The Perils of Fiscal Rules

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The Perils of Deficit Fiscal Rules[☆]

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Abstract

In a simple growth model with public debt, we introduce a flexible fiscal rule whereby part of the debt burden is sterilized by the primary fiscal surplus. The sterilization coefficient critically affects the behavior of the economy, producing multiple equilibria, indeterminacy, long-run cycles, and hysteresis. Contrary to the standard policy recommendation, a strong reaction of the primary surplus to the debt burden can condemn the economy to a low-growth/high-debt trap. In addition, the reaction of the primary balance may be a source of high-periodicity public debt cycles, and small variations in the sterilization coefficient may have dramatic consequences on the balanced growth path. Our findings are supported by a calibration based on OECD historical data. Finally, extending the model to include shocks allows illustrating the presence of stochastic limit cycles.

Keywords: deficit fiscal rules; economic growth; public debt; debt cycles; growth trap.
JEL Codes: E10, E62, H62, H63, N10, O40.
Declarations of interest: none

1. Introduction

High levels of public debts in developed and developing countries since at the 1990s have led to a growing interest in fiscal rules (Debrun et al., 2008).[1] According to the IMF Fiscal Rule Dataset, in 92 of the surveyed 96 countries there is at least one fiscal rule in 2015—against only 5 countries in 1985. However, the benefits of fiscal rules are not unanimous. Notably, the pro-cyclical nature of rules which are characterized by fixed deficit and/or debt targets has often been reported (Wyplosz 2012 and Bova et al. 2014).

Recently, several voices advocated for more flexibility in fiscal rules. Flexible (or "second-generation") rules, such as the use of cyclically-adjusted targets (see Eyraud

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[1]Fiscal rules are defined as legal restrictions upon fiscal indicators such as government debt, spending, or budget balance (see Kopits and Symansky, 1998).

et al., 2018), may be associated with less procyclicality and even result into higher economic growth (Aghion et al., 2007). To date, however, the growth and stabilizing effects of fiscal rules remain unclear, and the empirical evidence on their capacity to stabilize the debt-to-GDP ratios in the long-run is particularly fragile.

The goal of this paper is precisely to investigate theoretically the impact of flexible fiscal rules in a simple endogenous growth setup. This framework follows from two motivations. First, the adoption of a fiscal rule is a reform that is rarely reversed and its impact must be assessed during a sufficiently long period, hence the choice of a growth model. Second, as public indebtedness is a long-lasting process, assessing the (de)stabilizing properties of fiscal rules must be addressed both in the short and the long run, especially in economies with increasing public debt. Exogenous growth setups are irrelevant from this perspective, since there is no economic growth in the long run and public deficits are only transitory (as in Schmitt-Grohé and Uribe, 1997, section 3). In contrast, endogenous growth models provide a useful tool because their properties are compatible with the existence of growing public debt in the long-run.

Relying on these arguments, we build an endogenous growth model with elastic labor supply, close to Turnovsky (2000). However, while Turnovsky (2000) discusses the effects of various distortionary taxes, we introduce public debt and fiscal deficits. We choose to explore the effects of a particular type of fiscal rule, namely deficit rules. This choice arises from both empirical and theoretical considerations. On the empirical side, compared with other type of fiscal rules (i.e. debt, expenditure, or revenue rules), deficit rules are by far the most popular—as documented by Schaechter et al. (2012)—and are currently in place in more than 80 countries according to the IMF Fiscal Rules Dataset. On the theoretical side, debt rules such as those considered in Minea and Villieu (2012), Nishimura et al. (2015), and Cheron et al. (2019) would be too restrictive for our purpose, given that they impose an exogenous debt-to-GDP ratio in the long run.

To assess the role of the flexible design of fiscal rules, we consider a general deficit rule whereby the deficit-to-output ratio depends both on a fixed component (as in "first-generation" rules), and a time-varying component that captures the response of the primary deficit to the public-debt burden. The motivation of this last component drives from the literature on the so-called "fiscal reaction function" (see Bohn, 1998) that requires a positive reaction of the primary balance to public debt to ensure debt sustainability. As we will show, the reaction of the primary balance to the debt burden, namely how the government sterilizes the debt burden by increasing taxes or reducing primary expenditure, has a decisive influence on the behaviour of our model.

Our results are as follows.

(*1*) *Deficit rules lead to multiple equilibria.* Up to four equilibria appear in our model:

two high-growth equilibria, a low-growth trap, and a catastrophic equilibrium where the economy disappears. The mechanism leading to multiplicity comes from two ingredients: (i) the endogenous labor supply and (ii) the deficit fiscal rule. In our model, low economic growth produces a high public debt-to-GDP ratio, which generates a large public debt burden. Under the fiscal rule, the tax rate must rise to sterilize part of this burden. With an elastic labor supply, this additional tax bill discourages households from working, which reduces labor supply and output. This hurts the return of savings and economic growth. This mechanism, which reverses in the case of a high economic growth, generates multiple self-fulfilling equilibria, leading to a selection problem in the form of a global indeterminacy.

(*2*) *Deficit rules can lead to large and long-lasting fluctuations.* In the neighborhood of the low-growth trap, stable limit-cycles appear due to the occurrence of a (supercritical) Hopf bifurcation.[2] These cycles expand as the government sterilizes further the debt burden. Hence, contrary to the standard policy recommendation, a strong reaction of the primary surplus to the debt burden is not likely to stabilize the economy. At the end, the limit-cycle degenerates into a homoclinic orbit, i.e. a path that joins a steady state to itself.[3] Our model thus can reproduce the very long-lasting debt cycles that appear in the data.[4]

(*3*) *Deficit rules can generate hysteresis and strong sensitivity to changes in parameters.* A small change in the reaction of the primary deficit to the public debt burden irreversibly affects the balanced-growth path along the steady state. In particular, a tight fiscal policy (i.e. a high sterilization coefficient of the deficit rule) may condemn the economy to a low-growth/high-debt trap; while, for very close values of this coefficient, the economy can reach a high-growth steady state.

Quantitatively, our results are supported by a calibration using OECD historical data. Multiple equilibria, public debt cycles, and hysteresis emerge for reasonable values of the economic growth and public-debt ratio. Moreover, by extending our model to include stochastic shocks, we obtain irregular endogenous public-debt cycles with both high and low periodicity. Such long-cycles are experienced in several developed and developing countries, but let unexplained by New-Keynesian DSGE models that focus on short-term fluctuations.

Our model addresses several major topics in macroeconomics.

[2]The local stability of these cycles implies that a small perturbation of a parameter would not remove debt cycles. We numerically show that such cycles appear for a wide range of calibrations.

[3]The homoclinic orbit triggered by deficit rules mirrors the findings of Benhabib et al. (2001) regarding monetary (Taylor) rules. This orbit is associated to the occurrence of a Bogdanov-Takens bifurcation, recently used to characterize Beveridge cycles by Sniekers (2018).

[4]See e.g. Reinhart and Rogoff (2011), Abbas et al. (2011), or Poghosyan (2015).

From a theoretical perspective, compared to existing papers showing that indeterminacy and multiplicity appear in endogenous growth models with public debt (Futagami et al., 2008; Minea and Villieu, 2012; Cheron et al., 2019), our paper offers three main contributions.

First, we explore the implications of flexible deficit rules, fort the first time to the best of our knowledge, and highlight the critical role played by the sterilization of the debt burden.[5] Second, we provide a meticulous inspection of various types of bifurcations, such that the Hopf and Bogdanov-Takens ones, which are of crucial importance to establish the cyclical behavior of public debt and growth. Third, by introducing stochastic shocks, our setup is able to reproduce long-lasting debt cycles whose periodicity goes beyond usual short-period cycles detectable by standard New-Keynesian DSGE models.

Additionally, our paper joins two strands of literature on indeterminacy in growth models. *(i) Increasing returns with endogenous labor supply.* In the pioneer work of Benhabib and Farmer (1994), a positively-sloped labor demand is a necessary condition for indeterminacy to emerge.[6] Subsequent studies released this constraint by using e.g. multi-sectoral frameworks (see Benhabib et al., 2000). *(ii) Fiscal policy with distortionary taxation.* Following Schmitt-Grohé and Uribe (1997) a number of contributions illustrate that the use of distorsive taxes under a balanced-budget rule can generate local indeterminacy (see, e.g., Giannitsarou, 2007; Ghilardi and Rossi, 2014). These results have been refined using public capital externalities (Park and Philippopoulos, 2004; Chen, 2006), or auxiliary assumptions about preferences (Nourry et al., 2013) or the production function (Ghilardi and Rossi, 2014).

Contrary to these studies, our multiplicity and indeterminacy results emerge in a one-sector model with a decreasing labor demand, wasteful public spending, and an additive utility function. Our paper combines the effect of an elastic labor supply and distorsive taxes in the context of a growing public debt, producing a new channel of indeterminacy that relies on the degree of sterilization of the debt burden in the flexible fiscal rule.

From a policy perspective, the perils of deficit rules emerge from two key elements:

(1) *The sterilization of the debt burden in the rule.* We first show that, with positive public debt, a necessary condition for the existence of a steady state is that the primary surplus' reaction to the debt burden is higher than some threshold. This echoes a long-lasting literature going back to Blinder and Solow (1973) and Tobin and Buiter (1976), and revisited more recently by Bohn (1998, 2008), suggesting that "*a strictly positive response of primary surpluses to the debt ratio is sufficient for sustainability*".[7]

[5]Note that our results rely on elastic labor supply, without resorting to productive public spending, as the aforementioned papers.

[6]The survey of Benhabib and Farmer (1999) provides a thorough discussion of this condition.

[7]In our model, however, the response of the primary balance needs non only to be positive but higher than some critical level. This feature is consistent with recent works that establish lower bounds for the

Furthermore, we show that an *excessive* reaction of the primary balance also has undesirable effects, including indeterminacy, multiplicity, or long-lasting fluctuations.[8] In particular, the higher the sterilization of the debt burden, the larger the oscillations of public debt and economic growth, and the higher the area of indeterminacy. Overly-ambitious fiscal rules can thus be counterproductive.

(2) *Short- and long-run destabilizing effects.* In our setup, flexible deficit rules are not only associated with undesirable short-run fluctuations, but also with large oscillations of public debt and economic growth in the long-run. Moreover, such rules are fragile because very small changes in their parameters may produce radical shifts in the dynamics. These features do not come from inadequate responses to exogenous shocks (as, for example, in New Keynesian models) but from endogenous forces that govern the interaction between the households' behavior and the government's intertemporal budget constraint.

A usual criticism of endogenous fluctuations is that the cycles they produce are too regular compared with the data. We address this issue in two ways. On the one hand, the orbits we obtain in the neighborhood of the Bogdanov-Takens bifurcation lead to asymmetric cycles with long periods of nearby-stationary growth and sudden short-living recessions. On the other hand, by extending our setup to include random (technology) shocks, the limit-cycles are deformed both in frequency and amplitude, as in Beaudry et al. (2020). Thus, our setup can produce short- and long-run fluctuations in the public debt-to-GDP ratio that appear reasonable in relation to the observations.

The paper is organized as follows. Section 2 presents the model, section 3 analyzes the steady state(s), which are quantitatively characterized in section 4. Section 5 studies local and global dynamics, section 6 discusses public debt cycles, section 7 extends our model to include stochastic shocks, and section 8 delivers some concluding remarks.

2. The model

We consider a closed economy populated by a continuum of representative individuals whose total measure is 1, and a government. Each representative agent consists of a household and a competitive firm. All agents are infinitely-lived and have perfect foresight. For each variable, we denote individual quantities by lower case letters (x), and aggregate quantities by corresponding upper case letters (X), with $x = X$ at equilibrium since the continuum of agents has unit measure.

2.1. Households

The representative household starts at the initial period with a positive stock of capital (k_0), and chooses the path of consumption $\{c_t\}_{t\geq 0}$, hours worked $\{l_t\}_{t\geq 0}$, and

primary surplus' reaction, especially in high debt contexts (Mendoza and Ostry, 2008).

[8] Such undesirable features arise for values of sterilization consistent with the estimations of Bohn (1998), Mendoza and Ostry (2008), or Collignon (2012).

capital $\{k_t\}_{t>0}$, such as to maximize the present discount value of his lifetime utility

$$U = \int_0^{\infty} e^{-\rho t} \{u(c_t) - v(l_t)\} \, dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate.

The utility function is separable in consumption and leisure,[9] and we consider $u(c_t) = \log(c_t)$, and preferences for leisure are such that $v(l_t) = \frac{B}{1+\varepsilon} l_t^{1+\varepsilon}$, where $\varepsilon \geq 0$ is the constant elasticity of intertemporal substitution in labour, and $B > 0$ a scale parameter.

The household uses labor income ($w_t l_t$, where w_t is the hourly wage rate) and capital revenues ($q_t k_t$, where q_t is the rental rate of capital), to consume (c_t), invest ($\dot{k}_t$), and buy government bonds (d_t), which return the real interest rate r_t. As owners of firms, he receives profits (π_t) that are redistributed on the form of lump-sum transfers.[10] In addition, he pays taxes on wages ($\tau_t w_t l_t$, where τ_t is the wage tax rate) and other non-wage based taxes (z_t) that can be considered as lump-sum for simplicity;[11] hence the following budget constraint

$$\dot{k}_t + \dot{d}_t = r_t d_t + q_t k_t + (1 - \tau_t) w_t l_t - c_t - z_t + \pi_t. \tag{2}$$

The first order conditions for the maximization of the household's programme give rise to the dynamic Euler relation (with $q_t = r_t$ in competitive equilibrium)

$$\frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{3}$$

and to the static relation

$$\frac{(1 - \tau_t) w_t}{c_t} = B l_t^{\varepsilon}. \tag{4}$$

Eq. (3) is the familiar Keynes-Ramsey rule that governs intertemporal consumption choices. Eq. (4) shows that, at each period t, the marginal gain of hours worked (the net real wage $(1 - \tau_t) w_t$, expressed in terms of marginal utility of consumption $1/c_t$), just equals the marginal cost ($B l_t^{\varepsilon}$).

Finally, the optimal path of consumption has to verify the set of transversality conditions

$$\lim_{t \to +\infty} \{\exp(-\rho t) \; u'(c_t) \, k_t\} = 0 \text{ and } \lim_{t \to +\infty} \{\exp(-\rho t) \; u'(c_t) \, d_t\} = 0,$$

[9] In the literature, indeterminacy often relies upon a non-separable utility function (see, e.g., Benhabib et al., 2001; Nourry et al., 2013). We choose a separable utility function to build a new channel of indeterminacy based on fiscal rules, without resorting to specific restrictions on the utility function.

[10] In our calibration, thanks to constant returns-to-scale in private factors, pure profit will be zero, i.e. $\pi_t = 0$.

[11] Such taxes reflect government's resources that are not directly based on work (e.g., taxes on consumption, land, capital, residential taxes, seignorage, etc) that we do not model explicitly.

ensuring that lifetime utility U is bounded.

2.2. Firms

Output of the individual firm (y_t) is produced using a Cobb-Douglas production function with an aggregate knowledge externality (X_t) that generates positive technological spillovers onto firms' productivity

$$y_t = \tilde{A} X_t^{\eta} l_t^{\alpha} k_t^{1-\eta}, \tag{5}$$

where l_t and k_t respectively stand for labor and physical capital, $\tilde{A} > 0$ is a scale parameter, $\alpha \in (0,1)$ is the elasticity of output to labor, and $\eta \in (0,1)$ is the magnitude of the knowledge externality. Thus, the individual firm faces positive, but diminishing, marginal physical products in all factors. In addition, we assume non-increasing returns to scale in the private factors, capital and labor ($\alpha \leq \eta$), but constant returns to scale in private capital and knowledge.[12]

The production function (5) just corresponds to the one in Turnovsky (2000), except that the knowledge externality does not come from productive public spending but from human capital, as in Romer (1986). In this way, we assume that knowledge is produced by the economy-wide levels of physical capital (K_t) and labor (L_t), namely $X_t = K_t L_t$.[13]

The resulting individual production is then similar to the one used in Benhabib and Farmer (1994, p. 22) with aggregate capital and labor externalities, namely $y_t = \tilde{A} k_t^a l_t^b K_t^{a\theta_1} L_t^{b\theta_2}$, with $a := 1-\eta$, $b := \alpha$, $\theta_1 = \eta/(1-\eta)$ and $\theta_2 = \eta/\alpha$. Nevertheless, contrary to Benhabib and Farmer (1994), our indeterminacy results do not, as we will see, rely on increasing returns in aggregate labor (i.e. $b(1+\theta_2) > 1$), and we restrict our attention to the case where the aggregate labor demand is normal, namely with decreasing marginal returns on the whole (i.e. $\alpha + \eta < 1$).

The first order conditions for profit maximization (relative to private factors) are

$$w_t = \alpha \frac{y_t}{l_t}, \tag{6}$$

$$r_t = (1-\eta)\frac{y_t}{k_t}. \tag{7}$$

At the aggregate level, the knowledge externality will allow reaching an endogenous growth path, because the social return of capital is not decreasing. Effectively, the aggregate production function is

$$Y_t = \tilde{A} K_t L_t^{\sigma}, \tag{8}$$

[12] In order to ensure perpetual growth, the production function must be linearly homogeneous in the factors that are being accumulated (capital and knowledge).

[13] Human capital externalities, i.e. the fact that your coworkers' human capital makes you more productive, are well documented in empirical literature (see, e.g. Rauch, 1993; Moretti, 2004).

where $\sigma := \alpha + \eta$. The aggregate production function exhibits increasing returns-to-scale, as soon as $\sigma > 0$. However, as we will see in the calibration performed in section 4, our results are derived for small increasing returns-to-scale (i.e. low values of σ).

2.3. The government

The government provides public expenditure G_t, levies taxes ($\tau_t w_t L_t + Z_t$), and borrows from households. The fiscal deficit is financed by issuing debt ($\dot{D}_t$); hence, the following budget constraint

$$\dot{D}_t = r_t D_t - S_t, \tag{9}$$

where the primary fiscal surplus is

$$S_t = \tau_t w_t L_t + Z_t - G_t. \tag{10}$$

Let us now introduce the fiscal rule that governs the behaviour of the deficit-to-GDP ratio. As the literature has long acknowledged, targeting a constant primary deficit-to-GDP ratio does not preclude the snowball dynamics of public indebtedness. Then, a general deficit-to-GDP target is more likely to stabilize the public debt, because in this case the debt burden is sterilized by adjustments in the primary surplus (see Blinder and Solow, 1973; Liviatan, 1982). More recently, a number of authors have highlighted that the reaction of the primary surplus to changes of public debt is indeed decisive for the stability of the fiscal stance (see, e.g., Bohn, 1998; Ghosh et al., 2013; Mauro et al., 2015).

To take account of these findings, we consider the following deficit rule

$$\dot{D}_t = \theta Y_t + (1 - \phi) r_t D_t. \tag{11}$$

According to this rule, the deficit-to-output ratio ($\dot{D}_t / Y_t$) has two components: an autonomous component ($\theta \geq 0$), and a time-varying component that depends on the public debt burden relative to output ($(1 - \phi) r_t D_t / Y_t$).[14] Since the primary surplus writes $S_t = \phi r_t D_t - \theta Y_t$, the parameter $\phi \geq 0$ measures the reaction of the primary budget surplus (relative to GDP) to the public debt burden-to-GDP ratio. We call this parameter the "sterilization coefficient" of the debt burden. If $\phi = 0$, the government targets the primary deficit ratio ($-S_t / Y_t = \theta$); while if $\phi = 1$, it targets the general deficit ratio ($\dot{D}_t / Y_t = \theta$), in which case the primary deficit has to fully adjust to the debt burden ($-S_t / Y_t = \theta - r_t D_t / Y_t$).

As a first motivation, the fiscal rule (11) generalizes those considered in the previous literature on fiscal deficits and endogenous growth. Minea and Villieu (2012) and Menuet et al. (2018) use a simple deficit-to-GDP rule that corresponds to $\phi = 1$ in the present

[14] This rule can characterize a kind of "second-generation" fiscal rule with a flexible component, as opposed to a "first-generation" fiscal rule that would require a fixed deficit-to-GDP ceiling.

model. Futagami et al. (2008) and Cheron et al. (2019) assume deficit rules in which the public debt ratio gradually adjusts to a long-run target, without taking into account the impact of the debt burden. However, this impact is decisive for the dynamic behavior of public debt, and deserves to be carefully analyzed.

This leads to the second motivation of our fiscal rule (11). A large literature shows that a positive reaction of the primary surplus to public debt is a prerequisite for public debt sustainability (i.e. $\phi > 0$). Bohn (1998) finds that the U.S. government responses to increases in the debt-to-GDP ratio by raising the primary surplus (the estimated reaction coefficients are between 2.8% and 5.4%), and concludes that such a positive response is sufficient for sustainability. However, subsequent works (see e.g. Mendoza and Ostry, 2008; Collignon, 2012; Ghosh et al., 2013; Daniel and Shiamptanis, 2013; Mauro et al., 2015) challenge this claim and suggest minimum thresholds for the reaction of the primary balance to ensure public debt sustainability. While our findings are in some respects consistent with this literature,[15] the question of the extent to which the government should use the primary surplus to sterilize the debt burden is at the heart of our paper. We will notably show that too much sterilization (i.e. a too high ϕ) is not necessarily desirable for macroeconomic stability.

2.4. Equilibrium

We focus on the equilibrium in a decentralized economy in which all household-firm units behave similarly.

Definition 1. A competitive equilibrium is a path of endogenous variables $\{\tau_t, w_t, r_t, c_t, l_t, y_t, k_t, S_t, C_t, L_t, K_t, Y_t, D_t\}_0^\infty$ that solves Eqs. (3), (4), (6), (7), (8), (9), (10), and (11), equilibrium relations $c_t = C_t$, $k_t = K_t$, $l_t = L_t$, $y_t = Y_t$, and satisfies the set of transversality conditions and the goods market equilibrium $\dot{K}_t = Y_t - C_t - G_t$, given exogenous paths for G_t and Z_t.

In our endogenous growth model, in the long run the economy reaches a balanced-growth path (BGP), namely a competitive equilibrium where consumption, private capital, public debt, and output grow at the same (endogenous) rate. To obtain long-run stationary ratios, we deflate all growing endogenous variables by the capital stock, i.e. (we henceforth omit time indexes): $y_k := Y/K$, $c_k := C/K$, and $d_k := D/K$. In addition, exogenous variables must be homogenous of degree one relative to output to ensure the feasibility of a constant growth path. Thus, we assume that the government claims a part g of aggregate output for public spending ($G_t/Y_t = g \in [0, 1]$) and that non-wage

[15] From Eq. (11), we have $(\dot{D_t}/Y_t) = \theta + [(1-\phi)r_t - (\dot{Y_t}/Y_t)](D_t/Y_t)$; thus, along the long-run balanced-growth path where $\gamma = \dot{Y}_t/Y_t$ and $r = \gamma + \rho$, we find $(\dot{D_t}/Y_t) = \theta + (\rho - \phi r)(D_t/Y_t)$. Even if the stability of the public debt-to-output ratio depends on the whole evolution of the system and cannot be assessed from this partial relationship solely, the rate of growth of the public debt ratio is negatively linked to its level only if $\phi > \rho/r$.

taxes are a constant share of output ($Z_t/Y_t = \mu \in [0,1]$). Given the fiscal rule (11), the tax rate on wages is then the adjustment variable in the government's budget constraint, as in Schmitt-Grohé and Uribe (1997).

Using Eqs. (6), (7), (9) and (11), it follows that

$$\tau = \frac{(1-\eta)\phi d_k + g - \mu - \theta}{\alpha} = 1 - \left(\frac{\bar{d} - (1-\eta)\phi d_k}{\alpha} \right), \tag{12}$$

where $\bar{d} = \alpha + \mu + \theta - g$. This relation can be viewed as the tax reaction function that governs the response of the primary balance to the debt burden. To ensure that $\tau \in (0,1)$, we make the following assumption.

Assumption 1. $(1-\eta)\phi d_k < \bar{d} < \alpha + (1-\eta)\phi d_k$

From (4), (7), and (8), we obtain the equilibrium level of output

$$y_k = A \left(\frac{\alpha(1-\tau)}{c_k} \right)^{\psi}, \tag{13}$$

where $\psi := \frac{\sigma}{1+\varepsilon-\sigma}$, and $A := \tilde{A} \left(\frac{\tilde{A}}{B} \right)^{\psi}$.

Assumption 2 (Normal labor demand) $\sigma < 1$.

Under Assumption 2, labor demand is normal, i.e. decreasing with real wages.[16] This important feature means that, in our setup, a positively-sloped labor-demand curve in not needed to generate indeterminacy, as in Benhabib and Farmer (1994) and Farmer and Guo (1994).[17] Assumption 2 is a sufficient (unnecessary) condition for $\psi > 0$; hence there is an inverse relationship between the consumption ratio and the output ratio in Eq (13). This relation comes from the labor market equilibrium (4). Following an increase in the consumption ratio, the marginal utility of consumption decreases and induces households to substitute leisure for working hours (since $\varepsilon \geq 0$, leisure and consumption are complement in equilibrium). Then, the equilibrium labor supply and output are reduced (the same result arises following an increase in the tax rate on wages).

[16] Indeed, from (6) and (8), the aggregate labor demand writes $L_t = (w_t/\alpha \tilde{A} K_t)^{1/(\sigma-1)}$.

[17] In Benhabib and Farmer (1994, p. 30) a necessary condition for indeterminacy is that (using our notations): $\sigma > 1 + \varepsilon$. This implies that the aggregate labor demand must increase with real wages (see Eqs. (6) and (8) with $\sigma - 1 > \varepsilon \geq 0$). For the labor demand to slope up with real wages, increasing returns should be large, as discussed by Benhabib and Farmer (1994) and Schmitt-Grohé and Uribe (1997). In our model, as we have seen, we assume $\sigma < 1 < 1 + \varepsilon$, such that labor demand is normal, i.e. decreasing with real wages. We nevertheless obtain indeterminacy, thanks to the public debt dynamics. In addition, in our model, indeterminacy is consistent with lowly-increasing social returns, as illustrated by our quantitative analysis in section 4.

The optimal aggregate consumption behaviour is, from (3) and (7),

$$\frac{\dot{C}}{C} = (1-\eta)y_k - \rho, \tag{14}$$

and the path of the capital stock is given by the goods market equilibrium

$$\frac{\dot{K}}{K} = (1-g)y_k - c_k. \tag{15}$$

The path of the public debt follows the definition of the deficit ratio, namely

$$\frac{\dot{D}}{D} = \theta\frac{y_k}{d_k} + (1-\phi)(1-\eta)y_k. \tag{16}$$

Hence, the reduced-form of the model is obtained by Eqs. (14), (15) and (16)

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = (g-\eta)y_k - \rho + c_k, & \text{(a)} \\[2ex] \dfrac{\dot{d}_k}{d_k} = \dfrac{\theta y_k}{d_k} + (1-\eta)(\bar{\phi}-\phi)y_k + c_k, & \text{(b)} \end{cases} \tag{17}$$

where $\bar{\phi} := (g-\eta)/(1-\eta) > 0$. From Eqs. (12) and (13), we have

$$y_k = A\left(\frac{\bar{d} - (1-\eta)\phi d_k}{c_k}\right)^{\psi} =: y_k(c_k, d_k). \tag{18}$$

In equilibrium, any increase in the debt ratio (d_k) reduces the output ratio (y_k). Indeed, the growing interest-burden of public debt leads to more taxes on wages, which discourages labor supply. The same crowding-out effect applies in case of an increase in public spending, through coefficient $\bar{d}$.

Definition 2. A steady state i is a competitive equilibrium where consumption, capital, output, and public debt grow at the common (endogenous) rate γ^i, such that $\dot{c}_k = \dot{d}_k = 0$ in (17). At any steady state i, the economy is characterized by a BGP: $\gamma^i := \dot{C}/C = \dot{K}/K = \dot{Y}/Y = \dot{D}/D$, while the real interest rate (r^i) is constant.

To ensure long-run solutions with positive public debt we make the following assumption.

Assumption 3. $\phi > \bar{\phi}$

From (17.b), in steady state ($\dot{d}_k = 0$), the public debt ratio is $d_k = \theta y_k/[(1-\eta)(\phi - \bar{\phi})y_k - c_k]$. As public debt is positive in the data, $\phi > \bar{\phi}$ is an innocuous assumption, consistent with numerous empirical studies finding that the reaction of the primary balance

to the public debt must exceed some threshold to rule out unsustainable debt dynamics (see e.g. Mendoza and Ostry, 2008; Collignon, 2012; Ghosh et al., 2013; Daniel and Shiamptanis, 2013; Mauro et al., 2015).

The following section characterizes the steady-state solutions.

3. Long-run solutions

The long-run endogenous growth solutions are described by two relations between c_k and d_k. The first one is the $\dot{c}_k = 0$ locus, which comes from the Euler relation (14) and the goods market equilibrium (15)

$$d_k = \frac{1}{(1-\eta)\phi}\left\{\bar{d} - c_k\left(\frac{\rho - c_k}{(g-\eta)A}\right)^{1/\psi}\right\}. \tag{19}$$

The second relation is the $\dot{d}_k = 0$ locus, related to the government's budget constraint (9), and the deficit rule (11)

$$\theta y_k(c_k, d_k) = \left\{(1-\eta)(\phi - \bar{\phi})y_k(c_k, d_k) - c_k\right\} d_k. \tag{20}$$

Steady-state solutions are obtained as the crossing-point of Eqs. (19) and (20), as depicted in Figure 1.

Proposition 1. *There are three positive-growth candidates for a steady-state (points M, P and Q), and a no-growth degenerate solution (point D).*

Proof. (i) Eq. (19) depicts a U-shaped curve in the (c_k, d_k)-plane, with a minimum at $\hat{c}_k = \psi\rho/(1+\psi) < \rho$. (ii) Eq. (20) depicts a bell-shaped curve in the (d_k, c_k)-plane, with a maximum at $\bar{d}_k$. The maximal consumption ratio in this curve is denoted by $\bar{c}_k$.[18]

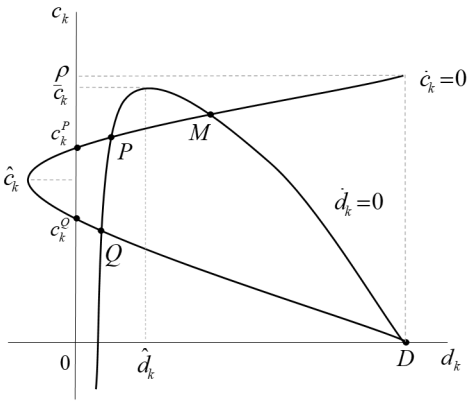


Figure 1: The steady states

[18] For small values of θ, we can approximate $\hat{d}_k$ and $\bar{c}_k$ by $\hat{d}_k \approx \frac{1}{1-\eta}\sqrt{\frac{\theta\bar{d}}{\psi(\phi-\bar{\phi})}}$ and $\bar{c}_k \approx [A\bar{d}^{\psi}(1-\eta)(\phi-\bar{\phi})]^{1/(1+\psi)}$, respectively.

As Figure 1 shows, four crossing-points between these two curves can emerge. A trivial solution, denoted by D, is associated to $c_k = 0 =: c_k^D$ and $d_k = \bar{d}/(\phi(1-\eta)) := d_k^D$ (in this case, $y_k^D = 0$). The couple (c_k^D, d_k^D) is such that the economy asymptotically vanishes. We will refer to this solution as a "harrodian" perspective. Although such a collapse solution is not economically attractive, it cannot be rejected without assessing the dynamics of the model, as we will see.[19] The areas for which the different positive-growth candidates emerge (or not) as steady-state solutions are discussed in subsection 3.2.

3.1. Some intuition

The multiplicity of steady states comes from the interaction of two non-monotonic relationships between the consumption (c_k) and the debt (d_k) ratios in system (17). Let us explain these two relationships.

The first form of multiplicity comes from Eq. (17a) that writes, at the steady state

$$\frac{\dot{c}_k}{c_k} = \underbrace{[(1-\eta)y_k - \rho]}_{\dot{C}/C} - \underbrace{[(1-g)y_k - c_k]}_{\dot{K}/K} = 0.$$

This relation depicts a non-monotonic linkage between c_k and d_k, through the effect of the consumption ratio. Indeed, any increase in c_k has two consequences.

(1) First, given y_k, a rise in the consumption ratio reduces private investment in the goods market equilibrium (namely, $\dot{K}/K$ decreases). As a result, through channel (1), $\dot{c}_k/c_k$ positively depends on c_k.

(2) Second, thanks to the endogenous labor supply, the rise of the consumption ratio reduces y_k, with two implications: (i) the return on private capital is weakened, thereby reducing the incentive for saving and the growth rate of consumption $\dot{C}/C$ in the Keynes-Ramsey rule (14); (ii) in the goods market equilibrium, the decrease in y_k causes private investment to decline, i.e. $\dot{K}/K$ falls. The strength of (i) depends on the elasticity of private capital $(1-\eta)$, while the strength of (ii) depends on the supply net of public expenditures $(1-g)$. If $1-\eta > 1-g \Leftrightarrow g > \eta$, following a decrease in y_k, consumption will decline more sharply than the stock of private capital ($\dot{C}/C - \dot{K}/K > 0$). As a result, through channel (2), $\dot{c}_k/c_k$ negatively depends on c_k.

Consequently, the consumption ratio exerts a non-monotonic impact on $\dot{c}_k/c_k$: for small values of c_k the channel (1) prevails, while for high values the channel (2) predominates. At the steady state, for the equality $\dot{c}_k/c_k = 0$ to be restored it is needed that the debt ratio d_k adjusts. As y_k negatively depends on d_k through the endogenous labor supply, the relation $\dot{c}_k = 0$ describes a U-shaped curve in the (c_k, d_k)-plane: c_k and d_k are positively (negatively) linked for $c_k > (<)\hat{c}_k$, as depicted in Figure 2a. In this way,

[19] Households' preferences are defined only for $c_t > 0$, but the steady state D can be asymptotically reached with $\lim_{t\to+\infty} c_t = 0^+$.

there are two values of the consumption ratio that are consistent with each value of the debt ratio, provided that $d_k > \hat{d}_k$.

Two points deserve particular attention. First, at this stage, multiplicity does not come from public debt, since the non-monotonic effect of c_k remains even if $d_k = 0$. Second, multiplicity has nothing to do with increasing returns in aggregate labor (namely, an increasing labor demand, as in Benhabib and Farmer 1994 or Schmitt-Grohé and Uribe 1997). In our setup, multiplicity comes from the differential effect of the consumption ratio c_k on the sphere of consumer goods (which runs via parameter η through the return on savings) and on the sphere of investment goods (which runs via parameter g through the supply net of public demand). In particular, if $g < \eta$, multiplicity vanishes in Eq. (17.a) because channels (1) and (2) both provide a positive linkage between c_k and d_k.

Our assumption that $g > \eta$ is in line with empirical works. In our model η is the elasticity of output to the stock of knowledge, which can be assimilated to human capital. On the one hand, a large empirical literature (see e.g. Moretti 2004 for a survey) found that the human capital externalities are roughly small. For example, Rauch (1993) and Ciccone and Peri (2006) suggested that these externalities are around 3-8%. Acemoglu and Angrist (2000) highlighted that external returns to education are close to 1%. In their new dataset, Cohen and Soto (2007) showed that the elasticity of output to the years of schooling are between 0.049 and 0.123 depending on the econometric specification. On the other hand, in OECD economies, the long-term average ratio of public spending to GDP is close to 23%. Consequently, our condition $g > \eta$ holds considering all the aforementioned empirical estimates.

a: Consumption ratio ($\dot{c}_k = 0$) b: Public debt ratio ($\dot{d}_k = 0$)

Figure 2: The stationary loci

The second form of multiplicity comes from Eq. (17.b) that writes, at the steady state

$$\frac{\dot{d}_k}{d_k} = \underbrace{\left[\theta\frac{y_k}{d_k} + (1-\phi)(1-\eta)y_k\right]}_{\dot{D}/D} - \underbrace{\left[(1-g)y_k - c_k\right]}_{\dot{K}/K} = 0.$$

This relation depicts a non-monotonic linkage between c_k and d_k, through the impact of the debt ratio. Any increase in d_k has two effects. (1) The associated rise in the debt burden leads to an increase in the tax rate that reduces the labor supply and the output ratio. This negatively affects capital accumulation $\dot{K}/K$. (2) Simultaneously, the growth rate of public debt ($\dot{D}/D = (1-\phi)r + \theta Y/D$) is also reduced, since the real interest rate falls downs (following the decrease in the output ratio) and the autonomous public deficit ($\theta Y/D = \theta y_k/d_k$) also declines.

This gives rise to a non-monotonous impact of the debt ratio on $\dot{d}_k/d_k$: for small values of d_k the effect (1) prevails, while for high values the effect (2) predominates. At the steady state, for the equality $\dot{d}_k/d_k = 0$ to be restored it is needed that the consumption ratio c_k adjusts. As c_k negatively affects the accumulation of private capital $(\dot{K}/K)$, we obtain a non-monotonic relation between d_k and c_k: the curve $\dot{d}_k = 0$ is first increasing until $d_k = \bar{d}_k$, then decreasing, as depicted in Figure 2b.

Hence, there are two values of the debt ratio that are consistent with each value of the consumption ratio, provided that $c_k < \bar{c}_k$. Interestingly, this multiplicity remains under a balanced-budget rule, which is characterized in our setup by $\theta = 0$ and $\phi = 1$ (i.e. $\dot{D} = 0$). Thus the adoption of stringent fiscal rules does not preclude the possibility of multiplicity (and indeterminacy, as we will see).

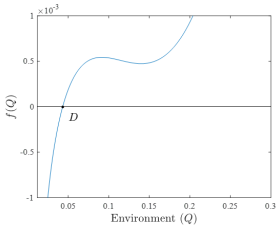
By grouping the two non-monotonic relations, we can obtain up to four long term equilibria, as shown in Figure 1. Depending on the value of parameters, the two curves can be positioned differently and show only 2 or 3 steady states. Let us now characterize the different regimes.

3.2. Characterization of regimes

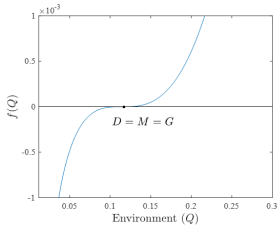
Figure 2 depicts the different configurations of long-run equilibria. In the most general case (denoted regime $\mathcal{R}_3$ in Figure 2c) the two steady-state curves intersect four times, giving rise to four steady-states: Q, P, M, and D. The associated BGPs are such that $0 = \gamma^D < \gamma^M < \gamma^P < \gamma^Q$, corresponding to the inverse ranking of the public debt ratio $d_k^Q < d_k^P < d_k^M < d_k^D$. This regime arises for intermediate values of the sterilization coefficient (ϕ). For small values of this coefficient (i.e. $\bar{\phi} < \phi < \phi_1$), only the steady states Q and D occur (regime $\mathcal{R}_1$ in Figure 2a). For high values of this parameter (i.e. $\phi > \phi_2$), only the steady states M and D arise (regime $\mathcal{R}_2$ in Figure 2e).

Hence, regimes $\mathcal{R}_1$ and $\mathcal{R}_2$ are located on either sides of two saddle-node bifurcations that surround regime $\mathcal{R}_3$.[20] The first bifurcation (labelled SN_1 in Figure 2b), appears at $\phi = \phi_1$, and is such that points P and M collide. For a small change in the sterilization coefficient, the economy switches from regime $\mathcal{R}_3$ into regime $\mathcal{R}_1$. The second bifurcation (labelled SN_2 in Figure 2d), appears at $\phi = \phi_2$, and is such that points P and Q collide. For a small change in ϕ the economy switches from regime $\mathcal{R}_3$ into regime $\mathcal{R}_2$.

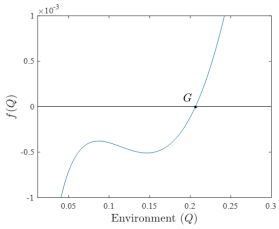
[20] A saddle-node bifurcation arises when two steady states of a dynamic system collide and annihilate each other.

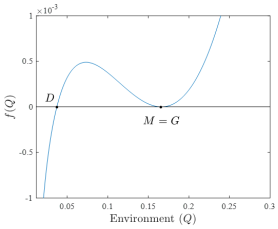


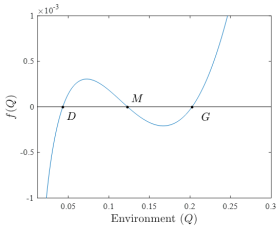
a. Regime $\mathcal{R}_1$

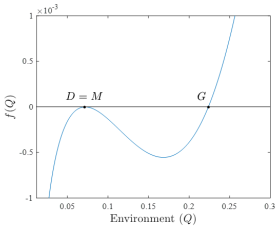


b. SN_1 Bifurcation









c. Regime $\mathcal{R}_3$

d. SN_2 Bifurcation

e. Regime $\mathcal{R}_2$

Figure 3: Topological regimes ($\hat{c}_k < \bar{c}_k$)

Apart from being the most general case, regime $\mathcal{R}_3$ is economically the most relevant one. On the one hand, as shown in our calibration below, the sterilization coefficient of the debt burden ϕ is likely to locate between ϕ_1 and ϕ_2 in most industrialized countries. On the other hand, long-lasting fluctuations of public debt and economic growth that are evidenced by several empirical studies (see e.g. Reinhart and Rogoff, 2011; Abbas et al., 2011; Poghosyan, 2015, and our section 6) appear in this regime only. Then, from a policy perspective, regime $\mathcal{R}_3$ allows studying the impact of sterilization on the magnitude of debt and growth cycles. We henceforth focus on this regime in the rest of the paper.

4. A quantitative assessment

This section provides a calibration based on historical data from industrialized countries showing that the different steady states arise for plausible parameters' values.[21] Thanks to this calibration, we study how changes in the sterilization coefficient (ϕ) affect the existence and value of steady states.

[21] Our quantitative results (and in particular the occurrence of various types of bifurcations) of are not only valid for one specific calibration, but can be obtained under (possibly very) different vectors of parameters. By varying continuously one (or several) parameters, the topological behaviour of the model (i.e. the number of steady states or their stability) may suddenly shift: these critical values of parameters define the bifurcations.

4.1. Calibration

In our benchmark calibration, we choose $\rho = 0.02$, corresponding to the long-run value of the risk free (real) interest rate. The labor elasticity of substitution is fixed at $\varepsilon = 0$, thus characterizing an infinite Frisch elasticity as usual in business cycle models (as in Schmitt-Grohé and Uribe, 1997). Regarding the technology, we set $A = 0.08$ to obtain realistic rates of economic growth, and the size of the knowledge externality in the production function is chosen from Turnovsky (2000), namely $\eta = 0.08$. As in Turnovsky (2000), the equality $\alpha = \eta$ in the production function means that technical progress is purely labor-augmenting. Our model being a one-sector AK technology, capital should be interpreted broadly as an amalgam of physical and human capital. Considering that the fraction of human capital is around 2/3, it implies that with $1 - \eta = 0.92$ the share of output going to physical capital is around 0.3.

Regarding fiscal policy, government's expenditure is chosen so that the fraction of GDP devoted to public spending on goods and services equals the historical average in the US ($g = 0.23$). Besides, the non-wage taxes are assumed to be 17% of GDP (namely, $\mu = 0.17$). For these parameters' values, the corresponding rate of wage taxation (in percent of GDP) is between 10% and 16.5%, depending on the equilibrium considered (historical average values in OECD data are around 15%). Finally, we scan the parameters of the deficit rule (θ and ϕ) over a wide range to verify the presence (or not) of different types of bifurcations. In all the configurations the resulting value of the deficit ratio ($\dot{D}/Y$) is between 0 and 0.065, namely consistent with US historical data.

PARAMETERS

Households		
S	1	Intertemporal elasticity of substitution
ρ	0.02	Discount rate
ε	0	Labor elasticity of substitution
Technology		
A	0.08	Productivity parameter
η	0.08	Size of the knowledge externality in the production function
α	0.08	Labor elasticity in the production function
Government		
g	0.23	Government spending on goods and services
θ	0 to 0.01	Fixed component of the fiscal rule (benchmark: 0.005)
ϕ	0 to 2	Sterilization coefficient
μ	0.17	Share of non-wage taxes in GDP

Table 1: The baseline calibration

	ϕ	γ^M	γ^P	γ^Q	$\dot{D}^M/Y^M$	$\dot{D}^P/Y^P$	$\dot{D}^Q/Y^Q$
SN_1	0.3826	0.0567		0.0923	0.0303		0.0201
regime $\mathcal{R}_3$	0.7	0.0159	0.0774	0.0871	0.0156	0.008	0.0079
SN_2	0.9539	0.0066	0.0827		0.0065	0.064	

Table 2: Economic growth and the public deficit ratio

Table 2 reproduces the different steady-state solutions in regime $\mathcal{R}_3$. The two saddle-node bifurcations SN_1 and SN_2 emerge at $\phi = \phi_1 \simeq 0.382$ and $\phi = \phi_2 \simeq 0.953$ respectively, and are associated to realistic long-run economic growth and public debt ratio.

Regime $\mathcal{R}_3$ occurs for $\phi \in (\phi_1, \phi_2)$. For, say, $\phi = 0.7$, the long-run economic growth rate equals 1.59%, 7.74%, and 8.71% at points M, P, and Q, respectively; and the long-run public deficit to output ratio is 1.56%, 0.8%, and 0.79%, respectively. These values are fairly realistic. In particular, point M is closely related to OECD data. This feature is of particular importance, since the cyclical dynamics in our model appear in the neighborhood of this steady state.

Noteworthy, values of the sterilization coefficient between ϕ_1 and ϕ_2 are in line with evidences in developed countries. In empirical studies, the reaction of the primary surplus to the debt/GDP ratio is between 2.8-5.4% (Bohn, 1998) and 9.4-12.1% (Bohn, 2008) in the US, between 2-3.8% in industrial countries (Mendoza and Ostry, 2008), and between 2.5% (Collignon, 2012) and 3-5% (Checherita-Westphal and Zyárek, 2017) in the Eurozone. In our model, the response of the primary surplus (S) to the debt/GDP ratio (d_y) is, from (11), $\partial S/\partial d_y = \phi r + \phi d_y \partial r/\partial d_y$. Ignoring the reaction of the real interest rate, it follows that $\partial S/\partial d_y$ is of the order 2.54-9.8%, therefore consistent with the above-mentioned empirical findings.[22]

Notice that regime $\mathcal{R}_3$ occurs without the need of high social returns-to-scale in the aggregate production function. With $\eta = 0.08$, social returns-to-scale are of 1.16. Therefore, in our setup multiplicity (and indeterminacy, as we will see) arise with relatively low increasing returns-to-scale (0.16), which are consistent with empirical evidences (see, e.g. Basu and Fernald, 1997).[23]

[22] At steady state, we have $r = \gamma + \rho$. Using $\rho = 0.02$ and Table 2, we compute $\phi r \in (2.93\%, 4.3\%)$ at SN_1, and $\phi r \in (2.54\%, 9.8\%)$ at SN_2. By adding the response of the interest rate to public debt ($\partial r/\partial d_y$), our results still fit with data. Indeed, the average long-run effect of debt on interest rates ranges from about 2 to 3 basis points for each increase of 1 percentage point in debt as a percentage of GDP (see Gamber and Seliski, 2019). Thus, if the long-run interest rate and the long-run debt ratio are roughly 5% and 100%, respectively, the elasticity of the interest rate to public debt is around $\epsilon = 0.5$; hence, as $\partial S/\partial d_y = \phi r + \phi r \left(\partial r/r\right) / \left(\partial d_y/d_y\right) = \phi r(1+\epsilon)$, the reaction of the primary surplus to public debt is multiplied by 1.5.

[23] In contrast, Benhabib and Farmer (1994) need increasing returns in excess of 0.5. It is possible to reduce the degree of increasing returns needed to generate indeterminacy in multi-sector models (see, e.g., Benhabib and Nishimura, 1998; Benhabib et al., 2000); for example, indeterminacy requires increasing returns of about 0.07 in Benhabib and Farmer (1996). In our one-sector model, the debt accumulation plays the role of an additional sector that allows reducing the requirement on increasing returns.

4.2. Effect of the sterilization coefficient in comparative statics

Using our calibration, we can compute the effect of changes in the sterilization coefficient (ϕ) on the emergence and the value of the different steady states.

Figure 3 depicts the steady-state locus of economic growth and public debt and shows how steady sates Q, P and M move in comparative statics following changes in ϕ. To explain the shape of these shifts, let us first note that, from Eq. (11) with $r = \gamma + \rho$ in steady state, the long-run public debt-to-output ratio writes

$$\frac{D}{Y} = \frac{\theta}{\phi(\gamma + \rho) - \rho}. \tag{21}$$

Any increase in ϕ exerts two effects.

(i) A direct effect. When the government further sterilises the debt burden, the debt-to-output ratio (D/Y) decreases in the steady state.

(ii) An indirect effect via the economic growth rate (γ). As the financing of the debt burden relies more on taxes, the tax rate jumps up, reducing the output ratio with two conflicting consequences. First, the incentive to save declines, so does economic growth. Second, in the goods market equilibrium, the reduction of the output ratio causes the consumption ratio to decrease, which positively affects economic growth.[24] Thus, the economic growth response to changes in ϕ is ambiguous.

At steady state P, this response is positive; hence both the direct and the indirect effects go together in the same direction. Consequently, any increase in ϕ enhances economic growth and reduces the debt-to-output ratio.

At steady state Q and M, the response of growth to sterilization is negative, giving birth to a conflict between the direct (i) and the indirect (ii) effects. At point Q, the direct effect outweighs the indirect one, so that both economic growth and the debt ratio decrease as sterilization enlarges. At point M, the level of public debt is so high that the associated tax rate strongly damages economic growth. As a result, following an increase in ϕ, the economic growth falls sharply. Hence, the indirect effect outweighs the direct one, and the debt ratio rises.

Figure 3: Bifurcation diagram as a function of ϕ

The location of these different long-run equilibria depends on the degree of sterilization. Ignoring point D, multiplicity appears for $\phi \in (\phi_1, \phi_2)$, with three possible steady states as established in Table 2. For $\bar{\phi} < \phi < \phi_1$, only Q-type steady states appear, associated with high growth and – at low values of ϕ – high debt ratios; while for $\phi > \phi_2$ only M-type steady states appear, associated with low economic growth and high debt ratios. Hence, the long-run public debt ratio is large both at low and high levels of ϕ.

[24] Notice that, the output ratio negatively depends on the consumption ratio, see Eq. (18).

The main policy message coming from this analysis is twofold and contrasts with traditional recommendations. On the one hand, a positive response of the primary surplus to the debt burden is not sufficient to stabilise the public debt ratio at a low level. On the other hand, a strong positive response is not necessarily desirable both from the perspective of the stability of the economy and the level of public debt.

Indeed, as we have seen, for a steady state with positive public debt to exist, it is not sufficient for ϕ to be positive, but it must be higher than the threshold $\bar{\phi}$. Furthermore, even if a loose fiscal rule such that $\bar{\phi} < \phi < \phi_1$ generates a unique high-growth Q-type steady state, it may produce large public debt and give birth to aggregate fluctuations, due to the local indeterminacy of Q, as we will see below. At last, a more stringent fiscal rule (i.e. $\phi_1 < \phi < \phi_2$) leads to multiplicity of steady states, while an even tighter policy (i.e. $\phi > \phi_2$) produces a unique M-type steady state, but associated with low growth and high public debt.

A last point deserves to be highlighted. The presence of two saddle-node bifurcations (at $\phi = \phi_1$ and $\phi = \phi_2$, respectively) is likely to generate a hysteresis phenomenon. To capture this phenomenon, let us suppose that the economy is initially located at point A in Figure 3, namely in regime $\mathcal{R}_3$ on the segment of P-type steady states. If ϕ increases until ϕ_2, the steady state moves along this segment. Economic growth improves and the public debt ratio decreases until point LP_2 at the border of regimes $\mathcal{R}_2$ and $\mathcal{R}_3$. If ϕ increases further (i.e., above ϕ_2) the economy switches into regime $\mathcal{R}_2$. This means that the steady state jumps to point B on the segment of M-type steady states, with a sudden cut in long-run growth and a sudden rise in the public debt ratio. Starting from this point B, if ϕ further increases, the economy stays on the segment of M-type steady states: economic growth decreases and the public debt ratio increases.

Hence, for very small changes in the sterilisation coefficient, the steady state warps in a non-reversible way: a too tight fiscal policy (namely, a too strong reaction of the primary surplus to the public-debt-burden) may condemn the economy to an irreversible steady state characterized by low economic growth and high debt. Hysteresis follows: if, departing from point B, the sterilization coefficient is reduced, the steady state does not necessarily come back to LP_2, but can move along the M-type steady-state curve with rising growth and declining public debt ratio.

Of course, such an analysis is only based on comparative statics, and the precise way the different steady sates can be – or not – reached has to be further investigated from a dynamic perspective. This is the objective of the following section.

5. Dynamics

This section is devoted to the analysis of local and global dynamics in regime $\mathcal{R}_3$, which is the more general one.

5.1. Local dynamics

The local dynamics is based on a linearization in the neighborhood of steady state i, $i \in \{D, M, P, Q\}$. In this case, system (17) behaves according to $(\dot{c}_k, \dot{d}_k) = \mathbf{J}^i(c_k - c_k^i, d_k - d_k^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_{k0}) and one pre-determined variable (the public-debt ratio d_{k0}, since the initial stocks of public debt D_0 and private capital K_0 are predetermined). Hence, for steady-state i to be well determined the Jacobian matrix must contain two opposite-sign eigenvalues. The following proposition establishes the behaviour of each steady state.

Proposition 2. *D and P are locally determinate (saddle-point stable); Q is locally indeterminate (stable); and M can be either locally indeterminate (stable) or unstable.*

Proof. See Appendix A.

As the stability of steady state M switches (moving from stable to unstable, or vice versa), a periodic solution can emerge through a Hopf bifurcation, as established in the following corollary.

Corollary 1. *There is a critical value $\phi^h \in (\phi_1, \phi_2)$, such that a Hopf bifurcation arises in the neighborhood of point $M(c_k^M, d_k^M)$ for $\phi = \phi^h$, provided that $d_k^M < \hat{d}_k$.*

Proof. See Appendix A.

Due to the Hopf bifurcation, in the neighborhood of the low-growth trap M, small changes in the sterilization coefficient ϕ can generate radical shifts in the dynamics, together with large oscillations of economic growth and public debt. This bifurcation can be supercritical, generating a stable limit cycle (if the first Lyapunov coefficient is negative), or subcritical, generating an unstable closed orbit (if the first Lyapunov coefficient is positive).[25] In our baseline calibration, the Hopf bifurcation occurs at $\phi^h \simeq 0.3839$ for an economic growth $\gamma^M = 5.43\%$ and a deficit ratio $\dot{D}^M/Y^M = 3.17\%$. The associated first Lyapunov coefficient is negative (-2.44), ensuring the emergence of stable limit-cycles.

5.2. Global dynamics

In light of the local dynamics analysis, point M is of peculiar importance. The following proposition sums up the global dynamics originating in the neighborhood of this point.

Proposition 3. *The dynamics exhibit indeterminacy as follows:*

[25] In our model, the two cases can emerge, depending on parameters. Indeed, a Generalized Hopf bifurcation occurs, at which the first Lyapunov coefficient is zero. The occurrence of such a Generalized Hopf bifurcation ensures the presence of stable limit cycles for nearby parameter values. Appendix C numerically characterizes this Generalized Hopf bifurcation.

i. *Trajectories departing from an initial debt ratio close to d_k^M can converge either to points D, P, or Q, depending on the initial jump of the consumption ratio. Points D and P can be reached by only one trajectory, while there is an infinite set of trajectories converging to Q.*
ii. *Additionally, if $d_k^M < \hat{d}_k$, the economy can join M or a periodic orbit around M.*

Proof. (i) Through an adequate jump of the initial consumption ratio (c_{k0}), the economy can converge towards saddle-points D or P. In addition, consistent with equilibrium relationships and transversality conditions, there is an infinite set of values of c_{k0} that lead to steady state Q, which is therefore locally indeterminate. However, if $d_k^M > \hat{d}_k$, point M is unstable (see Appendix A), and cannot be reached. (ii) If $d_k^M < \hat{d}_k$, a Hopf bifurcation arises at $\phi = \phi^h$ generating a stable limit-cycle (in the baseline calibration), as stated in corollary 1. For slightly higher values of ϕ the economy converges towards one of the periodic orbit that joins the limit-cycle around M, following an appropriate jump of the consumption ratio. As a consequence, any path that converges toward P, Q or D, or joins the limit-cycle, can be reached.[26] For slightly lower values of ϕ, in contrast, point M is stable and can be reached by an infinity of trajectories. □

The global dynamics is depicted in Figure 4. Our model is characterized by local (in the vicinity of Q and M, and of the possible limit-cycle that surrounds the low-growth trap M), and global indeterminacy. The short and long-run behavior of the economy is then subject to "animal spirits", in the form of self-fulfilling prophecies that generate multiple balanced growth paths in the future. Such indeterminacy is intuitive. Suppose, for example, that at the initial time households expect low public debt in the steady state. This implies that the expected tax rate is low, and the expected return of capital is high. Then, at the initial time households increase their savings, making the initial consumption ratio (c_{k0}) to be low and the initial hours worked to be high. In equilibrium, labor supply will also be high, generating large fiscal resources and low public debt in the future (along P and Q BGPs). Conversely, following the same mechanism, high expected public debt is self-fulfilling and may lead to the growth solutions M or D. In other words, by their consumption-leisure tradeoff at the initial time, forward-looking households can—in equilibrium—validate any expectation on the BGP that can be reached in the future.

[26] In our model, all steady states are consistent with the standard transversality conditions, by construction. When the steady state is saddle path, the transversality condition serves to characterize the initial jump of c_{k0} that puts the trajectory on the unique stable manifold. When the steady state is stable, all the trajectories verify the transversality condition; hence the local indeterminacy: for any predetermined value of d_{k0}, the transverslity condition cannot be used to select a particular path. The fact that several steady states (some locally determined and others locally undetermined), all consistent with the transversality condition, coexist for the same set of parameters characterizes the global indetermination of the system.

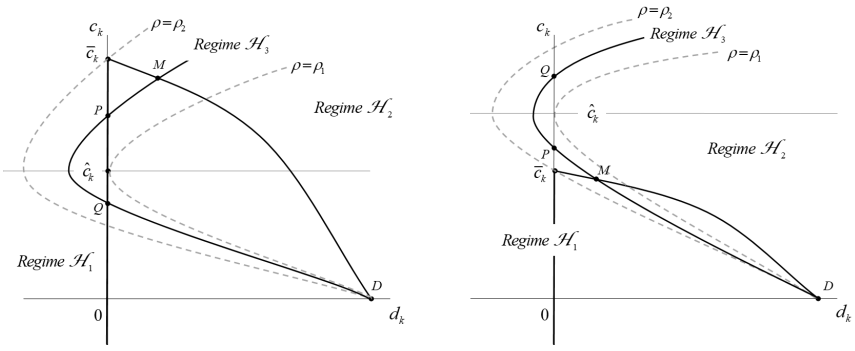


Figure 4: Global dynamics in regime $\mathcal{R}_3$

In our model, indeterminacy, either local or global, is the norm rather than the exception. As established in Appendix B, local and global indeterminacy also arise in regime $\mathcal{R}_1$, because both the (locally indeterminate) steady state Q and the (saddle-path) steady state D can be reached in the long-run. In regime $\mathcal{R}_2$, the only way to escape indeterminacy is when point M is unstable, with no limit-cycle (this is notably the case if $d_k^M > \hat{d}_k$). In this case, the economy converges to the harrodian equilibrium with zero economic growth. Indeterminacy and the associated aggregate instability in the form self-fulfilling fluctuations can thus be viewed as the price to be paid to avoid the poverty trap (D) in the long-run.

The following section further explores the possibility of long-lasting aggregate fluctuations in link with the emergence of endogenous limit-cycles in regime $\mathcal{R}_3$.

6. Long-run endogenous public debt cycle

According to corollary 1, our model produces a Hopf bifurcation in the neighborhood of the low-growth trap M in regime $\mathcal{R}_3$. In the baseline calibration, this bifurcation occurs at $\phi^h \simeq 0.3839$ and is supercritical, such that stable limit-cycles born for slightly higher values of ϕ. The existence of *stable* limit-cycles implies that a small perturbation to a parameter would not eliminate the cyclical dynamics of public debt and growth. As the fiscal rule becomes more active (i.e. ϕ increases), these limit cycles enlarge and generate large fluctuations in public debt and economic growth (Figure 6).[27] Therefore, contrasting with the standard policy recommendation, policies that aim at sterilizing a large part of the debt burden are likely to generate aggregate instability.

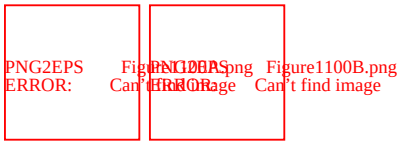


Figure 6: Public debt and growth cycles as a function of ϕ

[27] It should be pointed out that the cycles are not regular, but produce, for high values of ϕ, long periods of nearby-stationary growth and sudden short-living recessions.

However, the limit-cycles cannot expand without bound around M, due to the existence of the steady state P. As ϕ further increases, the limit-cycle enlarges until it coincides with the stable and unstable manifolds of P (at $\phi = \tilde{\phi} \simeq 0.387$ in the baseline calibration). At this value, there is a saddle-loop bifurcation: the limit-cycle gives rise to a homoclinic orbit connecting P to himself.[28] Figure 7 illustrates this bifurcation using a phase portrait of the dynamics. The stable limit-cycle around M (Figure 7a) degenerates into the homoclinic orbit as the sterilization coefficient get close to $\tilde{\phi}$ (Figure 7b).

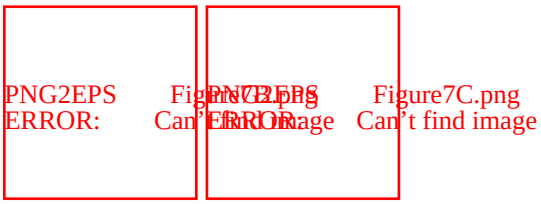


Figure 7: The degeneracy to a homoclinic orbit in the phase portrait

As in Sniekers (2018) and Benhabib et al. (2001), to analytically establish the existence of the homoclinic orbit, we shall refer to the Bogdanov-Takens (BT) bifurcation. In a two-(or more-)parameter system, such a bifurcation occurs when a Hopf bifurcation, a saddle-loop bifurcation, and a saddle-node bifurcation coincide in a single point of the parameter space. Focusing on our two policy parameters of interest ϕ and θ, the following proposition shows that a codimension-2 BT bifurcation[29] arises in our model, when steady states M and P collide at a point such that $d_k^M = d_k^P < \hat{d}_k$.

Proposition 4. *There are critical values of fiscal instruments ϕ^{bt} and θ^{bt}, such that a BT bifurcation appears at $(\phi, \theta) = (\phi^{bt}, \theta^{bt})$.*

Proof: See Appendix B.

The mechanism driving the homoclinic orbit is as follows. As the value of parameters (ϕ, θ) gets closer to (ϕ^{bt}, θ^{bt}), steady states M and P get closer from each other, and, in the end, the limit-cycle that surrounds M merges with the stable and unstable saddle-paths of P through a saddle-loop bifurcation generating the homoclinic orbit. This bifurcation is not only a technical detail but has deep consequences on the dynamics. Indeed, for parameter values close to (ϕ^{bt}, θ^{bt}), the economy can experiment large fluctuations in economic growth and public debt, or slowly converge towards the steady state P along this homoclinic orbit (i.e. a cycle with virtually infinite period). In this way, even in the absence of stochastic shocks, our model can generate extremely long debt and growth cycles, which is consistent with time series observations.

[28] For higher values of ϕ, periodic orbits no longer exist and the anti-saddle path of P now escapes point P, and moves eventually to points Q or D.

[29] The codimension of a bifurcation is the number of parameters that must be varied for the bifurcation to occur. We numerically characterize the BT bifurcation in Appendix C.

In this line, Figure 8 exemplifies an outstanding aspect of the perils of fiscal rules: a small change in the sterilization coefficient may produce radical shifts in long-run dynamics. If, e.g. ϕ moves from 0.385254 to 0.385255, the paths of economic growth and public debt are similar until $t = 5000$, but their dynamics suddenly change after this time. In the first case, the economy gradually converges towards Q, while in the second case it is characterized by periodic recessions with sharp increases in public debt, along a long-lasting cycle close to the homoclinic orbit.

Figure 8: Path-dependance to small changes in the deficit target

Therefore, even a fiscal rule that ensures sustainability by sterilizing a large part of the debt burden does not protect the economy from large fluctuations in growth and public debt, nor does it prevent the occurrence of sudden periodic debt crises. Moreover, the equilibrium that can be achieved in the long term is subject to extreme sensitivity to change in the degree of sterilisation of the deficit rule. These two features have never been raised in the literature on debt sustainability and the fiscal reaction function (Bohn, Ostry,...).

In our model, fluctuations occur in the absence of any stochastic shock. This contrasts with standard New-Keynesian DSGE models, where a stationary pattern of fluctuations is mainly produced by exogenous impulses, usually in the neighborhood of a unique well-determined steady state. Criticisms have been addressed to both types of setups. On the one hand, DSGE frameworks can replicate the observed short-run fluctuations but at the price of very persistent exogenous shocks. In addition, they cannot replicate the long-lasting fluctuations in growth and public debt that we consider in this paper.[30] On the other hand, the main shortcoming of limit-cycle models is that the obtained limit-cycles seem to be too regular both in frequency and amplitude compared to the data. In a recent paper, Beaudry et al. (2020) attempt to unify these two approaches through the notion of *stochastic limit cycles*. By introducing stochastic shocks, they obtain limit-cycles that are no longer regular because these shocks distort their frequency and amplitude. The next section follows this agenda.

7. Stochastic public debt limit-cycles

In this section, we extend our model to stochastic shocks by following the approach of Schmitt-Grohé and Uribe 1997 (section IV). First, we consider a discrete-time version of our model with a calibration based on Table 1, but with a stochastic technology $A_t = A + \upsilon_t$, where υ_t is an autoregressive AR(1) process. Second, we compute the

[30] See e.g. Reinhart and Rogoff (2011), Abbas et al. (2011), or Poghosyan (2015).

determinist equilibria, which are very close to those of the continuous-time version (at equilibrium M, we have $\gamma^M = 0.054$ and $D^M/Y^M = 0.6$, and the Hopf bifurcation occurs at $\phi^h = 0.383$, see Appendix C for details). Third, we linearize the discrete-time model in the neighborhood of steady state M. The equilibrium conditions can be reduced to the following first-order vector stochastic linear difference equation:

$$\begin{pmatrix} \mathbb{E}_t c_{t+1} \\ d_{t+1} \end{pmatrix} = \begin{pmatrix} c^M \\ d^M \end{pmatrix} + \mathbf{J} \begin{pmatrix} c_t - c^M \\ d_t - d^M \end{pmatrix} + \mathbf{V}\upsilon_t,$$

where $\mathbf{J}$ is the Jacobian matrix evaluated at M, $\mathbf{V}$ is a vector whose components are the partial derivative of $\mathbb{E}_t c_{t+1}$ and d_{t+1} with respect to A_t evaluated at M, and $\mathbb{E}_t$ is the expectation operator given the stock of information at time t. The shock υ_t follows the process $\upsilon_t = \lambda\upsilon_{t-1} + \varepsilon_t$, with $\{\varepsilon_t\}_t$ a set of *iid* normal random variables with variance ξ^2.

At first, we examine how stochastic shocks distort the limit-cycles. Figure 7 illustrates the dynamics of the public-debt-to-output ratio in the vicinity of the Hopf bifurcation for a starting point close to M. The left chart depicts dynamics over time: the grey curve is the determinist limit-cycle (i.e. $\varepsilon_t = 0$ for any t), and the black curve is one realisation of the stochastic limit-cycle. The right chart illustrates the spectral density of the two cycles for all periods p, which allows searching for recurrent p-period cyclical phenomena.

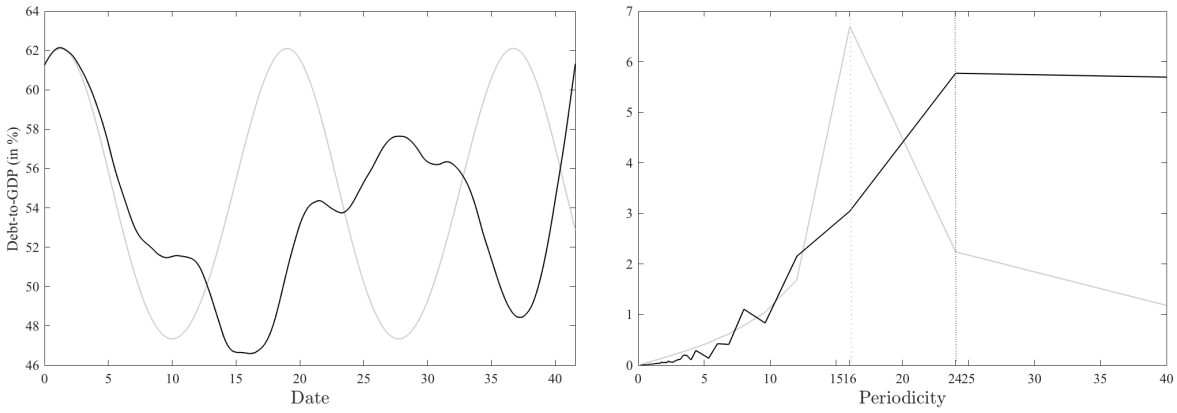


Figure 7: Dynamics of the public debt ratio, for $\phi = 0.385$, $\lambda = 0.9$, and $\xi^2 = 0.005$.

The left chart shows that the introduction of stochastic impulses does not simply add noise around an otherwise-deterministic cycle, but distorts both the amplitude and the frequency of fluctuations, and implies random shifts that accelerate/delay the cycle itself. Consequently, by taking stochastic shocks into account, we obtain irregular fluctuations (the black curve) rather than a smooth cycle (the grey curve). Our model thus displays paths consistent with observed macroeconomic variables; namely, boom-and-bust cycles that are stochastic in both amplitudes and durations.

The right chart compares the spectral density implied by the deterministic limit-cycle with respect to the one implied by the stochastic limit-cycle. The former (the grey curve) has an extreme peak, called the "fundamental period" (at about 16 years, if we interpret one time period as one year). This value corresponds to the period of the deterministic limit-cycle in the left chart. In the presence of stochastic shocks (the black curve), in contrast, the main peak is less marked (at about 22 years, for this realisation of the shock). Furthermore, other low-periodicity peaks appear that fall within the scope of DSGE models. Consequently, while the role of endogenous forces in establishing strong internal propagation mechanisms is preserved, our extended setup unifies short-run fluctuations and the presence of an oscillating trend of the public debt ratio.

To illustrate the occurrence of both high- and low-periodicity cycles in the data, we plot in Figure 8 the observed debt-to-GDP ratio (left-charts) and the associated spectral density (right-charts) for several developed and developing countries from 1950 to 2015.

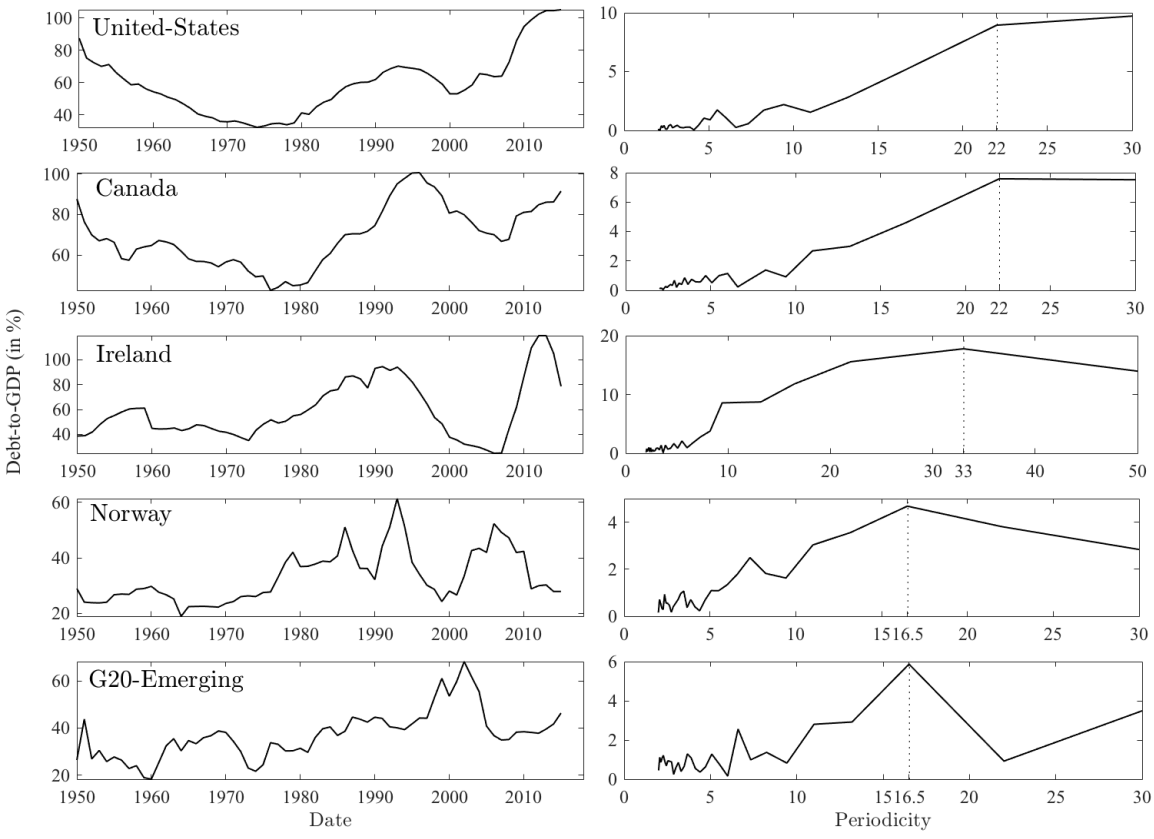


Figure 8: Properties of Debt-to-GDP ratio for some countries in the 1950-2015 period (source: IMF)

As shown by the right charts, in addition to the usual short-run cycles, all considered countries present a long-run debt cycle signaled by spectral densities characterized by

a main peak. For example, the long-run public debt cycle in US and Canada lasts roughly 22 years, while around 16.5 years in Norway and major developing countries.[31] These findings illustrate the capacity of a simple two-dimensional growth model with a stochastic limit-cycle to capture the main long-term cyclical properties of debt-to-GDP data in developed and developing countries.

8. Conclusion

This paper provides a theoretical setup to analyze the effects of flexible deficit rules. We emphasize various forms of perils. First, flexible rules can trigger multiple equilibria and indeterminacy through a new channel based on the sterilization of the debt burden. Second, flexible rules can produce high-periodicity endogenous public debt cycles. Third, flexible rules can generate hysteresis and sensitivity to changes in their key parameters, notably the sterilization of the public debt burden.

On the policy side, these findings call for a careful use of flexible deficit rules. Specifically, our results contrast with the central policy recommendation that to ensure sustainability fiscal surpluses must strongly respond to public debt. By contrast, excessive sterilization of the debt burden can condemn the economy to a low-growth/high-debt trap. On the methodological side, the stochastic version of our model is able to illustrate a key feature of the data in both developed and developing countries, namely the presence of long-lasting debt cycles whose periodicity goes beyond usual short-period cycles detectable by standard New-Keynesian DSGE models.

Our findings could motivate subsequent work on several grounds. First, the destabilizing role of fiscal rules could be analyzed in a more general model that looks at the interplay between fiscal rules and monetary policies (see, e.g, Combes et al. 2018, or Menuet et al. 2018). Second, our limit-cycle based theory of public debt fluctuations should be revisited in a multi-country framework *à la* Halac and Yared (2018) that can account for fiscal rules coordination among various governments. Third, the non-linear effects triggered by our deficit fiscal rules make the case for exploring alternative fiscal rules.

References

Abbas, S.A., Belhocine, N., El-Ganainy, A., Horton, M., 2011. Historical patterns and dynamics of public debt—evidence from a new database. IMF Economic Review 59, 717–742.

Acemoglu, D., Angrist, J., 2000. How large are human-capital externalities? Evidence from compulsory schooling laws. NBER macroeconomics annual 15, 9–59.

Aghion, P., Marinescu, I., Caballero, R.J., Kashyap, A.K., 2007. Cyclical budgetary policy and economic growth: What do we learn from oecd panel data? NBER Macroeconomics annual 22, 251–297.

[31] Our investigation is consistent with Poghosyan (2015) emphasizing that (*i*) debt cycles in 57 advanced and emerging economies over the 1960-2014 period last about 13 years on average, and (*ii*) debt cycles are on average longer in advanced economies than in developing ones.

Basu, S., Fernald, J.G., 1997. Returns to scale in US production: Estimates and implications. Journal of Political Economy 105, 249–283.

Beaudry, P., Galizia, D., Portier, F., 2020. Putting the cycle back into business cycle analysis. American Economic Review 110, 1–47.

Benhabib, J., Farmer, R., 1994. Indeterminacy and increasing returns. Journal of Economic Theory 63, 19–41.

Benhabib, J., Farmer, R., 1996. Indeterminacy and sector specific externalities. Journal of Monetary Economics 37, 397–419.

Benhabib, J., Farmer, R., 1999. Indeterminacy and sunspots in macroeconomics, in: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics. North Holland Publishing Co., Amsterdam, pp. 387–448.

Benhabib, J., Meng, Q., Nishimura, K., 2000. Indeterminacy under constant returns to scale in multisector economies. Econometrica 68, 1541–1548.

Benhabib, J., Nishimura, K., 1998. Indeterminacy and sunspots with constant returns. Journal of Economic Theory 81, 58–96.

Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001. The Perils of Taylor Rules. Journal of Economic Theory 96, 40–69.

Blinder, A.S., Solow, R.M., 1973. Does fiscal policy matter? Journal of Public Economics 2, 319–337.

Bohn, H., 1998. The behavior of U.S. public debt and deficits. The Quarterly Journal of Economics 113, 949–963.

Bohn, H., 2008. The sustainability of fiscal policy in the United States, in: Neck, R., Sturm, J.E. (Eds.), Sustainability of Public Debt. MIT Press. chapter 2.

Bova, M.E., Carcenac, N., Guerguil, M.M., 2014. Fiscal rules and the procyclicality of fiscal policy in the developing world. IMF Working Paper 14/22 .

Checherita-Westphal, C., Zyárek, V., 2017. Fiscal reaction function and fiscal fatigue: evidence for the euro area. ECB Working Paper No. 2036 .

Chen, B.L., 2006. Public capital, endogenous growth, and endogenous fluctuations. Journal of Macroeconomics 28, 768–774.

Cheron, A., Nishimura, K., Nourry, C., Seegmuller, T., Venditti, A., 2019. Growth and public debt: What are the relevant tradeoffs? Journal of Money, Credit and Banking 51, 655–682.

Ciccone, A., Peri, G., 2006. Identifying human-capital externalities: Theory with applications. The Review of Economic Studies 73, 381–412.

Cohen, D., Soto, M., 2007. Growth and human capital: good data, good results. Journal of Economic Growth 12, 51–76.

Collignon, S., 2012. Fiscal policy rules and the sustainability of public debt in Europe. International Economic Review 53, 539–567.

Combes, J.L., Debrun, X., Minea, A., Tapsoba, R., 2018. Inflation Targeting, Fiscal Rules and the Policy Mix: Crosseffects and Interactions. The Economic Journal 128, 2755–2784.

Daniel, B.C., Shiamptanis, C., 2013. Pushing the limit? Fiscal policy in the European Monetary Union. Journal of Economic Dynamics and Control 37, 2307–2321.

Debrun, X., Moulin, L., Turrini, A., Ayuso-i-Casals, J., Kumar, M.S., Drazen, A., Fuest, C., 2008. Tied to the mast? National fiscal rules in the European Union. Economic Policy 23, 297–362.

Eyraud, L., Debrun, X., Hodge, A., Lledó, V., Pattillo, C., 2018. Second-Generation Fiscal Rules: Balancing Simplicity, Flexibility, and Enforceability. IMF Staff Discussion Note 18/04.

Farmer, R., Guo, J.T., 1994. Real business cycles and the animal spirits hypothesis. Journal of Economic Theory 63, 42–72.

Futagami, K., Iwaisako, T., Ohdoi, R., 2008. Debt policy rule, productive government spending, and multiple growth paths. Macroeconomic Dynamics 12, 445–462.

Gamber, E., Seliski, J., 2019. The effect of government debt on interest rates. Congressional Budget Office Working Paper 2019-01 .

Ghilardi, M., Rossi, R., 2014. Aggregate stability and balanced-budget rules. Journal of Money, Credit and Banking 46, 1787–1809.

Ghosh, A.R., Kim, J.I., Mendoza, E.G., Ostry, J.D., Qureshi, M.S., 2013. Fiscal fatigue, fiscal space and debt sustainability in advanced economies. Economic Journal 123, F4–F30.

Giannitsarou, C., 2007. Balanced budget rules and aggregate instability: The role of consumption taxes. Economic Journal 117, 1423–1435.

Halac, M., Yared, P., 2018. Fiscal rules and discretion in a world economy. American Economic Review 108, 2305–34.

Kopits, M.G., Symansky, M.S.A., 1998. Fiscal policy rules. IMF Occasional Paper No. 162 .

Kuznetsov, Y.A., 1998. Elements of Applied Bifurcation Theory, Second Edition. Springer, NY.

Liviatan, N., 1982. Neutrality of government bonds reconsidered. Journal of Public Economics 19, 261–270.

Mauro, P., Romeu, R., Binder, A., Zaman, A., 2015. A modern history of fiscal prudence and profligacy. Journal of Monetary Economics 76, 55–70.

Mendoza, E.G., Ostry, J.D., 2008. International evidence on fiscal solvency: Is fiscal policy responsible? Journal of Monetary Economics 55, 1081–1093.

Menuet, M., Minea, A., Villieu, P., 2018. Deficit, Monetization, and Economic Growth: A Case for Multiplicity and Indeterminacy. Economic Theory 65, 819–853.

Minea, A., Villieu, P., 2012. Persistent Deficit, Growth, and Indeterminacy. Macroeconomic Dynamics 16, 267–283.

Moretti, E., 2004. Human capital externalities in cities. Handbook of regional and urban economics 4, 2243–2291.

Nishimura, K., Seegmuller, T., Venditti, A., 2015. Fiscal policy, debt constraint and expectations-driven volatility. Journal of Mathematical Economics 61, 305–316.

Nourry, C., Seegmuller, T., Venditti, A., 2013. Aggregate instability under balanced-budget consumption taxes: A re-examination. Journal of Economic Theory 148, 1977–2006.

Park, H., Philippopoulos, A., 2004. Indeterminacy and fiscal policies in a growing economy. Journal of Economic Dynamics and Control 28, 645–660.

Poghosyan, T., 2015. How do public debt cycles interact with financial cycles? IMF Working Paper 15/248 .

Rauch, J.E., 1993. Productivity gains from geographic concentration of human capital: evidence from the cities. Journal of Urban Economics 34, 380–400.

Reinhart, C.M., Rogoff, K.S., 2011. From financial crash to debt crisis. American Economic Review 201, 1676–1706.

Romer, P., 1986. Increasing returns and long-run growth. Journal of Political Economy 94, 1002–1037.

Schaechter, A., Kinda, T., Budina, N., Weber, A., 2012. Fiscal rules in response to the crisis–toward the "next-generation" rules. A new dataset. IMF working paper WP/12/187.

Schmitt-Grohé, S., Uribe, M., 1997. Balanced-budget rules, distortionary taxes and aggregate instability. Journal of Political Economy 105, 976–1000.

Sniekers, F., 2018. Persistence and volatility of Beveridge Cycles. International Economic Review 59, 665–698.

Tobin, J., Buiter, W., 1976. Long-run effects of fiscal and monetary policy on aggregate demand. Cowles Foundation Discussion Paper No. 384 .

Turnovsky, S., 2000. Fiscal policy, elastic labor supply, and endogenous growth. Journal of Monetary Economics 45, 185–210.

Wyplosz, C., 2012. Fiscal rules: Theoretical issues and historical experiences, in: Alesina, A., Giavazzi,

F. (Eds.), Fiscal policy after the financial crisis. University of Chicago Press, pp. 495–525.

SUPPLEMENTARY MATERIAL: to be published exclusively *on-line*

Appendix A. Proof of proposition 2

The Jacobian matrix evaluated at the steady-state i, $i \in \{D, M, P, Q\}$ is

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CD^i \\ DC^i & DD^i \end{pmatrix},$$

where, using (17),

$$CC^i = c_k^i[1 + (g - \eta)yc^i], \tag{A.1}$$

$$CD^i = c_k^i(g - \eta)yd^i, \tag{A.2}$$

$$DD^i = \theta yd^i - \theta\frac{y_k^i}{d_k^i} + (1 - \eta)(\phi - \bar{\phi})yd^i d_k^i, \tag{A.3}$$

$$DC^i = \theta yc^i - (1 - \eta)(\phi - \bar{\phi})yc^i d_k^i + d_k^i, \tag{A.4}$$

with, using (18),

$$yc^i := \frac{\partial y_k^i}{\partial c_k^i} = -\frac{\psi y_k^i}{c_k^i} < 0, \text{ and } yd^i := \frac{\partial y_k^i}{\partial d_k^i} = -\frac{(1 - \eta)\phi\psi y_k^i}{\bar{d} - (1 - \eta)\phi d_k^i} < 0. \tag{A.5}$$

As our model is based on a two-dimensional dynamics system, we can study the local stability of steady states by inspecting the slope of $\dot{c}_k = 0$ (denoted by s_c^i) and $\dot{d}_k = 0$ (denoted by s_d^i) in the neighbourhood of each steady-state i in Figure A1.

First, using the Implicit Function Theorem, we compute $s_c^i = -CD^i/CC^i$ and $s_d^i = -DD^i/DC^i$.

Second, the trace and the determinant of the jacobian matrix are $\text{Tr}(\mathbf{J}^i) = CC^i + DD^i$ and $\det(\mathbf{J}^i) = CC^i DD^i - CD^i DC^i = CC^i DC^i(s_c^i - s_d^i)$.

Third, from Eq. (A.1)-(A.4), as θ is small enough (see our baseline calibration in Table 1), we have $DC^i > 0$ and $CD^i < 0$.

Hence, we can divide the (d_k, c_k)-plan in four distinct areas, as depicted in Figure A1:

- *north-east* (NE): $s_c^i > 0$, and $s_d^i < 0 \Rightarrow \det(\mathbf{J}^i) > 0$, and $\text{Tr}(\mathbf{J}^i) > 0$.
- *south-east* (SE): $s_c^i < 0$, $s_d^i < 0$, and $|s_d^i| > |s_c^i| \Rightarrow CC^i < 0$, and $\det(\mathbf{J}^i) < 0$.
- *south-west* (SW): $s_c^i < 0$ and $s_d^i > 0 \Rightarrow \det(\mathbf{J}^i) > 0$ and $\text{Tr}(\mathbf{J}^i) < 0$.
- *north-west* (NW): $s_c^i > 0$, $s_d^i > 0$, and there are two configurations: (i) if $|s_d^i| > |s_c^i| \Rightarrow CC^i > 0$, and $\det(\mathbf{J}^i) < 0$; (ii) if $|s_d^i| < |s_c^i| \Rightarrow \det(\mathbf{J}^i) > 0$ and $\text{Tr}(\mathbf{J}^i)$ can be positive or negative.

As $D \in SE$, D is saddle-path stable. If P and Q exist, as $P \in NW$ with $|s_d^P| > |s_c^P|$ and $Q \in SW$, it follows that P is saddle-path stable and Q is locally indeterminate (stable). Regarding point M, two situations can arise: if $M \in NE$, M is unstable,

while if $M \in NW$ with $|s_d^M| < |s_c^M|$, a Hopf bifurcation can occur when $CC^M + DD^M = 0$. Corollary 1 comes directly from Eqs. (A.1) and (A.3).

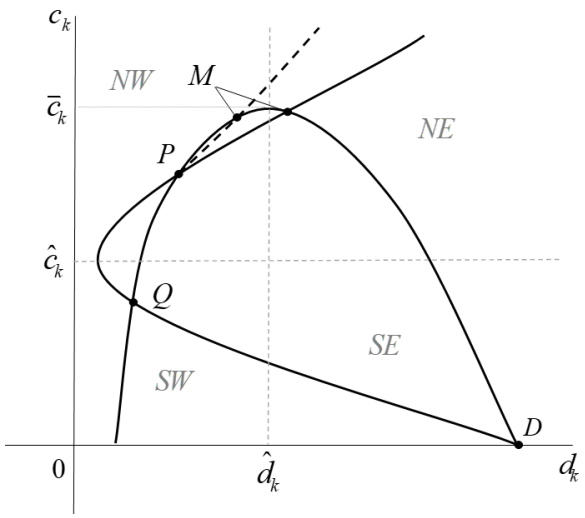


Figure A1: Local stability ($\hat{c}_k < \bar{c}_k$)

Appendix B. Global dynamics in regimes $\mathcal{R}_1$ and $\mathcal{R}_2$

Regime $\mathcal{R}_1$ – This regime is characterized by two steady states. One is associated to high economic growth Q and is stable, while the harrodian equilibrium D is saddle-path stable. Consequently, there is local indeterminacy in the vicinity of Q, and possibly global indeterminacy because the economy can move towards Q or D if the initial debt ratio is such that $d_{k0} > \hat{d}_k$ (see Figure 2a).

Regime $\mathcal{R}_2$ – In this regime there are also two steady states, M and D (see Figure 2e). The latter is still locally determinate, but the topological behavior of the low-growth trap M depends on its position relative to d_k, as stated in Appendix A.

If $d_k^M > \hat{d}_k$, M is unstable, and there is no local or global indeterminacy. Starting from an initial public debt ratio close to d_k^M, the economy converges towards the harrodian equilibrium D.

If $d_k^M < \hat{d}_k$, there is global indeterminacy: for initial public debt ratios originating in the neighborhood of M the economy can converge to the point D, to the point M, or can join a limit-cycle around point M, depending on the initial jump in consumption. Indeed, if $d_k^M < \hat{d}_k$ there is a Hopf bifurcation, which is supercritical in our baseline calibration, as we have seen. In addition, if $d_{k0} > \hat{d}_k$ there is a unique trajectory that goes towards the harrodian equilibrium D. As a consequence, given a predetermined debt ratio, any path that converges toward M or D, or joins the cycle, can be reached.

Appendix C. Location of regimes

Figure C1 depicts the different regimes and bifurcations in the (ϕ, θ)-plane. The two saddle-node bifurcations are depicted by the curves $SN_1(\phi)$ and $SN_2(\phi)$ that define the limit-points between regimes $\mathcal{R}_1$ and $\mathcal{R}_3$, and $\mathcal{R}_3$ and $\mathcal{R}_2$, respectively. In our benchmark calibration ($\theta = 0.005$) these limit-points are labelled LP_1 and LP_2, respectively. The dashed curve $\mathcal{H}(\phi)$ depicts the locus of Hopf bifurcations. In the benchmark calibration, the Hopf bifurcation (labelled H) is located in regime $\mathcal{R}_3$, and occurs at $\phi \simeq 0.3839$.

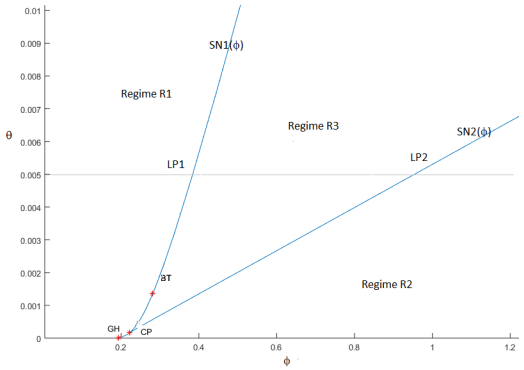


Figure C1: Location of regimes and bifurcation points in the (ϕ, θ) plane (plus que 1)

With two parameters allowed to vary, two codimention-2 bifurcations appear.[32] First, a Generalized Hopf (Bautin) bifurcation (labelled GH) occurs at $\theta \simeq 0$ and $\phi \simeq 0.19$. Second, a Bogdanov-Takens bifurcation (labelled BT) appears at $\theta \simeq 0.13\%$ and $\phi \simeq 0.28$. At this point, the saddle-node curve $SN_1(\phi)$ is tangent to the Hopf-curve $\mathcal{H}(\phi)$. This bifurcation is consistent with realistic values of economic growth ($\gamma^M \simeq 6.55\%$) and deficit ratio ($\dot{D}^M/Y^M \simeq 2.18\%$).

	ϕ	θ	γ^M	$\dot{D}^M/Y^M$
Hopf	0.3839	0.005	0.0543	0.0317
GH	0.1941	0.000001	0.0831	0.0053
BT	0.2813	0.001357	0.0655	0.0218

Table C1: Bifurcations, economic growth and the public deficit ratio

The mechanism driving the homoclinic orbit is as follows. The point where P and M collide defines the saddle-node bifurcation SN_1, while (if $\hat{c}_k < \bar{c}_k$ and $d_k^M < \hat{d}_k$) M undergoes a Hopf bifurcation generating a periodic orbit in regime $\mathcal{R}_3$, as we have seen. The BT bifurcation is then obtained as the collision of the saddle-node and the Hopf bifurcations. As (ϕ, θ) gets closer to (ϕ^{bt}, θ^{bt}), the non-saddle point M converges towards

[32] The codimension of a bifurcation is the number of parameters that must be varied for the bifurcation to occur.

P, so that the periodic orbit collides with the manifolds of the saddle equilibrium and degenerates into a homoclinic orbit.

Appendix D. Bogdanov-Takens bifurcation and homoclinic orbits

We prove the occurrence of a Bogdanov-Takens (BT) bifurcation and homoclinic orbits in the neighborhood of equilibrium M using a two-step proof. In the first step, we will show that there is a critical pair of fiscal instruments that characterizes the BT singularity. In the second step, we demonstrate the existence of a homoclinic orbit around point M, using the argument that points P and M collide at the BT bifurcation.

Step 1: Preliminary.
First, we compute the coordinates of point M for a small value of θ (formally $\theta \to 0$). By Eqs. (14), (15) and (16), it follows that $y_k^M = \frac{\rho}{\phi(1-\eta)}$, $\gamma^M = \frac{(1-\phi)\rho}{\phi}$, $c_k^M = \frac{\rho(\phi-\bar{\phi})}{\phi}$, and $d_k^M = \frac{1}{(1-\eta)\phi}\left\{\bar{d} - c_k^M \left(\frac{y_k^M}{A}\right)^{1/\psi}\right\}$. From Eq. (A.5), we compute: $yc^M = -\frac{\psi}{(1-\eta)(\phi-\bar{\phi})}$, and $y_d^M = -\frac{\phi\psi}{(\phi-\bar{\phi})(\rho/A\phi(1-\eta))^{1/\psi}}$.
Second, the trace and the determinant of the Jacobian matrix are, when $\theta \to 0$,

$$\mathrm{tr}(\mathbf{J}^M) = c_k^M[1 + (g-\eta)yc^M] + (1-\eta)(\phi-\bar{\phi})yd^M d_k^M, \tag{D.1}$$

$$\begin{aligned}\det(\mathbf{J}^M) = {} & c_k^M[1 + (g-\eta)yc^M](1-\eta)(\phi-\bar{\phi})yd^i d_k^M \\ & - c_k^M(g-\eta)yd^M\left(d_k^M - (1-\eta)(\phi-\bar{\phi})yc^M d_k^M\right).\end{aligned} \tag{D.2}$$

We need to find the values of parameters ϕ and θ, such that $\mathrm{Tr}(\mathbf{J}^M) = \det(\mathbf{J}^M) = 0$.

On the one hand, using (D.2), it follows that

$$\det(\mathbf{J}^M) = 0 \Leftrightarrow -2\psi(g-\eta) + (1-\eta)(\phi-\bar{\phi}) - (g-\eta) = 0$$

As $\bar{\phi} := (g-\eta)/(1-\eta)$, we conclude that $\det(\mathbf{J}^M) = 0 \Leftrightarrow \phi^{bt} = 2\bar{\phi}(\psi+1) > \bar{\phi}$.

On the other hand, using (D.1), it follows that

$$\mathrm{tr}(\mathbf{J}^M) = 0 \Leftrightarrow \frac{\rho}{\phi\psi}[\phi(1-\psi) - \bar{\phi}]\left(\frac{\rho}{\phi A(1-\eta)}\right)^{1/\psi} = \bar{d}$$

As $\bar{d} = \alpha + \mu + \theta - g$, we conclude that $\mathrm{tr}(\mathbf{J}^M) = 0 \Leftrightarrow$

$$\theta =: \theta^{bt} = g - \alpha - \mu + \left(\frac{\rho}{\phi^{bt}}\right)^{1+1/\psi}\left(\frac{\phi^{bt}(1-\psi) - \bar{\phi}}{\psi[A(1-\eta)]^{1/\psi}}\right).$$

Consequently, at $(\phi, \theta) = (\phi^{bt}, \theta^{bt})$, it follows that $\mathrm{Tr}(\mathbf{J}^M) = (\mathbf{J}^M) = 0$; hence, the Jacobian matrix $\mathbf{J}^M$ has a double zero eigenvalue.

Step 2: Homoclinic orbit.
We prove the occurrence of the BT bifurcation by applying a theorem that allows us to transform our system into a simpler, topologically equivalent planar system of differential equations with well-known bifurcation diagram. We conclude using a lemma that ensures the occurrence of homoclinic orbits.

Theorem (Kuznetsov, 1998, Theorem 8.4, p. 321) Suppose that a planar system

$$\dot{x} = f(x, \Lambda),\ x \in \mathbb{R}^2,\ \Lambda \in \mathbb{R}^2,$$

with smooth f, has at $\Lambda = 0$, the equilibrium $x = 0$ with a double zero eigenvalue:

$$\lambda_{1,2} = 0.$$

Assume the following generic conditions are satisfied:
(BT.0) the jacobian matrix $\mathbf{J}(0) = f_x(0, 0) \neq 0$;
(BT.1) $a_{20}(0) + b_{11}(0) \neq 0$;
(BT.2) $b_{20}(0) \neq 0$;
(BT.3) the map

$$(x, \Lambda) \mapsto \left(f(x, \Lambda), \mathrm{tr} \left(\frac{\partial f(x, \Lambda)}{\partial x} \right), \det \left(\frac{\partial f(x, \Lambda)}{\partial x} \right) \right)$$

is regular at point $(x, \Lambda) = (0, 0)$.

Then there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparametrization, and smooth invertible parameter changes, which together reduce the system to

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2 \eta_1 + \eta_1^2 + s\eta_1\eta_2 + O(||\eta||^3), \end{cases}$$

where $s := \mathrm{sgn}[b_{20}(a_{20}(0) + b_{11}(0))] = \pm 1$. □

Let $\Lambda := (\phi - \phi^{bt}, \theta - \theta^{bt})$ and $x := (c_k - c_k^M, d_k - d_k^M)$. Clearly, at $\Lambda = 0$, the equilibrium $x = 0$ has a double zero eigenvalue. We need to ensure conditions (BT.0)-(BT.3).

Condition (BT.0). Using Eq. (A.1), at point M, we have

$$CC^M = c_k^M[1 + (g - \eta)yc^M] = \frac{\rho}{\phi}[\phi(1 - \psi) - \bar{\phi}],$$

hence; $CC^M|_{\phi=\phi^{bt}} = \rho[\phi^{bt}(1 - \psi) - \bar{\phi}]/\phi^{bt} \neq 0$. Consequently, the jacobian matrix $\mathbf{J}^M$ evaluated at $(\phi, \theta) = (\phi^{bt}, \theta^{bt})$ is non-zero.

Conditions (BT.1) and (BT.2). Numerically, we compute the generic BT parameters,

and show that $a_{20}(0) + b_{11}(0) \neq 0$ and $b_{20}(0) \neq 0$ for a large constellation of parameters. Using our baseline calibration, we find $a_{20} = -0.0232$ and $b_{11} = 4.88$.

Conditions (BT.3). Let $\Phi : (x, \Lambda) \mapsto \big(f(x, \Lambda), \mathrm{Tr}(\mathbf{J}^M), \det(\mathbf{J}^M)\big)$. Numerically, we ensure that $\det(\Phi(0, 0)) \neq 0$ for a large space of parameters.

Finally, according to the above-mentioned theorem, our system is topological equivalent to the following two-differential-equations system in the neighborhood of M

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2 \eta_1 + \eta_1^2 \pm \eta_1 \eta_2, \end{cases} \tag{D.3}$$

where β_1 and β_2 are combinations of parameters. The coefficient on $\eta_1 \eta_2$ is -1, since the periodic orbit around point M is stable (the first Lyapunov coefficient is negative in our baseline calibration). Thus, the bifurcation diagram is usually depicted in the (β_1, β_2)-plane (Kuznetsov, 1998, section 8.4.2), where the origin corresponds to the BT bifurcation.

Against this background, the existence of homoclinic orbits directly derives from the properties of the bifurcation diagram and the following lemma.

Lemma (Kuznetsov, 1998, Lemma 8.7) There is a unique smooth curve $\mathcal{P}$ corresponding to a saddle homoclinic bifurcation in system (D.3) that originates at $\beta = 0$ and has the following local representation

$$\mathcal{P} = \left\{ (\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2),\ \beta_2 < 0 \right\}.$$

□

Consequently, in the neighborhood of equilibrium M, this lemma establishes that there is a combination of parameters such that there exists at least one bifurcation curve originating at $\beta = 0$ (i.e. $(\phi, \theta) = (\phi^{bt}, \theta^{bt})$), along which system (D.3) has a saddle homoclinic bifurcation. To sum up, if (c_k, d_k) is close to (c_k^M, d_k^M), and (ϕ, θ) is close to the BT bifurcation (ϕ^{bt}, θ^{bt}) the economy can experiment an homoclinic orbit (along the curve $\mathcal{P}$).

Appendix E. The discrete-time stochastic version of our model.

This Appendix presents the discrete-time stochastic version of our model. Compared to the continues-time deterministic version, we introduce a stochastic process $\{\upsilon_t\}_t$ that affects the productivity, namely $A_t = A + \upsilon_t$.

The Household's problem (1)-(2) takes the form

$$\max \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^t \left\{ \log(c_t) - \frac{B}{1+\varepsilon} l_t^{1+\varepsilon} \right\} \right]$$

subject to

$$k_{t+1} + d_{t+1} = (1 + r_t)d_t + (1 + q_t)k_t + (1 - \tau_t)w_t l_t - c_t - z_t + \pi_t,$$

where $\beta = 1/(1 + \rho)$ is the discount factor.

The first order conditions lead to

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1 + r_{t+1}}{c_{t+1}} \right], \text{ and } \frac{(1 - \tau_t)w_t}{c_t} = B l_t^{\varepsilon},$$

under the transversality conditions: $\lim_{t\to\infty}(\beta^t k_{t+1}/c_t) = 0$ and $\lim_{t\to\infty}(\beta^t d_{t+1}/c_t) = 0$.

As in the main text, the maximization of the firm's profit leads to: $w_t = \alpha y_t/l_t$ and $r_t = (1 - \eta)y_t/k_t$. The government is subject to the same inter-temporal budget constraint (9), and the fiscal rule (11); hence, the law of motion of the public debt level is $D_{t+1} = D_t + \theta Y_t + (1 - \phi)r_t D_t$.

To derive endogenous growth solutions, we deflate all growing variables by the capital stock to obtain long-run stationary ratios: $y_{k,t} := Y_t/K_t$, $c_{k,t} = C_t/K_t$, and $d_{k,t} = D_t/K_t$. At the competitive equilibrium, output is, using Eq. (18)

$$y_{k,t} = A_t \left(\frac{\bar{d} - (1 - \eta)\phi d_{k,t}}{c_{k,t}} \right)^{\psi}. \tag{E.1}$$

The capital stock is given by the goods market equilibrium

$$\frac{K_{t+1}}{K_t} = 1 + (1 - g)y_{k,t} - c_{k,t} =: 1 + \gamma_{k,t}.$$

The reduced-form of the model writes now

$$\begin{cases} \dfrac{1}{c_{k,t}} = \dfrac{\beta}{1 + \gamma_{k,t}} \mathbb{E}_t \left[\dfrac{1 + (1 - \eta)y_{k,t+1}}{c_{k,t+1}} \right] & \text{(a)}, \\ \\ d_{k,t} = \dfrac{d_{k,t} + \theta y_{k,t} + (1 - \phi)y_{k,t} d_{k,t}}{1 + \gamma_{k,t}} & \text{(b)}. \end{cases} \tag{E.2}$$

Looking at the determinist state (i.e. $\upsilon_t = 0$, $\forall t$), system (E.2) can be rewritten as

$$\begin{cases} c_{k,t+1} = \dfrac{c_{k,t}}{1+\gamma_{k,t}}[\beta(1+(1-\eta)y_{k,t+1}) - 1 - (1-g)y_{k,t} + c_{k,t}] + c_{k,t}, \\ \\ d_{k,t+1} = \dfrac{d_{k,t}}{1+\gamma_{k,t}}\left[\theta\dfrac{y_{k,t}}{d_{k,t}} - (1-\eta)(\phi-\bar{\phi})y_{k,t} + c_{k,t}\right] + d_{k,t}. \end{cases} \tag{E.3}$$

At the determinist equilibrium $c_{k,t+1} = c_{k,t}$, $d_{k,t+1} = d_{k,t}$, and $y_{k,t+1} = y_{k,t}$, so that the expressions in brackets need to be zero. Hence, the equilibrium conditions are very close to those in continues-time (see Eq. 17). Technically, if $\rho \to 0^+ \Leftrightarrow \beta \to 1$, the equilibrium values in the discrete-time and continues-time versions are the same.

We now linearize system (E.2) in the neighborhood of the determinist equilibrium M. This system behaves, for (c_k, d_k) close to (c^M, d^M), as

$$\mathbb{E}_t c_{k,t+1} \approx c^M + CC(c_{k,t} - c^M) + CD(d_{k,t} - d^M) + CA(A_t - A^M), \tag{E.4}$$
$$d_{k,t+1} \approx d^M + DC(c_{k,t} - c^M) + DD(d_{k,t} - d^M) + DA(A_t - A^M). \tag{E.5}$$

As $A^M = A$, it follows that $A_t - A^M = \upsilon_t$.

First, using Eq. (E.2b), we derive

$$\begin{aligned} DD &= \frac{d^M}{1+\gamma^M}\left[\theta\frac{y_d}{d^M} - \theta\frac{y^M}{(d^M)^2} - (1-\eta)(\phi-\bar{\phi})y_d\right] + 1, \\ DC &= \frac{d^M}{1+\gamma^M}\left[\theta\frac{y_c}{d^M} - (1-\eta)(\phi-\bar{\phi})y_c + 1\right], \\ DA &= \frac{d^M}{1+\gamma^M}\left[\theta\frac{y_A}{d^M} - (1-\eta)(\phi-\bar{\phi})y_A\right]. \end{aligned}$$

where, as in our continues-time model, using (E.1)

$$y_c = -\psi y^M/c^M,\ y_d = -(1-\eta)\phi\psi y^M/(\bar{d} - (1-\eta)\phi d^M),\ y_A = y^M/A.$$

Second, using the first order approximation property (namely, $\mathbb{E}_t[g(X_t)] = g(\mathbb{E}_t[X_t])$, for any non-linear function g, and random variable X_t), Eq. (E.2.a) can write as $f(\mathbb{E}_t c_{k,t+1}, d_{t+1}, c_t, d_t, A) \approx 0$, where

$$f(\mathbb{E}_t c_{k,t+1}, d_{k,t+1}, c_{k,t}, d_{k,t}, A) = \mathbb{E}_t c_{k,t+1} - \frac{c_{k,t}}{1+\gamma_{k,t}}[\beta(1+(1-\eta)\mathbb{E}_t y_{k,t+1}) - 1 - (1-g)y_{k,t} + c_{k,t}] - c_{k,t},$$

with $\mathbb{E}_t y_{k,t+1} \approx A\left(\frac{\bar{d} - (1-\eta)\phi d_{k,t+1}}{\mathbb{E}_t c_{k,t+1}}\right)^{\psi}$.

By denoting f_i the partial derivative, we compute

$$
\begin{aligned}
f_1 &= 1 - \frac{c^M}{1+\gamma^M}\beta(1-\eta)y_c, \\
f_2 &= -\frac{c^M}{1+\gamma^M}\beta(1-\eta)y_d, \\
f_3 &= -\frac{c^M}{1+\gamma^M}\left[1-(1-g)y_c\right] - 1, \\
f_4 &= -\frac{c^M}{1+\gamma^M}[\beta(1-\alpha)y_d - (1-g)y_d], \\
f_5 &= -\frac{c^M}{1+\gamma^M}\beta y_A.
\end{aligned}
$$

Hence, using the Intermediate Value Theorem, we derive that

$$
\begin{aligned}
CC &= -(f_3 + DCf_2)/f_1, \\
CD &= -(f_4 + DDf_2)/f_1, \\
CA &= -(f_5 + DAf_2)/f_1.
\end{aligned}
$$

By denoting the Jacobian matrix $\mathbf{J} = \begin{pmatrix} CC & DC \\ DC & DD \end{pmatrix}$, the Hopf bifurcation appears at $\phi = \phi^h$, such that $|\mathrm{tr}(\mathbf{J})| \leq 2$ and $|\det(\mathbf{J})| = 1$. In our simulation, we find $\phi^h \approx 0.383$.