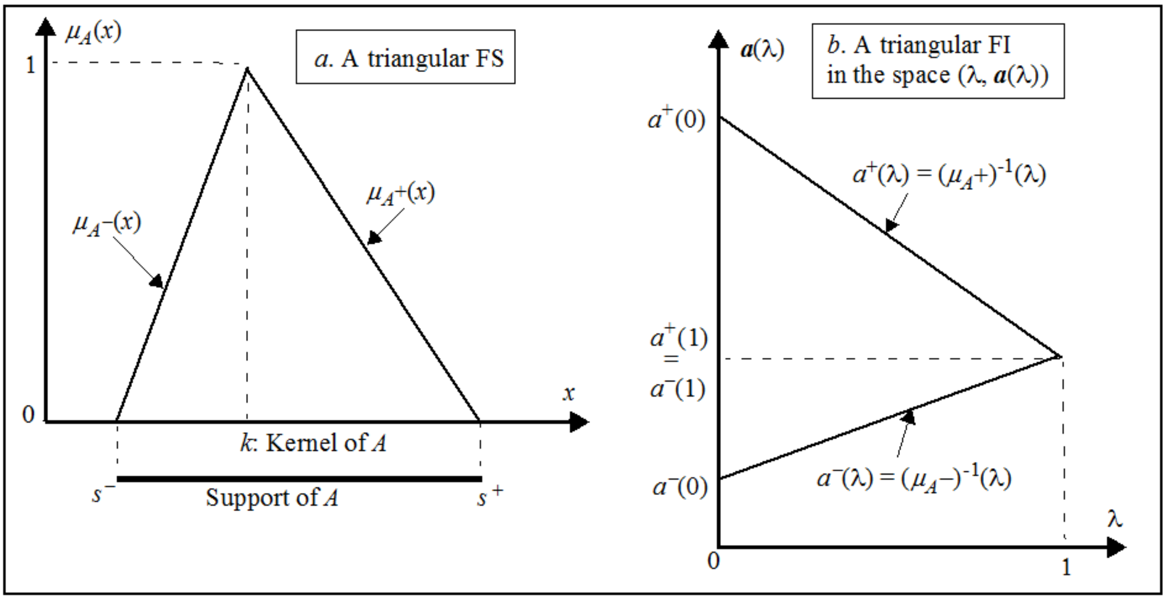

Fig. 1: Type-1 triangular FS and its representation with a type-1 FI

In a framework of equivalence, Fig. 2.b presents an FS and its representation as an FI in the same figure. The values of x and those attached to $\boldsymbol{a}(\lambda)$ are simultaneously shown on the horizontal axis. On the vertical axis, both the degrees of relevance λ and the degrees of belonging to A are shown.

A membership function representation can be moved to a representation by a pair of gradual numbers, and vice versa, without losing any information. Thus, the membership function μ_A of a normal and unimodal FS can be deduced from the gradual number bounds as follows:

$$\mu_A(x) = \begin{cases} \sup\{\lambda \mid a^-(\lambda) \leq x \text{ ; if: } a^-(0) \leq x \leq a^-(1) \\ \sup\{\lambda \mid a^+(\lambda) \geq x \text{ ; if: } a^+(1) \leq x \leq a^+(0) \end{cases} \Rightarrow \mu_A(x) = \begin{cases} (a^-)^{-1}(x) \text{ ; if: } a^-(0) \leq x \leq a^-(1) \\ (a^+)^{-1}(x) \text{ ; if: } a^+(1) \leq x \leq a^+(0) \end{cases}$$

Let us give an example to show the equivalence between an FS and an FI using the gradual number representation. Let us assume an FS A, which is defined by its membership function $\mu_A(x)$, given by:

$$\mu_A(x) = \begin{cases} \mu_{A^-}(x) = (x-1)/5\ ;\ 1 \le x \le 6 \\ \mu_{A^+}(x) = (11-x)/5\ ;\ 6 \le x \le 11 \end{cases}$$

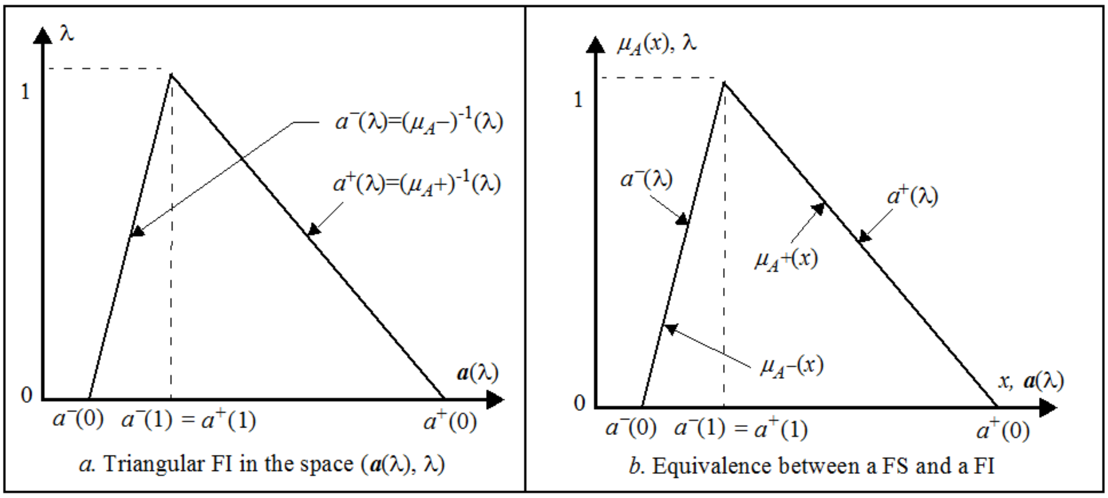

Fig. 2: Type-1 FI representation and equivalence

This FS can be represented as the FI $\boldsymbol{a}(\lambda) = [a^-(\lambda), a^+(\lambda)]$, where $a^-(\lambda)$ and $a^+(\lambda)$ are gradual numbers that are computed such that

$$\begin{cases} \lambda = \mu_{A^-}(x) = (x-1)/5 \Rightarrow a^-(\lambda) = (\mu_{A^-})^{-1}(\lambda) = 1+5\lambda \\ \lambda = \mu_{A^+}(x) = (11-x)/5 \Rightarrow a^+(\lambda) = (\mu_{A^+})^{-1}(\lambda) = 11-5\lambda \end{cases}$$

II.2. Type-2 fuzzy intervals

Generally, a type-m FS is an FS whose membership values are FSs of type m–1 (m>1). For example, a type-2 FS is an FS whose membership values are type-1 FSs. Similar to type-1 formalism, a type-2 FS is characterized by a type-2 membership function that is represented by two type-1 membership functions: the lower function (inf) and the upper function (sup). In this context, a type-2 FS, which is denoted $\tilde{A}$, is completely defined by these two type-1, FS A^{inf} and A^{sup}, which are defined by their membership functions, $\mu_A^{inf}(x)$ and $\mu_A^{sup}(x)$, and subject to the constraint $\mu_A^{inf}(x) < \mu_A^{sup}(x)$ (refer to Fig. 3.a for a particular case of triangular type-2 FS). Analogously, if the type-1 FS A can be viewed as the type-1 FI $\boldsymbol{a}(\lambda)$ represented by an ordered pair of its profiles, a type-2 FS $\tilde{A}$ can also be represented by the type-2 FI $\tilde{\boldsymbol{a}}(\lambda)$ (refer to Fig. 3.b). The latter is defined by two type-1 FIs—lower $\boldsymbol{a}^{inf}(\lambda)$ and upper $\boldsymbol{a}^{sup}(\lambda)$ FI—with the inclusion constraint $\boldsymbol{a}^{inf}(\lambda) \subseteq \boldsymbol{a}^{sup}(\lambda)$. In this context, a type-2 FI is defined by:

$$\tilde{\boldsymbol{a}}(\lambda) = \{\ \boldsymbol{a}_{inf}(\lambda), \boldsymbol{a}_{sup}(\lambda)\ |\ \boldsymbol{a}_{inf}(\lambda) \subseteq \boldsymbol{a}_{sup}(\lambda)\}$$

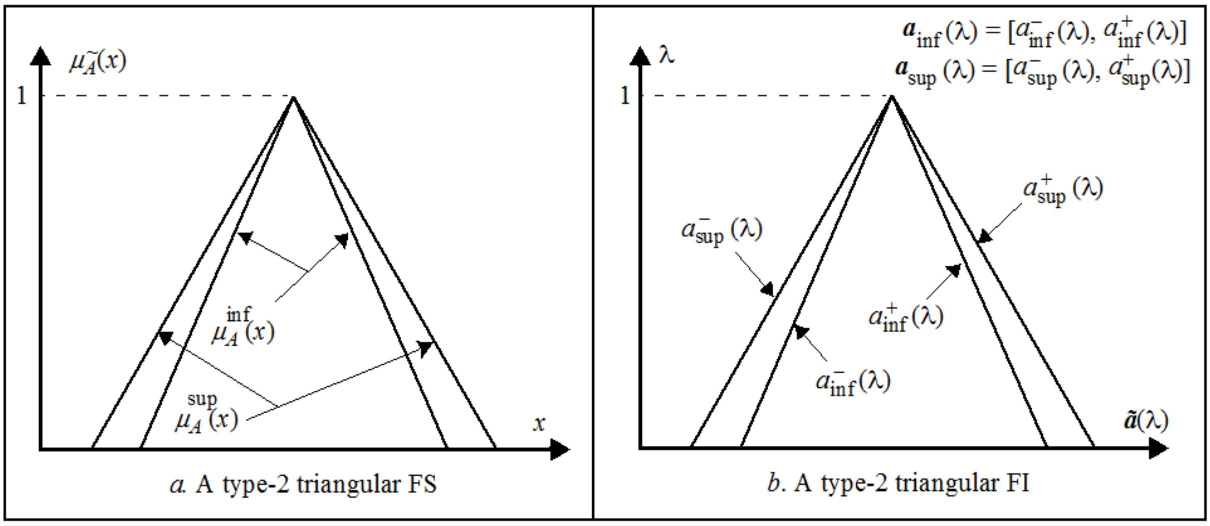

Fig. 3: Type-2 triangular FS and its representation with a type-2 FI

For illustration, Fig. 4.a represents a triangular type-2 FS and its representation as a type-2 FI on the same diagram to highlight the equivalence. Similar to type-1 formalism, the values of x and those attached to $\tilde{\boldsymbol{a}}(\lambda)$ are simultaneously shown on the horizontal axis. Both the degrees of λ and the degrees of belonging to type-2 FS are shown on the vertical axis. An illustration is given in Fig. 4.b.

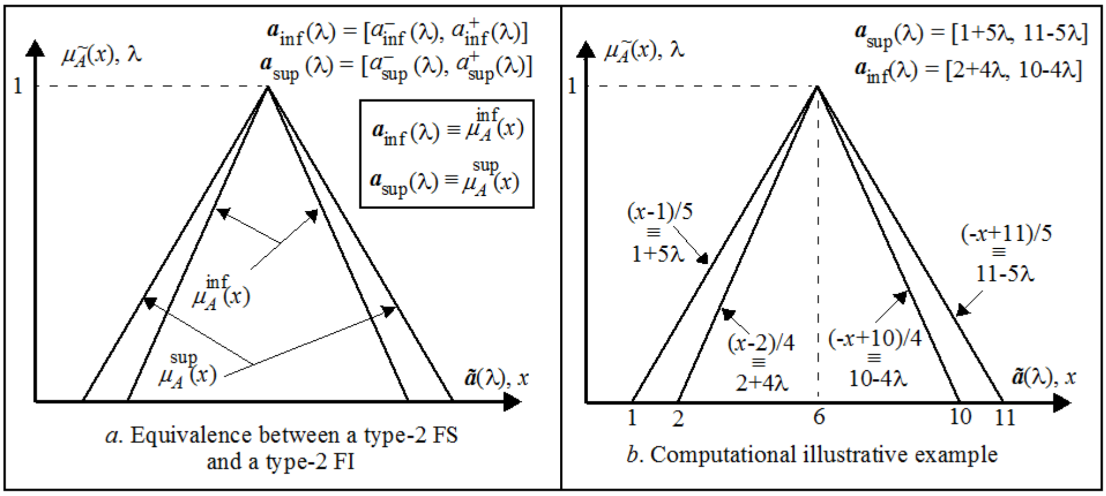

Fig. 4: Type-2 FI representation and equivalence

III. TYPE-2 FUZZY INTERVALS: SEMANTICS AND INTERPRETATIONS

III.1. Type-2 fuzzy intervals: ontic and/or epistemic

According to the meaning attributed to the conventional interval $\boldsymbol{a}$, two different interpretations can be distinguished: ontic and epistemic [9][15][18][28]. This vision can be extended to a type-2 FI interpretation and meaning. A type-2 FI $\tilde{\boldsymbol{a}}(\lambda)$ can be considered to be ontic or epistemic. In an ontic representation, $\tilde{\boldsymbol{a}}(\lambda)$ is considered to be a compact entity. Furthermore, $\tilde{\boldsymbol{a}}(\lambda)$ is viewed as a set of conjunctive type-1 FIs. All type-1 FIs $\boldsymbol{a}(\lambda)$ between the bounds $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)$ are considered to be conjunctive elements (refer to Fig. 5.a). This vision may be feasible in some applications, such as computing with words in imprecise and uncertain environments. In an epistemic interpretation, the type-2 FI $\tilde{\boldsymbol{a}}(\lambda)$ is considered to be a set of disjunctive type-1 FIs $\boldsymbol{a}(\lambda)$ between $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)$ (refer to Fig. 5.b). This vision is better adapted in situations where the exact shape of a type-1 FI may not be easily obtained. The exact FI $\boldsymbol{a}(\lambda)$ is unknown, and only the bounds $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)$, including FI $\boldsymbol{a}(\lambda)$, are available. The manipulated information is uncertain and cannot be precisely revealed by only a unique type-1 FI. This representation, which is extensively exploited in fuzzy literature, is well suited in experimental scenarios, such as modeling, control and decision-making applications. In this paper, the epistemic vision of type-2 FI is adopted.

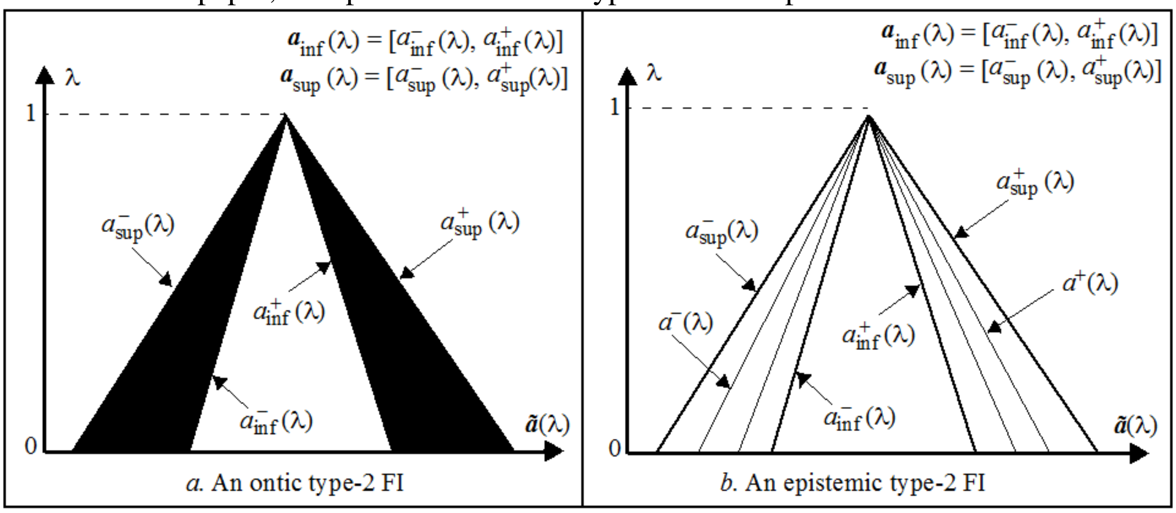

Fig. 5: Ontic and epistemic type-2 FI

III.2. Some semantics of type-2 fuzzy intervals

In the manipulation of type-2 FS (or FI) by the epistemic vision, different semantics can be associated with the interpretation of imprecision and uncertainty according to an interval representation and reasoning. These imprecise and/or uncertain semantics are interrelated.

The first semantic of a type-2 representation is connected with the fuzzy meaning given to the fuzzification procedure, i.e., a domain transformation, where crisp data (inputs) are transformed into fuzzy data. In this context, for a specific value $x = x_0$, unlike the type-1 membership function that yields a crisp membership grade (belonging degree), the type-2 function provides a membership grade that is represented by an interval (refer to Fig. 6). Belonging to the type-2 fuzzy membership function is not a crisp value but rather an interval. The fuzzification operation aims to identify an association between the crisp value $x = x_0$ and a belonging interval given by (refer to Fig. 6):

$$\mu_{\tilde{A}}(x = x_0) = [\mu_A^{\inf}(x_0),\ \mu_A^{\sup}(x_0)] = [\lambda_{\inf}, \lambda_{\sup}] \quad (1)$$

Another interesting semantic mentioned by Mendel in [34][35] consists of using the average values and standard deviations on the two bounds of a type-2 FI. According to this representation philosophy, the nominal (middle) type-1 FI can be considered with its left and right radii, which are interpreted as an upper bound of uncertainty. By the *MR* representation, if additional knowledge is provided for the type-2 FI, e.g., its best estimate is its middle (midpoint), and then the radius can be considered a measure of its dispersion compared with its midpoint (refer to Fig. 6). This vision is considered uncertain and can be justified and motivated by its proximity with a stochastic representation, where Gaussian random variables are assumed. As explained in [1], the Gaussian variable representation and arithmetic based on the mean and standard deviation) resemble the interval representation and arithmetic in the *MR* space.

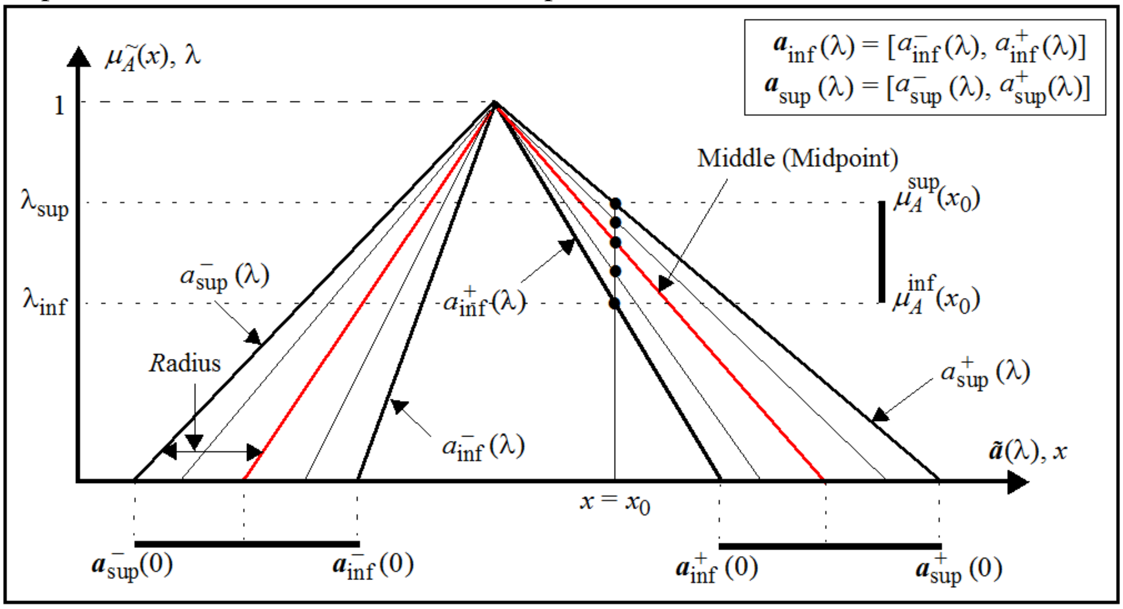

Fig. 6: Semantic interpretation of a type-2 FI

In another register, a third semantic interpretation that is inherent to some decision-making problems can be mentioned. For instance, when assuming a type-2 FI $\tilde{\boldsymbol{a}}(\lambda)$ derived from a compromise between some goals and constraints according to a decision-making strategy, the type-1 FI intervals $\boldsymbol{a}_{\sup}(\lambda)$ and $\boldsymbol{a}_{\inf}(\lambda)$ can be considered the higher and lower bounds of uncertainty in the decision, respectively. This type-2 FI is considered the footprint uncertainty of the decision. In this context, a decision-making strategy can be interpreted as choosing the optimal type-1 FI $\boldsymbol{a}_{cmp}(\lambda)$ between $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)$ according to a decision criterion. This decision-making problem has an epistemic nature, where the objective is to find a type-1 FI among a family of possible FIs, bounded by $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)$. This method is adopted in the follow-up paper.

IV. INTERSECTION OPERATOR BETWEEN FUZZY INTERVALS

This section proposes a method based on interval relations to compute the intersection operation between two type-2 FIs. The principle is provided for conventional intervals and subsequently extended to type-1 FIs and type-2 FIs.

IV.1. Intersection between intervals

In the interval arithmetic context, the intersection between the two intervals $\boldsymbol{a}$ and $\boldsymbol{b}$ is expressed as:

$$\boldsymbol{a} \cap \boldsymbol{b} = [a^-, a^+] \cap [b^-, b^+] = [\max(a^-, b^-), \min(a^+, b^+)] \quad (2)$$

Depending on the relative position of $\boldsymbol{a}$ and $\boldsymbol{b}$, three different cases are discussed: disjoint, inclusion and overlapping (refer to Fig. 7). The case of equality is a particular case of overlapping or inclusion. In this paper, equality is considered to be a special case of inclusion [7].

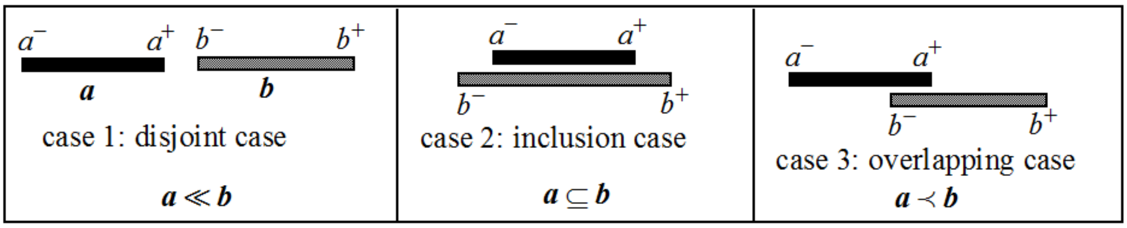

Fig. 7: Three possible cases between two intervals

According to the intersection results between $\boldsymbol{a}$ and $\boldsymbol{b}$, these three cases can be merged into the following two major situations.

• **Situation 1:** The intersection between $\boldsymbol{a}$ and $\boldsymbol{b}$ is empty (case 1).
In this situation, both intervals are disjoint. This disjoint case occurs when

$$\boldsymbol{a} \ll \boldsymbol{b} \Leftrightarrow a^+ < b^- \Leftrightarrow M_a + R_a < M_b - R_b \Leftrightarrow R_a + R_b < M_b - M_a \text{ } (\boldsymbol{a} \text{ is before } \boldsymbol{b}) \quad (3)$$

The disjoint case when $\boldsymbol{b}$ is before $\boldsymbol{a}$ can be easily obtained by the permuted intervals in (3), i.e.,

$$\boldsymbol{a} \gg \boldsymbol{b} \Leftrightarrow b^+ < a^- \Leftrightarrow R_a + R_b < M_a - M_b \quad (4)$$

By unifying conditions (3) and (4), the following Boolean disjoint indicator can be defined:

$$D(\boldsymbol{a},\boldsymbol{b}) = R_a + R_b < |M_b - M_a| \quad (5)$$

If $D(\boldsymbol{a}, \boldsymbol{b}) = 1$, then $\boldsymbol{a} \cap \boldsymbol{b} = \varnothing$. In the opposite case, when $D(\boldsymbol{a}, \boldsymbol{b}) = 0$, the intervals can be in overlapping cases or inclusion cases.

• **Situation 2:** The intersection between $\boldsymbol{a}$ and $\boldsymbol{b}$ is not empty (case 2 and case 3). This situation includes the overlapping case and the inclusion case.

- **Inclusion case**

$$\boldsymbol{a} \subseteq \boldsymbol{b} \Leftrightarrow \begin{cases} b^- \le a^- \\ a^+ \le b^+ \end{cases} \Leftrightarrow \begin{cases} M_b - R_b \le M_a - R_a \\ M_a + R_a \le M_b + R_b \end{cases} \Leftrightarrow \begin{cases} M_b - M_a \le R_b - R_a \\ M_a - M_b \le R_b - R_a \end{cases} \quad (6)$$

Equation (5) can be rewritten as follows:

$$\boldsymbol{a} \subseteq \boldsymbol{b} \Leftrightarrow |M_b - M_a| \le R_b - R_a \quad (7)$$

In this case, $\boldsymbol{a} \cap \boldsymbol{b} = [a^-, a^+]$. Similarly, the relation $\boldsymbol{b} \subseteq \boldsymbol{a}$ is obtained by permitting $\boldsymbol{a}$ and $\boldsymbol{b}$ in (7):

$$\boldsymbol{b} \subseteq \boldsymbol{a} \Leftrightarrow |M_a - M_b| \le R_a - R_b \text{ ; and } \boldsymbol{a} \cap \boldsymbol{b} = [b^-, b^+] \quad (8)$$

By unification of (7) and (8), the Boolean inclusion indicator can be defined as follows:

$$I(\boldsymbol{a},\boldsymbol{b}) = |M_b - M_a| \le |R_b - R_a| \quad (9)$$

The intersection between $\boldsymbol{a}$ and $\boldsymbol{b}$ in the inclusion case is expressed as:

$$\boldsymbol{a} \cap \boldsymbol{b} = \boldsymbol{\varphi}_I(\boldsymbol{a},\boldsymbol{b}) = [\varphi_I^-(\boldsymbol{a},\boldsymbol{b}), \varphi_I^+(\boldsymbol{a},\boldsymbol{b})] \quad (10)$$

where $\begin{cases} \varphi_I^-(\boldsymbol{a},\boldsymbol{b}) = a^- \cdot \gamma_I + b^- \cdot (1-\gamma_I) \\ \varphi_I^+(\boldsymbol{a},\boldsymbol{b}) = a^+ \cdot \gamma_I + b^+ \cdot (1-\gamma_I) \end{cases}$; with: $\begin{cases} \gamma_I = (1 + sign(R_b - R_a))/2 \\ sign(x) = 1 \text{ ; if } x \ge 0 \text{ and } -1 \text{ if } x < 0 \end{cases}$

• **Overlapping case**

The same reasoning presented in the inclusion case can be employed to obtain an overlapping condition. Based on the two previous situations, the case of overlapping can be simply deduced. The settings when $\boldsymbol{a} \prec \boldsymbol{b}$ ($\boldsymbol{a}$ is before $\boldsymbol{b}$) and when $\boldsymbol{a} \succ \boldsymbol{b}$ ($\boldsymbol{a}$ is after $\boldsymbol{b}$) can be defined by the Boolean indicator

$$O(\boldsymbol{a},\boldsymbol{b}) = \neg D(\boldsymbol{a},\boldsymbol{b}) \cdot \neg I(\boldsymbol{a},\boldsymbol{b}) \tag{11}$$

where $\neg$ represents the logical negation operator. Thus, the intersection between $\boldsymbol{a}$ and $\boldsymbol{b}$ is expressed as

$$\boldsymbol{a} \cap \boldsymbol{b} = \begin{cases} [b^-, a^+] \text{ ; if} : M_b > M_a \Leftrightarrow \boldsymbol{a} \prec \boldsymbol{b} \\ [a^-, b^+] \text{ ; if} : M_b < M_a \Leftrightarrow \boldsymbol{a} \succ \boldsymbol{b} \end{cases} \tag{12}$$

The expression (12) can be reformulated by the following expression:

$$\boldsymbol{a} \cap \boldsymbol{b} = \boldsymbol{\varphi}_O(\boldsymbol{a},\boldsymbol{b}) = [\varphi_O^-(\boldsymbol{a},\boldsymbol{b}), \varphi_O^+(\boldsymbol{a},\boldsymbol{b})] \text{ ; where } \begin{cases} \varphi_O^-(\boldsymbol{a},\boldsymbol{b}) = a^- \cdot \gamma_O + b^- \cdot (1-\gamma_O) \\ \varphi_O^+(\boldsymbol{a},\boldsymbol{b}) = a^+ \cdot (1-\gamma_O) + b^+ \cdot \gamma_O \\ \gamma_O = (1 - sign(M_b - M_a))/2 \end{cases} \tag{13}$$

The intersection for the inclusion and overlapping cases is merged into the following expression:

$$\boldsymbol{a} \cap \boldsymbol{b} = I(\boldsymbol{a},\boldsymbol{b}) \cdot \boldsymbol{\varphi}_I(\boldsymbol{a},\boldsymbol{b}) + O(\boldsymbol{a},\boldsymbol{b}) \cdot \boldsymbol{\varphi}_O(\boldsymbol{a},\boldsymbol{b}) \tag{14}$$

The intersection operator between two intervals $\boldsymbol{a}$ and $\boldsymbol{b}$ is expressed as

$$\boldsymbol{a} \cap \boldsymbol{b} = \begin{cases} \varnothing \text{ ;} & \text{if: } D(\boldsymbol{a},\boldsymbol{b}) = 1 \\ I(\boldsymbol{a},\boldsymbol{b}) \cdot \boldsymbol{\varphi}_I(\boldsymbol{a},\boldsymbol{b}) + O(\boldsymbol{a},\boldsymbol{b}) \cdot \boldsymbol{\varphi}_O(\boldsymbol{a},\boldsymbol{b}); & \text{if: } D(\boldsymbol{a},\boldsymbol{b}) = 0 \end{cases} \tag{15}$$

The expressions $D(\boldsymbol{a}, \boldsymbol{b})$, $I(\boldsymbol{a}, \boldsymbol{b})$ and $O(\boldsymbol{a}, \boldsymbol{b})$ are mutually exclusive Boolean indicators.

IV.2. Intersection between type-1 fuzzy intervals

Theoretically, the intersection operation between conventional intervals given in the previous section is directly transposable in the type-1 FI framework. When the two FIs, $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$, are considered, the intersection operation is elaborated by extending the expression (15) to the FI case. This extension generates the following FI expression:

$$\boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = \begin{cases} \varnothing \text{ ;} & \text{if: } D(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)) = 1 \\ I(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)) \cdot \boldsymbol{\varphi}_I(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)) + O(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)) \cdot \boldsymbol{\varphi}_O(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)); & \text{if: } D(\boldsymbol{a}(\lambda),\boldsymbol{b}(\lambda)) = 0 \end{cases} \tag{16}$$

In (16), the indicators D, I and O and $\boldsymbol{\varphi}_I$ and $\boldsymbol{\varphi}_O$ are FI versions of the expressions given by (15). All intervals in (15) are replaced by FI. However, in practical implementations, some differences exist. Unlike intervals where a unique horizontal dimension is employed, an FI is represented according to two dimensions: horizontal and vertical. In this context, specific attention must be given to the points of intersection between the FI profiles. In a simple way, a point of intersection characterizes a cross between two profiles. These break points must be determined beforehand to apply the expression of the intersection operator (16). The reasoning methodology regarding these intersection points is detailed in the following section.

A. Reasoning methodology principle

As in conventional intervals, according to the shapes and relative positions of the two type-1 FIs, two major situations can be distinguished:

• **Situation 1**: The intersection between the type-1 FI is empty.

In this situation, the FIs $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ are totally disjoint ($\boldsymbol{a}(\lambda)$□ $\boldsymbol{b}(\lambda)$ or $\boldsymbol{a}(\lambda)$□ $\boldsymbol{b}(\lambda)$) and $\boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = \varnothing$. Since the handled quantities are FIs, the disjunction condition $D(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda))$ can be limited to the interval supports, i.e., $\boldsymbol{a}(0)$ and $\boldsymbol{b}(0)$. If $D(\boldsymbol{a}(0), \boldsymbol{b}(0)) = 1$, then $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ are totally disjoint and an intersection between profiles is impossible.

• **Situation 2**: The intersection between the type-1 FIs is not empty.

When considering two conventional intervals, only one relation ∈ {disjoint, overlapping, inclusion} can occur. However, several relations may coexist between type-1 FIs according to the points of intersection between the profiles. Let us consider two FIs, $\boldsymbol{a}(\lambda)$, $\lambda \in [0, \lambda_a]$ and $\boldsymbol{b}(\lambda)$, $\lambda \in [0, \lambda_b]$. If the FIs are normalized, we obtain $\lambda_a = \lambda_b = 1$. Let us denote IP_{LL} and IP_{RR} as the intersection points (IPs) between two left (ascending) profiles or two right (descending) profiles, respectively. An intersection point between a left profile and a right profile (or between a right profile and a left profile) is denoted as IP_{RL}.
When crossing λ by starting from 0, at each intersection point, the relation between the FI changes according to a well-defined neighborhood relation. In this context, if a relation is considered a node and an intersection point between two profiles is considered an edge, then the relations between FIs $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ can be interpreted by the graph in Fig. 8.

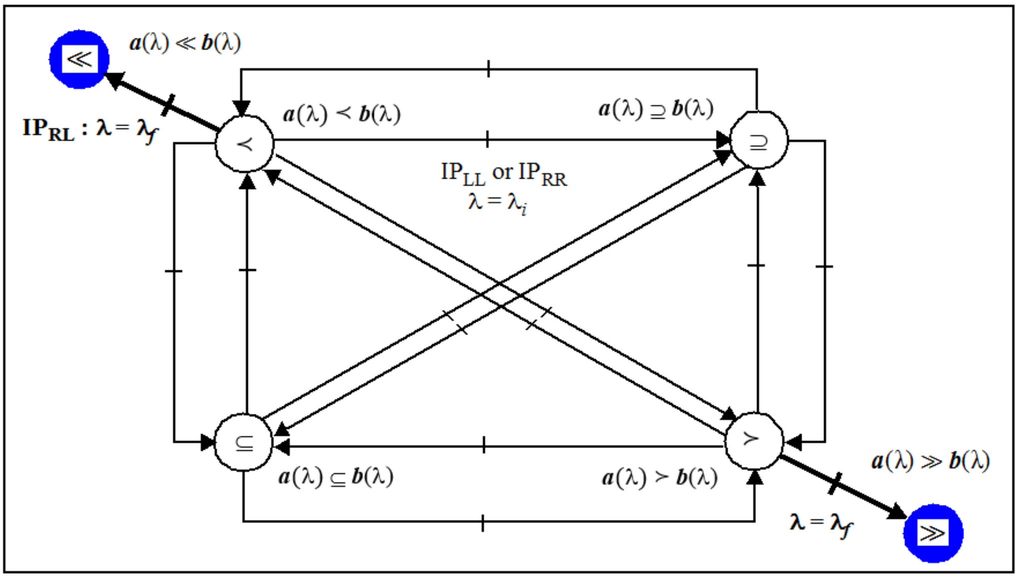

Fig. 8: Graph of relations between two fuzzy intervals

According to the shape of the employed FI, any state ∈ {≺, ≻, ⊆, ⊒} in the graph of Fig. 8 can be considered an initial state. When assuming n intersection points, n ordered λ-values can be identified, i.e., $\lambda_1 < \ldots < \lambda_n$. Each intersection point IP_{LL} or IP_{RR} causes an order relation change, and the realized relation holds until the next intersection point occurs. When the graph consists of a unique node, which is both the initial state and the final state, the relation is always valid for all λ. According to Fig. 8, when the unique intersection point IP_{RL} occurs, a passage toward the disjoint relation is produced. This passage to the relation □ (or □), which can only be derived from relations ≺ or ≻, shows that the intersection between $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ will become equal to ∅. In this situation, this point IP_{RL}, which corresponds to $\lambda = \lambda_f$, is the maximum value of λ for the operation $\boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda)$ (refer to Fig. 9).
When IP_{RL} does not exist (see Fig. 10), the maximum value of λ for the intersection operation is equal to $min(\lambda_a, \lambda_b)$.
For illustration, let us examine the two examples given in Figs. 9–10. The first example corresponds to normalized FI. In the second, the FIs are not normalized. In each situation, the intersection operation is computed according to expression (16). For visibility reasons, the result of the intersection is shown in gray in the figures.
In this example (refer to Fig. 9), three IP_{LL} ($\lambda_1, \lambda_3, \lambda_4$ with $\lambda_1<\lambda_3<\lambda_4$), two IP_{RR} (λ_2, λ_4 with $\lambda_2<\lambda_4$) and one IP_{RL} (λ_f) exist. Starting with λ=0, the initial relation is $\boldsymbol{a}(\lambda) \subseteq \boldsymbol{b}(\lambda)$. This inclusion relation holds until the first intersection point is touched at $\lambda = \lambda_1$, which modified the relation to $\boldsymbol{a}(\lambda) \prec \boldsymbol{b}(\lambda)$. Three additional changes of the relation occur at the values λ_2, λ_3 and λ_4, which sequentially yield the relations $\boldsymbol{a}(\lambda) \sqsupseteq \boldsymbol{b}(\lambda)$, $\boldsymbol{a}(\lambda) \succ \boldsymbol{b}(\lambda)$ and $\boldsymbol{a}(\lambda) \prec \boldsymbol{b}(\lambda)$. The presence of IP_{RL} ($\lambda=\lambda_f$) causes the passage to

the relation $\ll$. In this case, λ_f is the maximum λ value for $\boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda)$. This evolution of relations is described by the state graph in Fig. 11.a, where the initial state is circled twice.

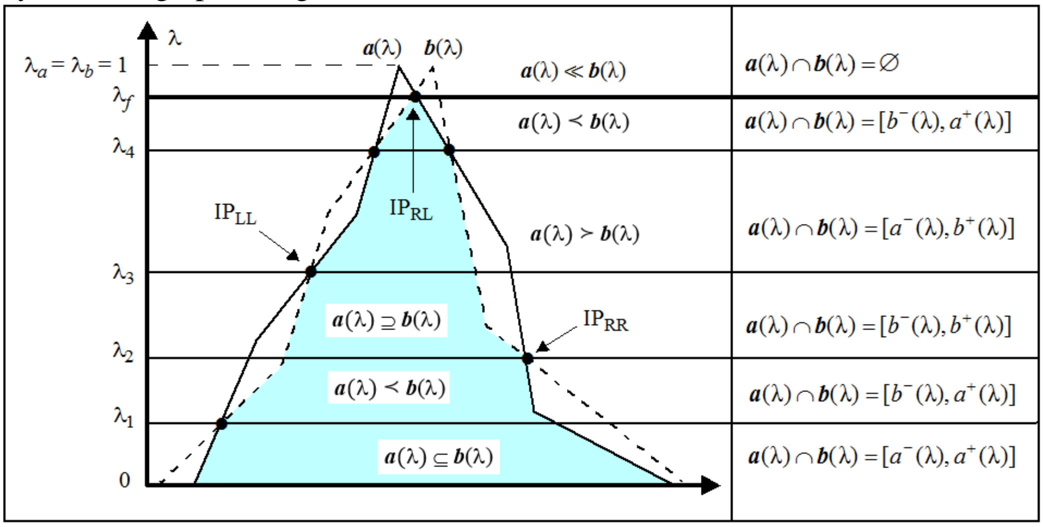

Fig. 9: Intersection operation between two normalized FIs (case 1)

The states represent the relations between $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$, and the transitions refer to λ_i, where an intersection point occurs. The same analysis can be performed for example 2 and includes Fig. 10 and the graph illustrated in Fig. 11.b.

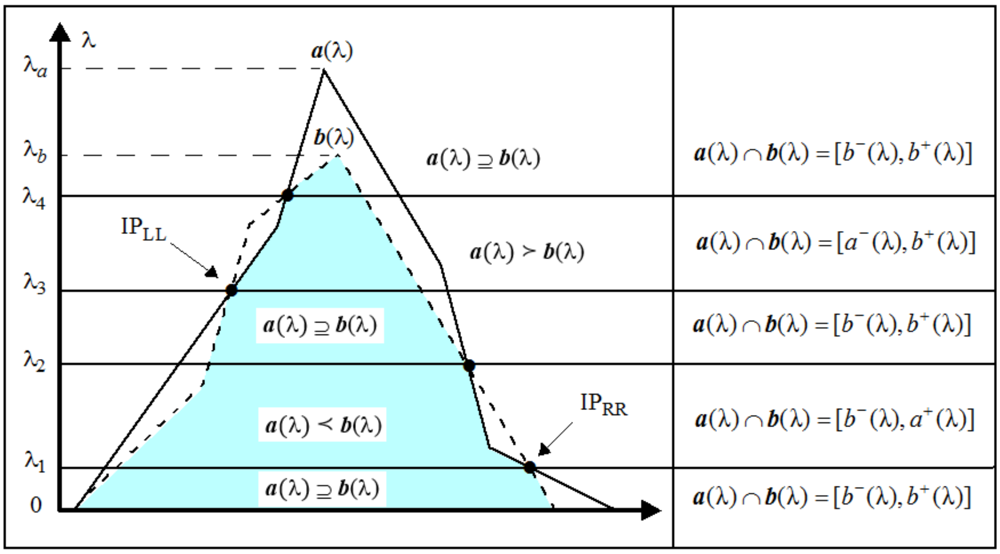

Fig. 10: Intersection operation between two nonnormalized FIs (case 2)

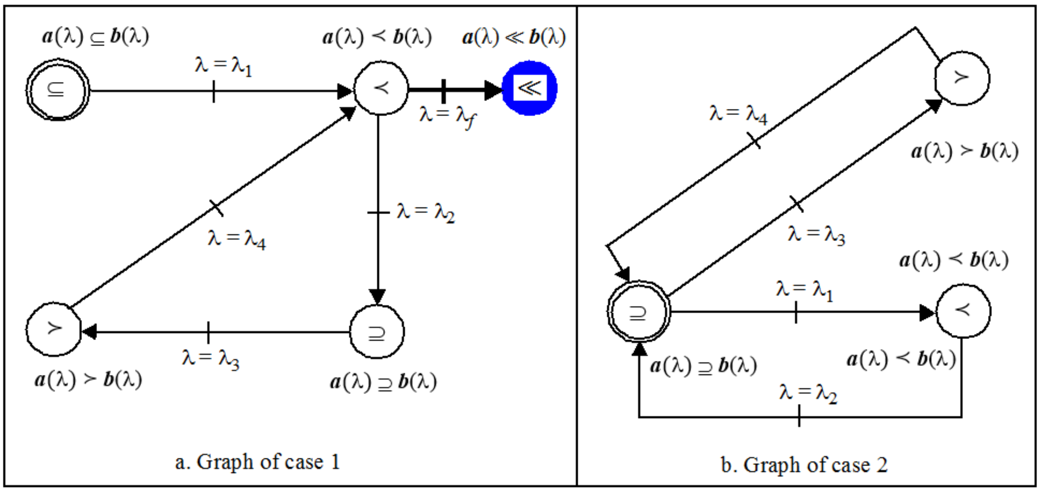

Fig. 11: Relation graph for the intersection operations

B. Numerical example

Let us consider the normalized type-1 FI $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ of Fig. 12.a., i.e.,

$$\boldsymbol{a}(\lambda) = [1+4\lambda, 13-8\lambda]; \text{ and: } \boldsymbol{b}(\lambda) = [2+5\lambda, 11-4\lambda]$$

The intersection points between the profiles are expressed as

$$\text{IP}_{\text{RR}}\text{: } a^{+}(\lambda) = b^{+}(\lambda) \Rightarrow \lambda = \lambda_1 = 0.5 \text{ ; IP}_{\text{RL}}\text{: } a^{+}(\lambda) = b^{-}(\lambda) \Rightarrow \lambda = \lambda_f = 11/13$$

The intersection operation is illustrated in Fig. 12.a. Fig. 12.b shows the graph of relations between $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$, where three different cases are presented. The computational mechanism is as follows:

- Case 1: if $0 \le \lambda \le 0.5$: $\boldsymbol{a}(\lambda) \supseteq \boldsymbol{b}(\lambda)$

$$\begin{cases} I(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = 1; \gamma_I = (1 + sign(R_b - R_a))/2 = 0 \\ \varphi_I^{-}(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = b^{-}(\lambda);\ \varphi_I^{+}(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = b^{+}(\lambda) \end{cases} \Leftrightarrow \boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = [b^{-}(\lambda), b^{+}(\lambda)]$$

- Case 2: if $0.5 < \lambda \le 11/13$: $\boldsymbol{a}(\lambda) \prec \boldsymbol{b}(\lambda)$

$$\begin{cases} O(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = 1; \gamma_O = (1 - sign(M_b - M_a))/2 = 0 \\ \varphi_O^{-}(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = b^{-}(\lambda);\ \varphi_O^{+}(\boldsymbol{a}(\lambda), \boldsymbol{b}(\lambda)) = a^{+}(\lambda) \end{cases} \Leftrightarrow \boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = [b^{-}(\lambda), a^{+}(\lambda)]$$

- Case 3: if $11/13 < \lambda \le 1$: $\boldsymbol{a}(\lambda) \ll \boldsymbol{b}(\lambda)$

$$D(\boldsymbol{a}, \boldsymbol{b}) = 1;\ \ \boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = \varnothing$$

For brevity of notation, the intersection operation between $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ is expressed as

$$\boldsymbol{a}(\lambda) \cap \boldsymbol{b}(\lambda) = \begin{cases} [2+5\lambda, 11-4\lambda] \text{ ; } & \text{if: } 0 \le \lambda \le 0.5 \\ [2+5\lambda, 13-8\lambda]; & \text{if: } 0.5 < \lambda \le 11/13 \end{cases}$$

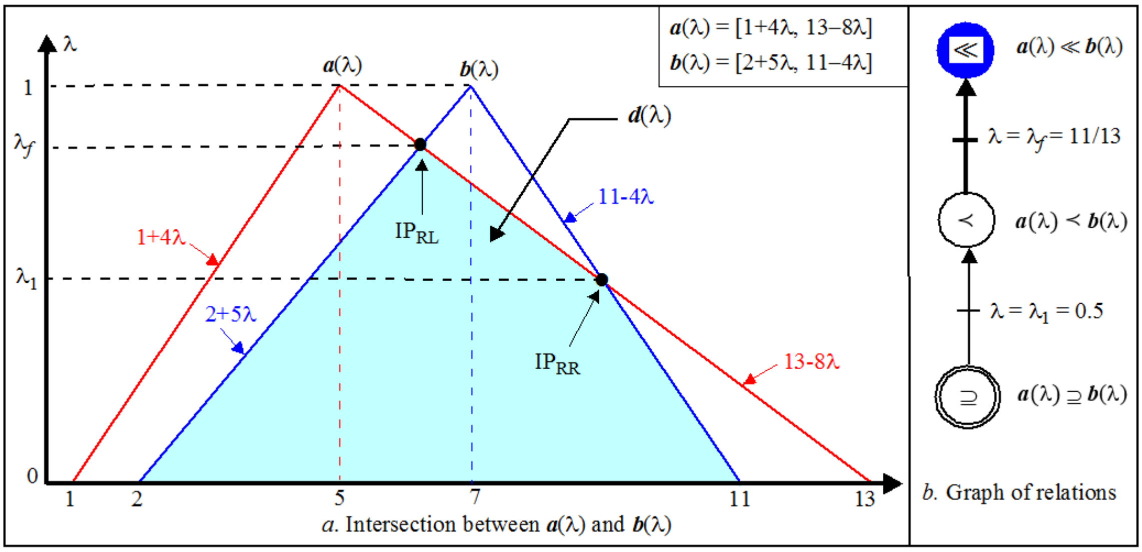

Fig. 12: Intersection between two type-1 FI $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$

C. Illustrative example

In a particular situation, when only linear triangular type-1 FIs are considered, a cartography of all possible relations between FIs is obtained. This cartography is divided into four categories according to whether the number of intersection points between profiles is 0, 1, 2 or 3. In each category (refer to Figs. 13–16) the relations between intervals and the obtained intersection area are provided.

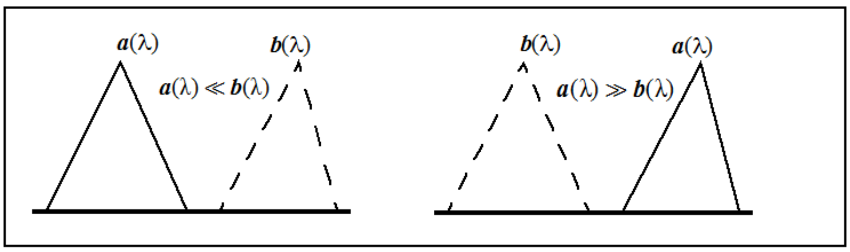

Fig. 13: Category 0 with no intersection point between FI profiles

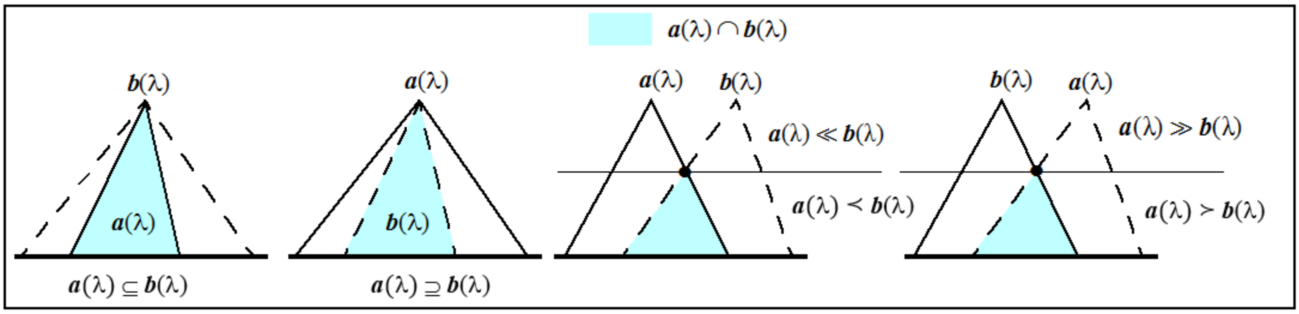

Fig. 14: Category 1: one intersection point between FI profiles

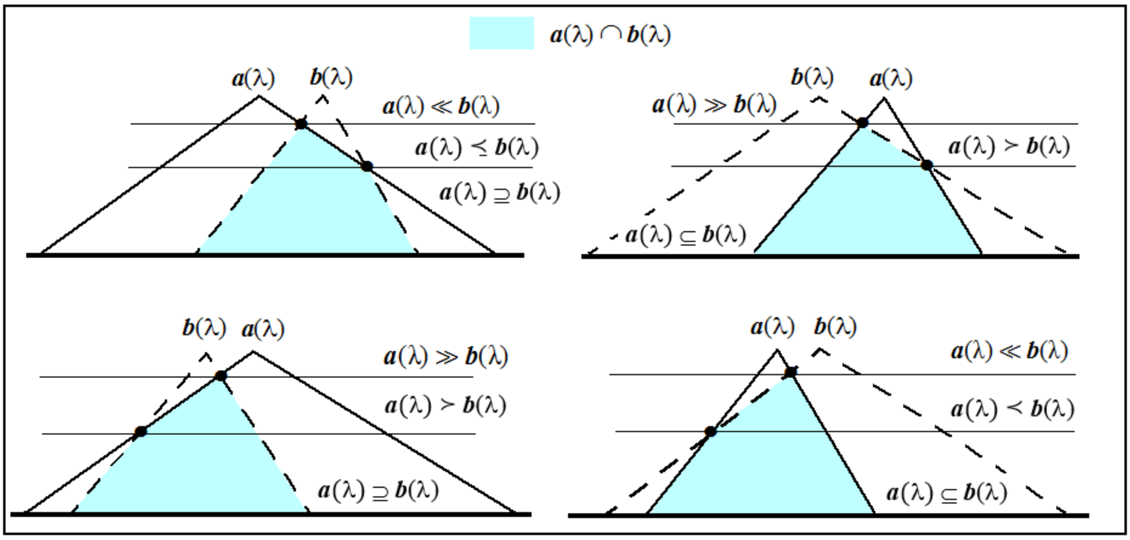

Fig. 15: Category 2: two intersection points between FI profiles

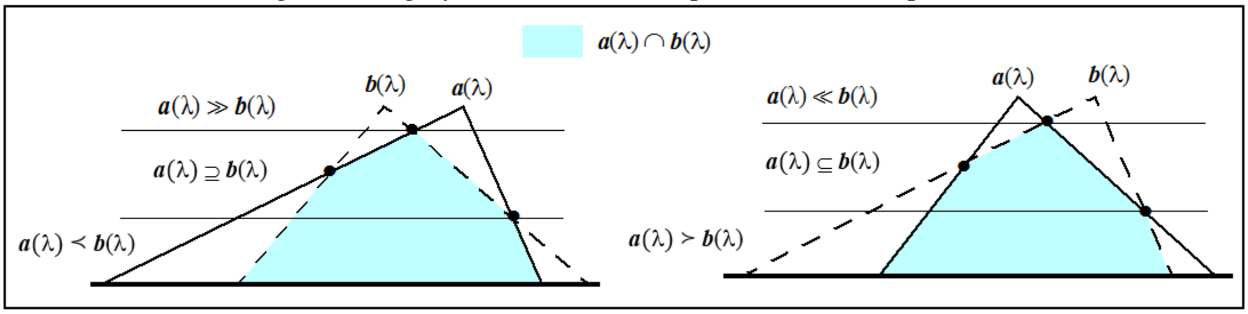

Fig. 16: Category 3: three intersection points between FI profiles

IV.3. Intersection between type-2 fuzzy intervals

A. Methodology principle

Let us consider two type-2 FIs, $\tilde{\boldsymbol{a}}(\lambda) = \{\boldsymbol{a}_{\inf}(\lambda), \boldsymbol{a}_{\sup}(\lambda) \mid \boldsymbol{a}_{\inf}(\lambda) \subseteq \boldsymbol{a}_{\sup}(\lambda)\}$ and $\tilde{\boldsymbol{b}}(\lambda) = \{\boldsymbol{b}_{\inf}(\lambda), \boldsymbol{b}_{\sup}(\lambda) \mid \boldsymbol{b}_{\inf}(\lambda) \subseteq \boldsymbol{b}_{\sup}(\lambda)\}$. The intersection between type-2 FI $\tilde{\boldsymbol{a}}(\lambda)$ and type-2 FI $\tilde{\boldsymbol{b}}(\lambda)$ is defined by the following expression:

$$\tilde{\boldsymbol{a}}(\lambda) \cap \tilde{\boldsymbol{b}}(\lambda) = \{\boldsymbol{a}_{\inf}(\lambda) \cap \boldsymbol{b}_{\inf}(\lambda), \boldsymbol{a}_{\sup}(\lambda) \cap \boldsymbol{b}_{\sup}(\lambda) \mid \boldsymbol{a}_{\inf}(\lambda) \cap \boldsymbol{b}_{\inf}(\lambda) \subseteq \boldsymbol{a}_{\sup}(\lambda) \cap \boldsymbol{b}_{\sup}(\lambda)\}$$

Because a type-2 FI is considered to be the concatenation of two type-1 FI bounds with the inclusion constraint, the same methodology that was previously presented for type-1 FIs can be used to separately compute the two intersection quantities $\boldsymbol{a}_{\inf}(\lambda) \cap \boldsymbol{b}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda) \cap \boldsymbol{b}_{\sup}(\lambda)$. The principle of the intersection between two type-2 FIs is illustrated in Figs. 17–18.

For instance, Fig. 17 separately illustrates this computational mechanism for the lower and upper type-1 FIs. Consequently, $\tilde{\boldsymbol{a}}(\lambda) \cap \tilde{\boldsymbol{b}}(\lambda)$ is depicted in Fig. 18. In the presence of several type-1 FIs, note that the intersection operator is associative. Knowing that the manipulated FI is unimodal and piecewise linear, the intersection operation is also unimodal and piecewise linear.

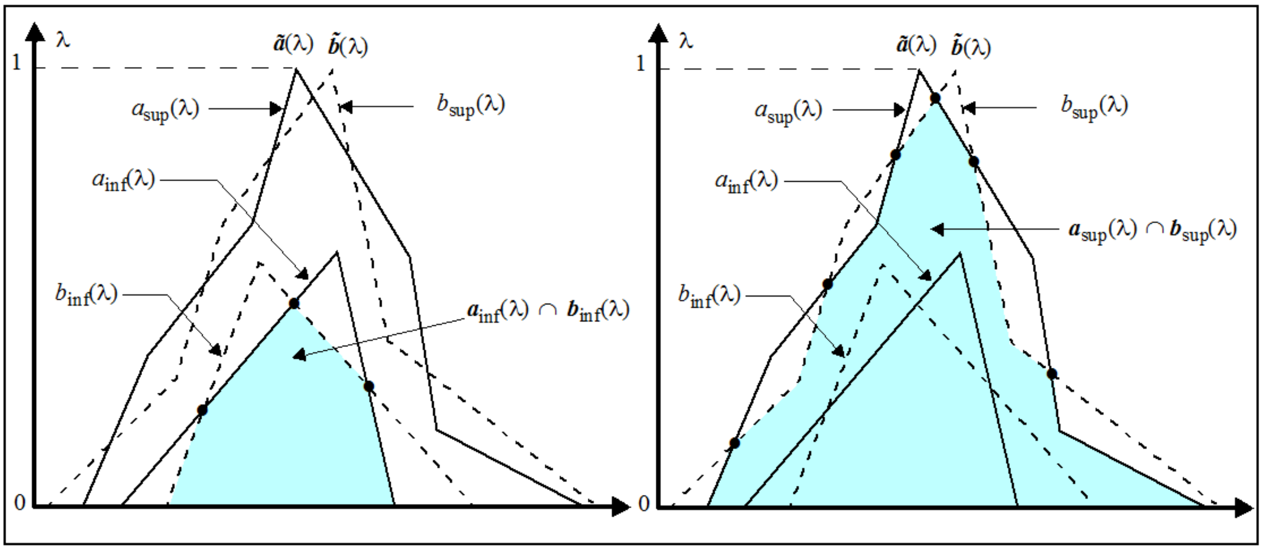

Fig. 17: Intersection type-1 FIs $\boldsymbol{a}_{\inf}(\lambda)\cap\boldsymbol{b}_{\inf}(\lambda)$ and $\boldsymbol{a}_{\sup}(\lambda)\cap\boldsymbol{b}_{\sup}(\lambda)$

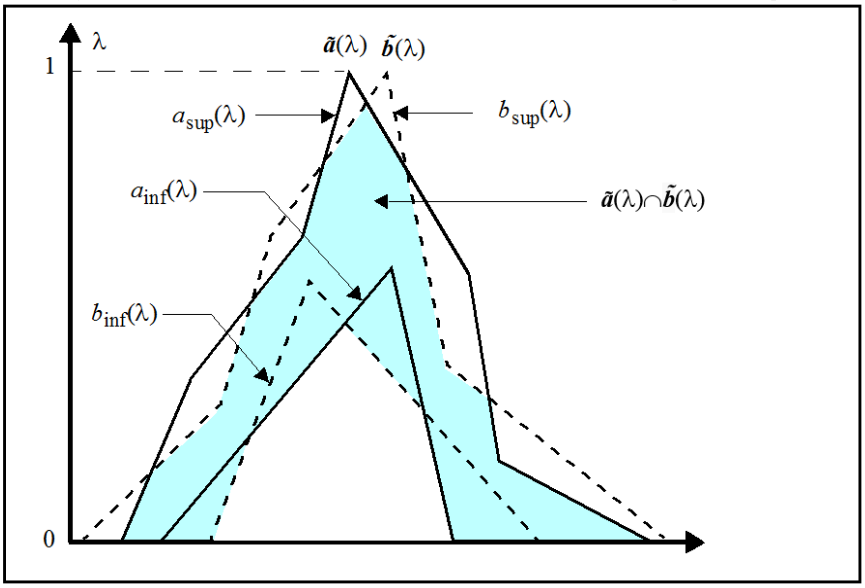

Fig. 18: Intersection operation result between two type-2 FIs

B. Numerical example

Let us consider the type-2 FIs $\tilde{\boldsymbol{a}}(\lambda)$ and $\tilde{\boldsymbol{b}}(\lambda)$ of Fig. 19, i.e.,

$$\tilde{\boldsymbol{a}}(\lambda)=\{\boldsymbol{a}_{\inf}(\lambda),\boldsymbol{a}_{\sup}(\lambda)\};\text{ where:}\begin{cases}\boldsymbol{a}_{\inf}(\lambda)=[3+3\lambda,10-6\lambda];\ \lambda\in[0,7/9]\\ \boldsymbol{a}_{\sup}(\lambda)=[1+4\lambda,13-8\lambda];\ \lambda\in[0,1]\end{cases};$$

$$\text{and }\tilde{\boldsymbol{b}}(\lambda)=\{\boldsymbol{b}_{\inf}(\lambda),\boldsymbol{b}_{\sup}(\lambda)\};\text{ where:}\begin{cases}\boldsymbol{b}_{\inf}(\lambda)=[5+4\lambda,10-4\lambda];\ \lambda\in[0,5/8]\\ \boldsymbol{a}_{\sup}(\lambda)=[2+5\lambda,11-4\lambda];\ \lambda\in[0,1]\end{cases}$$

The intersection points between the type-1 FI profiles of $\boldsymbol{a}_{\sup}(\lambda)$ and $\boldsymbol{b}_{\sup}(\lambda)$ are the same as those provided in section 2.B, i.e.,

$$\text{IP}_{\text{RR}}\text{: } a^{+}_{\sup}(\lambda)=b^{+}_{\sup}(\lambda)\Rightarrow\lambda=\lambda_{1}^{\sup}=0.5\text{; and: IP}_{\text{RL}}\text{: } a^{+}_{\sup}(\lambda)=b^{-}_{\sup}(\lambda)\Rightarrow\lambda=\lambda_{f}^{\sup}=11/13$$

In the same way, the intersection point between the type-1 FI profiles of $\boldsymbol{a}_{\inf}(\lambda)$ and $\boldsymbol{b}_{\inf}(\lambda)$ is

$$\text{IP}_{\text{RL}}\text{: } a^{+}_{\inf}(\lambda)=b^{-}_{\inf}(\lambda)\Rightarrow\lambda=\lambda_{f}^{\inf}=0.5$$

By adopting the computational mechanism detailed in section IV.2 for type-1 FIs (refer to Fig. 19), the type-2 FI from the intersection operation is expressed as

$$\tilde{\boldsymbol{d}}(\lambda)=\tilde{\boldsymbol{a}}(\lambda)\cap\tilde{\boldsymbol{b}}(\lambda)=\{\boldsymbol{d}_{\inf}(\lambda),\boldsymbol{d}_{\sup}(\lambda)\}\text{, where:}$$

$$d_{\inf}(\lambda) = a_{\inf}(\lambda) \cap b_{\inf}(\lambda) = [5+4\lambda, 10-6\lambda] \;;\; \text{if: } 0 \le \lambda \le 0.5$$

$$\text{and } d_{\sup}(\lambda) = a_{\sup}(\lambda) \cap b_{\sup}(\lambda) = \begin{cases} [2+5\lambda, 11-4\lambda] \;; & \text{if: } 0 \le \lambda \le 0.5 \\ [2+5\lambda, 13-8\lambda]; & \text{if: } 0.5 < \lambda \le 11/13 \end{cases}$$

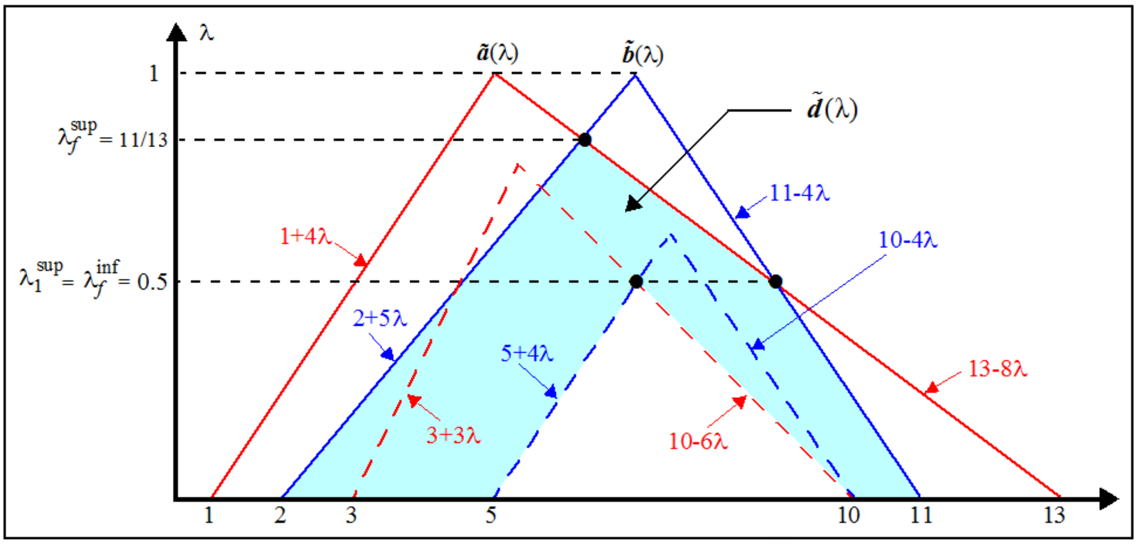

Fig. 19: Intersection between $\tilde{a}(\lambda)$ and $\tilde{b}(\lambda)$

V. TYPE-1 AND TYPE-2 FUZZY INTERVAL DECISION-MAKING METHODOLOGY

V.1. Type-1 decision-making strategy

The fuzzy decision-making principle proposed by Bellman and Zadeh [3], which is the basis of the fuzzy optimization, defines the fuzzy decision as a confluence of fuzzy goals and fuzzy constraints. Let us consider a set of type-1 FIs, which is considered to be the objectives (goals):

$$G = \{\boldsymbol{g_i}(\lambda) = [g_i^-(\lambda), g_i^+(\lambda)] \mid g_i^-(\lambda) \le g_i^+(\lambda), i = 0, \ldots, m\}$$

and a set of type-1 FI constraints:

$$C = \{\boldsymbol{c_i}(\lambda) = [c_i^-(\lambda), c_i^+(\lambda)] \mid c_i^-(\lambda) \le c_i^+(\lambda), i = 0, \ldots, n\}$$

By this decision-making formalism, the goals and constraints are represented and processed in the same way. From a practical and implementation point of view, the goals and constraints do not have to be distinguished. This method remains valid for handling decision methods using only fuzzy goals or fuzzy constraints. The type-1 FI $\boldsymbol{d}(\lambda)$, which represents the domain of the decision, is expressed by the intersection of goals and constraints and is defined as follows:

$$\boldsymbol{d}(\lambda) = \boldsymbol{g}_1(\lambda) \cap \ldots \cap \boldsymbol{g}_m(\lambda) \cap \boldsymbol{c}_1(\lambda) \cap \ldots \cap \boldsymbol{c}_n(\lambda) \quad (17)$$

In the decision domain $\boldsymbol{d}(\lambda)$, the values of λ quantify the degree of utility for the different decision options. The type-1 FI $\boldsymbol{d}(\lambda)$ is computed using the formalism detailed in section IV. According to $\boldsymbol{d}(\lambda)$, the Bellman-Zadeh decision-making methodology is given by the FI formalism, i.e.,

$$\lambda_* = \arg\max_{\lambda} \boldsymbol{d}(\lambda) = \arg\max_{\lambda} \{\boldsymbol{g}_1(\lambda) \cap \ldots \cap \boldsymbol{g}_m(\lambda) \cap \boldsymbol{c}_1(\lambda) \cap \ldots \cap \boldsymbol{c}_n(\lambda)\} \quad (18)$$

In (18), λ_* is a unique and crisp value that represents the λ value that corresponds to the decision, which reflects the optimal fulfillment degree of the confluence between fuzzy goals and constraints (inducing an optimal utility value). This method is equivalent to that developed with membership-based formalism. FI $\boldsymbol{d}(\lambda)$ can be represented by an FS D with a membership function $\mu_D(x)$. In this case, the optimal value λ_* of λ on the vertical dimension corresponds to the optimal value x_* of x on the horizontal dimension. Knowing the value of λ_*, the value of x_* can be directly deduced by the profile expressions. Due to the intersection computational mechanism of section IV, when considering two unimodal type-1 FIs, the value of λ_* is determined according to the following simple statements:

- If the intersection point IP_{RL} exists, then the optimal value $\lambda_* = \lambda_f$.
- If the intersection point IP_{RL} does not exist, then the optimal value $\lambda_* = min(\lambda_a, \lambda_b)$.

Due to the commutative and associative properties of the intersection operator, in the presence of several FIs, the computational mechanism is performed by pairs of FIs.

Let us reconsider the example in section IV.2.B. The FIs $\boldsymbol{a}(\lambda)$ and $\boldsymbol{b}(\lambda)$ are interpreted as a goal and a constraint. According to the expression of $\boldsymbol{d}(\lambda)$ (refer to section IV.2.b and Fig. 20), the optimal value $\lambda_* = \lambda_f = 11/13$. As illustrated in Fig. 20, the membership function μ_D, as deduced by a profile inversion procedure, is expressed as

$$\begin{cases} \mu_{D^-}(x) = (x-2)/5; & \text{if: } 2 \le x \le 81/13 \\ \mu_{D^+}(x) = \begin{cases} (13-x)/8; & \text{if: } 81/13 \le x \le 9 \\ (11-x)/4; & \text{if: } 9 \le x \le 11 \end{cases} \end{cases}$$

and the optimal value x_* is $x_* = 13 - 8\cdot\lambda_* = 2 + 5\cdot\lambda_* = 81/13 = 6.23$.

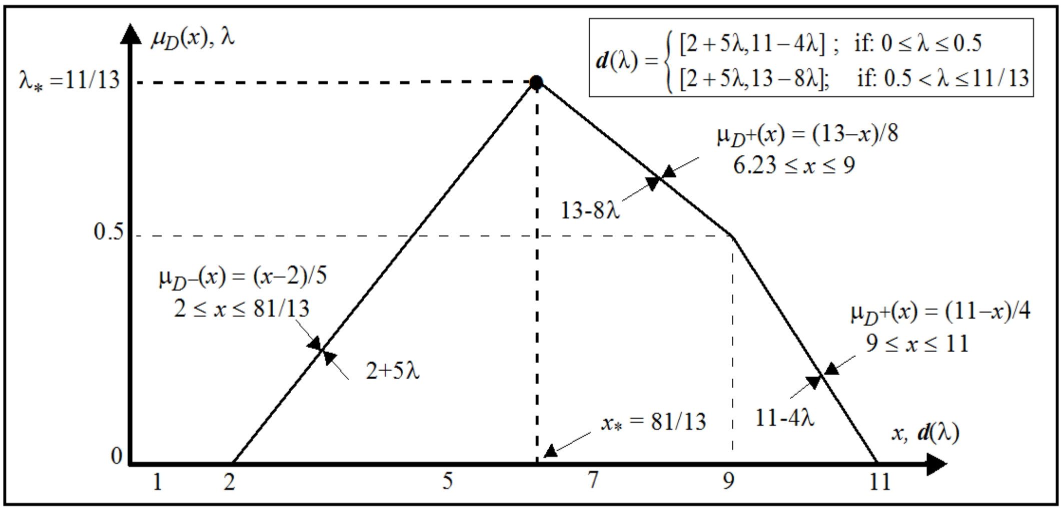

Fig. 20: Example of the optimal decision-making solution using type-1 FI

V.2. Type-2 decision-making strategy

V.2.1. Methodology principle

Due to the presence of uncertainty in the type-1 FI goal and constraint representation, type-2 FIs are considered. The utility represented by T1FIs is subject to uncertainty and considered to be type-2 FIs. Let us consider a set of goals represented by

$$\tilde{\boldsymbol{G}} = \{\tilde{\boldsymbol{g}}_i(\lambda) = \{\boldsymbol{g}^i_{\inf}(\lambda), \boldsymbol{g}^i_{\sup}(\lambda) \mid \boldsymbol{g}^i_{\inf}(\lambda) \subseteq \boldsymbol{g}^i_{\sup}(\lambda)\}, i = 0, \ldots, m\}$$

and a set of type-2 FI constraints:

$$\tilde{\boldsymbol{C}} = \{\tilde{\boldsymbol{c}}_i(\lambda) = \{\boldsymbol{c}^i_{\inf}(\lambda), \boldsymbol{c}^i_{\sup}(\lambda) \mid \boldsymbol{c}^i_{\inf}(\lambda) \subseteq \boldsymbol{c}^i_{\sup}(\lambda)\}, i = 0, \ldots, n\}$$

Let us assume that the lower (inf) type-1 FI bounds of goals and constraints and the upper (sup) type-1 FI bounds of goals and constraints represent the worst case (pessimistic) and the best case (optimistic), respectively. This choice is natural and corresponds to a situation that is typically exploited in uncertain decision problems. In some practical situations, according to corresponding utility, the meaning given to the bounds can be swapped; i.e., the lower bound becomes the best case, and the upper bound becomes the worst case.

The uncertain decision domain $\tilde{\boldsymbol{d}}(\lambda)$ defined by the intersection of the type-2 goals and constraints is defined as follows:

$$\tilde{\boldsymbol{d}}(\lambda) = \tilde{\boldsymbol{g}}_1(\lambda) \cap \ldots \cap \tilde{\boldsymbol{g}}_m(\lambda) \cap \tilde{\boldsymbol{c}}_1(\lambda) \cap \ldots \cap \tilde{\boldsymbol{c}}_n(\lambda) = \{\boldsymbol{d}_{\inf}(\lambda), \boldsymbol{d}_{\sup}(\lambda) \mid \boldsymbol{d}_{\inf}(\lambda) \subseteq \boldsymbol{d}_{\sup}(\lambda)\} \quad (19)$$

According to (19), two extreme situations can be distinguished. The first situation, which corresponds to the optimistic case, is expressed as follows:

$$\boldsymbol{d}_{\sup}(\lambda)=[d^-_{\sup}(\lambda),d^+_{\sup}(\lambda)]=\boldsymbol{g}^1_{\sup}(\lambda)\cap\ldots\cap\boldsymbol{g}^m_{\sup}(\lambda)\cap\boldsymbol{c}^1_{\sup}(\lambda)\cap\ldots\cap\tilde{\boldsymbol{c}}^n_{\sup}(\lambda) \quad (20)$$

The second situation, which refers to the pessimistic case, is expressed as follows:

$$\boldsymbol{d}_{\inf}(\lambda)=[d^-_{\inf}(\lambda),d^+_{\inf}(\lambda)]=\boldsymbol{g}^1_{\inf}(\lambda)\cap\ldots\cap\boldsymbol{g}^m_{\inf}(\lambda)\cap\boldsymbol{c}^1_{\inf}(\lambda)\cap\ldots\cap\boldsymbol{c}^n_{\inf}(\lambda) \quad (21)$$

Once the decision domain $\tilde{\boldsymbol{d}}(\lambda)$ is determined, an optimal FI solution in $\tilde{\boldsymbol{d}}(\lambda)$ can be selected with regard to a decision criterion.

In the approach proposed in [42], the goals and constraints are represented by type-2 membership functions. Their computations are generally performed using one of the two approaches introduced in the literature: the α-cut approach and the extension principle approach using different *t*-norms. While the extension principle can produce NP-Hard computations, approximation *via* α-cuts is relatively time-consuming. Conceptually, the proposed computational method differs from existing methods and enables the extension of interval arithmetic and reasoning in type-2 FI approaches while avoiding the discretization procedure, which is necessary for implementing the α-cuts principle. Unlike the approach proposed in [42], our method is not numerically limited to computing crisp optimal solutions, and it can analytically express all possible decision domains (all type-1 FI solutions). This property contributes to the applicability of our approach in uncertain decision-making methods and offers flexibility in the management of uncertainty.

In this context, if a risk coefficient $\beta\in[0,1]$ is specified by the decision makers, the compromise solution $\boldsymbol{d}_\beta(\lambda)$ between $\boldsymbol{d}_{\sup}(\lambda)$ and $\boldsymbol{d}_{\inf}(\lambda)$ can be chosen, i.e.,

$$\boldsymbol{d}_{\inf}(\lambda)\subseteq\boldsymbol{d}_\beta(\lambda)\subseteq\boldsymbol{d}_{\sup}(\lambda)$$

For instance, as detailed in [42], a linear combination between the pessimistic situation and the optimistic situation can be applied. Thus, the solution $\boldsymbol{d}_\beta(\lambda)$ is expressed as

$$\boldsymbol{d}_\beta(\lambda)=\beta\cdot\boldsymbol{d}_{\sup}(\lambda)+(1-\beta)\cdot\boldsymbol{d}_{\inf}(\lambda) \quad (22)$$

According to (22), when the risk level $\beta = 0$, $\boldsymbol{d}_\beta(\lambda)$ corresponds to the pessimistic decision $\boldsymbol{d}_{\inf}(\lambda)$. This situation is considered to be the worst (unfavorable) situation; however, the risk to be taken for its realization is the lowest risk. In the opposite case, when the risk level $\beta = 1$, $\boldsymbol{d}_\beta(\lambda)$ matches the optimistic (best) decision $\boldsymbol{d}_{\sup}(\lambda)$. This case is the most favorable case; however, the risk to be taken for its realization is larger. For any comprise risk level value $\beta\in[0,1]$, the type-1 FI $\boldsymbol{d}_\beta(\lambda)$ refers to a compromise risk situation. Fig. 21 shows this principle for a given decision domain, which has been previously determined. For instance, if $\beta = 0.5$, the middle situation is obtained, i.e.,

$$\boldsymbol{d}_{0.5}(\lambda)=0.5\cdot\boldsymbol{d}_{\sup}(\lambda)+0.5\cdot\boldsymbol{d}_{\inf}(\lambda)=(\boldsymbol{d}_{\sup}(\lambda)+\boldsymbol{d}_{\inf}(\lambda))/2$$

As detailed in section V.1., the Bellman-Zadeh principle can be applied to the type-1 FI $\boldsymbol{d}_\beta(\lambda)$ and yields the λ-optimal value, i.e.,

$$\lambda^\beta_*=\arg\max\boldsymbol{d}_\beta(\lambda)$$

This optimal decision-making solution is the maximum λ value of $\boldsymbol{d}_\beta(\lambda)$. Due to the convexity property of the FI, the optimal λ solution λ^β_* is always between the pessimistic value and the optimistic optimal value, i.e., $\lambda^{\inf}_*$ and $\lambda^{\sup}_*$, which are obtained by the application of the Bellman-Zadeh principle for the type-1 FI bounds $\boldsymbol{d}_{\inf}(\lambda)$ and $\boldsymbol{d}_{\sup}(\lambda)$, i.e.,

$$\lambda^\beta_*\in[\lambda^{\inf}_*,\lambda^{\sup}_*];\ \text{with: }\lambda^{\inf}_*=\arg\max\boldsymbol{d}_{\inf}(\lambda);\ \text{and: }\lambda^{\sup}_*=\arg\max\boldsymbol{d}_{\sup}(\lambda) \quad (23)$$

Once the optimal value λ^β_* is determined (on the vertical dimension), the optimal solution x^β_* for the referential X (on the horizontal dimension) can be deduced via the profiles (gradual number of bounds). To enable the solution interpretation on the vertical and horizontal dimensions, the optimal solution of the decision-making problem is represented by the couple $(x^\beta_*,\lambda^\beta_*)$ for the given risk level β.

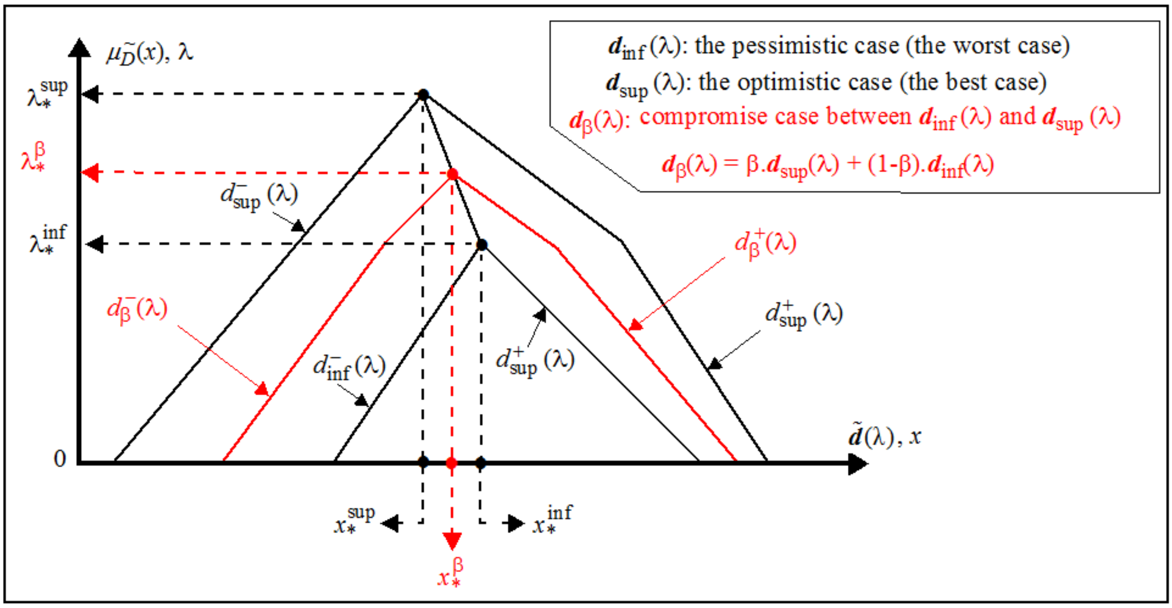

Fig. 21: Optimal behavior in decision-making according to risk level β

In this framework, knowing that FIs are unimodal and piecewise linear, according to the shape of the type-2 decision domain $\tilde{d}(\lambda)$, three cases can be distinguished (refer to Fig. 22).

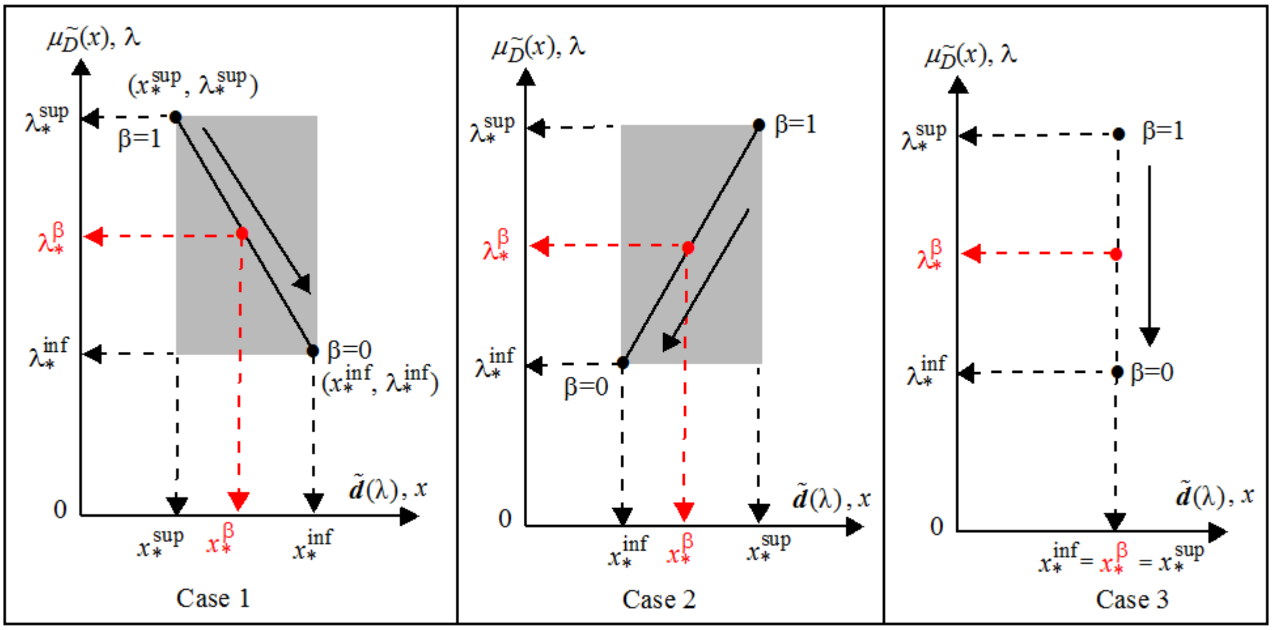

Fig. 22: Three configurations of the optimal solutions according to the shape of the decision domain

The first case occurs when the modal value of $\boldsymbol{d}_{sup}(\lambda)$ is before the modal value of $\boldsymbol{d}_{inf}(\lambda)$, i.e., x_*^{sup} is before x_*^{inf}. Reciprocally, case 2 occurs when the modal value of $\boldsymbol{d}_{sup}(\lambda)$ is after the modal value of $\boldsymbol{d}_{inf}(\lambda)$, i.e., x_*^{sup} is after x_*^{inf}. If these modal values are vertically aligned, case 3 occurs. In the latter case, regardless of the value of β and consequently for λ_*^{β}, the solution x_*^{β} remains unchanged. According to the risk level β, the evolution of the solution is restricted to a linear relationship (diagonal line) between the extreme optimal solutions $(x_*^{sup}, \lambda_*^{sup})$ and $(x_*^{inf}, \lambda_*^{inf})$. In this case, the decision solution is expressed as

$$\begin{pmatrix} x_*^{\beta} \\ \lambda_*^{\beta} \end{pmatrix} = \beta \cdot \begin{pmatrix} x_*^{sup} \\ \lambda_*^{sup} \end{pmatrix} + (1-\beta) \cdot \begin{pmatrix} x_*^{inf} \\ \lambda_*^{inf} \end{pmatrix} \tag{24}$$

This methodology enables decision makers to manage their choices according to a specified risk level.

If β=0, then $\lambda_*^\beta = \lambda_*^{inf}$ and $x_*^\beta = x_*^{inf}$. If β=1, then $\lambda_*^\beta = \lambda_*^{sup}$ and $x_*^\beta = x_*^{inf}$. For any value of β, the optimal solution of $(\lambda_*^\beta, x_*^\beta)$ is diagonally chosen between the solutions that correspond to β=0 and β=1. According to Fig. 22, the following expression can be deduced:

$$\lambda_*^\beta \in [\lambda_*^{inf}, \lambda_*^{sup}] \Rightarrow x_*^\beta \in [\min(x_*^{inf}, x_*^{sup}), \max(x_*^{inf}, x_*^{sup})]$$

This finding is in accordance with the result reported in [42], where membership formalism was applied. When β=0.5, the midpoints of the intervals $[\lambda_*^{inf}, \lambda_*^{sup}]$ and $[\min(x_*^{inf}, x_*^{sup}), \max(x_*^{inf}, x_*^{sup})]$ are obtained. Thus, if additional knowledge is provided by the decision makers, e.g., this midpoint solution is the best confluence between the goals and constraints, then the interval radius can be considered to be a measure of dispersion compared with the midpoint solution. For instance, if $\lambda_*^{0.5}$ (midpoint of $[\lambda_*^{inf}, \lambda_*^{sup}]$ is considered the best solution, the radius of $[\min(x_*^{inf}, x_*^{sup}), \max(x_*^{inf}, x_*^{sup})]$ is considered a dispersion measure.

V.2.2. Numerical illustrative and comparative example

Let us consider the numerical example of section IV.2.b. The decision domain is given by the type-2 FIs $\tilde{\boldsymbol{d}}(\lambda) = \{\boldsymbol{d}_{inf}(\lambda), \boldsymbol{d}_{sup}(\lambda)\}$. The configuration of this example is similar to that of case 1 given in Fig. 22. An example that illustrates the configuration of case 3 in Fig. 22 is provided in Appendix A. The decision-making strategy according to the risk level β = 0.5 is illustrated in Fig. 23.

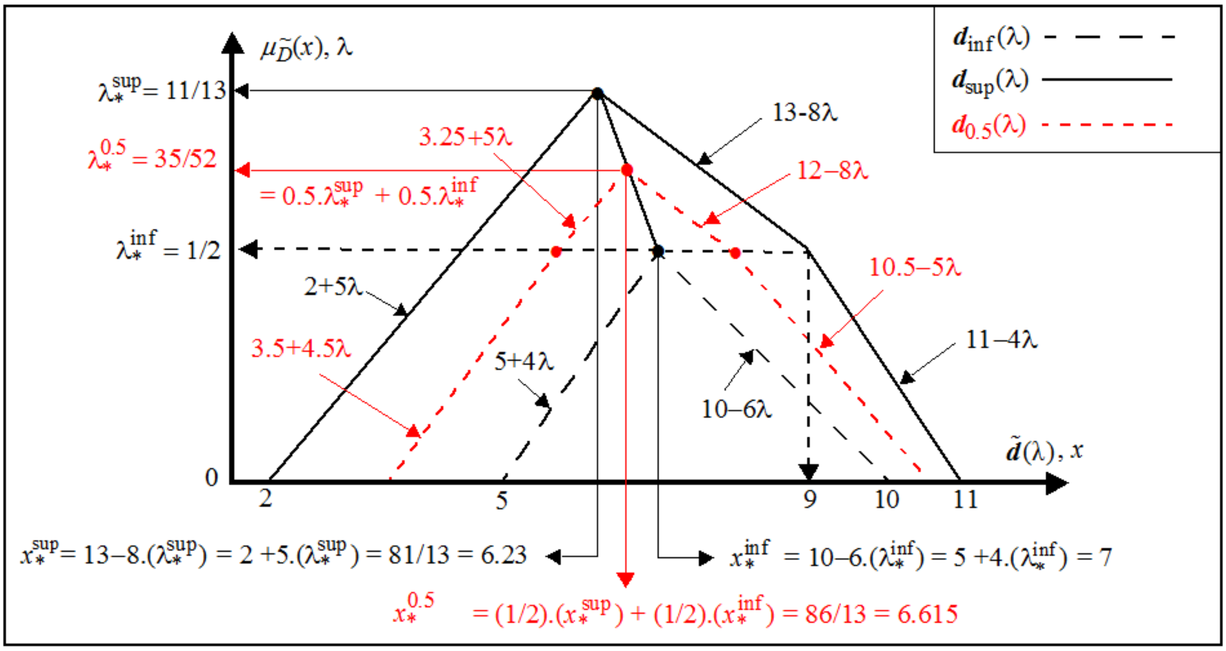

Fig. 23: Decision-making for β = 0.5

For each level β, the type-1 FI compromise $\boldsymbol{d}_\beta(\lambda)$ can be computed by using gradual numbers that represent the FI profiles. The optimal solution $(x_*^\beta, \lambda_*^\beta)$ can be computed as follows:

$$\begin{pmatrix} x_*^\beta \\ \lambda_*^\beta \end{pmatrix} = \beta \cdot \begin{pmatrix} 81/13 \\ 11/13 \end{pmatrix} + (1-\beta) \cdot \begin{pmatrix} 7 \\ 1/2 \end{pmatrix}$$

In the same way, if β is equal to 0.8, the results are depicted in Fig. 24.

To provide an overview of the difference between this study and the approach given in [42] and explain the reasoning behind our approach, a conceptual comparison between the two visions is provided. In this context, for the approach proposed in [42], the type-2 FIs $\tilde{\boldsymbol{a}}(\lambda)$ and $\tilde{\boldsymbol{b}}(\lambda)$ of Fig. 19 are represented by type-2 FSs $\tilde{A}$ and $\tilde{B}$. Each type-2 FS is characterized by its lower and upper membership functions.

For any given risk level β, the optimal solution using the method in [42] is expressed as follows:

$$x_*^\beta = \arg\max_{x \in X} \{((1-\beta) \cdot \mu_A^{inf}(x) + \beta \cdot \mu_A^{sup}(x)) \cap ((1-\beta) \cdot \mu_B^{inf}(x) + \beta \cdot \mu_B^{sup}(x))\}$$

The crisp optimal results obtained using the approach given in [42] are in accordance with those of our approach. For instance, if β = 0.8, the optimization problem yields $x_*^{0.8}$ = 83/13. However, our approach does not employ either the extension principle or the α-cuts principle. Unlike the method in [42], our approach can determine all decision domains $\tilde{d}(\lambda)$, which represent the uncertainty footprint of the decision. In addition to the optimal crisp value, for any value of β, the profiles of the type-1 FI that represents this decision are obtained. These remarks clearly indicate the advantage of our computing strategy using only standard interval relations and interval arithmetic operations while avoiding the iterative aspect inherent to optimization algorithms, especially in complex situations where several goals and constraints can be applied. As illustrated in the example, our method can permit important elasticity in the management of uncertainty and enables the propagation of this decision domain by using aggregation operators to perform the decision-making strategy (refer to Appendix *B* for an example).

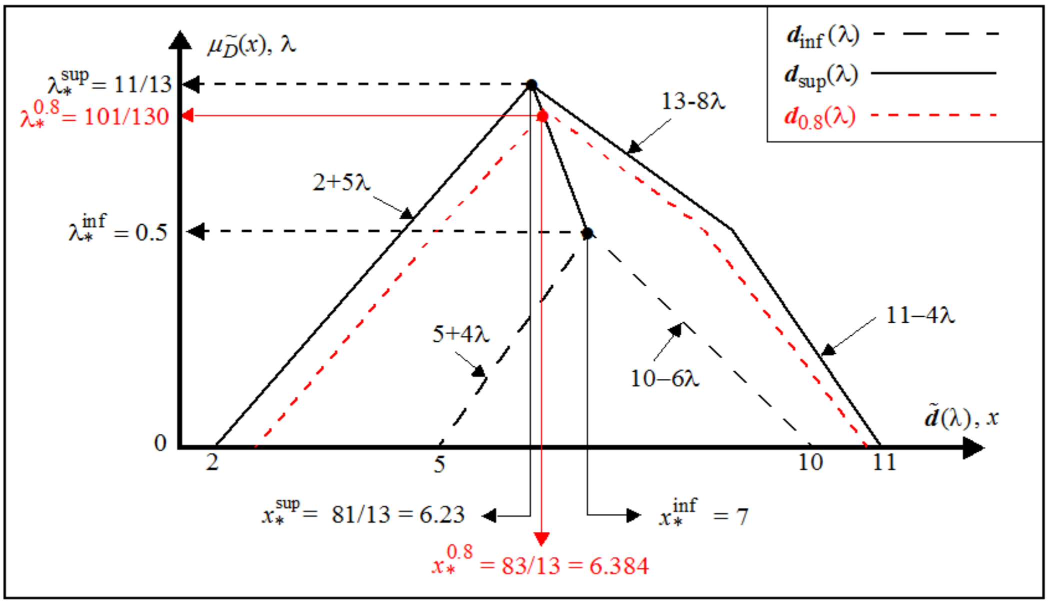

Fig. 24: Decision-making for β = 0.8

V.2.3. Application examples

A. Example 1

This example is inspired by [42]. We consider the temperature example, where the objective of a decision maker is to optimize the temperature of a room depending on the satisfaction levels of some people. Assume that you have invited two groups of friends (a group of men denoted by $\tilde{a}$ and a group of women named $\tilde{b}$). We know that the most demanding (pessimistic) man will be completely happy with 17°C and will be completely unhappy at temperatures less than 16°C or greater than 19°C. The degree of satisfaction between 16° and 19° is given by the type-1 FI illustrated in Fig. 25 (dashed triangle in $\tilde{a}$). The most tolerant (optimistic) man states that he will be completely satisfied at 17° and completely unsatisfied at temperatures less than 14° and greater than 25°. The satisfaction degree is also depicted in Fig. 25 (solid triangle in $\tilde{a}$). The satisfaction profiles of all other men range between the optimistic situation and the pessimistic situation. This statement can be represented by the type-2 FI $\tilde{a}(\lambda)$ shown in Fig. 25, where the upper type-1 FI (the optimistic or best case) is represented as a solid tringle and the lower type-1 FI (the pessimistic or worst case) is represented by a dashed triangle. By adopting the same reasoning, the requirements of the women's group in terms of room temperature are represented by the type-2 FI $\tilde{b}(\lambda)$, as shown in Fig. 25. The type-2 FIs and their intersection are represented in Fig. 25. The configuration of this example refers to case 2 in Fig. 22. In this example, the decision procedure aims to determine the temperature of the room while achieving the highest degree of satisfaction.

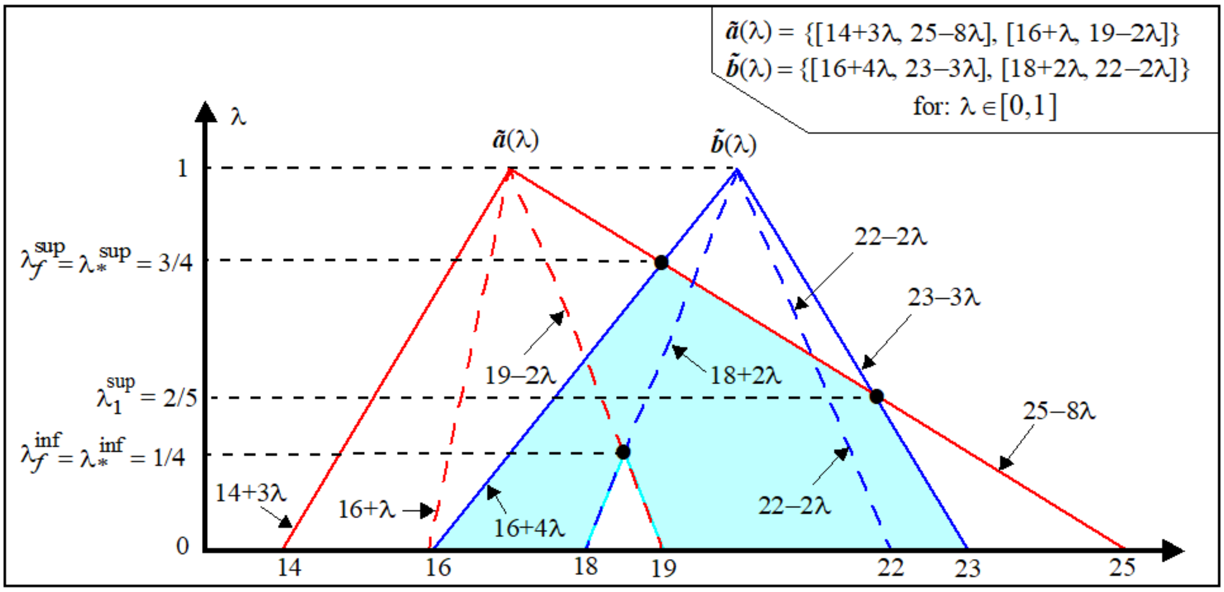

Fig. 25: Uncertain decision domain with pessimistic and optimistic type-2 FIs

The intersection points between the upper type-1 FI profiles are

$$\text{IP}_{\text{RR}}:\ \lambda_1^{\sup} = 2/5\,;\ \text{IP}_{\text{RL}}:\ \lambda_f^{\sup} = 3/4.$$

In the same way, the intersection point between the lower type-1 FI profiles is

$$\text{IP}_{\text{RL}}:\ \lambda_f^{\inf} = 1/4.$$

The decision domain is expressed as follows:

$$\tilde{\boldsymbol{d}}(\lambda) = \{\boldsymbol{d}_{\inf}(\lambda), \boldsymbol{d}_{\sup}(\lambda)\}\,;\ \text{where: } \boldsymbol{d}_{\inf}(\lambda) = [18+2\lambda, 19-2\lambda]\,;\ \text{ if: } 0 \le \lambda \le 1/4$$

$$\text{and: } \boldsymbol{d}_{\sup}(\lambda) = \begin{cases} [16+4\lambda, 23-3\lambda]\,; & \text{if: } 0 \le \lambda \le 2/5 \\ [16+4\lambda, 25-8\lambda]; & \text{if: } 2/5 < \lambda \le 3/4 \end{cases}$$

As previously detailed, the pessimistic and optimistic decision-making solutions using the Bellman-Zadeh principle are

$$\begin{cases} \lambda_*^{\inf} = \lambda_f^{\inf} = 1/4 \\ \lambda_*^{\sup} = \lambda_f^{\sup} = 3/4 \end{cases} \Rightarrow \begin{cases} x_*^{\inf} = d_{\inf}^{-}(\lambda_*^{\inf}) = 18 + 2\cdot(1/4) = d_{\inf}^{+}(\lambda_*^{\inf}) = 19 - 2\cdot(1/4) = 18.5 \\ x_*^{\sup} = d_{\sup}^{-}(\lambda_*^{\sup}) = 16 + 4\cdot(3/4) = d_{\sup}^{+}(\lambda_*^{\sup}) = 25 - 8\cdot(3/4) = 19 \end{cases}$$

In this example, the decision $(x_*^{\inf}, \lambda_*^{\inf}) = (18.5, 1/4)$ indicates that the temperature 18.5° corresponds to the lowest satisfaction degree of all persons in the two groups (none of the persons will be less satisfied than 25%). If the decision maker chooses this value, its decision is considered to be prudent with the lowest risk level, i.e., $\beta = 0$. In the opposite case, the decision $(x_*^{\sup}, \lambda_*^{\sup}) = (19, 3/4)$ is considered to be the riskiest case ($\beta = 1$) because all persons in the groups in the best (optimistic) case will be satisfied at 75%. In this context, for the given risk level β, if $\lambda_*^{\beta} \in [1/4, 3/4] \Rightarrow x_*^{\beta} \in [18.5, 19]$. For instance, Fig. 26 illustrates the decision-making mechanism for $\beta = 0.5$. For any value of β, the optimal solution $(x_*^{\beta}, \lambda_*^{\beta})$ is approximated as follows:

$$\begin{pmatrix} x_*^{\beta} \\ \lambda_*^{\beta} \end{pmatrix} = \beta \cdot \begin{pmatrix} 19 \\ 3/4 \end{pmatrix} + (1-\beta)\cdot \begin{pmatrix} 18.5 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 18.75 \\ 0.5 \end{pmatrix}$$

If the temperature 18.75° is selected by the decision maker, a 50% satisfaction level will be attained by all persons with a medium risk. Compared with the method in [42], all remarks and advantages discussed in the illustrative example (refer to section V2.2.) remain valid in this case.

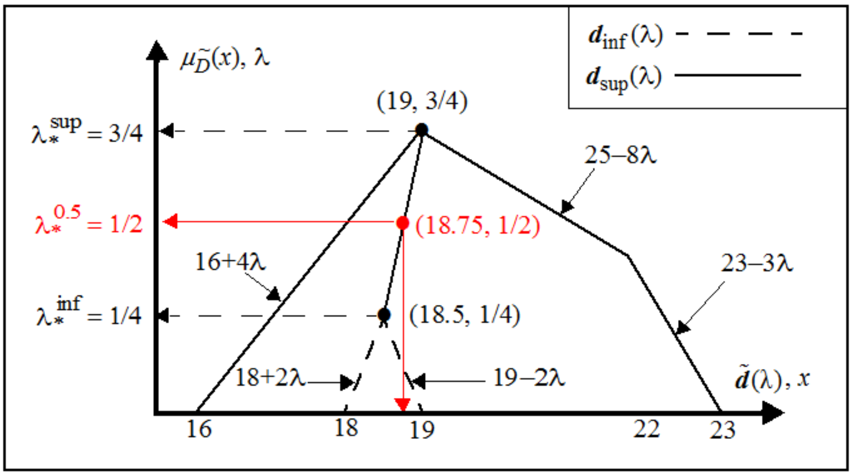

Fig. 26: Simplified decision according to a risk level β = 0.5

B. Example 2

Let us consider an employee who goes to work in the morning (by car between 7:00 and 13:00), works for 6 hours, and then returns home in the afternoon (between 13:00 and 19:00). The objective of this decision application is to help this employee choose the departure and return times by considering the density of road traffic. This example is inspired by [42].

The traffic density was measured every half hour between 7:00 and 19:00 for the previous 10 days (at each half hour, 10 measures were recorded). For illustration purposes, measurements taken between 7:00 and 13:00 are shown in Fig. 27. For example, at 8:00, the traffic road density falls between 0.7 and 0.8. At 7:00, the traffic remains stable at its highest level (approximately 0.8). In the same way, at 13:00, the traffic remains stable at its lowest level (approximately 0.2).

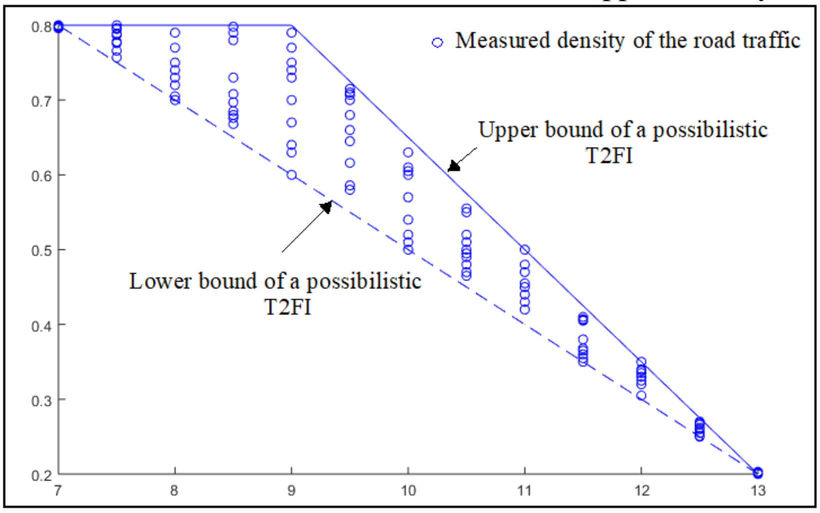

Fig. 27: Morning road traffic density and its representation by a type-2 FI

This uncertain data can be represented by a type-2 FI. In this context, a possibilistic approach is exploited to determine the type-2 FI (refer to Fig. 27). Thus, the objective is to determine the least uncertain linear (or piecewise linear) type-2 FI with respect to the inclusion constraints between the measured data and the type-2 FI bounds (all measured data are encapsulated in the type-2 FI). For additional details about the possibilistic approach, refer to [4][9]. The densities of the morning and afternoon traffic are represented by the type-2 FIs $\tilde{\boldsymbol{m}}(\lambda)$ and $\tilde{\boldsymbol{a}}(\lambda)$, as shown in Fig. 28 and expressed as

$$\tilde{\boldsymbol{m}}(\lambda) = \{\boldsymbol{m}_{\inf}(\lambda), \boldsymbol{m}_{\sup}(\lambda)\}; \text{where:} \begin{cases} \boldsymbol{m}_{\inf}(\lambda) = 15 - 10\lambda;\ \lambda \in [0.2, 0.8] \\ \boldsymbol{m}_{\sup}(\lambda) = 43/3 - (20/3)\lambda;\ \lambda \in [0.2, 0.8] \end{cases};$$

and: $\tilde{a}(\lambda) = \{\boldsymbol{a}_{\inf}(\lambda), \boldsymbol{a}_{\sup}(\lambda)\}$; where: $\begin{cases} \boldsymbol{a}_{\inf}(\lambda) = \begin{cases} 149/13 + (100/13)\lambda; & \lambda \in [0.2, 0.46] \\ 52/5 + 10\lambda; & \lambda \in [0.46, 0.66] \\ 53/7 + (100/7)\lambda; & \lambda \in [0.66, 0.8] \end{cases} \\ \boldsymbol{a}_{\sup}(\lambda) = 35/3 + (20/3)\lambda; \ \lambda \in [0.2, 0.8] \end{cases}$

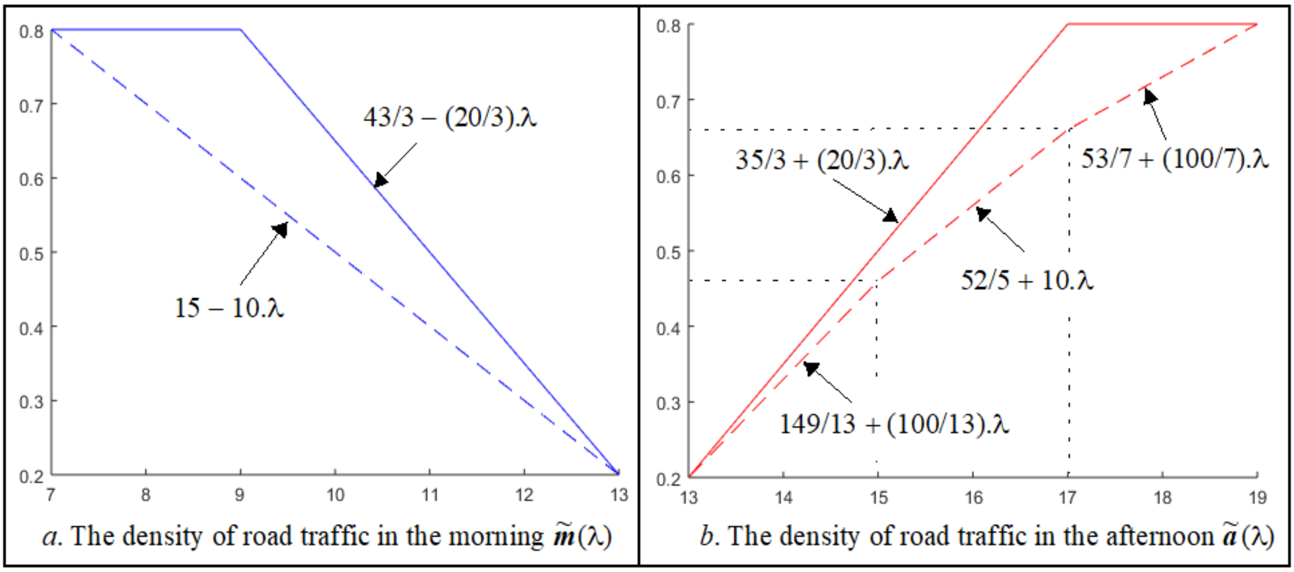

Fig. 28: Type-2 FIs that represent the road traffic densities in the morning and afternoon

To achieve a compromise between the departure times and the return times, the type-2 FI $\tilde{a}(\lambda)$ that represents the afternoon traffic is shifted 6 hours to the left. For instance, a morning trip at 8:00 corresponds to an afternoon trip at 14:00 (refer to Fig. 29).

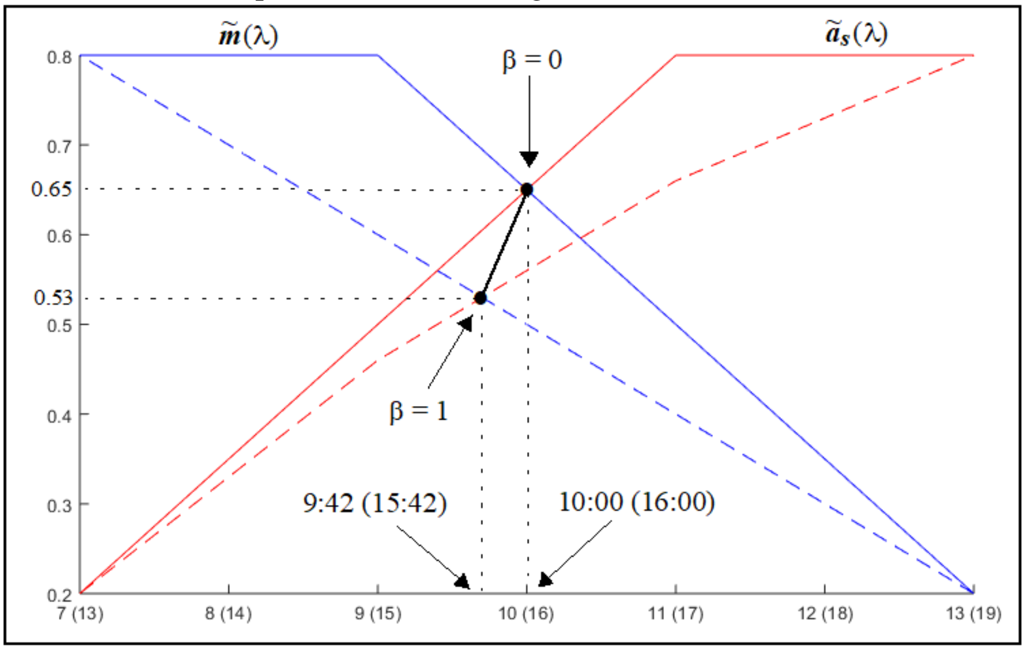

Fig. 29: Uncertain decision-making according to the road traffic densities

In this context, the type-2 FI $\tilde{a}(\lambda)$ becomes

$$\tilde{a}_s(\lambda) = \{\boldsymbol{a}^s_{\inf}(\lambda), \boldsymbol{a}^s_{\sup}(\lambda)\}; \text{where: } \begin{cases} \boldsymbol{a}^s_{\inf}(\lambda) = \begin{cases} 71/13 + (100/13)\lambda; & \lambda \in [0.2, 0.46] \\ 22/5 + 10\lambda; & \lambda \in [0.46, 0.66] \\ 11/7 + (100/7)\lambda; & \lambda \in [0.66, 0.8] \end{cases} \\ \boldsymbol{a}^s_{\sup}(\lambda) = 17/3 + (20/3)\lambda; \ \lambda \in [0.2, 0.8] \end{cases}$$

As previously detailed, the decision-making solutions using the Bellman-Zadeh principle are (refer to Fig. 29):

$$\begin{cases}\lambda_*^{\inf} = 0.53 \\ \lambda_*^{\sup} = 0.65\end{cases} \Rightarrow \begin{cases}x_*^{\inf} = 9.70 \text{ (9:42 for departure and 15:42 for return)} \\ x_*^{\sup} = 10 \text{ (10:00 for departure and 16:00 for return)}\end{cases}$$

In this application, because the best configuration (in terms of utility) is the configuration with low road traffic, the lower type-1 FI is considered to be the best case. In the same way, the upper type-1 FI is considered to be the worst case. In this context, a cautious decision maker (the least risky or the most risk-averse) will drive to work at 10:00 and return at 16:00 because the worst-case traffic is approximately 0.65. In the opposite case, the highest risky (the least risk-averse) decision maker will drive to work at 9:42 and return at 15:42 because the best traffic conditions occur at approximately 0.53. For intermediate risk levels, the optimal departure time ranges between 9:42 and 10:00 (the optimal return time is between 15:42 and 16:00).

For any value of β, the optimal solution $(x_*^{\beta}, \lambda_*^{\beta})$ for the departure is computed by the following approximation:

$$\begin{pmatrix} x_*^{\beta} \\ \lambda_*^{\beta} \end{pmatrix} = (1-\beta)\cdot\begin{pmatrix} x_*^{\sup} \\ \lambda_*^{\sup} \end{pmatrix} + \beta\cdot\begin{pmatrix} x_*^{\inf} \\ \lambda_*^{\inf} \end{pmatrix} = (1-\beta)\cdot\begin{pmatrix} 10 \\ 0.65 \end{pmatrix} + \beta\cdot\begin{pmatrix} 9.70 \\ 0.53 \end{pmatrix}$$

The optimal solution for the return can be deduced from the solution of the departure. For example, if the selected risk level is equal to 0.7, we can obtain $(x_*^{0.7}, \lambda_*^{0.7}) = (9.79, 0.56)$. The value 9.79 corresponds to 9:47. In this case, the decision with a risk level of 0.7 is a departure time at 9:47 and a return time at 15:47.

V2.4. Remarks and discussion

- In this paper, to utilize type-2 FIs in an analytically tractable way, particularly in a fuzzy decision-making context, a computational approach is proposed for the implementation of the Bellman-Zadeh principle. Because interval arithmetic operations on real numbers have to be extended to type-1 FIs, the motivation and reasoning behind our approach have been to extend this philosophy to type-2 FIs in a decision-making application. The proposed method has been applied to linear and piecewise linear type-2 FIs but remains transposable regardless of the shape of the FIs. In this context of nonlinear type-2 FI shapes, the approximated relation (24) becomes very restrictive, and application of the generic equation given by (22) is more reasonable.
- While the aim of this paper is to develop an interval-based computational mechanism for the Bellman-Zadeh decision-making principle, the proposed methodology can be transposed in several applications based on type-2 FIs, where guaranteed and analytical computations are possible. A reflection about the applicability of our approach for extending multicriteria and multiattribute decision-making approaches [26][27][39][50] to the type-2 fuzzy context should be mentioned. The proposed approach can motivate a certain interest in the frameworks of type-2 automatic control [10][29], type-2 regression and modeling [2][23], type-2 linear programming methods [19], etc. Our type-2 FI representation can be employed to implement several aggregation operators (conjunctive and disjunctive operators, weighted average and ordered weighted average operators, and the Choquet integral). As an example, Appendix B shows the potential use of our computational method through the 2-Additive Choquet integral (2-ACI) [7][30].
- Direct application of this approach in type-2 FI model predictive control is possible. The proposed approach can extend the type-1 FI model predictive control (MPC) strategies [16] for handling type-2 FIs [29]. In this framework, fuzzy goals and fuzzy constraints, which are defined using relevant system variables, are assumed to be uncertain and represented by type-2 FIs. By the MPC philosophy, fuzzy constraints are usually defined in the domain of the control actions, and fuzzy goals are usually defined in the domain of the outputs and/or state space variables. The control

strategy objective is to force the process to perform better based on a compromise between the goals and the constraints. In this case, the proposed computational method can be employed to obtain the control actions by a multistage fuzzy decision-making (FDM) approach based on the Bellman-Zadeh principle. The proposed formalism adapts to decision-making applications in control, especially when the models are parametric regressive models with inputs, outputs and parameters represented by type-2 FIs.

• If this approach can take advantage of the flexibility, rigor and guaranteed results of interval arithmetic and reasoning, it can be criticized for its accumulation of fuzziness (such as type-1 FI approaches). This phenomenon causes overestimation of uncertainties in the resulting type-2 FIs. This overestimation is derived from the decorrelation phenomenon of interval arithmetic, which is also known as a dependency problem. Because interval arithmetic guarantees the set of all possible results, the pessimistic independence property between the intervals is implicitly assumed. This overestimation problem can be reduced by implementing some extensions and hybridizations of interval arithmetic [38][44].

VI. CONCLUSION

In this paper, an interval-based approach of the Bellman-Zadeh decision-making methodology, where the goals and constraints are represented by unimodal and piecewise liner type-2 FIs, is proposed. First, the intersection operator between type-2 FIs was investigated. This operation uses the gradual number representation to extend interval relations to a type-2 FI. This extension facilitates the implementation of the intersection operator, which creates an uncertain decision area bounded by two type-1 FIs. In a second part of this paper, a decision-making strategy is proposed to select the optimal solution within this decision area according to a specified criterion. According to the Bellman-Zadeh methodology, the risk criterion was exploited as an example. Other decision criteria can be employed. The proposed approach has been presented for unimodal FIs but can be adapted to a nonunimodal FI, such as a trapezoidal FI. The intersection operator is based on the relations $\{\prec, \succ, \subseteq, \supseteq\}$, which verify the conditions of partial-order relations [25]. This paper provides illustrative examples only. However, several types of decision-making applications can be imagined. The ideas developed in this paper can be most likely used as a basis for studying the order relations between type-2 FIs and their rankings in multiple attribute decision-making problems.

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Appendix A: Decision-making computational example

Let us consider the type-2 FI $\tilde{\boldsymbol{g}}(\lambda)$ and $\tilde{\boldsymbol{c}}(\lambda)$ of Fig. 27, i.e.,

$$\tilde{\boldsymbol{g}}(\lambda)=\{\boldsymbol{g}_{\inf}(\lambda),\boldsymbol{g}_{\sup}(\lambda)\};\text{where:}\begin{cases}\boldsymbol{g}_{\inf}(\lambda)=[5+2\lambda,10-3\lambda];\ \lambda\in[0,1]\\ \boldsymbol{g}_{\sup}(\lambda)=[4+3\lambda,16-9\lambda];\ \lambda\in[0,1]\end{cases};\text{ and:}$$

$$\tilde{\boldsymbol{c}}(\lambda)=\{\boldsymbol{c}_{\inf}(\lambda),\boldsymbol{c}_{\sup}(\lambda)\};\text{where:}\begin{cases}\boldsymbol{c}_{\inf}(\lambda)=[8+5\lambda,13-2\lambda];\ \lambda\in[0,5/7]\\ \boldsymbol{c}_{\sup}(\lambda)=[4+7\lambda,15-4\lambda];\ \lambda\in[0,1]\end{cases}$$

By adopting the same method employed in the previous examples, the intersection points between the profiles and the decision domain are as computed and illustrated in Fig. 30.
The optimal pessimistic and optimistic values $(x_*^{\inf},\lambda_*^{\inf})$ and $(x_*^{\sup},\lambda_*^{\sup})$ are

$$\begin{cases}\lambda_*^{\inf}=\lambda_f^{\inf}=1/4\\ \lambda_*^{\sup}=\lambda_f^{\sup}=3/4\end{cases}\Rightarrow\begin{cases}x_*^{\inf}=8+5\cdot(1/4)=10-3\cdot(1/4)=9.25\\ x_*^{\sup}=4+7\cdot(3/4)=16-9\cdot(3/4)=9.25\end{cases}$$

The configuration of this example is consistent with that of case 3 given in Fig. 22.

In this context, regardless of the value of β, the solution x_*^{β} remains unchanged. All FI $\boldsymbol{d}_{\beta}(\lambda)$ yield a unique solution x_*^{β} on the horizontal dimension.

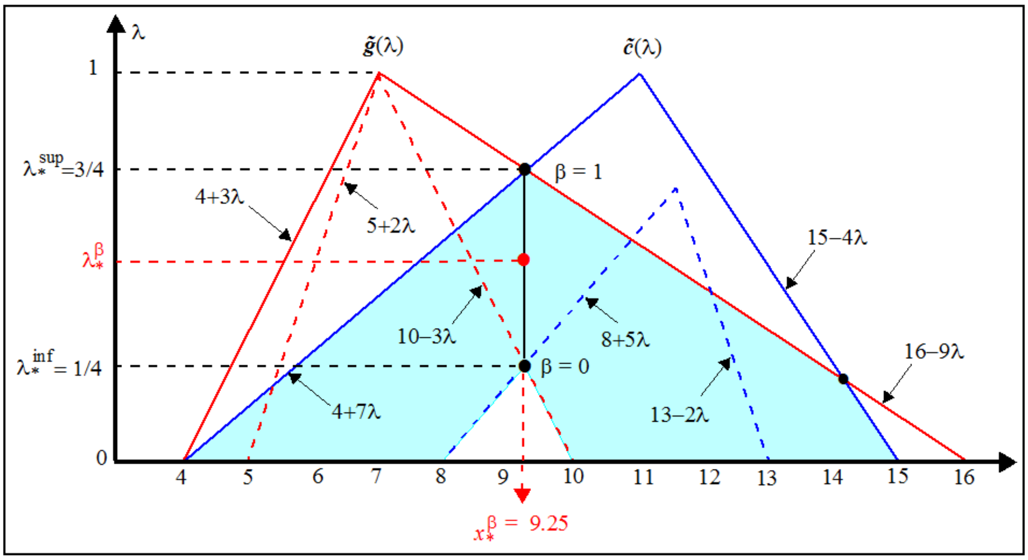

Fig. 30: Uncertain decision domain and decision strategy

Appendix B: 2-Additive type-2 FI Choquet integral computational example

Let us consider the set of crisp alternatives $\{a_1, \ldots, a_n\}$ to be aggregated and associated with a set of n criteria. The 2-ACI is expressed as

$$CI = \sum_{I_{ij}>0} min(a_i, a_j) \cdot I_{ij} + \sum_{I_{ij}<0} max(a_i, a_j) \cdot |I_{ij}| + \sum_{i=1}^{n} a_i \cdot (v_i - \frac{1}{2} \cdot \sum_{j \neq i} |I_{ij}|) \quad (25)$$

In (25), the coefficient I_{ij} represents the mutual interaction between criteria i and j and can be interpreted as follows:

- A positive I_{ij} indicates that the criteria are complementary (positive synergy).
- A negative I_{ij} indicates that the criteria are redundant (negative synergy).
- A null I_{ij} indicates that no interaction between criteria exists (the criteria are independent).

The coefficients v_i in (25) are the Shapley indices that represent the relative importance of each elementary criterion in relation to all other criterion, with $\Sigma_{i=1}^{n} v_i = 1$. The 2-ACI has been extended to the fuzzy context in which the alternatives are represented by type-1 FIs [7][30]. This extension resulted in the following expression:

$$\boldsymbol{CI}(\lambda) = \sum_{I_{ij}>0} min([\boldsymbol{a}_i(\lambda)], [\boldsymbol{a}_j(\lambda)]) \times I_{ij} + \sum_{I_{ij}<0} max([\boldsymbol{a}_i(\lambda)], [\boldsymbol{a}_j(\lambda)]) \times |I_{ij}| + \sum_{i=1}^{n} [\boldsymbol{a}_i(\lambda)] \times (v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}|) \quad (26)$$

In (26), $\boldsymbol{a}_1(\lambda), \ldots, \boldsymbol{a}_n(\lambda)$ are type-1 FI alternatives. Our objective is to extend (26) to the situation in which the alternatives are uncertain and represented by type-2 FIs. In this context, the 2-ACI given by (26) becomes:

$$\tilde{\boldsymbol{CI}}(\lambda) = \sum_{I_{ij}>0} min(\tilde{\boldsymbol{a}}_i(\lambda), \tilde{\boldsymbol{a}}_j(\lambda)) \times I_{ij} + \sum_{I_{ij}<0} min(\tilde{\boldsymbol{a}}_i(\lambda), \tilde{\boldsymbol{a}}_j(\lambda)) \times |I_{ij}| + \sum_{i=1}^{n} \tilde{\boldsymbol{a}}_i(\lambda) \times (v_i - \frac{1}{2} \cdot \sum_{j \neq i} |I_{ij}|) \quad (27)$$

where $\tilde{\boldsymbol{CI}}(\lambda) = \{\boldsymbol{CI}_{\inf}(\lambda), \boldsymbol{CI}_{\sup}(\lambda) \mid \boldsymbol{CI}_{\inf}(\lambda) \subseteq \boldsymbol{CI}_{\sup}(\lambda)\}$.

In (27), $\tilde{\boldsymbol{a}}_i(\lambda) = \{\boldsymbol{a}_{\inf}^i(\lambda), \boldsymbol{a}_{\sup}^i(\lambda) \mid \boldsymbol{a}_{\inf}^i(\lambda) \subseteq \boldsymbol{a}_{\sup}^i(\lambda)\}, i = 1, \ldots, n\}$ are type-2 FI alternatives. In the implementation of the 2-ACI given by (27), the *min* and *max* between two type-2 FIs is realized using the same methodology as the intersection operator. Thus, the *min* and *max* between two type-2 FIs $\tilde{\boldsymbol{a}}(\lambda)$ and $\tilde{\boldsymbol{b}}(\lambda)$ is defined by the expressions

$$min(\tilde{\boldsymbol{a}}(\lambda), \tilde{\boldsymbol{b}}(\lambda)) = \{min(\boldsymbol{a}_{\inf}(\lambda), \boldsymbol{b}_{\inf}(\lambda)), min(\boldsymbol{a}_{\sup}(\lambda), \boldsymbol{b}_{\sup}(\lambda))\}$$

$$max(\tilde{a}(\lambda),\tilde{b}(\lambda)) = \{max(a_{inf}(\lambda), b_{inf}(\lambda)), max(a_{sup}(\lambda), b_{sup}(\lambda))\}$$

where the *min* and *max* operators between the type-1 FIs are computed using the methodology given in [7]. Let us illustrate the computation of the 2-ACI for aggregating the four type-2 FI alternatives $\tilde{a}_1(\lambda)$, …, $\tilde{a}_4(\lambda)$ illustrated in Fig. 31 and listed in Table 1; with $\nu_1 = 0.4$, $\nu_2 = 0.35$, $\nu_3 = 0.1$, and $\nu_4 = 0.15$ and $I_{14} = -0.35$, $I_{34} = -0.3$, $I_{13} = 0.15$, and $I_{23} = 0.6$. According to these data values, the 2-ACI (27) is expressed as follows:

$$\tilde{CI}(\lambda) = 0.15\times min(\tilde{a}_1(\lambda),\tilde{a}_3(\lambda)) + 0.6\times min(\tilde{a}_2(\lambda),\tilde{a}_3(\lambda)) + 0.35\times max(\tilde{a}_1(\lambda),\tilde{a}_4(\lambda))$$
$$+ 0.3\times max(\tilde{a}_3(\lambda),\tilde{a}_4(\lambda)) + \sum_{i=1}^{n}\tilde{a}_i(\lambda)\times(v_i - 0.5\sum_{j\neq i}|I_{ij}|)$$

T2FIs	Lower T1FI	Upper T1FI
$\tilde{a}_1(\lambda)$	[1+λ, 4–2λ]	[2λ, 6–4λ]
$\tilde{a}_2(\lambda)$	[3+2λ, 8–3λ]	[2+3λ, 9–4λ]
$\tilde{a}_3(\lambda)$	[3+3λ, 7–λ]	[1+5λ, 8–2λ]
$\tilde{a}_4(\lambda)$	[1+2λ, 4–λ]	[3λ, 5–2λ]

Table 1: Expressions of the four type-2 FI alternatives

The *min* and *max* operators between the type-2 FIs are computed as follows:

$$min(\tilde{a}_1(\lambda),\tilde{a}_3(\lambda)) = \tilde{a}_1(\lambda);$$

In this case, $\tilde{a}_1(\lambda)$ and $\tilde{a}_3(\lambda)$ are totally ordered.

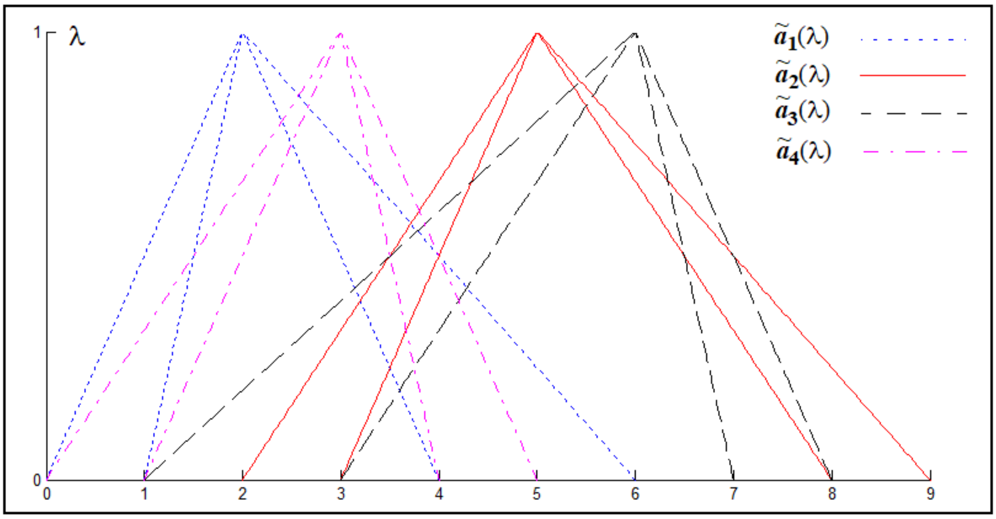

Fig. 31: Shape of the four type-2 FI alternatives

The order relation between the type-2 FIs $\tilde{a}_2(\lambda)$ and $\tilde{a}_3(\lambda)$ is not total and cannot be totally ordered. In this case, the *min* operator is computed as follows (refer to Fig. 32):

$$min(\tilde{a}_1(\lambda),\tilde{a}_3(\lambda)) = \tilde{\theta}(\lambda)\text{; with: } \theta_{inf}(\lambda) = min(a^2_{inf}(\lambda), a^3_{inf}(\lambda)) = \begin{cases}[3+2\lambda, 7-\lambda]; \text{if: } 0 \le \lambda \le 0.5\\ [3+2\lambda, 8-3\lambda]; \text{if: } 0.5 < \lambda \le 1\end{cases}$$

$$\text{and: } \theta_{sup}(\lambda) = min(a^2_{sup}(\lambda), a^3_{sup}(\lambda)) = \begin{cases}[1+5\lambda, 8-2\lambda]; \text{if: } 0 \le \lambda \le 0.5\\ [2+3\lambda, 9-4\lambda]; \text{if: } 0.5 < \lambda \le 1\end{cases}$$

By applying the same methodology, the *max* operator is computed as follows (refer to Fig. 33):

$$max(\tilde{a}_3(\lambda),\tilde{a}_4(\lambda)) = \tilde{a}_3(\lambda)\text{; and:}$$

$$max(\tilde{a}_1(\lambda),\tilde{a}_4(\lambda)) = \tilde{\omega}(\lambda)\text{; with: } \omega_{inf}(\lambda) = max(a^1_{inf}(\lambda), a^4_{inf}(\lambda)) = [1+2\lambda, 4-\lambda]\text{; and:}$$

$$\omega_{sup}(\lambda) = max(a^4_{sup}(\lambda), a^4_{sup}(\lambda)) = \begin{cases}[3\lambda, 6-4\lambda]; \text{if: } 0 \le \lambda \le 0.5\\ [3\lambda, 5-2\lambda]; \text{if: } 0.5 < \lambda \le 1\end{cases}$$

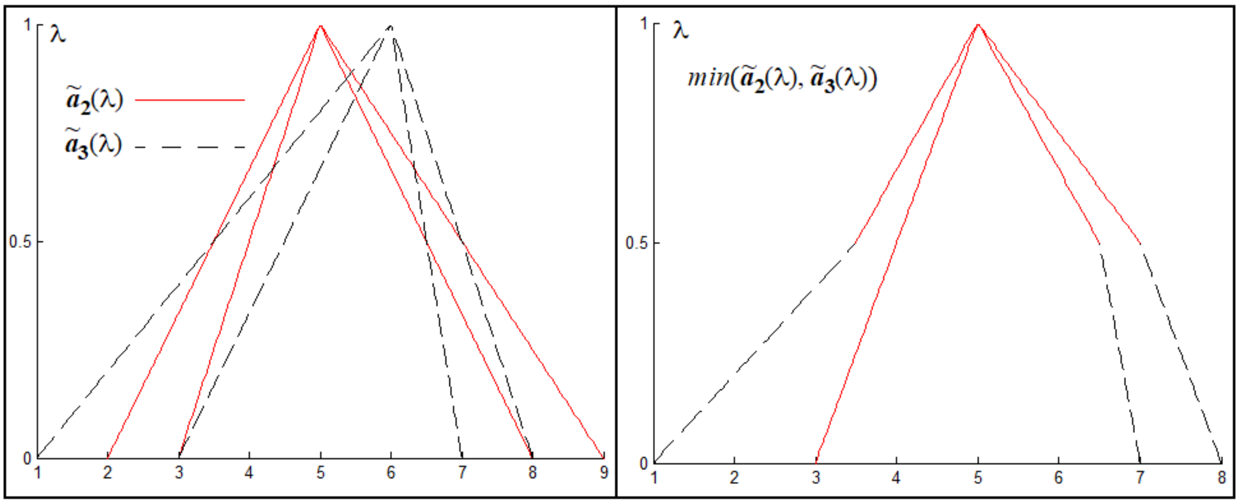

Fig. 32: *min* operator between $\tilde{\boldsymbol{a}}_2(\lambda)$ and $\tilde{\boldsymbol{a}}_3(\lambda)$

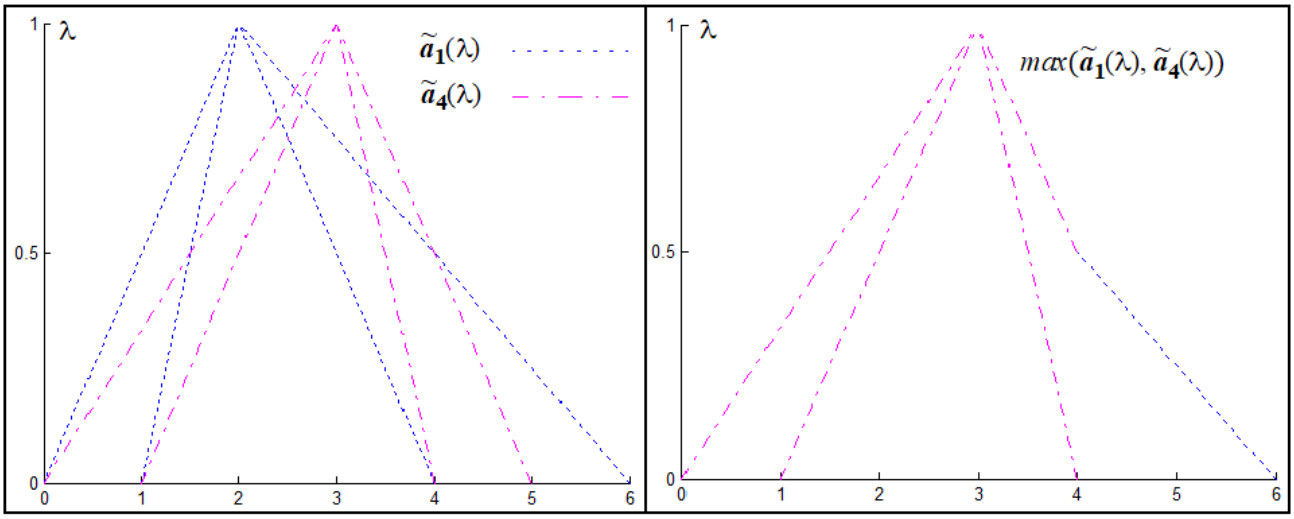

Fig. 33: *max* operator between $\tilde{\boldsymbol{a}}_1(\lambda)$ and $\tilde{\boldsymbol{a}}_4(\lambda)$

Using the proposed computational method, the final 2-ACI aggregation operator (27) is expressed by the following analytical expressions (refer to Fig. 34):

$$\boldsymbol{CI}_{\inf} = \begin{cases} [3.35+3.2\lambda, 9.2-2.05\lambda]; \text{if: } 0 \le \lambda \le 0.5 \\ [3.35+3.20\lambda, 9.8-3.25\lambda]; \text{if: } 0.5 < \lambda \le 1 \end{cases}; \quad \boldsymbol{CI}_{\sup} = \begin{cases} [0.9+6.25\lambda, 11.5-4.7\lambda]; \text{if: } 0 \le \lambda \le 0.5 \\ [1.5+5.05\lambda, 911.75-5.2\lambda]; \text{if: } 0.5 < \lambda \le 1 \end{cases}$$

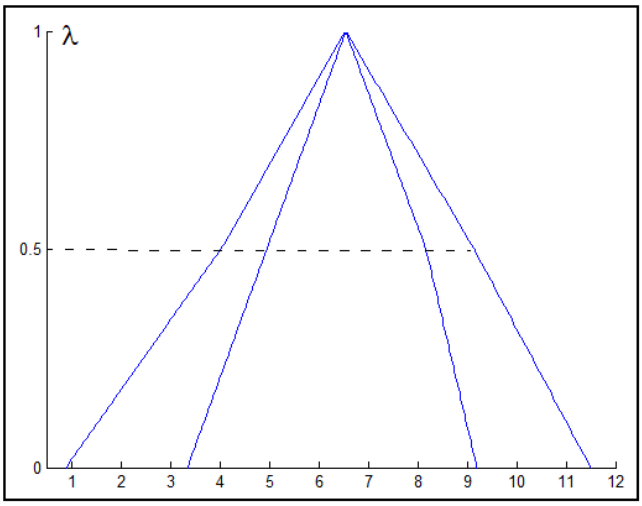

Fig. 34: 2-ACI result as a type-2 FI