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# Characterization of the Undesirable Global Minima of the Godard Cost Function: Case of Noncircular Symmetric Signals

Sébastien Houcke and Antoine Chevreuil

**Abstract**—The deconvolution of a filtered version of a zero-mean normalized independent and identically distributed (i.i.d.) signal  $(s_n)_{n \in \mathbb{Z}}$  having a strictly negative Kurtosis  $\gamma_2 = \mathbb{E}[|s_n|^4] - 2(\mathbb{E}[|s_n|^2])^2 - |\mathbb{E}[s_n^2]|^2$  is addressed. This correspondence focuses on the global minimizers of the Godard function. A well-known result states that these minimizers achieve deconvolution at least if the input signal shows the symmetry  $\mathbb{E}[s^2] = 0$ . When this constraint is relaxed,  $(s_n)_{n \in \mathbb{Z}}$  is said to be noncircular symmetric; it is shown that the minimizers achieve deconvolution if and only if  $2|\mathbb{E}[s_n^2]|^2 < -\gamma_2(s)$ . If this condition is not met, the global minimizers are found to be finite-impulse-response filters with two taps.

**Index Terms**—Deconvolution, constant modulus algorithm, contrast function, Godard algorithm.

## I. INTRODUCTION

In recent years, much attention has been paid to source deconvolution, mainly for the wide range of potential applications, among which blind equalization problems in digital communications play a central role. The principles of deconvolution have been set forth by Donoho [1] and Benveniste *et al.* [2], who introduced the notion of objective functions for stationary time series. The Godard [3], [4] and Shalvi–Weinstein [5] objective functions are by far the most popular of the literature (there are intrinsic links between both of them, as proved by Regalia [6]).

The optimization of these functions achieves deconvolution under the condition that the source is circularly symmetric complex. In this correspondence, we investigate whether the deconvolution is achieved or not when this assumption on the source is relaxed.

We briefly justify why considering such a problem is not just a mental exercise. The first point is that many communication sources are not circular (real sources of course, but also complex sources having unusual constellations—see Benedetto *et al.* [7, ch. 5.4]): in this case, no general result has been presented—as far as we know—concerning the minimization of the Godard function. The second point is that, instead of considering the constant modulus objective function, one could of course achieve the deconvolution by maximizing the Donoho objective function (say, the modulus of the fourth-order cumulant), but the estimation of this latter requires us to estimate an extra term as compared with the circular case that may impair the accuracy of the estimation. Moreover, this extra term cannot be consistently estimated in certain cases: this occurs when the received signal is corrupted by

a multiplicative time-varying complex exponential (Doppler shift or frequency offset in the communication framework).

This problem seems to have been neglected in the literature. As a matter of fact, only very partial results have been obtained so far. Ding *et al.* [8] have shown that any filtered version of a modulus one sequence (*except a binary source*) has modulus one if and only if the filter in question has a single tap, thus proving that the minimizers of Godard function achieve deconvolution. However, this result applies *only* to modulus one sequences [circular or not, independent and identically distributed (i.i.d.) or not]. On the contrary, Papadias [9] showed that for *binary* sources, the minimization of the Godard function does not allow one to achieve deconvolution directly: indeed, it is easy to show off certain filtered versions of a binary source having the modulus one property. This short survey proves that the determination of the minimizers of the Godard function is still an open problem in general.

Section II describes the problem: the minimization of a functional is defined. The optimization is easily carried out for certain sources (a binary source for instance): this point is proved in Section III. The general case, more involved, is worked out in Section IV. In Section V, we recast the results obtained into the deconvolution framework. Illustrations of the results are provided in Section VI

## II. SETTING THE PROBLEM

Let  $(s_n)_{n \in \mathbb{Z}}$  denote a (complex or real-valued) i.i.d. sequence, the moments of which exist up to the fourth order. Without loss of generality, we assume in the sequel that  $s_n$  is zero mean and normalized, i.e.,  $\mathbb{E}[|s_n|^2] = 1$ . We will denote by  $\gamma_2$  its Kurtosis:  $\gamma_2 = \mathbb{E}[|s_n|^4] - |\mathbb{E}[s_n^2]|^2 - 2$ . As a main assumption, we consider along the paper that the Kurtosis is strictly negative (the source is said to be platykurtic: this is not restrictive in the framework of digital communications). For any nonnull  $f = (f_k)_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$  we consider the time series

$$z_n = \sum_{k=-\infty}^{+\infty} f_k s_{n-k}. \quad (1)$$

On the one hand, the standard Shalvi–Weinstein function [5] is

$$\frac{\mathbb{E}[|z_n|^4]}{(\mathbb{E}[|z_n|^2])^2} \quad (2)$$

which may be expanded as

$$J(f) = \frac{-y \sum_k |f_k|^4 + x |\sum_k f_k|^2}{(\sum_k |f_k|^2)^2} + 2. \quad (3)$$

where we set  $y = -\gamma_2 > 0$  and  $x = |\mathbb{E}[s_n^2]|^2$ . If the source is symmetric (implying that  $\mathbb{E}[s_n^2] = 0$ ), hence  $J(f)$  reduces to  $2 - y(\sum_k |f_k|^4 / (\sum_k |f_k|^2)^2)$ . Since  $y > 0$ , we obviously have  $J(f) \geq 2 - y$ , with equality if and only if the series  $f$  has a single nonzero coefficient. For  $x > 0$ , however, the minimization of  $J$  is not that straightforward. We will denote by  $\kappa$  the lower bound of  $J$ :

$$\kappa = \inf_{f \in \ell_2(\mathbb{Z})} J(f). \quad (4)$$

On the other hand, the Godard objective function is defined as

$$\tilde{J}(f) = \mathbb{E}[ (|z_n|^2 - 1)^2 ]. \quad (5)$$

S. Houcke is with Département Signal et Communication PRACOM TAMCI, ENST Bretagne, BP 832, 29285 Brest Cedex, France (e-mail: Sebastien.Houcke@enst-bretagne.fr).

A. Chevreuil is with the Institut Gaspard Monge UMR 8049, CNRS IGM-LabInfo, “Équipe Signal Pour les Communications” Institut Gaspard Monge, Université de Marne-la-Vallée, 77454 Marne-la-Vallée Cedex 2, France (e-mail: Antoine.Chevreuil@univ-mlv.fr).

Expanding (5) as  $\tilde{J}(f) = \mathbb{E}[|z_n|^4] - 2\mathbb{E}[|z_n|^2] + 1$  gives

$$\tilde{J}(f) \geq J(f)(\mathbb{E}[|z_n|^2])^2 - 2\mathbb{E}[|z_n|^2] + 1 = \kappa\rho^2 - 2\rho + 1$$

where we have set  $\rho = \mathbb{E}[|z_n|^2]$ . As  $\kappa \geq 1$  (this is due to the Jensen inequality applied to (2)), the right-hand side (RHS) of (6) is a convex second-order polynomial in  $\rho$  and is minimum for  $\rho = 1/\kappa$ . Setting  $\tilde{\kappa} = \inf_{f \in \ell_2(\mathbb{Z})} \tilde{J}(f)$ , we have  $\tilde{\kappa} \geq 1 - (1/\kappa)$ . If  $f$  reaches the lower bound of  $\tilde{J}$ , then  $\tilde{J}(1/\|f\|_2\sqrt{\kappa})f = 1 - (1/\kappa)$ . This proves that  $\tilde{\kappa} = 1 - (1/\kappa)$ . Moreover, if  $f$  reaches the lower bound  $\tilde{\kappa}$  of  $\tilde{J}$ , then it reaches the lower bound  $\kappa$  of  $J$ .

With this link between  $J$  and  $\tilde{J}$  being specified, we may now concentrate on the minimization of  $J$ , which is more tractable than this of  $\tilde{J}$ . Actually, we answer the three points: 1) compute  $\kappa$  with regard to  $x, y$ ; 2) show that  $\kappa$  is reached; 3) characterize the series  $f$  such that  $J(f) = \kappa$ .

As the subsequent developments prove, it is convenient to introduce the following subset of  $\ell_2(\mathbb{Z})$ .

*Definition 1:* for  $\theta \in (-\pi/2, \pi/2)$  and every  $p, q \in \mathbb{Z}, p \neq q$ , we define  $(e_k^{(p,q)}(\theta))_{k \in \mathbb{Z}}$  as

$$\begin{cases} e_p^{(p,q)}(\theta) = \cos \theta \\ e_q^{(p,q)}(\theta) = i \sin \theta \\ e_k^{(p,q)}(\theta) = 0 \text{ if } k \neq p, k \neq q \end{cases} \quad (6)$$

and we set  $\mathcal{F}(\theta) = \bigcup_{p \neq q} \{\lambda e_k^{(p,q)}(\theta) \mid \lambda \in \mathbb{C}\}$ .

For instance,  $\mathcal{F}(0)$  is the set of Kronecker series. For a general  $\theta, \mathcal{F}(\theta)$  is clearly a set of series having at most two nonnull coefficients.

### III. SOME STRAIGHTFORWARD ALGEBRA: WHEN $2x \leq y$

Keeping in mind that for null  $x$ , the set of global minima of  $J$  is  $\mathcal{F}(0)$ , it may be understood that  $\mathcal{F}(0)$  remains the set of global minima of  $J$  when  $x$  is ‘‘small.’’

*Result 1:* If  $2x \leq y$ , we have  $\kappa = 2 + x - y$ . Besides, if  $2x < y$ , equality  $J(f) = \kappa$  holds if and only if  $f \in \mathcal{F}(0)$ . If  $2x = y$ ,  $J(f) = \kappa$  holds if and only if there exists  $\theta \in (-\pi/2, \pi/2)$  such that  $f \in \mathcal{F}(\theta)$ .

*Proof:* Setting  $\eta = y - 2x$ , and

$$\Delta = x \left| \sum_k f_k^2 \right|^2 - y \sum_k |f_k|^4 + (y - x) \left( \sum_k |f_k|^2 \right)^2$$

we deduce that  $J(f) \geq 2 + x - y$  holds if and only if  $\Delta \geq 0$  holds. Expanding  $\Delta$  gives

$$\Delta = 2x \sum_{k < l} |f_k f_l|^2 (1 + \cos(\theta_k - \theta_l)) + \eta \left( \left( \sum_k |f_k|^2 \right)^2 - \sum_k |f_k|^4 \right) \quad (7)$$

where we have set  $f_k^2 = |f_k|^2 e^{i\theta_k}$ . By definition,  $\eta \geq 0$ , hence  $\Delta \geq 0$  as the sum of two nonnegative terms. This gives  $\kappa \geq 2 + x - y$ . As far as the equality is concerned, we notice that  $\Delta = 0$  occurs if and only if the two terms of the RHS of (7) equal zero.

- If  $\eta > 0$ , the second term of the RHS of (7) equals zero if and only if  $f \in \mathcal{F}(0)$ ; conversely,  $f \in \mathcal{F}(0)$  makes the first term of the RHS of (7) be zero. Hence,  $\Delta = 0$ .
- If  $\eta = 0$ ,  $\Delta = 0$  if and only if the first term of the RHS of (7) is zero. The latter being a sum of nonnegative terms, this reads

$$\text{for all } k < l, |f_k f_l|^2 (1 + \cos(\theta_k - \theta_l)) = 0. \quad (8)$$

Assume that  $f$  has at least three nonnull coefficients, i.e., there exist  $k_1 < k_2 < k_3$  such that  $f_{k_1} f_{k_2} f_{k_3} \neq 0$ . Without restriction, we may set  $\theta_{k_1} = 0$ . Condition (8) implies that

$$\theta_{k_2} = \frac{\pi}{2}[\pi] \quad (9)$$

$$\theta_{k_3} = \frac{\pi}{2}[\pi] \quad (10)$$

$$\theta_{k_3} = \theta_{k_2} + \frac{\pi}{2}[\pi]. \quad (11)$$

From (9) and (11), it may be deduced that  $\theta_{k_3} = \pi[\pi]$ : This is in contradiction with (10). Hence,  $f$  has two nonnull coefficients at most: let them be  $f_{k_1}$  and  $f_{k_2}$ ; then (9) holds, and there exists a unique  $\theta \in (-\pi/2, \pi/2)$  such that  $\cos \theta = (|f_{k_1}| / \sqrt{|f_{k_1}|^2 + |f_{k_2}|^2})$  and  $\sin \theta = -(|f_{k_2}| / \sqrt{|f_{k_1}|^2 + |f_{k_2}|^2})$ . We obviously have  $f \in \mathcal{F}(\theta)$ . Conversely, if  $f \in \mathcal{F}(\theta)$  for any  $\theta$ , then  $\Delta = 0$ . ■

### IV. NEW RESULTS: WHEN $2x > y$

The case  $y < 2x$  prevents one from using the same direct arguments. The first point consists of restricting the minimization of  $J$  over  $\mathbb{C}^N$  for a given integer  $N \geq 2$ . This has to be rigorously justified. In this section, we set

$$\kappa_N = \inf_{f \in \mathbb{C}^N} J(f).$$

The series  $(\kappa_N)_{N \in \mathbb{N}}$  is decreasing, and we have Lemma 1.

*Lemma 1:*

$$\kappa = \lim_{N \rightarrow \infty} \kappa_N. \quad (12)$$

The proof is relegated to Appendix I.

Owing to this result, we focus on the minimization of  $J$  over  $\mathbb{C}^N$ . For the sake of readability, the restriction of  $J$  over  $\mathbb{C}^N$  is still denoted  $J$ . The proof of the following result lies in Appendix II.

*Lemma 2:* Let  $f^{(s)}$  be a stationary point of  $J$ . Necessarily, the following condition is fulfilled: 1) either there exist two nonnull integers  $K_1$  and  $K_2$  and two strictly positive constants  $\alpha_1$  and  $\alpha_2, \alpha_1 \neq \alpha_2$  such that  $f^{(s)}$  has exactly  $K_1 + K_2$  nonnull components:  $K_1$  components of  $f^{(s)}$  have a modulus equal to  $\sqrt{\alpha_1}$  and  $K_2$  components of  $f^{(s)}$  have a modulus equal to  $\sqrt{\alpha_2}$  or there exist a nonnull integer  $K$  and a strictly positive constant such that  $f^{(s)}$  has exactly  $K$  nonnull components, all with modulus  $\sqrt{\alpha}$ .

We first notice that for any  $f \in \mathcal{F}(\pm(\pi/4))$ , we have  $J(f) = 2 - (y/2)$ ; hence

$$\kappa_N \leq 2 - \frac{y}{2}. \quad (13)$$

We are going to prove that (13) is actually an equality.

$J$  is homogenous, i.e., for any  $\lambda \in \mathbb{C}$  and any  $f \in \mathbb{C}^N, J(\lambda f) = J(f)$ . Hence, we consider the minimization of  $J$  over the unit sphere of  $\mathbb{C}^N$ :  $\mathcal{S}^N = \{f \text{ such that } \|f\|_2 = 1\}$ . As  $J$  is continuous and  $\mathcal{S}^N$  is compact, the lower bound  $\kappa_N$  is reached at one of the stationary points specified in Lemma 2. Denote it by  $f^{(\min)}$ .

Assume that  $f^{(\min)}$  belongs to the first type given in Lemma 2. After possible reordering, we may suppose that coefficients  $f_1^{(\min)}, \dots, f_{K_1}^{(\min)}$  have a square modulus equal to a certain  $\alpha_1 \neq 0$ , coefficients  $f_{K_1+1}^{(\min)}, \dots, f_{K_1+K_2}^{(\min)}$  have a square modulus equal to a certain  $\alpha_2 \neq 0$ . We first show by contradiction that  $\min(K_1, K_2) < 2$ . Suppose that  $\min(K_1, K_2) \geq 2$ . We have

$$J(f^{(\min)}) = \frac{x \left| \sum_k (f_k^{(\min)})^2 \right|^2 - y (K_1 \alpha_1^2 + K_2 \alpha_2^2)}{(K_1 \alpha_1 + K_2 \alpha_2)^2} + 2$$

hence

$$J\left(f^{(\min)}\right) \geq -y \frac{K_1 \alpha_1^2 + K_2 \alpha_2^2}{(K_1 \alpha_1 + K_2 \alpha_2)^2} + 2. \quad (14)$$

It is easily shown that for any  $K_1, K_2 > 0$ , and any positive  $\alpha_1, \alpha_2$  such that  $\alpha_1 \alpha_2 \neq 0$

$$\frac{K_1 \alpha_1^2 + K_2 \alpha_2^2}{(K_1 \alpha_1 + K_2 \alpha_2)^2} < \frac{1}{\min(K_1, K_2)} \quad (15)$$

(actually (15) is an equality if and only if  $\alpha_1 = 0$  or  $\alpha_2 = 0$ , which is impossible due to the hypothesis). Both this result and (14) yield

$$J\left(f^{(\min)}\right) > 2 - \frac{y}{\min(K_1, K_2)} \geq 2 - \frac{y}{2}. \quad (16)$$

This is in contradiction with (13); we have proved that  $\min(K_1, K_2) \leq 1$ . By assumption, neither  $K_1$  nor  $K_2$  can be zero. Hence,  $\min(K_1, K_2) = 1$ ; we may set  $K_2 = 1$ . We show by contradiction that  $K_1 = 1$ . Suppose that  $K_1 \geq 2$ . We set for every  $k = 1, 2, \dots, K_1$   $(f_k^{(\min)})^2 = \alpha_1 e^{i\theta_k}$ . The minimum of  $J$  may be expanded as

$$J\left(f^{(\min)}\right) = \frac{x|\alpha_2 + \alpha_1 \sum_k e^{i\theta_k}|^2 - y(K_1 \alpha_1^2 + \alpha_2^2)}{(K_1 \alpha_1 + \alpha_2)^2} + 2. \quad (17)$$

As  $J(\lambda f^{(\min)}) = J(f^{(\min)})$ , we may assume that  $K_1 \alpha_1 + \alpha_2 = 1$ . We deduce that  $\varphi : (0, \frac{1}{K_1}) \times \mathbb{R}^{K_1+1} \rightarrow \mathbb{R}$  defined by

$$\varphi(\alpha, \omega_1, \dots, \omega_{K_1}) = x \left| 1 - K_1 \alpha + \alpha \sum_k e^{i\omega_k} \right|^2 - y((1 - K_1 \alpha)^2 + K_1 \alpha^2) \quad (18)$$

has a global minimum at  $(\alpha_1, \theta_1, \dots, \theta_{K_1})$ . In the sequel, we consider any stationary vector of  $\varphi$ , denoted by  $(\alpha^{(s)}, \omega_1^{(s)}, \dots, \omega_{K_1}^{(s)})$ . From the computation of the partial derivatives  $(\frac{\partial \varphi}{\partial \omega_l})_l$ , we directly deduce that

$$\sum_k \frac{\partial \varphi}{\partial \omega_k} = -2\alpha x(1 - K_1 \alpha) \sum_k \sin \omega_k \quad (19)$$

hence we have

$$\sum_k \sin \omega_k^{(s)} = 0. \quad (20)$$

In view of (18) and (20), we define  $\beta$  as

$$\beta = K_1 - \sum_{k=1}^{K_1} \cos \omega_k^{(s)} \geq 0 \quad (21)$$

and consider the mapping

$$\psi_\beta(\alpha) = x(1 - \alpha\beta)^2 - y((1 - K_1 \alpha)^2 + K_1 \alpha^2). \quad (22)$$

Due to (21), we may restrict the analysis of  $\psi_\beta$  for  $0 \leq \beta \leq 2K_1$ . In Appendix III, we show Lemma 3.

*Lemma 3:* For any  $0 \leq \beta \leq 2K_1$  and any  $\alpha \in (0, (1/K_1))$ , we have

$$\psi_\beta(\alpha) > -\frac{y}{2}. \quad (23)$$

In particular, for any stationary point of  $\varphi$ , we have  $\varphi(\alpha^{(s)}, \omega_1^{(s)}, \dots, \omega_{K_1}^{(s)}) > -(y/2)$ ; hence  $J(f^{(\min)}) > 2 - (y/2)$ , which is clearly a wrong statement due to (13). By contradiction, we have shown that  $K_1 \leq 1$ ; hence  $K_1 = 1$ . This implies that  $f^{(\min)}$  has two nonnull coefficients.

Assume now that  $f^{(\min)}$  belongs to the second type of the stationary points given in Lemma 2:  $f^{(\min)}$  has  $K$  nonnull components, each of modulus  $\sqrt{\alpha}$ . If  $K = 1$ , we would have  $J(f^{(\min)}) = x - y + 2 > 2 - y/2$ , which is impossible, owing to (13). Hence,  $K \geq 2$ . We

may set  $K_1 = K - 1$ , and  $\alpha_1 = \alpha, \alpha_2 = \alpha$ . This notation allows us to write  $J(f^{(\min)})$  as in (17). The same conclusion comes:  $K_1 = 1$ . Once gain, we have proved that  $f^{(\min)}$  has exactly two nonnull components.

Conversely, it remains to prove that  $f^{(\min)} \in \mathcal{F}(\pm(\pi/4))$ . We denote by  $f_1$  and  $f_2$  the nonnull coefficients of  $f^{(\min)}$ . With  $J$  being homogeneous, we may set  $f_1 = \sqrt{1 - \alpha}$  and  $f_2 = \sqrt{\alpha} e^{i\theta}$ . We hence have to study the minimum of

$$(\theta, \alpha) \mapsto x((1 - \alpha + \alpha \cos(\theta))^2 + \alpha^2 \sin(\theta)^2) - y((1 - \alpha)^2 + \alpha^2).$$

Setting to zero the first derivative with regard to  $\theta$  gives  $\theta = 0$  or  $2\theta = \pi[2\pi]$ . The case  $\theta = 0$  is not associated with the minimum. Hence,  $\theta = \pm(\pi/2)$ . This allows us to prove finally that  $\alpha = 1/2$ . Hence,  $f^{(\min)} \in \mathcal{F}(\pm(\pi/4))$ .

We hence have shown that for any  $f \in \mathbb{C}^N$ ,  $J(f) = \kappa_N$  iff  $f \in \mathcal{F}(\pm(\pi/4))$ . This implies that  $\kappa_N = \kappa_2$  for every  $N \geq 2$ ; owing to Lemma 1, this says that the lower bound  $\kappa$  of  $J$  is reached and, moreover,  $\kappa = \kappa_2 = 2 - (y/2)$ . Take any  $f$  such that  $J(f) = 2 - (y/2)$ . If  $f$  has a finite number of nonnull components, the previous developments show that  $f \in \mathcal{F}(\pm(\pi/4))$ . On the contrary, we know that  $f$  is a stationary point of the Lagrangian (27) (this time, the sums are infinite) [10, ch. 7]. The moduli of the components of  $f$  belong to a set of one or two elements. Now,  $f \in \ell_2(\mathbb{Z})$ : we deduce that  $f$  has a finite number of nonnull components; hence  $f \in \mathcal{F}(\pm(\pi/4))$ . We have shown Result 2.

*Result 2:* If  $2x > y$ ,  $\kappa = 2 - (y/2)$ . Moreover,  $J(f) = \kappa$  holds if and only if  $f \in \mathcal{F}(\pm(\pi/4))$ .

## V. DISCUSSION

Let us recast the results into the deconvolution framework: suppose that  $s_n$  is real-valued. Hence,  $x = 1$ ; and condition  $2x = y$  reads  $\mathbb{E}[s^4] = 1$ : this occurs when the sequence  $(s_n)$  is a binary sequence: the lower-bound is  $\kappa = 1$  (hence  $\tilde{\kappa} = 0$ ) and a  $f$  for which  $J$  is minimum belongs to  $\mathcal{F}(\theta), \theta \in (-\pi/2), (\pi/2)$ ; this result is not in contradiction with [9] but is more precise since we proved that no other  $f$  can reach the minimum. For real nonbinary sources, the fact that the global minimizers belong to  $\mathcal{F}(\pm\pi/4)$  is a pertinent information that allows one to retrieve the source  $s_n$  (a rotation of the equalized output has so be considered).

As far as (complex-valued) modulus one sequences  $s_n$  (but not binary) are concerned [8], we obviously have

$$\begin{aligned} \eta &= y - 2x \\ &= -(\mathbb{E}[|s_n|^4] - 2(\mathbb{E}[|s_n|^2])^2 - |E[s_n^2]|^2) - 2|E[s_n^2]|^2 \\ &= 1 - |E[s_n^2]|^2 \\ &> 0. \end{aligned}$$

This is exactly the context of result 1, and the minimizers of  $\tilde{J}$  are "trivial filters," i.e., elements of  $\mathcal{F}(0)$ . This result is consistent with Result 1 in [8].

## VI. APPLICATION TO BLIND EQUALIZATION

In this section, we consider an i.i.d. sequence  $s_n$ , where  $s_n$  belongs to the alphabet  $(1/\sqrt{5})\{1, -1, 3i, -3i\}$ . The points of the alphabet are equilikely; hence  $x = 64/100, y = 1$  and  $2x > y$ .

We consider the ARMA series given by  $3y_n - y_{n-1} = s_n + 2s_{n-1}$ . We have  $y_n = \sum_{k=0}^{\infty} h_k s_{n-k}$ , where  $\sum_{k=0}^{\infty} h_k z^{-k}$  is the stable and causal expansion of  $H(z) = (1 + 2z^{-1})/(3 - z^{-1})$ . The sequence  $z_n$  is obtained after passing  $y_n$  through a causal filter with finite-impulse-

response  $g$ . This means that  $g_k = 0$  if  $k \notin \{0, \dots, K-1\}$ . Hence,  $z_n$  follows model (1), with  $f_n = \sum_{k=0}^{K-1} g_k h_{n-k}$ .

As  $K$  is finite (we chose  $K = 15$ ), this in general prevents  $\tilde{J}$  from reaching its lower bound since no finite-order filter  $g$  may invert the system, i.e.,  $f$  can, at best, be "close" to  $\mathcal{F}(\pm(\pi/4))$ . We consider for a given  $f$  the distance to the set  $\bigcup_{\theta} \mathcal{F}(\theta) : d(f) = \min_{p,q} \inf_{\theta,\lambda} \|f - \lambda e^{(p,q)}(\theta)\|_2$ . The computation of  $d(f)$  is a nonlinear minimum least-square problem. As a least-square problem in  $\lambda$  (parameters  $p, q, \theta$  are supposed to be fixed), the  $\lambda_*$  minimizing  $\|f - \lambda e^{(p,q)}(\theta)\|_2^2$  is

$$\lambda_* = (f | e^{(p,q)}(\theta)) = f_p \cos \theta - i f_q \sin \theta.$$

Plugging this expression into  $\|f - \lambda e^{(p,q)}(\theta)\|_2^2$  eventually gives a function of  $\theta$  to be minimized (this time,  $p, q$  are fixed). The stationary  $\theta$  are given by the equation

$$2(-a_p^2 - b_p^2 + a_q^2 + b_q^2) \sin(2\theta) + (a_p b_q + a_q b_p) \cos(2\theta) = 0$$

where we set  $f_k = a_k + i b_k$ . There are two solutions in  $[-\pi, \pi]$ : one of them corresponds to the minimum of  $\|f - \lambda e^{(p,q)}(\theta)\|_2^2$ , the other to the maximum. In order to compute  $d(f)$ , it remains to consider the minimum previously found for every  $(p, q), p < q$ . This distance is achieved for a certain angle denoted by  $\theta(f)$ .

The number of observed data is finite: we observe  $y_0, \dots, y_{N-1}$ ; hence the cost function  $\tilde{J}$  is not perfectly known. We hence consider a consistent estimate

$$\tilde{J}_{\text{est}}(\mathbf{g}) = \frac{1}{N} \sum_{n=0}^{N-1} (|z_n|^2 - 1)^2$$

where we have set  $\mathbf{g} = (g_0, \dots, g_{K-1})$ . At the  $n+1$ th iteration of the steepest descent algorithm, we have

$$\mathbf{g}^{(n+1)} = \mathbf{g}^{(n)} - \mu_n \nabla^{(n)}$$

where

$$\begin{aligned} \nabla^{(n)} &= 2 \frac{\partial \tilde{J}_{\text{est}}}{\partial \mathbf{g}}(\mathbf{g}^{(n)}) \\ &= 4 \sum_{k=0}^{N-1} (z_k - 1) z_k Y_k. \end{aligned}$$

In this equation,  $Y_k = (y_k, y_{k-1}, \dots, y_{k-K+1})$ . The step  $\mu_n$  may be taken as the solution that minimizes the fourth-order polynomial  $\mu \mapsto \tilde{J}_{\text{est}}(\mathbf{g}^{(n)} - \mu \nabla^{(n)})$ .

Fig. 1 provides the constellation after convergence: as expected, this one is "close" to a 16-state constellation—the noise around each point is due to the fact that  $f$  does not exactly belong to  $\mathcal{F}(\pm(\pi/4))$ .

We consider 1000 Monte Carlo experiments. For every experiment, we compute the estimate  $\tilde{J}_{\text{est}}$ . After convergence of the steepest descent algorithm, we obtain  $g_{\text{est}}$  expected to be an argument minimum of  $\tilde{J}_{\text{est}}$ . This provides  $f_{\text{est}}$ , the convolution of  $g_{\text{est}}$  with  $h$ . For every  $f_{\text{est}}$ , both the distance  $d(f_{\text{est}})$  and the angle  $\theta(f_{\text{est}})$  are computed. The empirical mean and variance of these quantities are listed in Table I. As expected, the estimate  $f_{\text{est}}$  is close to  $\mathcal{F}(\pm(\pi/4))$ .

#### APPENDIX I PROOF OF LEMMA 1

Let  $m, M$  be two real numbers such that  $0 < m < M$  and denote by  $\mathcal{B}(m, M)$  the set

$$\mathcal{B}(m, M) = \{f \in \ell_2(\mathbb{Z}) \mid m \leq \|f\| \leq M\}. \quad (24)$$

We first show that  $J$  is uniformly continuous over  $\mathcal{B}(m, M)$ .

We set  $c_1(f) = \sum_k |f_k|^4$ ,  $c_2(f) = |\sum_k f_k^2|^2$ ,  $c_3(f) = (\sum_k |f_k|^2)^2$  and  $c_4(f) = -y c_1(f) + x c_2(f)$ . Notice that for any  $r, s \in \mathcal{B}(m, M)$

$$|J(s) - J(r)| \leq \frac{1}{|c_3(s)c_3(r)|} \times \frac{(|c_3(s)||c_4(s) - c_4(r)| + |c_4(s)||c_3(s) - c_3(r)|)}{|c_3(s)c_3(r)|};$$

as  $c_3$  has a nonnull lower-bound over  $\mathcal{B}(m, M)$ , the inequality implies that  $J$  is uniformly continuous over  $\mathcal{B}(m, M)$  as soon as  $c_1, c_2$ , and  $c_3$  are uniformly continuous.

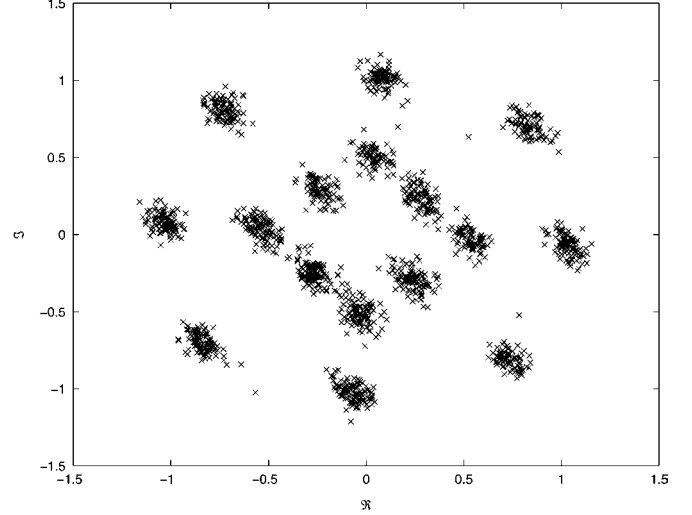


Fig. 1. Constellation after convergence.

TABLE I  
ESTIMATED MEAN/VARIANCE AFTER CONVERGENCE

criterion $\tilde{J}_{\text{est}}$		distance $d(f_{\text{est}})$		angle $\theta(f_{\text{est}})$	
mean	variance	mean	variance	mean	variance
$\frac{98}{100} \tilde{\kappa}$	$13 \cdot 10^{-5}$	$82 \cdot 10^{-4}$	$15 \cdot 10^{-5}$	$\frac{92}{100} \frac{\pi}{4}$	$61 \cdot 10^{-4}$

Take  $\epsilon > 0$  and consider  $r, s \in \mathcal{B}(m, M)$ . We have

$$\begin{aligned} |c_1(s) - c_1(r)| &\leq \sum_k |s_k^4 - r_k^4| \\ &\leq \sum_k |s_k^4 - r_k^4| \\ &\leq \sum_k |s_k^2 - r_k^2| |s_k^2 + r_k^2| \\ &\leq 2M^2 \sum_k |s_k - r_k| |s_k + r_k|. \end{aligned}$$

The Cauchy–Schwartz inequality finally yields

$$|c_1(s) - c_1(r)| \leq 2M^2 \sqrt{\sum_k |s_k - r_k|^2} \sqrt{\sum_k |s_k + r_k|^2}$$

and therefore, if  $\|s - r\|_2 \leq (\epsilon/4M^3)$ ,  $|c_1(s) - c_1(r)| \leq \epsilon$ , hence proving the uniform continuity of  $c_1$ .

As far as  $c_2$  is concerned, notice that

$$\begin{aligned} |c_2(s) - c_2(r)| &\leq \left| \left( \sum_k s_k^2 - r_k^2 \right) \left( \sum_k s_k^2 + r_k^2 \right) \right| \\ &\leq 2M^2 \left| \sum_k s_k^2 - r_k^2 \right| \\ &\leq 4M^3 \|s - r\|_2 \end{aligned}$$

hence proving the uniform continuity of  $c_2$ . As far as  $c_3$  is concerned, we let the reader show its uniform continuity. We now focus on the second part of the lemma. Of course,  $\kappa_N \geq \kappa$ . Suppose that  $\hat{\kappa} = \lim_{N \rightarrow \infty} \kappa_N > \kappa$ .

There exists an  $f$  such that

$$J(f) = \hat{\kappa} - \rho \quad (25)$$

for a certain  $\rho > 0$ .

With  $J$  being homogeneous, we may set  $\|f\|_2 = 1$ . We propose to split  $f$  as  $f = p^{(k)} + q^{(k)}$ , where  $p_i^{(k)} = f_i$  if  $|i| < k$  and  $p_i^{(k)} = 0$  if  $|i| \geq k$ . We consider  $1 > \epsilon > 0$ : we set  $m = 1 - \epsilon$  and  $M = 1 + \epsilon$ .

There exists an integer  $K_1$  such that for every  $k \geq K_1$  we have:  $p^{(k)} \in \mathcal{B}(m, M)$ . Let then  $k$  be such that  $k \geq K_1$ .

As  $J$  is uniformly continuous over  $\mathcal{B}(m, M)$ , there exists  $\alpha$  such that  $|J(s) - J(r)| \leq (\rho/2)$  as soon as  $\|s - r\|_2 \leq \alpha$ ; on the other hand,  $q^{(k)}$  may be chosen as small as desired, i.e., there exists  $K_2$  such that for every  $k \geq K_2$   $\|q^{(k)}\|_2 \leq \alpha$ . Taking any  $k \geq \max(K_1, K_2)$ , and choosing  $s = f, r = p^{(k)}$ , it yields

$$J(f) \geq J\left(p^{(k)}\right) - \frac{\rho}{2} \geq \hat{\kappa} - \frac{\rho}{2} \quad (26)$$

which clearly contradicts (25). We have proved Lemma 1.

## APPENDIX II PROOF OF LEMMA 2

For every integer  $k$ , we let  $a_k, b_k$  be  $a_k = \Re(f_k), b_k = \Im(f_k)$ , and define  $a = (a_k)_{k \in \mathbb{Z}}, b = (b_k)_{k \in \mathbb{Z}}$ . With  $J$  being homogeneous, the minimization of  $J$  over  $\mathbb{C}^N$  is equivalent to the minimization of  $J$  over the unit-sphere  $\mathcal{S}^N$ . A stationary point  $f^{(s)}$  of  $J$  is then a stationary point of the Lagrangian

$$\begin{aligned} \mathcal{L}(a, b, \lambda) = & x \left( \sum_k a_k^2 - b_k^2 \right)^2 \\ & + 4x \left( \sum_k a_k b_k \right)^2 - y \sum_k (a_k^2 + b_k^2)^2 \\ & - \lambda \left( \left( \sum_k a_k^2 + b_k^2 \right)^2 - 1 \right). \end{aligned} \quad (27)$$

We may define  $\alpha$  and  $\beta$  as  $\alpha = 2x \sum_k (a_k^{(s)2} - b_k^{(s)2})$  and  $\beta = 4x \sum_k a_k^{(s)} b_k^{(s)}$ . Forcing to zero the partial derivatives  $(\partial \mathcal{L} / \partial a_i)$  and  $(\partial \mathcal{L} / \partial b_i)$  implies that the vector  $X_l = (a_l, b_l)^T$  is a solution of the linear system

$$(M - (2y\nu_l + 2\lambda)I_2)X_l = 0 \quad (28)$$

where  $I_2$  is the identity matrix of size 2 and

$$M = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

and  $\nu_l = (a_l^{(s)})^2 + (b_l^{(s)})^2$ . Equation (28) implies, in particular that, for every index  $l$ , either  $X_l$  is null or belongs to the Kernel of  $M - (2y\nu_l + 2\lambda)I_2$ .

On the other hand, the eigenvalues of  $M$  are  $\pm \sqrt{\alpha^2 + \beta^2}$ :  $M - (2y\nu_l + 2\lambda)I_2$  has a nonnull kernel if and only if either  $2y\nu_l + 2\lambda = \sqrt{\alpha^2 + \beta^2}$  or  $2y\nu_l + 2\lambda = -\sqrt{\alpha^2 + \beta^2}$ . This implies that for every index  $l$ ,  $\nu_l$  is either zero or belongs to a set having either one or two elements.

## APPENDIX III PROOF OF LEMMA 3

As  $x, y > 0$ , we may without any restriction suppose that  $x = 1$ . As a function of  $\alpha$ ,  $\psi$  is a second-order polynomial

$$\psi_\beta(\alpha) = (\beta^2 - yK_1(K_1 + 1))\alpha^2 + 2\alpha(yK_1 - \beta) + 1 - y.$$

We set  $\beta_1 = \sqrt{yK_1(K_1 + 1)}$ .

1. **If  $\beta \leq \beta_1$** , then  $\psi_\beta$  is concave, hence the lower bound of  $\psi$  over  $(0, 1/K_1)$  is on one of the edges of this interval, that is

$$\inf_{\alpha \in (0, 1/K_1)} \psi_\beta(\alpha) = \min \left( \psi_\beta(0), \psi_\beta \left( \frac{1}{K_1} \right) \right). \quad (29)$$

TABLE II  
VARIATIONS OF  $\chi$ : CASE OF  $yK_1 < K_1 + 1$

$\beta$	$yK_1$		$K_1 + 1$		$2K_1$
$\chi(\beta)$		$\nearrow$		$\searrow$	

TABLE III  
VARIATIONS OF  $\chi$ : CASE OF  $yK_1 > K_1 + 1$

$\beta$	$K_1 + 1$		$yK_1$		$2K_1$
$\chi$		$\nearrow$		$\searrow$	

However,  $\psi_\beta(0) = 1 - y$  and, by assumption,  $1 - y > -y/2$ , hence

$$\psi_\beta(0) > -\frac{y}{2}. \quad (30)$$

Moreover,  $\psi_\beta((1/K_1)) = (1 - \beta/K_1)^2 - y/K_1 \geq -y/K_1$ . By assumption,  $K_1 \geq 2$ , hence

$$\psi_\beta \left( \frac{1}{K_1} \right) > -\frac{y}{2}. \quad (31)$$

Equations (29), (30), and (31) imply (23).

2. **If  $\beta > \beta_1$** , the minimum of  $\psi_\beta$  over  $\mathbb{R}$  is reached for a certain  $\alpha_0$  defined as

$$\alpha_0 = \frac{\beta - \beta_2}{\beta^2 - \beta_1^2}$$

where  $\beta_2 = yK_1$ . We set

$$\begin{aligned} \chi(\beta) &= \psi_\beta(\alpha_0) \\ &= -\frac{(\beta - \beta_2)^2}{\beta^2 - \beta_1^2} + 1 - y. \end{aligned}$$

The derivative of  $\chi$  writes

$$\chi'(\beta) = \frac{2(\beta - \beta_2)}{(\beta^2 - \beta_1^2)^2} (K_1 + 1 - \beta)yK_1.$$

- **If  $yK_1 < K_1 + 1$** , Table II shows the variations of  $\chi$ . We hence have for every  $0 \leq \beta \leq 2K_1$

$$\begin{aligned} \inf_{\alpha \in (0, 1/K_1)} \psi_\beta(\alpha) &\geq \inf_{\alpha \in \mathbb{R}} \psi_\beta(\alpha) \\ &\geq \chi(\beta) \\ &\geq \min(\chi(yK_1), \chi(2K_1)). \end{aligned} \quad (32)$$

Now

$$\chi(yK_1) = 1 - y > -y/2 \quad (33)$$

by assumption. On the other hand

$$\chi(2K_1) = 1 - y - \frac{K_1(2 - y)^2}{K_1(4 - y) - y}$$

hence

$$\chi(2K_1) > -y/2 \quad (34)$$

if and only if  $K_1 > 1$ ; this condition being verified by assumption, (33) and (34) hold true, hence (32) implies (23).

- **If  $yK_1 > K_1 + 1$** , Table III shows the variations of  $\chi$ . It yields

$$\inf_{\alpha \in (0, 1/K_1)} \psi_\beta(\alpha) \geq \min(\chi(K_1 + 1), \chi(2K_1)). \quad (35)$$

Notice that

$$\chi(K_1 + 1) > -y/2 \quad (36)$$

holds if and only if  $(yK_1 - K_1 - 1/K_1 + 1) + 1 - y > -y/2$ , that is  $K_1 > 1$ , which holds by assumption. As (34) and (36) are true, hence (35) implies (23).

- if  $yK_1 = K_1 + 1$ ,  $\chi$  is decreasing; this yields  $\chi(\beta) \geq \chi(2K_1) > -y/2$  and this proves (23).

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