



Finite-Volume approximation of the invariant measure of a viscous stochastic scalar conservation law

Sébastien Boyaval, Sofiane Martel, Julien Reygner

► To cite this version:

Sébastien Boyaval, Sofiane Martel, Julien Reygner. Finite-Volume approximation of the invariant measure of a viscous stochastic scalar conservation law. 2019. hal-02291253v1

HAL Id: hal-02291253

<https://hal.science/hal-02291253v1>

Preprint submitted on 19 Sep 2019 (v1), last revised 26 May 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

FINITE-VOLUME APPROXIMATION OF THE INVARIANT MEASURE OF A VISCOUS STOCHASTIC SCALAR CONSERVATION LAW

SÉBASTIEN BOYAVAL, SOFIANE MARTEL, AND JULIEN REYGNER

ABSTRACT. We aim to give a numerical approximation of the invariant measure of a viscous scalar conservation law, one-dimensional and periodic in the space variable, and stochastically forced with a white-in-time but spatially correlated noise. The flux function is assumed to be locally Lipschitz and to have at most polynomial growth. The numerical scheme we employ discretises the SPDE according to a finite volume method in space, and a split-step backward Euler method in time. As a first result, we prove the well-posedness as well as the existence and uniqueness of an invariant measure for both the spatial semi-discretisation and the fully discrete scheme. Our main result is then the convergence of the invariant measures of the discrete approximations, as the space and time steps go to zero, towards the invariant measure of the SPDE, with respect to the second-order Wasserstein distance. A few numerical experiments are performed to illustrate these results.

1. INTRODUCTION

1.1. Viscous scalar conservation law with random forcing. We consider the following viscous scalar conservation law with stochastic forcing

$$du = -\partial_x A(u)dt + \nu \partial_{xx} u dt + \sum_{k \geq 1} g_k dW^k(t), \quad x \in \mathbb{T}, \quad t \geq 0. \quad (1)$$

Periodic boundary conditions are assigned over the space variable x as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the one-dimensional torus, and $(W^k)_{k \geq 1}$ is a family of independent real Brownian motions. The viscosity coefficient ν is assumed to be positive. In the companion paper [27], we have shown the well-posedness in a strong sense of Equation (1), as well as the existence and uniqueness of an invariant measure for its solution. These results are recalled in Proposition 1.2 below. In this work, we aim to provide a numerical scheme, based on the finite-volume method, that allows to approximate this invariant measure. In this perspective, we place ourselves in the setting of [27] and recall our main notations and assumptions.

Notations. For all $p \in [1, +\infty]$, we denote by $L_0^p(\mathbb{T})$ the set of functions $f \in L^p(\mathbb{T})$ such that $\int_{\mathbb{T}} f(x)dx = 0$. We write $\|\cdot\|_{L_0^p(\mathbb{T})}$ the L^p -norm induced on $L_0^p(\mathbb{T})$ and $\langle \cdot, \cdot \rangle_{L_0^2(\mathbb{T})}$ the L^2 -scalar product induced on $L_0^2(\mathbb{T})$. In a similar manner, for any integer $m \geq 0$ and any $p \in [1, +\infty]$, we introduce the Sobolev space $W_0^{m,p}(\mathbb{T}) := L_0^p(\mathbb{T}) \cap W^{m,p}(\mathbb{T})$ which we equip with the norm $\|\cdot\|_{W_0^{m,p}(\mathbb{T})} := \|\partial_x^m \cdot\|_{L_0^p(\mathbb{T})}$. Incidentally, we will denote by $H_0^m(\mathbb{T})$ the space $W_0^{m,2}(\mathbb{T})$, which we recall is separable and Hilbert when endowed with the norm $\|\cdot\|_{H_0^m(\mathbb{T})} := \|\cdot\|_{W_0^{m,2}(\mathbb{T})}$ and the associated scalar product $\langle \cdot, \cdot \rangle_{H_0^m(\mathbb{T})}$. We recall the following inequalities: for all $1 \leq p \leq q \leq +\infty$,

$$\|u\|_{L_0^p(\mathbb{T})} \leq \|u\|_{L_0^q(\mathbb{T})}, \quad \forall u \in L_0^q(\mathbb{T}), \quad (2)$$

and

$$\|u\|_{L_0^\infty(\mathbb{T})} \leq \|u\|_{W_0^{1,1}(\mathbb{T})}, \quad \forall u \in W_0^{1,1}(\mathbb{T}). \quad (3)$$

In the sequel, we denote by \mathbb{N} the set of non-negative integers, and by \mathbb{N}^* the set of positive integers.

Assumption 1.1. The function $A : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^2 , its first derivative has at most polynomial growth:

$$\exists C_A > 0, \quad \exists p_A \in \mathbb{N}^*, \quad \forall v \in \mathbb{R}, \quad |A'(v)| \leq C_A (1 + |v|^{p_A}), \quad (4)$$

and its second derivative A'' is locally Lipschitz continuous on \mathbb{R} . Furthermore, for all $k \geq 1$, $g_k \in H_0^2(\mathbb{T})$ and

$$D_0 := \sum_{k \geq 1} \|g_k\|_{H_0^2(\mathbb{T})}^2 < +\infty. \quad (5)$$

The assumptions (4) and (5) will be needed in the arguments contained in this paper while the local Lipschitz continuity of A'' is only necessary for Proposition 1.2, whose proof is done in [27].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a normal filtration $(\mathcal{F}_t)_{t \geq 0}$ in the sense of [14, Section 3.3], on which $(W^k)_{k \geq 1}$ is a family of independent Brownian motions. Under Assumption 1.1, the series $\sum_k g_k W^k$ converges

2010 *Mathematics Subject Classification.* 60H15, 30R60, 65M08, 60H35.

Key words and phrases. Stochastic conservation laws, Invariant measure, Finite Volume schemes.

This work is partially supported by the French National Research Agency (ANR) under the programs QuAMProcs and ANR-17-CE40-0030 - EFI - Entropy, flows, inequalities.

in $L^2(\Omega, C([0, T], H_0^2(\mathbb{T})))$, for any $T > 0$, towards an $H_0^2(\mathbb{T})$ -valued Wiener process $(W^Q(t))_{t \in [0, T]}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, defined in the sense of [14, Section 4.2], with the trace class covariance operator Q defined by

$$Q : \begin{cases} H_0^2(\mathbb{T}) & \longrightarrow H_0^2(\mathbb{T}) \\ v & \longmapsto \sum_{k \geq 1} g_k \langle v, g_k \rangle_{H_0^2(\mathbb{T})}. \end{cases}$$

Given a normed vector space E , $\mathcal{B}(E)$ denotes the Borel sets of E , $\mathcal{P}(E)$ denotes the set of Borel probability measures over E , and for $p \in [1, +\infty)$, $\mathcal{P}_p(E)$ denotes the subset of $\mathcal{P}(E)$ of probability measures with finite p -th order moment. The well-posedness of (1) as well as the existence and uniqueness of an invariant measure for its solution is proved in [27, Theorem 1, Theorem 2]:

Proposition 1.2. *Let $u_0 \in H_0^2(\mathbb{T})$. Under Assumption 1.1, there exists a unique strong solution $(u(t))_{t \geq 0}$ to Equation (1) with initial condition u_0 . That is, an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $(u(t))_{t \geq 0}$ with values in $H_0^2(\mathbb{T})$ such that, almost surely:*

- (1) *the mapping $t \mapsto u(t)$ is continuous from $[0, +\infty)$ to $H_0^2(\mathbb{T})$;*
- (2) *for all $t \geq 0$, the following equality holds:*

$$u(t) = u_0 + \int_0^t (-\partial_x A(u(s)) + \nu \partial_{xx} u(s)) ds + W^Q(t). \quad (6)$$

Furthermore, the process $(u(t))_{t \geq 0}$ admits a unique invariant measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$. Besides, if v is a random variable with distribution μ , then $\mathbb{E}[\|v\|_{H_0^2(\mathbb{T})}^2] < +\infty$ and for all $p \in [1, +\infty)$, $\mathbb{E}[\|v\|_{L^p_0(\mathbb{T})}^p] < +\infty$.

Let us precise that for any $t \geq 0$, $u(t)$ will always refer to an element of the space $H_0^2(\mathbb{T})$. The scalar values taken by this function are denoted by $u(t, x)$, for $x \in \mathbb{T}$.

1.2. Space discretisation. In order to discretise (1) with respect to the space variable, we first define a regular mesh \mathcal{T} on the torus:

$$\mathcal{T} := \left\{ \left(\frac{i-1}{N}, \frac{i}{N} \right], i \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Averaging in (1) over each cell of \mathcal{T} , we get

$$\begin{aligned} d \left(N \int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t, x) dx \right) &= -N \left(A \left(u \left(t, \frac{i}{N} \right) \right) - A \left(u \left(t, \frac{i-1}{N} \right) \right) \right) dt \\ &\quad + \nu N \left(\partial_x u \left(t, \frac{i}{N} \right) - \partial_x u \left(t, \frac{i-1}{N} \right) \right) dt + \sum_{k \geq 1} N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x) dx dW^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}. \end{aligned} \quad (7)$$

Finite-volume schemes aim to approximate the dynamics of the average value of the solution over each cell of the mesh. This leads to the introduction of a *numerical flux* function $\bar{A}(u, v)$ approximating the flux of the conserved quantity at the interface between two adjacent cells. As regards the viscous term in (7), we replace the space derivatives by their finite difference approximations. As for the noise coefficients, we introduce the shorthand notation

$$\sigma_i^k := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x) dx, \quad k \geq 1, \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

These operations result in the following stochastic differential equation

$$\begin{aligned} dU_i(t) &= -N \left(\bar{A}(U_i(t), U_{i+1}(t)) - \bar{A}(U_{i-1}(t), U_i(t)) \right) dt \\ &\quad + \nu N^2 (U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)) dt + \sum_{k \geq 1} \sigma_i^k dW^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}, \quad t \geq 0, \end{aligned} \quad (8)$$

as a semi-discrete finite-volume approximation of (1) in the sense that $U_i(t)$ is meant to be an approximation of the spatial average $N \int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t, x) dx$. We may interpret the noise term (*i.e.* the last term in (8)) as a discrete version of the Q -Wiener process $(W^Q(t))_{t \geq 0}$ introduced in Section 1.1. Let us notice that the \mathbb{R}^N -valued stochastic process $(\mathbf{W}^{Q,N}(t))_{t \geq 0}$ whose components are defined by

$$W_i^{Q,N}(t) := \sum_{k \geq 1} \sigma_i^k W^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}, \quad t \geq 0,$$

is a Wiener process with the covariance

$$\mathbb{E} \left[W_i^{Q,N}(t) W_j^{Q,N}(t) \right] = t \sum_{k \geq 1} \sigma_i^k \sigma_j^k,$$

which is finite as the Jensen inequality, Assumption 1.1 and (3) ensure that for all $i \in \mathbb{Z}/N\mathbb{Z}$,

$$\sum_{k \geq 1} |\sigma_i^k|_2^2 = \sum_{k \geq 1} \left| N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x) dx \right|^2 \leq \sum_{k \geq 1} N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \|g_k\|_{L_0^\infty(\mathbb{T})}^2 dx \leq \sum_{k \geq 1} \|g_k\|_{H_0^2(\mathbb{T})}^2 \leq D_0. \quad (9)$$

Furthermore, each vector $\sigma^k = (\sigma_1^k, \dots, \sigma_N^k)$ satisfies a discrete cancellation condition:

$$\sum_{i=1}^N \sigma_i^k = N \int_{\mathbb{T}} g_k(x) dx = 0.$$

Thus, denoting

$$\mathbb{R}_0^N := \{\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{R}^N : u_1 + \dots + u_N = 0\},$$

we get that $(\mathbf{W}^{Q,N}(t))_{t \geq 0}$ is an \mathbb{R}_0^N -valued process. We equip the space \mathbb{R}_0^N with the renormalised L^p norm $\|\cdot\|_p$ and scalar product $\langle \cdot, \cdot \rangle$: for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$ and any $p \in [1, +\infty)$,

$$\|\mathbf{u}\|_p^p := \frac{1}{N} \sum_{i=1}^N |u_i|^p, \quad \langle \mathbf{u}, \mathbf{v} \rangle := \frac{1}{N} \sum_{i=1}^N u_i v_i.$$

Furthermore, for any $\mathbf{u} \in \mathbb{R}_0^N$, we set by convention $\|\mathbf{u}\|_0^0 = 1$. Besides, notice that for any $1 \leq p \leq q < +\infty$, we have

$$\|\mathbf{u}\|_p \leq \|\mathbf{u}\|_q, \quad \forall \mathbf{u} \in \mathbb{R}_0^N. \quad (10)$$

The drift function in (8) is the function \mathbf{b} defined on \mathbb{R}_0^N by the components

$$b_i(\mathbf{v}) := -N (\bar{A}(v_i, v_{i+1}) - \bar{A}(v_{i-1}, v_i)) + \nu N^2 (v_{i+1} - 2v_i + v_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

These notations being set, we can write the SDE (8) in the vectorised form

$$d\mathbf{U}(t) = \mathbf{b}(\mathbf{U}(t))dt + d\mathbf{W}^{Q,N}(t), \quad t \geq 0. \quad (11)$$

It appears that \mathbf{b} takes values in \mathbb{R}_0^N . As a consequence, Equation (11) is conservative in the following sense: if $\mathbf{U}_0 \in \mathbb{R}_0^N$, then for all $t \geq 0$, $\mathbf{U}(t) \in \mathbb{R}_0^N$.

We may now state our assumptions on the numerical flux:

Assumption 1.3. The function \bar{A} belongs to $C^1(\mathbb{R}^2, \mathbb{R})$, its first derivatives $\partial_1 \bar{A}$ and $\partial_2 \bar{A}$ are locally Lipschitz continuous on \mathbb{R}^2 , and it satisfies the following properties:

(i) Consistency:

$$\forall u \in \mathbb{R}, \quad \bar{A}(u, u) = A(u); \quad (12)$$

(ii) Monotonicity:

$$\forall u, v \in \mathbb{R}, \quad \partial_1 \bar{A}(u, v) \geq 0, \quad \partial_2 \bar{A}(u, v) \leq 0; \quad (13)$$

(iii) Polynomial growth:

$$\exists C_{\bar{A}} > 0, \quad \exists p_{\bar{A}} \in \mathbb{N}^*, \quad \forall u, v \in \mathbb{R}, \quad |\partial_1 \bar{A}(u, v)| \leq C_{\bar{A}}(1 + |u|^{p_{\bar{A}}}), \quad |\partial_2 \bar{A}(u, v)| \leq C_{\bar{A}}(1 + |v|^{p_{\bar{A}}}). \quad (14)$$

Note in particular that the flux function, and therefore the non-linearity of Equation (1), is not subject to a global Lipschitz continuity assumption. Nevertheless, we will prove in Proposition 2.6 below that (11) is well-posed under Assumption 1.3.

Remark 1.4 (Engquist-Osher numerical flux). A notable class of numerical fluxes satisfying the monotonicity and polynomial growth conditions (under Assumption 1.1) are the flux-splitting schemes [23, Example 5.2], among which a commonly employed example is the *Engquist-Osher flux* [22] defined by

$$\bar{A}_{\text{EO}}(u, v) := \frac{A(u) + A(v)}{2} - \frac{1}{2} \int_u^v |A'(z)| dz.$$

1.3. Space and time discretisation. The second stage in constructing a numerical scheme for (1) is the time discretisation of the SDE (11). Considering a time step $\Delta t > 0$ and a positive integer n , we introduce the notation $\Delta \mathbf{W}_n^{Q,N} := \mathbf{W}^{Q,N}(n\Delta t) - \mathbf{W}^{Q,N}((n-1)\Delta t)$.

As it was already noticed in [28], explicit numerical schemes for SDEs with non-globally Lipschitz continuous coefficients do not preserve in general the large time stability, whereas implicit schemes are more robust. Therefore, since our main focus in this paper is to approximate invariant measures, we propose the following *split-step stochastic backward Euler method*:

$$\begin{cases} \mathbf{U}_{n+\frac{1}{2}} = \mathbf{U}_n + \Delta t \mathbf{b}(\mathbf{U}_{n+\frac{1}{2}}), \\ \mathbf{U}_{n+1} = \mathbf{U}_{n+\frac{1}{2}} + \Delta \mathbf{W}_{n+1}^{Q,N}. \end{cases} \quad (15)$$

The well-posedness of the scheme, *i.e.* the existence and uniqueness of the value $\mathbf{U}_{n+\frac{1}{2}}$ in the first line of (15), is ensured by Proposition 2.14.

1.4. Main results. Our first focus is on the large-time behaviour of the processes $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{U}_n)_{n \in \mathbb{N}}$. In this context, we state our first result:

Theorem 1.5. *Under Assumptions 1.1 and 1.3, the following two statements hold:*

- (i) *for any $N \geq 1$, the process $(\mathbf{U}(t))_{t \geq 0}$ solution of the SDE (11) admits a unique invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$;*
- (ii) *for any $N \geq 1$ and any $\Delta t > 0$, the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$ defined by (15) admits a unique invariant measure $\nu_{N, \Delta t} \in \mathcal{P}(\mathbb{R}_0^N)$.*

Moreover, for any $N \geq 1$ and any $\Delta t > 0$, the measures ν_N and $\nu_{N, \Delta t}$ belong to $\mathcal{P}_2(\mathbb{R}_0^N)$.

The proofs for these two statements are given separately in Section 2. The structure of the proof is the same as for [27, Theorem 2] where we derived the existence and uniqueness of an invariant measure for the solution of (1) from two important properties: respectively the dissipativity of the solution and an L^1 -contraction property. In Lemma 2.4 below, we show that both of these properties are preserved at the discrete level. Therefore, we will address the existence part with a tightness argument (the Krylov-Bogoliubov theorem) and the uniqueness with a coupling argument. To compare two different probability measures, we will make use of the following metric:

Definition 1.6 (Wasserstein distance). Let $(E, \|\cdot\|_E)$ be a normed vector space and let $\alpha, \beta \in \mathcal{P}_2(E)$. The *second order Wasserstein distance* between α and β is defined by

$$W_2(\alpha, \beta) := \inf_{\pi \in \Pi(\alpha, \beta)} \left(\int_{E \times E} \|u - v\|_E^2 d\pi(u, v) \right)^{1/2},$$

where $\Pi(\alpha, \beta)$ is the set of probability measures on $E \times E$ with marginals α and β :

$$\Pi(\alpha, \beta) := \{ \pi \in \mathcal{P}_2(E \times E) : \forall B \in \mathcal{B}(E), \pi(B \times E) = \alpha(B) \text{ and } \pi(E \times B) = \beta(B) \}.$$

The reader is referred to [29, Chapter 6] for further details on the Wasserstein distance, and in particular for the proof that it actually defines a distance on $\mathcal{P}_2(E)$. From now on, the space $\mathcal{P}_2(E)$ will be endowed with the distance W_2 : convergence of elements of $\mathcal{P}_2(E)$ will always be meant in the sense of the Wasserstein distance. The only cases we will address in this paper are $E = L_0^2(\mathbb{T})$ and $E = \mathbb{R}_0^N$.

As a first step to approximate numerically the measure μ , we start to embed the measures ν_N and $\nu_{N, \Delta t}$ into $\mathcal{P}(L_0^2(\mathbb{T}))$. For $m = 0, 1, 2$, let $\Psi_N^{(m)} : \mathbb{R}_0^N \rightarrow H_0^m(\mathbb{T})$ denote embedding functions in such a way that for any $\mathbf{u} \in \mathbb{R}_0^N$, $\Psi_N^{(0)} \mathbf{u}$, $\Psi_N^{(1)} \mathbf{u}$ and $\Psi_N^{(2)} \mathbf{u}$ correspond respectively to piecewise constant, piecewise affine, and piecewise quadratic interpolations of the vector \mathbf{u} on the mesh \mathcal{T} . The functions $\Psi_N^{(m)}$ will be precisely defined at the beginning of Section 3.

For $m = 0, 1, 2$, we define the pushforward measures $\mu_N^{(m)} := \nu_N \circ (\Psi_N^{(m)})^{-1}$, and $\mu_{N, \Delta t}^{(m)} := \nu_{N, \Delta t} \circ (\Psi_N^{(m)})^{-1}$. In particular, the measures $\mu_N^{(m)}$ and $\mu_{N, \Delta t}^{(m)}$ give full weight to $H_0^m(\mathbb{T})$. Sections 3 and 4 are devoted to the proof of our main result:

Theorem 1.7. *Under Assumptions 1.1 and 1.3, we have for all $m = 0, 1, 2$,*

$$\lim_{N \rightarrow \infty} \mu_N^{(m)} = \mu \quad \text{in } \mathcal{P}_2(L_0^2(\mathbb{T})), \quad (16)$$

and moreover, for any $N \geq 1$,

$$\lim_{\Delta t \rightarrow 0} \mu_{N, \Delta t} = \nu_N \quad \text{in } \mathcal{P}_2(\mathbb{R}_0^N). \quad (17)$$

In short, we have the following approximation result for all $m = 0, 1, 2$:

$$\lim_{N \rightarrow \infty} \lim_{\Delta t \rightarrow 0} \mu_{N, \Delta t}^{(m)} = \mu \quad \text{in } \mathcal{P}_2(L_0^2(\mathbb{T})).$$

Remark 1.8. In Theorem 1.7, μ is seen as a probability measure of $\mathcal{P}_2(L_0^2(\mathbb{T}))$ giving full weight to $H_0^2(\mathbb{T})$, as opposed to Proposition 1.2 where μ was directly seen as a measure of the space $\mathcal{P}_2(H_0^2(\mathbb{T}))$.

Remark 1.9 (Ergodicity). As the invariant measure μ of the process $(u(t))_{t \geq 0}$ is unique from Proposition 1.2, it is ergodic. In particular, a consequence of Birkhoff's ergodic theorem (see for instance [14, Theorem 1.2.3]) is that for any $\varphi \in L^1(\mu)$ and for μ -almost every initial condition $u_0 \in H_0^2(\mathbb{T})$, almost surely,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi(u(s)) ds = \mathbb{E}[\varphi(v)], \quad \text{where } v \sim \mu.$$

By virtue of Theorem 1.5, this property also holds at the discrete level: the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$ satisfies for any $\varphi \in L^1(\nu_{N, \Delta t})$ and for $\nu_{N, \Delta t}$ -almost every initial condition $\mathbf{U}_0 \in \mathbb{R}_0^N$, almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=0}^{n-1} \varphi(\mathbf{U}_l) = \mathbb{E}[\varphi(\mathbf{V})], \quad \text{where } \mathbf{V} \sim \nu_{N, \Delta t}.$$

Thanks to this property, it is possible to approximate numerically expectations of functionals under the invariant measure by averaging in time the simulated process. We used this method to perform the numerical experiments presented in Section 5.

1.5. Review of literature. Many results are found concerning the numerical approximation *in finite time* of stochastic conservation laws. A particular case of interest is the stochastic Burgers equation which corresponds to the case of the flux function $A(u) = u^2/2$. Finite difference schemes are presented in [1, 24] to approximate its solution. When the viscosity coefficient is equal to zero, the SPDE falls into a different framework. Convergence of finite-volume schemes in this *hyperbolic* case have been established both under the kinetic [18, 19, 17] and the entropic formulations [2, 3].

As regards the numerical approximation of the invariant measure of an SPDE, we may start by mentioning [11] concerning the damped stochastic non-linear Schrödinger equation, where a spectral Galerkin method is used for the space discretisation and a modified implicit Euler scheme for the temporal discretisation. Several works of Bréhier are devoted to the numerical approximation of the invariant measures of semi-linear SPDEs in Hilbert spaces perturbed with white noise [8, 9, 10], where spectral Galerkin and semi-implicit Euler methods are used. Those results hold under a global Lipschitz assumption on the nonlinearity. In the more recent works [12, 13], non-Lipschitz nonlinearities are considered, but they still need to satisfy a one-sided Lipschitz condition.

In the present work, our assumptions on the flux function do not imply that the non-linear term is globally Lipschitz in $L_0^2(\mathbb{T})$ nor even one-sided Lipschitz. In particular, the case of the Burgers' equation is covered. However, Equation (1) satisfies an L^1 -contraction property [27, Proposition 5] which may be viewed as a one-sided Lipschitz condition in the Banach space $L_0^1(\mathbb{T})$.

1.6. Outline of the paper. The existence and uniqueness of an invariant measure for the solution of (11), *i.e.* the first part of Theorem 1.5, is proved in Section 2.2, and for the split-step scheme (15), *i.e.* the second part of Theorem 1.5, it is proved in Section 2.3.

The proof of Theorem 1.7 is also split in two separate parts. The convergence in space (16) is proved in Section 3 and then, in Section 4, we prove the convergence with respect to the time step, *i.e.* Equation (17).

We performed numerical experiments to test the stationarity and convergence results in the Burgers case. The results of these experiments are exposed in Section 5, where we furthermore illustrate some properties regarding the turbulent behaviour of the process in its stationary regime.

2. SEMI-DISCRETE AND SPLIT-STEP SCHEMES: WELL-POSEDNESS AND INVARIANT MEASURE

Preliminary results are given in Subsection 2.1. In Subsection 2.2, we prove the well-posedness of (11), and after establishing some properties for the solution, we prove the existence of an invariant measure at Proposition 2.10. Then, Lemmas 2.12 and 2.13 lead to the proof of uniqueness of this invariant measure, *i.e.* to the proof of the first assertion of Theorem 1.5.

2.1. Notations and properties. All the lemmas stated in this section will be proved in the appendix. We define the discrete differential operators $\mathbf{D}_N^{(m)} : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $m = 0, 1, 2$, by

$$\left(\mathbf{D}_N^{(0)}\mathbf{u}\right)_i := u_i, \quad \left(\mathbf{D}_N^{(1)}\mathbf{u}\right)_i := N(u_{i+1} - u_i), \quad \left(\mathbf{D}_N^{(2)}\mathbf{u}\right)_i := N^2(u_{i+1} - 2u_i + u_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

We will often make use, in this paper, of the summation by parts identity

$$\left\langle \mathbf{D}_N^{(1)}\mathbf{u}, \mathbf{D}_N^{(1)}\mathbf{v} \right\rangle = - \left\langle \mathbf{D}_N^{(2)}\mathbf{u}, \mathbf{v} \right\rangle, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N. \quad (18)$$

These operators satisfy furthermore the following properties:

Lemma 2.1 (Discrete Poincaré inequality). *Let $\mathbf{u} \in \mathbb{R}^N$. If there exist $i_-, i_+ \in \mathbb{Z}/N\mathbb{Z}$ such that $u_{i_-} \leq 0 \leq u_{i_+}$, then for any $m = 0, 1$, any $p \in [1, +\infty)$,*

$$\left\| \mathbf{D}_N^{(m)}\mathbf{u} \right\|_p \leq \left\| \mathbf{D}_N^{(m+1)}\mathbf{u} \right\|_p.$$

It should be noted that this discrete Poincaré inequality holds in particular for $\mathbf{u} \in \mathbb{R}_0^N$.

Several times in this paper, we will establish estimates uniformly in N (resp. in Δt) over the moment of the discrete Sobolev norm $\mathbb{E}[\|\mathbf{D}_N^{(m)}\mathbf{V}\|_p^p]$ where \mathbf{V} is an invariant measure for the semi-discrete scheme (resp. the fully discrete scheme). Whenever this situation appears, we will denote by $C^{m,p}$ (resp. $\bar{C}^{m,p}$) the uniform upper bound.

Lemma 2.2. *For any $\mathbf{u} \in \mathbb{R}_0^N$ and $p \in 2\mathbb{N}^*$, we have*

$$\left\langle \mathbf{D}_N^{(1)}(\mathbf{u}^{p-1}), \mathbf{D}_N^{(1)}\mathbf{u} \right\rangle \geq \frac{4(p-1)}{p^2} \|\mathbf{u}\|_p^p,$$

where $\mathbf{u}^{p-1} := (u_1^{p-1}, \dots, u_N^{p-1})$.

Lemma 2.3. *Under Assumption 1.3, for any $\mathbf{u} \in \mathbb{R}_0^N$ and any $q \in 2\mathbb{N}^*$, we have*

$$\sum_{i=1}^N u_i^{q-1} (\bar{A}(u_i, u_{i+1}) - \bar{A}(u_{i-1}, u_i)) \geq 0.$$

For any $z \in \mathbb{R}$, we write $\text{sign}(z) := \mathbf{1}_{z \geq 0} - \mathbf{1}_{z < 0}$. By extension, for $\mathbf{u} \in \mathbb{R}_0^N$, $\mathbf{sign}(\mathbf{u})$ denotes the vector of $\{-1, +1\}^N$ defined by $(\mathbf{sign}(\mathbf{u}))_i = \text{sign}(u_i)$.

The discretised drift \mathbf{b} preserves some nice properties of Equation (1) that we will use repeatedly throughout this paper:

Lemma 2.4. *Under Assumption 1.3, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$, the function \mathbf{b} satisfies*

- (i) $\langle \mathbf{sign}(\mathbf{u} - \mathbf{v}), \mathbf{b}(\mathbf{u}) - \mathbf{b}(\mathbf{v}) \rangle \leq 0$ (L^1 -contraction);
- (ii) $\langle \mathbf{u}, \mathbf{b}(\mathbf{u}) \rangle \leq -\nu \|\mathbf{D}_N^{(1)} \mathbf{u}\|_2^2$ (dissipativity).

Remark 2.5. The dissipativity property actually holds for the family of E-fluxes [26], a larger family than the class of monotone numerical fluxes. The monotonicity assumption (13) seems however necessary as regards the L^1 -contraction property.

2.2. The semi-discrete scheme. Before addressing the invariant measure of the solution of (11), we first ensure the existence and uniqueness of this solution:

Proposition 2.6. *Let \mathbf{U}_0 be an \mathbb{R}_0^N -valued, \mathcal{F}_0 -measurable random variable. Under Assumptions 1.1 and 1.3, the stochastic differential equation (11) admits a unique strong solution $(\mathbf{U}(t))_{t \geq 0}$ taking values in \mathbb{R}_0^N and with initial condition \mathbf{U}_0 .*

Proof. Let $\mathbf{u}_0 \in \mathbb{R}_0^N$. Since the function \mathbf{b} is locally Lipschitz continuous, there exists a unique strong solution $(\mathbf{U}(t))_{t \in [0, T^*)}$ to Equation (11) with initial condition \mathbf{u}_0 defined up to an explosion time T^* , i.e. a stopping time taking values in $(0, +\infty]$ such that almost surely, if $T^* < +\infty$, then

$$\lim_{t \rightarrow T^*} \|\mathbf{U}(t)\|_2 = +\infty$$

(see for instance [25, Theorem 2.3, Lemma 2.1, Theorem 3.1]). In particular, if we define the stopping times

$$\tau_M := \inf \{t \geq 0 : \|\mathbf{U}(t)\|_2^2 \geq M\}, \quad (19)$$

then for all $M \geq 0$, we have $\tau_M \leq T^*$ almost surely.

From Dynkin's formula applied to Equation (11), we get for all $t \geq 0$,

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{U}(t \wedge \tau_M)\|_2^2 \right] &= \|\mathbf{u}_0\|_2^2 + 2\mathbb{E} \left[\int_0^{t \wedge \tau_M} \langle \mathbf{U}(s), \mathbf{b}(\mathbf{U}(s)) \rangle ds \right] + \mathbb{E}[t \wedge \tau_M] \sum_{k \geq 1} \|\sigma^k\|_2^2 \\ &\leq \|\mathbf{u}_0\|_2^2 - 2\nu \mathbb{E} \left[\int_0^{t \wedge \tau_M} \left\| \mathbf{D}_N^{(1)} \mathbf{U}(s) \right\|_2^2 ds \right] + \mathbb{E}[t \wedge \tau_M] \sum_{k \geq 1} \|\sigma^k\|_2^2, \end{aligned}$$

where the inequality comes from Lemma 2.4.(ii). As a consequence, using (9),

$$\mathbb{E} \left[\|\mathbf{U}(t \wedge \tau_M)\|_2^2 \right] \leq \|\mathbf{u}_0\|_2^2 + tD_0.$$

From Markov's inequality, we now derive

$$\mathbb{P}(\tau_M \leq t) = \mathbb{P} \left(\|\mathbf{U}(t \wedge \tau_M)\|_2^2 \geq M \right) \leq \frac{\mathbb{E} \left[\|\mathbf{U}(t \wedge \tau_M)\|_2^2 \right]}{M} \leq \frac{\|\mathbf{u}_0\|_2^2 + tD_0}{M} \xrightarrow{M \rightarrow \infty} 0.$$

As the random variable $\mathbf{1}_{\tau_M \leq t}$ is almost surely non-decreasing as M increases, it admits an almost sure limit as $M \rightarrow +\infty$. From the dominated convergence theorem, this limit is actually zero:

$$\mathbb{E} \left[\lim_{M \rightarrow \infty} \mathbf{1}_{\tau_M \leq t} \right] = \lim_{M \rightarrow \infty} \mathbb{P}(\tau_M \leq t) = 0.$$

As a consequence, almost surely, $\lim_{M \rightarrow \infty} \tau_M = +\infty$, and then $T^* = +\infty$ almost surely, meaning that $(\mathbf{U}(t))_{t \geq 0}$ is a global solution of (11).

Now, if the initial condition of $(\mathbf{U}(t))_{t \geq 0}$ is an \mathcal{F}_0 -measurable random variable \mathbf{U}_0 distributed under some probability measure α on \mathbb{R}_0^N , then we have

$$\mathbb{P}(T^* = +\infty) = \int_{\mathbb{R}_0^N} \mathbb{P}_{\mathbf{u}_0}(T^* = +\infty) d\alpha(\mathbf{u}_0) = 1, \quad (20)$$

where $\mathbb{P}_{\mathbf{u}_0}$ is the conditional probability given the event $\mathbf{U}(0) = \mathbf{u}_0$. \square

We now turn to the proof of existence of an invariant measure. The following lemma, Proposition 2.8, and Corollary 2.9 are preliminary results.

Lemma 2.7 (Moment estimates on the semi-discrete approximation). *Let $p \in 2\mathbb{N}^*$ and let \mathbf{U}_0 be an \mathcal{F}_0 -measurable random variable such that $\mathbb{E}[\|\mathbf{U}_0\|_p^p] < +\infty$. Then, under Assumptions 1.1 and 1.3, the strong solution $(\mathbf{U}(t))_{t \geq 0}$ of (11) with initial condition \mathbf{U}_0 satisfies:*

(i) *For all $t \geq 0$,*

$$\mathbb{E} \left[\|\mathbf{U}(t)\|_p^p \right] + \nu p \mathbb{E} \left[\int_0^t \left\langle \mathbf{D}_N^{(1)}(\mathbf{U}(s)^{p-1}), \mathbf{D}_N^{(1)} \mathbf{U}(s) \right\rangle ds \right] \leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + D_0 \frac{p(p-1)}{2} \mathbb{E} \left[\int_0^t \|\mathbf{U}(s)\|_{p-2}^{p-2} ds \right] \quad (21)$$

where $\mathbf{U}(s)^p$ denotes the vector $(U_1(s)^p, \dots, U_N(s)^p)$ and when $p = 2$, we recall the convention $\|\cdot\|_{p-2}^{p-2} = 1$.

(ii) *There exist six positive constants $c_0^{(p)}, c_1^{(p)}, c_2^{(p)}, \theta_0^{(p)}, \theta_1^{(p)}$ and $\theta_2^{(p)}$ depending only on D_0, ν and p such that we have*

$$\forall t > 0, \quad \mathbb{E} \left[\int_0^t \|\mathbf{U}(s)\|_p^p ds \right] \leq \theta_0^{(p)} + \theta_1^{(p)} \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + \theta_2^{(p)} t, \quad (22)$$

and

$$\forall T > 0, \quad \sup_{t \in [0, T]} \mathbb{E} \left[\|\mathbf{U}(t)\|_p^p \right] \leq c_0^{(p)} + c_1^{(p)} \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + c_2^{(p)} T. \quad (23)$$

Proof. Let τ_M be the stopping time defined at (19). Applying Dynkin's formula to Equation (11), we get the following dynamics for the p -th order moment: for all $t \geq 0$ and all $M \geq 0$,

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{U}(t \wedge \tau_M)\|_p^p \right] &= \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] - p \mathbb{E} \left[\int_0^{t \wedge \tau_M} \sum_{i=1}^N U_i(s)^{p-1} (\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s))) ds \right] \\ &\quad + \nu N p \mathbb{E} \left[\int_0^{t \wedge \tau_M} \left\langle \mathbf{U}(s)^{p-1}, \mathbf{D}_N^{(2)} \mathbf{U}(s) \right\rangle ds \right] + \frac{p(p-1)}{2N} \mathbb{E} \left[\int_0^{t \wedge \tau_M} \sum_{i=1}^N U_i(s)^{p-2} \sum_{k \geq 1} (\sigma_i^k)^2 ds \right]. \end{aligned} \quad (24)$$

From (9), we have for all $i = 1, \dots, N$,

$$\sum_{k \geq 1} (\sigma_i^k)^2 \leq D_0.$$

On the other hand, the second term of the right-hand side in (24) is non-positive thanks to Lemma 2.3. Hence, using (18) in the viscous term, we get

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{U}(t \wedge \tau_M)\|_p^p \right] &\leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] - \nu N p \mathbb{E} \left[\int_0^{t \wedge \tau_M} \left\langle \mathbf{D}_N^{(1)}(\mathbf{U}(s)^{p-1}), \mathbf{D}_N^{(1)} \mathbf{U}(s) \right\rangle ds \right] \\ &\quad + \frac{p(p-1)}{2} D_0 \mathbb{E} \left[\int_0^{t \wedge \tau_M} \|\mathbf{U}(s)\|_{p-2}^{p-2} ds \right]. \end{aligned} \quad (25)$$

Letting M go to $+\infty$, applying the monotone convergence theorem on the right-hand side and Fatou's lemma on the left-hand side yields the first assertion of the lemma.

From the first assertion and Lemma 2.2, we have

$$\frac{4\nu(p-1)}{p} \mathbb{E} \left[\int_0^t \|\mathbf{U}(s)\|_p^p ds \right] \leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + \frac{D_0 p(p-1)}{2} \mathbb{E} \left[\int_0^t \|\mathbf{U}(s)\|_{p-2}^{p-2} ds \right]. \quad (26)$$

Noticing that $\|\cdot\|_{p-2}^{p-2} \leq 1 + \|\cdot\|_p^p$, by (26) and an induction argument, we can show that for all $p \in 2\mathbb{N}^*$, (22) holds. Now, from the first assertion once again and (22), we have for all $p \in 2\mathbb{N}^*$ and all $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{U}(t)\|_p^p \right] &\leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + \frac{D_0 p(p-1)}{2} \mathbb{E} \left[\int_0^T \|\mathbf{U}(s)\|_{p-2}^{p-2} ds \right] \\ &\leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + \frac{D_0 p(p-1)}{2} \left(\theta_0^{(p-2)} + \theta_1^{(p-2)} \mathbb{E} \left[\|\mathbf{U}_0\|_{p-2}^{p-2} \right] + \theta_2^{(p-2)} T \right) \\ &\leq \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + \frac{D_0 p(p-1)}{2} \left(\theta_0^{(p-2)} + \theta_1^{(p-2)} \left(1 + \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] \right) + \theta_2^{(p-2)} T \right) \\ &=: c_0^{(p)} + c_1^{(p)} \mathbb{E} \left[\|\mathbf{U}_0\|_p^p \right] + c_2^{(p)} T. \end{aligned}$$

Since the right-hand side does not depend on t , we get (23). \square

Proposition 2.8 (L^1 -contraction). *Under Assumptions 1.1 and 1.3, two strong solutions $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ of (11) (driven by the same Wiener process $\mathbf{W}^{Q,N}$) with possibly different initial conditions satisfy almost surely:*

$$\|\mathbf{U}(t) - \mathbf{V}(t)\|_1 \leq \|\mathbf{U}(s) - \mathbf{V}(s)\|_1, \quad 0 \leq s \leq t.$$

Proof. Since $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ are driven by the same Wiener process, then $(\mathbf{U}(t) - \mathbf{V}(t))_{t \geq 0}$ is an absolutely continuous process:

$$d(\mathbf{U}(t) - \mathbf{V}(t)) = (\mathbf{b}(\mathbf{U}(t)) - \mathbf{b}(\mathbf{V}(t))) dt.$$

In particular, we can write for all $t \geq 0$,

$$\frac{d}{dt} \|\mathbf{U}(t) - \mathbf{V}(t)\|_1 = \langle \mathbf{sign}(\mathbf{U}(t) - \mathbf{V}(t)), \mathbf{b}(\mathbf{U}(t)) - \mathbf{b}(\mathbf{V}(t)) \rangle \leq 0,$$

where the inequality comes from Lemma 2.4.(i), and the result follows by integrating in time. \square

This last property ensures the following result which we state without a proof:

Corollary 2.9 (Feller property). *Under Assumption 1.3, the strong solution $(\mathbf{U}(t))_{t \geq 0}$ of Equation (11) satisfies the Feller property, i.e. for any continuous and bounded function $\varphi : \mathbb{R}_0^N \rightarrow \mathbb{R}$ and any $t \geq 0$, the mapping*

$$\mathbf{u}_0 \in \mathbb{R}_0^N \mapsto \mathbb{E}_{\mathbf{u}_0} [\varphi(\mathbf{U}(t))] \in \mathbb{R}$$

is continuous and bounded, where $\mathbb{E}_{\mathbf{u}_0}$ is the conditional expectation given the event $\mathbf{U}(0) = \mathbf{u}_0$.

Proposition 2.10 (Existence of an invariant measure for the semi-discrete scheme). *Under Assumptions 1.1 and 1.3, the strong solution $(\mathbf{U}(t))_{t \geq 0}$ of (11) admits an invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$. Moreover, for all $p \in [1, +\infty)$, there exists a constant $C^{0,p}$ not depending on N such that if \mathbf{V} is a random variable with distribution ν_N , then*

$$\mathbb{E} [\|\mathbf{V}\|_p^p] \leq C^{0,p}.$$

Proof. Let $(\mathbf{U}(t))_{t \geq 0}$ be the solution of (11) with a deterministic initial condition $\mathbf{u}_0 \in \mathbb{R}_0^N$. From Lemma 2.7.(ii), we have for all $t > 0$ and all $p \in 2\mathbb{N}^*$,

$$\frac{1}{t} \int_0^t \mathbb{E}_{\mathbf{u}_0} [\|\mathbf{U}(s)\|_p^p] ds \leq \frac{1}{t} \theta_0^{(p)} + \frac{1}{t} \theta_1^{(p)} \|\mathbf{u}_0\|_2^2 + \theta_2^{(p)}. \quad (27)$$

Let us take $p = 2$. Applying the Markov inequality and taking the limit superior in t , we get for all $\varepsilon > 0$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}_{\mathbf{u}_0} \left(\|\mathbf{U}(s)\|_2^2 > \frac{1}{\varepsilon} \right) ds \leq \varepsilon \theta_2^{(2)}.$$

Since from Corollary 2.9, $(\mathbf{U}(t))_{t \geq 0}$ is Feller, the existence of an invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$ for $(\mathbf{U}(t))_{t \geq 0}$ is now a consequence of the Krylov-Bogoliubov theorem [15, Corollary 3.1.2].

Let \mathbf{V} be a random variable with distribution ν_N . We will derive now from Equation (27) that \mathbf{V} has finite moments. A computation of the same kind as the one below is found for instance in the proof of [?, Proposition 4.24]. For any $M > 0$ and any $p \in 2\mathbb{N}^*$,

$$\begin{aligned} \mathbb{E} [\|\mathbf{V}\|_p^p \wedge M] &= \frac{1}{t} \int_0^t \int_{\mathbb{R}_0^N} \mathbb{E}_{\mathbf{u}_0} [\|\mathbf{U}(s)\|_p^p \wedge M] d\nu_N(\mathbf{u}_0) ds \\ &= \int_{\mathbb{R}_0^N} \frac{1}{t} \int_0^t \mathbb{E}_{\mathbf{u}_0} [\|\mathbf{U}(s)\|_p^p \wedge M] ds d\nu_N(\mathbf{u}_0) \\ &\leq \int_{\mathbb{R}_0^N} \left(\frac{1}{t} \int_0^t \mathbb{E}_{\mathbf{u}_0} [\|\mathbf{U}(s)\|_p^p] ds \right) \wedge M d\nu_N(\mathbf{u}_0) \\ &\leq \int_{\mathbb{R}_0^N} \left(\frac{1}{t} \theta_0^{(p)} + \theta_1^{(p)} \frac{\|\mathbf{u}_0\|_p^p}{t} + \theta_2^{(p)} \right) \wedge M d\nu_N(\mathbf{u}_0). \end{aligned}$$

Now, letting $t \rightarrow +\infty$, we get from the dominated convergence theorem,

$$\mathbb{E} [\|\mathbf{V}\|_p^p \wedge M] \leq \int_{\mathbb{R}_0^N} \lim_{t \rightarrow \infty} \left(\frac{1}{t} \theta_0^{(p)} + \theta_1^{(p)} \frac{\|\mathbf{u}_0\|_p^p}{t} + \theta_2^{(p)} \right) \wedge M d\nu_N(\mathbf{u}_0) = \theta_2^{(p)} \wedge M \leq \theta_2^{(p)},$$

and the result for $p \in 2\mathbb{N}^*$ follows by letting $M \rightarrow +\infty$ and using the monotone convergence theorem. This result extends readily to the general case $p \in [1, +\infty)$ by using for instance the Jensen inequality. \square

Corollary 2.11. *Under Assumptions 1.1 and 1.3, let ν_N be an invariant measure for the solution $(\mathbf{U}(t))_{t \geq 0}$ of (11) and let \mathbf{V} be a random variable with distribution ν_N . Then, for all $p \in 2\mathbb{N}^*$, \mathbf{V} satisfies*

$$\mathbb{E} [\langle \mathbf{D}_N^{(1)}(\mathbf{V}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{V} \rangle] \leq \frac{D_0(p-1)}{2\nu} C^{0,p-2},$$

where we set $C^{0,0} = 1$.

Proof. Let $(\mathbf{U}(t))_{t \geq 0}$ be a solution of (11) whose initial condition \mathbf{U}_0 has distribution ν_N . According to Proposition 2.10, one has $\mathbb{E}[\|\mathbf{U}_0\|_p^p] < +\infty$. Thus, one can apply Lemma 2.7.(i) which in the stationary case, writes

$$\nu_N \mathbb{E} \left[\left\langle \mathbf{D}_N^{(1)}(\mathbf{V}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{V} \right\rangle \right] \leq D_0 \frac{p(p-1)}{2} \mathbb{E} \left[\|\mathbf{V}\|_{p-2}^{p-2} \right],$$

and it remains to apply Proposition 2.10 to conclude. \square

We now turn to the proof of uniqueness of the invariant measure ν_N . We will first need some preliminary results:

Lemma 2.12. *Let Assumptions 1.1 and 1.3 hold and let $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ be two strong solutions of (11) driven by the same Wiener process. Then, for all $M > 0$ and all $\varepsilon > 0$, there exists $t_{\varepsilon, M} > 0$ such that*

$$p_{\varepsilon, M} := \inf_{\|\mathbf{u}_0\|_2 \vee \|\mathbf{v}_0\|_2 \leq M} \mathbb{P}_{(\mathbf{u}_0, \mathbf{v}_0)} (\|\mathbf{U}(t_{\varepsilon, M})\|_1 + \|\mathbf{V}(t_{\varepsilon, M})\|_1 \leq \varepsilon) > 0.$$

Proof. We recall that $\mathbf{b} : \mathbb{R}_0^N \rightarrow \mathbb{R}_0^N$ is locally Lipschitz continuous (for every norm over \mathbb{R}_0^N). Let $M > 0$ and $\varepsilon > 0$. Let us also fix the deterministic values $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$ satisfying $\|\mathbf{u}_0\|_2 \vee \|\mathbf{v}_0\|_2 \leq M$, along with the following constants:

$$t_{\varepsilon, M} := -\frac{1}{2\nu} \log \frac{\varepsilon^2}{16M^2};$$

$$L_{M+\varepsilon} := \text{Lipschitz constant of } b \text{ over the ball } \{\mathbf{u} \in \mathbb{R}_0^N : \|\mathbf{u}\|_1 \leq M + \varepsilon\};$$

$$\delta_\varepsilon := \frac{\varepsilon}{4} e^{-L_{M+\varepsilon} t_{\varepsilon, M}}.$$

Let $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ denote two solutions of (11) with the initial conditions \mathbf{u}_0 and \mathbf{v}_0 . We introduce the stopping times

$$\tilde{\tau}^{\mathbf{U}} := \inf \{t \geq 0 : \|\mathbf{U}(t)\|_1 \geq M + \varepsilon\};$$

$$\tilde{\tau}^{\mathbf{V}} := \inf \{t \geq 0 : \|\mathbf{V}(t)\|_1 \geq M + \varepsilon\}.$$

Furthermore, we denote by $(\mathbf{u}(t))_{t \geq 0}$ and $(\mathbf{v}(t))_{t \geq 0}$ the noiseless counterparts of $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$:

$$\frac{d}{dt} \mathbf{u}(t) = \mathbf{b}(\mathbf{u}(t)), \quad \frac{d}{dt} \mathbf{v}(t) = \mathbf{b}(\mathbf{v}(t)),$$

with respective initial conditions \mathbf{u}_0 and \mathbf{v}_0 .

By the dissipativity property (Lemma 2.4.(ii)) and Lemma 2.1, we have

$$\begin{aligned} \frac{d}{dt} (\|\mathbf{u}(t)\|_2^2 + \|\mathbf{v}(t)\|_2^2) &= 2(\langle \mathbf{u}(t), \mathbf{b}(\mathbf{u}(t)) \rangle + \langle \mathbf{v}(t), \mathbf{b}(\mathbf{v}(t)) \rangle) \\ &\leq -2\nu \left(\left\| \mathbf{D}_N^{(1)} \mathbf{u}(t) \right\|_2^2 + \left\| \mathbf{D}_N^{(1)} \mathbf{v}(t) \right\|_2^2 \right) \\ &\leq -2\nu (\|\mathbf{u}(t)\|_2^2 + \|\mathbf{v}(t)\|_2^2), \end{aligned}$$

so that Grönwall's lemma yields the upper bound

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{v}(t)\|_2^2 \leq (\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2) e^{-2\nu t},$$

meaning that for $t \geq t_{\varepsilon, M}$, we have

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{v}(t)\|_2^2 \leq \frac{\varepsilon^2}{8},$$

and consequently, by (10),

$$\|\mathbf{u}(t)\|_1 + \|\mathbf{v}(t)\|_1 \leq \|\mathbf{u}(t)\|_2 + \|\mathbf{v}(t)\|_2 \leq \frac{\varepsilon}{2}.$$

We now restrict ourselves to the situation where

$$\omega \in \left\{ \sup_{t \in [0, t_{\varepsilon, M}]} \|\mathbf{W}^{Q, N}(t)\|_1 \leq \delta_\varepsilon \right\}.$$

For any $t \leq \tilde{\tau}^{\mathbf{U}} \wedge \tilde{\tau}^{\mathbf{V}} \wedge t_{\varepsilon, M}$, the four vectors $\mathbf{U}(t)$, $\mathbf{V}(t)$, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ stay in the ball $\{\|\cdot\|_1 \leq M + \varepsilon\}$, and thanks to the local Lipschitz continuity assumption on \mathbf{b} we have

$$\begin{aligned} \|\mathbf{U}(t) - \mathbf{u}(t)\|_1 + \|\mathbf{V}(t) - \mathbf{v}(t)\|_1 &= \left\| \int_0^t (\mathbf{b}(\mathbf{U}(s)) - \mathbf{b}(\mathbf{u}(s))) ds + \mathbf{W}^{Q, N}(t) \right\|_1 + \left\| \int_0^t (\mathbf{b}(\mathbf{V}(s)) - \mathbf{b}(\mathbf{v}(s))) ds + \mathbf{W}^{Q, N}(t) \right\|_1 \\ &\leq \int_0^t (\|\mathbf{b}(\mathbf{U}(s)) - \mathbf{b}(\mathbf{u}(s))\|_1 + \|\mathbf{b}(\mathbf{V}(s)) - \mathbf{b}(\mathbf{v}(s))\|_1) ds + 2 \|\mathbf{W}^{Q, N}(t)\|_1 \\ &\leq L_{M+\varepsilon} \int_0^t (\|\mathbf{U}(s) - \mathbf{u}(s)\|_1 + \|\mathbf{V}(s) - \mathbf{v}(s)\|_1) ds + 2\delta_\varepsilon, \end{aligned}$$

so by Grönwall's lemma, we have

$$\|\mathbf{U}(t) - \mathbf{u}(t)\|_1 + \|\mathbf{V}(t) - \mathbf{v}(t)\|_1 \leq 2\delta_\varepsilon e^{L_{M+\varepsilon}t} \leq 2\delta_\varepsilon e^{L_{M+\varepsilon}t_{\varepsilon,M}} = \frac{\varepsilon}{2}, \quad (28)$$

for every $t \in [0, \tilde{\tau}^{\mathbf{U}} \wedge \tilde{\tau}^{\mathbf{V}} \wedge t_{\varepsilon,M}]$. But it appears that the case $\tilde{\tau}^{\mathbf{U}} \wedge \tilde{\tau}^{\mathbf{V}} < t_{\varepsilon,M}$ is impossible for small values of ε . Indeed, it would either imply $\|(\mathbf{U} - \mathbf{u})(\tilde{\tau}^{\mathbf{U}})\|_1 \leq \varepsilon/2$ or $\|(\mathbf{V} - \mathbf{v})(\tilde{\tau}^{\mathbf{V}})\|_1 \leq \varepsilon/2$ which is impossible because we have on the one hand

$$\|\mathbf{U}(\tilde{\tau}^{\mathbf{U}})\|_1 \geq M + \varepsilon \quad (\text{or } \|\mathbf{V}(\tilde{\tau}^{\mathbf{V}})\|_1 \geq M + \varepsilon),$$

and on the other hand

$$\|\mathbf{u}(\tilde{\tau}^{\mathbf{U}})\|_1 \leq \|\mathbf{u}(\tilde{\tau}^{\mathbf{U}})\|_2 \leq \|\mathbf{u}_0\|_2 \leq M \quad (\text{or } \|\mathbf{v}(\tilde{\tau}^{\mathbf{V}})\|_1 \leq M).$$

Therefore, Inequality (28) holds for all $t \in [0, t_{\varepsilon,M}]$. Thus,

$$\|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \|\mathbf{U}(t_{\varepsilon,M}) - \mathbf{u}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M}) - \mathbf{v}(t_{\varepsilon,M})\|_1 + \|\mathbf{u}(t_{\varepsilon,M})\|_1 + \|\mathbf{v}(t_{\varepsilon,M})\|_1 \leq \varepsilon,$$

and we have just shown that

$$\left\{ \sup_{t \in [0, t_{\varepsilon,M}]} \|\mathbf{W}^{Q,N}(t)\|_1 \leq \delta_\varepsilon \right\} \subset \{ \|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \varepsilon \}.$$

and therefore,

$$\mathbb{P}_{(\mathbf{u}_0, \mathbf{v}_0)} (\|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \varepsilon) \geq \mathbb{P} \left(\sup_{t \in [0, t_{\varepsilon,M}]} \|\mathbf{W}^{Q,N}(t)\|_1 \leq \delta_\varepsilon \right).$$

Notice that the right-hand side does not depend on \mathbf{u}_0 nor \mathbf{v}_0 . Furthermore, it is positive since $\mathbf{W}^{Q,N}$ is an \mathbb{R}^N -valued Wiener process. Hence, taking the infimum over \mathbf{u}_0 and \mathbf{v}_0 on the left-hand side yields the wanted result. \square

For two solutions $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ of (11) driven by the same Wiener process $\mathbf{W}^{Q,N}$, we define the following entrance time for all $M \geq 0$:

$$\tau_M := \inf \{ t \geq 0 : \|\mathbf{U}(t)\|_2 \vee \|\mathbf{V}(t)\|_2 \leq M \}. \quad (29)$$

Lemma 2.13. *Under Assumptions 1.1 and 1.3, there exists $M > 0$ such that for any deterministic initial conditions $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$, $\tau_M < +\infty$ almost surely.*

Proof. From Itô's formula, we have for all $t \geq 0$,

$$\begin{aligned} \|\mathbf{U}(\tau_M \wedge t)\|_2^2 + \|\mathbf{V}(\tau_M \wedge t)\|_2^2 &= \|\mathbf{U}_0\|_2^2 + \|\mathbf{V}_0\|_2^2 + \int_0^{\tau_M \wedge t} \langle \mathbf{b}(\mathbf{U}(s)), \mathbf{U}(s) \rangle ds + \int_0^{\tau_M \wedge t} \langle \mathbf{b}(\mathbf{V}(s)), \mathbf{V}(s) \rangle ds \\ &\quad + \sum_{k \geq 1} \int_0^{\tau_M \wedge t} \langle \mathbf{U}(s) + \mathbf{V}(s), \boldsymbol{\sigma}^k \rangle dW^k(s) + 2 \sum_{k \geq 1} \int_0^{\tau_M \wedge t} \|\boldsymbol{\sigma}^k\|_2^2 ds. \end{aligned} \quad (30)$$

The fifth term of the right-hand side is a martingale. Indeed, by the Cauchy-Schwarz inequality, Inequality (9), and the bound (22), we have

$$\begin{aligned} \mathbb{E} \left[\sum_{k \geq 1} \int_0^{\tau_M \wedge t} |\langle \mathbf{U}(s) + \mathbf{V}(s), \boldsymbol{\sigma}^k \rangle|^2 ds \right] &\leq \left(\sum_{k \geq 1} \|\boldsymbol{\sigma}^k\|_2^2 \right) \mathbb{E} \left[\int_0^t \|\mathbf{U}(s) + \mathbf{V}(s)\|_2^2 ds \right] \\ &\leq 2D_0 \left(\mathbb{E} \left[\int_0^t \|\mathbf{U}(s)\|_2^2 ds \right] + \mathbb{E} \left[\int_0^t \|\mathbf{V}(s)\|_2^2 ds \right] \right) \\ &\leq 2D_0 \left(2\theta_0^{(2)} + \theta_1^{(2)} \left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 \right) + 2\theta_2^{(2)} t \right) \\ &< +\infty. \end{aligned}$$

Thus, taking the expectation in (30), applying Lemma 2.4.(ii), Inequality (9), Lemma 2.1 and (29), we get

$$\begin{aligned}
& \mathbb{E} \left[\|\mathbf{U}(\tau_M \wedge t)\|_2^2 + \|\mathbf{V}(\tau_M \wedge t)\|_2^2 \right] \\
&= \|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 + 2\mathbb{E} \left[\int_0^{\tau_M \wedge t} (\langle \mathbf{b}(\mathbf{U}(s)), \mathbf{U}(s) \rangle + \langle \mathbf{b}(\mathbf{V}(s)), \mathbf{V}(s) \rangle) ds \right] + 2\mathbb{E} \left[\int_0^{\tau_M \wedge t} \sum_{k \geq 1} \|\boldsymbol{\sigma}^k\|_2^2 ds \right] \\
&\leq \|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 - 2\nu \mathbb{E} \left[\int_0^{\tau_M \wedge t} \left(\|\mathbf{D}_N^{(1)} \mathbf{U}(s)\|_2^2 + \|\mathbf{D}_N^{(1)} \mathbf{V}(s)\|_2^2 \right) ds \right] + 2\mathbb{E}[\tau_M \wedge t] D_0 \\
&\leq \|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 - 2\nu \mathbb{E} \left[\int_0^{\tau_M \wedge t} (\|\mathbf{U}(s)\|_2^2 + \|\mathbf{V}(s)\|_2^2) ds \right] + 2\mathbb{E}[\tau_M \wedge t] D_0 \\
&\leq \|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 + 2(D_0 - \nu M^2) \mathbb{E}[\tau_M \wedge t]
\end{aligned}$$

So if we choose $M > \sqrt{D_0/\nu}$, we get

$$\mathbb{E}[\tau_M \wedge t] \leq \frac{\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2}{2(\nu M^2 - D_0)},$$

so that we can apply the monotone convergence theorem:

$$\mathbb{E}[\tau_M] = \lim_{t \rightarrow \infty} \mathbb{E}[\tau_M \wedge t] < +\infty,$$

which concludes the proof. \square

Proof of Theorem 1.5, Assertion (i). We start to fix $\varepsilon > 0$ to which we associate the quantities $t_{\varepsilon, M}$ and $p_{\varepsilon, M}$ defined at Lemma 2.12, where M has been defined at Lemma 2.13. Let $(\mathbf{U}(t))_{t \geq 0}$ and $(\mathbf{V}(t))_{t \geq 0}$ start respectively from arbitrary deterministic initial conditions \mathbf{u}_0 and \mathbf{v}_0 and be driven by the same Wiener process. We define the increasing stopping time sequence

$$\begin{aligned}
T_1 &:= \tau_M \\
T_2 &:= \inf \{t \geq T_1 + t_{\varepsilon, M} : \|\mathbf{U}(t)\|_2 \vee \|\mathbf{V}(t)\|_2 \leq M\} \\
T_3 &:= \inf \{t \geq T_2 + t_{\varepsilon, M} : \|\mathbf{U}(t)\|_2 \vee \|\mathbf{V}(t)\|_2 \leq M\} \\
&\vdots
\end{aligned}$$

By the strong Markov property and Lemma 2.13, each term of this sequence is finite almost surely. We claim that

$$\forall J \in \mathbb{N}^*, \quad \mathbb{P}(\forall j = 1, \dots, J, \quad \|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon) \leq (1 - p_{\varepsilon, M})^J. \quad (31)$$

Indeed, it is true for $J = 1$ thanks to the strong Markov property and Lemma 2.12:

$$\mathbb{P}(\|\mathbf{U}(\tau_M + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(\tau_M + t_{\varepsilon, M})\|_1 > \varepsilon) = \mathbb{E}[\mathbb{P}(\|\mathbf{U}(\tau_M + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(\tau_M + t_{\varepsilon, M})\|_1 > \varepsilon | \mathcal{F}_{\tau_M})] \leq 1 - p_{\varepsilon, M},$$

and the general case follows by induction: assuming that Inequality (31) is true for some $J \in \mathbb{N}^*$, we have

$$\begin{aligned}
& \mathbb{P}(\forall j = 1, \dots, J+1, \|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon) \\
&= \mathbb{E}[\mathbb{P}(\forall j = 1, \dots, J+1, \|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon | \mathcal{F}_{T_{J+1}})] \\
&= \mathbb{E} \left[\left(\prod_{j=1}^J \mathbf{1}_{\|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon} \right) \mathbb{P}(\|\mathbf{U}(T_{J+1} + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_{J+1} + t_{\varepsilon, M})\|_1 > \varepsilon | \mathcal{F}_{T_{J+1}}) \right] \\
&\leq (1 - p_{\varepsilon, M})^J \times (1 - p_{\varepsilon, M}) = (1 - p_{\varepsilon, M})^{J+1}.
\end{aligned}$$

Letting $J \rightarrow +\infty$, we get

$$\begin{aligned}
\mathbb{P}(\forall j \in \mathbb{N}^*, \quad \|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon) &= \lim_{J \rightarrow \infty} \mathbb{P}(\forall j = 1, \dots, J, \quad \|\mathbf{U}(T_j + t_{\varepsilon, M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon, M})\|_1 > \varepsilon) \\
&\leq \lim_{J \rightarrow \infty} (1 - p_{\varepsilon, M})^J = 0,
\end{aligned}$$

and consequently,

$$\mathbb{P}(\exists t \geq 0, \quad \|\mathbf{U}(t)\|_1 + \|\mathbf{V}(t)\|_1 \leq \varepsilon) = 1,$$

meaning that almost surely,

$$\exists t \geq 0, \quad \|\mathbf{U}(t) - \mathbf{V}(t)\|_1 \leq \varepsilon.$$

Now recall that thanks to Proposition 2.8, $\|\mathbf{U}(t) - \mathbf{V}(t)\|_1$ is non-increasing in time almost surely. Since ε has been chosen arbitrarily, the above inequality actually indicates that $\|\mathbf{U}(t) - \mathbf{V}(t)\|_1$ converges almost surely to 0 as $t \rightarrow +\infty$ when the initial conditions are deterministic. However, this almost sure convergence extends naturally to random and

\mathcal{F}_0 -measurable initial conditions using the same argument as for (20). Let $\phi : \mathbb{R}_0^N \rightarrow \mathbb{R}$ be a Lipschitz continuous and bounded test function, with Lipschitz constant L_ϕ . We have in particular, almost surely,

$$\lim_{t \rightarrow \infty} |\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))| \leq L_\phi \lim_{t \rightarrow \infty} \|\mathbf{U}(t) - \mathbf{V}(t)\|_1 = 0. \quad (32)$$

To conclude the proof, assume that there exist two invariant measures $\nu_N^{(1)}$ and $\nu_N^{(2)}$ for the solution of (11), and take random initial conditions \mathbf{U}_0 and \mathbf{V}_0 with distributions $\nu_N^{(1)}$ and $\nu_N^{(2)}$ respectively. We have for all $t \geq 0$,

$$|\mathbb{E}[\phi(\mathbf{U}_0)] - \mathbb{E}[\phi(\mathbf{V}_0)]| = |\mathbb{E}[\phi(\mathbf{U}(t))] - \mathbb{E}[\phi(\mathbf{V}(t))]| \leq \mathbb{E}[|\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))|].$$

Letting t go to $+\infty$, by (32) and the dominated convergence theorem, we have

$$|\mathbb{E}[\phi(\mathbf{U}_0)] - \mathbb{E}[\phi(\mathbf{V}_0)]| \leq \lim_{t \rightarrow \infty} \mathbb{E}[|\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))|] = 0.$$

As a consequence, \mathbf{U}_0 and \mathbf{V}_0 have the same distribution, meaning that $\nu_N^{(1)} = \nu_N^{(2)}$. \square

2.3. Invariant measure for the split-step scheme. In this subsection, we aim to prove the existence and uniqueness of an invariant measure for the discrete time process $(\mathbf{U}_n)_{n \in \mathbb{N}}$ defined by (15). The general argument is the same as the one used in Subsection 2.2 for the semi-discrete case and the intermediary results are stated in the same order. Therefore, the proofs which are not affected by the time discretisation are omitted.

As the time step Δt is meant to converge towards 0, we may consider that it will always lie in an interval $(0, \Delta t_{\max}]$ for some arbitrarily chosen $\Delta t_{\max} > 0$.

The following preliminary result ensures that the scheme (15) is well-posed.

Proposition 2.14. *Under Assumption 1.3, given $\Delta t > 0$ and $\mathbf{v} \in \mathbb{R}_0^N$, there exists a unique $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t \mathbf{b}(\mathbf{w})$.*

Proof. Uniqueness. It is a straightforward consequence of Lemma 2.4.(i): if \mathbf{w}_1 and \mathbf{w}_2 are two solutions, then

$$\begin{aligned} \|\mathbf{w}_1 - \mathbf{w}_2\|_1 &= \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle \\ &= \Delta t \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{b}(\mathbf{w}_1) - \mathbf{b}(\mathbf{w}_2) \rangle \leq 0. \end{aligned}$$

Existence. The mapping $\mathbf{Id} - \Delta t \mathbf{b} : \mathbb{R}_0^N \rightarrow \mathbb{R}_0^N$ is continuous. Furthermore, by Lemmas 2.4.(ii) and 2.1, we have for all $\mathbf{w} \in \mathbb{R}_0^N$,

$$\begin{aligned} \frac{\langle (\mathbf{Id} - \Delta t \mathbf{b})(\mathbf{w}), \mathbf{w} \rangle}{\|\mathbf{w}\|_2} &= \|\mathbf{w}\|_2 - \Delta t \frac{\langle \mathbf{b}(\mathbf{w}), \mathbf{w} \rangle}{\|\mathbf{w}\|_2} \\ &\geq \|\mathbf{w}\|_2 + \nu \Delta t \frac{\|\mathbf{D}_N^{(1)} \mathbf{w}\|_2^2}{\|\mathbf{w}\|_2} \geq (1 + \nu \Delta t) \|\mathbf{w}\|_2. \end{aligned}$$

Thus, as a consequence of [16, Theorem 3.3], $\mathbf{Id} - \Delta t \mathbf{b}$ is surjective in \mathbb{R}_0^N and, for any $\mathbf{v} \in \mathbb{R}_0^N$, there exists $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t \mathbf{b}(\mathbf{w})$. \square

Lemma 2.15 (L^1 -contraction). *Let Assumptions 1.1 and 1.3 hold and let $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ be two solutions of (15) (driven by the same Wiener process $\mathbf{W}^{Q,N}$). Then, almost surely and for any $n \in \mathbb{N}$,*

$$\|\mathbf{U}_{n+1} - \mathbf{V}_{n+1}\|_1 \leq \|\mathbf{U}_n - \mathbf{V}_n\|_1.$$

Proof. From Equations (15) and Lemma 2.4.(ii), we write

$$\begin{aligned} \|\mathbf{U}_{n+1} - \mathbf{V}_{n+1}\|_1 &= \left\| \mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right\|_1 \\ &= \left\langle \mathbf{sign} \left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right), \mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right\rangle \\ &= \left\langle \mathbf{sign} \left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right), \mathbf{U}_n - \mathbf{V}_n \right\rangle + \Delta t \left\langle \mathbf{sign} \left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right), \mathbf{b} \left(\mathbf{U}_{n+\frac{1}{2}} \right) - \mathbf{b} \left(\mathbf{V}_{n+\frac{1}{2}} \right) \right\rangle \\ &\leq \left\langle \mathbf{sign} \left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}} \right), \mathbf{U}_n - \mathbf{V}_n \right\rangle \\ &\leq \|\mathbf{U}_n - \mathbf{V}_n\|_1. \end{aligned} \quad \square$$

Remark 2.16. Note that the choice of the split-step backward Euler scheme is essential for the L^1 -contraction property to hold. Indeed, consider for instance two processes $(\tilde{\mathbf{U}}_n)_{n \in \mathbb{N}}$ and $(\tilde{\mathbf{V}}_n)_{n \in \mathbb{N}}$ built via a explicit Euler method, that is,

$$\tilde{\mathbf{U}}_{n+1} = \tilde{\mathbf{U}}_n + \Delta t \mathbf{b}(\tilde{\mathbf{U}}_n) + \Delta \mathbf{W}_{n+1}^{Q,N}$$

(and naturally, the same construction for $(\tilde{\mathbf{V}}_n)_{n \in \mathbb{N}}$), then the expansion of the L^1 distance gives

$$\left\| \tilde{\mathbf{U}}_{n+1} - \tilde{\mathbf{V}}_{n+1} \right\|_1 = \mathbf{sign} \left\langle \left(\tilde{\mathbf{U}}_{n+1} - \tilde{\mathbf{V}}_{n+1} \right), \tilde{\mathbf{U}}_n - \tilde{\mathbf{V}}_n \right\rangle + \Delta t \left\langle \mathbf{sign} \left(\tilde{\mathbf{U}}_{n+1} - \tilde{\mathbf{V}}_{n+1} \right), \mathbf{b} \left(\tilde{\mathbf{U}}_n \right) - \mathbf{b} \left(\tilde{\mathbf{V}}_n \right) \right\rangle.$$

Thus, we would need to control the second term of the right-hand side in the above equation, which is delicate given that \mathbf{b} is not globally Lipschitz.

As for the semi-discrete scheme, Lemma 2.15 induce the following property:

Corollary 2.17. *Under Assumptions 1.1 and 1.3, the solution $(\mathbf{U}_n)_{n \in \mathbb{N}}$ of (15) is a Feller process.*

Proposition 2.18. *Under Assumptions 1.1 and 1.3, for any time step $\Delta t > 0$, the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$ solution of the split-step backward Euler method (15) admits an invariant measure $\nu_{N, \Delta t}$. Moreover, if \mathbf{V} is a random variable with distribution $\nu_{N, \Delta t}$, then*

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V} \right\|_2^2 \right] \leq D_0 \left(\frac{1}{2\nu} + \Delta t_{\max} \right) =: \bar{C}^{1,2} \quad (33)$$

and

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V}_{\frac{1}{2}} \right\|_2^2 \right] \leq \frac{D_0}{2\nu}, \quad (34)$$

where $\mathbf{V}_{\frac{1}{2}}$ denotes the solution of $\mathbf{V}_{\frac{1}{2}} = \mathbf{V} + \Delta t \mathbf{b}(\mathbf{V}_{\frac{1}{2}})$.

Proof. Let $\mathbf{u}_0 \in \mathbb{R}_0^N$ be the deterministic initial condition of the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$. Starting from the first equation in (15), we have

$$\left\| \mathbf{U}_{n+\frac{1}{2}} - \Delta t \mathbf{b} \left(\mathbf{U}_{n+\frac{1}{2}} \right) \right\|_2^2 = \left\| \mathbf{U}_n \right\|_2^2,$$

by expanding the left-hand side, we derive the inequality

$$\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 \leq \left\| \mathbf{U}_n \right\|_2^2 + 2\Delta t \left\langle \mathbf{b} \left(\mathbf{U}_{n+\frac{1}{2}} \right), \mathbf{U}_{n+\frac{1}{2}} \right\rangle.$$

Using the dissipativity inequality (Lemma 2.4. (ii)), we get

$$\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 \leq \left\| \mathbf{U}_n \right\|_2^2 - 2\nu \Delta t \left\| \mathbf{D}_N^{(1)} \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2. \quad (35)$$

Now, from the second equation in (15), we have

$$\left\| \mathbf{U}_{n+1} \right\|_2^2 = \left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 + 2 \left\langle \mathbf{U}_{n+\frac{1}{2}}, \Delta \mathbf{W}_{n+1}^{Q,N} \right\rangle + \left\| \Delta \mathbf{W}_{n+1}^{Q,N} \right\|_2^2. \quad (36)$$

Injecting Inequality (35) into Equation (36), we get

$$\left\| \mathbf{U}_{n+1} \right\|_2^2 - \left\| \mathbf{U}_n \right\|_2^2 \leq -2\nu \Delta t \left\| \mathbf{D}_N^{(1)} \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 + 2 \left\langle \mathbf{U}_{n+\frac{1}{2}}, \Delta \mathbf{W}_{n+1}^{Q,N} \right\rangle + \left\| \Delta \mathbf{W}_{n+1}^{Q,N} \right\|_2^2. \quad (37)$$

By definition of $\mathbf{W}^{Q,N}$ and from (9), we have

$$\mathbb{E} \left[\left\| \Delta \mathbf{W}_{n+1}^{Q,N} \right\|_2^2 \right] = \frac{1}{N} \Delta t \sum_{i=1}^N \sum_{k \geq 1} (\sigma_i^k)^2 \leq D_0 \Delta t. \quad (38)$$

On the other hand, the variables $\mathbf{U}_{n+\frac{1}{2}}$ and $\Delta \mathbf{W}_{n+1}^{Q,N}$ are independent, so that taking the expectation in (37) yields

$$\mathbb{E} \left[\left\| \mathbf{U}_{n+1} \right\|_2^2 \right] - \mathbb{E} \left[\left\| \mathbf{U}_n \right\|_2^2 \right] \leq -2\nu \Delta t \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 \right] + D_0 \Delta t,$$

which is valid for any $n \in \mathbb{N}$, so that we can get a telescopic sum:

$$\begin{aligned} \mathbb{E} \left[\left\| \mathbf{U}_n \right\|_2^2 \right] - \left\| \mathbf{u}_0 \right\|_2^2 &= \sum_{l=0}^{n-1} \left(\mathbb{E} \left[\left\| \mathbf{U}_{l+1} \right\|_2^2 \right] - \mathbb{E} \left[\left\| \mathbf{U}_l \right\|_2^2 \right] \right) \\ &\leq -2\nu \Delta t \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] + n \Delta t D_0. \end{aligned} \quad (39)$$

Hence,

$$2\nu \Delta t \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] \leq \left\| \mathbf{u}_0 \right\|_2^2 + n \Delta t D_0. \quad (40)$$

Besides,

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+1} \right\|_2^2 \right] = \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] + \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \Delta \mathbf{W}_{l+1}^{Q,N} \right\|_2^2 \right], \quad (41)$$

and

$$\begin{aligned}
\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \Delta \mathbf{W}_{l+1}^{Q,N} \right\|_2^2 \right] &= N \mathbb{E} \left[\sum_{i=1}^N \left(\sum_{k \geq 1} (\sigma_{i+1}^k - \sigma_i^k) (W^k((l+1)\Delta t) - W^k(l\Delta t)) \right)^2 \right] \\
&= N \sum_{i=1}^N \sum_{k \geq 1} (\sigma_{i+1}^k - \sigma_i^k)^2 \mathbb{E} \left[(W^k((l+1)\Delta t) - W^k(l\Delta t))^2 \right] \\
&= \Delta t \sum_{k \geq 1} \left\| \mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k \right\|_2^2.
\end{aligned}$$

Now, from the definition of $\boldsymbol{\sigma}^k$, the Jensen inequality and (5), we have

$$\begin{aligned}
\left\| \mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k \right\|_2^2 &= N \sum_{i=1}^N \sum_{k \geq 1} \left(N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(g_k \left(x + \frac{1}{N} \right) - g_k(x) \right) dx \right)^2 \\
&\leq N^2 \sum_{i=1}^N \sum_{k \geq 1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(g_k \left(x + \frac{1}{N} \right) - g_k(x) \right)^2 dx \\
&= N^2 \sum_{i=1}^N \sum_{k \geq 1} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\int_x^{x+\frac{1}{N}} \partial_x g_k(y) dy \right)^2 dx \\
&\leq N^2 \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{1}{N} \int_x^{x+\frac{1}{N}} \sum_{k \geq 1} \partial_x g_k(y)^2 dy dx \\
&\leq D_0.
\end{aligned} \tag{42}$$

Thus, we have

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \Delta \mathbf{W}_{l+1}^{Q,N} \right\|_2^2 \right] \leq \Delta t D_0. \tag{43}$$

Injecting (43) into (41), and (41) into (40), we get

$$\frac{1}{n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+1} \right\|_2^2 \right] \leq \frac{1}{2n\nu\Delta t} \|\mathbf{u}_0\|_2^2 + \frac{D_0}{2\nu} + \Delta t D_0. \tag{44}$$

Since $\|\mathbf{D}_N^{(1)} \cdot\|_2$ defines a norm on \mathbb{R}_0^N and since from Corollary 2.17, the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$ is Feller, the result follows from Markov's inequality and the Krylov-Bogoliubov theorem [15, Theorem 3.1.1].

Using the same arguments as for the end of the proof of Proposition 2.10, Inequalities (44) and (40) yield respectively (33) and (34). \square

We now proceed to the proof of uniqueness of the invariant measure $\nu_{N,\Delta t}$.

Lemma 2.19 (Hitting any neighbourhood of 0 with positive probability). *Let Assumptions 1.1 and 1.3 hold. Let $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ be two solutions of (15) driven by the same Wiener process $\mathbf{W}^{Q,N}$. For any $\varepsilon > 0$ and any $M > 0$, there exists $n_{\varepsilon,M} \in \mathbb{N}$ such that*

$$p_{\varepsilon,M} := \inf_{\|\mathbf{u}_0\|_2 \vee \|\mathbf{v}_0\|_2 \leq M} \mathbb{P}_{(\mathbf{u}_0, \mathbf{v}_0)} (\|\mathbf{U}_{n_{\varepsilon,M}}\|_1 + \|\mathbf{V}_{n_{\varepsilon,M}}\|_1 \leq \varepsilon) > 0.$$

Proof. First, let $\varepsilon > 0$ and let us fix $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$ such that $\|\mathbf{u}_0\|_2 \leq M$ and $\|\mathbf{v}_0\|_2 \leq M$.

Let $(\mathbf{u}_n)_{n \in \mathbb{N}}$ and $(\mathbf{v}_n)_{n \in \mathbb{N}}$ denote the noiseless counterparts of the processes $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$, i.e.

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \mathbf{b}(\mathbf{u}_{n+1}) \\ \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t \mathbf{b}(\mathbf{v}_{n+1}), \end{cases} \tag{45}$$

with initial conditions \mathbf{u}_0 and \mathbf{v}_0 . Then $(\mathbf{u}_n)_{n \in \mathbb{N}}$ and $(\mathbf{v}_n)_{n \in \mathbb{N}}$ are subject to non-perturbed dissipativity, and consequently the sum of their energies decreases to 0 over time. Indeed, we have

$$\begin{aligned}
\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2 &= \|\mathbf{u}_{n+1} - \Delta t \mathbf{b}(\mathbf{u}_{n+1})\|_2^2 + \|\mathbf{v}_{n+1} - \Delta t \mathbf{b}(\mathbf{v}_{n+1})\|_2^2 \\
&= \|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 + (\Delta t)^2 (\|\mathbf{b}(\mathbf{u}_{n+1})\|_2^2 + \|\mathbf{b}(\mathbf{v}_{n+1})\|_2^2) - 2\Delta t (\langle \mathbf{u}_{n+1}, \mathbf{b}(\mathbf{u}_{n+1}) \rangle + \langle \mathbf{v}_{n+1}, \mathbf{b}(\mathbf{v}_{n+1}) \rangle)
\end{aligned}$$

therefore, using successively Lemma 2.4.(ii) and Lemma 2.1,

$$\begin{aligned}\|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 - \left(\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2\right) &\leq 2\Delta t \left(\langle \mathbf{u}_{n+1}, \mathbf{b}(\mathbf{u}_{n+1}) \rangle + \langle \mathbf{v}_{n+1}, \mathbf{b}(\mathbf{v}_{n+1}) \rangle\right) \\ &\leq -2\Delta t \nu \left(\left\|\mathbf{D}_N^{(1)} \mathbf{u}_{n+1}\right\|_2^2 + \left\|\mathbf{D}_N^{(1)} \mathbf{v}_{n+1}\right\|_2^2\right) \\ &\leq -2\Delta t \nu \left(\|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2\right)\end{aligned}$$

so that

$$\|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 \leq \frac{1}{1 + 2\Delta t \nu} \left(\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2\right),$$

by induction, we get for all $n \in \mathbb{N}$,

$$\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2 \leq \left(\frac{1}{1 + 2\Delta t \nu}\right)^n \left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2\right).$$

It appears now that if we fix the value

$$n_{\varepsilon, M} := \left\lceil \frac{-1}{\log(1 + 2\Delta t \nu)} \log\left(\frac{\varepsilon^2}{16M^2}\right) \right\rceil,$$

we get for all $n \geq n_{\varepsilon, M}$,

$$\|\mathbf{u}_n\|_1 + \|\mathbf{v}_n\|_1 \leq \|\mathbf{u}_n\|_2 + \|\mathbf{v}_n\|_2 \leq \frac{\varepsilon}{2}.$$

Now, we fix $\delta_\varepsilon := \varepsilon/(4n_{\varepsilon, M})$ and we restrict ourselves to the event

$$\left\{ \sup_{n=1, \dots, n_{\varepsilon, M}} \|\Delta \mathbf{W}_n^{Q, N}\|_1 \leq \delta_\varepsilon \right\}. \quad (46)$$

Let $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ be two solutions of (15) with the deterministic initial conditions \mathbf{u}_0 and \mathbf{v}_0 respectively. With similar arguments as for the proof of Proposition 2.8, we get from (15), (45) and Lemma 2.4.(ii), for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\mathbf{U}_{n+1} - \mathbf{u}_{n+1}\|_1 + \|\mathbf{V}_{n+1} - \mathbf{v}_{n+1}\|_1 &\leq \left\|\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right\|_1 + \left\|\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right\|_1 + 2\left\|\Delta \mathbf{W}_{n+1}^{Q, N}\right\|_1 \\ &= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right), \mathbf{U}_n - \mathbf{u}_n \right\rangle \\ &\quad + \Delta t \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right), \mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right) - \mathbf{b}(\mathbf{u}_{n+1}) \right\rangle \\ &\quad + \left\langle \mathbf{sign}\left(\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right), \mathbf{V}_n - \mathbf{v}_n \right\rangle \\ &\quad + \Delta t \left\langle \mathbf{sign}\left(\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right), \mathbf{b}\left(\mathbf{V}_{n+\frac{1}{2}}\right) - \mathbf{b}(\mathbf{v}_{n+1}) \right\rangle + 2\left\|\Delta \mathbf{W}_{n+1}^{Q, N}\right\|_1 \\ &\leq \|\mathbf{U}_n - \mathbf{u}_n\|_1 + \|\mathbf{V}_n - \mathbf{v}_n\|_1 + 2\left\|\Delta \mathbf{W}_{n+1}^{Q, N}\right\|_1.\end{aligned}$$

On the event (46), we have for all $n = 1, \dots, n_{\varepsilon, M}$,

$$\|\mathbf{U}_{n+1} - \mathbf{u}_{n+1}\|_1 + \|\mathbf{V}_{n+1} - \mathbf{v}_{n+1}\|_1 \leq \|\mathbf{U}_n - \mathbf{u}_n\|_1 + \|\mathbf{V}_n - \mathbf{v}_n\|_1 + 2\delta_\varepsilon.$$

In particular, by induction, we have

$$\|\mathbf{U}_{n_{\varepsilon, M}} - \mathbf{u}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{V}_{n_{\varepsilon, M}} - \mathbf{v}_{n_{\varepsilon, M}}\|_1 \leq 2n_{\varepsilon, M}\delta_\varepsilon = \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}\|\mathbf{U}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{V}_{n_{\varepsilon, M}}\|_1 &\leq \|\mathbf{U}_{n_{\varepsilon, M}} - \mathbf{u}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{V}_{n_{\varepsilon, M}} - \mathbf{v}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{u}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{v}_{n_{\varepsilon, M}}\|_1 \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We just have shown that

$$\mathbb{P}_{(\mathbf{u}_0, \mathbf{v}_0)} \left(\|\mathbf{U}_{n_{\varepsilon, M}}\|_1 + \|\mathbf{V}_{n_{\varepsilon, M}}\|_1 \leq \varepsilon \right) \geq \mathbb{P} \left(\sup_{n=1, \dots, n_{\varepsilon, M}} \|\Delta \mathbf{W}_n^{Q, N}\|_1 \leq \delta_\varepsilon \right) > 0.$$

Since the event (46) does not depend on \mathbf{u}_0 nor \mathbf{v}_0 , we get the result. \square

For two solutions $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$ of (15) driven by the same Wiener process $\mathbf{W}^{Q,N}$, we define the entrance time

$$\eta_M := \inf \{n \in \mathbb{N} : \|\mathbf{U}_{n+1}\|_2 \vee \|\mathbf{V}_{n+1}\|_2 \leq M\}.$$

The following lemma is the time-discrete version of Lemma 2.13. The proof is omitted as it is very similar to its time-continuous counterpart.

Lemma 2.20 (Almost sure entrance in some ball). *Under Assumptions 1.1 and 1.3, there exists $M > 0$ such that for any initial conditions $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$ for the processes $(\mathbf{U}_n)_{n \in \mathbb{N}}$ and $(\mathbf{V}_n)_{n \in \mathbb{N}}$, $\eta_M < +\infty$ almost surely.*

Proof of Theorem 1, Assertion (ii). Given Lemmas 2.15, 2.19 and 2.20, the proof is done in exactly the same way as for Assertion (i).

Finally, the fact that ν_N and $\nu_{N,\Delta t}$ belong to $\mathcal{P}_2(\mathbb{R}_0^N)$ come from Propositions 2.10 and 2.18 respectively (for $\nu_{N,\Delta t}$, we use in particular the fact that $\|\mathbf{D}_N^{(1)} \cdot\|_2$ defines a norm on \mathbb{R}_0^N). \square

3. CONVERGENCE OF INVARIANT MEASURES: SEMI-DISCRETE SCHEME TOWARDS SPDE

The purpose of this section is to prove that $W_2(\mu_N^{(m)}, \mu) \rightarrow 0$, $N \rightarrow +\infty$, $m = 0, 1, 2$, which will be the first part of the proof of Theorem 1.7. In Subsection 3.1, we provide a result ensuring that it is sufficient that the convergence holds for only one $m \in \{0, 1, 2\}$, in which case it will hold for the three of them. Then, we show that $(\mu_N^{(m)})_{N \geq 1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$, and in Subsection 3.2, we present a procedure to identify any subsequential limit of $(\mu_N^{(m)})_{N \geq 1}$ as the invariant measure μ for the solution of (1), which leads to the proof the first assertion of Theorem 1.7. Subsection 3.3 contain the proofs of the lemmas from Subsections 3.1 and 3.2.

3.1. Notations and preliminary results. For $m = 0, 1, 2$, we define the *interpolation operators* $\Psi_N^{(m)} : \mathbb{R}_0^N \rightarrow W_0^{m,\infty}(\mathbb{T})$ by

$$\Psi_N^{(m)} \mathbf{v}(x) = \sum_{i=1}^N v_i \phi_N^{(m)} \left(x - \frac{i}{N} \right), \quad \mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}_0^N,$$

where

$$\phi_N^{(0)}(x) = \mathbf{1}_{(-\frac{1}{N}, 0]}(x),$$

$$\phi_N^{(1)}(x) = N \left(x + \frac{1}{N} \right) \mathbf{1}_{(-\frac{1}{N}, 0]}(x) + N \left(\frac{1}{N} - x \right) \mathbf{1}_{(0, \frac{1}{N}]}(x),$$

$$\phi_N^{(2)}(x) = \frac{N^2}{2} \left(x + \frac{1}{N} \right) \left(x + \frac{2}{N} \right) \mathbf{1}_{(-\frac{2}{N}, -\frac{1}{N}]}(x) - N^2 \left(x + \frac{1}{N} \right) \left(x - \frac{1}{N} \right) \mathbf{1}_{(-\frac{1}{N}, 0]}(x) + \frac{N^2}{2} \left(x - \frac{1}{N} \right) \left(x - \frac{2}{N} \right) \mathbf{1}_{(0, \frac{1}{N}]}(x),$$

so that $\Psi_N^{(0)} \mathbf{v}$, $\Psi_N^{(1)} \mathbf{v}$ and $\Psi_N^{(2)} \mathbf{v}$ are the respectively piecewise constant, linear and quadratic interpolations of the values v_i at the points i/N . In this regard, note that for $\mathbf{v} \in \mathbb{R}_0^N$, $i \in \mathbb{Z}/N\mathbb{Z}$ and $m = 0, 1, 2$, we have $\Psi_N^{(m)} \mathbf{v}(\frac{i}{N}) = v_i$.

We recall that these operators allowed to define the sequences of embedded invariant measures $\mu_N^{(m)} = \nu_N \circ (\Psi_N^{(m)})^{-1}$, where $\mu_N^{(m)}$ is here considered as an element of $\mathcal{P}(L_0^2(\mathbb{T}))$.

We prove the following lemma in the appendix:

Lemma 3.1. *The following properties hold:*

(i) *for $\mathbf{v} \in \mathbb{R}_0^N$, any $p \in [1, +\infty]$ and any $m = 0, 1, 2$,*

$$\left\| \Psi_N^{(m)} \mathbf{v} \right\|_{W_0^{m,p}(\mathbb{T})} = \left\| \mathbf{D}_N^{(m)} \mathbf{v} \right\|_p,$$

(ii) *for any $\mathbf{v} \in \mathbb{R}_0^N$,*

$$\left\| \Psi_N^{(1)} \mathbf{v} - \Psi_N^{(0)} \mathbf{v} \right\|_{L_0^2(\mathbb{T})}^2 = \frac{1}{3N^2} \left\| \mathbf{D}_N^{(1)} \mathbf{v} \right\|_2^2$$

and

$$\left\| \Psi_N^{(2)} \mathbf{v} - \Psi_N^{(0)} \mathbf{v} \right\|_{L_0^2(\mathbb{T})}^2 \leq \frac{3}{20N^4} \left\| \mathbf{D}_N^{(2)} \mathbf{v} \right\|_2^2 + \frac{1}{2N^2} \left\| \mathbf{D}_N^{(1)} \mathbf{v} \right\|_2^2.$$

The proof of the following Lemma is given below in Subsection 3.3.

Lemma 3.2 (Discrete H_0^1 and H_0^2 bounds). *Let Assumptions 1.1 and 1.3 hold and let \mathbf{V} be a random variable in \mathbb{R}_0^N with distribution ν_N , then \mathbf{V} satisfies*

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V} \right\|_2^2 \right] \leq \frac{D_0}{2\nu} =: C^{1,2}. \quad (47)$$

Furthermore, there exists a positive constant $C^{2,2}$ not depending on N such that,

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(2)} \mathbf{V} \right\|_2^2 \right] \leq C^{2,2}. \quad (48)$$

The following result ensures in particular that the estimates obtained at Lemma 3.2 and Proposition 2.10 remain true when passing to the limit $N \rightarrow +\infty$.

Lemma 3.3 (Relative compactness). *Under Assumptions 1.1 and 1.3, the three families of probability measures $(\mu_N^{(m)})_{N \geq 1}$, $m = 0, 1, 2$, are relatively compact in the space $\mathcal{P}_2(L_0^2(\mathbb{T}))$. Moreover, for any $m = 0, 1, 2$ and for any subsequential limit $\mu^* \in \mathcal{P}_2(L_0^2(\mathbb{T}))$ of $(\mu_N^{(m)})_{N \geq 1}$, a random variable $v \sim \mu^*$ satisfies*

$$\mathbb{E} \left[\|v\|_{H_0^1(\mathbb{T})}^2 \right] \leq C^{1,2}, \quad \mathbb{E} \left[\|v\|_{H_0^2(\mathbb{T})}^2 \right] \leq C^{2,2} \quad \text{and} \quad \mathbb{E} \left[\|v\|_{L_0^p(\mathbb{T})}^p \right] \leq C^{0,p}, \quad p \in [1, +\infty). \quad (49)$$

Proof. Step 1. Relative compactness of $(\mu_N^{(1)})_{N \geq 1}$ in $\mathcal{P}(L_0^2(\mathbb{T}))$. Let \mathbf{V} be an \mathbb{R}_0^N -valued random variable with distribution ν_N . Then, $\Psi_N^{(1)} \mathbf{V}$ has distribution $\mu_N^{(1)}$. Thanks to Lemmas 3.1 and 3.2, we have

$$\mathbb{E} \left[\left\| \Psi_N^{(1)} \mathbf{V} \right\|_{H_0^1(\mathbb{T})}^2 \right] = \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V} \right\|_2^2 \right] \leq C^{1,2}.$$

Thus, Markov's inequality implies

$$\forall \varepsilon > 0, \quad \mathbb{P} \left(\left\| \Psi_N^{(1)} \mathbf{V} \right\|_{H_0^1(\mathbb{T})}^2 > \frac{1}{\varepsilon} \right) \leq \varepsilon C^{1,2}.$$

The space $H_0^1(\mathbb{T})$ is compactly embedded in $L_0^2(\mathbb{T})$, so this last inequality means that the sequence $(\mu_N^{(1)})_{N \in \mathbb{N}^*}$ is tight in the space $\mathcal{P}(L_0^2(\mathbb{T}))$. As a consequence of Prokhorov's theorem [5, Theorem 5.1], any subsequence of $(\mu_N^{(1)})_{N \geq 1}$ admits itself a weakly converging subsequence in $\mathcal{P}(L_0^2(\mathbb{T}))$. In this respect, let $\mu^* \in \mathcal{P}(L_0^2(\mathbb{T}))$ be a subsequential limit of $(\mu_N^{(1)})_{N \geq 1}$ and let $(\mu_{N_j}^{(1)})_{j \in \mathbb{N}}$ be the associated subsequence.

Step 2. Relative compactness of $(\mu_N^{(0)})_{N \geq 1}$ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$. Let v be a random variable with distribution μ^* . On the one hand, the sequence of random variables $(\Psi_{N_j}^{(1)} \mathbf{V})_{j \in \mathbb{N}}$ converges in distribution towards v . On the other hand, by Lemmas 3.1.(ii) and 3.2, we have

$$\mathbb{E} \left[\left\| \Psi_{N_j}^{(0)} \mathbf{V} - \Psi_{N_j}^{(1)} \mathbf{V} \right\|_{L_0^2(\mathbb{T})}^2 \right] \leq \frac{1}{3N_j^2} \mathbb{E} \left[\left\| \mathbf{D}_{N_j}^{(1)} \mathbf{V} \right\|_2^2 \right] \leq \frac{C^{1,2}}{3N_j^2} \xrightarrow{j \rightarrow \infty} 0,$$

so that $\Psi_{N_j}^{(0)} \mathbf{V} - \Psi_{N_j}^{(1)} \mathbf{V}$ converges in probability towards 0 as $j \rightarrow +\infty$. As a consequence, by Slutsky's theorem [5, Theorem 3.9], the couple $(\Psi_{N_j}^{(1)} \mathbf{V}, \Psi_{N_j}^{(0)} \mathbf{V} - \Psi_{N_j}^{(1)} \mathbf{V})$ converges in distribution towards $(v, 0)$ as $j \rightarrow +\infty$. In particular, $\Psi_{N_j}^{(0)} \mathbf{V}$ converges in distribution towards v , which means that $\mu_{N_j}^{(0)}$ converges weakly in $\mathcal{P}(L_0^2(\mathbb{T}))$ towards μ^* . Moreover, $\mu_{N_j}^{(0)}$ has uniform moment bounds with respect to j thanks to (2), Lemma 3.1.(i) and Proposition 2.10:

$$\mathbb{E} \left[\left\| \Psi_{N_j}^{(0)} \mathbf{V} \right\|_{L_0^p(\mathbb{T})}^p \right] \leq \mathbb{E} \left[\left\| \Psi_{N_j}^{(0)} \mathbf{V} \right\|_{L_0^p(\mathbb{T})}^p \right] = \mathbb{E} \left[\left\| \mathbf{V} \right\|_p^p \right] \leq C^{0,p}, \quad \forall p \geq 2. \quad (50)$$

As a consequence, $(\mu_{N_j}^{(0)})_{j \in \mathbb{N}}$ satisfies a uniform integrability condition in the sense of [29, Definition 6.8.(iii)] and thus, is converging for the Wasserstein distance in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ towards μ^* .

Step 3. Relative compactness of $(\mu_{N_j}^{(1)})_{j \in \mathbb{N}}$ and $(\mu_{N_j}^{(2)})_{j \in \mathbb{N}}$ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$. The sequences $(\mu_{N_j}^{(1)})_{j \in \mathbb{N}}$ and $(\mu_{N_j}^{(2)})_{j \in \mathbb{N}}$ also converge towards μ^* in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ by Lemmas 3.1 and 3.2. Indeed, we have

$$W_2 \left(\mu_{N_j}^{(1)}, \mu_{N_j}^{(0)} \right)^2 \leq \mathbb{E} \left[\left\| \Psi_{N_j}^{(1)} \mathbf{V} - \Psi_{N_j}^{(0)} \mathbf{V} \right\|_{L_0^2(\mathbb{T})}^2 \right] \leq \frac{C^{1,2}}{3N_j^2}, \quad (51)$$

and

$$\begin{aligned} W_2 \left(\mu_{N_j}^{(2)}, \mu_{N_j}^{(0)} \right)^2 &\leq \mathbb{E} \left[\left\| \Psi_{N_j}^{(2)} \mathbf{V} - \Psi_{N_j}^{(0)} \mathbf{V} \right\|_{L_0^2(\mathbb{T})}^2 \right] \\ &\leq \mathbb{E} \left[\frac{3}{20N_j^4} \left\| \mathbf{D}_{N_j}^{(2)} \mathbf{V} \right\|_2^2 + \frac{1}{2N_j^2} \left\| \mathbf{D}_{N_j}^{(1)} \mathbf{V} \right\|_2^2 \right] \\ &\leq \frac{3C^{2,2}}{20N_j^4} + \frac{C^{1,2}}{2N_j^2}. \end{aligned} \quad (52)$$

Step 4. Moment estimates. Finally, the estimates (49) follow from Portemanteau's theorem: since $\|\cdot\|_{H_0^1(\mathbb{T})}^2$ is lower semi-continuous on the space $L_0^2(\mathbb{T})$, we have

$$\mathbb{E} \left[\|v\|_{H_0^1(\mathbb{T})}^2 \right] \leq \liminf_{j \rightarrow \infty} \mathbb{E} \left[\left\| \Psi_{N_j}^{(1)} \mathbf{V} \right\|_{H_0^1(\mathbb{T})}^2 \right] \leq C^{1,2},$$

and the same argument applies for $\|\cdot\|_{H_0^2(\mathbb{T})}^2$ and $\|\cdot\|_{L_0^p(\mathbb{T})}^p$ using respectively the sequences of random variables $(\Psi_{N_j}^{(2)} \mathbf{V})_{j \in \mathbb{N}}$ and $(\Psi_{N_j}^{(0)} \mathbf{V})_{j \in \mathbb{N}}$. \square

The three following lemmas will be useful for the proof of finite time convergence stated in the next subsection, namely Proposition 3.7. The proofs are given in Subsection 3.3.

Lemma 3.4 (Discrete $W_0^{1,3}$ bound). *Let Assumptions 1.1 and 1.3 hold and let \mathbf{V} be an \mathbb{R}_0^N -valued random variable distributed according to ν_N . Then, there exists a constant $C^{1,3}$ depending only on ν , $p_{\bar{A}}$ and D_0 such that*

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V} \right\|_3^3 \right] \leq C^{1,3}.$$

Lemma 3.5 (Discrete H_0^1 bound in finite time). *Let Assumptions 1.1 and 1.3 hold and let $(\mathbf{U}(t))_{t \geq 0}$ be the solution of (11) with an initial condition $\mathbf{U}_0 \sim \nu_N$. For every $T > 0$, there exists a constant $C_T^{1,2}$ not depending on N such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left\| \mathbf{D}_N^{(1)} \mathbf{U}(t) \right\|_2^2 \right] \leq C_T^{1,2}.$$

Lemma 3.6 (Moments on the solution of (1)). *Under Assumption 1.1, for all $p \in [2, +\infty)$ and $T > 0$, there are constants $\tilde{C}_T^{0,p}$ and $\tilde{C}_T^{1,2}$ such that the solution $(u(t))_{t \geq 0}$ of (1) with initial condition $u_0 \sim \mu^*$ satisfies for all $t \in [0, T]$:*

$$\sup_{t \in [0, T]} \mathbb{E} \left[\|u(t)\|_{L_0^p(\mathbb{T})}^p \right] \leq \tilde{C}_T^{0,p} \quad \text{and} \quad \sup_{t \in [0, T]} \mathbb{E} \left[\|u(t)\|_{H_0^1(\mathbb{T})}^2 \right] \leq \tilde{C}_T^{1,2}.$$

3.2. Characterisation of the limit. In Lemma 3.3 we proved the existence of subsequential limits μ^* for the sequences of embedded invariant measures $(\mu_N^{(m)})_{N \geq 1}$. Our convergence argument now consists in identifying any such limit μ^* with the unique invariant measure μ of the solution $(u(t))_{t \geq 0}$ of (1) (see Proposition 1.2). We proved in Lemma 3.3 that μ^* gives full weight to $H_0^2(\mathbb{T})$. As a consequence of this result, the measure μ^* can be considered as an initial distribution for $(u(t))_{t \geq 0}$. The weak convergence of the subsequence $\mu_{N_j}^{(1)}$ towards μ^* can be represented, by virtue of the Skorokhod theorem, by the almost sure $L_0^2(\mathbb{T})$ -convergence, on some particular probability space, of a sequence of random variables $u_{j,0}^{(1)}$ towards u_0 , where $u_{j,0}^{(1)} \sim \mu_{N_j}^{(1)}$, $\forall j \in \mathbb{N}$, and $u_0 \sim \mu^*$. We may define on this probability space, up to enlarging it, a Q -Wiener process $(W^Q(t))_{t \geq 0}$ defined as in Section 1.1, independent of u_0 and $u_{j,0}^{(1)}$, along with a normal filtration. In such a way, we may consider u_0 and $u_{j,0}^{(1)}$ as initial conditions for the solution of (1) and the embedded solutions of (11) respectively. More precisely, if we denote by $\mathbf{U}_0 = (U_{1,0}, \dots, U_{N_j,0})$ the $\mathbb{R}_0^{N_j}$ -valued random variable such that $U_{i,0} = u_{j,0}(i/N)$, and if we define $(\mathbf{U}(t))_{t \geq 0}$ the solution of (11) starting at \mathbf{U}_0 , then we define the process $(u_{N_j}^{(1)}(t))_{t \geq 0}$ by $u_{N_j}^{(1)}(t) = \Psi_{N_j}^{(1)} \mathbf{U}(t)$, for all $t \geq 0$.

Given that $(u_{N_j}^{(1)}(t))_{t \geq 0}$ is a numerical approximation of $(u(t))_{t \geq 0}$, convergence at time 0 shall lead to convergence at every finite time t :

Proposition 3.7. *Under Assumptions 1.1 and 1.3, for every $t \geq 0$, we have*

$$\lim_{j \rightarrow \infty} \mathbb{E} \left[\left\| u_{N_j}^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \right] = 0.$$

This result is proved in Section 3.3 below. Let us explain how this finite time result leads to the convergence of $(\mu_N^{(m)})_{N \geq 1}$ towards μ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$.

Proof of Theorem 1.7: part 1/2. The measure $\mu_{N_j}^{(1)}$ is invariant for the process $(u_{N_j}^{(1)}(t))_{t \geq 0}$. For all $t \geq 0$, let $\mu_t^* \in \mathcal{P}(L_0^2(\mathbb{T}))$ denote the probability distribution of $u(t)$. By Definition 1.6 and Proposition 3.7, we have

$$\forall t \geq 0, \quad \lim_{j \rightarrow \infty} W_2 \left(\mu_{N_j}^{(1)}, \mu_t^* \right) = 0.$$

By continuity of the Wasserstein distance [29, Corollary 6.11] and Lemma 3.3, this leads to

$$\forall t \geq 0, \quad W_2(\mu^*, \mu_t^*) = 0. \tag{53}$$

From [27, Lemma 6], there exists a unique probability measure in $\mathcal{P}(H_0^2(\mathbb{T}))$ coinciding with μ^* on the Borel sets of $H_0^2(\mathbb{T})$ (for convenience, we still call this measure μ^*). The meaning of (53) is that this measure $\mu^* \in \mathcal{P}(H_0^2(\mathbb{T}))$ is invariant for

the process $(u(t))_{t \geq 0}$. However, this process already has a unique invariant measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$, so that necessarily $\mu^* = \mu$. As a consequence, μ is the only subsequential limit in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ of the sequence $(\mu_N^{(1)})_{N \geq 1}$, and since from Lemma 3.3, $(\mu_N^{(1)})_{N \geq 1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$, we just proved (16) (see for instance [5, Theorem 2.6]) in the case $m = 1$. The cases $m = 0$ and $m = 2$ follow from the bounds (51) and (52). \square

Remark 3.8. This last proof shows in particular that μ^* is invariant for $(u(t))_{t \geq 0}$. Therefore it provides a second proof for the existence part in [27, Theorem 2].

3.3. Proofs. Proof of Lemma 3.2. Let us decompose the function \mathbf{b} as the sum of \mathbf{b}_1 and \mathbf{b}_2 , defined by

$$\forall \mathbf{v} \in \mathbb{R}_0^N, \quad b_i^1(\mathbf{v}) := -N(\bar{A}(v_i, v_{i+1}) - \bar{A}(v_{i-1}, v_i)) \quad \text{and} \quad \mathbf{b}^2 := \nu \mathbf{D}_N^{(2)}. \quad (54)$$

To prove the first inequality of the lemma, apply Corollary 2.11 with $p = 2$ and recall that we may take $C^{0,0} = 1$.

We focus now on the discrete H_0^2 estimate. Let $\mathbf{V} \sim \nu_N$ and let $(\mathbf{U}(t))_{t \geq 0}$ be the solution of (11) with initial distribution ν_N . We may compute the dynamics of the discrete H_0^1 -norm of $(\mathbf{U}(t))_{t \geq 0}$ by using Itô's formula: for all $t \geq 0$,

$$\|\mathbf{D}_N^{(1)} \mathbf{U}(t)\|_2^2 = \|\mathbf{D}_N^{(1)} \mathbf{U}_0\|_2^2 + 2 \int_0^t \langle \mathbf{D}_N^{(1)} \mathbf{b}(\mathbf{U}(s)), \mathbf{D}_N^{(1)} \mathbf{U}(s) \rangle ds + 2 \int_0^t \langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N})(s) \rangle + t \sum_{k \geq 1} \|\mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k\|_2^2. \quad (55)$$

It appears that the third term of the right-hand side is a martingale since

$$\begin{aligned} \sum_{k \geq 1} \mathbb{E} \left[\int_0^t \langle \mathbf{D}_N^{(1)} \mathbf{U}(s), \mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k \rangle^2 ds \right] &\leq t \left(\sum_{k \geq 1} \|\mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k\|_2^2 \right) \mathbb{E} \left[\|\mathbf{D}_N^{(1)} \mathbf{V}\|_2^2 \right] \\ &\leq t D_0 C^{1,2} < +\infty, \end{aligned} \quad (56)$$

where we used the stationarity of $(\mathbf{U}(t))_{t \geq 0}$, Inequality (42), and the first inequality from this lemma. Thus, taking the expectation and expanding the drift term, we get

$$\mathbb{E} \left[\|\mathbf{D}_N^{(1)} \mathbf{U}(t)\|_2^2 \right] = \mathbb{E} \left[\|\mathbf{D}_N^{(1)} \mathbf{U}_0\|_2^2 \right] + 2 \int_0^t \mathbb{E} \left[\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), \mathbf{D}_N^{(1)} \mathbf{b}(\mathbf{U}(s)) \rangle \right] ds + t \sum_{k \geq 1} \|\mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k\|_2^2.$$

Since the process starts from its invariant measure, the left-hand side cancels with the first term of the right-hand side. Besides, we may drop the time index. Using the decomposition $\mathbf{b} = \mathbf{b}^1 + \mathbf{b}^2$, we may sum by parts both the viscous and the flux term, and after dividing by t , it remains

$$2\nu \mathbb{E} \left[\|\mathbf{D}_N^{(2)} \mathbf{V}\|_2^2 \right] = -2\mathbb{E} \left[\langle \mathbf{D}_N^{(2)} \mathbf{V}, \mathbf{b}^1(\mathbf{V}) \rangle \right] + \sum_{k \geq 1} \|\mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k\|_2^2 \leq 2\sqrt{\mathbb{E} \left[\|\mathbf{D}_N^{(2)} \mathbf{V}\|_2^2 \right]} \sqrt{\mathbb{E} \left[\|\mathbf{b}^1(\mathbf{V})\|_2^2 \right]} + D_0, \quad (57)$$

where we used in particular the Cauchy-Schwarz inequality. We can bound the term in the second square root thanks to Assumption 1.3:

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{b}^1(\mathbf{V})\|_2^2 \right] &= \mathbb{E} \left[N \sum_{i=1}^N (\bar{A}(V_i, V_{i+1}) - \bar{A}(V_{i-1}, V_i))^2 \right] \\ &= \mathbb{E} \left[N \sum_{i=1}^N \left(\int_{V_i}^{V_{i+1}} \partial_2 \bar{A}(V_i, z) dz + \int_{V_{i-1}}^{V_i} \partial_1 \bar{A}(z, V_i) dz \right)^2 \right] \quad (\text{by (13)}) \\ &\leq 2\mathbb{E} \left[N \sum_{i=1}^N (V_{i+1} - V_i) \int_{V_i}^{V_{i+1}} \partial_2 \bar{A}(V_i, z)^2 dz \right] + 2\mathbb{E} \left[N \sum_{i=1}^N (V_i - V_{i-1}) \int_{V_{i-1}}^{V_i} \partial_1 \bar{A}(z, V_i)^2 dz \right] \quad (\text{by Jensen}) \\ &\leq 4C_{\bar{A}}^2 \mathbb{E} \left[N \sum_{i=1}^N (V_i - V_{i-1}) \int_{V_{i-1}}^{V_i} (1 + |z|^{p_{\bar{A}}})^2 dz \right] \quad (\text{by (14)}) \\ &\leq 8C_{\bar{A}}^2 \left(\mathbb{E} \left[N \sum_{i=1}^N (V_i - V_{i-1})^2 \right] + \mathbb{E} \left[N \sum_{i=1}^N (V_i - V_{i-1}) \int_{V_{i-1}}^{V_i} |z|^{2p_{\bar{A}}} dz \right] \right) \\ &= 8C_{\bar{A}}^2 \left(\mathbb{E} \left[\|\mathbf{D}_N^{(1)} \mathbf{V}\|_2^2 \right] + \frac{1}{2p_{\bar{A}} + 1} \mathbb{E} \left[\langle \mathbf{D}_N^{(1)} (\mathbf{V}^{2p_{\bar{A}}+1}), \mathbf{D}_N^{(1)} \mathbf{V} \rangle \right] \right) \\ &\leq 8C_{\bar{A}}^2 \left(C^{1,2} + \frac{D_0}{2\nu} C^{0,2p_{\bar{A}}} \right), \end{aligned} \quad (58)$$

where we used Lemma 3.2 and Corollary 2.11 with $p = 2p_{\bar{A}} + 2$ at the last line.

Injecting (58) into (57), we get

$$2\nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]\leq 2\sqrt{\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]}\sqrt{4C_A^2\frac{D_0}{\nu}(1+C^{0,2p_A})+D_0}.$$

Applying Young's inequality on the right-hand side, we get

$$2\nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]\leq \nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]+4C_A^2\frac{D_0}{\nu^2}(1+C^{0,2p_A})+D_0,$$

which rewrites

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]\leq 4C_A^2\frac{D_0}{\nu^3}(1+C^{0,2p_A})+\frac{D_0}{\nu}.$$

Since the right-hand side does not depend on N , we get the result. \square

Proof of Lemma 3.4. From summations by parts and Hölder's inequality, we establish a discrete Gagliardo-Nirenberg inequality in the following way (similar inequalities in the multi-dimensional case are given for instance in [7, Lemma 6] or [4, Theorem 4]):

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^N|V_{i+1}-V_i|^3\right]&=\mathbb{E}\left[\sum_{i=1}^N(V_{i+1}-V_i)^2|V_{i+1}-V_i|\right] \\ &=-\mathbb{E}\left[\sum_{i=1}^NV_i((V_{i+1}-V_i)|V_{i+1}-V_i|-(V_i-V_{i-1})|V_i-V_{i-1}|)\right] \\ &=-\mathbb{E}\left[\sum_{i=1}^NV_i((V_{i+1}-2V_i+V_{i-1})|V_{i+1}-V_i|+(V_i-V_{i-1})(|V_{i+1}-V_i|-|V_i-V_{i-1}|))\right] \\ &\leq\mathbb{E}\left[\sum_{i=1}^N|V_i||V_{i+1}-V_i||V_{i+1}-2V_i+V_{i-1}|\right]+\mathbb{E}\left[\sum_{i=1}^N|V_i||V_i-V_{i-1}||V_{i+1}-2V_i+V_{i-1}|\right] \\ &\leq 2\mathbb{E}\left[\sum_{i=1}^N|V_i|^6\right]^{\frac{1}{6}}\mathbb{E}\left[\sum_{i=1}^N|V_{i+1}-V_i|^3\right]^{\frac{1}{3}}\mathbb{E}\left[\sum_{i=1}^N|V_{i+1}-2V_i+V_{i-1}|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Dividing on both sides by $\mathbb{E}[\sum_i|V_{i+1}-V_i|^3]^{1/3}$ and then passing to the power 3/2, we obtain

$$\mathbb{E}\left[\sum_{i=1}^N|V_{i+1}-V_i|^3\right]\leq 2^{3/2}\mathbb{E}\left[\sum_{i=1}^N|V_i|^6\right]^{\frac{1}{4}}\mathbb{E}\left[\sum_{i=1}^N|V_{i+1}-2V_i+V_{i-1}|^2\right]^{\frac{3}{4}}.$$

Multiplying on both sides by N^2 , we derive the inequality

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_3^3\right]\leq 2\sqrt{2}\mathbb{E}\left[\left\|\mathbf{V}\right\|_6^6\right]^{1/4}\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}\right\|_2^2\right]^{3/4},$$

and we conclude thanks to Proposition 2.10 and Lemma 3.2:

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_3^3\right]\leq 2\sqrt{2}(C^{0,6})^{1/4}(C^{2,2})^{3/4}=:C^{1,3}.$$

\square

Proof of Lemma 3.5. Let $\mathbf{U}_0 \sim \nu_N$. Using the decomposition $\mathbf{b} = \mathbf{b}^1 + \mathbf{b}^2$ introduced at (54), we may expand Equation (55):

$$\begin{aligned}\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2&=\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2+2\int_0^t\left\langle\mathbf{D}_N^{(1)}\mathbf{b}^1(\mathbf{U}(s)),\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle ds+2\nu\int_0^t\left\langle\mathbf{D}_N^{(1)}\mathbf{D}_N^{(2)}\mathbf{U}(s),\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle ds \\ &\quad +2\int_0^t\left\langle\mathbf{D}_N^{(1)}\mathbf{U}(s),d\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle+t\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2.\end{aligned}\quad (59)$$

We shall address the second term of the right-hand side by applying (18) and Young's inequality:

$$\begin{aligned}2\int_0^t\left\langle\mathbf{D}_N^{(1)}\mathbf{b}^1(\mathbf{U}(s)),\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle ds&=-2\int_0^t\left\langle\mathbf{b}^1(\mathbf{U}(s)),\mathbf{D}_N^{(2)}\mathbf{U}(s)\right\rangle ds \\ &\leq\frac{1}{2\nu}\int_0^t\left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 ds+2\nu\int_0^t\left\|\mathbf{D}_N^{(2)}\mathbf{U}(s)\right\|_2^2 ds.\end{aligned}\quad (60)$$

As for the viscous term in (59), Equation (18) leads to

$$2\nu \int_0^t \left\langle \mathbf{D}_N^{(1)} \mathbf{D}_N^{(2)} \mathbf{U}(s), \mathbf{D}_N^{(1)} \mathbf{U}(s) \right\rangle ds = -2\nu \int_0^t \left\| \mathbf{D}_N^{(2)} \mathbf{U}(s) \right\|_2^2 ds. \quad (61)$$

Thus, injecting (61) and (60) into (59) and using the bound (42) results in:

$$\left\| \mathbf{D}_N^{(1)} \mathbf{U}(t) \right\|_2^2 \leq \left\| \mathbf{D}_N^{(1)} \mathbf{U}_0 \right\|_2^2 + \frac{1}{2\nu} \int_0^t \left\| \mathbf{b}^1(\mathbf{U}(s)) \right\|_2^2 ds + 2 \int_0^t \left\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d \left(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N} \right) (s) \right\rangle + t D_0. \quad (62)$$

Taking the supremum in time and the expectation over the second term of the right-hand side, by stationarity of $(\mathbf{U}(t))_{t \geq 0}$, we get the bound

$$\mathbb{E} \left[\sup_{t \in [0, T]} \frac{1}{2\nu} \int_0^t \left\| \mathbf{b}^1(\mathbf{U}(s)) \right\|_2^2 ds \right] \leq \frac{1}{2\nu} \mathbb{E} \left[\int_0^T \left\| \mathbf{b}^1(\mathbf{U}(s)) \right\|_2^2 ds \right] = \frac{1}{2\nu} \int_0^T \mathbb{E} \left[\left\| \mathbf{b}^1(\mathbf{U}(s)) \right\|_2^2 \right] ds = \frac{T}{2\nu} \mathbb{E} \left[\left\| \mathbf{b}^1(\mathbf{U}_0) \right\|_2^2 \right].$$

Applying now inequality (58), we get

$$\mathbb{E} \left[\sup_{t \in [0, T]} \frac{1}{2\nu} \int_0^t \left\| \mathbf{b}^1(\mathbf{U}(s)) \right\|_2^2 ds \right] \leq \frac{2C_A^2 T D_0}{\nu^2} (1 + C^{0, 2p_A}). \quad (63)$$

We now turn our attention to the third term of the right-hand side in (62). Recall that by (56), the process $(\int_0^t \langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N})(s) \rangle)_{t \geq 0}$ is a martingale. Therefore, applying successively the Jensen and the Doob inequalities, the Itô isometry, the Cauchy-Schwarz inequality and Lemma 3.2, we get

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \left\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d \left(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N} \right) (s) \right\rangle \right| \right] &\leq \mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \left\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d \left(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N} \right) (s) \right\rangle \right|^2 \right]^{1/2} \\ &\leq 2 \mathbb{E} \left[\left| \int_0^T \left\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), d \left(\mathbf{D}_N^{(1)} \mathbf{W}^{Q,N} \right) (s) \right\rangle \right|^2 \right]^{1/2} \\ &= 2 \mathbb{E} \left[\sum_{k \geq 1} \int_0^T \left\langle \mathbf{D}_N^{(1)} \mathbf{U}(s), \mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k \right\rangle^2 ds \right]^{1/2} \\ &\leq 2\sqrt{T} \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_0 \right\|_2^2 \right]^{1/2} \left(\sum_{k \geq 1} \left\| \mathbf{D}_N^{(1)} \boldsymbol{\sigma}^k \right\|_2^2 \right)^{1/2} \\ &\leq 2\sqrt{TC^{1,2} D_0} = D_0 \sqrt{\frac{2T}{\nu}}. \end{aligned} \quad (64)$$

Now, taking the supremum in time and the expectation in (62) and injecting (47), (63) and (64), we end up with

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left\| \mathbf{D}_N^{(1)} \mathbf{U}(t) \right\|_2^2 \right] \leq C^{1,2} + \frac{2C_A^2 T D_0}{\nu^2} (1 + C^{0, 2p_A}) + D_0 \sqrt{\frac{2T}{\nu}} + T D_0.$$

Since the right-hand side of the above inequality does not depend on N , the result follows. \square

Proof of Lemma 3.6. Let $p \in 2\mathbb{N}^*$ and let us repeat the proof of [27, Lemma 3] up to [27, Equation (23)]. When the initial condition u_0 is random and has distribution μ^* , this equation writes

$$\begin{aligned} \mathbb{E} \left[\|u(t \wedge T_r)\|_{L_0^p(\mathbb{T})}^p \right] &= \mathbb{E} \left[\|u_0\|_{L_0^p(\mathbb{T})}^p \right] - p \mathbb{E} \left[\int_0^{t \wedge T_r} \int_{\mathbb{T}} \partial_x A(u(s)) u(s)^{p-1} dx ds \right] \\ &\quad - \nu p(p-1) \mathbb{E} \left[\int_0^{t \wedge T_r} \int_{\mathbb{T}} \partial_x u(s)^2 u(s)^{p-2} dx ds \right] + \frac{p(p-1)}{2} \sum_{k \geq 1} \mathbb{E} \left[\int_0^{t \wedge T_r} \int_{\mathbb{T}} u(s)^{p-2} g_k^2 dx ds \right], \end{aligned}$$

for all $t \in [0, T]$ and $r \geq 0$, where T_r is a stopping time converging almost surely towards $+\infty$ as $r \rightarrow +\infty$ (by [27, Corollary 2]). Using [27, Equation (24)], the non-positivity of the third term of the right-hand side, and bounding the g_k 's by their supremum, we get the inequality

$$\mathbb{E} \left[\|u(t \wedge T_r)\|_{L_0^p(\mathbb{T})}^p \right] \leq \mathbb{E} \left[\|u_0\|_{L_0^p(\mathbb{T})}^p \right] + \frac{p(p-1)}{2} \left(\sum_{k \geq 1} \|g_k\|_{L_0^\infty(\mathbb{T})}^2 \right) \mathbb{E} \left[\int_0^{t \wedge T_r} \|u(s)\|_{L_0^{p-2}(\mathbb{T})}^{p-2} ds \right].$$

Using now Lemma 3.3, (3), (5), and [27, Equation (18)], we get

$$\mathbb{E} \left[\|u(t \wedge T_r)\|_{L_0^p(\mathbb{T})}^p \right] \leq C^{0,p} + \frac{p(p-1)}{2} D_0 \left(C_5^{(p-2)} \left(1 + \mathbb{E} \left[\|u_0\|_{L_0^{p-2}(\mathbb{T})}^{p-2} \right] \right) + C_6^{(p-2)} t \right),$$

where the constants $C_5^{(p-2)}$ and $C_6^{(p-2)}$, defined in [27], depend only on ν , p and D_0 . Using once again Lemma 3.3, letting $r \rightarrow +\infty$ and bounding t by T , we obtain

$$\limsup_{r \rightarrow \infty} \mathbb{E} \left[\|u(t \wedge T_r)\|_{L_0^p(\mathbb{T})}^p \right] \leq C^{0,p} + \frac{p(p-1)}{2} D_0 \left(C_5^{(p-2)} (1 + C^{0,p-2}) + C_6^{(p-2)} T \right) =: \tilde{C}_T^{0,p}.$$

Applying Fatou's lemma on the left-hand side, we get

$$\mathbb{E} \left[\|u(t)\|_{L_0^p(\mathbb{T})}^p \right] \leq \tilde{C}_T^{0,p},$$

and since the right-hand side does not depend on t , we get the first wanted inequality in the case $p \in 2\mathbb{N}^*$. The general case $p \in [2, +\infty)$ then follows from the Jensen inequality.

To prove the second inequality, we start from [27, Lemma 4] which, when u_0 is random, gives the estimate

$$\mathbb{E} \left[\|u(t \wedge T_r)\|_{H_0^1(\mathbb{T})}^2 \right] \leq \mathbb{E} \left[\|u_0\|_{H_0^1(\mathbb{T})}^2 \right] + C_7 \left(1 + \mathbb{E} \left[\|u_0\|_{L_0^{2p_A+2}(\mathbb{T})}^{2p_A+2} \right] \right) + C_8 t,$$

from which we deduce, by applying Fatou's lemma on the left-hand side and Lemma 3.3 on the right-hand side:

$$\mathbb{E} \left[\|u(t)\|_{H_0^1(\mathbb{T})}^2 \right] \leq C^{1,2} + C_7 (1 + C^{0,2p_A+2}) + C_8 T =: \tilde{C}_T^{1,2}. \quad \square$$

We now turn to the proof of the main lemma in this section:

Proof of Proposition 3.7. In all this proof, for notational convenience, the subsequence $(u_{N_j}^{(1)})_{j \in \mathbb{N}}$ will be denoted by $(u_N^{(1)})_{N \geq 1}$.

Step 0. Decomposition of the error. Let us fix a time horizon $T > 0$. We introduce the stopping time

$$\tau_{M,N} := \inf \left\{ t \geq 0 : \|u(t)\|_{H_0^1(\mathbb{T})} \vee \|u_N^{(1)}(t)\|_{H_0^1(\mathbb{T})} \geq M \right\},$$

and we split the expectation in two parts: for all $t \in [0, T]$,

$$\mathbb{E} \left[\|u_N^{(1)}(t) - u(t)\|_{L_0^2(\mathbb{T})}^2 \right] = \mathbb{E} \left[\|u_N^{(1)}(t) - u(t)\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t \leq \tau_{M,N}} \right] + \mathbb{E} \left[\|u_N^{(1)}(t) - u(t)\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t > \tau_{M,N}} \right]. \quad (65)$$

We will address the first term of the RHS in the steps 1 to 6, and the second one in the step 7.

We have

$$\mathbb{E} \left[\|u_N^{(1)}(t) - u(t)\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t \leq \tau_{M,N}} \right] \leq \mathbb{E} \left[\|u_N^{(1)}(\tau_{M,N} \wedge t) - u(\tau_{M,N} \wedge t)\|_{L_0^2(\mathbb{T})}^2 \right], \quad (66)$$

and we will use this localization argument to take benefit from the local Lipschitz continuity of the non-linear term which, by use of the Grönwall lemma, will lead us to show that for any fixed $M > 0$,

$$\mathbb{E} \left[\|u_N^{(1)}(t) - u(t)\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t \leq \tau_{M,N}} \right] \xrightarrow{N \rightarrow \infty} 0.$$

Applying Itô's formula [14, Theorem 4.32] and taking the expectation, we have

$$\begin{aligned} & \mathbb{E} \left[\|u_N^{(1)}(\tau_{M,N} \wedge t) - u(\tau_{M,N} \wedge t)\|_{L_0^2(\mathbb{T})}^2 \right] \\ &= \mathbb{E} \left[\|u_N^{(1)}(0) - u(0)\|_{L_0^2(\mathbb{T})}^2 \right] \\ & - 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(u_N^{(1)}(s, x) - u(s, x) \right) \left(\sum_{i=1}^N N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) \phi_N^{(1)} \left(x - \frac{i}{N} \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right) dx ds \right] \\ & - 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(u_N^{(1)}(s, x) - u(s, x) \right) \left(\partial_x A \left(u_N^{(1)}(s, x) \right) - \partial_x A(u(s, x)) \right) dx ds \right] \\ & + 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(u_N^{(1)}(s, x) - u(s, x) \right) \left(N^2 \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \phi_N^{(1)} \left(x - \frac{i}{N} \right) - \partial_{xx} u(s, x) \right) dx ds \right] \\ & + \sum_{k \geq 1} \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(\sum_{i=1}^N \sigma_i^k \phi_N^{(1)} \left(x - \frac{i}{N} \right) - g_k(x) \right)^2 dx ds \right] \\ & =: I_1^N + I_2^N(t) + I_3^N(t) + I_4^N(t) + I_5^N(t), \end{aligned}$$

where the local martingale term vanished thanks to the localisation. From Step 1 to Step 5, we will get an upper bound over each of the terms I_l^N , $l = 1, \dots, 5$. More precisely, for all $l = 1, 2, 4, 5$, we will show that there exists a sequence $(\varepsilon_l^N)_{N \in \mathbb{N}}$ of non-negative real numbers not depending on t such that $\lim_{N \rightarrow \infty} \varepsilon_l^N = 0$ and such that the following inequalities are satisfied for all $N \in \mathbb{N}$ and all $t \in [0, T]$:

$$I_1^N \leq \varepsilon_1^N, \quad (67)$$

$$I_2^N(t) \leq \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 ds \right] + \varepsilon_2^N, \quad (68)$$

$$I_4^N(t) \leq -2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 ds \right] + \varepsilon_4^N, \quad (69)$$

$$I_5^N(t) \leq \varepsilon_5^N. \quad (70)$$

In the case $l = 3$, we will show that there exists a constant $\gamma_M > 0$ not depending on N nor t such that

$$I_3^N(t) \leq 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 ds \right] + \gamma_M \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 ds \right].$$

Step 1. The initial condition. By the construction of the sequence $(u_N^{(1)})_{N \geq 1}$ in Section 3.2, we have almost surely

$$\lim_{N \rightarrow \infty} \left\| u_N^{(1)}(0) - u(0) \right\|_{L_0^2(\mathbb{T})} = 0. \quad (71)$$

Moreover, Proposition 2.10 and Lemma 3.3 ensure the uniform bound with respect to N over the following fourth order moment:

$$\begin{aligned} \mathbb{E} \left[\left\| u_N^{(1)}(0) - u(0) \right\|_{L_0^2(\mathbb{T})}^4 \right] &\leq 8\mathbb{E} \left[\left\| u_N^{(1)}(0) \right\|_{L_0^2(\mathbb{T})}^4 \right] + 8\mathbb{E} \left[\left\| u(0) \right\|_{L_0^2(\mathbb{T})}^4 \right] \\ &\leq 8\mathbb{E} \left[\left\| u_N^{(1)}(0) \right\|_{L_0^4(\mathbb{T})}^4 \right] + 8\mathbb{E} \left[\left\| u(0) \right\|_{L_0^4(\mathbb{T})}^4 \right] \quad (\text{by (2)}) \\ &\leq 16C^{0,4} \quad (\text{by Proposition 2.10 and Lemma 3.3}), \end{aligned} \quad (72)$$

and the convergence of I_1^N towards 0 follows (see for instance [5, Theorem 3.5]). Thus, since I_1^N does not depend on t , we may take $\varepsilon_1^N := I_1^N$.

Step 2. The flux-numerical flux approximation. Using Young's inequality, we have

$$\begin{aligned} I_2^N(t) &\leq \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 ds \right] \\ &\quad + \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(\sum_{i=1}^N N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) \phi_N^{(1)} \left(x - \frac{i}{N} \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right)^2 dx ds \right]. \end{aligned} \quad (73)$$

We focus on the second term of the right-hand side which we can rewrite by

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\left(N \left(\bar{A}(U_{i-1}(s), U_i(s)) - \bar{A}(U_{i-2}(s), U_{i-1}(s)) \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right) \phi_N^{(1)} \left(x - \frac{i-1}{N} \right) \right. \right. \\ \left. \left. + \left(N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right) \phi_N^{(1)} \left(x - \frac{i}{N} \right) \right)^2 dx ds \right], \end{aligned}$$

which we control by the following upper bound

$$\begin{aligned} 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \left(\bar{A}(U_{i-1}(s), U_i(s)) - \bar{A}(U_{i-2}(s), U_{i-1}(s)) \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right)^2 \right. \\ \left. + \left(N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) - \partial_x A \left(u_N^{(1)}(s, x) \right) \right)^2 dx ds \right]. \end{aligned}$$

By definition of $u_N^{(1)}$, we have for all $s \geq 0$ and all $x \in (\frac{i-1}{N}, \frac{i}{N}]$, $\partial_x u_N^{(1)}(s, x) = N(U_i(s) - U_{i-1}(s))$. Let us now focus on the second term of the above integrand and observe that by symmetry, the left one may be treated in exactly the same

way. We have thanks to (12):

$$\begin{aligned}
& \left(N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) - N(U_i(s) - U_{i-1}(s)) A' \left(u_N^{(1)}(s, x) \right) \right)^2 \\
&= \left(N \left(\bar{A}(U_i(s), U_{i+1}(s)) - \bar{A}(U_{i-1}(s), U_i(s)) \right) - N(U_i(s) - U_{i-1}(s)) \left(\partial_1 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) + \partial_2 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right) \right)^2 \\
&\leq 2 \left(N \int_{U_{i-1}(s)}^{U_i(s)} \left(\partial_1 \bar{A}(z, U_i(s)) - \partial_1 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right) dz \right)^2 \\
&\quad + 2N^2 \left(\int_{U_i(s)}^{U_{i+1}(s)} \partial_2 \bar{A}(U_i(s), z) dz - (U_i(s) - U_{i-1}(s)) \partial_2 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right)^2.
\end{aligned}$$

In the following computations, we will get upper bounds on the terms

$$I_{2.1}^N(t) := \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \int_{U_{i-1}(s)}^{U_i(s)} \left(\partial_1 \bar{A}(z, U_i(s)) - \partial_1 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right) dz \right)^2 dx ds \right],$$

$$I_{2.2}^N(t) := \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 \left(\int_{U_i(s)}^{U_{i+1}(s)} \partial_2 \bar{A}(U_i(s), z) dz - (U_i(s) - U_{i-1}(s)) \partial_2 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right)^2 dx ds \right].$$

We first look at the term $I_{2.1}^N(t)$. The function $\partial_1 \bar{A}$ is uniformly continuous on $[-M, M]^2$. In particular,

$$\forall \varepsilon > 0, \exists \delta_{M,\varepsilon} > 0, \max(|w - x|, |y - z|) \leq \delta_{M,\varepsilon}, \max\{|w|, |x|, |y|, |z|\} \leq M \Rightarrow |\partial_1 \bar{A}(w, y) - \partial_1 \bar{A}(x, z)| \leq \varepsilon.$$

Furthermore, for every $s \leq \tau_{M,N}$, by Lemma 3.1 and (3), we have $\sup_{i=1, \dots, N} |U_i(s)| = \|u_N^{(1)}(s)\|_{L_0^\infty(\mathbb{T})} \leq \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})} \leq M$. Let $C_\partial^{(M)}$ be a local bound of $\partial_1 \bar{A}$ and $\partial_2 \bar{A}$ over the square $[-M, M]^2$.

Let $\varepsilon > 0$. We have

$$\begin{aligned}
I_{2.1}^N(t) &= \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \int_{U_{i-1}(s)}^{U_i(s)} \left(\partial_1 \bar{A}(z, U_i(s)) - \partial_1 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right) dz \mathbf{1}_{|U_i(s) - U_{i-1}(s)| \leq \delta_{M,\varepsilon}} \right)^2 dx ds \right] \\
&\quad + \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \int_{U_{i-1}(s)}^{U_i(s)} \left(\partial_1 \bar{A}(z, U_i(s)) - \partial_1 \bar{A} \left(u_N^{(1)}(s, x), u_N^{(1)}(s, x) \right) \right) dz \mathbf{1}_{|U_i(s) - U_{i-1}(s)| > \delta_{M,\varepsilon}} \right)^2 dx ds \right] \\
&\leq \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \int_{U_{i-1}(s)}^{U_i(s)} \varepsilon dz \mathbf{1}_{|U_i(s) - U_{i-1}(s)| \leq \delta_{M,\varepsilon}} \right)^2 dx ds \right] \\
&\quad + \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(N \int_{U_{i-1}(s)}^{U_i(s)} 2C_\partial^{(M)} dz \frac{|U_i(s) - U_{i-1}(s)|^{1/2}}{\delta_{M,\varepsilon}^{1/2}} \right)^2 dx ds \right] \\
&\leq \varepsilon^2 \mathbb{E} \left[\int_0^t N \sum_{i=1}^N |U_i(s) - U_{i-1}(s)|^2 ds \right] + \frac{4 \left(C_\partial^{(M)} \right)^2}{\delta_{M,\varepsilon}} \int_0^t \mathbb{E} \left[N \sum_{i=1}^N |U_i(s) - U_{i-1}(s)|^3 \right] ds \\
&= \varepsilon^2 t \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_0 \right\|_2^2 \right] + \frac{4 \left(C_\partial^{(M)} \right)^2}{\delta_{M,\varepsilon} N} t \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_0 \right\|_3^3 \right] \\
&\leq \varepsilon^2 T C^{1,2} + \frac{4 \left(C_\partial^{(M)} \right)^2}{\delta_{M,\varepsilon} N} T C^{1,3} \quad (\text{by Lemmas 3.2 and 3.4}).
\end{aligned}$$

Since ε was chosen arbitrarily, it follows that $\varepsilon_{2.1}^N := \sup_{t \in [0, T]} I_{2.1}^N(t)$ satisfies $\lim_{N \rightarrow \infty} \varepsilon_{2.1}^N = 0$.

As for the term $I_{2.2}^N(t)$ we have

$$\begin{aligned}
I_{2.2}^N(t) &= \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 \left(\int_{U_i(s)}^{U_{i+1}(s)} \left(\partial_2 \bar{A}(U_i(s), z) - \partial_2 \bar{A}(u_N^{(1)}(s, x), u_N^{(1)}(s, x)) \right) dz \right. \right. \\
&\quad \left. \left. + (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \partial_2 \bar{A}(u_N^{(1)}(s, x), u_N^{(1)}(s, x)) \right)^2 dx ds \right] \\
&\leq 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 \left(\int_{U_i(s)}^{U_{i+1}(s)} \left(\partial_2 \bar{A}(U_i(s), z) - \partial_2 \bar{A}(u_N^{(1)}(s, x), u_N^{(1)}(s, x)) \right) dz \right)^2 dx ds \right] \\
&\quad + 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 \partial_2 \bar{A}(u_N^{(1)}(s, x), u_N^{(1)}(s, x))^2 dx ds \right] =: I_{2.2.1}^N(t) + I_{2.2.2}^N(t)
\end{aligned}$$

Now, the term $I_{2.2.1}^N(t)$ can be treated the same way as $I_{2.1}^N(t)$. In particular, $\varepsilon_{2.2.1}^N := \sup_{t \in [0, T]} I_{2.2.1}^N(t)$ satisfies $\lim_{N \rightarrow \infty} \varepsilon_{2.2.1}^N = 0$. As for $I_{2.2.2}^N(t)$, we have

$$\begin{aligned}
I_{2.2.2}^N(t) &\leq \left(C_{\partial}^{(M)} \right)^2 \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} N \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 ds \right] \\
&\leq 2 \left(C_{\partial}^{(M)} \right)^2 \int_0^t \mathbb{E} \left[N \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 \right] ds \\
&= 2 \left(C_{\partial}^{(M)} \right)^2 \frac{t}{N^2} \mathbb{E} \left[\left\| \mathbf{D}_N^{(2)} \mathbf{U}_0 \right\|_2^2 \right] \\
&\leq 2 \left(C_{\partial}^{(M)} \right)^2 T \frac{C^{2,2}}{N^2} \quad (\text{by Lemma 3.2}) \\
&=: \varepsilon_{2.2.2}^N.
\end{aligned}$$

At last, the sum of all these error terms amounts to an error term ε_2^N satisfying the requirements of Step 0, so that the inequality (73) reduces to (68).

Step 3. The flux term. Integrating by parts and applying Young's inequality, we get

$$\begin{aligned}
I_3^N(t) &= 2\mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \partial_x \left(u_N^{(1)}(s, x) - u(s, x) \right) \left(A(u_N^{(1)}(s, x)) - A(u(s, x)) \right) dx ds \right] \\
&\leq 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(\partial_x u_N^{(1)}(s, x) - \partial_x u(s, x) \right)^2 dx ds \right] + \frac{1}{2\nu} \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \left(A(u_N^{(1)}(s, x)) - A(u(s, x)) \right)^2 dx ds \right].
\end{aligned}$$

Denoting by L_M a local Lipschitz constant of A over the interval $[-M, M]$, we get

$$I_3^N(t) \leq 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 ds \right] + \frac{L_M^2}{2\nu} \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 ds \right],$$

and we set therefore $\gamma_M := L_M^2/(2\nu)$.

Step 4. The viscous term. We shall compare the term

$$J^N(s) = \int_{\mathbb{T}} \left(u_N^{(1)}(s, x) - u(s, x) \right) \left(\sum_{i=1}^N N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \phi_N^{(1)} \left(x - \frac{i}{N} \right) - \partial_{xx} u(s, x) \right) dx,$$

with

$$\tilde{J}^N(s) := - \int_{\mathbb{T}} (\partial_x u_N^{(1)}(s, x) - \partial_x u(s, x))^2 dx \leq 0.$$

Expanding the product in the definition of J^N , we first write

$$J^N(s) = J_1^N(s) + J_2^N(s) + J_3^N(s) + J_4^N(s),$$

where

$$\begin{aligned}
J_1^N(s) &:= \int_{\mathbb{T}} u_N^{(1)}(s, x) \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \phi_N^{(1)}\left(x - \frac{i}{N}\right) dx, \\
J_2^N(s) &:= - \int_{\mathbb{T}} u_N^{(1)}(s, x) \partial_{xx} u(s, x) dx, \\
J_3^N(s) &:= - \int_{\mathbb{T}} u(s, x) \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \phi_N^{(1)}\left(x - \frac{i}{N}\right) dx, \\
J_4^N(s) &:= \int_{\mathbb{T}} u(s, x) \partial_{xx} u(s, x) dx;
\end{aligned}$$

likewise,

$$\tilde{J}^N(s) = \tilde{J}_1^N(s) + \tilde{J}_2^N(s) + \tilde{J}_3^N(s) + \tilde{J}_4^N(s),$$

where

$$\begin{aligned}
\tilde{J}_1^N(s) &:= - \int_{\mathbb{T}} \partial_x u_N^{(1)}(s, x) \partial_x u_N^{(1)}(s, x) dx, \\
\tilde{J}_2^N(s) &:= \int_{\mathbb{T}} \partial_x u_N^{(1)}(s, x) \partial_x u(s, x) dx, \\
\tilde{J}_3^N(s) &:= \int_{\mathbb{T}} \partial_x u(s, x) \partial_x u_N^{(1)}(s, x) dx, \\
\tilde{J}_4^N(s) &:= - \int_{\mathbb{T}} \partial_x u(s, x) \partial_x u(s, x) dx.
\end{aligned}$$

Integration by parts shows that $J_2^N(s) = \tilde{J}_2^N(s)$ and $J_4^N(s) = \tilde{J}_4^N(s)$. We now focus on the computation of $J_1^N(s) - \tilde{J}_1^N(s)$ and $J_3^N(s) - \tilde{J}_3^N(s)$.

Using the definition of $u_N^{(1)}$, we get

$$\begin{aligned}
J_1^N(s) - \tilde{J}_1^N(s) &= \sum_{i,k=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) U_k(s) \int_{\mathbb{T}} \phi_N^{(1)}\left(x - \frac{i}{N}\right) \phi_N^{(1)}\left(x - \frac{k}{N}\right) dx \\
&\quad + \sum_{i,k=1}^N U_i(s) U_k(s) \int_{\mathbb{T}} \partial_x \phi_N^{(1)}\left(x - \frac{i}{N}\right) \partial_x \phi_N^{(1)}\left(x - \frac{k}{N}\right) dx.
\end{aligned}$$

Direct computation yields

$$\int_{\mathbb{T}} \phi_N^{(1)}\left(x - \frac{i}{N}\right) \phi_N^{(1)}\left(x - \frac{k}{N}\right) dx = \begin{cases} \frac{2}{3N} & \text{if } k = i, \\ \frac{1}{6N} & \text{if } k = i \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{\mathbb{T}} \partial_x \phi_N^{(1)}\left(x - \frac{i}{N}\right) \partial_x \phi_N^{(1)}\left(x - \frac{k}{N}\right) dx = \begin{cases} 2N & \text{if } k = i, \\ -N & \text{if } k = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence,

$$\begin{aligned}
J_1^N(s) - \tilde{J}_1^N(s) &= \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \left(\frac{1}{6N} U_{i-1}(s) + \frac{2}{3N} U_i(s) + \frac{1}{6N} U_{i+1}(s) \right) \\
&\quad + \sum_{i=1}^N U_i(s) (-N U_{i-1}(s) + 2N U_i(s) - N U_{i+1}(s)) \\
&= \frac{N}{6} \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 = \frac{1}{6N^2} \left\| \mathbf{D}_N^{(2)} \mathbf{U}(s) \right\|_2^2.
\end{aligned}$$

By Lemma 3.2, we deduce that

$$\mathbb{E} \left[\left| J_1^N(s) - \tilde{J}_1^N(s) \right| \right] \leq \frac{C^{2,2}}{6N^2}.$$

In order to compute $J_3^N(s) - \tilde{J}_3^N(s)$, we first rewrite

$$\begin{aligned}
\tilde{J}_3^N(s) &= \int_{\mathbb{T}} \partial_x u(s, x) \partial_x u_N^{(1)}(s, x) dx \\
&= \sum_{i=1}^N U_i(s) \int_{\mathbb{T}} \partial_x u(s, x) \partial_x \phi_N^{(1)} \left(x - \frac{i}{N} \right) dx \\
&= \sum_{i=1}^N N U_i(s) \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \partial_x u(s, x) dx - \int_{\frac{i}{N}}^{\frac{i+1}{N}} \partial_x u(s, x) dx \right) \\
&= - \sum_{i=1}^N N U_i(s) \left(u \left(s, \frac{i+1}{N} \right) - 2u \left(s, \frac{i}{N} \right) + u \left(s, \frac{i-1}{N} \right) \right) \\
&= - \sum_{i=1}^N N u \left(s, \frac{i}{N} \right) (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
J_3^N(s) - \tilde{J}_3^N(s) &= - \sum_{i=1}^N N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \left(N \int_{\mathbb{T}} u(s, x) \phi_N^{(1)} \left(x - \frac{i}{N} \right) dx - u \left(s, \frac{i}{N} \right) \right) \\
&= - \sum_{i=1}^N N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \int_{\mathbb{T}} \left(u(s, x) - u \left(s, \frac{i}{N} \right) \right) \phi_N^{(1)} \left(x - \frac{i}{N} \right) dx \\
&= - \sum_{i=1}^N N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \int_{\mathbb{T}} \int_{\frac{i-1}{N}}^x \partial_x u(s, y) dy \phi_N^{(1)} \left(x - \frac{i}{N} \right) dx.
\end{aligned}$$

Using the rough bound

$$|\phi_N^{(1)}(x)| \leq \mathbf{1}_{\{-\frac{1}{N} \leq x \leq \frac{1}{N}\}},$$

we write

$$\left| \int_{\mathbb{T}} \int_{\frac{i-1}{N}}^x \partial_x u(s, y) dy \phi_N^{(1)} \left(x - \frac{i}{N} \right) dx \right| \leq \frac{2}{N} \int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s, y)| dy,$$

whence

$$\left| J_3^N(s) - \tilde{J}_3^N(s) \right| \leq 2 \sum_{i=1}^N N |U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)| \int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s, y)| dy.$$

By the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[\left| J_3^N(s) - \tilde{J}_3^N(s) \right| \right] \leq 2N \sqrt{\mathbb{E} \left[\sum_{i=1}^N |U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)|^2 \right]} \sqrt{\mathbb{E} \left[\sum_{i=1}^N \left(\int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s, y)| dy \right)^2 \right]}.$$

By Jensen's inequality and Lemma 3.6,

$$\mathbb{E} \left[\sum_{i=1}^N \left(\int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s, y)| dy \right)^2 \right] \leq \frac{4}{N} \mathbb{E} \left[\|u(s)\|_{H_0^1(\mathbb{T})}^2 \right] \leq \frac{4\tilde{C}_T^{1,2}}{N}.$$

As a conclusion,

$$\mathbb{E} \left[\left| J_3^N(s) - \tilde{J}_3^N(s) \right| \right] \leq \frac{4\sqrt{C^{2,2}\tilde{C}_T^{1,2}}}{N}.$$

Coming back to the expression of $I_4^N(t)$, we have

$$\begin{aligned}
I_4^N(t) &\leq 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} J^N(s) ds \right] \\
&\leq 2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \tilde{J}^N(s) ds \right] + 2\nu \int_0^t \mathbb{E} \left[\left| J_1^N(s) - \tilde{J}_1^N(s) \right| \right] ds + 2\nu \int_0^t \mathbb{E} \left[\left| J_3^N(s) - \tilde{J}_3^N(s) \right| \right] ds \\
&\leq -2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 ds \right] + \frac{TC^{2,2}\nu}{3N^2} + \frac{8T\nu\sqrt{C^{2,2}\tilde{C}_T^{1,2}}}{N} \\
&=: -2\nu \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 ds \right] + \varepsilon_4^N.
\end{aligned}$$

Step 5. The noise term. We have

$$\begin{aligned}
I_5^N(t) &= \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \int_{\mathbb{T}} \sum_{k \geq 1} \left(\sum_{i=1}^N \sigma_i^k \phi_N^{(1)} \left(x - \frac{i}{N} \right) - g_k(x) \right)^2 dx ds \right] \\
&\leq t \int_{\mathbb{T}} \sum_{k \geq 1} \left(\sum_{i=1}^N \sigma_i^k \phi_N^{(1)} \left(x - \frac{i}{N} \right) - g_k(x) \right)^2 dx.
\end{aligned}$$

Using the fact that $\phi_N^{(1)} \left(x - \frac{i-1}{N} \right) + \phi_N^{(1)} \left(x - \frac{i}{N} \right) = 1$, for all $x \in [(i-1)/N, i/N]$, we get

$$I_5^N(t) \leq t \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k \geq 1} \left((\sigma_{i-1}^k - g_k(x)) \phi_N^{(1)} \left(x - \frac{i-1}{N} \right) + (\sigma_i^k - g_k(x)) \phi_N^{(1)} \left(x - \frac{i}{N} \right) \right)^2 dx,$$

and by periodicity of the indexes, we can say that

$$\begin{aligned}
I_5^N(t) &\leq 2t \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k \geq 1} \left(N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(y) dy - g_k(x) \right)^2 \phi_N^{(1)} \left(x - \frac{i}{N} \right)^2 dx \\
&\leq 2t \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k \geq 1} \left(N \int_{\frac{i-1}{N}}^{\frac{i}{N}} (g_k(y) - g_k(x)) dy \right)^2 \phi_N^{(1)} \left(x - \frac{i}{N} \right)^2 dx \\
&\leq 2t \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \sum_{k \geq 1} (g_k(y) - g_k(x))^2 dy dx \quad (\text{since } |\phi_N^{(1)}| \leq 1 \text{ and by Jensen}) \\
&\leq 2tN \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \frac{D_0}{N^2} dy dx = \frac{2tD_0}{N^2} \leq \frac{2TD_0}{N^2} =: \varepsilon_5^N.
\end{aligned}$$

Step 6. Conclusion for the “bounded” event. Summing all the I_i^N terms, we get for all $t \in [0, T]$

$$\mathbb{E} \left[\left\| u_N^{(1)}(\tau_{M,N} \wedge t) - u(\tau_{M,N} \wedge t) \right\|_{L_0^2(\mathbb{T})}^2 \right] \leq (1 + \gamma_M) \mathbb{E} \left[\int_0^{\tau_{M,N} \wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 ds \right] + \varepsilon_1^N + \varepsilon_2^N + \varepsilon_4^N + \varepsilon_5^N.$$

We set $\varepsilon^N := \varepsilon_1^N + \varepsilon_2^N + \varepsilon_4^N + \varepsilon_5^N$. By (66), we have

$$\mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t \leq \tau_{M,N}} \right] \leq (1 + \gamma_M) \int_0^t \mathbb{E} \left[\left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{s \leq \tau_{M,N}} \right] ds + \varepsilon^N.$$

Thus, Grönwall’s lemma applies and gives

$$\mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t \leq \tau_{M,N}} \right] \leq \varepsilon^N e^{(1+\gamma_M)t}.$$

Step 7. Conclusion of the proof. We want now to take care of the second term of the right-hand side in (65). Using in particular the Cauchy-Schwarz inequality, Proposition 2.10, the Markov inequality, Lemma 3.6 and Lemma 3.5,

we have

$$\begin{aligned}
& \mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t > \tau_{M,N}} \right] \\
&= \mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})} \vee \sup_{s \in [0,t]} \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})} > M} \right] \\
&\leq \mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^4 \right]^{1/2} \mathbb{P} \left(\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})} \vee \sup_{s \in [0,t]} \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})} > M \right)^{1/2} \\
&\leq \sqrt{8} \left(\mathbb{E} \left[\|u_N^{(1)}(t)\|_{L_0^4(\mathbb{T})}^4 \right] + \mathbb{E} \left[\|u(t)\|_{L_0^4(\mathbb{T})}^4 \right] \right)^{1/2} \left(\mathbb{P} \left(\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})} > M \right) + \mathbb{P} \left(\sup_{s \in [0,t]} \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})} > M \right) \right)^{1/2} \\
&\leq \sqrt{8} \left(C^{0,4} + \tilde{C}_T^{0,4} \right)^{1/2} \left(\mathbb{P} \left(\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})} > M \right) + \frac{1}{M} \mathbb{E} \left[\sup_{s \in [0,t]} \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})} \right] \right)^{1/2} \\
&\leq \sqrt{8} \left(C^{0,4} + \tilde{C}_T^{0,4} \right)^{1/2} \left(\mathbb{P} \left(\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})} > M \right) + \frac{1}{M} C_T^{1,2} \right)^{1/2}
\end{aligned}$$

Furthermore, as $(u(t))_{t \geq 0}$ is continuous from $[0, +\infty)$ to $H_0^1(\mathbb{T})$ (see Proposition 1.2), the random variable $\sup_{s \in [0,t]} \|u(s)\|_{H_0^1(\mathbb{T})}$ is finite almost surely. As a consequence,

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E} \left[\left\| u_N^{(1)}(t) - u(t) \right\|_{L_0^2(\mathbb{T})}^2 \mathbf{1}_{t > \tau_{M,N}} \right] = 0.$$

Combining this inequality with the conclusion of the step 6 yields the wanted result. \square

4. CONVERGENCE OF INVARIANT MEASURES: SPLIT-STEP SCHEME TOWARDS SEMI-DISCRETE SCHEME

In this section, we aim to prove the second part of Theorem 1.7, namely Equation (17). The structure of the proof is the same as for the first part of Theorem 1.7. In Subsection 4.1, we show that the family of probability measures $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is tight in $\mathcal{P}(\mathbb{R}_0^N)$ and then relatively compact in $\mathcal{P}_2(\mathbb{R}_0^N)$. In Subsection 4.2, in a similar manner as in Subsection 3.2 for the semi-discrete case, we identify each subsequential limit of the family $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$, when $\Delta t \rightarrow 0$, as the invariant measure of the process $(\mathbf{U}_n)_{n \in \mathbb{N}}$, which leads to the final part of the proof of Theorem 1.7. Subsection 4.3 contain the proofs of the lemmas stated in Subsections 4.1 and 4.2.

4.1. Tightness, relative compactness and some estimates.

Lemma 4.1 (Tightness). *Under Assumptions 1.1 and 1.3, for any $N \geq 1$, the family of probability measures $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is tight in the space $\mathcal{P}(\mathbb{R}_0^N)$.*

Proof. We established at Proposition 2.18 that a random variable $\mathbf{V} \sim \nu_{N,\Delta t}$ satisfies the discrete H_0^1 estimate:

$$\mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{V} \right\|_2^2 \right] \leq \bar{C}^{1,2}.$$

Since $\|\mathbf{D}_N^{(1)} \cdot\|_2$ defines a norm on \mathbb{R}_0^N , the result follows from the Markov inequality. \square

Lemma 4.2 (Fourth-order moment). *Under Assumptions 1.1 and 1.3, there exists a constant $\bar{C}^{0,4} > 0$, depending only on D_0 , ν and Δt_{\max} , such that for any time step $\Delta t \in (0, \Delta t_{\max}]$ and any random variable $\mathbf{V} \sim \nu_{N,\Delta t}$, we have*

$$\mathbb{E} \left[\|\mathbf{V}\|_4^4 \right] \leq \bar{C}^{0,4}.$$

Corollary 4.3 (Relative compactness). *Let $N \geq 1$. Under Assumptions 1.1 and 1.3, the family $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is relatively compact in $\mathcal{P}_2(\mathbb{R}_0^N)$.*

Proof. By virtue of the Prokhorov theorem [5, Theorem 5.1] and Lemma 4.1, any sequence extracted from $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ admits a weakly converging subsequence in $\mathcal{P}(\mathbb{R}_0^N)$. Let ν^* be a subsequential weak limit and let $(\nu_{N,\Delta t_j})_{j \in \mathbb{N}}$ be a sequence weakly converging towards ν^* . Let $(\mathbf{V}_j)_{j \in \mathbb{N}}$ be a sequence of \mathbb{R}_0^N -valued random variables such that $\mathbf{V}_j \sim \nu_{N,\Delta t_j}$. By virtue of the Portemanteau theorem, since $\|\cdot\|_4^4$ is continuous (and thus lower semi-continuous) on \mathbb{R}_0^N , we have

$$\mathbb{E} \left[\|\mathbf{V}\|_4^4 \right] \leq \liminf_{j \rightarrow \infty} \mathbb{E} \left[\|\mathbf{V}_j\|_4^4 \right] \leq \bar{C}^{0,4},$$

so that ν^* admits a fourth-order moment and thus belongs to $\mathcal{P}_4(\mathbb{R}_0^N)$. Moreover, it follows also from Lemma 4.1 that the sequence $(\nu_{N,\Delta t_j})_{j \in \mathbb{N}}$ satisfies a uniform integrability condition in the sense of [29, Definition 6.8], and the result is now a consequence of [29, Theorem 6.9]. \square

Lemma 4.4 (Finite time bound). *Let Assumptions 1.1 and 1.3 hold, let $T > 0$ be a time horizon and let $(\mathbf{U}_n)_{n \in \mathbb{N}}$ be a solution of (15) with an initial condition $\mathbf{U}_0 \sim \nu_{N, \Delta t}$. There exists a constant $\bar{C}_T^{0,2}$ depending only on D_0 , T , ν and Δt_{\max} , such that for any time step $\Delta t \in (0, \Delta t_{\max}]$, we have*

$$\mathbb{E} \left[\sup_{n=0,1,\dots, \lfloor \frac{T}{\Delta t} \rfloor} \|\mathbf{U}_n\|_2^2 \right] \leq \bar{C}_T^{0,2}.$$

4.2. Characterisation of the limit. As in Subsection 3.2 for the semi-discrete scheme, we want to use a result of convergence in finite time of the numerical scheme (15) in order to identify each subsequential limit of the family $\{\nu_{N, \Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$, when $\Delta t \rightarrow 0$, as the invariant measure ν_N for the solution of Equation (11). By virtue of Corollary 4.3, let $\nu^* \in \mathcal{P}_2(\mathbb{R}_0^N)$ and let $(\Delta t_j)_{j \in \mathbb{N}}$ be a sequence of time steps decreasing to zero such that $W_2(\nu_{N, \Delta t_j}, \nu^*)$ converges to zero as $j \rightarrow +\infty$. By virtue of the Skorokhod representation theorem, let $(\mathbf{U}_0^{(j)})_{j \in \mathbb{N}}$ be a sequence of \mathbb{R}_0^N -valued random variables converging almost surely to \mathbf{U}_0 such that

$$\mathbf{U}_0^{(j)} \sim \nu_{N, \Delta t_j}, \quad \forall j \in \mathbb{N} \quad \text{and} \quad \mathbf{U}_0 \sim \nu^*.$$

As we described in Subsection 3.2, up to an extension of the probability space, these random variables may be considered as initial conditions for the equations (11) and (15) (driven by the same Wiener process).

Lemma 4.5 (Finite time convergence). *Let Assumptions 1.1 and 1.3 hold. Let $(\mathbf{U}(t))_{t \geq 0}$ be the solution of (11) with initial condition \mathbf{U}_0 and for all $j \in \mathbb{N}$, let $(\mathbf{U}_n^{(j)})_{n \in \mathbb{N}}$ be the solution of (15) with initial condition $\mathbf{U}_0^{(j)}$. For any $j \in \mathbb{N}$, we define the piecewise constant approximation $(\bar{\mathbf{U}}^{(j)}(t))_{t \geq 0}$ of $\mathbf{U}_n^{(j)}$ by $\bar{\mathbf{U}}^{(j)}(t) = \mathbf{U}_n^{(j)}$ if $t \in [n\Delta t_j, (n+1)\Delta t_j)$. Then, for all $T > 0$,*

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \right] = 0.$$

Proof of Theorem 1.7: Part 2/2. The arguments are identical to the proof of the first part. \square

4.3. Proofs. Proof of Lemma 4.2. Let $(\mathbf{U}_n)_{n \in \mathbb{N}} = (U_{1,n}, \dots, U_{N,n})_{n \in \mathbb{N}}$ be a solution of (15) with a deterministic initial condition \mathbf{u}_0 . By convexity of the function $v \mapsto v^4$, for any $\alpha, \beta \in \mathbb{R}$, we have $(\alpha - \beta)^4 \geq \alpha^4 - 4\alpha^3\beta$. In particular, for any $i \in \mathbb{Z}/N\mathbb{Z}$, taking $\alpha = U_{i, n+\frac{1}{2}}$ and $\beta = \Delta t b_i(\mathbf{U}_{n+\frac{1}{2}})$, we have

$$U_{i,n}^4 = \left(U_{i, n+\frac{1}{2}} - \Delta t b_i(\mathbf{U}_{n+\frac{1}{2}}) \right)^4 \geq U_{i, n+\frac{1}{2}}^4 - 4U_{i, n+\frac{1}{2}}^3 \Delta t b_i(\mathbf{U}_{n+\frac{1}{2}}).$$

Hence, expanding the drift function and summing over i , we get

$$\|\mathbf{U}_n\|_4^4 \geq \left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 + \sum_{i=1}^N 4\Delta t U_{i, n+\frac{1}{2}}^3 \left(\bar{A}(U_{i, n+\frac{1}{2}}, U_{i+1, n+\frac{1}{2}}) - \bar{A}(U_{i-1, n+\frac{1}{2}}, U_{i, n+\frac{1}{2}}) \right) - 4\nu\Delta t \left\langle \mathbf{U}_{n+\frac{1}{2}}^3, \mathbf{D}_N^{(2)} \mathbf{U}_{n+\frac{1}{2}} \right\rangle.$$

We know thanks to Lemma 2.3 that the second term of the right-hand side is non-negative. Summing by parts the third term, we get

$$\|\mathbf{U}_n\|_4^4 \geq \left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 + 4\nu\Delta t \left\langle \mathbf{D}_N^{(1)} \mathbf{U}_{n+\frac{1}{2}}^3, \mathbf{D}_N^{(1)} \mathbf{U}_{n+\frac{1}{2}} \right\rangle.$$

From Lemma 2.2, we get

$$\|\mathbf{U}_n\|_4^4 \geq \left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 + 3\nu\Delta t \left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4. \quad (74)$$

On the other hand, let us look at the second step of the scheme (15). By the construction of the split-step scheme, the random variables $U_{i, n+\frac{1}{2}}$ and $\Delta W_{i, n+1}^{Q, N}$ are independent. Since $\Delta W_{i, n+1}^{Q, N} \sim \mathcal{N}(0, \Delta t \sum_{k \geq 1} (\sigma_i^k)^2)$ and $\sum_{k \geq 1} (\sigma_i^k)^2 \leq D_0$ (by (9)), we write

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{U}_{n+1}\|_4^4 \right] &= \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} + \Delta \mathbf{W}_{n+1}^{Q, N} \right\|_4^4 \right] \\ &= \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 \right] + \frac{6}{N} \mathbb{E} \left[\sum_{i=1}^N U_{i, n+\frac{1}{2}}^2 \left(\Delta W_{i, n+1}^{Q, N} \right)^2 \right] + \mathbb{E} \left[\left\| \Delta \mathbf{W}_{n+1}^{Q, N} \right\|_4^4 \right] \\ &\leq \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 \right] + 6D_0\Delta t \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 \right] + 3D_0^2\Delta t^2 \end{aligned} \quad (75)$$

Combining Inequalities (74) and (75), we get

$$\mathbb{E} \left[\|\mathbf{U}_n\|_4^4 \right] \geq \mathbb{E} \left[\|\mathbf{U}_{n+1}\|_4^4 \right] - 6D_0\Delta t \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_2^2 \right] - 3D_0^2\Delta t^2 + 3\nu\Delta t \mathbb{E} \left[\left\| \mathbf{U}_{n+\frac{1}{2}} \right\|_4^4 \right]$$

from which we get a telescopic sum:

$$3\nu\Delta t \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_4^4 \right] \leq \|\mathbf{u}_0\|_4^4 - \mathbb{E} \left[\left\| \mathbf{U}_n \right\|_4^4 \right] + 6D_0\Delta t \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] + 3nD_0^2\Delta t^2.$$

Thus,

$$\frac{1}{n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_4^4 \right] \leq \frac{1}{3\nu\Delta tn} \|\mathbf{u}_0\|_4^4 + \frac{2D_0}{\nu n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] + \frac{D_0^2\Delta t}{\nu}. \quad (76)$$

Recall that from Lemma 2.1 and Equation (40), we have

$$\frac{1}{n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] \leq \frac{1}{n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 \right] \leq \frac{\|\mathbf{u}_0\|_2^2}{2\nu n\Delta t} + \frac{D_0}{2\nu}. \quad (77)$$

Injecting (77) into (76), we get

$$\frac{1}{n} \sum_{l=0}^{n-1} \mathbb{E} \left[\left\| \mathbf{U}_{l+\frac{1}{2}} \right\|_4^4 \right] \leq \frac{1}{3\nu\Delta tn} \|\mathbf{u}_0\|_4^4 + \frac{2D_0}{\nu} \left(\frac{\|\mathbf{u}_0\|_2^2}{2\nu n\Delta t} + \frac{D_0}{2\nu} \right) + \frac{D_0^2\Delta t}{\nu}.$$

Using now the same arguments as for the end of the proof of Proposition 2.10, letting $\mathbf{V} \sim \nu_{N,\Delta t}$, we get

$$\mathbb{E} \left[\left\| \mathbf{V}_{\frac{1}{2}} \right\|_4^4 \right] \leq \frac{D_0^2}{\nu^2} + \frac{D_0^2\Delta t}{\nu} = \frac{D_0^2}{\nu} \left(\frac{1}{\nu} + \Delta t \right).$$

To conclude, we use Inequality (75) once again:

$$\begin{aligned} \mathbb{E} \left[\left\| \mathbf{V} \right\|_4^4 \right] &\leq \mathbb{E} \left[\left\| \mathbf{V}_{\frac{1}{2}} \right\|_4^4 \right] + 6D_0\Delta t \mathbb{E} \left[\left\| \mathbf{V}_{\frac{1}{2}} \right\|_2^2 \right] + 3D_0^2\Delta t^2 \\ &\leq \frac{D_0^2}{\nu} \left(\frac{1}{\nu} + \Delta t \right) + \frac{3D_0^2\Delta t}{\nu} + 3D_0^2\Delta t^2 \\ &\leq D_0^2 \left(\frac{1}{\nu} + 3\Delta t_{\max} \right) \left(\frac{1}{\nu} + \Delta t_{\max} \right). \end{aligned} \quad \square$$

Proof of Lemma 4.4. Let us repeat the proof of Proposition 2.18 up to Equation (37). For all $n = 0, 1, \dots, \lfloor \frac{T}{\Delta t} \rfloor$, we write

$$\begin{aligned} \|\mathbf{U}_n\|_2^2 &= \|\mathbf{U}_0\|_2^2 + \sum_{l=0}^{n-1} \left(\|\mathbf{U}_{l+1}\|_2^2 - \|\mathbf{U}_l\|_2^2 \right) \\ &\leq \|\mathbf{U}_0\|_2^2 - 2\nu\Delta t \sum_{l=0}^{n-1} \left\| \mathbf{D}_N^{(1)} \mathbf{U}_{l+\frac{1}{2}} \right\|_2^2 + 2 \sum_{l=0}^{n-1} \left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \right\rangle + \sum_{l=0}^{n-1} \left\| \Delta \mathbf{W}_{l+1}^{Q,N} \right\|_2^2. \end{aligned}$$

The viscous term may be removed from the inequality. Taking the supremum in time and the expectation, we get

$$\mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \|\mathbf{U}_n\|_2^2 \right] \leq \mathbb{E} \left[\|\mathbf{U}_0\|_2^2 \right] + 2\mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \left| \sum_{l=0}^{n-1} \left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \right\rangle \right| \right] + \mathbb{E} \left[\sum_{l=0}^{\lfloor T/\Delta t \rfloor - 1} \left\| \Delta \mathbf{W}_{l+1}^{Q,N} \right\|_2^2 \right]. \quad (78)$$

First, by Lemma 2.1 and Proposition 2.18, we have

$$\mathbb{E} \left[\|\mathbf{U}_0\|_2^2 \right] \leq \mathbb{E} \left[\left\| \mathbf{D}_N^{(1)} \mathbf{U}_0 \right\|_2^2 \right] \leq \bar{C}^{1,2}.$$

Noticing that the process $(\sum_{l=0}^{n-1} \langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \rangle)_{n \geq 1}$ is a martingale, we get by applying successively Jensen's and Doob's inequalities to the second term of the right-hand side,

$$\begin{aligned} \mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \left| \sum_{l=0}^{n-1} \left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \right\rangle \right| \right] &\leq \mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \left| \sum_{l=0}^{n-1} \left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \right\rangle \right|^2 \right]^{1/2} \\ &\leq 2\mathbb{E} \left[\left| \sum_{l=0}^{\lfloor T/\Delta t \rfloor - 1} \left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \right\rangle \right|^2 \right]^{1/2}. \end{aligned}$$

From (15), we may observe that each increment $\Delta \mathbf{W}_{l+1}^{Q,N}$ is independent of the family $(\mathbf{U}_{m+\frac{1}{2}}, \Delta \mathbf{W}_m^{Q,N})_{m=0,\dots,l}$. Therefore, letting $\mathbf{V} \sim \nu_{N,\Delta t}$ and letting $\mathbf{V}_{\frac{1}{2}}$ be the random variable satisfying $\mathbf{V}_{\frac{1}{2}} = \mathbf{V} + \mathbf{b}(\mathbf{V}_{\frac{1}{2}})$, we have

$$\begin{aligned} \mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \left| \sum_{l=0}^{n-1} \langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \rangle \right| \right] &\leq 2 \mathbb{E} \left[\sum_{l=0}^{\lfloor T/\Delta t \rfloor - 1} \left| \langle \mathbf{U}_{l+\frac{1}{2}}, \Delta \mathbf{W}_{l+1}^{Q,N} \rangle \right|^2 \right]^{1/2} \\ &\leq 2 \left(\sum_{l=0}^{\lfloor T/\Delta t \rfloor - 1} \mathbb{E} \left[\|\mathbf{U}_{l+\frac{1}{2}}\|_2^2 \right] \mathbb{E} \left[\|\Delta \mathbf{W}_{l+1}^{Q,N}\|_2^2 \right] \right)^{1/2} \\ &\leq 2 \sqrt{D_0 \Delta t} \left(\left\lfloor \frac{T}{\Delta t} \right\rfloor \mathbb{E} \left[\|\mathbf{V}_{\frac{1}{2}}\|_2^2 \right] \right)^{1/2} \quad (\text{by (38)}) \\ &\leq D_0 \sqrt{\frac{2T}{\nu}} \quad (\text{by Lemma 2.1 and (34)}). \end{aligned}$$

Injecting this bound into (78), we get

$$\mathbb{E} \left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t} \rfloor} \|\mathbf{U}_n\|_2^2 \right] \leq \bar{C}^{1,2} + 2D_0 \sqrt{\frac{2T}{\nu}} + TD_0. \quad \square$$

Proof of Lemma 4.5. Step 0. Decomposition of the error in two events. Let $T > 0$. We start by introducing the exit time of some ball for the time-continuous and the time-discretised processes:

$$\rho_M^{(j)} := \inf \left\{ t \geq 0 : \|\mathbf{U}(t)\|_2^2 \vee \|\bar{\mathbf{U}}^{(j)}(t)\|_2^2 \geq M \right\}.$$

Then, we decompose the approximation error according to whether or not the discrete and continuous processes stay in the ball of radius M : for all $t \in [0, T]$,

$$\mathbb{E} \left[\|\bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\|_2^2 \right] = \mathbb{E} \left[\|\bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\|_2^2 \mathbf{1}_{t < \rho_M^{(j)}} \right] + \mathbb{E} \left[\|\bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\|_2^2 \mathbf{1}_{\rho_M^{(j)} \leq t} \right]. \quad (79)$$

Step 1. Decomposition of the error for the bounded trajectories. For any $t \in [0, T]$ and any $j \in \mathbb{N}$, let $n_{[t,M,j]} := \left\lfloor \frac{t \wedge \rho_M^{(j)}}{\Delta t_j} \right\rfloor$. From (15) and (11), we write

$$\begin{aligned} \bar{\mathbf{U}}^{(j)}(t \wedge \rho_M^{(j)}) - \mathbf{U}(t \wedge \rho_M^{(j)}) &= \mathbf{U}_0^{(j)} - \mathbf{U}_0 + \sum_{l=0}^{n_{[t,M,j]}-1} \left(\Delta t_j \mathbf{b}(\mathbf{U}_{l+\frac{1}{2}}^{(j)}) - \int_{l\Delta t_j}^{(l+1)\Delta t_j} \mathbf{b}(\mathbf{U}(s)) ds \right) \\ &\quad + \left(t \wedge \rho_M^{(j)} - n_{[t,M,j]} \Delta t_j \right) \mathbf{b}(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}) - \int_{n_{[t,M,j]} \Delta t_j}^{t \wedge \rho_M^{(j)}} \mathbf{b}(\mathbf{U}(s)) ds. \end{aligned} \quad (80)$$

From this dynamics, we can decompose then again the first term of the right-hand side in (79) in three terms:

$$\begin{aligned} \mathbb{E} \left[\|\bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\|_2^2 \mathbf{1}_{t < \rho_M^{(j)}} \right] &\leq \mathbb{E} \left[\|\bar{\mathbf{U}}^{(j)}(t \wedge \rho_M^{(j)}) - \mathbf{U}(t \wedge \rho_M^{(j)})\|_2^2 \right] \\ &\leq 2 \mathbb{E} \left[\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\|_2^2 \right] \\ &\quad + 2 \mathbb{E} \left[\left\| \sum_{l=0}^{n_{[t,M,j]}-1} \left(\Delta t_j \mathbf{b}(\mathbf{U}_{l+\frac{1}{2}}^{(j)}) - \int_{l\Delta t_j}^{(l+1)\Delta t_j} \mathbf{b}(\mathbf{U}(s)) ds \right) + \left(t \wedge \rho_M^{(j)} - n_{[t,M,j]} \Delta t_j \right) \mathbf{b}(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}) - \int_{n_{[t,M,j]} \Delta t_j}^{t \wedge \rho_M^{(j)}} \mathbf{b}(\mathbf{U}(s)) ds \right\|_2^2 \right] \\ &\leq 2 \mathbb{E} \left[\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\|_2^2 \right] \end{aligned} \quad (81)$$

$$+ 4 \mathbb{E} \left[\left\| \sum_{l=0}^{n_{[t,M,j]}-1} \Delta t_j \left(\mathbf{b}(\mathbf{U}_{l+\frac{1}{2}}^{(j)}) - \mathbf{b}(\mathbf{U}_l^{(j)}) \right) + \left(t \wedge \rho_M^{(j)} - n_{[t,M,j]} \right) \left(\mathbf{b}(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}) - \mathbf{b}(\mathbf{U}_{n_{[t,M,j]}}^{(j)}) \right) \right\|_2^2 \right] \quad (82)$$

$$+ 4 \mathbb{E} \left[\left\| \sum_{l=0}^{n_{[t,M,j]}-1} \int_{l\Delta t_j}^{(l+1)\Delta t_j} \left(\mathbf{b}(\mathbf{U}_l^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right) ds + \int_{n_{[t,M,j]} \Delta t_j}^{t \wedge \rho_M^{(j)}} \left(\mathbf{b}(\mathbf{U}_{n_{[t,M,j]}}^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right) ds \right\|_2^2 \right]. \quad (83)$$

Step 2. Bound over the term (83). Let L_M be a Lipschitz constant of \mathbf{b} over the ball $\{\|\cdot\|_2^2 \leq M\}$. From Jensen's inequality, we have

$$\begin{aligned}
& 4\mathbb{E} \left[\left\| \sum_{l=0}^{n_{[t,M,j]}-1} \int_{l\Delta t_j}^{(l+1)\Delta t_j} \left(\mathbf{b}(\mathbf{U}_l^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right) ds + \int_{n_{[t,M,j]}\Delta t_j}^{t \wedge \rho_M^{(j)}} \left(\mathbf{b}(\mathbf{U}_{n_{[t,M,j]}}^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right) ds \right\|_2^2 \right] \\
& \leq 4T\mathbb{E} \left[\sum_{l=0}^{n_{[t,M,j]}-1} \int_{l\Delta t_j}^{(l+1)\Delta t_j} \left\| \mathbf{b}(\mathbf{U}_l^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right\|_2^2 ds + \int_{n_{[t,M,j]}\Delta t_j}^{t \wedge \rho_M^{(j)}} \left\| \mathbf{b}(\mathbf{U}_{n_{[t,M,j]}}^{(j)}) - \mathbf{b}(\mathbf{U}(s)) \right\|_2^2 ds \right] \\
& \leq 4TL_M^2\mathbb{E} \left[\sum_{l=0}^{n_{[t,M,j]}-1} \int_{l\Delta t_j}^{(l+1)\Delta t_j} \left\| \mathbf{U}_l^{(j)} - \mathbf{U}(s) \right\|_2^2 ds + \int_{n_{[t,M,j]}\Delta t_j}^{t \wedge \rho_M^{(j)}} \left\| \mathbf{U}_{n_{[t,M,j]}}^{(j)} - \mathbf{U}(s) \right\|_2^2 ds \right] \\
& = 4TL_M^2\mathbb{E} \left[\int_0^{t \wedge \rho_M^{(j)}} \left\| \bar{\mathbf{U}}^{(j)}(s) - \mathbf{U}(s) \right\|_2^2 ds \right] \\
& = 4TL_M^2 \int_0^t \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(s) - \mathbf{U}(s) \right\|_2^2 \mathbf{1}_{s < \rho_M^{(j)}} \right] ds.
\end{aligned}$$

Step 3. Decomposition of the term (82). Let C_M be a supremum of $\|\mathbf{b}\|_2^2$ over the ball $\{\|\cdot\|_2^2 \leq M\}$. Using the Jensen inequality and the locally Lipschitz continuity of \mathbf{b} , we write

$$\begin{aligned}
& 4\mathbb{E} \left[\left\| \sum_{l=0}^{n_{[t,M,j]}-1} \Delta t_j \left(\mathbf{b}(\mathbf{U}_{l+\frac{1}{2}}^{(j)}) - \mathbf{b}(\mathbf{U}_l^{(j)}) \right) + (t \wedge \rho_M^{(j)} - n_{[t,M,j]}\Delta t_j) \left(\mathbf{b}(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}) - \mathbf{b}(\mathbf{U}_{n_{[t,M,j]}}^{(j)}) \right) \right\|_2^2 \right] \\
& \leq 4TL_M^2\Delta t_j\mathbb{E} \left[\sum_{l=0}^{n_{[t,M,j]}} \left\| \mathbf{U}_{l+\frac{1}{2}}^{(j)} - \mathbf{U}_l^{(j)} \right\|_2^2 \right] \\
& = 4T\Delta t_j^3L_M^2\mathbb{E} \left[\sum_{l=0}^{n_{[t,M,j]}} \left\| \mathbf{b}(\mathbf{U}_{l+\frac{1}{2}}^{(j)}) \right\|_2^2 \right] \\
& \leq 4T(T + \Delta t_j)\Delta t_j^2L_M^2C_M,
\end{aligned}$$

where at the last line, we have used the fact that for any $l = 0, \dots, n_{[t,M,j]}$, by (35), we have $\|\mathbf{U}_{l+\frac{1}{2}}\|_2^2 \leq \|\mathbf{U}_l\|_2^2 \leq M$.

Step 4. Conclusion for the bounded event. Summing the estimates obtained from Step 1 to Step 3, we get

$$\begin{aligned}
\mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{t < \rho_M^{(j)}} \right] & \leq 2\mathbb{E} \left[\left\| \mathbf{U}_0^{(j)} - \mathbf{U}_0 \right\|_2^2 \right] \\
& \quad + 4TL_M^2 \int_0^t \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(s) - \mathbf{U}(s) \right\|_2^2 \mathbf{1}_{s < \rho_M^{(j)}} \right] ds \\
& \quad + 4T(T + \Delta t_j)\Delta t_j^2L_M^2C_M.
\end{aligned}$$

By construction, the random variable $\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\|_2$ tends to 0 almost surely as $j \rightarrow +\infty$. Furthermore, it has a fourth order moment uniform in j thanks to Lemma 4.2 and the Portemanteau theorem:

$$\begin{aligned}
\mathbb{E} \left[\left\| \mathbf{U}_0^{(j)} - \mathbf{U}_0 \right\|_2^4 \right] & \leq 8\mathbb{E} \left[\left\| \mathbf{U}_0^{(j)} \right\|_4^4 \right] + 8\mathbb{E} \left[\left\| \mathbf{U}_0 \right\|_4^4 \right] \\
& \leq 8\bar{C}^{0,4} + 8 \liminf_{j \rightarrow \infty} \mathbb{E} \left[\left\| \mathbf{U}_0^{(j)} \right\|_4^4 \right] \leq 16\bar{C}^{0,4}.
\end{aligned}$$

Thus, $\mathbb{E}[\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\|_2^2]$ tends to 0 as $j \rightarrow +\infty$. Now, by Grönwall's lemma, we have

$$\mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{t < \rho_M^{(j)}} \right] \leq \left(2\mathbb{E} \left[\left\| \mathbf{U}_0^{(j)} - \mathbf{U}_0 \right\|_2^2 \right] + 4T(T + \Delta t_j)\Delta t_j^2L_M^2C_M \right) e^{4T^2L_M^2}.$$

As a consequence,

$$\lim_{j \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{t < \rho_M^{(j)}} \right] = 0. \tag{84}$$

Step 5. The exiting trajectories. We want to bound the second term of the RHS in (79) uniformly in j . Observing that

$$\left\{ \rho_M^{(j)} \leq t \right\} = \left\{ \sup_{s \in [0, t]} \|\mathbf{U}(s)\|_2^2 \vee \sup_{s \in [0, t]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \geq M \right\},$$

we get from the Cauchy-Schwarz inequality, Lemma 4.2 and Lemma 2.7.(ii):

$$\begin{aligned} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{\rho_M^{(j)} \leq t} \right] &\leq \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^4 \right]^{1/2} \mathbb{P} \left(\sup_{s \in [0, t]} \|\mathbf{U}(s)\|_2^2 \vee \sup_{s \in [0, t]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \geq M \right)^{1/2} \\ &\leq 2 \left(\bar{C}^{0,4} + c_0^{(4)} + c_1^{(4)} C^{0,4} + c_2^{(4)} T \right)^{1/2} \mathbb{P} \left(\sup_{s \in [0, T]} \|\mathbf{U}(s)\|_2^2 \vee \sup_{s \in [0, T]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \geq M \right)^{1/2}. \end{aligned}$$

As for the second term, we have thanks to the Markov inequality and Lemma 4.4,

$$\begin{aligned} \mathbb{P} \left(\sup_{s \in [0, T]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \vee \sup_{s \in [0, T]} \|\mathbf{U}(s)\|_2^2 \geq M \right) &\leq \mathbb{P} \left(\sup_{s \in [0, T]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \geq M \right) + \mathbb{P} \left(\sup_{s \in [0, T]} \|\mathbf{U}(s)\|_2^2 \geq M \right) \\ &\leq \frac{1}{M} \mathbb{E} \left[\sup_{s \in [0, T]} \|\bar{\mathbf{U}}^{(j)}(s)\|_2^2 \right] + \mathbb{P} \left(\sup_{s \in [0, T]} \|\mathbf{U}(s)\|_2^2 \geq M \right) \\ &\leq \frac{\bar{C}_T^{0,2}}{M} + \mathbb{P} \left(\sup_{s \in [0, T]} \|\mathbf{U}(s)\|_2^2 \geq M \right) \xrightarrow{M \rightarrow \infty} 0. \end{aligned}$$

As a consequence,

$$\lim_{M \rightarrow \infty} \limsup_{j \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{\rho_M^{(j)} \leq t} \right] = 0. \quad (85)$$

Step 6. Conclusion. In the end, from (79) and (84), we have for every $M > 0$,

$$\limsup_{j \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \right] \leq \limsup_{j \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E} \left[\left\| \bar{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t) \right\|_2^2 \mathbf{1}_{\rho_M^{(j)} \leq t} \right].$$

Thus, letting $M \rightarrow +\infty$ and applying (85) yields the wanted result. \square

5. NUMERICAL EXPERIMENTS

In this section, we provide numerical experiments to illustrate Theorems 1.5 and 1.7. All the experiments in this section are performed on the Burgers equation, *i.e.* the flux function is set to be $A(u) = u^2/2$. Moreover, we will also fix the following set of parameters: $\nu = 10^{-5}$, $u_0 \equiv 0$, $g_k(x) = \cos(2\pi kx)$ for $k = 1, \dots, 4$ and $g_k \equiv 0$ for $k \geq 5$. The implicit equation in (15) is solved numerically by use of the Newton-Raphson method.

5.1. Stationarity. We seek here to give a numerical illustration of the stationarity of the Markov chain $(\mathbf{U}_n^{N, \Delta t})_{n \in \mathbb{N}}$ defined by (15) (in all this section, the number of cells and the time step will always appear as a superscript in the solutions). As already mentioned in Remark 1.9, by virtue of Birkhoff's ergodic theorem, for any test function $\varphi : \mathbb{R}_0^N \rightarrow \mathbb{R}$ such that $\varphi \in L^1(\nu_{N, \Delta t})$ and any random variable $\mathbf{V} \sim \nu_{N, \Delta t}$, the process $(\mathbf{U}_n^{N, \Delta t})_{n \in \mathbb{N}}$ shall satisfy

$$Y_n := \frac{1}{n} \sum_{l=0}^{n-1} \varphi \left(\mathbf{U}_l^{N, \Delta t} \right) \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{E} [\varphi(\mathbf{V})].$$

In Figure 5.1, we record the values of the sequence $(Y_n)_{n \geq 1}$ up to the iteration $\bar{n} = 10^4$, with the following set of parameters: $\Delta t = 10^{-3}$, $N = 512$, $\varphi = \cos(\|\cdot\|_2)$. In particular, the time interval considered here is the interval $[0, 10]$.

The stationary state seems to be reached approximately at time $t = 3$.

5.2. Convergence in space. In the following experiment, we aim to retrieve numerically the convergence result of Theorem 1.7 as N tends to infinity and for a fixed time step. Instead of computing directly the Wasserstein distance, we compute the strong L^2 error with respect to a reference solution computed with $N_{\text{ref}} = 2^{11}$. More precisely, we record in Figure 5.2 the values of

$$\left(\frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \frac{1}{N_{\text{ref}}} \sum_{i=1}^{N_{\text{ref}}} \left(U_{\left\lfloor \frac{N_{\text{ref}}(i-1)}{N} \right\rfloor + 1, n}^{N, \Delta t} - U_{i, n}^{N_{\text{ref}}, \Delta t} \right)^2 \right)^{1/2} \quad (86)$$

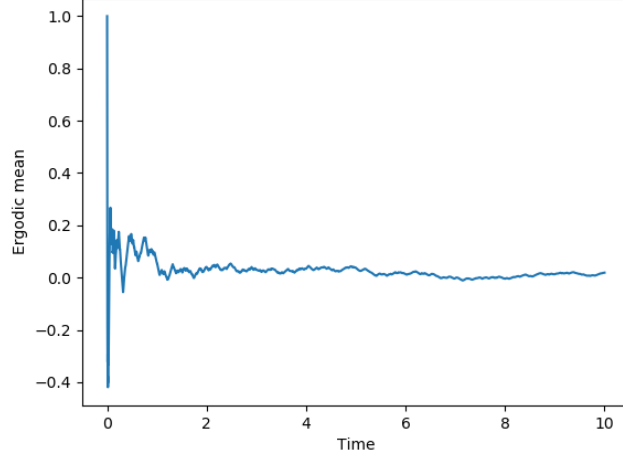


FIGURE 1. Ergodic mean of the process $(\varphi(\mathbf{U}_n^{N,\Delta t}))_{n \in \mathbb{N}}$.

as N takes values in $\{2^3, 2^4, \dots, 2^{10}\}$. For \bar{n} sufficiently large, the discrete processes aim to be close to their stationary state and thus, the value (86) is meant to be an upper bound of the Wasserstein error approximation of the invariant measure μ :

$$\begin{aligned} \left(\frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \frac{1}{N_{\text{ref}}} \sum_{i=1}^{N_{\text{ref}}} \left(U_{\lceil \frac{N_{\text{ref}}(i-1)}{N} \rceil + 1, n}^{N, \Delta t} - U_{i, n}^{N_{\text{ref}}, \Delta t} \right)^2 \right)^{1/2} &\approx \mathbb{E} \left[\left\| u_N^{(0)}(\bar{n}\Delta t) - u(\bar{n}\Delta t) \right\|_{L_0^2(\mathbb{T})}^2 \right]^{1/2} \\ &\geq W_2 \left(\mathcal{L} \left(u_N^{(0)}(\bar{n}\Delta t) \right), \mathcal{L} \left(u(\bar{n}\Delta t) \right) \right) \\ &\approx W_2 \left(\mu_N^{(0)}, \mu \right). \end{aligned}$$

Here, the other parameters are set to: $\Delta t = 10^{-3}$, $\bar{n} = 10^4$.

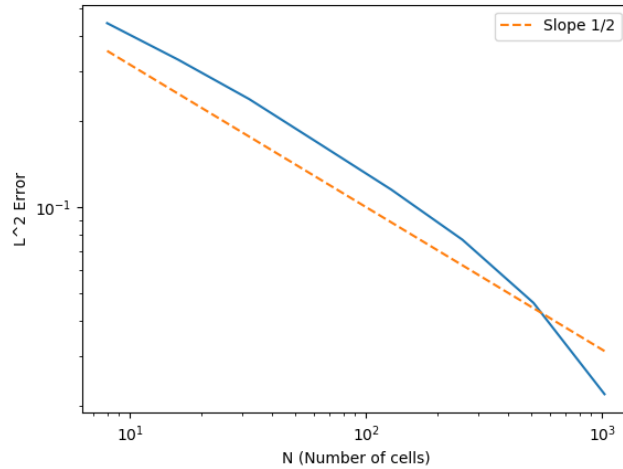


FIGURE 2. Strong error convergence at a large time with respect to N

The result shows that the convergence in space happens at an order of at least 1/2.

5.3. Convergence in time. We apply the same procedure to study the convergence with respect to the time step Δt . A reference solution is computed for the time step $\Delta t_{\text{ref}} = 2^{-11}$, and for a number $\bar{n} = 10 \times 2^{11}$ of iterations, we compute

the following L^2 error

$$\left(\frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \left\| \mathbf{U}_{\lfloor n \Delta t_{\text{ref}} \Delta t^{-1} \rfloor}^{N, \Delta t} - \mathbf{U}_{\bar{n}}^{N, \Delta t_{\text{ref}}} \right\|_2^2 \right)^{1/2},$$

which is supposed to be an upper bound of the Wasserstein distance error between the respective invariant measures for (11) and for (15):

$$\begin{aligned} \left(\frac{1}{\bar{n}} \sum_{n=1}^{\bar{n}} \left\| \mathbf{U}_{\lfloor n \Delta t_{\text{ref}} \Delta t^{-1} \rfloor}^{N, \Delta t} - \mathbf{U}_{\bar{n}}^{N, \Delta t_{\text{ref}}} \right\|_2^2 \right)^{1/2} &\approx \mathbb{E} \left[\left\| \mathbf{U}_{\lfloor \bar{n} \Delta t_{\text{ref}} \Delta t^{-1} \rfloor}^{N, \Delta t} - \mathbf{U}_{\bar{n}}^{N, \Delta t_{\text{ref}}} \right\|_2^2 \right]^{1/2} \\ &\geq W_2 \left(\mathcal{L} \left(\mathbf{U}_{\lfloor \bar{n} \Delta t_{\text{ref}} \Delta t^{-1} \rfloor}^{N, \Delta t} \right), \mathcal{L} \left(\mathbf{U}_{\bar{n}}^{N, \Delta t_{\text{ref}}} \right) \right) \\ &\approx W_2 \left(\nu_{N, \Delta t}, \nu_N \right). \end{aligned}$$

This error is evaluated when Δt takes values in $\{2^{-4}, 2^{-5}, \dots, 2^{-10}\}$ and for $N = 256$.

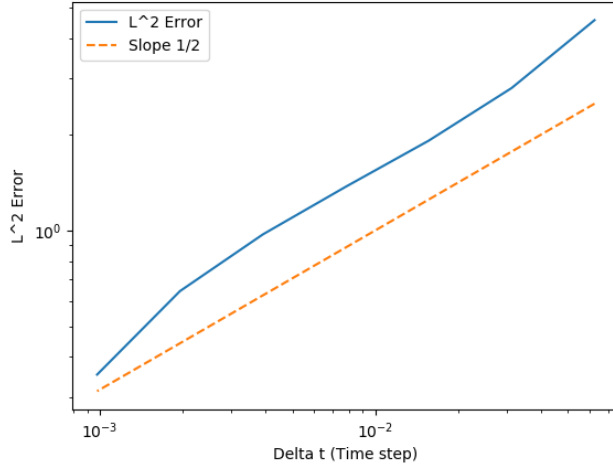


FIGURE 3. Strong error convergence at a large time with respect to Δt

A rate of convergence of 1/2 also stems from this experiment.

5.4. Burgulence estimates. Endowed with the Burgers flux function, Equation (1) may be interpreted as a one-dimensional and simplified version of the Navier-Stokes system, and as such, it is considered a toy model for turbulence (the so-called *burgulence*, see for instance [6, Chapter 1] or [20, 21] in this prospect). According to the turbulence theory dating back to Kolmogorov, universal properties emerge as a turbulent dynamical system approaches its stationary state. Here, we will try to recover numerically two of these properties which, in a framework close to Equation (1), have been proved rigorously in Boritchev's work [6]. The first one concerns the decay rate of the energy spectrum. Let u be an $L_0^2(\mathbb{T})$ -valued random variable whose distribution is the invariant measure μ of the process associated to (1), and let \hat{u}_k denote, for any $k \geq 1$, the k -th Fourier coefficient of u :

$$\hat{u}_k := \int_{\mathbb{T}} u(x) e^{-2i\pi kx} dx.$$

Here, we call *energy spectrum* the function E defined for all $k \geq 1$ by $E(k) := \mathbb{E}[|\hat{u}_k|^2]$. This function satisfies a specific decay rate [6, Theorem 4.7.3] up to some averaging around the neighbour coefficients of k . For any $k \geq 1$ and any $M > 1$, we set $\mathcal{S}_{k,M} := [M^{-1}k, Mk] \cap \mathbb{N}$ which defines a set of neighbours of k . We now state the result contained in [6, Theorem 4.7.3]:

Proposition 5.1. *There exists an interval $I \subset [0, 1]$, called the inertial range, such that*

$$\frac{1}{|\mathcal{S}_{k,M}|} \sum_{j \in \mathcal{S}_{k,M}} E(j) \sim k^{-2}, \quad k^{-1} \in I.$$

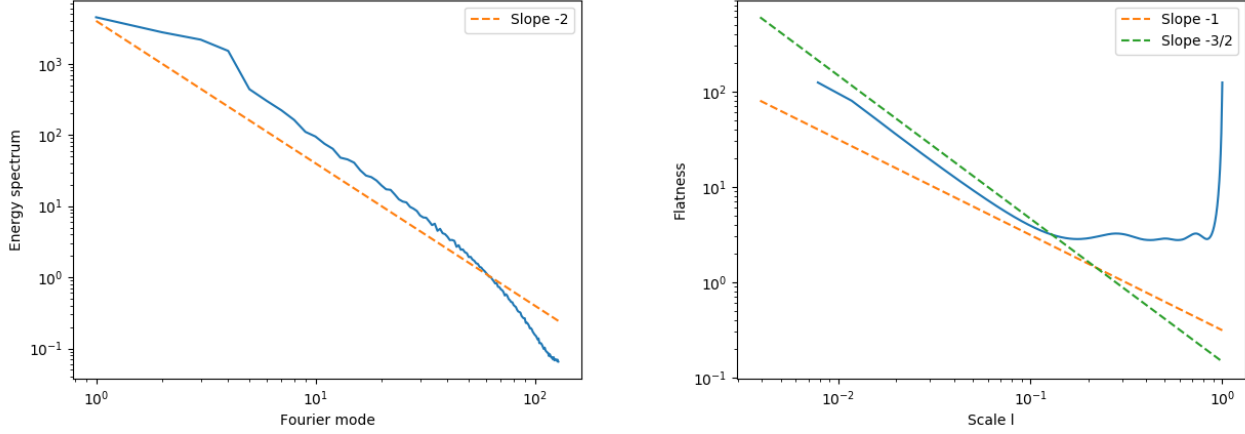


FIGURE 4. Decay rates for the energy spectrum (left) and for the flatness (right).

Here, $x \sim y$ means that there exists a constant $C > 0$ such that $C^{-1}y \leq x \leq Cy$. The inertial range I is defined with more details in [6, Section 4.6]. To give a physical interpretation of this interval, it corresponds to the range of scales in which the energy of the system is transported from large scales to smaller ones. If we write $I = [\alpha, \beta]$, the inertial range is positioned between the *energy range* $[\beta, 1]$ containing the large scales, which in our case are generated by the stochastic forcing, and the *dissipation range* $[0, \alpha]$ containing the small scales dissipated by the viscous term. In particular, α depends linearly on ν .

The second universal property of interest concerns the *flatness*, that is the function F defined by

$$F(l) := \frac{\mathbb{E} \left[\int_{\mathbb{T}} |u(x+l) - u(x)|^4 dx \right]}{\mathbb{E} \left[\int_{\mathbb{T}} |u(x+l) - u(x)|^2 dx \right]^2}, \quad l \in \mathbb{T},$$

where u is a random variable with distribution μ . The flatness aims to be an indicator of the spatial intermittency in the turbulent system described by (1). A decay rate for F in the inertial range is provided in [6, Corollary 4.6.9]:

Proposition 5.2. *Let I be the inertial range from Proposition 5.1. Then,*

$$F(l) \sim l^{-1}, \quad l \in I.$$

From (15), we computed the numerical approximations of the energy spectrum and the flatness. These computations are plotted in Figure 4. More precisely, we used the following respective approximations:

$$E(k) \approx \frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \left| \hat{U}_{k,n}^{N,\Delta t} \right|^2, \quad k = 1, \dots, \left\lceil \frac{N}{2} \right\rceil;$$

$$F\left(\frac{j}{N}\right) \approx \frac{\frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \frac{1}{N} \sum_{i=1}^N \left| U_{i+j,n}^{N,\Delta t} - U_{i,n}^{N,\Delta t} \right|^4}{\left(\frac{1}{\bar{n}} \sum_{n=0}^{\bar{n}-1} \frac{1}{N} \sum_{i=1}^N \left| U_{i+j,n}^{N,\Delta t} - U_{i,n}^{N,\Delta t} \right|^2 \right)^2}, \quad j \in \mathbb{Z}/N\mathbb{Z}.$$

Here, $\hat{U}_n^{N,\Delta t}$ is the discrete Fourier transform of $\mathbf{U}_n^{N,\Delta t}$, which we computed using an FFT algorithm from the Python library `numpy.fft` (v1.17). In both experiments, we have taken $N = 256$, $\Delta t = 10^{-3}$, and $\bar{n} = 10000$.

The result of Proposition 5.1 seems recovered as the slope of the energy spectrum tends to behave like k^{-2} in some sub-interval of $[0, 1]$. As regards the flatness, for intermediate scales, a decay rate varying from the order $-3/2$ to the order -1 is observed.

Proof of Lemma 2.1. Let $m = 0$. Let $p \in [1, +\infty)$ and let $\mathbf{u} \in \mathbb{R}^N$ be as in the statement. We have

$$\begin{aligned}
 \left\| \mathbf{D}_N^{(0)} \mathbf{u} \right\|_p^p &= \frac{1}{N} \sum_{u_i \geq 0} |u_i|^p + \frac{1}{N} \sum_{u_i < 0} |u_i|^p \\
 &\leq \frac{1}{N} \sum_{u_i \geq 0} |u_i - u_{i-}|^p + \frac{1}{N} \sum_{u_i < 0} |u_i - u_{i+}|^p \\
 &\leq \frac{1}{N} \sum_{u_i \geq 0} N^{p-1} \sum_{j=1}^N |u_{j+1} - u_j|^p + \frac{1}{N} \sum_{u_i < 0} N^{p-1} \sum_{j=1}^N |u_{j+1} - u_j|^p \\
 &= \frac{1}{N} \sum_{j=1}^N |N(u_{j+1} - u_j)|^p = \left\| \mathbf{D}_N^{(1)} \mathbf{u} \right\|_p^p,
 \end{aligned}$$

where we used the Jensen inequality passing from the third to the fourth line. We just have proved the wanted inequality when $m = 0$ but the case $m = 1$ is proved in the same way. \square

Proof of Lemma 2.2. For $\mathbf{u} \in \mathbb{R}_0^N$ and $p \in 2\mathbb{N}^*$, we have

$$\begin{aligned}
 \left\langle \mathbf{D}_N^{(1)}(\mathbf{u}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{u} \right\rangle &= N \sum_{i=1}^N \left(u_{i+1}^{p-1} - u_i^{p-1} \right) (u_{i+1} - u_i) \\
 &= N(p-1) \sum_{i=1}^N (u_{i+1} - u_i) \int_{u_i}^{u_{i+1}} |z|^{p-2} dz \\
 &= N(p-1) \sum_{i=1}^N (u_{i+1} - u_i) \int_{u_i}^{u_{i+1}} \left(|z|^{p/2-1} \right)^2 dz \\
 &\geq N(p-1) \sum_{i=1}^N \left(\int_{u_i}^{u_{i+1}} |z|^{p/2-1} dz \right)^2 \quad (\text{by Jensen's inequality}) \\
 &= \frac{4N(p-1)}{p^2} \sum_{i=1}^N \left(\int_{u_i}^{u_{i+1}} \frac{d}{dz} \left(\text{sign}(z) |z|^{p/2} \right) dz \right)^2 \\
 &= \frac{4N(p-1)}{p^2} \sum_{i=1}^N \left(\text{sign}(u_{i+1}) |u_{i+1}|^{p/2} - \text{sign}(u_i) |u_i|^{p/2} \right)^2.
 \end{aligned}$$

We can now apply Lemma 2.1 to the vector $(\text{sign}(u_i) |u_i|^{p/2})_{i=1}^N$ and eventually, we obtain

$$\left\langle \mathbf{D}_N^{(1)}(\mathbf{u}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{u} \right\rangle \geq \frac{4(p-1)}{Np^2} \sum_{i=1}^N \left(\text{sign}(u_i) |u_i|^{p/2} \right)^2 = \frac{4(p-1)}{p^2} \|\mathbf{u}\|_p^p.$$

\square

Proof of Lemma 2.3. Let $\mathbf{u} \in \mathbb{R}_0^N$ and $q \in 2\mathbb{N}^*$. Summing by parts and using (12) and (13), we have

$$\begin{aligned}
 \sum_{i=1}^N u_i^{q-1} (\bar{A}(u_i, u_{i+1}) - \bar{A}(u_{i-1}, u_i)) &= - \sum_{i=1}^N \left(u_{i+1}^{q-1} - u_i^{q-1} \right) \bar{A}(u_i, u_{i+1}) \\
 &= - \sum_{i=1}^N \int_{u_i^{q-1}}^{u_{i+1}^{q-1}} \bar{A}(u_i, u_{i+1}) dz \\
 &\geq - \sum_{i=1}^N \int_{u_i^{q-1}}^{u_{i+1}^{q-1}} \bar{A} \left(z^{1/(q-1)}, z^{1/(q-1)} \right) dz \\
 &= - \sum_{i=1}^N \int_{u_i^{q-1}}^{u_{i+1}^{q-1}} A \left(z^{1/(q-1)} \right) dz \\
 &= - \sum_{i=1}^N \int_{u_i^{q-1}}^{u_{i+1}^{q-1}} \frac{d}{dz} (\mathcal{A}_q(z)) dz = 0,
 \end{aligned}$$

where \mathcal{A}_q denotes a function defined on \mathbb{R} such that $\mathcal{A}'_q(z) = A(z^{1/(q-1)})$. □

Proof of Lemma 2.4. (i) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$. From the definition of \mathbf{b} , we write

$$\begin{aligned} \langle \mathbf{sign}(\mathbf{u} - \mathbf{v}), \mathbf{b}(\mathbf{u}) - \mathbf{b}(\mathbf{v}) \rangle &= - \sum_{i=1}^N \text{sign}(u_i - v_i) (\bar{A}(u_i, u_{i+1}) - \bar{A}(u_{i-1}, u_i) - \bar{A}(v_i, v_{i+1}) + \bar{A}(v_{i-1}, v_i)) \\ &\quad + \nu N \sum_{i=1}^N \text{sign}(u_i - v_i) (u_{i+1} - 2u_i + u_{i-1} - v_{i+1} + 2v_i - v_{i-1}). \end{aligned}$$

By periodicity, both terms of the right-hand side can be summed by parts, which leads to

$$\begin{aligned} \langle \mathbf{sign}(\mathbf{u} - \mathbf{v}), \mathbf{b}(\mathbf{u}) - \mathbf{b}(\mathbf{v}) \rangle &= \sum_{i=1}^N (\text{sign}(u_{i+1} - v_{i+1}) - \text{sign}(u_i - v_i)) (\bar{A}(u_i, u_{i+1}) - \bar{A}(v_i, v_{i+1})) \\ &\quad - \nu N \sum_{i=1}^N (\text{sign}(u_{i+1} - v_{i+1}) - \text{sign}(u_i - v_i)) ((u_{i+1} - v_{i+1}) - (u_i - v_i)). \end{aligned}$$

Observe that since the function $\text{sign} : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing, each term of the second sum is non-negative. As for the first sum, it follows from the monotonicity property of \bar{A} that each term is non-positive. Let us address for instance the case where $u_{i+1} \geq v_{i+1}$ and $u_i \leq v_i$. Then, on the one hand, we have $\text{sign}(u_{i+1} - v_{i+1}) - \text{sign}(u_i - v_i) = 2$. On the other hand, we have

$$\begin{aligned} \bar{A}(u_i, u_{i+1}) - \bar{A}(v_i, v_{i+1}) &= (\bar{A}(u_i, u_{i+1}) - \bar{A}(u_i, v_{i+1})) + (\bar{A}(u_i, v_{i+1}) - \bar{A}(v_i, v_{i+1})) \\ &= \int_{v_{i+1}}^{u_{i+1}} \partial_2 \bar{A}(u_i, z) dz - \int_{u_i}^{v_i} \partial_1 \bar{A}(z, v_{i+1}) dz \leq 0. \end{aligned}$$

The case where $u_{i+1} \leq v_{i+1}$ and $u_i \geq v_i$ is treated symmetrically.

(ii) Let $\mathbf{u} \in \mathbb{R}_0^N$. We have

$$\langle \mathbf{u}, \mathbf{b}(\mathbf{u}) \rangle = - \sum_{i=1}^N u_i (\bar{A}(u_i, u_{i+1}) - \bar{A}(u_{i-1}, u_i)) + \nu N \sum_{i=1}^N u_i (u_{i+1} - 2u_i + u_{i-1}).$$

Lemma 2.3 with $q = 2$ shows that the first term of the above decomposition is non-positive. Summing by parts the second term yields the result. □

Proof of Lemma 3.1. (i) The wanted equality follows from standard computations.

(ii) Let us start with the first inequality. We have

$$\begin{aligned} \left\| \Psi_N^{(1)} \mathbf{v} - \Psi_N^{(0)} \mathbf{v} \right\|_{L_0^2(\mathbb{T})}^2 &= \int_{\mathbb{T}} \left(\sum_{i=1}^N v_i N \left(x - \frac{i-1}{N} \right) \mathbf{1}_{(\frac{i-1}{N}, \frac{i}{N}]}(x) + \sum_{i=1}^N v_i N \left(\frac{i+1}{N} - x \right) \mathbf{1}_{(\frac{i}{N}, \frac{i+1}{N}]}(x) - \sum_{i=1}^N v_i \mathbf{1}_{(\frac{i-1}{N}, \frac{i}{N}]}(x) \right)^2 dx \\ &= \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(v_i N \left(x - \frac{i-1}{N} \right) + v_{i-1} N \left(\frac{i}{N} - x \right) - v_i \right)^2 dx \\ &= \sum_{i=1}^N (v_i - v_{i-1})^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (Nx - i)^2 dx \\ &= \frac{1}{3N} \sum_{i=1}^N (v_i - v_{i-1})^2 = \frac{1}{3N^2} \left\| \mathbf{D}_N^{(1)} \mathbf{v} \right\|_2^2. \end{aligned}$$

As for the second inequality, we have

$$\begin{aligned}
& \left\| \Psi_N^{(2)} \mathbf{v} - \Psi_N^{(0)} \mathbf{v} \right\|_{L_0^2(\mathbb{T})}^2 \\
&= \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\frac{1}{2} v_{i+1} (Nx - i) (Nx - (i-1)) - v_i (Nx - (i-1)) (Nx - (i+1)) + \frac{1}{2} v_{i-1} (Nx - i) (Nx - (i+1)) - v_i \right)^2 dx \\
&= \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\frac{1}{2} v_{i+1} (Nx - i)^2 + \frac{1}{2} v_{i+1} (Nx - i) - v_i (Nx - i)^2 + \frac{1}{2} v_{i-1} (Nx - i)^2 - \frac{1}{2} v_{i-1} (Nx - i) \right)^2 dx \\
&\leq 3 \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\left(\frac{1}{2} v_{i+1} - v_i + \frac{1}{2} v_{i-1} \right) (Nx - i)^2 \right)^2 dx \\
&\quad + 3 \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\left(\frac{1}{2} v_{i+1} - \frac{1}{2} v_i \right) (Nx - i) \right)^2 dx \\
&\quad + 3 \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\left(\frac{1}{2} v_i - \frac{1}{2} v_{i-1} \right) (Nx - i) \right)^2 dx \\
&= \frac{3}{4} \sum_{i=1}^N (v_{i+1} - 2v_i + v_{i-1})^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (Nx - i)^4 dx + \frac{3}{2} \sum_{i=1}^N (v_{i+1} - v_i)^2 \int_{\frac{i-1}{N}}^{\frac{i}{N}} (Nx - i)^2 dx \\
&= \frac{3}{20N^4} \left\| \mathbf{D}_N^{(2)} \mathbf{v} \right\|_2^2 + \frac{1}{2N^2} \left\| \mathbf{D}_N^{(1)} \mathbf{v} \right\|_2^2. \quad \square
\end{aligned}$$

Acknowledgements. The authors are thankful to Charles-Édouard Bréhier for insightful discussions.

REFERENCES

- [1] Aureli Alabert and István Gyongy. *On Numerical Approximation of Stochastic Burgers' Equation*, pages 1–15. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.
- [2] Caroline Bauzet, Julia Charrier, and Thierry Gallouët. Convergence of flux-splitting finite volume schemes for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation. *Mathematics of Computation*, 85:2777–2813, November 2016.
- [3] Caroline Bauzet, Julia Charrier, and Thierry Gallouët. Convergence of monotone finite volume schemes for hyperbolic scalar conservation laws with multiplicative noise. *Stochastics and Partial Differential Equations Analysis and Computations*, 4(1):150–223, 2016.
- [4] Marianne Bessemoulin-Chatard, Claire Chainais-Hillairet, and Francis Filbet. On discrete functional inequalities for some finite volume schemes. *IMA Journal of Numerical Analysis*, pages 10–32, July 2014.
- [5] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [6] Alexandre Boritchev. *Generalised Burgers equation with random force and small viscosity*. Theses, Ecole Polytechnique X, October 2012.
- [7] François Bouchut, Robert Eymard, and Alain Prignet. Finite volume schemes for the approximation via characteristics of linear convection equations with irregular data. *Journal of Evolution Equations*, 11(3):687–724, Sep 2011.
- [8] Charles-Édouard Bréhier. Approximation of the invariant measure with an Euler scheme for Stochastic PDE's driven by Space-Time White Noise. *Potential Analysis*, 1(40):1–40, January 2014.
- [9] Charles-Édouard Bréhier and Marie Kopec. Approximation of the invariant law of SPDEs: error analysis using a Poisson equation for a full-discretization scheme. *IMA Journal of Numerical Analysis*, 2016.
- [10] Charles-Édouard Bréhier and Gilles Vilmart. High order integrator for sampling the invariant distribution of a class of parabolic stochastic PDEs with additive space-time noise. *SIAM Journal on Scientific Computing*, 38(4):A2283–A2306, 2016.
- [11] Chuchu Chen, Jialin Hong, and Xu Wang. Approximation of invariant measure for damped stochastic nonlinear Schrödinger equation via an ergodic numerical scheme. *Potential Analysis*, 46(2):323–367, Feb 2017.
- [12] Ziheng Chen, Siqing Gan, and Xiaojie Wang. A full-discrete exponential Euler approximation of the invariant measure for parabolic stochastic partial differential equations. November 2018.
- [13] Jianbo Cui, Jialin Hong, and Liying Sun. Weak convergence and invariant measure of a full discretization for non-globally Lipschitz parabolic SPDE. arXiv:1811.04075, 2018.
- [14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1992.
- [15] Giuseppe Da Prato and Jerzy Zabczyk. *Ergodicity for Infinite Dimensional Systems*. Cambridge Monographs on Partic. Cambridge University Press, 1996.
- [16] Klaus Deimling. *Nonlinear functional analysis*. Springer-Verlag, 1985.
- [17] Sylvain Dotti. *Numerical approximation of hyperbolic stochastic scalar conservation laws*. Theses, Aix-Marseille Université (AMU), December 2017.
- [18] Sylvain Dotti and Julien Vovelle. Convergence of approximations to stochastic scalar conservation laws. *Archive for Rational Mechanics and Analysis*, 230(2):539–591, Nov 2018.
- [19] Sylvain Dotti and Julien Vovelle. Convergence of the finite volume method for scalar conservation laws with multiplicative noise: an approach by kinetic formulation. *Stochastics and Partial Differential Equations: Analysis and Computations*, Jul 2019.
- [20] Weinan E. Stochastic hydrodynamics, 2000.
- [21] Weinan E. Stochastic PDEs in turbulence theory, 2000.

- [22] Bjorn Engquist and Stanley Osher. Stable and entropy satisfying approximations for transonic flow calculations. *Mathematics of Computation*, 34(149):45–75, 1980.
- [23] Robert Eymard, Thierry Gallouët, and Raphaële Herbin. Finite volume methods. In *Solution of Equation in \mathbb{R}^n (Part 3), Techniques of Scientific Computing (Part 3)*, volume 7 of *Handbook of Numerical Analysis*, pages 713 – 1018. Elsevier, 2000.
- [24] Martin Hairer and Jochen Voss. Approximations to the stochastic Burgers equation. *Journal of Nonlinear Science*, 21(6):897–920, Dec 2011.
- [25] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*. Kodansha scientific books. North-Holland, 1989.
- [26] Charalambos Makridakis and Benoît Perthame. Sharp CFL, discrete kinetic formulation, and entropic schemes for scalar conservation laws. *SIAM J. Numer. Anal.*, 41(3):1032–1051, 2003.
- [27] Sofiane Martel and Julien Reygner. Viscous scalar conservation law: strong solution and invariant measure. arXiv:1905.07908, 2019.
- [28] Jonathan C. Mattingly, Andrew M. Stuart, and Desmond J. Higham. Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise. *Stochastic Processes and their Applications*, 101(2):185 – 232, 2002.
- [29] Cédric Villani. *Optimal Transport: Old and New*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2008.

UNIVERSITÉ PARIS-EST, LABORATOIRE D’HYDRAULIQUE SAINT-VENANT (ENPC – EDF R&D – CEREMA), 78401 CHATOU CEDEX, FRANCE.

INRIA PARIS, MATHÉRIALS

E-mail address: `sebastien.boyaaval@enpc.fr`

UNIVERSITÉ PARIS-EST, CERMICS (ENPC), 77455 MARNE-LA-VALLÉE, FRANCE.

UNIVERSITÉ PARIS-EST, LABORATOIRE D’HYDRAULIQUE SAINT-VENANT (ENPC – EDF R&D – CEREMA), 78401 CHATOU CEDEX, FRANCE.

E-mail address: `sofiane.martel@enpc.fr`

UNIVERSITÉ PARIS-EST, CERMICS (ENPC), 77455 MARNE-LA-VALLÉE, FRANCE.

E-mail address: `julien.reygner@enpc.fr`