The Perils of Fiscal Rules

Maxime Menuet, Alexandru Minea, Patrick Villieu

The Perils of Fiscal Rules

Maxime Menuet[a], Alexandru Minea[a,b,1], Patrick Villieu[c]

[a] *CERDI, UMR 6583, CNRS, IRD, Université Clermont-Auvergne*
[b] *Department of Economics, Carleton University, Ottawa, Canada*
[c] *Univ. Orléans, CNRS, LEO, FA45067, Orléans, France*

Abstract

This paper develops a limit-cycle-based theory of debt fluctuations through a simple endogenous growth model. Public debt and deficit are introduced by relaxing the balanced-budget rule hypothesis, and assuming a simple fiscal rule. Our main result is that fiscal rules can be destabilizing, leading to (i) multiple equilibria—four balanced-growth paths can emerge—, (ii) endogenous public debt cycles, which appear both in the short and the long run, and (iii) hysteresis phenomena arising from extreme sensitivity of changes in parameters. We also reveal that a balanced-budget rule does not preclude large aggregate fluctuations. Finally, our calibration exercise highlights that our model produces asymmetric cycles consistent with observed stylized facts.

Keywords: Fiscal rules; Indeterminacy; Limit cycle; Public Debt; Bifurcation.

1. Introduction

The public-debt-to-GDP ratio is characterized by oscillating fluctuations, both in developed and developing countries. Based on historical time series, public debt paths follow long-lasting and regular cycles (Reinhart and Rogoff, 2010; Abbas et al., 2011; Poghosyan, 2015). In pioneer works, Reinhart and Rogoff (2010, 2011) analyzed episodes of debt cycles for 70 countries spanning an exceptionally long time period, and concluded that *"public debt follows a lengthy and repeated boom-bust cycle"*. Between 1880 and 2009, Abbas et al. (2011) identified regular debt cycles, with a total of 66 episodes of debt decline and 63 episodes of debt increase. Focusing on the 1950-2015 period (IMF data), the HP-detrended cyclical component of the public-debt-to-GDP ratio for developed and developing countries (see Figures 1a-2a) and its spectral density (see Figures 1b-2b) highlight the presence of debt cycles of a certain regularity, around 12 (20) years for developing (developed) countries.[2]

[1] Corresponding author: alexandru.minea@uca.fr

[2] This is consistent with Poghosyan (2015), who shows that debt cycles during the period 1960-2014 last about 13 years.

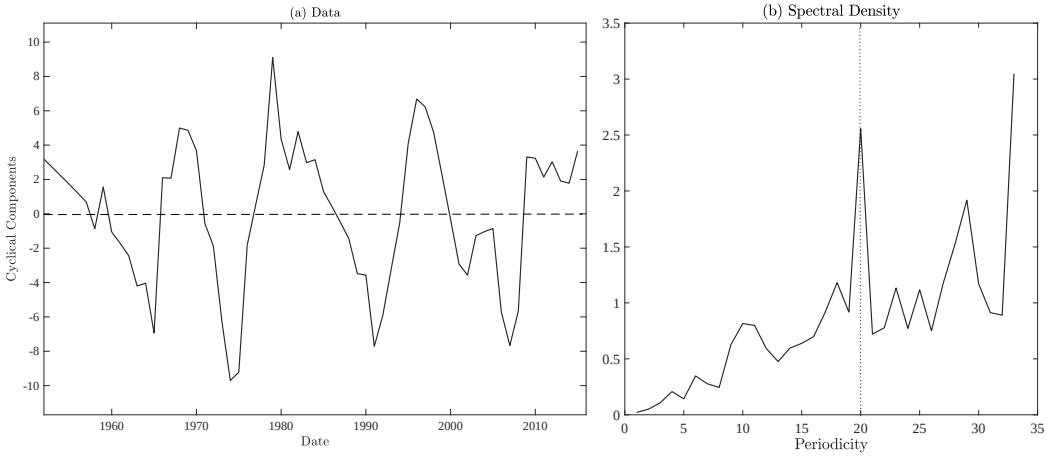


Figure 1: Cyclical Component and Spectral Densities for Developed Countries

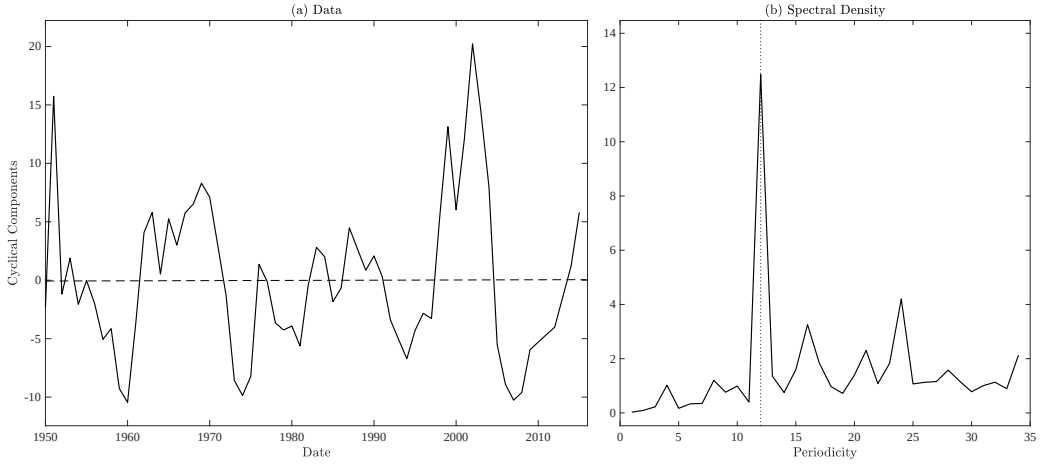


Figure 2: Cyclical Component and Spectral Densities for Developing Countries

Long-lasting theoretical explanations of debt cycles lay on *exogenous factors*, such as wars or majors recessions, as in the tax-smoothing theory of Barro (1979) and Bohn (1998). However, this exogenous-shock-based perspective does not adequately replicate the regularity and frequency of observed cycles, especially in peacetime periods. In this paper, we suggest that debt cycles have *endogenous origins* where boom and bust are tightly linked. In this case, the internal mechanisms of the economy are sufficient to generate fluctuations even in absence of any stochastic shocks.

The main argument we develop is that a simple deficit rule can generate long-lasting public debt cycles, driven by the interaction between the optimal saving behaviour of households and the government's budget constraint, without the need of exogenous per-

turbations. Such cycles can appear both in the short and long run through the occurrence of limit-cycles and complex dynamics.

In light of the existing literature, little is known regarding the implications of fiscal deficits for aggregate fluctuations. Preceding studies rest so far on a balanced-budget rule, and do not account for public debt. Yet, the persistence of public deficits and debt characterizes most developed countries since the mid-1970.[3] In addition, starting the 1980s, many economies adopted fiscal rules constraining deficit and/or debt.[4] As such, assessing the impact of deficit rules on endogenous fluctuations is a major challenge facing economic theory.

This paper addresses this challenge. For reporting on the continuous growth of public debt in the long run, we build a Romer (1986)-type endogenous growth model, with endogenous labor supply, in the spirit of Schmitt-Grohé and Uribe (1997) (hereafter, SGU).[5] Public debt and deficit are introduced by relaxing the balanced-budget rule hypothesis, and assuming a simple fiscal rule characterized by a constant deficit-to-output ratio. The study of the dynamics of the economy, both analytically and using a graphical analysis, reveals the occurrence of local and global bifurcations.

Our results are as follows.

First, our model exhibits multiplicity of equilibria. Consistent with US or OECD historical data, our calibration shows that four equilibria can appear: two high-growth equilibria, a low-growth trap, and a "catastrophic" equilibrium in which the economy disappears. Intuitively, this multiplicity comes from two non-monotonic relations between consumption and public debt. The first relation is driven by the government's budget constraint. Any increase in debt generates a rise in taxes leading to a crowding-out effect on output and growth; if this crowing-out effect is low (high), consumption and public debt are positively (negatively) linked, hence the first non-monotonic relationship. The second relation comes from the behavior of households. Any increase in public debt reduces both output and the real interest rate, exerting an adverse effect on consumption and on the inducement to save; depending on parameters, consumption can increase or decline, hence the second non-monotonic relationship.

Second, endogenous public debt cycles can emerge in the neighborhood of the low-

[3]The deficit-to-GDP ratio was around 2.5% on average in OECD countries in the period 1970-2005, and this ratio increased since the Great Recession (according to the 2017 IMF's *World Economic Outlook*, average general government gross debt in ratio of GDP in developed countries rose from around 72% in 2007 to roughly 103% in 2019; and the imbalances triggered by the public debt were at the core of the 2012 Eurozone debt crisis).

[4]Among all types of fiscal rules, debt and deficit rules were enacted in more than 60 countries by 2012, namely roughly three times more than expenditure rules, and more than six times more than revenue rules (see Schaechter et al., 2012; Combes et al., 2017).

[5]In exogenous growth setups, public debt is only transitory—see the section 3 in SGU. With endogenous growth, in contrast, public debt can grow in the long run; see, e.g. Minea and Villieu (2012), Boucekkine et al. (2015), Nishimura et al. (2015), Menuet et al. (2018), Cheron et al. (2019), among others.

growth trap. These cycles are (locally) stable for a range of parameter values through the occurrence of a (supercritical) *Hopf bifurcation*. This implies that a small perturbation of a parameter would not remove debt cycles. We show that these limit-cycles get larger as the deficit target is reduced. At the limit, long-lasting endogenous fluctuations can take the form of a homoclinic orbit, which defines a path that joins a steady-state to itself. From a global dynamics perspective, the existence of such an orbit is ensured by the occurrence of a generic *Bogdanov-Takens (BT) bifurcation*.

Third, fiscal rules can generate hysteresis and extreme sensitivity to changes in parameters. For small changes in the deficit-to-output ratio, the steady-state warps in a non-reversible way through the occurrence of a *Cusp bifurcation*: a tight deficit ratio may irreversibly condemn the economy to a low-growth/high-debt trap. Consequently, a balanced-budget rule does not preclude large aggregate fluctuations.

Quantitatively, our calibration exercise reveals the realism of our findings. The different (local and global) bifurcations occur for reasonable values of economic growth and public-debt ratio, and the asymmetric cycles that our model produces, with long periods of nearby-stationary growth and sudden short-living recessions, is consistent with observed stylized facts. Thus, our model can roughly replicate long-lasting fluctuations by endogenous mechanisms, reducing the need for exogenous shocks.

All in all, our analysis illustrates the *perils of fiscal rules*. Fiscal rules—including the balanced-budget rule, hereafter BBR—can lead to indeterminacy, multiple equilibria, and public debt cycles, with undesirable consequences on economic growth. These features arise in our general macroeconomic setup with standard assumptions, and are confirmed by calibration. Although stylized, our setup addresses major long-lasting topics in macroeconomics.

From a theoretical perspective, our paper is close to a rich and expanding literature aiming at identifying the different channels of fiscal policy-driven (in)determinacy. These channels can be roughly divided into three categories. The first one relates to the way taxes are modeled; examples include taxes on consumption, instead of labor (Giannitsarou, 2007), or progressive taxation (Christiano and Harrison, 1999). Second, the way public spending are modeled is also of importance; growth- or utility-enhancing, instead of wasteful public spending, can either support determinacy (Chen, 2006) or indeterminacy (Guo and Harrison, 2008, with exogenous growth, and Palivos et al., 2003; Park and Philippopoulos, 2004 with endogenous growth). Third, the (de)stabilizing effects of fiscal policy may significantly differ depending on alternative assumptions on taxes and public spending.[6]

With respect to this literature, we adopt wasteful public spending, endogenous flat-

[6]Such departures include non-separable utility function (Linnemann, 2008; Nourry et al., 2013; Abad et al., 2017), CES production function (Guo and Lansing, 2009; Ghilardi and Rossi, 2014), or two-sector models (Nishimura et al., 2013; Chang et al., 2015).

rate taxes on endogenous labor, and an additive utility function, as in SGU. Therefore, our (in)determinacy results are not triggered by the channels previously emphasized. On the methodological side, moving from exogenous to endogenous growth dramatically changes SGUs conclusions regarding the effects of labor taxes. Indeed, their findings must be amended on two grounds: (i) aggregate instability only occurs if public spending is high, and (ii) our aggregate-instability result covers a broader class of mechanisms than in SGU, because it may rely on local *and* global indeterminacy.

Importantly, in the existing literature, global indeterminacy comes from positive externalities associated to increasing returns or two-sector frameworks. In contrast, in our model, global indeterminacy is established in a one-sector model, and does not fundamentally rest on increasing returns in production. Instead, it rather resorts to the non-trivial dynamics of the debt-to-capital ratio that give rise to complex interactions between the government's budget constraint and the households' saving behavior. Against this background, although based on a one-sector model, indeterminacy does not depend on the famous Benhabib-Farmer condition (Benhabib and Farmer, 1994), namely that the increasing labor demand must be positively sloped and steeper than the labor supply.[7] In our model with constant returns-to-scale and decreasing returns in all private factors, the labor demand is a decreasing function of the wage, but nonetheless consistent with indeterminacy.[8]

From a policy perspective, the main message of our paper is that fiscal rules can be destabilizing, because they open the door for multiple equilibria and complex cyclical dynamics. This finding may be related to empirical studies that highlight the destabilizing role of fiscal rules due to pro-cyclical biases of fiscal policy in reaction to adverse supply or demand shocks (see, e.g. Alt and Lowry, 1994; Alesina and Bayoumi, 1996; Fatás and Mihov, 2006, among others). In our paper, however, the destabilizing effect of fiscal rules is not based on inadequate responses to exogenous shocks but on endogenous fluctuations resulting from a tight deficit target.

The paper is organized as follows. Section 2 presents the model, section 3 analyzes the steady-state, section 4 performs a calibration exercise, section 5 studies local and global dynamics, section 6 discusses public debt cycles, section 7 extends our results to balanced-budget rules, and section 8 concludes the paper.

[7] The survey of Benhabib and Farmer (1999) provides a thorough discussion of this condition.

[8] Technically, our paper is also related to applications of the Bogdanov-Takens (BT) bifurcation in economics. By applying the result of Kopell and Howard (1975) in a monetary model, Benhabib et al. (2001) first used this bifurcation to prove the destabilizing role of Taylor rules. Subsequent works employed the BT bifurcation to demonstrate the occurrence of Beveridge cycles (Sniekers, 2018), or homoclinic orbits in growth models (Benhabib et al., 2008, with exogenous growth and variable capacity utilization, and Mattana et al., 2009, with endogenous growth and two-sectors). In our paper, the BT bifurcation serves to prove the existence of homoclinic orbits associated with large public debt cycles.

2. The model

We consider a simple continuous-time endogenous-growth model with N representative individuals and a government. Each representative agent consists of a household and a competitive firm. All agents are infinitely-lived and have perfect foresight. Population is constant, and we denote individual quantities by lower case letters and aggregate quantities by corresponding upper case letters, namely $X = Nx$ for all variable X.

2.1. Households

The representative household starts at the initial period with a positive stock of capital (k_0), and chooses the path of consumption $\{c_t\}_{t\geq 0}$, hours worked $\{l_t\}_{t\geq 0}$, and capital $\{k_t\}_{t>0}$, such as to maximize the present discount value of its lifetime utility, which is assumed to be separable in consumption and leisure[9]

$$U = \int_0^\infty e^{-\rho t} \left\{u(c_t) - v(l_t)\right\} dt, \tag{1}$$

where $\rho > 0$ is the subjective discount rate. We consider a logarithm utility function ($u(c_t) = \log(c_t)$), and preferences for leisure are such that $v(l_t) = \frac{B}{1+\varepsilon} l_t^{1+\varepsilon}$, where $\varepsilon \geq 0$ is the constant elasticity of intertemporal substitution in labour, and $B > 0$ a scale parameter.

Households use labor income ($w_t l_t$, where w_t is the hourly wage rate) and capital revenues ($q_t k_t$, where q_t is the rental rate of capital), to consume (c_t), invest ($\dot{k}_t$), and buy government bonds (d_t), which return the real interest rate r_t. They pay taxes on wages ($\tau_t w_t l_t$, where τ_t is the wage tax rate), on consumption ($\tau_{ct} c_t$, where τ_{ct} is the tax rate on consumption), and perceive (positive or negative) lump-sum transfers π_t (in equilibrium, π_t is the share Π_t/N of total lump-sump transfer Π_t); hence the following budget constraint

$$\dot{k}_t + \dot{d}_t = r_t d_t + q_t k_t + (1-\tau_t) w_t l_t - (1+\tau_{ct}) c_t + \pi_t. \tag{2}$$

The first order conditions for the maximization of the household's programme give rise to the dynamic Euler relation (with $q_t = r_t$ in competitive equilibrium)

$$\frac{\dot{c}_t}{c_t} = r_t - \rho - \frac{\dot{\tau}_{ct}}{1+\tau_{ct}}, \tag{3}$$

and to the static relation

$$\frac{(1-\tau_t) w_t}{(1+\tau_{ct}) c_t} = B l_t^\varepsilon. \tag{4}$$

[9]The use of a separable utility function allows neutralizing an important source of indeterminacy in the form of non-separable preferences (see, e.g., Benhabib et al., 2001; Nourry et al., 2013).

Eq. (3) is the familiar Keynes-Ramsey rule that governs intertemporal consumption choices, here in the presence of varying consumption taxes. Eq. (4) shows that, at each period t, the marginal gain of hours worked (the net real wage $(1-\tau_t)w_t$, expressed in terms of marginal utility of consumption, including the effect of consumption taxes $1/((1+\tau_{ct})c_t)$), just equals the marginal cost (Bl_t^{ε}).

Finally, the optimal path of consumption has to verify the set of transversality conditions

$$\lim_{t\to+\infty}\{\exp(-\rho t)\ u'(c_t)\,k_t\}=0 \text{ and } \lim_{t\to+\infty}\{\exp(-\rho t)\ u'(c_t)\,d_t\}=0,$$

ensuring that lifetime utility U is bounded.[10]

2.2. Firms

Output (y_t) is produced using a constant returns-to-scale technology with a knowledge externality, namely $y_t=\tilde{A}h_t^{\alpha}k_t^{1-\alpha}$, where h_t and k_t respectively stand for human and physical capital, $\tilde{A}>0$ is a scale parameter, and $\alpha\in(0,1)$ is the elasticity of output to human capital.

Human capital is produced both by raw labor (or training activity) l_t, and by the economy-wide stock of knowledge X_t that generates positive technological spillovers onto firms' productivity (as Romer, 1986), namely $h_t=X_t l_t$. We assume that knowledge is produced by a simple Cobb-Douglas technology depending on aggregate levels of physical and human capital: $X_t=H_t^{\beta}K_t^{1-\beta}$, where $\beta\in(0,1)$ is a measure of human capital efficiency in the accumulation of knowledge. At aggregate level, we then obtain $H_t=K_tL_t^{1/(1-\beta)}=K_tL_t^{1+\phi}$, with $1+\phi=1/(1-\beta)\geq 1$.[11]

The production function exhibits constant returns-to-scale at the individual level, and decreasing returns in all private factors. The first order conditions for profit maximization (relative to private factors) are

$$w_t=\alpha\frac{y_t}{l_t}, \tag{5}$$

$$r_t=(1-\alpha)\frac{y_t}{k_t}. \tag{6}$$

At the aggregate level, the knowledge externality will allow reaching an endogenous growth path, because the social return of capital is not decreasing. Effectively, the aggregate production function is

$$Y_t=\tilde{A}K_tL_t^{\alpha(1+\phi)}. \tag{7}$$

[10] On the BGP associated to constant growth and interest rates (γ^* and r^*, respectively), these transversality conditions correspond to the no-Ponzi game constraint $\gamma^*<r^*$. This condition ensures that public debt will be repaid in the long run, and does not preclude the possibility that $\gamma>r$ in the short run.

[11] Human capital externalities, i.e. the fact that your coworkers' human capital makes you more productive, are well documented in empirical literature (see, e.g. Rauch, 1993; Moretti, 2004, who find very significant estimates of human capital externalities). Alternative models of endogenous growth, based on the Lucas (1988) archetype, consider the formation of human capital through individual training decisions that compete with productive activities.

As we will see in section 4, devoted to the quantitative assessment of our results, the main results of this paper appear even if the aggregated returns-to-scale are close to constant (i.e. $1+\alpha(1+\phi)$ close to one).

2.3. The government

The government provides public expenditure G_t and total transfers Π_t, levies taxes, and borrows from households. The fiscal deficit is financed by issuing debt ($\dot{D}_t$); hence, the following budget constraint

$$\dot{D}_t = r_t D_t + G_t + \Pi_t - \tau_t w_t L_t - \tau_{ct} c_t. \tag{8}$$

We shall assume that the government (i) claims a part g of aggregate output for public spending ($G_t = gY_t$),[12] (ii) retrieves a part μ of aggregate output from consumption taxes net from lump-sum transfers ($\tau_{ct} c_t - \Pi_t = \mu Y_t$), and (iii) fixes a constant tax rate on consumption ($\tau_{ct} = \tau_c < 1$).

At this stage, there are three exogenous parameters (g, μ, and τ_c) and two endogenous policy instruments in Eq. (8): public debt (D_t), and the tax rate on wages (τ_t). To close the model, one instrument has to be exogenously specified. To this end, we suppose that the government follows a fiscal rule characterized by a constant deficit-to-output ratio, namely[13]

$$\frac{\dot{D}_t}{Y_t} = \theta. \tag{9}$$

In this way, the tax-rate on wages will serve to adjust the government's budget constraint, as in SGU.

The deficit target $\theta \geq 0$ is consistent with current institutional frameworks (for example, a 3% deficit ceiling was adopted by the EU, the West African Economic and Monetary Union, or the East African Community, see IMF, 2018). As we will see, Eq. (9) conveys the main message of the paper in a direct and transparent way: a basic fiscal rule (including the balanced budget rule—BBR—, $\theta = 0$) may have destabilizing effects, because the interaction between the households' optimal saving behavior and the government's budget constraint opens the door to indeterminacy and complex dynamics, including long-run public debt cycles.

2.4. Equilibrium

We focus on the symmetric equilibrium in which all household-firm behave similarly.

[12] As in SGU, public expenditure has no effect on utility or production (i.e. wasteful public spending). For a recent model with productive expenditure, see, e.g. Menuet et al. (2018).

[13] Such a deficit rule is discussed in Menuet et al. (2018). We can also specify a gradual rule $\dot{\delta}_t = \eta(\theta - \delta_t)$, with $\delta_t = \dot{D}_t/Y_t$, and $\eta > 0$ a parameter reflecting the speed of adjustment of the deficit ratio to its long-run target. Such a rule does not qualitatively change our results. The fiscal rule (9) corresponds to $\eta = +\infty$, i.e. $\delta_t = \theta$.

Definition 1. A symmetric competitive equilibrium is a path $\{C_t, L_t, K_t, D_t, Y_t\}_0^\infty$ that solves Eqs. (3), (4), (5), (6), (8), (9), and satisfies the set of transversality conditions and the IS equilibrium $\dot{K}_t = Y_t - C_t - G_t$.

To find endogenous growth solutions, we deflate all growing variables by the capital stock to obtain long-run stationary ratios, namely (we henceforth omit time indexes): $y_k := Y/K$, $c_k := C/K$, and $d_k = D/K$.

Given the fiscal rule (9), the tax rate on wages is the adjustment variable in the government's budget constraint. Using Eqs. (5), (6), (8) and (9), it follows that

$$\tau = \frac{(1-\alpha)d_k + g - \mu - \theta}{\alpha} = 1 - \left(\frac{\bar{d} - (1-\alpha)d_k}{\alpha}\right), \tag{10}$$

where $\bar{d} = \alpha + \mu + \theta - g$.

Assumption 1. $(1-\alpha)d_k < \bar{d} < \alpha + (1-\alpha)d_k$

Assumption 1 ensures that $\tau \in (0,1)$. From (4), (6), and (7), we obtain the equilibrium level of output

$$y_k = A\left(\frac{\alpha(1-\tau)}{c_k}\right)^\psi, \tag{11}$$

where $\psi := \frac{\alpha(1+\phi)}{1+\varepsilon-\alpha(1+\phi)}$, and $A := \tilde{A}\left(\frac{\tilde{A}N^\varepsilon}{B(1+\tau_c)}\right)^\psi$.

Assumption 2 (Normal labor demand) $\alpha(1+\phi) < 1$.

Assumption 2 is a sufficient (unnecessary) condition for $\psi > 0$, and is verified under the plausible condition $\beta < 1 - \alpha$. Under Assumption 2, labor demand is *normal*, i.e. decreasing with real wages.[14] This is an important feature, because our indeterminacy results do not rest on a positively-sloped labor-demand curve, contrasting with Benhabib and Farmer (1994) and Farmer and Guo (1994).[15]

[14]Indeed, from (5) and (7), the aggregate labor demand writes $L_t = [\frac{w_t}{(1-\alpha)\tilde{A}K_t}]^{1/(\alpha(1+\phi)-1)}$.

[15]In Benhabib and Farmer, 1994, p.30, a necessary condition for indeterminacy is that (using our notations): $\alpha(1+\phi) > 1+\varepsilon$. This implies that the aggregate labor demand has to be increasing with real wages (see Eqs. (5) and (7) with $\alpha(1+\phi)-1 > \varepsilon \geq 0$). For the labor demand to slope up with real wages, increasing returns must be important, as discussed by Benhabib and Farmer (1994) and Schmitt-Grohé and Uribe (1997). In our model, as we have seen, we assume $\alpha(1+\phi) < 1 < 1+\varepsilon$, such that labor demand is normal, i.e. decreasing with real wages. We nevertheless obtain indeterminacy, thanks to the public debt dynamics. In addition, in our model, indeterminacy is consistent with lowly-increasing social returns, as illustrated by our quantitative analysis in section 4 (see Benhabib and Farmer, 1999, for a synthesis of several ways to obtain indeterminacy with small increasing returns).

The inverse relationship between the consumption ratio and the output ratio in Eq (11) comes from the labor market equilibrium (4). Following an increase in the consumption ratio, the marginal utility of consumption decreases, thus inducing households to substitute leisure for working hours (since $\varepsilon \geq 0$, leisure and consumption are complement in equilibrium). Then, the equilibrium labor supply and output are reduced. The same result arises if the tax rate on wages increases.

With constant consumption taxes, the optimal aggregate consumption behaviour is, from (3) and (6),

$$\frac{\dot{C}}{C} = (1-\alpha)y_k - \rho, \tag{12}$$

and the path of the capital stock is given by the goods market equilibrium

$$\frac{\dot{K}}{K} = (1-g)y_k - c_k. \tag{13}$$

The path of public debt follows the definition of the deficit ratio, namely

$$\frac{\dot{D}}{D} = \theta \frac{y_k}{d_k}. \tag{14}$$

Hence, the reduced-form of the model is obtained by Eqs. (12), (13) and (14)

$$\begin{cases} \dfrac{\dot{c}_k}{c_k} = (g-\alpha)y_k - \rho + c_k, \\ \\ \dfrac{\dot{d}_k}{d_k} = \dfrac{\theta y_k}{d_k} - (1-g)y_k + c_k, \end{cases} \tag{15}$$

where, from Eqs. (10) and (11)

$$y_k = A\left(\frac{\bar{d} - (1-\alpha)d_k}{c_k}\right)^{\psi} =: y_k(c_k, d_k). \tag{16}$$

In equilibrium, any increase in the debt ratio (d_k) reduces the output ratio (y_k). Indeed, the growing interest-burden of public debt leads to more taxes on wages, which discourages labor supply. The same crowding-out effect applies in case of an increase in public spending or a decrease in the deficit target, through coefficient $\bar{d}$.

Definition 2. A steady-state i is a symmetric competitive equilibrium where consumption, capital, output, and public debt grow at the common (endogenous) rate γ^i, such that $\dot{c}_k = \dot{d}_k = 0$ in (15). At any steady state i, the economy is characterized by a balanced-growth path (BGP): $\gamma^i := \dot{C}/C = \dot{K}/K = \dot{Y}/Y = \dot{D}/D$, while the real interest rate (r^i) is constant.

We determine the steady-state solutions of the model in section 3, section 4 provides a quantitative assessment, and section 5 discusses local and global dynamics.

3. Long-run solutions and the deficit target

In this section, we consider strictly positive deficit targets ($\theta > 0$).[16] The long-run endogenous growth solutions are described by two relations between c_k and d_k. The first one is the $\dot{c}_k = 0$ locus, which comes from the Euler relation (12) and the IS equilibrium (13)

$$d_k^* = \frac{1}{1-\alpha}\left\{\bar{d} - c_k^*\left(\frac{\rho - c_k^*}{(g-\alpha)A}\right)^{1/\psi}\right\}, \tag{17}$$

where a star denotes steady-state values.

The second relation is the $\dot{d}_k = 0$ locus, related to the government's budget constraint (8), and the deficit rule (9)

$$\theta y_k(c_k^*, d_k^*) = [(1-g)y_k(c_k^*, d_k^*) - c_k^*]d_k^*. \tag{18}$$

Steady-state solutions are obtained as the crossing-point of Eqs. (17) and (18).

Theorem 1. *The long-run equilibria are characterized by the following regimes.*

- *Regime $\mathcal{L}$ (low public spending): $g < \alpha$. There are two positive-growth candidates for a steady-state: a high-growth solution (point P) and a low-growth solution (point M).*
- *Regime $\mathcal{H}$ (high public spending): $g > \alpha$. There are three positive-growth candidates for a steady-state (points M, P and Q), and a no-growth degenerate solution (point D).*

Proof. First, Eq. (17) corresponds to $d_k^* = d_k(c_k^*)$. There are two cases. On the one hand, if $g < \alpha$, we have $d_k'(c_k^*) < 0$, $\forall c_k^* > \rho$, hence a monotonic decreasing relation between c_k^* and d_k^*. On the other hand, if $g > \alpha$, we have $d_k'(c_k^*) < 0$ if $c_k^* \in [0, \hat{c}_k)$, and $d_k'(c_k^*) > 0$ if $c_k^* \in (\hat{c}_k, \rho)$, where

$$\hat{c}_k = \frac{\psi\rho}{1+\psi}, \tag{19}$$

hence a U-shaped curve in the (c_k, d_k)-plane, with a minimum at $\hat{c}_k$.

Second, Eq. (18) leads to $c_k^* = c_k(d_k^*)$, where

$$c_k(d_k^*) = \left\{\frac{A}{d_k^*}\left[\bar{d} - (1-\alpha)d_k^*\right]^{\psi}\left[(1-g)d_k^* - \theta\right]\right\}^{1/(1+\psi)}, \tag{20}$$

[16] The balanced-budget rule (BBR) case is studied in section 7.

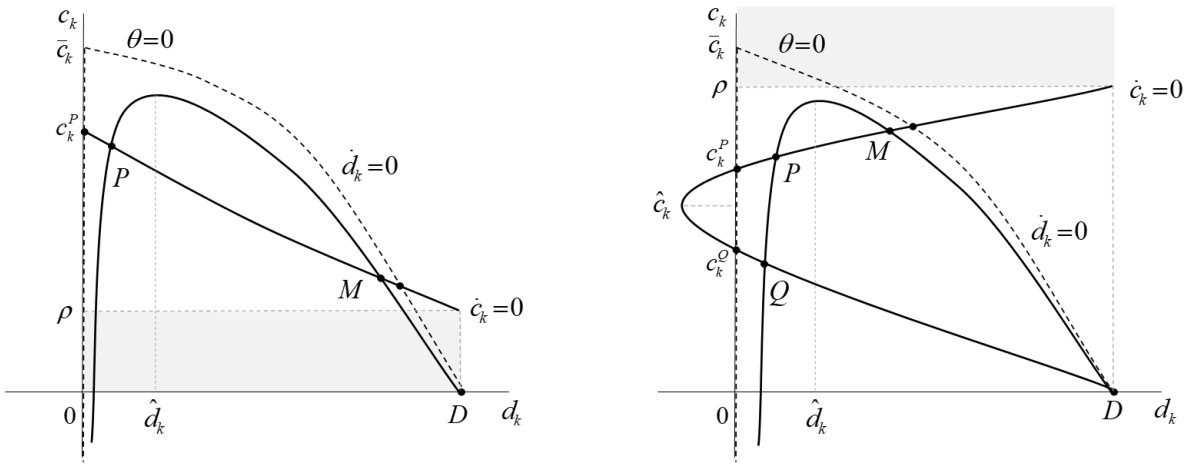
hence $c_k'(d_k^*) \geq 0 \Leftrightarrow -(d_k^*)^2\psi(1-\alpha)(1-g) - d_k^*\theta(1-\alpha)(1-\psi) + \theta\bar{d} \geq 0$. Focusing on $d_k^* \geq 0$, $c_k(\cdot)$ depicts a bell-shaped curve in the (d_k, c_k)-plane, with a maximum at

$$\hat{d}_k := \frac{\sqrt{(1-\alpha)\theta[\theta(1-\alpha)(1-\psi)^2 + 4\psi(1-g)\bar{d}]} - \theta(1-\alpha)(1-\psi)}{2\psi(1-\alpha)(1-g)} \geq 0.$$

Notice that, for small deficit targets (the term $\theta(1-\alpha)(1-\psi)$ is small enough, see our calibrations in section 4), this maximum can be approximated by

$$\hat{d}_k \approx \sqrt{\frac{\theta\bar{d}}{\psi(1-g)(1-\alpha)}}. \tag{21}$$

We define $\bar{c}_k$ as the maximal consumption ratio, namely $\bar{c}_k := c_k(\hat{d}_k)$. If $\theta = 0$, this value corresponds to $\bar{c}_k = [A\bar{d}^{\psi}(1-g)]^{1/(1+\psi)}$.



3a: Regime $\mathcal{L}$ $(g < \alpha)$

3b: Regime $\mathcal{H}_2$ $(g > \alpha)$

Figure 3: The steady states

In Regime $\mathcal{L}$ $(g < \alpha)$, there are at most two BGPs (P and M, as in Figure 3a). As θ increases, the curve $\dot{c}_k = 0$ moves upwards (see Eq. 17), while the curve $\dot{d}_k = 0$ moves backwards (see Eq. 20). Hence, there is a critical value $\bar{\theta}$, such that points the P and M collide, defining a *saddle-node bifurcation*: for slightly higher values of θ, there is no equilibrium.

In Regime $\mathcal{H}$ $(g > \alpha)$, four BGPs are feasible, as in Figure 3b. A trivial solution, denoted by point D, is associated with $c_k^* = 0 =: c_k^D$ and $d_k^* = \bar{d}/(1-\alpha) := d_k^D$ (in this case, we have $y_k^D = 0$). The couple (c_k^D, d_k^D) is such that the economy asymptotically vanishes. We will refer to this solution as close to the "harrodian" perspective. Although such a "collapse" solution is not economically attractive, it cannot be rejected without

assessing the local and global dynamics of the model, as we will see.[17] The areas for which the different positive-growth candidates emerge (or not) as steady-state solutions are discussed in the following subsection. □

3.1. Behaviour of Regime $\mathcal{H}$

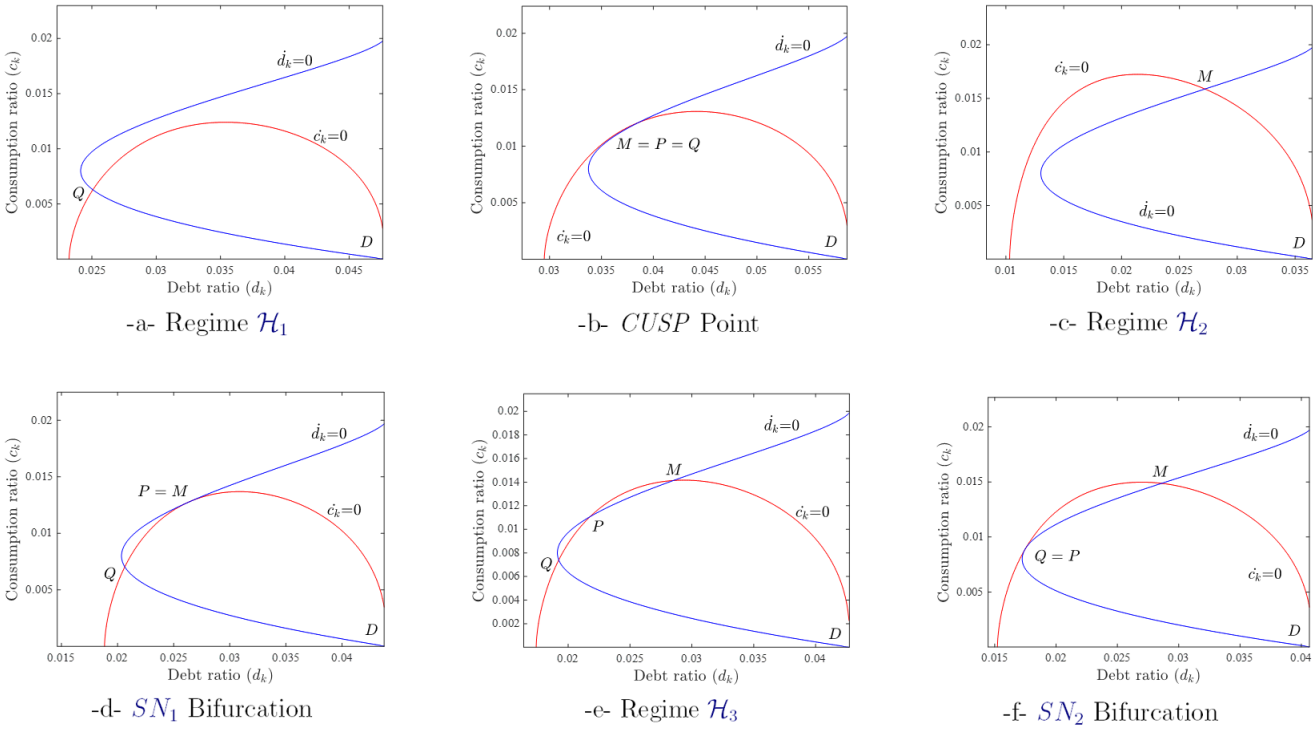
Figure 4 depicts the different configurations of long-run equilibria. The most general case is Regime $\mathcal{H}_3$ (Figure 4e), in which the two steady-state curves intersect four times, at points Q, P, M, and D. The associated BGPs are such that $0 = \gamma^D < \gamma^M < \gamma^P < \gamma^Q$, corresponding to the inverse ranking of the public debt ratio $d_k^Q < d_k^P < d_k^M < d_k^D$. Regime $\mathcal{H}_3$ occurs only if $\hat{c}_k < \bar{c}_k$, where $\hat{c}_k$ and $\bar{c}_k$ respectively correspond to the leftmost point of $\dot{c}_k = 0$ and to the maximum of $\dot{d}_k = 0$, and we focus on this configuration in the following.[18]

Ignoring point D for the moment, the areas of structural invariance are located on either sides of the two saddle-node global bifurcations that surround Regime $\mathcal{H}_3$. At the first bifurcation, labelled SN_1 (Figure 4d), points P and M collide. For a small change in parameters, the economy switches from Regime $\mathcal{H}_3$ into Regime $\mathcal{H}_1$, which is characterized by a unique BGP (point Q in Figure 4a). At the second bifurcation, labelled SN_2 (Figure 4f), the situation is symmetric: points P and Q collide, such that for a small change in parameters, the economy switches from Regime $\mathcal{H}_3$ into Regime $\mathcal{H}_2$, where point M is the unique BGP (Figure 4c). The two saddle-node bifurcations eventually collide for some parameters' values, defining a *CUSP point*, such that steady-states M, P, and Q merge (Figure 4b). In this case, Regime $\mathcal{H}_3$ vanishes, and the long-run solution is unique. For any arbitrarily small change in parameters, on both sides of this CUSP bifurcation, the number of long-run solutions goes from one to three. As we will see, this CUSP bifurcation may generate hysteresis.

Our quantitative illustration of the model highlights that these different configurations are realistic. In section 4 below, using policy instruments θ and g as parameters of interest, we show that all types of bifurcations are consistent with OECD or US historical data.

[17] Households' preferences are defined only for $c_t > 0$, but the steady state D can be asymptotically reached with $\lim_{t\to+\infty} c_t = 0^+$.

[18] If $\hat{c}_k > \bar{c}_k$, Regime $\mathcal{H}_3$ cannot appear, since only two BGPs are feasible: points Q and D in Regime $\mathcal{H}_1$, or points M and D in Regime $\mathcal{H}_2$. Since these regimes also appear if $\hat{c}_k < \bar{c}_k$, we can restrict our analysis to this configuration (the alternative configuration $\hat{c}_k > \bar{c}_k$ will be extensively addressed in the BBR case, in section 7).



-a- Regime $\mathcal{H}_1$ -b- *CUSP* Point -c- Regime $\mathcal{H}_2$

-d- SN_1 Bifurcation -e- Regime $\mathcal{H}_3$ -f- SN_2 Bifurcation

Figure 4: Topological regimes in Regime $\mathcal{H}$ ($\hat{c}_k < \bar{c}_k$)

3.2. Some intuition

Fundamentally, the multiplicity of equilibria comes from the government budget constraint. In steady state, the fiscal rule leads to $\theta y_k^* = \gamma^* d_k^*$, where $\gamma^* = (1-g)y_k^* - c_k^*$. Yet, any increase in the debt ratio d_k^* generates a rise in taxes to finance the additional government debt burden, leading to an adverse effect on output (y_k^*) and growth (γ^*). It follows that the deficit rule is consistent with two stationary solutions: a high-growth/low-debt solution and a high-debt/low-growth solution. This explains the hump-shaped relationship between the debt ratio and the consumption ratio. An increase in the debt ratio d_k^* generates first a positive effect on consumption, provided that $d_k^* < \hat{d}_k$, because the crowding-out effect of public debt on economic growth is low. On the opposite, as soon as $d_k^* > \hat{d}_k$, economic growth is strongly affected by the government debt interest burden, and any further rise in d_k^* reduces consumption. Consequently, due to the fiscal rule, c_k^* is non-monotonically associated to d_k^*.

In the BBR case ($\theta = 0$), the mechanism is similar, except that $\hat{d}_k = 0$ in Eq. (21), and the $\dot{d}_k = 0$ curve becomes a degenerate hump-shaped curve, composed of the two branches $\gamma^* = 0$ and $d_k^* = 0$, as in Figure 3. Indeed, the BBR is consistent with a positive (constant) stock of public debt and does not preclude multiplicity, as shown in section 7.

Against this background, the mere presence of public debt with a fiscal rule explains the multiplicity in Regimes $\mathcal{L}$ and $\mathcal{H}$. In Regime $\mathcal{H}$, there is an additional source of

multiplicity, due to high public spending. Indeed, in the long-run, the BGP requires $\dot{K}/K = \dot{C}/C = \gamma^*$. According to the Euler equation (12) and the IS equilibrium (13), this condition amounts to $(1-g)y_k^* - c_k^* = (1-\alpha)y_k^* - \rho$, or

$$c_k^* - \rho = (\alpha - g)y_k(c_k^*, d_k^*). \tag{22}$$

As we have seen, y_k negatively depends on d_k and c_k through the labour market equilibrium (see Eq. 16). Then, the relation between c_k^* and d_k^* crucially depends on the sign of $\alpha - g$. If $g < \alpha$ (regime $\mathcal{L}$), an increase of c_k^* decreases the RHS of (22) and rises the LHS, thus generating an unambiguously monotonic decreasing relation between c_k^* and d_k^*, as depicted in Figure 3a. If $g > \alpha$ (regime $\mathcal{H}$), both sides of Eq. (22) positively depend on c_k^*, which produces a non-monotonic association between c_k^* and d_k^*. Consequently, any debt ratio is associated with two consumption ratios, as in Figure 3b.

The role of the condition $g > (<)\alpha$ is intuitive. As d_k^* increases, the growth rates of consumption $(\dot{C}/C)$ and private capital $(\dot{K}/K)$ decrease, through an adverse effect on output. However, the impact of the output ratio on the growth rate of consumption depends on the return of capital $(1-\alpha)$—see Eq. (12), while its impact on the growth rate of capital depends on public spending $(1-g)$—see Eq. (13). Therefore, if $g < (>)\alpha$, the investment-goods sector is more (less) sensitive than the consumption-goods sector to a change of d_k, and the consumption ratio c_k must adjust in order to restore the equality $\dot{K}/K = \dot{C}/C$ along the BGP.

Two results deserve particular attention. First, as we will see, although the Regime $\mathcal{L}$ is well-determined, the higher-growth solution (Q) cannot be reached. This explain why governments can be induced to increase public spending until reaching Regime $\mathcal{H}$. Second, in regime $\mathcal{H}_3$ indeterminacy cannot be avoided, unless the positive long-run solution disappears. Effectively, one cannot obtain the positive BGP P without the undetermined solution Q. *Thus, local and global indeterminacy can be viewed as the price that must be paid to generate a positive long-run growth solution.*

The following section presents a calibration exercise showing that the different configurations arise for plausible parameters' values.

4. A quantitative assessment

Our numerical results are based on reasonable values of parameters. In our benchmark calibration, we choose $\rho = 0.02$, corresponding to the long-run value of the risk free (real) interest rate, and the labor elasticity of substitution is fixed at $\varepsilon = 0$, thus characterizing an infinite Frisch elasticity as usual in business cycle models (see SGU). Regarding the technology, we set $A = 0.05$ to obtain realistic rates of economic growth, and the size of the knowledge externality in the production function is $\alpha = 0.1$, close to its value $(\alpha = 0.08)$ in Turnovsky (2000). The measure of human capital intensity in the accumulation of

knowledge is set to $\beta = 0.75$ in our baseline calibration, since human capital is probably the most important factor in the production of knowledge.

Regarding the government's behavior, we consider a range of deficit ratios $\theta \in (0, 0.03)$, in line with long-run average values in US or OECD from 1950 to 2019. In the baseline calibration, governments expenditure is chosen so that the fraction of net national production devoted to public spending on goods and services equals the historical average in the United States ($g = 0.225$). Besides, the sum of consumption taxes and lump-sum taxes is assumed to be 15% of GDP (namely, $\mu = 0.15$). For these parameters' values, the corresponding rate of wage taxation (in percent of GDP) is between 10% and 16.5%, depending on the equilibrium considered (the average value in OECD data is roughly 15%).

PARAMETERS

Households		
S	1	Intertemporal elasticity of substitution
ρ	0.02	Discount rate
ε	0	Labor elasticity of substitution
Technology		
A	0.05	Productivity parameter
α	0.1	Size of the knowledge externality in the production function
β	0.75	Share of human capital in the production of knowledge
Government		
g	0.225	Government spending on goods and services
θ	0 to 0.03	Long-run deficit-ratio target
μ	0.15	Share of consumption taxes plus lump-sum taxes in GDP

Table 1: The baseline calibration

In our benchmark calibration, regime $\mathcal{L}$ appears for $g < 0.1$, while, in the opposite case, regime $\mathcal{H}$ prevails. As $g \simeq 22.5\%$ in historical data, we particularly focus from now on the latter regime. Table 2 reproduces the different steady-state solutions in the more general case (Regime $\mathcal{H}_3$), under a deficit target $\theta = 1.3\%$ (consistent with the existence of Regime $\mathcal{H}_3$). The long-run economic growth rate equals 2%, 4.7%, and 6.8% at points M, P, and Q, respectively; and the long-run public debt to output ratio is 64%, 27%, and 19%, respectively. These numbers are fairly realistic. In particular, point M is closely related to OECD data. This feature is of particular importance, since the cyclical dynamics in our model appear in the neighborhood of this steady state. Moreover, the two saddle-node bifurcations SN_1 and SN_2 emerge for reasonable values of the deficit target, namely $\theta \simeq 1.46\%$ and $\theta \simeq 1.18\%$ respectively, and are associated to realistic long-run economic growth and public debt ratio.

Table 2 also computes different types of bifurcations of codimension 1 and 2 (the codimension of a bifurcation is the number of parameters that must be varied for the bifurcation to occur). In our benchmark calibration, a *Hopf bifurcation* occurs at $\theta \simeq 1.42\%$ (namely in Regime $\mathcal{H}_3$); while, if two parameters (here, the deficit target θ and the public pending ratio g) are allowed to vary, a *CUSP bifurcation* arises at $\theta \simeq 2.3\%$ and $g \simeq 22\%$, and a *Bogdanov-Takens bifurcation* (labelled BT) appears at $\theta \simeq 0.7\%$ and $g \simeq 23\%$. For lower values of the discount rate (e.g. $\rho = 0.01$), a *Generalized Hopf (Bautin) bifurcation* (labelled GH) can also occur (at $\theta \simeq 1.8\%$ and $g \simeq 24\%$ in our calibration). The interpretation of these bifurcations will be provided in the following sections. What is of particularly importance here is that all types of bifurcations are consistent with economic conditions experienced by developed countries during the last decades.

	θ	g	γ^M	γ^P	γ^Q	D^M/Y^M	D^P/Y^P	D^Q/Y^Q
SN_1	0.0146	0.225	0.032		0.08	0.46		0.15
Regime $\mathcal{H}_3$	0.013	0.225	0.02	0.047	0.068	0.64	0.27	0.19
SN_2	0.0118	0.225	0.017	0.058		0.70	0.20	
CUSP	0.023	0.22	0.049			0.47		
Hopf	0.0142	0.225	0.026	0.039	0.072	0.55	0.36	0.20
GH	0.018	0.24	0.025	$--$	$--$	0.71	$--$	$--$
BT	0.007	0.23	0.02	0.033	0.08	0.33	0.21	0.09

Table 2: Economic growth and the public debt ratio under different configurations

The three regimes established in Figure 4 appear on either parts of the two saddle-node bifurcations (SN_1 and SN_2). These bifurcations can be generated by particular values of the deficit target (θ_1 and θ_2, respectively). Ignoring the harrodian equilibrium D, if $\theta > \theta_1 \simeq 1.46\%$ only the high-growth solution Q exists (Regime $\mathcal{H}_1$); if $\theta < \theta_2 \simeq 1.18\%$ only the low-growth trap M appears (Regime $\mathcal{H}_2$); while in the intermediate regime $\theta_2 < \theta < \theta_1$, three positive-growth BGPs prevail as we have seen (Regime $\mathcal{H}_3$).

Interestingly, Regime $\mathcal{H}_3$ can occur without the need of high social returns-to-scale in the aggregate production function. With $g = 0.15$, for example, regime $\mathcal{H}_3$ is consistent with $\beta = 0$ and $\alpha = 0.01$, namely for almost constant returns-to-scale ($1+\alpha(1+\phi) = 1.01$). Indeed, for these values, regime $\mathcal{H}_3$ prevails as soon as $0 < \theta < 1.345\%$. Therefore, in our setup multiplicity (and indeterminacy) can arise even if returns to scale are close to constant, as empirical evidence suggests (see, e.g. Basu and Fernald, 1997).[19]

[19]In contrast, Benhabib and Farmer (1994) need increasing returns in excess of 0.5. Following the empirical works of Burnside (1996) or Basu and Fernald (1997), suggesting that the U.S. manufacturing industry displays roughly constant returns with no external effects, a theoretical research agenda was opened by Benhabib and Farmer (1996) in order to reduce the degree of increasing returns needed to

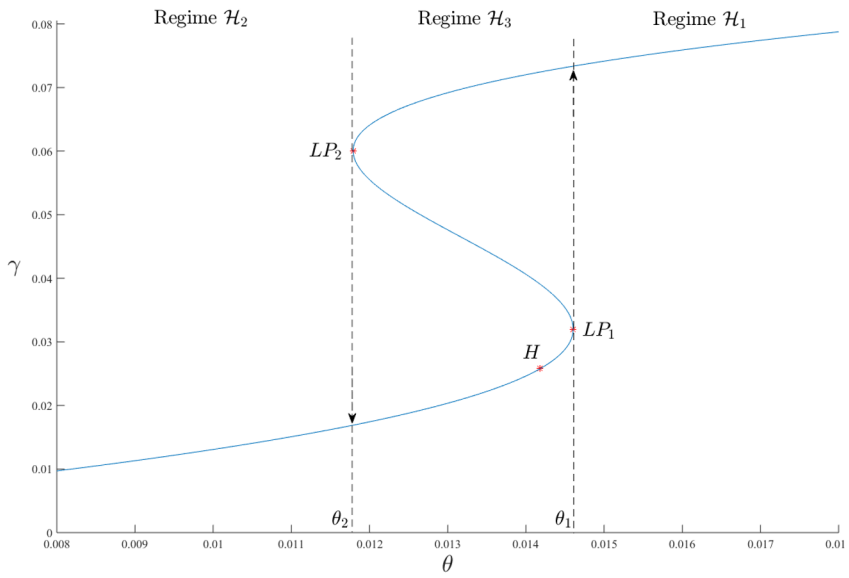


Figure 5: Bifurcation diagram as a function of θ (regime $\mathcal{H}$)

The presence of two saddle-node bifurcations in the neighborhood of the CUSP singularity can generate a hysteresis phenomenon, as described in Figure 5. Suppose, e.g., that the deficit target is $\theta_2 < \theta < \theta_1$, and that the economy locates at the low-growth steady-state M in Regime $\mathcal{H}_3$ (starting, e.g., at point H). If θ increases until θ_1, the economy moves along steady state M until point LP_1. If θ increases further, the economy switches into Regime $\mathcal{H}_1$ and the steady state suddenly jumps from M to Q. However, if from this point θ is decreased, the economy does not come back to steady-state M, but public debt decreases along steady-state Q until point LP_2 at $\theta = \theta_2$. If we further decrease the value of θ, the economy switches in Regime $\mathcal{H}_2$, and the steady state suddenly jumps to M. Hence, for small changes in the deficit target, the steady state warps in a non-reversible way: decreasing too much the long-term deficit target may condemn the economy to an irreversible steady-state with low economic growth (and high debt). Of course, such an analysis is only based on comparative statics of the steady states, and must be further investigated from a dynamics perspective. This is the goal of the following sections.

5. Analysis of dynamics

This section is devoted to the analysis of local dynamics, followed by global dynamics.

generate indeterminacy. The conclusion was that indeterminacy can arise with low increasing returns, or even constant returns, in multi-sector models (see, e.g., Benhabib and Nishimura, 1998; Benhabib et al., 2000); for example, indeterminacy requires increasing returns of about 0.07 in Benhabib and Farmer (1996). In our one-sector model, indeterminacy can arise with very low increasing returns around 0.01.

5.1. Local dynamics

By linearization, in the neighborhood of steady-state i, $i \in \mathscr{S} = \{D, M, P, Q\}$, the system (15) behaves according to $(\dot{c}_k, \dot{d}_k) = \mathbf{J}^i(c_k - c_k^i, d_k - d_k^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_{k0}) and one pre-determined variable (the public-debt ratio d_{k0}, since the initial stocks of public debt D_0 and private capital K_0 are predetermined). Hence, for BGP i to be well determined, the Jacobian matrix

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CD^i \\ DC^i & DD^i \end{pmatrix},$$

must contain two opposite-sign eigenvalues, where, using (15),

$$CC^i = c_k^i[1 + (g - \alpha)yc^i], \tag{23}$$
$$CD^i = c_k^i(g - \alpha)yd^i, \tag{24}$$
$$DD^i = \theta yd^i - \gamma^i - (1 - g)yd^i d_k^i, \tag{25}$$
$$DC^i = \theta yc^i - (1 - g)yc^i d_k^i + d_k^i, \tag{26}$$

with, using (16),

$$yc^i := \frac{\partial y_k^i}{\partial c_k^i} = -\frac{\psi y_k^i}{c_k^i} < 0, \text{ and } yd^i := \frac{\partial y_k^i}{\partial d_k^i} = -\frac{(1 - \alpha)\psi y_k^i}{\bar{d} - (1 - \alpha)d_k^i} < 0. \tag{27}$$

Hence, the trace and the determinant of the Jacobian matrix are, respectively

$$\mathrm{Tr}(\mathbf{J}^i) = \theta yd^i + c_k^i[1 + (g - \alpha)yc^i] - \gamma^i - (1 - g)yd^i d_k^i, \tag{28}$$
$$\det(\mathbf{J}^i) = -c_k^i[\gamma^i - \theta yd^i + (1 - g)d_k^i yd^i + (g - \alpha)(\gamma^i yc^i + d_k^i yd^i)]. \tag{29}$$

The following theorem establishes the topological behaviour of each steady state.

Theorem 2. *(Local Stability)*

- *In regime $\mathcal{L}$, M is locally unstable and P is locally determinate (saddle-point stable).*
- *In regime $\mathcal{H}$, points D and P are locally determinate (saddle-point stable); point Q is locally indeterminate (stable); and M can be either locally indeterminate (stable) or unstable.*

Corollary 1. *A Hopf bifurcation can emerge in the neighborhood of $M(c_k^M, d_k^M)$, if its coordinates verify $d_k^M < \hat{d}_k$. In particular, for small values of θ, the Hopf bifurcation is reached at θ^h such that*

$$\theta^h := (1 - g)d_k^M - \frac{CC^M}{yd^M}. \tag{30}$$

Proof. See Appendix A.

The major difference between regimes $\mathcal{L}$ and $\mathcal{H}$ is related to the monotonicity of the $\dot{c}_k = 0$ curve. As we have seen (see Theorem 1), the two regimes are characterized by the saddle-node bifurcation SN_1, at which points M and P collide. But, in regime $\mathcal{H}$, another bifurcation SN_2 arises, which gives birth, for nearby parameters' values, to the positive-growth solutions P and Q. Thus, the interplay between the consumption ratio and the public debt ratio can give rise to complex dynamics, depending on the location of steady states in the plane (d_k, c_k).

The local stability analysis points out that, due to the possibility of a Hopf bifurcation, small changes in the deficit target can generate radical shifts in the dynamics, together with large oscillations of economic growth and public debt in the neighborhood of the low-growth trap M. Sections 6 and 7 explicitly characterize this bifurcation. Beforehand, we turn to global dynamics.

5.2. Global dynamics and (in)determinacy

According to the analysis of local dynamics, we can distinguish four cases depending on the fiscal policy parameters—the deficit target and the public spending ratio.

Regime $\mathcal{L}$ – There are two steady states, but only the higher BGP (P) can be reached in the long-run, as the low-growth trap (M) is unstable. Hence, there is no local or global indeterminacy in this regime (Figure 6a).[20]

Regime $\mathcal{H}_1$ – This regime is also characterized by two steady states. One is associated to high economic growth (Q, with low consumption and deficit ratios) and is stable, while the harrodian equilibrium D is saddle-path stable. Consequently, there is local indeterminacy in the vicinity of Q, and possibly global indeterminacy, because the economy can move towards Q or D, if the initial debt ratio is such that $d_{k0} > \hat{d}_k$ (see Figure 6b).

Regime $\mathcal{H}_2$ – In this regime there are equally two steady states, M and D. The latter is still locally determinate, but the topological behavior of the low-growth trap M depends on its position relative to d_k. If $d_k^M > \hat{d}_k$, M is unstable, and there is no local or global indeterminacy. Starting from an initial public debt ratio close to d_k^M, the economy converges towards the harrodian equilibrium D. In contrast, if $d_k^M < \hat{d}_k$, there is global indeterminacy, as states the following proposition.

Proposition 1. *If $d_k^M < \hat{d}_k$, the steady state exhibits indeterminacy as follows: for initial public debt ratios originating in the neighborhood of M, the economy can converge to the point D, to the point M, or can join a periodic orbit around point M, depending on the initial jump in consumption.*

[20] The initial public debt ratio exerts a threshold effect: if $d_{k0} < d_k^M$, for any predetermined d_{k0} the consumption ratio c_{k0} jumps to place the economy on the saddle-path that converges towards P, which defines the unique long-run equilibrium. In contrast, if $d_{k0} > d_k^M$ there is no long-run solution.

Proof. If $d_k^M < \hat{d}_k$ there is a Hopf bifurcation at $CC^M = -DD^M \Leftrightarrow \theta = \theta^h$, as we have seen. This Hopf bifurcation can be supercritical, generating a stable limit cycle (if the first Lyapunov coefficient is negative), or subcritical, generating an unstable closed orbit (if the first Lyapunov coefficient is positive), depending on parameters. Indeed, a Generalized (Bautin) Hopf bifurcation may arise for realistic values of parameters, as stated in Table 2. At this Generalized Hopf bifurcation the first Lyapunov coefficient is zero, ensuring the presence of stable limit cycles for nearby parameter values. Hence, the economy can converge towards a periodic orbit around M if the bifurcation is supercritical or directly jump on this orbit if the bifurcation is subcritical as in Figure 6c. In addition, if $d_{k0} > \hat{d}_k$, there is a unique trajectory that goes towards the harrodian equilibrium D. As a consequence, given a predetermined debt ratio any path that converges toward M or D, or joins the cycle, can be reached. □

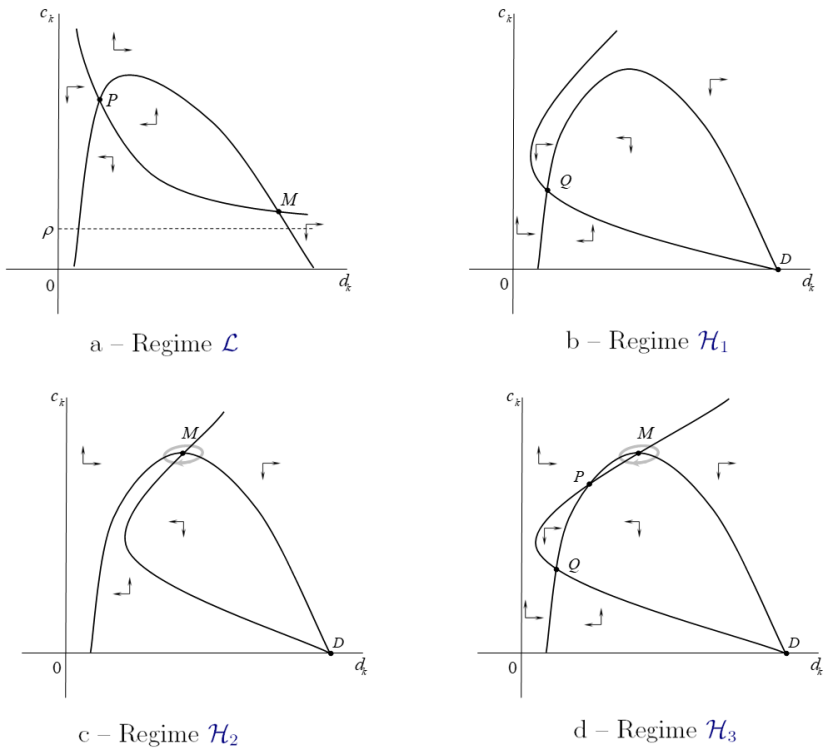


Figure 6: Global dynamics

Regime $\mathcal{H}_3$ – There are four steady states: P and D are still saddle-path stable, Q is still stable, but the local stability of point M depends on its location relative to $\hat{d}_k$, as in Regime $\mathcal{H}_2$. The following proposition summarizes the global dynamics.

Proposition 2. *In regime $\mathcal{H}_3$, the dynamics exhibit indeterminacy as follows:*

i. *Trajectories originating in the neighborhood of M can converge either to points D, P, or Q. Points D and P can be reached by only one trajectory, while there is an infinite set of trajectories converging to Q.*

ii. *Additionally, if* $d_k^M < \hat{d}_k$*, the economy can join* M *or a periodic orbit around* M*.*

Proof. The proof of (i) directly results from Theorem 1. If $d_k^M > \hat{d}_k$, point M is unstable, and the economy can converge towards saddle-points D or P, or the (locally indeterminate) steady state Q. If $d_k^M < \hat{d}_k$, a Hopf bifurcation arises at $\theta = \theta^h$, as stated in Corollary 1. As previously, this bifurcation can be subcritical or supercritical, depending on parameters. Thus, point M can be stable, associated to an instable periodic orbit, or unstable with a stable limit cycle around M; hence (ii). □

Regime $\mathcal{H}$ is thus characterized by local (in the vicinity of Q and M, and of the possible limit-cycle that surrounds the low-growth trap M), and global indeterminacy. The short-run and long-run behavior of the economy is then subject to "animal spirits", in the form of self-fulfilling prophecies that generate multiple balanced growth paths in the future. Such indeterminacy is intuitive. Suppose, for example, that at the initial time households expect low public debt in the steady-state. This implies that the expected tax rate is low, and the expected net return of capital is high. Then, at the initial time households increase their savings, such that the initial consumption ratio (c_{k0}) is low and the initial hours worked will be high. In equilibrium, labor supply will also be high, generating large fiscal resources and low public debt in the future (along P and Q BGPs). Conversely, following the same mechanism, high expected public debt is self-fulfilling, and may lead to the growth solutions M or D. In other words, by their consumption-leisure tradeoff at the initial time, forward-looking households can, in equilibrium, validate any expectation on the BGP that can be reached in the future.

Figure 7 synthesizes our results in the (θ, g) plane. The two saddle-node bifurcations are depicted by the curves $SN_1(\theta)$ and $SN_2(\theta)$ that represent the limit-points between regimes $\mathcal{H}_1$ and $\mathcal{H}_3$, and $\mathcal{H}_3$ and $\mathcal{H}_2$, respectively. For our benchmark calibration $g = 0.225$, and these limit-points are labelled LP_1 and LP_2, respectively. The CUSP point (labelled CP) occurs at the intersection of these two bifurcation curves, such that, for higher values of the deficit target or lower values of the public spending ratio, Regime $\mathcal{H}_3$ vanishes. The dashed curve $\mathcal{H}(\theta)$ depicts the locus of Hopf bifurcations. For the benchmark calibration, the Hopf bifurcation point (labelled H) is located in Regime $\mathcal{H}_3$, but it can be located in Regime $\mathcal{H}_2$ for lower values of the public spending ratio, as shows Figure 7.

It must be emphasized that the area consistent with Regime $\mathcal{H}_3$ enlarges as the deficit target is reduced. Therefore, ceteris paribus, a small deficit target is likely to increase the risk of aggregate fluctuations. Yet, as shows section 7, local and global indeterminacy do not vanish under a BBR.

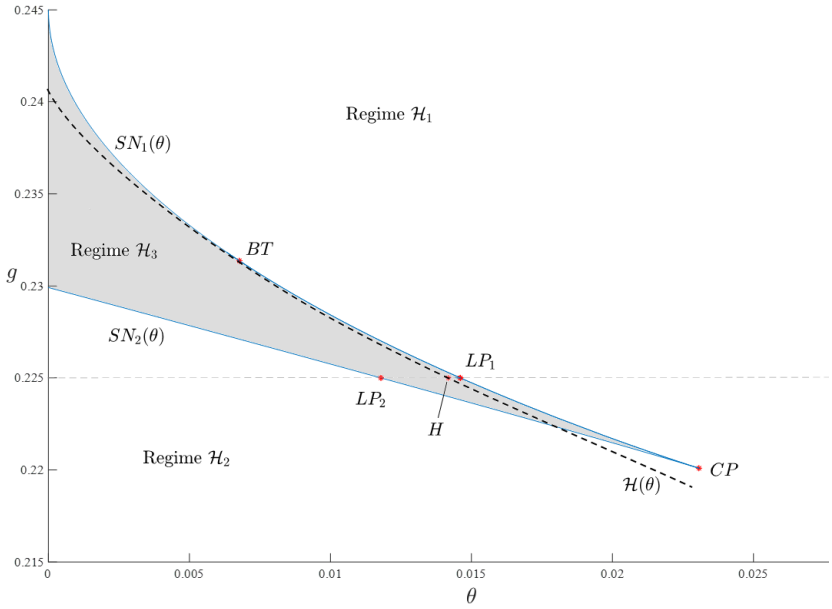


Figure 7: Location of regimes and bifurcation points in the (θ, g) plane

6. Long-run endogenous public debt cycles

According to Corollary 1 and Proposition 2, our model produces a (local) Hopf bifurcation in the neighborhood of the low-growth trap M in Regime $\mathcal{H}_3$. Moreover, the occurrence of a (global) Generalized Hopf bifurcation ensures that this local bifurcation can be supercritical, hence ensuring the presence of stable limit-cycles for nearby parameters' values. In this section we characterize these limit-cycles, and show that they get larger as the deficit target is reduced. At the limit, when steady states M and P are close, the limit-cycle that surrounds M merges with the stable and unstable saddle-paths of P, through a saddle-loop bifurcation generating an homoclinic orbit. The existence of such a homoclinic orbit follows from the occurrence of a Bogdanov-Takens bifurcation. In a two (or more) parameter system, such a bifurcation occurs when a Hopf bifurcation, a saddle-loop bifurcation, and a saddle-node bifurcation coincide in a single point of the parameter space.

At first, we compute the family of limit-cycles that emerge when the deficit target θ is reduced in the vicinity of the Hopf bifurcation. As we have seen, in the benchmark calibration the Hopf bifurcation occurs at $\theta^h \simeq 1.418\%$, and (as the bifurcation is supercritical in the benchmark calibration) stable limit-cycles born for slightly lower values. As the deficit target becomes more stringent, these limit cycles enlarge and generate large fluctuations in public debt and economic growth (Figure 8).

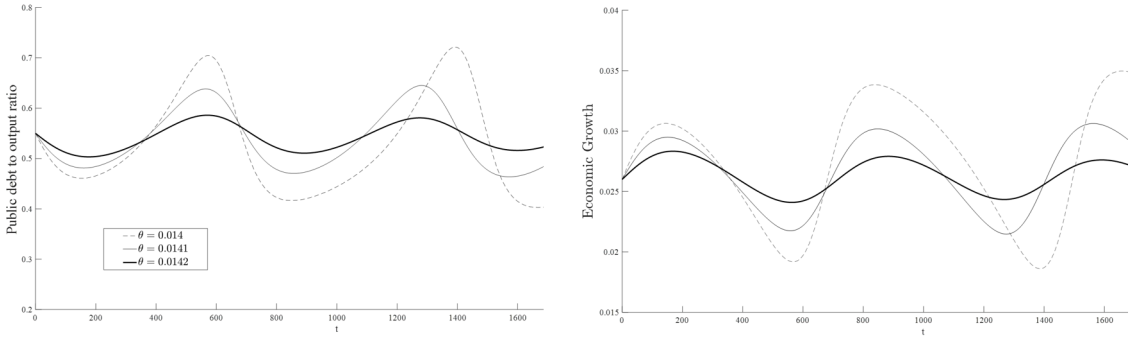
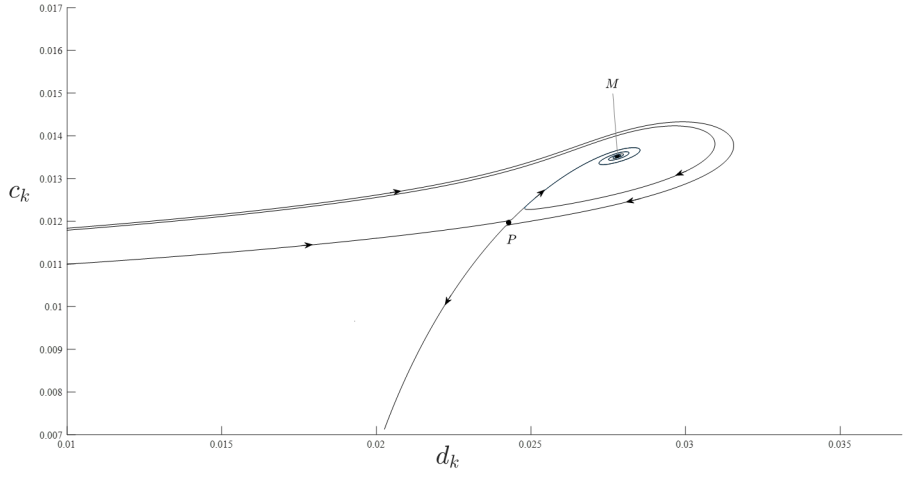


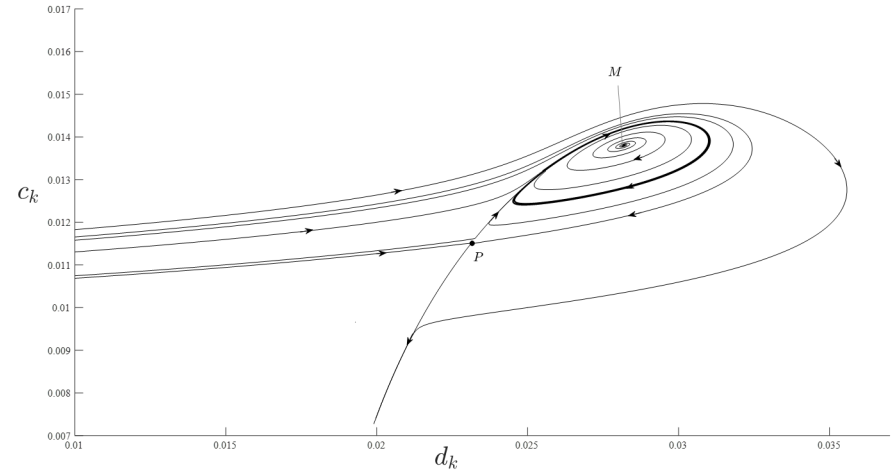
Figure 8: Public debt and growth cycles as a function of θ

Figures 9 describes the dynamics in the neighborhood of points M and P, depending on the deficit target. If $\theta > \theta^h$, the low-growth trap is locally sable (Figure 9a). At $\theta = \theta^h$ the Hopf bifurcation occurs, and stable limit-cycles arise for $\tilde{\theta} < \theta < \theta^h$ (Figure 9b). The existence of stable limit cycles for a range of parameters such that $\tilde{\theta} < \theta < \theta^h$ implies that a small perturbation to a parameter would not eliminate the cyclical dynamics of public debt and growth. The limit-cycle enlarge as θ decreases, until it coincides with the stable and unstable manifolds of the saddle point P, at $\theta = \tilde{\theta} \simeq 1.3885\%$. At this point, there is a saddle-loop bifurcation, which is depicted in Figure 9c. At the bifurcation, the periodic orbit connects P to himself, producing a homoclinic orbit (i.e. the limit cycle degenerates into an orbit homoclinic to the saddle). For lower values of θ, as in Figure 9d, periodic orbits no longer exist and the anti-saddle path of P now escapes point P and moves eventually to points Q or D.

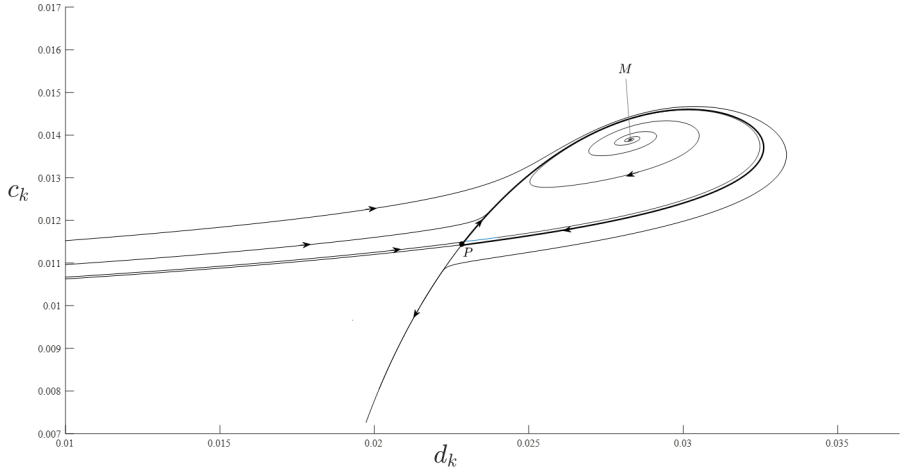
9a: M is stable ($\theta^h < \theta$)



9b: Stable limit cycle around M ($\tilde{\theta} < \theta < \theta^h$)



9c: Saddle-loop bifurcation ($\theta = \tilde{\theta}$)



9d: Limit-cycles vanish ($\theta < \tilde{\theta}$)

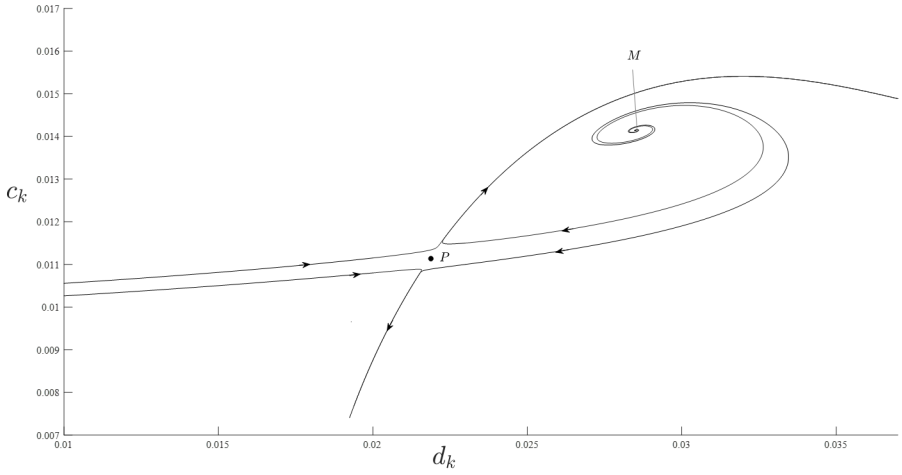


Figure 9: A typology of global dynamics in the neighborhood of the low-growth trap

To prove the existence of the homoclinic orbit, we shall refer to the Bogdanov-Takens (BT) bifurcation. This co-dimension two bifurcation occurs in a dynamic two-parameter system when a critical point has a zero eigenvalue of multiplicity two. The following proposition shows that a generic BT bifurcation arises in our model, when steady states M and P collide at a point such that $d_k^M = d_k^P < \hat{d}_k$.

Proposition 3. *Define the BT singularity as a steady state with two zero eigenvalues but a nonzero Jacobian matrix. There is a critical pair of fiscal instruments (g^{bt}, θ^{bt}) that satisfies this singularity.*

Proof. We prove Proposition 3 for small economic growth ($\gamma \to 0$), as it is the case in the neighborhood of the low-growth trap M (see Appendix B). There is a pair (g^{bt}, θ^{bt}), such that $(\det(\mathbf{J}^M), \mathrm{Tr}(\mathbf{J}^M)) = (0, 0)$, where

$$g^{bt} = \frac{\alpha + \mu - A\left(\frac{\rho}{(1-\alpha)A}\right)^{(1+\psi)/\psi}}{1 - A\left(\frac{\rho}{(1-\alpha)A}\right)^{(1+\psi)/\psi}},$$

$$\theta^{bt} = \frac{(1 + \alpha\psi - g^{bt}(1+\psi))(\alpha + \mu - g^{bt})}{\psi(g^{bt} - \alpha)}.$$

□

The Bogdanov-Takens bifurcation is obtained at point BT in Table 2, at $(g^{bt}, \theta^{bt}) \simeq (0.23, 0.007)$. At this point, as Figure 7 shows, the saddle-node curve $SN_1(\theta)$ is tangent with the Hopf-curve $\mathcal{H}(\theta)$. The mechanism driving the homoclinic orbits is as follows. The point where P and M collide defines the saddle-node bifurcation SN_1, while (if $\hat{c}_k < \bar{c}_k$ and $d_k^M < \hat{d}_k$) M undergoes a Hopf bifurcation generating a periodic orbit in Regime $\mathcal{H}_3$, as we have seen. The BT bifurcation is then obtained as the collision of the saddle-node and the Hopf bifurcations. As (g, θ) gets closer to (g^{bt}, θ^{bt}), the non-saddle point M converges towards P, so that the periodic orbit collides with the manifolds of the saddle equilibrium and degenerates into a homoclinic orbit.

The presence of a BT bifurcation has important implications. Indeed, for parameter values close to (g^{bt}, θ^{bt}), the economy can experiment large fluctuations in economic growth and public debt, or slowly converge towards the steady state P along the homoclinic orbit (a cycle with virtually infinite period). For nearby parameter values, the economy escapes point P and converges towards the stable steady state Q.

Figure 10 exemplifies a spectacular aspect of the perils of fiscal rules: a very small change in the target may produce radical shifts in long-run dynamics. If, e.g., θ passes from 1.3888% to 1.3889%, the paths of economic growth and public debt are similar until $t = 3000$, but their dynamics suddenly change after this time. In the first case, the economy gradually converges towards Q, while in the second case it is characterized by periodic recessions with sharp increases in public debt, or by a homoclinic loop

that reaches the saddle steady state P (this saddle-loop bifurcation defines the borderline between the trajectories that converge to Q and those that join P). Beyond this extreme sensitivity to changes in parameters, the asymmetric cyclical dynamics that our model produces—with long periods of nearby-stationary growth and sudden short-living recessions—are consistent with observed stylized facts.

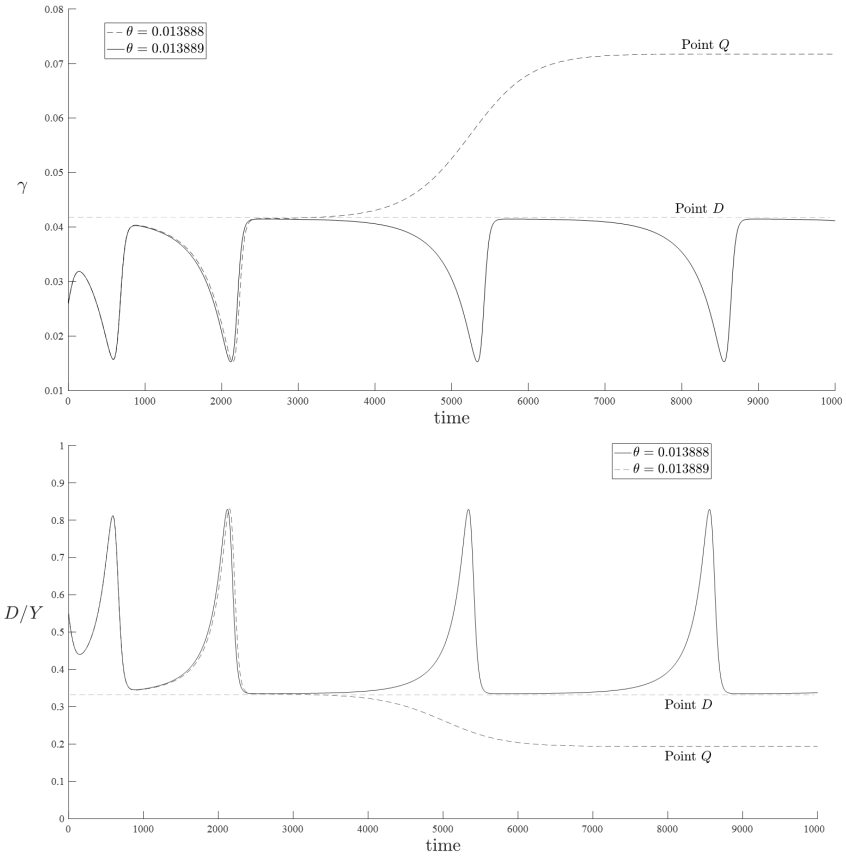


Figure 10: Path-dependance to small changes in the deficit target

The fact that a public deficit target gives birth to large fluctuations and possible local and global indeterminacy might plead in favor of the adoption of balanced-budget rules. However, in our model, the tighter the deficit target, the larger oscillations of public debt and economic growth, and the higher the area of indeterminacy, as we have seen. Furthermore, as shown in the following section, the BBR is also likely to produce indeterminacy and large fluctuations.

7. The BBR special case

In this section, we study the case $\theta = 0$, $\forall t$, which characterizes the balanced-budget rule (BBR) associated with no deficit (but possibly positive inherited public debt, i.e.

$d_{k0} \geq 0$). In this case, Eq. (18) leads to

$$\gamma(c_k^*, d_k^*) d_k^* = 0,$$

where economic growth is $\gamma^* := (1-g)y_k(c_k^*, d_k^*) - c_k^*$. Disregarding negative public debt, this condition implies that either $\gamma = 0$ and $d_k > 0$, or $\gamma > 0$ and $d_k = 0$.

Compared with the previous section, the $\dot{d}_k = 0$ locus now becomes a degenerate hump-shaped curve which peaks at $(0, \bar{c}_k)$, as depicted by the dashed lines in Figure 3. The first part of this curve corresponds to the $d_k = 0$ locus associated to non-zero growth; and the second part corresponds to $\gamma(c_k, d_k) = 0$, describing a decreasing relation between d_k and c_k. This means that the maximum of $\dot{d}_k = 0$ is now located at $\hat{d}_k = 0$.

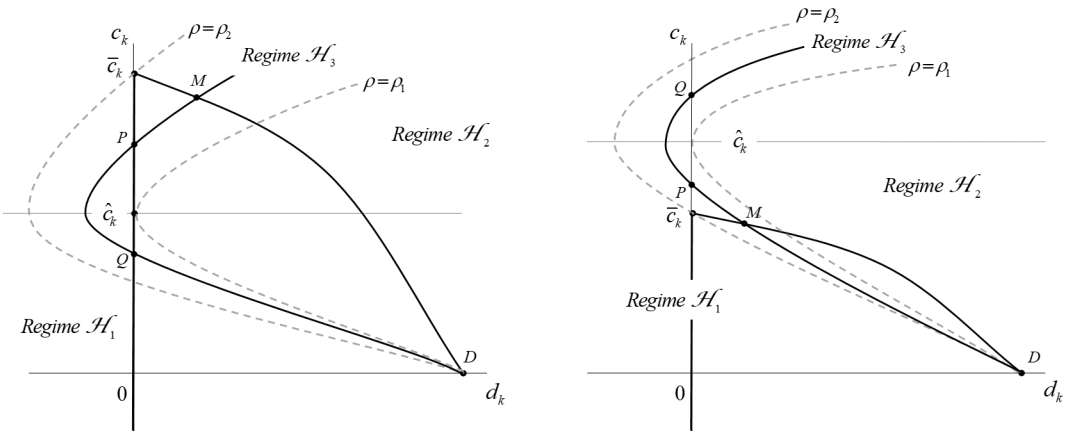
Regime $\mathcal{L}$ is qualitatively unchanged, except that point M now qualifies a zero-growth trap (see Appendix C for the analytical proof). In Regime $\mathcal{H}$, under the BBR the parameter sets that give rise to the saddle-node bifurcations can be found analytically, as establishes the following theorem.

Theorem 3. *If $\theta = 0$ and $g > \alpha$, there are two saddle-node bifurcations at ρ_1 and ρ_2 (with $0 < \rho_1 < \rho_2$), where*

$$\rho_1 := (1+\psi)\left[\frac{\bar{d}[(g-\alpha)A]^{1/\psi}}{\psi}\right]^{\psi/(1+\psi)},$$
$$\rho_2 := (g-\alpha)A\left(\frac{\bar{d}}{\bar{c}_k}\right)^{\psi} + \bar{c}_k.$$

Proof. See Appendix C.

The critical values ρ_1 and ρ_2 correspond respectively to points $(0, \bar{c}_k)$ (bifurcation point SN_1) and to $(0, \hat{c}_k)$ (bifurcation point SN_2) in Figure 11, such that regime $\mathcal{H}$ can be subdivided between three cases. If $\rho > \rho_2$ (Regime $\mathcal{H}_1$), there is one positive growth solution (corresponding to point Q in the previous sections), and one negative-growth solution. If $\rho < \rho_1$ (Regime $\mathcal{H}_2$), there is one no-growth solution (corresponding to point M in the previous sections). If $\rho_1 < \rho < \rho_2$ (Regime $\mathcal{H}_3$), there are two positive-growth solutions (points P and Q) and the no-growth solution (point M) if $\bar{c}_k > \hat{c}_k$ (Figure 11a). If $\bar{c}_k < \hat{c}_k$, the two positive-growth solutions turn into negative-growth solutions (Figure 11b). Hence, with a BBR, Regime $\mathcal{H}_3$ can appear in the case $\bar{c}_k < \hat{c}_k$, in contrast with the preceding sections, but points P and Q are now associated to negative-growth BGPs. Even if these solutions are not economically attractive, they play an important role in the dynamics around point M, as we will see. In all configurations, there is also the degenerate solution (D). According to our calibration exercise, ρ_1 and ρ_2 take reasonable values, namely $\rho_1 = 0.0213$ and $\rho_2 = 0.0378$.



-a- $\bar{c}_k > \hat{c}_k$ -b- $\bar{c}_k < \hat{c}_k$

Figure 11: Topological behaviour of Regime $\mathcal{H}$ ($\theta = 0$)

The following proposition states the topological properties of the different equilibria.

Proposition 4.

i. *If $c_k^M > \hat{c}_k$, P is locally determinate (saddle-path stable), Q is locally indeterminate, and the no-growth trap M is unstable.*

ii. *If $c_k^M < \hat{c}_k$, the no-growth trap M can become stable, with the presence of a Hopf bifurcation, and, since economic growth is negative, Q is determinate and P becomes indeterminate.*

Proof: See Appendix D.

Under the BBR, our model is similar to SGU, but in an endogenous growth context.[21] In a neoclassical exogenous growth model, SGU show that aggregate instability (defined as the local indeterminacy of the unique perfect-foresight steady-state) occurs when taxes are levied on labor income, irrespective of the level of public spending. In our endogenous growth setup, their analysis needs to be amended on two grounds. First, in the case with low public spending ($g < \alpha$, Regime $\mathcal{L}$), the perfect-foresight BGP is unique and well-determined, such that there is no aggregate instability, as in section 3. Second, in the case with high public spending ($g > \alpha$, Regime $\mathcal{H}$), there are multiple steady states, of which one is locally indeterminate. Hence, in this regime, our aggregate-instability result covers a broader class of mechanisms than in SGU, because it relies both on local *and global* indeterminacy. Furthermore, cyclical dynamics can appear around the no-growth trap, in the vicinity of the Hopf bifurcation.

[21] Contrary to SGU who consider a BBR with no debt ($D_t = 0, \forall t$), our BBR can be associated to a (strictly) positive public debt level. Indeed, for an economy starting with an initial public debt $D_0 > 0$, the BBR implies that public debt is constant over time: $\dot{D}_t = 0 \Leftrightarrow D_t = D_0$, $\forall t$. This is an important point, because we exhibit complex dynamics of the public debt *ratio* ($d_{kt} = D_0/K_t > 0$) even if the public debt *level* is constant.

Under the BBR, the Hopf-bifurcation parameter can no longer be the deficit target (as $\theta = 0$), and another bifurcation parameter must be found. The following proposition establishes the existence of a Hopf bifurcation as a function of the public spending ratio.

Proposition 5. *In the BBR case, a Hopf bifurcation occurs if* $c_k^M < \hat{c}_k$ *at the unique value* g^h*, such that*

$$g^h = \frac{A_0(1-(1-\alpha)\psi)+\psi(\alpha+\mu)}{A_0+\psi},$$

where $A_0 := A(\rho/(1-\alpha)A)^{(1+\psi)/\psi}$.

Proof: The proof directly results from corollary 1. By Eq. (30), if $\theta = 0$ the Hopf bifurcation is such that

$$(1-g^h)d_k^M = \frac{CC^M}{yd^M}. \tag{31}$$

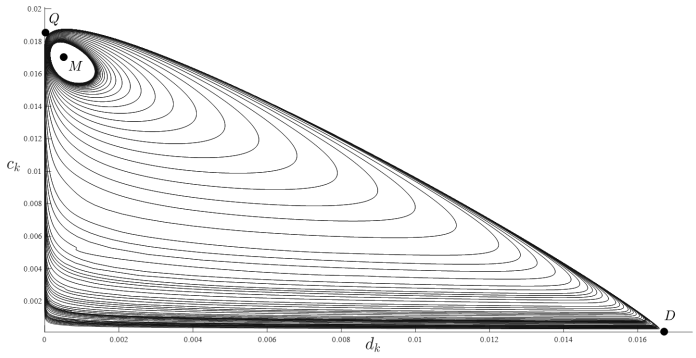
As $yd^M < 0$, Eq. (31) can be verified only if $CC^M < 0$, namely if $c_k^M < \hat{c}_k$. Indeed, in the BBR case, the Hopf bifurcation cannot arise if $c_k^M > \hat{c}_k$, because $\hat{d}_k = 0 \Rightarrow DD^M > 0$, $\forall d_k \geq 0$. Since $DD^M > 0$, a necessary condition for $\mathrm{Tr}(\mathbf{J}^M)$ to change sign is $CC^M < 0$. From Appendix B, we have

$$CC^M = \frac{\rho(1+\psi)}{1-\alpha}(g^m - g), \text{ where } g^m := \frac{1+\alpha\psi}{1+\psi} > \alpha,$$

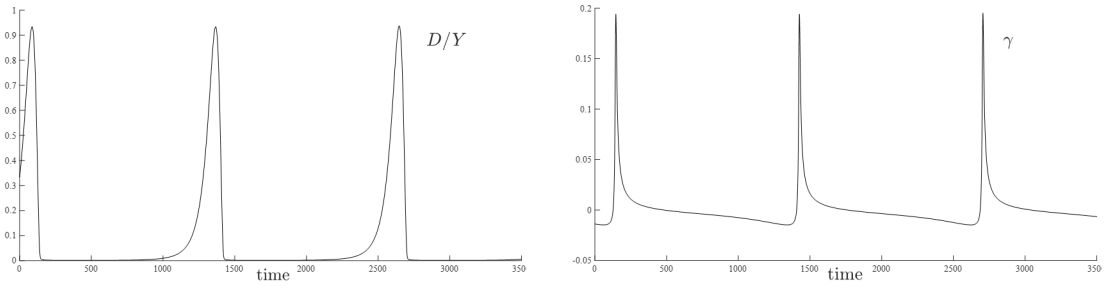
hence the public spending ratio must be higher than g^m.[22] By inspection of (23) and (27) follows Proposition 3. □

The case $c_k^M < \hat{c}_k$ occurs when human capital externalities are high enough (in our calibration exercise below, we consider $\beta = 0.885$ with otherwise the baseline calibration). The public spending ratio that corresponds to the Hopf bifurcation is $g^h = 0.2345$ (the associated Lyapunov coefficient is equal to 587, defining a sub-critical Hopf bifurcation). For all reasonable configurations of parameters, no Generalized Hopf bifurcation has been found, excluding the existence of stable limit-cycles under the BBR. Therefore, there is a periodic orbit around the no-growth trap M that can be reached by a jump in the consumption ratio. As the cycle is unstable, inside the orbit all trajectories converge to the low-growth trap M, while outside the orbit the economy eventually goes to the catastrophic equilibrium D (if we exclude the negative-growth steady state Q). This generates both local (in the vicinity of M) and global indeterminacy: since the initial consumption ratio is a free jumpable variable, starting with a predetermined public debt ratio $d_{k0} < \bar{d}/(1-\alpha)$, the economy can converge towards the no-growth trap M, the catastrophic equilibrium D, or a periodic orbit around M, depending on households' views on the future.

[22] To ensure that $g^h \geq g^m$, we assume that $\alpha(1+\psi A_0) \leq \mu(1+\psi)$.



12a: The family of periodic orbits as g increases



12b: Behavior of the debt ratio and economic growth for $g = 0.2348$

Figure 12: Dynamics under the BBR

The magnitude of the periodic orbit that encloses point M enlarges as g increases. The economy then fluctuates between the harrodian BGP (D) and the negative-growth steady state Q, as in Figure 12a. The largest orbit (obtained for $\bar{g} \simeq 0.2349$) is the heteroclinic connection between D and Q (Q is saddle-point stable with negative growth), defining the envelope of an elliptic sector, which contains an infinite number of stable orbits that converge to point M. Inside the periodic orbit, self-fulfilling prophecies can generate a great variety of transitory endogenous cycles along which the economy experiments large fluctuations before being trapped in secular stagnation. For a high public spending ratio, i.e. close to $\bar{g}$, the periodic orbit passes "near" points D and Q, such that the economy is subject to sudden "heart attacks" of public debt (see Figure 12b), as emphasized by Rogoff (2015).

Compared with the the main analysis with a positive deficit rule, under the BBR the economy is equally subject to large oscillations in public debt and economic growth. However, two major differences emerge. First, the closed orbit that turns around point M is unstable and contains all stable trajectories that converge towards M. Consequently, the cycle can be reached only by an adequate jump in the initial consumption ratio, following households' expectations. Do households expect that the steady state is M (or D), the economy leaves the cyclical orbit to jump on a path that joins this point.

This "knife-edge" property of expectations was avoided in the preceding section, because the limit-cycle was stable.[23] Second, in the vicinity of the heteroclinic orbit, stationary growth is negative at point Q, while it was positive previously at point P in the vicinity of the saddle-loop bifurcation. Then, beyond large public debt fluctuations, the dynamics of economic growth radically differ. With a positive deficit target, the economy experiences long periods of positive growth, interrupted by brief slowdown episodes (see Figure 10). In contrast, the BBR leads the economy into a secular recession, interspersed with periods of exuberant growth as in Figure 12b.

8. Concluding remarks

In this paper we developed an original framework for assessing the role of fiscal rules for aggregate fluctuations. Our model illustrates the *perils of fiscal rules*. First, fiscal rules are destabilizing because multiple equilibria and complex dynamics can emerge even when the deficit target is low (including the balanced-budget rule). Second, the economy can experiment (possibly large) public debt and growth fluctuations both in the short and the long run without the need of exogenous shocks. Indeed, the interaction between households' optimal saving behaviour and the government's budget constraint gives birth to Hopf and Bogdanov-Takens bifurcations that ensure the stability of public debt cycles. The calibration of the model—consistent with OECD or US data—reveals the realism of our findings: the various bifurcations occur for reasonable values of parameters, and the cycles that our model are consistent with observed stylized facts.

Finally, our model opens the door for a limit-cycles-based theory of public debt fluctuations. From a methodological perspective, a fruitful extension would be to examine a stochastic version of our model, as in Beaudry et al. (2016). From a policy perspective, another extension could analyze the consequences of fiscal policy in terms of aggregate fluctuations. On the one hand, some of the conclusions of the existing literature may have to be revisited in the presence of deficit and debt. Reassessing the role of progressive taxes, endogenous public spending, or alternative specifications of preferences as drivers of indeterminacy are some handful examples. On the other hand, the complex effects triggered by our simple deficit rule make the case for exploring alternative fiscal rules, all the more given their increased popularity since the recent crisis (Combes et al., 2017; Menuet et al., 2018). These possible directions are left for future research.

[23] Therefore, if $\theta > 0$ cycles are not associated to a particular expectation, but are consistent with an infinity of expectations. There is no knife-edge, and the cycles emerge even if consumption was a backward variable. Importantly, the jump of consumption depends on the the tax rate, therefore on wages: if the adjustment of wages cannot take place—for example, if there is a resistance to increases in wage taxes—then consumption is no longer a jump variable (or the adjustment will be sluggish).

References

Abad, N., Seegmuller, N., Venditti, A., 2017. Nonseparable preferences do not rule out aggregate instability under balanced-budget rules: A note. Macroeconomic Dynamics 21, 259–277.

Abbas, S.A., Belhocine, N., El-Ganainy, A., Horton, M., 2011. Historical patterns and dynamics of public debt—evidence from a new database. IMF Economic Review 59, 717–742.

Alesina, A., Bayoumi, T., 1996. The costs and benefits of fiscal rules: evidence from US states. NBER Working Papers No. w5614 .

Alt, J., Lowry, R., 1994. Divided government, fiscal institutions, and deficits: Evidence from the States. American Political Science Review 88.

Barro, R., 1979. On the determination of the public debt. Journal of Political Economy 87, 940–971.

Basu, S., Fernald, J.G., 1997. Returns to scale in US production: Estimates and implications. Journal of Political Economy 105, 249–283.

Beaudry, P., Galizia, D., Portier, F., 2016. Putting the cycle back into business cycle analysis. NBER Working Papers No. 22825 .

Benhabib, J., Farmer, R., 1994. Indeterminacy and increasing returns. Journal of Economic Theory 63, 19–41.

Benhabib, J., Farmer, R., 1996. Indeterminacy and sector specific externalities. Journal of Monetary Economics 37, 397–419.

Benhabib, J., Farmer, R., 1999. Indeterminacy and sunspots in macroeconomics, in: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics. North Holland Publishing Co., Amsterdam, pp. 387–448.

Benhabib, J., Meng, Q., Nishimura, K., 2000. Indeterminacy under constant returns to scale in multisector economies. Econometrica 68, 1541–1548.

Benhabib, J., Nishimura, K., 1998. Indeterminacy and sunspots with constant returns. Journal of Economic Theory 81, 58–96.

Benhabib, J., Nishimura, K., Shigoka, T., 2008. Bifurcation and sunspots in the continuous time equilibrium model with capacity utilization. International Journal of Economic Theory 4, 337–355.

Benhabib, J., Schmitt-Grohe, S., Uribe, M., 2001. The perils of taylor rules. Journal of Economic Theory 96, 40–69.

Bohn, H., 1998. The behavior of u. s. public debt and deficits. The Quarterly Journal of Economics 113, 949–963.

Boucekkine, R., Nishimura, K., Venditti, A., 2015. Introduction to financial frictions and debt constraints. Journal of Mathematical Economics 61, 271–275.

Burnside, C., 1996. Production function regressions, returns to scale, and externalities. Journal of Monetary Economics 37, 177–201.

Chang, J.J., Guo, J.T., Shieh, J.Y., Wang, W.N., 2015. Sectoral composition of government spending and macroeconomic (in)stability. Economic Inquiry 53, 23–33.

Chen, B.L., 2006. Public capital, endogenous growth, and endogenous fluctuations. Journal of Macroeconomics 28, 768–774.

Cheron, A., Nishimura, K., Nourry, C., Seegmuller, T., Venditti, A., 2019. Growth and public debt: What are the relevant tradeoffs? Journal of Money, Credit and Banking 51, 655–682.

Christiano, L., Harrison, S., 1999. Chaos, sunspots and automatic stabilizers. Journal of Monetary Economics 44, 3–31.

Combes, J., Debrun, X., Minea, A., Tapsoba, R., 2017. Inflation Targeting, Fiscal Rules and the Policy Mix: Crosseffects and Interactions. The Economic Journal forthcoming, https://doi.org/10.1111/ecoj.12538.

Farmer, R., Guo, J.T., 1994. Real business cycles and the animal spirits hypothesis. Journal of Economic

Theory 63, 42–72.
Fatás, A., Mihov, I., 2006. The macroeconomic effects of fiscal rules in the US States. Journal of Public Economics 90, 101–117.
Ghilardi, M., Rossi, R., 2014. Aggregate stability and balanced-budget rules. Journal of Money, Credit and Banking 46, 1787–1809.
Giannitsarou, C., 2007. Balanced budget rules and aggregate instability: The role of consumption taxes. Economic Journal 117, 1423–1435.
Guo, J.T., Harrison, S., 2008. Useful government spending and macroeconomic (in)stability under balanced-budget rules. Journal of Public Economic Theory 10, 383–397.
Guo, J.T., Lansing, K., 2009. Capital-labor substitution and equilibrium indeterminacy. Journal of Economic Dynamics and Control 33, 1991–2000.
IMF, 2018. How to calibrate fiscal rules: A primer. Note 8 - Fiscal Affairs Department .
Kopell, N., Howard, L.N., 1975. Bifurcations and trajectories joining critical points. Advances in Mathematics 18, 306–58.
Kuznetsov, Y.A., 1998. Elements of Applied Bifurcation Theory, Second Edition. Springer, NY.
Linnemann, L., 2008. Balanced budget rules and macroeconomic stability with non-separable utility. Journal of Macroeconomics 30, 119–215.
Lucas, R.E., 1988. On the mechanics of economic development. Journal of Monetary Economics 22, 3–42.
Mattana, P., Nishimura, K., Shigoka, T., 2009. Homoclinic bifurcation and global indeterminacy of equilibrium in a twosector endogenous growth model. International Journal of Economic Theory 5, 25–47.
Menuet, M., Minea, A., Villieu, P., 2018. Deficit, Monetization, and Economic Growth: A Case for Multiplicity and Indeterminacy. Economic Theory 65, 819–853.
Minea, A., Villieu, P., 2012. Persistent Deficit, Growth, and Indeterminacy. Macroeconomic Dynamics 16, 267–283.
Moretti, E., 2004. Human capital externalities in cities. Handbook of regional and urban economics 4, 2243–2291.
Nishimura, K., Nourry, C., Seegmuller, T., Venditti, A., 2013. Destabilizing balanced-budget consumption taxes in multi-sector economies. International Journal of Economic Theory 9, 113–130.
Nishimura, K., Seegmuller, T., Venditti, A., 2015. Fiscal Policy, Debt Constraint and Expectations-Driven Volatility. Journal of Mathematical Economics 61, 305–316.
Nourry, C., Seegmuller, T., Venditti, A., 2013. Aggregate instability under balanced-budget consumption taxes: A re-examination. Journal of Economic Theory 148, 1977–2006.
Palivos, T., Yip, C., Zhang, J., 2003. Transitional dynamics and indeterminacy of equilibria in an endogenous growth model with a public input. Review of Development Economics 7, 86–98.
Park, H., Philippopoulos, A., 2004. Indeterminacy and fiscal policies in a growing economy. Journal of Economic Dynamics and Control 28, 645–660.
Poghosyan, T., 2015. How do public debt cycles interact with financial cycles? IMF Working Paper 15/248 .
Rauch, J.E., 1993. Productivity gains from geographic concentration of human capital: evidence from the cities. Journal of Urban Economics 34, 380–400.
Reinhart, C., Rogoff, K., 2010. Growth in a Time of Debt. American Economic Review 100, 573–578.
Reinhart, C.M., Rogoff, K.S., 2011. From financial crash to debt crisis. American Economic Review 201, 1676–1706.
Rogoff, K., 2015. Debt supercycle, not secular stagnation. Voxeu column .
Romer, P., 1986. Increasing returns and long-run growth. Journal of political economy 94, 1002–1037.
Schaechter, A., Kinda, T., Budina, N., Weber, A., 2012. Fiscal rules in response to the crisis–toward the

"next-generation" rules. A new dataset. IMF working paper WP/12/187.

Schmitt-Grohé, S., Uribe, M., 1997. Balanced-budget rules, distortionary taxes and aggregate instability. Journal of Political Economy 105, 976–1000.

Sniekers, F., 2018. Persistence and volatility of Beveridge Cycles. International Economic Review 59, 665–698.

Turnovsky, S., 2000. Fiscal policy, elastic labor supply, and endogenous growth. Journal of Monetary Economics 45, 185–210.

Appendix A. Proof of Theorem 2

We study the local stability of steady-states by inspecting the slope of $\dot{c}_k = 0$ (denoted by s_c^i) and $\dot{d}_k = 0$ (denoted by s_d^i) in the neighbourhood of each BGP i.

First, using the Implicit Function Theorem, we compute $s_c^i = -CD^i/CC^i$ and $s_d^i = -DD^i/DC^i$.

Second, the trace and the determinant of the jacobian matrix are $\text{Tr}(\mathbf{J}^i) = CC^i + DD^i$ and $\det(\mathbf{J}^i) = CC^i DD^i - CD^i DC^i = CC^i DC^i(s_c^i - s_d^i)$.

Third, from Eq. (23)-(26), as θ is small enough, we have (i) $DC^i > 0$; (ii) $CC^i > 0$ in regime $\mathcal{L}$; (iii) $CD^i > 0$ in regime $\mathcal{L}$ and $CD^i < 0$ in regime $\mathcal{H}$.

Hence, we deduce the following results.

- **In regime $\mathcal{L}$**, we have $s_d^M < 0$, with $|s_d^M| > |s_c^M|$, as shown in Figure 3a. Thus, it follows that $DD^M > 0$, i.e. $\det(\mathbf{J}^M) > 0$, and $\text{Tr}(\mathbf{J}^M) > 0$, hence point M is unstable. Regarding point P, there are two possible cases:

  (a) $DD^P < 0$ (as illustrated in Figure 3a). In this case, $\det(\mathbf{J}^P) < 0$, i.e. P is a saddle point.

  (b) $DD^P > 0$. In this case, we have $s_d^P < 0$, with $|s_d^P| < |s_c^P|$, thus: $\det(\mathbf{J}^P) < 0$, and P is still saddle.

- **In regime $\mathcal{H}$**, we can divide the (c_k, d_k)-plan in four distinct areas, as depicted in Figure A1:
  - *north-east* (NE): $s_c^i > 0$, and $s_d^i < 0 \Rightarrow \det(\mathbf{J}^i) > 0$, and $\text{Tr}(\mathbf{J}^i) > 0$.
  - *south-east* (SE): $s_c^i < 0$, $s_d^i < 0$, and $|s_d^i| > |s_c^i| \Rightarrow CC^i < 0$, and $\det(\mathbf{J}^i) < 0$.
  - *south-west* (SW): $s_c^i < 0$ and $s_d^i > 0 \Rightarrow \det(\mathbf{J}^i) > 0$ and $\text{Tr}(\mathbf{J}^i) < 0$.
  - *north-west* (NW): $s_c^i > 0$, $s_d^i > 0$, and there are two configurations: (i) if $|s_d^i| > |s_c^i| \Rightarrow CC^i > 0$, and $\det(\mathbf{J}^i) < 0$; (ii) if $|s_d^i| < |s_c^i| \Rightarrow \det(\mathbf{J}^i) > 0$ and $\text{Tr}(\mathbf{J}^i)$ can be positive or negative.

  As $D \in SE$, D is saddle-path stable. If P and Q exist, as $P \in NW$ with $|s_d^P| > |s_c^P|$ and $Q \in SW$, it follows that P is saddle-path stable and Q is locally indeterminate (stable). Regarding point M, two situations can arise: if $M \in NE$, M is unstable, while if $M \in NW$ with $|s_d^M| < |s_c^M|$, a Hopf bifurcation can occur when $CC^M + DD^M = 0$. Corollary 1 comes directly from Eqs. (23) and (25).

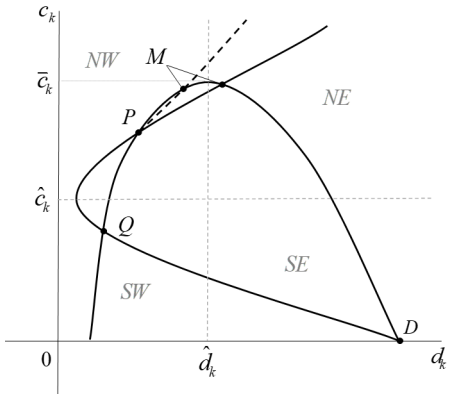


Figure A1: Local stability ($\hat{c}_k < \bar{c}_k$)

Appendix B. Bogdanov-Takens bifurcation and homoclinic orbits

We prove the occurrence of a Bogdanov-Takens (BT) bifurcation and homoclinic orbits in the neighborhood of equilibrium M (formally, $\gamma \to 0$) using a two-step proof. In the first step, we will show that there is a critical pair of fiscal instruments that characterizes the BT singularity. In the second step, we demonstrate the existence of a homoclinic orbit around point M, using the argument that points P and M collide at the BT bifurcation.

Step 1: Preliminary.
First of all, we compute the coordinates of point M when $\gamma^M \to 0$. By Eq. (12), when $\gamma^M \to 0$, it follows that $y_k^M = \rho/(1-\alpha)$. From Theorem 1, we deduce that $c_k^M = (1-g)\rho/(1-\alpha)$, and $d_k^M = \frac{1}{(1-\alpha)}\left\{\bar{d} - c_k^M \left(\frac{\rho}{(1-\alpha)A}\right)^{1/\psi}\right\}$. From Eq. (27), we compute: $yc^M = -\psi/(1-g)$, $yd^M = -\frac{(1-\alpha)\psi}{(1-g)(\rho/(1-\alpha)A)^{1/\psi}}$.

Second, when $\gamma^M \to 0$, we obtain, using Eqs. (28) and (29)

$$\mathrm{Tr}(\mathbf{J}^M) = \theta y d^M + c_k^M[1 + (g-\alpha) y c^M] - (1-g) y d^M d_k^M, \tag{B.1}$$

$$\det(\mathbf{J}^M) = -c_k^M y d^M[(1-\alpha) d_k^M - \theta]. \tag{B.2}$$

We need to find the values of parameters g and θ, such that $\mathrm{Tr}(\mathbf{J}^M) = \det(\mathbf{J}^M) = 0$. On the one hand, using (B.2), it follows that $\det(\mathbf{J}^M) = 0 \Leftrightarrow \bar{d} - (1-g)A\left(\frac{\rho}{(1-\alpha)A}\right)^{1+1/\psi} = \theta$. As $\bar{d} = \alpha + \mu + \theta - g$, we conclude that $\det(\mathbf{J}^M) = 0 \Leftrightarrow g = g^{bt}$, where

$$g^{bt} := \frac{\alpha + \mu - A\left(\frac{\rho}{(1-\alpha)A}\right)^{1+1/\psi}}{1 - A\left(\frac{\rho}{(1-\alpha)A}\right)^{1+1/\psi}}.$$

On the other hand, at $g = g^{bt}$, we have, using (B.1), $\text{Tr}(\mathbf{J}^M) = 0 \Leftrightarrow \theta = \theta^{bt}$, where

$$\theta^{bt} := \frac{(1 + \alpha\psi - g^{bt}(1+\psi))(\alpha + \mu - g^{bt})}{\psi(g^{bt} - \alpha)}.$$

As ρ is small enough, we ensure that $g^{bt} > 0$ and $\theta^{bt} > 0$ under the mild condition $1 - \alpha > \mu(1 + \psi)$ (in our baseline calibration, we find $g^{bt} \approx 0.23$ and $\theta^{bt} \approx 0.007$). Consequently, at $(g, \theta) = (g^{bt}, \theta^{bt})$, it follows that $\text{Tr}(\mathbf{J}^M) = (\mathbf{J}^M) = 0$; hence, the Jacobian matrix $\mathbf{J}^M$ has a double zero eigenvalue.

Step 2: Homoclinic orbit.
We prove the occurrence of the BT bifurcation by applying a theorem that allows us to transform our system into a simpler, topologically equivalent planar system of differential equations with well-known bifurcation diagram. We conclude using a Lemma that ensures the occurrence of homoclinic orbits.

Theorem (Kuznetsov, 1998, Theorem 8.4, p. 321) Suppose that a planar system

$$\dot{x} = f(x, \alpha),\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}^2,$$

with smooth f, has at $\alpha = 0$, the equilibrium $x = 0$ with a double zero eigenvalue:

$$\lambda_{1,2} = 0.$$

Assume the following generic conditions are satisfied:
(BT.0) the jacobian matrix $A(0) = f_x(0, 0) \neq 0$;
(BT.1) $a_{20}(0) + b_{11}(0) \neq 0$;
(BT.2) $b_{20}(0) \neq 0$;
(BT.3) the map

$$(x, \alpha) \mapsto \left(f(x, \alpha), \text{tr}\left(\frac{\partial f(x, \alpha)}{\partial x}\right), \det\left(\frac{\partial f(x, \alpha)}{\partial x}\right)\right)$$

is regular at point $(x, \alpha) = (0, 0)$.

Then there exist smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparametrization, and smooth invertible parameter changes, which together reduce the system to

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2 + O(||\eta||^3), \end{cases}$$

where $s := \text{sgn}[b_{20}(a_{20}(0) + b_{11}(0))] = \pm 1$. □

Let $\alpha := (g - g^{bt}, \theta - \theta^{bt})$ and $x := (c_k - c_k^M, d_k - d_k^M)$. Clearly, at $\alpha = 0$, the equilibrium $x = 0$ has a double zero eigenvalue. We need to ensure conditions (BT.0)-(BT.3).

Condition (BT.0). Using Eq. (23), at point M, we have

$$CC^M = c_k^M[1 + (g - \alpha)yc^M] = \frac{\rho}{1-\alpha}\left(1 + \psi\alpha - g(1+\psi)\right),$$

hence; $CC^M|_{g=g^{bt}} = \frac{\rho}{1-\alpha}\left(1 + \psi\alpha - g^{bt}(1+\psi)\right) \neq 0$. Consequently, the jacobian matrix $\mathbf{J}^M$ evaluated at $(g, \theta) = (g^{bt}, \theta^{bt})$ is non-zero.

Conditions (BT.1) and (BT.2). Numerically, we compute the generic BT parameters, and show that $a_{20}(0) + b_{11}(0) \neq 0$ and $b_{20}(0) \neq 0$ for a large constellation of parameters. Using our baseline calibration, we find $a_{20} = -0.0232$ and $b_{11} = 4.88$.

Conditions (BT.3). Let $\phi : (x, \alpha) \mapsto \left(f(x, \alpha), \mathrm{Tr}(\mathbf{J}^M), \det(\mathbf{J}^M)\right)$. Numerically, we ensure that $\det(\phi(0, 0)) \neq 0$ for a large space of parameters.

Finally, according to the above-mentioned theorem, our system is topological equivalent to the following two-differential-equations system in the neighborhood of equilibrium M

$$\begin{cases} \dot{\eta}_1 = \eta_2, \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2, \end{cases} \tag{B.3}$$

where β_1 and β_2 are combinations of parameters. The coefficient on $\eta_1\eta_2$ is -1, since the periodic orbit around point M is stable (the first Lyapunov coefficient is negative in our baseline calibration). Thus, the bifurcation diagram is usually depicted in the (β_1, β_2)-plane (Kuznetsov, 1998, section 8.4.2), where the origin corresponds to the BT bifurcation.

Against this background, the existence of homoclinic orbits directly derives from the properties of the bifurcation diagram and the following lemma.

Lemma (Kuznetsov, 1998, Lemma 8.7) There is a unique smooth curve $\mathcal{P}$ corresponding to a saddle homoclinic bifurcation in system (B.3) that originates at $\beta = 0$ and has the following local representation

$$\mathcal{P} = \left\{(\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2),\ \beta_2 < 0\right\}.$$

□

Consequently, in the neighborhood of equilibrium M, this lemma establishes that there is a combination of parameters such that there exists at least one bifurcation curve originating at $\beta = 0$ (i.e. $(g, \theta) = (g^{bt}, \theta^{bt})$), along which system (B.3) has a saddle homoclinic bifurcation. To sum up, if (c_k, d_k) is close to (c_k^M, d_k^M), and (g, θ) is close to the BT bifurcation (g^{bt}, θ^{bt}) the economy can experiment an homoclinic orbit (along the

curve $\mathcal{P}$).

Appendix C. Proof of Theorem 3

Using the Keynes-Ramsey relationship (17) with $\theta = 0$, positive growth solutions are given by the implicit function $\Phi(c_k) = 0$, where

$$\Phi(c_k) := \bar{d} - c_k \left(\frac{c_k - \rho}{(\alpha - g)A} \right)^{1/\psi}. \tag{C.1}$$

(a) *Case* $g < \alpha$. Clearly, $\Phi \in C^1((\rho, +\infty))$ and Φ is a decreasing function, hence a decreasing relationship between c_k and d_k, as depicted in Figure 3a. As $\Phi(\rho) = \bar{d} > 0$, and $\lim_{c_k \to +\infty} \Phi(c_k) = -\infty$, according to the Intermediate Value Theorem, there is a unique point $\check{c}_k \in (\rho, +\infty)$, such that $\Phi(\check{c}_k) = 0$. The point $P = (0, \check{c}_k)$ characterizes a steady-state if and only if $c_k^P < \bar{c}_k = [A(1-g)\bar{d}^{\psi}]^{1/(1+\psi)}$, which is true for a small discount rate.[24]

(b) *Case* $g > \alpha$. In this case, $\Phi \in C^1([0, \rho))$, and

$$\Phi'(c_k) = \left[\frac{\rho - c_k}{(g - \alpha)A} \right]^{-1+1/\psi} \left[\frac{1 + \psi}{\psi(g - \alpha)A} \right] (c_k - \hat{c}_k),$$

where $\hat{c}_k = \psi\rho/(1 + \psi) < \rho$ is the minimum of Φ on $[0, \rho)$, as depicted in Figure 3b.

Consequently, $\Phi'(c_k) < 0$ if $c_k \in [0, \hat{c}_k)$ and $\Phi'(c_k) > 0$ if $c_k \in (\hat{c}_k, \rho)$. As $\Phi(0) = \Phi(\rho) = \bar{d} > 0$, according to the Intermediate Value Theorem, there are two roots: $c_k^Q \in (0, \hat{c}_k)$ and $c_k^P \in (\hat{c}_k, \rho)$ if and only if

$$\Phi(\hat{c}_k) = \bar{d} - \hat{c}_k \left(\frac{\rho - \hat{c}_k}{(g - \alpha)A} \right)^{1/\psi} < 0. \tag{C.2}$$

As shown by the following lemma, the existence conditions can be expressed according to the value of ρ.

Lemma 1. *Let* $g > \alpha$. *There are two critical levels* ρ_1 *and* ρ_2 $(0 < \rho_1 < \rho_2)$ *such that:*

- *If* $\rho < \rho_1$, $\Phi(\cdot)$ *has no root.*
- *If* $\rho_1 < \rho < \rho_2$, $\Phi(\cdot)$ *has two roots, and* $c_k^P < \bar{c}_k$.
- *If* $\rho > \rho_2$, $\Phi(\cdot)$ *has two roots, and* $c_k^P > \bar{c}_k$.

Proof. First, from (C.2), we have $\Phi(\hat{c}_k) \geq 0 \Leftrightarrow \rho \leq \rho_1 := (1 + \psi) \left[\frac{\bar{d}[(g-\alpha)A]^{1/\psi}}{\psi} \right]^{\psi/(1+\psi)}$. As Figure 11 depicts, the value ρ_1 is such that the two positive growth solutions (P and Q)

[24] Indeed, if $\rho \to 0$, using Eq. (C.1) we have $c_k^P \approx [A(\alpha - g)\bar{d}^{\psi}]^{1/(1+\psi)} < [A(1-g)\bar{d}^{\psi}]^{1/(1+\psi)} = \bar{c}_k$.

coincide ($c_k^P = c_k^Q = \hat{c}_k$). For $\rho > \rho_1$, there are two roots (regime $\mathcal{H}_3$), while for $\rho < \rho_1$, there is no root (regimes $\mathcal{H}_1$ or $\mathcal{H}_2$).

Second, from (C.1), we compute $\Phi(\bar{c}_k) \geq 0 \Leftrightarrow \rho \leq \rho_2 := (g-\alpha)A\left(\frac{\bar{d}}{\bar{c}_k}\right)^{\psi} + \bar{c}_k$; hence if $\rho \leq \rho_2 \Leftrightarrow \bar{c}_k \leq c_k^Q < \hat{c}_k$ or $\bar{c}_k \geq c_k^P > \hat{c}_k$. There are two cases.

(i) $\bar{c}_k > \hat{c}_k$: in this case, we have $\rho \leq \rho_2 \Leftrightarrow c_k^P \leq \bar{c}_k$. As Figure 11a shows, ρ_2 is such that the higher positive growth solution (P) and the no growth solution coincide ($c_k^P = c_k^M = \bar{c}_k$).

(ii) $\bar{c}_k < \hat{c}_k$: in this case, we have $\rho \leq \rho_2 \Leftrightarrow c_k^Q \geq \bar{c}_k$. As Figure 11b shows, ρ_2 is such that the lower positive growth solution (Q) and the no growth solution coincide ($c_k^Q = c_k^M = \bar{c}_k$).

Finally, we ensure that $\rho_1 < \rho_2$ for a large constellation of parameters. Using our baseline calibration (Table 2), we find $\rho_1 = 0.0213$ and $\rho_2 = 0.0378$. □

Consequently, we sum up the two different cases.

(i) $\bar{c}_k > \hat{c}_k$. If $\rho > \rho_2 > \rho_1$, solutions P and Q are present, but there is only one positive-growth steady-state: Q (regime $\mathcal{H}_1$). If $\rho_1 < \rho < \rho_2$, P and Q characterize positive-growth solutions (regime $\mathcal{H}_3$). Finally, if $\rho < \rho_1$, P and Q do not exist, and there is no positive-growth solution (regime $\mathcal{H}_2$). In this way, there is a bifurcation at $\rho = \rho_1$ and $\rho = \rho_2$, as depicted in Figure 11a. Indeed, at $\rho = \rho_2$, the system changes from regime $\mathcal{H}_3$ to regime $\mathcal{H}_1$, and at $\rho = \rho_1$, the system changes from $\mathcal{H}_2$ to $\mathcal{H}_3$.

(ii) $\bar{c}_k < \hat{c}_k$. If $\rho > \rho_2 > \rho_1$, solutions P and Q are present, but Q is the only positive-growth solution – P is a negative-growth solution – (regime $\mathcal{H}_1$). If $\rho_1 < \rho < \rho_2$, P and Q both characterize negative-growth solutions (regime $\mathcal{H}_3$). Finally, if $\rho < \rho_1$, P and Q do not exist, and there is no non-zero-growth solution (regime $\mathcal{H}_2$). In this way, there is a bifurcation at $\rho = \rho_1$ and $\rho = \rho_2$, as depicted in Figure 11b: at $\rho = \rho_2$, the system changes from regime $\mathcal{H}_3$ to regime $\mathcal{H}_1$, and at $\rho = \rho_1$, the system changes from $\mathcal{H}_2$ to $\mathcal{H}_3$.

Appendix D. Local stability ($\theta = 0$)

(i) Regime $\mathcal{L}$.

At steady-state P, we have $d_k^P = 0$, thus, using Eqs. (28)-(29): $\text{Tr}(\mathbf{J}^P) = c_k^P[1 + (g-\alpha)yc^P] - \gamma^P$, and $\det(\mathbf{J}^P) = -c_k^P\gamma^P[1 + (g-\alpha)yc^P]$. As $g < \alpha$ and $yc^P < 0$, $\det(\mathbf{J}^P) < 0$, namely there are two opposite-sign eigenvalues. Consequently, P is saddle-point stable.

At steady-state M, $c_k^M > 0$, $d_k^M > 0$ and $\gamma^M = (1-\alpha)y_k^M - \rho = 0$, namely $\text{Tr}(\mathbf{J}^M) = c_k^M[1 + (g-\alpha)yc^M] - (1-g)yd^M d_k^M$, and $\det(\mathbf{J}^M) = -(1-\alpha)c_k^M d_k^M yd^M > 0$. As $yc^M < 0$ and $yd^M < 0$, we have $\det(\mathbf{J}^M) > 0$, and $\text{Tr}(\mathbf{J}^M) > 0$, and there are two positive eigenvalues. Consequently, M is locally unstable.

(ii) Regime $\mathcal{H}$.

First, let us consider the two solutions with positive economic growth. At steady-states P and Q, we have $d_k^i = 0$, thus: $\mathrm{Tr}(\mathbf{J}^i) = c_k^i[1 + (g - \alpha)yc^i] - \gamma^i$, and $\det(\mathbf{J}^P) = -c_k^i\gamma^i[1 + (g - \alpha)yc^i]$ for $i = P, Q$. Since $DC^i = 0$, there is one negative eigenvalue ($\lambda_1^i = -\gamma^i$) and one eigenvalue that changes sign, depending on the considered equilibrium ($\lambda_2^i = c_k^i[1 + (g - \alpha)yc^i]$). With $yc^i = -\psi y_k^i/c_k^i$ and $c_k^i = \rho - (g - \alpha)y_k^i$ at steady states $i = P, Q$, we obtain $\lambda_2^i := \lambda_2(c_k^i) = c_k^i + \psi(c_k^i - \rho)$. Thus $\lambda_2^i(\hat{c}_k) = 0$, where $\hat{c}_k := \psi\rho/(1+\psi)$. Since $c_k^Q < \hat{c}_k$ and $c_k^P > \hat{c}_k$, it follows that $\lambda_2^Q < 0$ and $\lambda_2^P > 0$. Consequently, P is characterized by two opposite-sign eigenvalues and is locally determined (saddle-point stable), while Q is characterized by two negative eigenvalues and is locally undetermined (stable). Regarding point M, as $\gamma^M = 0$, we have: $\det(\mathbf{J}^M) = -(1 - \alpha)c_k^M d_k^M yd^M > 0$, and $\mathrm{Tr}(\mathbf{J}^M) = c_k^M[1 + (g - \alpha)yc^M] - (1 - g)yd^M d_k^M = -(1 - g)yd^M d_k + (c_k^M - \hat{c}_k)/(1 + \psi)$. As $c_k^M > \hat{c}_k$, and $yd^M < 0$, it follows that $\mathrm{Tr}(\mathbf{J}^M) > 0$, hence M is unstable. This analysis generalizes the simple case of section 3.

Second, let us consider the two solution with negative economic growth. As $\lambda_1^i = -\gamma^i > 0$, $\lambda_2^Q < 0$, and $\lambda_2^P > 0$, we deduce that P is indeterminable (stable) and Q is saddle-point stable. Regarding point M, we still have $\det(\mathbf{J}^M) > 0$. As $c_k^M < \hat{c}_k$, the trace $\mathrm{Tr}(\mathbf{J}^M)$ can now change sign, depending on parameters, as proposition 5 states.