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The new International System of Units / *Le nouveau Système international d'unités*

A new dual system for the fundamental units including and going beyond the newly revised SI



Un nouveau système dual pour les unités fondamentales incluant et dépassant le nouveau SI

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ABSTRACT

We propose a new system for the fundamental units, which includes and goes beyond the present redefinition of the SI, by choosing also $c = \hbar = 1$. By fixing $c = c_0$, $m/s = 1$, $\hbar = \hbar_0$, $J s = 1$ and $\underline{\mu}_0 = \mu_0$, $N/A^2 = 1$, it allows us to define the metre, the joule, and the ampere as equal to $(1/299\,792\,458)$ s, $(1/\hbar_0 = .948 \dots \times 10^{34})$ s⁻¹ and $\sqrt{\mu_0 N} = \sqrt{\mu_0 c_0/\hbar_0}$ s⁻¹ = $1.890 \dots \times 10^{18}$ s⁻¹. It presents at the same time the advantages and elegance of a system with $\hbar = c = \underline{\mu}_0 = \underline{\epsilon}_0 = k = N_A = 1$, where the vacuum magnetic permeability, electric permittivity, and impedance are all equal to 1.

All units are rescaled from the natural ones and proportional to the s, s⁻¹, s⁻², ... or just 1, as for the coulomb, ohm and weber, now dimensionless. The coulomb is equal to $\sqrt{\mu_0 c_0/\hbar_0} = 1.890 \dots \times 10^{18}$, and the elementary charge to $e = 1.602 \dots \times 10^{-19}$ C = $\sqrt{4\pi\alpha} = .3028 \dots$. The ohm is equal to $1/\mu_0 c_0$ so that the impedance of the vacuum is $Z_0 = 376.730 \dots \Omega = 1$. The volt is $1/\sqrt{\mu_0 c_0 \hbar_0}$ s⁻¹ = $5.017 \dots \times 10^{15}$ s⁻¹, and the tesla c_0 V/m = $\sqrt{c_0^3/\mu_0 \hbar_0}$ s⁻² = $4.509 \dots \times 10^{32}$ s⁻².

The weber is $1/\sqrt{\mu_0 c_0 \hbar_0} = 5.017 \dots \times 10^{15}$. The flux quantum is $\Phi_0 = h/2e = 2.0678 \dots \times 10^{-15}$ Wb = $\pi/e = 10.374 \dots$, with $K_J = 483\,597 \dots$ GHz/V = $e/\pi = .096\,39 \dots$, and $R_K = 25\,812 \dots \Omega = 1/2\alpha \simeq 68.518$. One can also fix $e = 1.602\,176\,634 \times 10^{-19}$ C, at the price of adjusting the coulomb and all electrical units with $\mu_0 = 4\pi \times 10^{-7} \eta^2$, where η^2 , $\propto \alpha$, is very close to 1.

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R É S U M É

Nous proposons un nouveau système pour les unités fondamentales, qui inclut la redéfinition en cours du SI et va au-delà, en choisissant aussi $c = \hbar = 1$. En fixant $c = c_0$, $m/s = 1$, $\hbar = \hbar_0$, $J s = 1$ et $\underline{\mu}_0 = \mu_0$, $N/A^2 = 1$, il permet de définir le mètre, le joule et l'ampère comme égaux à $(1/299\,792\,458)$ s, $(1/\hbar_0 = 0,948 \dots \times 10^{34})$ s⁻¹ et

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$\sqrt{\mu_0 N} = \sqrt{\mu_0 c_0 / \hbar_0} s^{-1} = 1,890 \dots \times 10^{18} s^{-1}$. Il présente en même temps les avantages et l'élégance d'un système où $\hbar = c = \mu_0 = \epsilon_0 = k = N_A = 1$, où la perméabilité magnétique, la permittivité électrique et l'impédance du vide sont égales à 1.

Toutes les unités sont redéfinies à partir des unités naturelles et proportionnelles à la seconde, s^{-1} , s^{-2} , ... ou à 1, comme pour le coulomb, l'ohm et le weber, sans dimensions. Le coulomb est égal à $\sqrt{\mu_0 c_0 / \hbar_0} = 1,890 \dots \times 10^{18}$, et la charge élémentaire à $e = 1,602 \dots \times 10^{-19} C = \sqrt{4\pi\alpha} = 0,3028 \dots$. L'ohm est égal à $1/\mu_0 c_0$, et l'impédance du vide à $Z_0 = 376,730 \dots \Omega = 1$. Le volt est $1/\sqrt{\mu_0 c_0 \hbar_0} s^{-1} = 5,017 \dots \times 10^{15} s^{-1}$, et le tesla $c_0 V/m = \sqrt{c_0^3 / \mu_0 \hbar_0} s^{-2} = 4,509 \dots \times 10^{32} s^{-2}$.

Le weber est $1/\sqrt{\mu_0 c_0 \hbar_0} = 5,017 \dots \times 10^{15}$, et le quantum de flux $\Phi_0 = h/2e = 2,0678 \dots \times 10^{-15} Wb = \pi/e = 10,374 \dots$, avec $K_J = 483 597 \dots GHz/V = e/\pi = 0,096 39 \dots$, et $R_K = 25 812 \dots \Omega = 1/2\alpha \simeq 68,518$. On peut aussi fixer $e = 1,602 176 634 \times 10^{-19} C$ à condition d'ajuster le coulomb et toutes les unités électriques avec $\mu_0 = 4\pi \times 10^{-7} \eta^2$ où η^2, α , est très proche de 1.

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1. Presentation of the new system

Measuring geometrical or physical quantities, such as intervals of time, space distances, masses, energies, electric charges and currents, etc., requires the definition of appropriate units. It is desirable that these units be the same everywhere, and do not change with time. To realize this, they must be defined as much as possible in a universal way, from invariant physical objects or phenomena. This is how the metre was initially defined as “la dix-millionième partie du quart du méridien terrestre” [1]. The unit of weight followed, as the weight of a dm^3 of water at the temperature of melting ice [2]. This was at the origin of the metric system in France at the end of the 18th century, and later of the “Convention du mètre” in 1875 [3], leading to the International System of Units (SI) [4,5].

As of today, the second is defined from the period of a specific transition of the caesium-133 atom [6]. The metre is derived from the second by fixing the value of the speed of light in vacuum, c , to be exactly 299 792 458 m/s [7]. This is made possible thanks to the theory of relativity, verified to a very high degree of precision, according to which time and space are related entities of similar nature, the speed of light being the same, always and everywhere, independently of the reference frame in which it is measured.

But the unit of mass, the kilogram, still remains defined from a physical artefact, as the mass of the international prototype of the kilogram (\mathcal{K}), a cylinder of platinum–iridium alloy stored at the International Bureau of Weights and Measures (BIPM) in the “Pavillon de Breteuil” at Sèvres, France [8]. Its mass remains constant and equal to 1 kg by definition and according to an international agreement, even if the corresponding quantity of matter cannot remain exactly constant due to surface effects and does inevitably vary very slightly over the years [9]. This is not a satisfactory situation, and it would be desirable to get a universal definition of the kilogram, based on reproducible universal phenomena rather than on a single material object.

This is the main purpose of the present redefinition of the kilogram, which will soon be derived from the second, or the metre, using quantum mechanics, by fixing the value of the Planck constant h [10]. This one relates, through the relation $E = h\nu (= \hbar\omega)$, the energy E of a photon to the frequency ν (or angular frequency $\omega = 2\pi\nu$) of the corresponding electromagnetic wave. Fixing h at a certain number of joule-seconds, in agreement with its presently measured value ($h \simeq 6.626 \dots \times 10^{-34} J s$, or $\hbar = h/2\pi \simeq 1.054 \dots \times 10^{-34} J s$) will allow us to define the unit of energy, the joule, as corresponding to a very large number of s^{-1} ($.948 \dots \times 10^{34} s^{-1}$). The definition of the kilogram follows, such that 1 joule = 1 $kg m^2 s^{-2}$ as usual, rendering obsolete the international prototype stored at BIPM. Of course, the value of h gets fixed in agreement with its present best-determined value ($6.626 070 15 \times 10^{-34} J s$ [11]), so that the mass of the international prototype will still be practically equal to 1 kg at the time of the change, within experimental uncertainties. The change will not affect in any practical way the values of masses at the time it is performed, while allowing for more precise measurements in the future.

We could choose to measure distances directly in seconds by fixing the speed of light at $c = 1$, the natural choice in relativity; and energies and masses directly in s^{-1} by fixing the value of $\hbar = h/2\pi$ also at 1, the natural choice in quantum mechanics. This would be conceptually much simpler. But considering the second as being also a unit of distance, close to $3 \times 10^5 km$, and the second $^{-1}$ as a common unit for energy and mass (with $c = 1$), would not be very practical. Indeed, the energy unit of $1 s^{-1}$ would be very small, $1,054 \dots \times 10^{-34}$ joule. Furthermore, the kilogram being itself associated with a very large energy $E = mc^2 = .898 \dots \times 10^{17} J$, the s^{-1} as a unit of mass would be comparatively even smaller, $(1.054 \dots \times 10^{-34}) / (.898 \dots \times 10^{17}) kg$, i.e. $1.173 \dots \times 10^{-51} kg$. This is also understood from $\hbar/c^2 \simeq 1.173 \dots \times 10^{-51} kg s$, which fixes the mass unit of $1 s^{-1}$ at $1.173 \dots \times 10^{-51} kg$.

While such units are often very convenient in relativistic and quantum physics, it would seem unpractical to replace square metres by $\approx 10^{-17} s^2$ when measuring surfaces, or to ask for $10^{51} s^{-1}$ of potatoes on the market. For historical and

practical reasons, we remain attached to measuring distances in metres rather than in seconds, and masses and energies in kilograms and joules rather than in s^{-1} . Fortunately, we still have the freedom to define the metre, kilogram, and joule as corresponding to fixed submultiples or multiples of the second, or $second^{-1}$. This is how the units of length, and very soon energy and mass, are or will be derived from the unit of time by fixing c and \hbar at a certain number of m/s and Js according to $c = c_0$ m/s, $\hbar = \hbar_0$ Js, allowing one to define the metre and the joule, and subsequently the kilogram, the newton, etc., from the second. This is what the present redefinition of the SI, to become effective from 20 May 2019, is going to achieve [10].

But we can still go further. Within relativity space and time, related by the Lorentz symmetry group, become quantities of similar nature, which may be measured with a common unit. It is the same for mass and energy. Within quantum mechanics, energies and angular frequencies are directly related, allowing us to measure energies in s^{-1} , as for angular frequencies. In a conceptually ideal system, we ought to choose $c = 1$ as suggested by relativity, and $\hbar = 1$ as suggested by quantum mechanics so that the intrinsic angular momentum, or spin, of the electron is simply 1/2. This is in fact implicitly supposed when we say that the electron is a spin-1/2 particle. Such a system, however, is usually viewed as an idealized one for theoreticians, requiring to move back to an ordinary system in which time, space, energies, and masses are all measured in terms of their own units, in seconds, metres, joules, and kilograms, as in the traditional or newly revised SI.

We shall show here that the two points of view, although apparently antagonist, may be reconciled within a *single unified system including the new SI*, in which *both c and \hbar are also equal to 1*. Then *the metre, joule and kilogram get identified as specific submultiples or multiples of the second or $second^{-1}$* , the natural units of length, energy, and mass in an ideal system with $c = \hbar = 1$, once the second is chosen as the unit of time. We can thus have, simultaneously,

$$\left\{ \begin{array}{ll} c = 299\,792\,458 \text{ m/s} & \text{and} \quad c = 1, \\ h = 6.626\,070\,15 \times 10^{-34} \text{ Js} & \text{and} \quad \hbar = h/2\pi = 1 \end{array} \right. \quad (1)$$

These equalities will provide as desired the appropriate new SI definitions for the metre, the joule, and the kilogram, also made compatible with the advantages of working in a system with $c = \hbar = 1$, in which *the fundamental laws of physics do not even depend on these parameters*. These were previously referred to as “fundamental constants of nature”. Once identified with unity, they get downgraded to simply providing numerical conversion factors between related units.

2. The new system for the electrical units

This new picture can be extended to the electrical units, up to now defined from the mechanical ones by fixing the value of the parameter μ_0 defining the magnetic permeability of vacuum and entering the traditional definition of the ampere [4,12,13]. The new approach we propose here remains applicable, as we shall see, in the context of the new SI [10], in which the rigid connection between electrical and mechanical units gets somewhat loosened. This is a consequence of the recent decision taken at the last CGPM to define the coulomb (and thus the ampere and all electrical units) so that the numerical value of the elementary charge is exactly fixed at $e = 1.602\,176\,634 \times 10^{-19}$ C, requiring to turn μ_0 into an adjustable parameter [14]. It is indeed the price to pay for having decided to fix the numerical value of e , rather than measuring it as it has been done up to now.

Let us return to our new proposed approach. The present conventional choice $\mu_0 = 4\pi \times 10^{-7}$ N/A², or equivalently H/m, before the upcoming 2019 redefinition of electrical units, may be reconciled with the ideal one $\underline{\mu}_0 = 1$ by fixing the value of $\underline{\mu}_0$, as done for c and \hbar , at a certain number of N/A². We shall thus write $\underline{\mu}_0 = \mu_0$ N/A², where $\underline{\mu}_0$, now dimensionless, is initially fixed at $4\pi \times 10^{-7}$. But, as from 2019 the coulomb will no longer be obtained from the traditional definition of the ampere [4,12,13] but by fixing the value of the elementary charge [10], the value of μ_0 will still have been very slightly adjusted, from $4\pi \times 10^{-7}$ into $\mu_0 = 4\pi \times 10^{-7} \eta^2$.

In both cases, and thus independently of the on-going 2018–2019 change for the definition of the electrical units, we can ask for $\underline{\mu}_0 = 1$, which provides an ideal system with $\hbar = c = \underline{\mu}_0 = \underline{\epsilon}_0 = 1$. Fixing $\underline{\mu}_0 = \mu_0$ N/A² = 1 determines the ampere, which must verify

$$1 \text{ A}^2 = \mu_0 \text{ N} \quad (2)$$

The ampere appears as *proportional to a square root of the newton*, which is conveniently expressed as $1 \text{ A} = \sqrt{\mu_0 \text{ N}}$.

Altogether the set of three equations

$$\left\{ \begin{array}{ll} c = c_0 \text{ m/s} = 1, & \text{which fixes the metre,} \\ \hbar = \hbar_0 \text{ Js} = 1, & \text{which fixes the joule,} \\ \underline{\mu}_0 = \mu_0 \text{ N/A}^2 = 1, & \text{which fixes the ampere} \end{array} \right. \quad (3)$$

leads to a system that includes and goes beyond the present redefinition of the SI. It allows us to reconcile, within a single system of units,

1) the advantages of a conceptually desirable system in which $\hbar = c = \frac{\mu_o}{\epsilon_o} = 1$, the vacuum magnetic permeability, electric permittivity, and impedance being all equal to 1, with the elementary charge $e = \sqrt{4\pi\alpha} = .3028\dots$, $K_J = 2e/h = e/\pi$ and $R_K = h/e^2 = (376.730\dots \Omega)/2\alpha = 1/2\alpha$;

2) the convenience of our familiar SI units, second, metre, kilogram, joule, newton, ampere, coulomb, volt, etc. appropriately defined or redefined very much as usual;

3) with all these units also expressed in terms of the second, s^{-1} , s^{-2} , ..., or even as pure numbers as for the ohm, the coulomb, and the weber;

4) furthermore, as the numerical value of e will also get fixed, the coulomb, the ampere and all electrical units get redefined accordingly, using an adapted value of μ_o equal to $4\pi \times 10^{-7} \eta^2$ where η is very close to 1; this adjustment of electrical units is automatically and explicitly taken care of by expressing the ampere and coulomb proportionally to $\sqrt{\mu_o}$, as indicated above.

With the values of \hbar and c getting fixed at \hbar_o Js and c_o m/s, respectively, we get the expression of the newton

$$1 \text{ N} = \frac{1 \text{ J s}}{1 \text{ m/s}} s^{-2} = \frac{c_o}{\hbar_o} s^{-2} \quad (4)$$

This leads, most notably, to the following dual expressions for the ampere A and the coulomb C (at the moment still evaluated with $\mu_o = 4\pi \times 10^{-7}$), the elementary charge e , and the impedance of vacuum Z_o :

$$\left\{ \begin{array}{l} 1 \text{ A} = \sqrt{\mu_o \text{ N}} = \sqrt{\mu_o c_o / \hbar_o} s^{-1} = 1.890\,067\,014\,853\dots \times 10^{18} s^{-1}, \\ 1 \text{ C} = \sqrt{\mu_o \text{ kg m}} = 1/\sqrt{\epsilon_o c_o \hbar_o} = 1.890\,067\,014\,853\dots \times 10^{18}, \\ e = 1.602\,176\,634 \times 10^{-19} \text{ C} = \sqrt{4\pi\alpha} \simeq .302\,822\,1208, \\ Z_o = \mu_o c_o \Omega = 376.730\,313\,461\dots \Omega = 1 \end{array} \right. \quad (5)$$

The elementary charge e remains as an independent dimensionless free parameter equal to $\sqrt{4\pi\alpha}$, to be determined experimentally. However, it will get numerically fixed at exactly $1.602\,176\,634 \times 10^{-19}$ C in the new SI, at the price of suitably adjusting the coulomb and thus the value of μ_o , now to be taken as $4\pi \times 10^{-7} \eta^2$.

The electron-volt, $1 \text{ eV} = 1.602\dots \times 10^{-19}$ J, gets exactly fixed at $(e_o/\hbar_o) s^{-1} = 1.519\dots \times 10^{15} s^{-1}$. The volt, equal to $1/\sqrt{\mu_o c_o \hbar_o} s^{-1}$, is also recovered as $1 \text{ eV}/e = 1.519\dots \times 10^{15} s^{-1}/.3028\dots = 5.017\dots \times 10^{15} s^{-1}$. The Josephson and von Klitzing constants are related, respectively, to the size of the electron-volt and to the fine structure constant α . They get fixed numerically in SI units (GHz/V and ohms) [14] thanks to the choice of $h = \hbar_o$ Js, $e = e_o$ C. At the same time they also become pure numbers equal to e/π and $1/2\alpha$, respectively, thanks to our choice of $\hbar = 1$ with $c = \frac{\mu_o}{\epsilon_o} = 1$ so that $e = \sqrt{4\pi\alpha}$:

$$\left\{ \begin{array}{l} K_J = 2e/h = 2e_o/h_o \text{ Wb}^{-1} = 483\,597.848\,416\dots \text{ GHz/V} = e/\pi \simeq .0963\,912\,748, \\ R_K = h/e^2 = \frac{\hbar_o}{e_o^2} \Omega = \frac{\mu_o c_o \Omega}{2\alpha} = 25\,812.807\,459\dots \Omega = 1/2\alpha \simeq 68.517\,999\,58 \end{array} \right. \quad (6)$$

$K_J = e/\pi$ is equivalent to saying that $1 \text{ eV} = \pi(2e_o/h_o) s^{-1} = \pi \times .483\,597\dots \times 10^{15} s^{-1} = 1.519\dots \times 10^{15} s^{-1}$. The SI value in ohms of R_K is the vacuum impedance, $Z_o = \mu_o c_o \Omega = 376.730\,313\dots \Omega$ (now also equal to 1), multiplied by $137.035\,999\dots/2$, providing the well-known $25\,812.807\dots \Omega$. It is now, at the same time, also dimensionless and equal to $137.035\,999\dots/2$.

The resulting expressions of the various mechanical and electrical units are given in Table 1. We leave aside for the time being the question of a more fundamental definition for the unit of time, which one might like to obtain in the future from the electron mass m_e using $\hbar/m_e c^2 \simeq 1.288\,088\,667 \times 10^{-21}$ s.

3. The implications of fixing the value of the elementary charge

Let us discuss further the consequences of fixing exactly, in the planned revision of the SI to become effective from 20 May 2019 [10], the numerical value of the elementary charge e when expressed in coulombs, according to the additional requirement

$$e = e_o \text{ C} = 1.602\,176\,634 \times 10^{-19} \text{ C} \quad (7)$$

Although this may be conceptually questionable, this is intended to provide a very precise way to fix the coulomb, and the other electrical units, allowing for more precise measurements.

However, adopting the new expression (7) of the elementary charge e requires adjusting suitably the size of the coulomb, at present defined as 1 As, and thus the size of the ampere. Writing the two expressions of e according to the old and new definitions of the coulomb, $e = e_{\text{old}} C_{\text{old}} = e_o \eta C_{\text{old}} = e_o \text{ C}$, we see that the redefined coulomb C may be “larger” (or possibly smaller) than the earlier one by a factor

Table 1

The new dual system: $c = c_0 \text{ m/s} = 1$ and $\hbar = \hbar_0 \text{ Js} = 1$ fix the metre and the joule. $\mu_0 = \mu_0 \text{ N/A}^2 = 1$ allows us to express the ampere and coulomb as $1 \text{ A} = \sqrt{\mu_0 \text{ N}}$ and $1 \text{ C} = \sqrt{\mu_0 \text{ kg m}}$. With $\hbar/c = (\hbar_0/c_0) \text{ kg m} = 1$, the coulomb, dimensionless, is equal to $\sqrt{\mu_0 c_0/\hbar_0} = 1.890 \dots \times 10^{18}$. The elementary charge is $e = 1.602 \dots \times 10^{-19} \text{ C} = \sqrt{4\pi\alpha} = .3028 \dots$, and the impedance of vacuum $Z_0 = \mu_0 c_0 \Omega = 376.730 \dots \Omega = 1$, with $K_j = 2e/\hbar = e/\pi$ and $R_K = \hbar/e^2 = 1/2\alpha$. The dimensionless coulomb, ohm and weber are related by $1 \text{ C} \times 1 \Omega = 1 \text{ Wb}$.

The left column refers to the units and remarkable quantities expressed in a conventional way in terms of the s, m, and kg, with the electrical units derived from the ampere expressed as $\sqrt{\mu_0 \text{ N}}$. The two other columns give their expressions in s, s^{-1} , s^{-2} or as dimensionless constants. This is the case for the coulomb, ohm, and weber, and for $e = \sqrt{4\pi\alpha}$, the impedance of vacuum $Z_0 = 1$, and flux quantum $\Phi_0 = \pi/e$. The numerical values are evaluated here for $\mu_0 = 4\pi \times 10^{-7}$, and the corresponding benchmark value $\alpha = \alpha_0 = 1/137.035999158713 \dots$ as in (14). They should be rescaled using the parameter $\eta = \sqrt{\alpha/\alpha_0}$ according to (18), and may change by a few 10^{-10} , depending on α .

“Conventional expressions”	Expressions in terms of s, s^{-1} , s^{-2} or as constants ($\hbar = c = \underline{\mu_0} = \underline{\epsilon_0} = 1$)		
1 m	=	$\frac{1}{c_0} \text{ s}$	= $\frac{1}{299\,792\,458} \text{ s}$
1 J	=	$\frac{1}{\hbar_0} \text{ s}^{-1}$	= $.948\,252\,156\,246 \dots \times 10^{34} \text{ s}^{-1}$
1 kg	=	$\frac{c_0^2}{\hbar_0} \text{ s}^{-1}$	= $.852\,246\,536\,175 \dots \times 10^{51} \text{ s}^{-1}$
1 N	=	$\frac{c_0}{\hbar_0} \text{ s}^{-2}$	= $2.842\,788\,447\,250 \dots \times 10^{42} \text{ s}^{-2}$
1 A = $\sqrt{\mu_0 \text{ N}}$	=	$\sqrt{\frac{\mu_0 c_0}{\hbar_0}} \text{ s}^{-1}$	= $1.890\,067\,014\,853 \dots \times 10^{18} \text{ s}^{-1}$
1 C = $\sqrt{\mu_0 \text{ kg m}}$	=	$\sqrt{\frac{\mu_0 c_0}{\hbar_0}}$	= $1.890\,067\,014\,853 \dots \times 10^{18}$
1 V = 1 J/C	=	$\frac{1}{\sqrt{\mu_0 c_0 \hbar_0}} \text{ s}^{-1}$	= $5.017\,029\,284\,119 \dots \times 10^{15} \text{ s}^{-1}$
1 V/m = 1 N/C = $\frac{1}{\sqrt{\mu_0}} \sqrt{\text{J/m}^3} \text{ m/s}$	=	$\sqrt{\frac{c_0}{\mu_0 \hbar_0}} \text{ s}^{-2}$	= $1.504\,067\,540\,944 \dots \times 10^{24} \text{ s}^{-2}$
1 T = 1 N/(Am) = $\frac{1}{\sqrt{\mu_0}} \sqrt{\text{J/m}^3}$	=	$\sqrt{\frac{c_0^3}{\mu_0 \hbar_0}} \text{ s}^{-2}$	= $4.509\,081\,050\,976 \dots \times 10^{32} \text{ s}^{-2}$
1 Wb = 1 Vs	=	$\frac{1}{\sqrt{\mu_0 c_0 \hbar_0}}$	= $5.017\,029\,284\,119 \dots \times 10^{15}$
1 F = 1 C/V = 1 s/Ω = $\mu_0 \text{ s}^2/\text{m}$	=	$\mu_0 c_0 \text{ s}$	= $376.730\,313\,461 \dots \text{ s}$
1 H = 1 J/A ² = 1 Ωs = $\frac{1}{\mu_0} \text{ m}$	=	$\frac{1}{\mu_0 c_0} \text{ s}$	= $1/376.730\,313\,461 \dots \text{ s}$
1 Ω = 1 V/A = 1 W/A ² = $\frac{1}{\mu_0} \text{ m/s}$	=	$\frac{1}{\mu_0 c_0}$	= $1/376.730\,313\,461 \dots$
$Z_0 = \mu_0 c_0 \Omega = 376.730\,313\,461 \dots \Omega$	=	$\underline{\mu_0} c$	= 1
$e = 1.602\,176\,634 \times 10^{-19} \text{ C}$	=	$\sqrt{4\pi\alpha}$	= $.302\,822\,120\,789 \dots$
1 eV = $1.602\,176\,634 \times 10^{-19} \text{ J}$	=	$\frac{e_0}{\hbar_0} \text{ s}^{-1}$	= $1.519\,267\,447\,878 \dots \times 10^{15} \text{ s}^{-1}$
$\Phi_0 = \hbar/2e = 2.067\,833\,848\,461 \dots \times 10^{-15} \text{ Wb}$	=	$\sqrt{\frac{\pi}{4\alpha}} = \frac{\pi}{e}$	= $10.374\,382\,972 \dots$
$K_j = 2e/\hbar = 483\,597.848\,416\,983 \dots \text{ GHz/V}$	=	$\sqrt{\frac{4\alpha}{\pi}} = \frac{e}{\pi}$	= $.0963\,912\,748\,023 \dots$
$R_K = \frac{\hbar}{e^2} = \frac{\mu_0 c_0 \Omega}{2\alpha} = 25\,812.807\,459\,304 \dots \Omega$	=	$\frac{\mu_0 c}{2\alpha} = \frac{1}{2\alpha}$	= $68.517\,999\,579 \dots$

$$\eta = \frac{C}{C_{\text{old}}} = \frac{e_{\text{old}}}{e_0} \tag{8}$$

very close to 1. The coulomb, and thus the ampere, equal to $\sqrt{\mu_0 \text{ N}}$, get multiplied by η , with

$$1 \text{ A} = \sqrt{\mu_0 \text{ N}} = \sqrt{4\pi \times 10^{-7} \text{ N}} \times \eta \tag{9}$$

so that μ_0 should be multiplied by η^2 , becoming $\mu_0 = 4\pi \times 10^{-7} \eta^2$.

We must ensure that the two concurrent definitions of the coulomb and the ampere are compatible. The ampere is expressed as $1 \text{ A} = \sqrt{\mu_o \text{ N}}$ as in (5). The compatibility of the four-equation set in (3), (7) requires that the third equation in (3), $\mu_o = \mu_o \text{ N/A}^2 = 1$, be used, no longer to fix the exact size of the ampere from a fixed $\mu_o = 4\pi \times 10^{-7}$, but instead to determine a new “floating” value for μ_o , from now on equal to $4\pi \times 10^{-7} \eta^2$, from an ampere already defined as 1 C/s. The two definitions for the coulomb (and ampere) from (5) and (7) should be equalized as follows

$$1 \text{ A} = \sqrt{\mu_o \text{ N}} = \sqrt{\mu_o c_o / \hbar_o} \text{ s}^{-1} \implies 1 \text{ C} = 1 / \sqrt{\epsilon_o c_o \hbar_o} = e / e_o \iff e = e_o \text{ C} \quad (10)$$

This provides two equivalent expressions for the fine-structure constant,

$$\alpha = \frac{e_o^2}{4\pi \epsilon_o \hbar_o c_o} = \frac{\mu_o c_o e_o^2}{4\pi \hbar_o} = \frac{e^2}{4\pi} \quad (11)$$

$\alpha = e^2/4\pi$ corresponds to the choice $\hbar = c = \underline{\mu_o} = \underline{\epsilon_o} = 1$ also made here, leading to the dual expression of the elementary charge

$$e = e_o \text{ C} = 1.602 \, 176 \, 634 \times 10^{-19} \text{ C} = \sqrt{4\pi \alpha} \simeq .302 \, 822 \, 1208 \quad (12)$$

The price to pay for fixing e_o as above is that μ_o should now be multiplied by η^2 , getting adjusted to

$$\mu_o = \frac{4\pi \hbar_o}{c_o e_o^2} \alpha = 4\pi \times 10^{-7} \eta^2 \quad (13)$$

so that α stays unchanged. This leads to evaluate the benchmark value for α corresponding to keeping an unchanged $\mu_o = 4\pi \times 10^{-7}$ value with the new fixed choice of e_o ,

$$\alpha_o = \frac{4\pi \times 10^{-7} c_o e_o^2}{4\pi \hbar_o} = 7.297 \, 352 \, 565 \, 305 \dots \times 10^{-3} = 1/137.035 \, 999 \, 158 \, 713 \dots \quad (14)$$

It is very close to the present best-determined value $\alpha = 1/137.035 \, 999 \, 139(31)$, with a relative standard uncertainty of 2.3×10^{-10} [11,15]. It corresponds to

$$\sqrt{4\pi \alpha_o} = .302 \, 822 \, 120 \, 789 \, 201 \dots \quad (15)$$

μ_o is no longer rigidly fixed, but proportional to α . Fixing e_o as in (7), (12) leads to a very small rescaling of all electrical units depending on the parameter

$$\eta = \sqrt{\alpha/\alpha_o} \simeq 1 \quad (\text{up to } \lesssim \text{ a few } 10^{-10}) \quad (16)$$

Thus

$$\text{fixing } e = e_o \text{ C} \implies \text{all electrical units become dependent on } \alpha \quad (17)$$

with

$$(A, C) \propto \eta, \quad (V, T, \text{Wb}) \propto \eta^{-1}, \quad F \propto \eta^2, \quad (H, \Omega) \propto \eta^{-2}; \quad \mu_o \propto \eta^2, \quad \epsilon_o \propto \eta^{-2}; \quad \text{with } \eta = \sqrt{\alpha/\alpha_o}. \quad (18)$$

The sizes of the electrical units themselves become dependent on future experimental measurements of the fine structure constant α .

In particular the value in ohms of the “impedance of the vacuum”, $\mu_o c_o \propto \alpha$, now depends on α in the new SI, i.e. on what was previously the value of the elementary charge e ! Of course, Z_o itself, equal to $\mu_o c_o \Omega$, does not, as one can verify from (18). Still this artificially introduced dependence of the measure of the impedance of vacuum may look strange, especially for a quantity that is basically 1, with an appropriate choice of fundamental units (see below, subsection 10.2). Its measure $\mu_o c_o$ should actually be viewed as a parameter characterising the size of the ohm (unfortunately dependent on the experimentally measured value of α in the new SI), and not as a measure of a physical property of the vacuum.

This conceptual inconvenience of the new SI associated with the floating character of the electrical units gets alleviated within the unified framework proposed here, in which the impedance of vacuum, $Z_o = \mu_o c_o \Omega$, while still equal to 376.730... Ω , is also identical to 1. The magnetic permeability of vacuum, now expressed as $\underline{\mu_o} = \mu_o \text{ N/A}^2 = 1$, remains also equal to 1 with $1 \text{ A} = \sqrt{\mu_o \text{ N}}$, even if the ampere A, as well as the numerical value μ_o of the vacuum permeability (expressed in N/A^2 or H/m), is dependent on α , as seen in (18).

In case α were to vary with time as the possibility is occasionally considered, *all electrical units, including the dimensionless ohm, coulomb, and weber, would also vary with time*. This may be expressed through the set of equations

$$\frac{\dot{A}}{A} = \frac{\dot{C}}{C} = -\frac{\dot{V}}{V} = -\frac{\dot{T}}{T} = -\frac{\dot{Wb}}{Wb} = \frac{1}{2} \frac{\dot{\alpha}}{\alpha}, \quad \frac{\dot{F}}{F} = -\frac{\dot{H}}{H} = -\frac{\dot{\Omega}}{\Omega} = \frac{\dot{\alpha}}{\alpha} \quad (19)$$

as seen from Eqs. (18) and Table 1. This is a not-so-attractive consequence of the new SI choice of fixing e_0 as in (7), in principle not compatible with the requirement that units should not change with time. Fortunately, there are very strong limits on a possible time variation of α (with $|\dot{\alpha}/\alpha|$ constrained to be at present $< 10^{-16}/y$), so that in practice this does not appear as a limitation.

Let us now return for some time to the mechanical units, defining the metre, the joule, and the kilogram by fixing the numerical values of c and h .

4. Defining the metre from the second, by fixing c

The unit of time, the second, has long been defined as the fraction 1/86 400 of the “mean solar day”. It was then redefined from a physical phenomenon involving the period of a specific atomic transition of the caesium-133 atom. More precisely, since 1967 [6],

“The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.” (20)

The unit of length, the metre, first introduced as “la dix-millionième partie du quart du méridien terrestre” [1], was realized as the “Mètre des Archives” in 1799, then defined from 1889 to 1960 as the length of a specific object, the international prototype of platinum-iridium stored in the “Pavillon de Breteuil” at Sèvres. It was replaced in 1960 by a definition involving the wavelength of a krypton-86 radiation [16]. The theory of relativity, based on the invariance of the speed of light in vacuum, c , allowed for a new definition of the metre by relating it to the unit of time. This was done in 1983 by fixing the value of c at, exactly, $c = 299\,792\,458$ m/s [7]. Since this date,

“The metre is the length of the path travelled by light in vacuum during a time interval of 1/ 299 792 458 of a second.” (21)

This is now reformulated, in an equivalent way, by stating that [10]

“the speed of light in vacuum c is 299 792 458 m/s.” (22)

Indeed, relativity relates intimately the concepts of time and space, describing events in a 4-dimensional spacetime with coordinates $x^\mu = (ct, \vec{x})$. It also relates the energy E and the momentum \vec{p} of a particle of mass m into the components of a 4-vector $p^\mu = (E/c, \vec{p})$, with $p_\mu p^\mu = E^2/c^2 - \vec{p}^2 = m^2 c^2$. This reduces to the famous $E = mc^2$ for a massive particle at rest, and $E = pc$ for a massless photon travelling at the speed of light. The unit of length may thus be derived from the unit of time by fixing the value of c .

5. Fixing $c = 299\,792\,458$ m/s, or $c = 1$, or both at the same time

Choosing $c = 1$ would be the most natural choice, leading to measure space distances directly in seconds, and energies and momenta in units of mass. The unit of length u_1 would then be the distance travelled by light during the unit of time u_t . Once this one is defined as the second, u_1 is the length travelled by light in vacuum during 1 s (sometimes called a light-second), namely in SI units

$$\text{unit of length } u_1 = c s = 299\,792\,458 \text{ m} \quad (23)$$

This is almost the distance to the Moon. Choosing $c = 1$ could lead to abandon the metre to replace it by a unit of length almost 300 million times larger, which is usually rejected as unpractical.

In spite of that, we remain free to continue using the metre, now defined as a submultiple of the above natural unit of length according to $1 \text{ m} = 1/299\,792\,458 u_1$. This expresses that the speed of light is chosen to be $c = 1 u_1/1 u_t = 299\,792\,458$ m/s.

Even better, we can do both things at the same time by requiring $c = 299\,792\,458$ m/s and, “en même temps”, $c = 1$. We can then complete the official formulation (22) as follows

“the speed of light in vacuum is $c = 299\,792\,458$ m/s $\equiv 1$.” (24)

It expresses that the natural unit of length according to relativity, $u_1 = 299\,792\,458$ m, is not independent of the unit of time, and may be identified with it. We then get

$$1 \text{ m} = \frac{1}{299\,792\,458} \text{ s} \quad (25)$$

Expressing that $c = c_0$ m/s $\equiv 1$ allows for the identification $1 \text{ m} = (1/c_0) \text{ s}$.

This unconventional formulation defines directly the metre as a submultiple of the second. It may be taken as an improved definition of the metre, reconciling the present one in (21), (22) with the choice $c = 1$ suggested by relativity. Conversely, the metre also appears as a unit of time, as is natural owing to the symmetry between space and time provided by relativity. We may then say that

*“The time interval taken by light to travel in vacuum along a path of one metre long
is 1/ 299 792 458 of a second, or ... one metre.”* (26)

We may even define a common subunit for distance and time appearing as a (metric) foot, equal to $1 \text{ ns} = .299 792 458 \text{ m} \simeq 29.98 \text{ cm}$, again illustrating that the speed of light is $c = 1$.

We still need a unit of mass. This one is defined, since the first “Conférence générale des poids et mesures” in 1889, as the mass of the international prototype of the kilogram (IPK), also known as “le grand K”, \mathcal{K} , conserved at the BIPM. This was formulated in 1901 as [8]:

“The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.” (27)

The unit of force is then the newton, equal to 1 kg m s^{-2} , and the unit of energy the joule, equal to $1 \text{ N m} = 1 \text{ kg m}^2 \text{ s}^{-2}$. The unit of angular momentum, or action, is the $\text{kg m}^2 \text{ s}^{-1}$, or Js. But the quantity of matter of the international prototype of the kilogram cannot remain exactly constant, and varies very slightly over the years. In fact, the differences in mass between the IPK and supposedly identical copies average to a few tens of μg per century [9]. Could we avoid having to resort to such a physical object to define the unit of mass, and relate it instead to a reproducible universal phenomenon, as for the second and the metre? This is where quantum physics comes in, as angular momenta and actions are quantized in units of $\hbar = h/2\pi$, whose value in SI units is $\simeq 1.054 \dots \times 10^{-34} \text{ Js}$.

6. Quantum physics and the correspondence principle

Quantum physics allows us to relate energies with frequencies, and momenta with wavelengths. It involves the Planck constant h , which relates, through $E = h\nu = \hbar\omega$, the energy E carried by a photon with the frequency $\nu = \omega/2\pi$ of the corresponding electromagnetic wave, and similarly for the other particles. A particle of momentum \vec{p} is associated with a wave of wave vector \vec{k} given by $\vec{p} = \hbar\vec{k}$, and wavelength λ given by the De Broglie relation $p = \hbar k = h/\lambda$, where $\hbar = h/2\pi$ is the reduced Planck constant.

Quantum mechanics thus relates, through the constants h or \hbar , energies with inverses of times, and momenta with inverses of lengths, $E \propto t^{-1}$, $p \propto l^{-1}$. This is an expression of the *correspondence principle*, which associates with each particle (including the photon as a quantum of light) of energy E and momentum \vec{p} a wave

$$\psi(\vec{x}, t) \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)} = e^{-i(Et - \vec{p}\cdot\vec{x})/\hbar}, \quad \text{with } E = h\nu = \hbar\omega, \quad \vec{p} = \hbar\vec{k} \quad (28)$$

It may be formulated in a Lorentz-invariant way in terms of the spacetime coordinate $x^\mu = (ct, \vec{x})$, as

$$\psi(x^\mu) \propto e^{-i k_\mu x^\mu} = e^{-i p_\mu x^\mu / \hbar}, \quad \text{with } p^\mu = (E/c, \vec{p}) = \hbar k^\mu = \hbar (\omega/c, \vec{k}) \quad (29)$$

Thus a measurement of energy, or momentum, can be replaced by an associated measurement for the corresponding angular frequency, or wavenumber, once the numerical value of \hbar (in Js) is experimentally determined. For a free particle of mass m , $p_\mu x^\mu = mc^2 \tau$, where τ is the proper time experienced by the particle, so that $\psi(x^\mu)$ in (29) may also be expressed as a function of the proper time, proportionally to $e^{-imc^2 \tau / \hbar}$.

Even better, a measurement of energy (or momentum) becomes identical to a measurement of angular frequency (expressed in s^{-1}) or wave number (expressed in m^{-1}) if the value of \hbar gets fixed, in agreement with its present best experimental determination, within uncertainties. Units of energies or momenta get then derived from the corresponding units of angular frequencies or wavenumbers. The present definition of the kilogram, taken since 1889 as the mass of the international prototype stored at the BIPM, is no longer necessary, and may be abandoned in favour of a new definition based on quantum physics.

7. Defining the joule and kilogram from the inverse of the second, through quantum mechanics

\hbar is the fundamental quantum of action, or angular momentum, in quantum mechanics. It has the dimensions of (energy \times time), or (momentum \times space), or angular momentum (\times angle, dimensionless). It is measured in Js, or equivalently $\text{kg m}^2 \text{ s}^{-1}$, as also seen from the expressions of the operators hamiltonian, momentum, and orbital angular momentum,

$$H = i\hbar \frac{\partial}{\partial t}, \quad \vec{P} = -i\hbar \frac{\partial}{\partial \vec{x}}, \quad L_z = -i\hbar \frac{\partial}{\partial \varphi} \quad (30)$$

in agreement with the correspondence principle. In particular, the plane waves $\psi \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ in (28) are eigenfunctions of H and \vec{P} with the eigenvalues $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$. They are also eigenfunctions of the operator $i\hbar \frac{d}{d\tau}$ where τ is the proper

time experienced by the particle, with its mass m as eigenvalue.

We make no difference between the so-called “inertial” and “gravitational” masses of a particle or macroscopic object, m_i and m_g . They may be taken as identical, as investigated long ago by Eötvös and his collaborators [17]. The resulting Equivalence Principle, formulated by Einstein, is at the basis of General Relativity, which provides the theory of gravitation, at the classical level. The *MICROSCOPE* experiment provides at present the most stringent test on the validity of this principle, allowing for the identification of inertial and gravitational masses at a level of precision of 2×10^{-14} , for the pair of materials tested [18].

The reduced Planck constant, $\hbar = h/2\pi$, is such that the intrinsic angular momentum of the electron, or spin, is $\hbar/2$. The electron is a spin-1/2 particle, meaning that its spin is $S = 1/2$, when \hbar is taken equal to 1. Fixing $\hbar = 1$ also defines the unit of energy from (or even identical to) the unit of angular frequency, namely the s^{-1} (in principle $\times \hbar$, which is 1). The expression of \hbar (close to $1.054\,571\,818 \times 10^{-34}$ Js if $h = h_o$ Js is fixed at $6.626\,070\,15 \times 10^{-34}$ Js) implies that the natural unit of energy is

$$\text{unit of energy } u_e = \hbar s^{-1} = \hbar_o J = \frac{6.626\,070\,15 \times 10^{-34}}{2\pi} J = 1.054\,571\,817\,646\dots \times 10^{-34} J \quad (31)$$

(or $\text{kg m}^2 \text{s}^{-2}$). Fixing $c = 1$ also determines, as in (23), the unit of length $u_l = c s = c_o \text{ m}$ as the distance travelled by light in vacuum during the unit of time, here the second. This ultimately allows for identifying the unit of length with the unit of time, and thus to measure distances directly in seconds.

The unit of mass is then the same as for energy (as $c = 1$), namely $u_m = u_e = 1 s^{-1}$. This single unit may be reexpressed in conventional units using the joule as unit of energy or the kilogram as unit of mass, so that

$$\text{unit of mass } u_m = \hbar/c^2 s^{-1} = \hbar_o/c_o^2 \text{ kg} = 1.173\,369\,392\,016\dots \times 10^{-51} \text{ kg} \quad (32)$$

The unit of angular momentum, now dimensionless, is 1. This may be verified from $u_m = (\hbar/c^2) s^{-1}$, $u_l = c s$, $u_t = 1 s$, combining into *unit of angular momentum* $u_j = u_m u_l^2 / u_t = \hbar$. It may be reexpressed, in conventional units, as $(\hbar_o/2\pi) Js = 1.054\,571\,817\,646\dots \times 10^{-34} Js$.

For a theoretician, measuring space directly in seconds and masses and energies in s^{-1} would be the natural thing to do. This is not very practical, however, the second as a unit of distance being too large, while the s^{-1} as a unit of energy (close to 10^{-34} J) or unit of mass (close to 10^{-51} kg) is too small. The unit of angular momentum when $\hbar = 1$, now the dimensionless number 1, is suitable for individual photons, electrons or nucleons, but too small for macroscopic objects.

It is thus convenient to resize consistently the above units (23), (31), (32) of length, energy, and mass to turn them into more convenient ones, still keeping the familiar names of metre, joule, and kilogram in order not to disrupt long habits and use easily former measurements. The quantities c and h or \hbar , often previously referred to as “fundamental constants of nature”, get fixed, according to the resolution [10]

$$\begin{aligned} & \text{“the speed of light in vacuum } c \text{ is } 299\,792\,458 \text{ m/s”}, \\ & \text{“the Planck constant } h \text{ is } 6.626\,070\,15 \times 10^{-34} \text{ Js.”} \end{aligned} \quad (33)$$

and used as normalization constants in the redefinition of the system of units.

Just as we can use the above choice of c to redefine the metre from the natural unit of length u_l as in (23), we can use the choice of a fixed value of h (or \hbar), namely $h = h_o Js$, to redefine the joule and the kilogram from the natural units of energy and mass in (31), (32). This leads to

$$\begin{cases} 1 \text{ m} = 1/c_o \text{ unit of length } u_l, \\ 1 \text{ J} = 1/\hbar_o \text{ unit of energy } u_e, \\ 1 \text{ kg} = c_o^2/\hbar_o \text{ unit of mass } u_m \end{cases} \quad (34)$$

which define the metre, the joule, and the kilogram.

8. Fixing $h = 6.626\,070\,15 \times 10^{-34}$ Js, and $\hbar = 1$

Beyond that, we can identify $u_l = u_t = 1 s$, and $u_m = u_e = 1 s^{-1}$. This leads to *identify* the metre, the joule, the kilogram, and the newton, etc., as fixed numbers of s , s^{-1} or s^{-2} , etc. Indeed, beyond the official definitions $c = c_o \text{ ms}^{-1}$ with c_o fixed at 299 792 458, $h = h_o Js$ with h_o fixed at $6.626\,070\,15 \times 10^{-34}$, we now identify

$$\begin{cases} c = c_o \text{ ms}^{-1} \equiv 1, \\ \hbar = \hbar_o \text{ Js} \equiv 1, \end{cases} \implies \begin{cases} \hbar/c^2 = \hbar_o/c_o^2 \text{ kg s} & \equiv 1, \\ \hbar/c = \hbar_o/c_o \text{ kg m} = \hbar_o/c_o \text{ N s}^2 & \equiv 1, \\ \hbar c = \hbar_o c_o \text{ J m} & \equiv 1 \end{cases} \quad (35)$$

This leads to the identifications

$$1 \text{ metre} = \frac{1}{c_o} \text{ s}, \quad 1 \text{ joule} = \frac{1}{\hbar_o} \text{ s}^{-1}, \quad 1 \text{ kilogram} = \frac{c_o^2}{\hbar_o} \text{ s}^{-1}, \quad 1 \text{ newton} = \frac{c_o}{\hbar_o} \text{ s}^{-2} \quad (36)$$

or, explicitly,

$$\left\{ \begin{array}{l} 1 \text{ m} = (1/299\,792\,458) \text{ s}, \\ 1 \text{ J} = \frac{2\pi}{6.626\,070\,15 \times 10^{-34}} \text{ s}^{-1} = .948\,252\,156\,246\dots \times 10^{34} \text{ s}^{-1}, \\ 1 \text{ kg} = \frac{2\pi \times (299\,792\,458)^2}{6.626\,070\,15 \times 10^{-34}} \text{ s}^{-1} = .852\,246\,536\,175\dots \times 10^{51} \text{ s}^{-1}, \\ 1 \text{ N} = 1 \text{ kg m s}^{-2} = 1 \text{ J m}^{-1} = \frac{2\pi \times 299\,792\,458}{6.626\,070\,15 \times 10^{-34}} \text{ s}^{-2} = 2.842\,788\,447\,250\dots \times 10^{42} \text{ s}^{-2} \end{array} \right. \quad (37)$$

We can then say that the electron is a spin-1/2 particle, with $\hbar = \hbar_o \text{ Js} = 1$, the speed of light being $c = 299\,792\,458 \text{ m/s}$, and, at the same time, $c = 1$, with $\underline{\mu_o} = \mu_o \text{ N/A}^2 = 1$.

Should we fix h , or rather \hbar ?

The above equations (34)–(37) also indicate that, at least from a conceptual point of view, *it could have been preferable to fix numerically the value of \hbar* (possibly to $1.054\,571\,82 \times 10^{-34} \text{ Js}$), which determines directly the angular momentum of the photon (\hbar), or of the electron ($\hbar/2$), rather than fixing h . This would also have led to simpler definitions for the joule, the kilogram, and the newton in (34)–(37).

Anyhow, once such a choice has been made, the definitions of the joule, and of the associated kilogram, can also be formulated through one of the sentences

$$\left\{ \begin{array}{l} \text{the angular momentum of an electron is } \frac{1}{2} \frac{6.626\,070\,15 \times 10^{-34}}{2\pi} \text{ Js (or kg m}^2 \text{ s}^{-1}), \\ \text{the angular momentum of a circularly polarized photon is } \frac{6.626\,070\,15 \times 10^{-34}}{2\pi} \text{ Js (or kg m}^2 \text{ s}^{-1}), \end{array} \right. \quad (38)$$

which give a direct physical meaning to the fixing of Planck's constant h in (1), (33).

Still, defining in this way the metre, the joule, the kilogram, and the newton from the second, the s^{-1} , or the s^{-2} also requires a *mise en pratique* allowing one to apply these definitions to macroscopic objects. The Kibble balance, in particular, appeals to electrical measurements relying on electromagnetic interactions in the quantum regime, implying the Josephson and von Klitzing constants $K_J = 2e/h$ and $R_K = h/e^2$ [19]. One can also count a larger number of atoms in a crystal, and determine the mass of a silicon sphere as $m = (8V/a^3)m(\text{Si})$, where $m(\text{Si})$ is the mean mass of a silicon atom in the crystal, and a^3 the volume of the unit cell, with eight atoms on average [20].

The redefinition of the kilogram from quantum mechanics does not rely on interactions

The definition of the kilogram from quantum mechanics through fixing the value of h or \hbar , however, does not in principle appeal to interactions, and in particular does not require the explicit consideration of electromagnetic interactions. This is true even at the macroscopic level, as we shall see with the Casimir force, which depends only on \hbar and c , and whose expression is fixed once $\hbar c$ is fixed.

Indeed, the momentum of a particle, and thus its mass, or the energy of a photon, may be determined from the dimension associated with the diffraction or interference pattern associated with it. This involves only free particles. Indeed, the Dirac equation for a free spin-1/2 field or particle, the Klein–Gordon equation for a free spin-0 field or particle, and the Maxwell equations for a free massless spin-1 field, expressed as

$$\left\{ \begin{array}{l} \text{Dirac:} \quad (i\hbar \gamma^\mu \partial_\mu - mc) \psi = 0, \\ \text{Klein–Gordon:} \quad (\hbar^2 \partial^\mu \partial_\mu + m^2 c^2) \varphi = 0, \\ \text{Maxwell:} \quad \partial_\mu F^{\mu\nu} = 0 \end{array} \right. \quad (39)$$

involve the constants c and \hbar associated with the definition of the system of units, but not the interactions. With $c = \hbar = 1$ (a choice compatible with $c = 299\,792\,458 \text{ m/s}$ and $h = 6.626\,070\,15 \times 10^{-34} \text{ Js}$ as we saw), these equations get simply written as

$$(i\partial - m) \psi = 0, \quad (\square + m^2) \varphi = 0, \quad \partial_\mu F^{\mu\nu} = 0 \quad (40)$$

These fundamental equations no longer involve the “fundamental constants” c and \hbar ! But their numerical values in SI units, c_o and \hbar_o , now serve in the definition of the fundamental units of length and mass.

The consideration of the Casimir force between two conducting plates allows in principle for passing from the microscopic to the macroscopic level for the measurement of a force and thus also a mass, defined as earlier by fixing h or \hbar . Ideally, the Casimir force depends only on geometry, and associates with a distance l between two conducting plates a force per unit surface, i.e. a pressure. Its expression involves c and \hbar . Once c is fixed it may be used, in principle, to determine the value of \hbar in ordinary units of J s, even if only with a very modest precision. Or conversely, if \hbar and thus $\hbar c$ is fixed, it allows one to realize the joule from the metre. Considered as a force per unit surface, it reads

$$\mathcal{P} = \frac{F}{S} = -\frac{\pi^2}{240} \frac{\hbar c}{l^4} \simeq -\frac{.013 \text{ dyn/cm}^2}{l(\mu\text{m})^4} \simeq \frac{1.3 \text{ mN/m}^2}{l(\mu\text{m})^4} \quad (41)$$

It indicates how a very small force might be realized experimentally at the macroscopic level, even if not precisely, once \hbar is fixed and the unit of length has been defined by fixing c . This allows us, in principle, to pass at the macroscopic level from a geometric unit of distance to a mechanical unit of force and, subsequently, energy and mass, without having to consider explicitly the value of the elementary charge e (but taking into account the boundary conditions for the electromagnetic field between the plates).

In practice, the precision of quantum electromagnetic effects is essential to get a precise determination of h through $K_J = 2e/h$ and $R_K = h/e^2$, and taking full advantage of fixing h [19]. This motivates a more precise definition of electrical units than the usual one through the traditional definition of the ampere [10,14,21].

9. Electrical units as tied to the mechanical ones

9.1. The ampere and the other electrical units, as obtained from μ_o

The electrical units are at the moment rigidly tied to the mechanical ones through a fixed μ_o as seen in Section 2. Since 1946 and up to now in 2018, the ampere A has been defined as follows [12,13]:

“The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 m apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length.” (42)

This originates from Ampère’s force law stating that the force F per length L between two such parallel conductors at distance r , through which a current of intensity I passes, is $F/L = (\mu_o/2\pi) I^2/r$. This definition involves the fixed number 2×10^{-7} associated with the factor μ_o , referred to as the magnetic permeability of free space, conventionally taken as $4\pi \times 10^{-7} \text{ N/A}^2$ in the present SI (but soon to become $4\pi \times 10^{-7} \eta^2 \text{ N/A}^2$).

Ampère’s force law, however, is a fundamental law of physics, and it seems more logical to write it independently of the somewhat arbitrary parameter μ_o , which determines the sizes chosen for the electrical units (very much as done for Eqs. (40) when writing the free equations of motion independently of c and \hbar). We shall thus discard the factor μ_o from its expression (or write it with an implicit parameter $\underline{\mu_o} = 1$), to include it as a numerical dimensionless coefficient ($4\pi \times 10^{-7}$ in the present SI) within the man-made definition of the ampere. This allows for a new equivalent formulation, in which $1 \text{ A}^2 = \mu_o \text{ N}$, the ampere being obtained as proportional to a square root of the newton as in (2),

$$\text{Ampère's force law } \frac{F}{L} = \frac{I^2}{2\pi r} \implies 1 \text{ A} = \sqrt{\mu_o \text{ N}} \quad (43)$$

This provides the force per unit length between two conductors carrying currents of 1 A at a distance of 1 m in vacuum, obtained from (43) as $F = 1 \text{ A}^2/2\pi = (\mu_o/2\pi) \text{ N}$, i.e. at the moment, before the redefinition of the SI, as $2 \times 10^{-7} \text{ N}$ as in (42).

Writing as above the fundamental laws with $\underline{\mu_o} = 1$, the ampere, the coulomb, and the volt get given by the new expressions

$$\left\{ \begin{array}{l} 1 \text{ ampere} \\ 1 \text{ coulomb} = 1 \text{ A s} \\ 1 \text{ volt} = 1 \text{ J/C} = 1 \text{ W/A} = \frac{1}{\sqrt{\mu_o}} \text{ W}/\sqrt{\text{N}} = \sqrt{\frac{\text{J m s}^{-2}}{\mu_o}} \end{array} \right. = \left\{ \begin{array}{l} \sqrt{\mu_o \text{ N}} , \\ \sqrt{\mu_o \text{ kg m}} , \\ \sqrt{\frac{\text{J m s}^{-2}}{\mu_o}} \end{array} \right. \quad (44)$$

The coulomb, in particular, is proportional to a geometric mean of the kilogram and the metre.

The definitions (44), and (45) below, of the various electrical units follow, in agreement with the (μ_o -independent) expressions for the electrostatic energy (qV) or the electrical power (UI) for the volt; Laplace’s magnetic force on a wire ($dF = I \vec{dl} \times \vec{B}$) for the tesla; energy stored in a coil or capacitor ($LI^2/2$ or $CV^2/2$) for the henry and farad; and power dissipated into a resistance (RI^2) for the ohm. A magnetic field also appears as proportional to the square root of the

corresponding energy density (expressed as $B^2/2$, with $\underline{\mu}_o = 1$). An inductance is proportional to a length and a resistance to a velocity, etc., with

$$\left\{ \begin{array}{l} 1 \text{ tesla} = 1 \text{ N/(Am)} = \frac{1}{\sqrt{\mu_o}} \sqrt{\text{N/m}^2} = \frac{1}{\sqrt{\mu_o}} \sqrt{\text{J/m}^3}, \\ 1 \text{ henry} = 1 \text{ J/A}^2 = \frac{1}{\mu_o} \text{ J/N} = \frac{1}{\mu_o} \text{ m}, \\ 1 \text{ farad} = 1 \text{ C/V} = 1 \text{ J/V}^2 = \mu_o \text{ s}^2/\text{m}, \\ 1 \text{ ohm} = 1 \text{ V/A} = 1 \text{ W/A}^2 = \frac{1}{\mu_o} \text{ W/N} = \frac{1}{\mu_o} \text{ m/s} \end{array} \right. \quad (45)$$

The resulting expressions of electrical units are given in Table 1.

We note the usual relations

$$1 \text{ H} = 1 \Omega \text{ s}, \quad 1 \text{ F} = 1 \text{ s}/\Omega \quad (46)$$

in agreement with the characteristic times $\tau = L/R$ of an inductance and $\tau = RC$ of a capacitor in association with a resistance (with $1 \text{ H} \times 1 \text{ F} = 1 \text{ s}^2$, as from the relation $LC\omega^2 = 1$ for an LC oscillator). These definitions and relations are by construction compatible with the rescaling of μ_o by a factor η^2 , acting on the various units as in (18).

9.2. Fixing $c = c_o$ m/s, $\hbar = \hbar_o$ Js, then also $c = \hbar = 1$

We can then take advantage of fixing $c = c_o$ m/s = 1, $\hbar = \hbar_o$ Js = 1, to express the ampere, and all electrical units, in terms of the second, s^{-1} or s^{-2} , or even as fixed numerical constants, as displayed in Table 1. An impedance, $Z = R + j(L\omega - 1/C\omega)$, gets dimensionless, and may be expressed in ohms, a dimensionless unit, equal to $1/\mu_o c_o$, with the “impedance of vacuum” being now $Z_o = \mu_o c_o \Omega = 1$.

To understand things better, we decompose the fixing of c and \hbar in two steps, first by fixing their values in m/s and Js as in the new SI [10], then by choosing also $c = \hbar = 1$ as proposed here. With $c = c_o$ m/s, $\hbar = \hbar_o$ Js, we express the ampere and coulomb as

$$\left\{ \begin{array}{l} 1 \text{ A} = \sqrt{\mu_o \text{ N}} = \sqrt{\frac{\mu_o \text{ Js}}{\text{m/s}}} \text{ s}^{-1} = \sqrt{\frac{\mu_o c_o}{\hbar_o}} \sqrt{\frac{\hbar}{c}} \text{ s}^{-1}, \\ 1 \text{ C} = \sqrt{\mu_o \text{ kg m}} = \sqrt{\frac{\mu_o \text{ Js}}{\text{m/s}}} = \sqrt{\frac{\mu_o c_o}{\hbar_o}} \sqrt{\frac{\hbar}{c}} \end{array} \right. \quad (47)$$

Electric charges, including the coulomb, still have a dimension, and may be evaluated at this stage in $\sqrt{\text{kg m}}$, with μ_o taken as dimensionless.

With the additional choice $c = \hbar = 1$, the scale for all electrical units gets fixed (in terms of μ_o) and formulas (47) simplify into

$$\left\{ \begin{array}{l} 1 \text{ A} = \sqrt{\mu_o \text{ N}} = \sqrt{\frac{\mu_o c_o}{\hbar_o}} \text{ s}^{-1} = 1.890 \dots \times 10^{18} \text{ s}^{-1}, \\ 1 \text{ C} = \sqrt{\mu_o \text{ kg m}} = \sqrt{\frac{\mu_o c_o}{\hbar_o}} = \sqrt{\frac{1}{\epsilon_o c_o \hbar_o}} = 1.890 \dots \times 10^{18} \end{array} \right. \quad (48)$$

the coulomb becoming a dimensionless unit. Electrical charges are now dimensionless, as for angles, their natural unit being 1 in a system with $\hbar = c = \underline{\mu}_o = \underline{\epsilon}_o = 1$. The coulomb is an artificial unit, resulting from the definition (42) of the ampere (from μ_o), and of the metre and joule through the choices of c_o and \hbar_o . The elementary charge e , now dimensionless, is expressed (as seen from (15)) as

$$e = e_o \text{ C} = \sqrt{e_o^2 / \epsilon_o c_o \hbar_o} = \sqrt{4\pi\alpha} = \sqrt{4\pi\alpha_o} \eta = .302 822 120 789 \dots \times \eta \quad (49)$$

thanks to the choice $\hbar = c = \underline{\mu}_o = \underline{\epsilon}_o = 1$. The numerical value of e is thus to rescaled by the factor $\eta = \sqrt{\alpha/\alpha_o}$, very close to 1.

At the same time, with $1 \Omega = 1 \text{ W/A}^2 = (1/\mu_o) \text{ m/s} = 1/\mu_o c_o = \epsilon_o c_o$, the impedance of vacuum is

$$Z_o = \mu_o c_o \Omega = 376.730 313 461 \dots \times \eta^2 \Omega = 1 \quad (50)$$

as it should with $c = \frac{\mu_o}{\epsilon_o} = \underline{\epsilon_o} = 1$. (The above numerical value in ohms has been evaluated with $\mu_o = 4\pi \times 10^{-7}$, to be rescaled by η^2 , with Z_o remaining equal to 1.) The impedance being now a dimensionless quantity, its natural unit, 1, is simply what is also referred to as the “impedance of vacuum”.

From (45) and using $c = c_o \text{ m/s} = 1$, we find, with $1 \text{ F} = \mu_o c_o^2 \text{ m} = (1/\epsilon_o) \text{ m}$, the now symmetric expressions for the farad and the henry, given by

$$\boxed{\mu_o \text{ H} = \epsilon_o \text{ F} = 1 \text{ m}} \quad (51)$$

or equivalently

$$\mu_o c_o \text{ H} = \epsilon_o c_o \text{ F} = 1 \text{ s} \quad (52)$$

With $1 \Omega = 1/\mu_o c_o = \epsilon_o c_o$, we recover Eqs. (46),

$$1 \text{ H}/\Omega = 1 \text{ F}\Omega = 1 \text{ s} \quad (53)$$

For the tesla and V/m, we get from (44), (45) the analogous expressions

$$\boxed{\begin{cases} 1 \text{ T} = \frac{1}{\sqrt{\mu_o}} \sqrt{\text{J/m}^3}, \\ 1 \text{ V/m} = \frac{1}{\sqrt{\mu_o}} \sqrt{\text{J/m}^3} \text{ m/s} = \sqrt{\epsilon_o} \sqrt{\text{J/m}^3} = \frac{1}{c_o} \text{ T} \end{cases}} \quad (54)$$

thanks in the last case to the identification $1 \text{ m} = (1/c_o) \text{ s}$. This will be interpreted in subsection 10.2 in terms of the electric and magnetic field energy densities, now simply given by $E^2/2$ and $B^2/2$, corresponding to their standard SI expressions as $\mathcal{E} = \epsilon_o E^2/2$ and $B^2/2\mu_o$, respectively.

9.3. A direct characterization of the coulomb

Coulomb’s law is now written with $\underline{\epsilon_o} = 1$ as $F = q^2/4\pi r$. With $1 \text{ A} = \sqrt{\mu_o \text{ N}}$ so that $1 \text{ C} = \sqrt{\mu_o c_o^2 \text{ N}} \text{ m} = \sqrt{\text{N}/\epsilon_o} \text{ m}$, one has $1 \text{ C}^2 = 1 \text{ N m}^2/\epsilon_o$. The force between two charges of 1 C at 1 m apart in vacuum is recovered as $(1/4\pi\epsilon_o) \text{ N}$. We can duplicate (43) for the ampere, writing the parallel expression for the coulomb,

$$\text{Coulomb's law: } F = \frac{q^2}{4\pi r^2} \implies 1 \text{ C} = \sqrt{\frac{\text{N}}{\epsilon_o}} \text{ m} \quad (55)$$

This provides a physical understanding for the coulomb. The electrostatic force between two charges $q = 1 \text{ C}$, at $r = 1 \text{ m}$ apart in vacuum, is obtained as $(1/4\pi\epsilon_o) \text{ N} = (\mu_o c_o^2/4\pi) \text{ N} = 10^{-7} c_o^2 \eta^2 \text{ N} = 8.987\,551\,787 \dots \times 10^9 \text{ N}$. Their electrostatic energy $V = \mu_o c_o^2 q^2/4\pi r$ is

$$E = \frac{\mu_o}{4\pi} c_o^2 \text{ J} = 10^{-7} \eta^2 (299\,792\,458)^2 \text{ J} = 8.987\,551\,787 \dots \times 10^9 \eta^2 \text{ J} \quad (56)$$

This energy is no longer exactly known, owing to the new definition of the coulomb. The equivalent mass, according to relativity, is

$$\frac{E}{c^2} = \frac{1}{c^2} \frac{q^2}{4\pi\epsilon_o r} = \frac{\mu_o}{4\pi} \frac{q^2}{r} = 10^{-7} \eta^2 \frac{q^2}{r} \quad (57)$$

The coulomb may be characterized as follows:

“The coulomb is such that the electrostatic energy of two charges of 1 C, 1 m apart in vacuum, is equal to $(1/4\pi\epsilon_o) \text{ J}$.” (58)

or:

“The coulomb is such that the mass equivalent to the electrostatic energy of two charges of 1 C, 1 m apart in vacuum, is equal to $\mu_o/4\pi$ (i.e. $10^{-7} \eta^2$) kg.” (59)

The factor η , at the moment equal to 1 within the present SI, takes into account the fact that the numerical value of the elementary charge e is getting redefined as in (7), with $\mu_o/4\pi$ being very slightly shifted away from 10^{-7} , proportionally to η^2 .

10. Properties of the new system

10.1. Invariance of the fine structure constant α under a rescaling of the fundamental units

If we reintroduce explicitly the constant $\underline{\mu}_o = \mu_o \text{N/A}^2$ (momentarily considered as non-necessarily equal to 1), the ampere and coulomb become expressed as

$$1 \text{ A} = \sqrt{(\mu_o/\underline{\mu}_o)} \text{ N}, \quad 1 \text{ C} = \sqrt{(\mu_o/\underline{\mu}_o)} \text{ kg m} \quad (60)$$

With $\hbar = \hbar_o \text{ Js}$, $c = c_o \text{ m/s}$, so that $\hbar/c = (\hbar_o/c_o) \text{ kg m}$, we get, for the elementary charge,

$$e^2 = e_o^2 C^2 = \frac{\mu_o e_o^2}{\underline{\mu}_o} \frac{\hbar/c}{\hbar_o/c_o}, \quad \text{so that} \quad \frac{\underline{\mu}_o c e^2}{\hbar} = \frac{\mu_o c_o e_o^2}{\hbar_o} \quad (61)$$

or

$$\alpha = \frac{e^2}{4\pi \underline{\epsilon}_o \hbar c} = \frac{e_o^2}{4\pi \epsilon_o \hbar_o c_o} \quad (62)$$

This provides an identity between the two expressions of the fine structure constant, associated, for any set of values of \hbar , c , $\underline{\mu}_o$ (and corresponding $\underline{\epsilon}_o$), with the rescaling of the fundamental units through the choice $c = c_o \text{ m/s}$, $\hbar = \hbar_o \text{ Js}$, $\underline{\mu}_o = \mu_o \text{ N/A}^2$.

If as a further step we choose $\hbar = c = \underline{\mu}_o = \underline{\epsilon}_o = 1$, we recover

$$\alpha = \frac{e^2}{4\pi} = \frac{e_o^2}{4\pi \epsilon_o \hbar_o c_o} \quad (63)$$

leading to $e = \sqrt{4\pi\alpha} = .3028\dots$ as in (49).

10.2. Invariance of the impedance of vacuum Z_o under a rescaling of the fundamental units

Let us return to the SI for a short moment. The energy density associated with an electromagnetic field,

$$\mathcal{E} = \frac{\epsilon_o E^2}{2} + \frac{B^2}{2\mu_o} = \frac{E^2 + c^2 B^2}{2\mu_o c^2} = \sqrt{\frac{\epsilon_o}{\mu_o}} \left(\frac{E^2}{2c} + \frac{cB^2}{2} \right) \quad (64)$$

with $\epsilon_o \mu_o c^2 = 1$, originates from the Lagrangian density $\mathcal{L} = -(1/4\mu_o) F_{\mu\nu} F^{\mu\nu}$. The quantity

$$Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} = \mu_o c \quad (65)$$

equal to the ratio $E/(B/\mu_o)$ for a plane electromagnetic wave in vacuum, is referred to as the impedance of vacuum. It appears as a normalisation coefficient in Maxwell's equations $\text{rot } \vec{B} = \mu_o c (\vec{j}/c) + \dots$, $\text{div } \vec{E}/c = \rho/\epsilon_o c$, which fix the relative scale of $(\vec{E}/c, \vec{B})$ compared to that of the current density $(\rho, \vec{j}/c)$, source of the electromagnetic field. This may also be seen from Poisson's equation for a static field, expressed as $\Delta(V/c, \vec{A}) = -\mu_o c (\rho, \vec{j}/c)$, with $\text{div } \vec{A} = 0$.

With μ_o expressed in N/A^2 or H/m and c in m/s , Z_o is obtained in the SI in Ω . We have at present

$$Z_o = \mu_o c = (4\pi \times 10^{-7} \text{ H/m}) (c_o \text{ m/s}) = 4\pi \times 10^{-7} c_o \Omega = 376.730\dots \Omega \quad (66)$$

But, if the elementary charge gets fixed as in (7), μ_o gets changed into $4\pi \times 10^{-7} \eta^2$. The measure in ohms of the impedance of vacuum is multiplied by η^2 , to become in the new SI

$$Z_o = 4\pi \times 10^{-7} \eta^2 c_o \Omega = 376.730\dots \eta^2 \Omega \quad (67)$$

But this does not mean that the impedance of vacuum has been multiplied by η^2 ! At the same time, with $1 \Omega = 1 \text{ W/A}^2$, the size of the ohm is divided by η^2 as in (18), according to

$$\Omega = \Omega_{\text{old}}/\eta^2 \quad (68)$$

so that Z_o itself, equal to $\mu_o c_o \Omega$, remains unchanged. As for the elementary charge e , expressed either as $e_{\text{old}} C_{\text{old}}$ or as $e_o C$, Z_o , which is an intrinsic quantity, may be expressed equally well as

$$Z_o = \underbrace{4\pi \times 10^{-7} c_o}_{376.730\dots} \Omega_{\text{old}} = 4\pi \times 10^{-7} \eta^2 c_o \Omega \quad (69)$$

This ensures that the impedance of vacuum Z_o (but not its measure in ohms!), is indeed insensitive to the change in electrical units following from the fixing of $e = e_o C$ within the new SI, and remains insensitive to the value of the elementary charge e – i.e. to the value of α to be measured experimentally in the future. This is fortunate, as Z_o is an intrinsic quantity that should not depend on the chosen system of units, nor on future measurements of α . But it also illustrates how fixing the numerical value of the electrical charge in the new SI as in (7), while practically convenient, makes life more complicated as far as explaining what the impedance of the vacuum really is, and for which reason its measure should now depend on α .

In the new proposed system, there is no such concern, as $1 \Omega = 1/\mu_o c_o$ and $Z_o = \mu_o c_o \Omega = 1$, independently of any rescaling of μ_o in agreement with (18). This is a consequence of $\hbar = c = \frac{\mu_o}{\epsilon_o} = \frac{\epsilon_o}{\mu_o} = 1$ so that the impedance of the vacuum is the unit of impedance, namely simply 1. The Lagrangian and energy densities for the free electromagnetic field get simply expressed as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \mathcal{E} = \frac{E^2 + B^2}{2} \quad (70)$$

as usual in relativistic quantum field theory, and its equation of motion reads $\partial_\nu F^{\mu\nu} = j^\nu$.

10.3. The volt/m, the tesla, and the electromagnetic energy

Expressions (54) of the tesla and volt/metre read

$$1 \text{ T} = \frac{1}{\sqrt{\mu_o}} \sqrt{\text{J/m}^3}, \quad 1 \text{ V/m} = \sqrt{\epsilon_o} \sqrt{\text{J/m}^3} = \frac{1}{c_o} \text{ T} \quad (71)$$

We reobtain correctly the energy densities for unit magnetic and electric fields of 1 T and 1 V/m, obtained from (70) as

$$\mathcal{E} = \frac{B^2}{2} = \frac{1 \text{ T}^2}{2} = \frac{1}{2\mu_o} \text{ J/m}^3, \quad \text{and} \quad \mathcal{E} = \frac{E^2}{2} = \frac{1 (\text{V/m})^2}{2} = \frac{\epsilon_o}{2} \text{ J/m}^3 \quad (72)$$

This also provides a simple characterization of the tesla and volt/metre, as follows:

$$\left\{ \begin{array}{l} \text{“The tesla is the magnetic field corresponding to an energy density in vacuum } 1/2\mu_o \text{ (or } 10^7/8\pi\eta^2) \text{ J/m}^3\text{.”} \\ \text{“The volt/metre is the electric field corresponding to an energy density in vacuum } \epsilon_o/2 \text{ (or } 10^7/8\pi c_o^2\eta^2) \text{ J/m}^3\text{.”} \end{array} \right. \quad (73)$$

The energy density for a magnetic field of 1 T is larger than for an electric field of 1 V/m by a large factor $1/\mu_o\epsilon_o = c_o^2 \simeq 9 \times 10^{16}$, as easily understood since $1 \text{ T} = c_o \text{ V/m}$.

Let us also consider the Poynting vector, now simply expressed as $\vec{P} = \vec{E} \times \vec{B}$. For orthogonal unit electric and magnetic fields of 1 V/m and 1 T, the energy flux density per unit time would be

$$1 \text{ V/m} \times 1 \text{ T} = \sqrt{c_o/\mu_o\hbar_o} \text{ s}^{-2} \times \sqrt{c_o^3/\mu_o\hbar_o} \text{ s}^{-2} = \frac{1}{\mu_o} \text{ W/m}^2 = \frac{1}{4\pi \times 10^{-7} \eta^2} \text{ W/m}^2 \quad (74)$$

with $c_o^2 \text{ s}^{-2} = 1 \text{ m}^{-2}$, and $1/\hbar_o = 1 \text{ Js}$. This is evaluated in a more conventional way in the SI with $\vec{P} = \vec{E} \times \vec{B}/\mu_o$, as

$$\frac{1 \text{ V/m} \times 1 \text{ T}}{\mu_o} = \frac{1 (\text{W/A m}) (\text{N/A m})}{4\pi \times 10^{-7} \eta^2 \text{ N/A}^2} = \frac{1}{4\pi \times 10^{-7} \eta^2} \text{ W/m}^2 \quad (75)$$

Eq. (74) illustrates how the factor $4\pi \times 10^{-7} \eta^2$, instead of being present in the expression of the Poynting vector as in the standard formulation, gets now included within the new expressions of the electrical units as in (71).

Electrical units thus remain strongly tied to mechanical ones and completely fixed by them, up to the scale factor $\mu_o = 4\pi \times 10^{-7} \eta^2$, where η is very close to 1. These formulas reflect the tight connection between electrical and mechanical units, originating from the traditional definition of the ampere. This connection gets somewhat weakened as μ_o is no longer fixed if the value of the elementary charge gets fixed as in (7). Electrical units are then no longer rigidly tied to mechanical ones, but in a more flexible way, proportionally to $\sqrt{\mu_o}$ i.e. to $\eta = \sqrt{\alpha/\alpha_o}$ for the ampere and the coulomb, as described by Eqs. (18).

10.4. Inductances and capacitances

The henry may be defined from the magnetic energy stored in a coil, $E = LI^2/2$, so that

$$1 \text{ H} = 1 \text{ J/A}^2 = \frac{1}{\mu_o} \text{ J/N} = \frac{1}{\mu_o} \text{ m} = \frac{1}{\mu_o c_o} \text{ s} = \frac{1}{(376.730 \dots \eta^2)} \text{ s} \quad (76)$$

The inductance L for a long coil of length l , inner area S and N turns is now simply given, with $\underline{\mu}_o = 1$, by $L = N^2 S/l$. For a coil of unit length $l = 1$ m, the inductance per unit inner core area S (supposed small) and unit N^2 is

$$L = 1 \text{ m} = \mu_o \text{ H} \quad (77)$$

as found with the usual SI formula $L = \mu_o N^2 S/l$. Similarly, the capacitance of a plane capacitor with empty space between the plates is given, with $\underline{\epsilon}_o = 1$, by $C = S/l$. For a unit distance between the plates $l = 1$ m, the capacitance per unit surface S is

$$C = 1 \text{ m} = \epsilon_o \text{ F} \quad (78)$$

as seen from (51), and as found with the usual SI formula $C = \epsilon_o S/l$. The impedance $\sqrt{L/C}$ evaluated with the above values $L = 1 \text{ m} = \mu_o \text{ H}$ and $C = 1 \text{ m} = \epsilon_o \text{ F}$ is the impedance of free space, which is 1:

$$\sqrt{L/C} = \sqrt{\mu_o \text{ H} / \epsilon_o \text{ F}} = \mu_o c_o \sqrt{\text{H}/\text{F}} = \mu_o c_o \Omega = Z_o = 1 \quad (79)$$

10.5. The weber and the quantum of flux

The weber is the magnetic flux induced by a current of 1 A circulating in a coil of inductance $L = 1$ H,

$$1 \text{ Wb} = 1 \text{ H} \times 1 \text{ A} = 1/\sqrt{\mu_o c_o \hbar_o} = 1 \text{ Vs} = 5.017 029 284 119 \dots \times 10^{15} \eta^{-1} \quad (80)$$

It is a dimensionless unit, as for the coulomb and the ohm, equal to 1 Vs (as $1 \text{ H} = 1 \Omega \text{ s}$). These three dimensionless units,

$$\begin{cases} 1 \text{ C} = \sqrt{\mu_o c_o / \hbar_o} = 1.890 \dots \times 10^{18} \eta, \\ 1 \Omega = 1/\mu_o c_o = 1/(376.7 \dots \eta^2), \\ 1 \text{ Wb} = 1/\sqrt{\mu_o c_o \hbar_o} = 5.017 \dots \times 10^{15} \eta^{-1} \end{cases} \quad (81)$$

satisfy the relation, independent of c_o , μ_o and \hbar_o ,

$$1 \text{ C} \times 1 \Omega = 1 \text{ Wb} \quad (82)$$

reflecting that $1 \text{ A} \times 1 \Omega = 1 \text{ V}$, or $1 \text{ F} \times 1 \Omega = 1 \text{ s}$.

The magnetic flux quantum, now exactly fixed in webers, is thus

$$\Phi_o = \frac{h}{2e} = \frac{\pi \hbar_o}{e_o} \underbrace{\text{Js/C}}_{\text{Wb}} = \pi \sqrt{\frac{\hbar_o}{\mu_o c_o e_o^2}} = \sqrt{\frac{\pi}{4\alpha}} = \frac{\pi}{e} \simeq 10.374 382 97 \quad (83)$$

or

$$\Phi_o = h/2e = 2.067 833 848 461 \dots \times 10^{-15} \text{ Wb} = \frac{\pi}{e} \simeq 10.374 382 97 \quad (84)$$

We simply recover π/e with $e = \sqrt{4\pi\alpha}$, at no surprise since $\hbar = c = \underline{\mu}_o = \underline{\epsilon}_o = 1$. The inverse of the flux quantum is the Josephson constant,

$$K_J = \frac{1}{\Phi_o} = \frac{2e}{h} = \frac{2e_o}{\hbar_o} \text{ Wb}^{-1} = 483 597.848 416 \dots \text{ GHz/V} = \frac{e}{\pi} \simeq .0963 912 748 \quad (85)$$

It is related to the expression of 1 eV = $\pi (2e_o/h_o) \text{ s}^{-1} = 1.519 \dots \times 10^{15} \text{ s}^{-1}$, as we shall see in (91)–(93).

The expression of the physical laws, now written without reference to \hbar , c , μ_o , and ϵ_o , and the new expressions for the corresponding units, are given in Table 2.

10.6. Relating energies with distances and times

The spectrum of the hydrogen atom, for example, may be obtained with $\hbar = c = \underline{\epsilon}_o = 1$ and $e = \sqrt{4\pi\alpha}$, which leads directly to the Rydberg energy

$$1 \text{ Ry} = \frac{1}{2} m_e (e^2/4\pi)^2 = \frac{1}{2} m_e \alpha^2 \simeq 13.605 6930 \text{ eV} \quad (86)$$

We can use the fine structure constant $\alpha = e^2/4\pi \simeq 1/137.035 9991 \dots$, and express the mass of the electron as $m_e \simeq .510 998 946 \text{ MeV}$, with no need to refer to the velocity of light.

Table 2

The physical laws written in a universal way, with no reference to \hbar , c , μ_o , and ϵ_o . This includes Maxwell's equations. \hbar_o , c_o , μ_o and ϵ_o define our usual units by scaling appropriately the natural ones obtained from the second. The usual formulas including μ_o and ϵ_o are shown in the last column. Due to the fixing of $e = e_o$, μ_o can no longer be fixed at $4\pi \times 10^{-7}$, but should be multiplied by $\eta = \sqrt{\alpha/\alpha_o}$, which is very close to 1 (cf. Eqs. (14)–(16)).

Physical law or expression	New expression				Usual expression
Ampère's force law	$\frac{F}{L} = \frac{I^2}{2\pi r}$	with	$1 \text{ A} = \sqrt{\mu_o \text{ N}}$	\leftrightarrow	$\frac{F}{L} = \frac{\mu_o I^2}{2\pi r}$
Coulomb's law	$F = \frac{q^2}{4\pi r^2}$	"	$1 \text{ C} = \sqrt{\frac{\text{N}}{\epsilon_o}} \text{ m}$	\leftrightarrow	$F = \frac{q^2}{4\pi\epsilon_o r^2}$
electric energy density	$\mathcal{E} = \frac{E^2}{2}$	"	$1 \text{ V/m} = \sqrt{\epsilon_o \text{ J/m}^3}$	\leftrightarrow	$\mathcal{E} = \frac{\epsilon_o E^2}{2}$
magnetic energy density	$\mathcal{E} = \frac{B^2}{2}$	"	$1 \text{ T} = \sqrt{\frac{\text{J/m}^3}{\mu_o}}$	\leftrightarrow	$\mathcal{E} = \frac{B^2}{2\mu_o}$
Poynting vector	$\vec{P} = \vec{E} \times \vec{B}$	"	$1 (\text{V/m}) \text{ T} = \frac{\text{W/m}^2}{\mu_o}$	\leftrightarrow	$\vec{P} = \frac{\vec{E} \times \vec{B}}{\mu_o}$
Lagrangian density	$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$			\leftrightarrow	$\mathcal{L} = \frac{\epsilon_o E^2}{2} - \frac{B^2}{2\mu_o}$
cap. of plane capacitor	$C = \frac{S}{l}$	"	$1 \text{ m} = \epsilon_o F$	\leftrightarrow	$C = \frac{\epsilon_o S}{l}$
inductance of coil	$L = \frac{N^2 S}{l}$	"	$1 \text{ m} = \mu_o H$	\leftrightarrow	$L = \frac{\mu_o N^2 S}{l}$
impedance of vacuum	$Z_o = 1$			\leftrightarrow	$Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \Omega \simeq 377 \Omega$
fine structure constant	$\alpha = \frac{e^2}{4\pi} = \frac{e_o^2}{4\pi} c^2$	"	$1 \text{ C} = \sqrt{\frac{1}{\epsilon_o c_o \hbar_o}}$	\leftrightarrow	$\alpha = \frac{e_o^2}{4\pi\epsilon_o \hbar_o c_o} \simeq \frac{1}{137}$

A useful formula to relate energies and distances is obtained by evaluating the product $1 \text{ eV} \times 1 \text{ m}$, from

$$1 = (\hbar_o \text{ J s}) (c_o \text{ m/s}) = \frac{\hbar_o c_o}{e_o} \text{ eV m} = 10^9 \frac{\hbar_o c_o}{e_o} \text{ MeV fm} \tag{87}$$

using

$$10^9 \frac{(6.626\ 070\ 15 \times 10^{-34}) \times (2.997\ 924\ 58 \times 10^8)}{2\pi \times 1.602\ 176\ 634 \times 10^{-19}} = 197.326\ 980\ 459\ 302\dots \text{ (now exactly known)} \tag{88}$$

This may be remembered, not surprisingly, as $197.327 \text{ MeV fermi} \simeq 1$, or

$$1 \text{ MeV}^{-1} = 197.326\ 980\ 459\dots \text{ fm} \tag{89}$$

To give some illustrative examples, the Lyman α wavelength, Bohr radius, reduced Compton wavelength of the electron, range of weak interactions and Planck length may be expressed directly using (89), with no reference to \hbar and c factors, as

$$\left\{ \begin{array}{l} \text{Lyman } \alpha \text{ wavelength} \\ r_B \\ \chi_e \\ \chi_W \\ l_p \end{array} \right. \left\{ \begin{array}{l} \left(\approx \frac{4}{3} \left(1 + \frac{m_e}{m_p} \right) \frac{2\pi}{13.606 \text{ eV}} \right) \simeq 1215.67 \text{ \AA} , \\ = \frac{1}{m_e \alpha} \simeq \frac{137.035\ 999}{.510\ 998\ 946 \text{ MeV}} \simeq .529\ 177\ 21 \text{ \AA} , \\ = \frac{1}{m_e} \simeq \frac{1}{.510\ 998\ 946 \text{ MeV}} \simeq 3.861\ 5927 \times 10^{-13} \text{ m} , \\ = \frac{1}{m_W} \simeq \frac{1}{80.38 \text{ GeV}} \simeq 2.455 \times 10^{-18} \text{ m} , \\ = \frac{1}{m_p} = \sqrt{G_N} \simeq \frac{1}{1.2209 \times 10^{19} \text{ GeV}} \simeq 1.6162 \times 10^{-35} \text{ m} \end{array} \right. \tag{90}$$

The relation between energy and time comes from $1 \text{ J} = (1/\hbar_o) \text{ s}^{-1} = .948\ 252\ 156\dots \times 10^{34} \text{ s}^{-1}$, so that

$$1 \text{ eV} = e_o \text{ J} = \frac{e_o}{\hbar_o} \text{ s}^{-1} = 1.519\ 267\ 447\ 878\ 626\dots \times 10^{15} \text{ s}^{-1} = 1/(6.582\ 119\ 569\ 509\ 065\dots \times 10^{-16} \text{ s}) \tag{91}$$

also equivalent to

$$1 \text{ MeV}^{-1} = 197.326\ 980\ 459\ 302\dots \text{ fm} = 6.582\ 119\ 569\ 509\dots \times 10^{-22} \text{ s} \tag{92}$$

$1 \text{ eV} s/\pi = 2e_o/h_o = .483\,597\,848\,416\,983\dots \times 10^{15}$ involves the same factor as for $K_J = (2e_o/h_o) \text{ Hz/V}$. We have the equivalence between the new expression of K_J as equal to e/π , and the well-known expression of \hbar , precisely evaluated as $(\hbar_o/e_o) \text{ eV s} = 6.582\,119\,569\,509\dots \times 10^{-22} \text{ MeVs}$, and now equal to $\hbar = 1$:

$$\boxed{\begin{aligned} K_J &= .483\,597\dots \times 10^{15} \text{ Hz/V} = e/\pi \\ \iff 1 \text{ eV} &= \pi \times .483\,597\,848\,416\dots \times 10^{15} \text{ s}^{-1} = 1.519\,267\,447\,878\dots \times 10^{15} \text{ s}^{-1} \end{aligned}} \quad (93)$$

10.7. The kelvin and the Boltzmann constant

In the new SI, the unit of thermodynamic temperature, the kelvin K, will be derived from the unit of energy by fixing the numerical value k_o of the Boltzmann constant k [10], so that

$$k = 1.380\,649 \times 10^{-23} \text{ J/K} \quad (94)$$

We can then go one step further. Very much as space and time are related and may both be measured in seconds, or as energy may be measured in s^{-1} when time is measured in seconds, thermodynamic temperature and energy are related and may both be measured with the same unit, the s^{-1} , by choosing a unit value of the Boltzmann constant k . The Maxwell–Boltzmann distribution, for example, will then be simply expressed with $k = 1$, proportionally to $e^{-E/T}$.

Choosing $k = 1$ is compatible with fixing its value in SI units according to (94), leading to

$$k = 1.380\,649 \times 10^{-23} \text{ J/K} = 1 \quad (95)$$

This allows us to identify the kelvin K with a certain number k_o of joules according to

$$\boxed{1 \text{ K} = 1.380\,649 \times 10^{-23} \text{ J}} \quad (96)$$

We can also express the energies in eV, rather than in joules. We then have the exact relation

$$1 \text{ eV} = 1.602\,176\,634 \times 10^{-19} \text{ J} = \frac{1.602\,176\,634 \times 10^{-19}}{1.380\,649 \times 10^{-23}} \text{ K} = 11\,604.518\,121\dots \text{ K} \quad (97)$$

remembered as $1 \text{ eV} \simeq 11\,605 \text{ K}$.

10.8. A more symmetric treatment between electricity and magnetism

In this discussion, and within the SI, μ_o has been given a favoured treatment as compared to ϵ_o , owing to the original definition of the ampere. But there is no special reason to do so, as they contribute in similar ways to the expressions of the speed of light $c = 1/\sqrt{\epsilon_o \mu_o}$ and impedance of vacuum $Z_o = \sqrt{\mu_o/\epsilon_o}$. To get a system with $c = \underline{\mu_o} = \underline{\epsilon_o} = 1$, as proposed here, we may use, equivalently, two of the four sets of conditions

$$\left\{ \begin{array}{l} c = c_o \text{ m/s} \\ Z_o = z_o \Omega \\ \underline{\mu_o} = \mu_o \text{ H/m} = \mu_o c_o \text{ H/s} \\ \underline{\epsilon_o} = \epsilon_o \text{ F/m} = \epsilon_o c_o \text{ F/s} \end{array} \right. = 1, \quad (98)$$

where the last three are associated with Eqs. (50)–(52). For example, we already have the equivalences

$$\left\{ \begin{array}{l} c = c_o \text{ m/s}, \\ Z_o = z_o \Omega, \end{array} \right. \iff \left\{ \begin{array}{l} \underline{\mu_o} = \frac{Z_o}{c} = \frac{z_o}{c_o} \frac{\Omega \text{ s}}{\text{m}} = \mu_o \text{ H/m} = \mu_o c_o \text{ H/s}, \\ \underline{\epsilon_o} = \frac{1}{c Z_o} = \frac{1}{c_o z_o} \frac{\text{s}}{\Omega \text{ m}} = \epsilon_o \text{ F/m} = \epsilon_o c_o \text{ F/s} \end{array} \right. \quad (99)$$

using relations (46) between units, $1 \Omega \text{ s} = 1 \text{ H}$, $1 \text{ s}/\Omega = 1 \text{ F}$, together with

$$\left\{ \begin{array}{l} c_o = 1/\sqrt{\mu_o \epsilon_o}, \\ Z_o = \sqrt{\mu_o/\epsilon_o}, \end{array} \right. \iff \left\{ \begin{array}{l} \mu_o = Z_o/c_o, \\ \epsilon_o = 1/c_o Z_o \end{array} \right. \quad (100)$$

This leads to the equivalence between the two sets of conditions in (98), where electricity and magnetism play similar roles,

$$\left\{ \begin{array}{l} c = c_o \text{ m/s} = 1, \\ Z_o = z_o \Omega = 1, \end{array} \right. \iff \left\{ \begin{array}{l} \underline{\mu}_o = \mu_o c_o \text{ H/s} = 1, \\ \underline{\epsilon}_o = \epsilon_o c_o \text{ F/s} = 1 \end{array} \right. \quad (101)$$

The fixing of $\underline{\mu}_o$ or $\underline{\epsilon}_o$ to 1 may also be viewed, equivalently, as a fixing of the impedance of vacuum at 1, next to $c = 1$.

We have formulated this analysis by imposing $c = 1$ and $\underline{\mu}_o = 1$. But we could also have selected, equivalently, any two of the above sets of conditions. With $1 \text{ F/m} = 1 \text{ C}^2/\text{Jm}$, the supplementary set of equations in (101) may be used (optionally) as follows:

$$\left\{ \begin{array}{l} Z_o = z_o \Omega = 1, \text{ to fix } 1 \Omega = 1/z_o = 1/\mu_o c_o = \epsilon_o c_o, \\ \underline{\epsilon}_o = \epsilon_o \text{ C}^2/\text{Jm} = 1, \text{ to fix } 1 \text{ C} = \sqrt{\text{Jm}/\epsilon_o} = 1/\sqrt{\epsilon_o \hbar_o c_o} \end{array} \right. \quad (102)$$

with the same results as from the fixings of c and/or $\underline{\mu}_o$.

However, due to the fixing of the elementary charge as $e = e_o \text{ C}$, the quantity $z_o = \mu_o c_o = 1/\epsilon_o c_o$ must be kept “floating” to adapt to the new value of the ohm, just as for μ_o to adapt to the size of the ampere, and ϵ_o to that of the coulomb, with $z_o \propto \eta^2$, $\mu_o \propto \eta^2$, and $\epsilon_o \propto \eta^{-2}$ as in (18).

10.9. The classical limit

The fundamental laws of nature may now be written in a universal way, no longer referring to the parameters \hbar , c , $\underline{\mu}_o$, $\underline{\epsilon}_o$, k , ..., which can all be taken equal to 1, nor to the specific numerical values \hbar_o , c_o , μ_o , ϵ_o , k_o , ... which appear in their expressions when SI units are used. Maxwell’s equations, in particular, no longer involve μ_o and ϵ_o , now included within the definitions of the electrical units. The Lorentz transformations relating space and time (or magnetism and electricity) within the framework of relativity can also be written without reference to the speed of light c , now equal to 1, through the relation $c = c_o \text{ m/s} = 1$. The numerical parameter c_o gets included within the definition of the metre as obtained from the second, providing the same results as with the usual formalism, thanks to the relation $1 \text{ m} = (1/c_o) \text{ s}$.

In such a framework with $c = \hbar = 1$, Einstein’s formula $E = mc^2$ for a free particle at rest further simplifies into $E = m$. For a particle of mass m and momentum $p = k$, we have

$$E = \sqrt{m^2 + p^2} = \sqrt{m^2 + k^2} = \omega \quad (103)$$

Now that $c = 1$, one may enquire about “taking the classical limit” to recover a non-relativistic situation. This can no longer be done by taking a limit “ $c \rightarrow \infty$ ”, but is now obtained in the limit of small velocities as compared to 1. For small $v = d\omega/dk = dE/dp = p/E$, Eq. (103) provides back the non-relativistic expansion of the mechanical energy, as $E = m + p^2/2m + \dots$, without having to consider a limit in which the speed of light would become very large. This is similar for quantum effects. We can no longer consider a limit for which “ $\hbar \rightarrow 0$ ”. Still, with $\hbar = 1$ as the quantum of action (or angular momentum), the classical limit now corresponds to situations involving large values of the action, as compared to the unit quantum.

10.10. The mole and the Avogadro constant

Let us now mention the SI unit of “amount of substance” or mole. In the new SI, the mole gets defined by fixing the Avogadro constant at $N_A = 6.022\,140\,76 \times 10^{23} \text{ mol}^{-1}$ [10]. If n is the amount of substance (in moles) in a sample of X, the number of elementary entities is $N = n N_A$.

But the most natural unit for counting “entities” is 1, which fits well in a system of units with $\hbar = c = \underline{\mu}_o = \underline{\epsilon}_o = Z_o = k = 1$. We shall then fix the Avogadro constant at $N_A = 1$, as we did for \hbar , c , $\underline{\mu}_o$, $\underline{\epsilon}_o$, Z_o and k by fixing their numerical values in SI units to be \hbar_o , c_o , μ_o , ϵ_o , z_o , and k_o , respectively. The Avogadro constant N_A may then be fixed at $N_{A_o} \text{ mol}^{-1}$ as in the SI, and, at the same time to $N_A = 1$. This implies

$$N_A = \underbrace{6.022\,140\,76 \times 10^{23}}_{N_{A_o}} \text{ mol}^{-1} = 1 \quad (104)$$

This leads us to identify the mole with a pure number, namely the Avogadro number,

$$\boxed{1 \text{ mol} = N_{A_o} = 6.022\,140\,76 \times 10^{23}} \quad (105)$$

The number of elementary entities of a substance X in a sample of n moles is simply $N = n N_{A_o}$.

The Avogadro constant N_A , normally expressed in mol^{-1} , no longer appears as a fundamental constant of nature, but rather as a counting device. Fixing it at the natural unit, 1, as for the other constants, makes the mole appear as a fixed number, like saying that there are twelve eggs in a dozen of eggs. But it is a very large number, the Avogadro number, fixed for consistency with past definitions at $N_{A_o} = 6.022\,140\,76 \times 10^{23}$.

We may also consider the candela, SI unit of luminous intensity in a given direction, even if we cannot really view it as a fundamental unit. It is now defined and normalized by taking the luminous efficacy of a monochromatic radiation of frequency 540×10^{12} Hz (and wavelength $\lambda \simeq .555 \mu$) to be $K_{\text{cd}} = 683 \text{ cd sr W}^{-1}$ [4,10]. Fixing it at 1 as for the other constants, we get $K_{\text{cd}} = 683 \text{ lm W}^{-1} = 683 \text{ cd sr W}^{-1} = 1$. The candela and the lumen appear, for the specific radiation considered, as a certain number of watts per steradian or watts, with $1 \text{ cd} = (1/683) \text{ W/sr}$, $1 \text{ lm} = (1/683) \text{ W}$.

11. Conclusions

The International System of units is getting redefined, with the joule and kilogram obtained by fixing the Planck constant at $h = 6.626\,070\,15 \times 10^{-34}$ Js. This makes obsolete the international prototype of the kilogram stored at the BIPM. No longer having to rely on such a single material object for the definitions of mechanical and electrical units is a huge progress.

The coulomb and the other electrical units are also redefined, so that the elementary charge is fixed at $e = 1.602\,176\,634 \times 10^{-19}$ C in the new SI. This requires that the vacuum magnetic permeability μ_0 be slightly adapted, into $\mu_0 = 4\pi \times 10^{-7} \eta^2 \text{ N/A}^2$. Fixing the values of h and e in joule seconds and coulombs will allow for more precise measurements, thanks to quantum electrical metrology based on the Josephson and quantum Hall effects. Still one may regret that, with μ_0 no longer exactly fixed but proportional to α , all electrical units, together with the numerical values of vacuum magnetic permeability (in N/A^2 or H/m), electric permittivity (in F/m), and vacuum impedance (in ohms) become dependent on α .

An appealing system should have $c = \mu_0 = \epsilon_0 = 1$ and $\hbar = 1$, as suggested by relativity and quantum mechanics; but this is usually considered as unpractical as it would naturally lead to units of space and mass $\simeq 3 \times 10^8 \text{ m}$ and 10^{-51} kg , not very convenient. Still it is possible to reconcile and unite both systems, allying the practical interest and convenience of the normal SI units to the advantages and elegance of a symmetric system with $c = \mu_0 = \epsilon_0 = \hbar = 1$.

To this end, the fundamental laws of physics, universal, may be expressed without referring to c and \hbar , nor to the convention-dependent parameters c_0 , \hbar_0 , μ_0 , and ϵ_0 . These now serve to define and resize appropriately our fundamental units of length, energy and mass, and intensity, all defined from the second or s^{-1} , so that we recover our usual units, metre, joule and kilogram, ampere, suitably normalized.

This is done, first, through a rewriting of the laws of electromagnetism by eliminating μ_0 , and subsequently ϵ_0 , from their expressions. μ_0 becomes a conventional dimensionless normalisation coefficient for the electrical units, getting included within their expressions. The ampere and the coulomb get given by

$$1 \text{ A} = \sqrt{\mu_0 \text{ N}}, \quad 1 \text{ C} = \sqrt{\mu_0 \text{ kg m}} \quad (106)$$

This is sufficient to derive all electrical units from a choice of μ_0 . This one is initially fixed at $4\pi \times 10^{-7}$, multiplied by η^2 , very close to 1, to allow for the necessary adjustment of the ampere and the coulomb, if the elementary charge gets fixed as in (7). This new formulation is well adapted to take in charge that the ampere and the coulomb are no longer rigidly tied to the newton, but allowed to slightly “float” proportionally to $\sqrt{\mu_0}$, i.e. to $\eta = \sqrt{\alpha/\alpha_0}$.

By demanding that c and \hbar , in addition to being numerically fixed in SI units according to the official definitions, be also equal to 1, we can unite the advantages of both systems through the equations

$$\begin{cases} c = c_0 \text{ m/s} = 1 & \Rightarrow 1 \text{ m} = (1/299\,792\,458) \text{ s}, \\ \hbar = \hbar_0 \text{ Js} = 1 & \Rightarrow 1 \text{ J} = (1/\hbar_0) \text{ s}^{-1} = .948\,252 \dots \times 10^{34} \text{ s}^{-1}, \\ \mu_0 = \mu_0 \text{ N/A}^2 = 1 & \Rightarrow 1 \text{ A} = \sqrt{\mu_0 \text{ N}} = \sqrt{\mu_0 c_0 / \hbar_0} \text{ s}^{-1} = 1.890\,067 \dots \times 10^{18} \text{ s}^{-1} \end{cases} \quad (107)$$

One can also, equivalently, define directly the dimensionless ohm and coulomb through one of the conditions

$$\begin{cases} Z_0 = z_0 \Omega = 1 & \Rightarrow 1 \Omega = 1/Z_0 = 1/\mu_0 c_0 = \epsilon_0 c_0 = 1/376.730 \dots, \\ \epsilon_0 = \epsilon_0 \text{ C}^2/\text{Nm}^2 = 1 & \Rightarrow 1 \text{ C} = \sqrt{\text{Nm}^2/\epsilon_0} = \sqrt{1/\epsilon_0 \hbar_0 c_0} = 1.890\,067 \dots \times 10^{18} \end{cases} \quad (108)$$

allowing for a more symmetric treatment between electricity and magnetism.

The coulomb, the ohm, and the weber, related by $1 \text{ C} \times 1 \Omega = 1 \text{ Wb}$, become dimensionless, with the elementary charge e , the impedance of vacuum Z_0 , and the flux quantum Φ_0 expressed as pure numbers:

$$\begin{cases} e = 1.602\,176\,634 \times 10^{-19} \text{ C} & = \sqrt{4\pi\alpha} \simeq .302\,822\,1208, \\ Z_0 = \mu_0 c_0 \Omega = 376.730\,313 \dots \Omega & = 1, \\ \Phi_0 = h/2e = 2.067\,833\,848 \dots \times 10^{-15} \text{ Wb} & = \pi/e \simeq 10.374\,382\,97 \end{cases} \quad (109)$$

The impedance of vacuum is 1, as it should with $\epsilon_0 = \mu_0 = 1$. This is an improvement over the new SI description, in which $Z_0 = \mu_0 c_0 \Omega$ refers to an ohm dependent on μ_0 and thus now on α , hiding that $Z_0 = 1$ is the natural unit of impedance. Fixing the impedance of vacuum to $Z_0 = 376.730 \dots \Omega = 1$ also determines the von Klitzing constant, both in

ohms and in terms of the fine-structure constant, as $R_K = h/e^2 = 1/2\alpha$. Its SI expression in ohms is simply recovered as $376.730\ 313\dots\Omega$ (vacuum impedance) $\times 137.035\ 9991\dots/2$, i.e.

$$R_K = h/e^2 = 376.730\ 313\dots\ \Omega/2\alpha = 25\ 812.807\ 459\dots\ \Omega = 1/2\alpha \simeq 68.517\ 999\ 57 \quad (110)$$

The Josephson constant, now also dimensionless, is

$$K_J = 1/\Phi_0 = 2e/h = 483\ 597.848\ 416\dots\ \text{GHz/V} = e/\pi \simeq .0963\ 912\ 748 \quad (111)$$

Inductances and capacitances are naturally expressed in metres owing to their geometric origin. They can be converted in ordinary SI units, with inductances measured in henrys proportionally to μ_0 , and capacitances in farads proportionally to ϵ_0 , thanks to the symmetric relations

$$1\ \text{m} = \mu_0\ \text{H} = \epsilon_0\ \text{F}, \quad \text{or} \quad 1\ \text{s} = \mu_0 c_0\ \text{H} = \epsilon_0 c_0\ \text{F} \quad (112)$$

These provide back the equalities $1\ \text{s} = 1\ \text{H}/\Omega = 1\ \text{F}\Omega$, and $1\ \text{Wb} = 1\ \text{C} \times 1\ \Omega$ relating the dimensionless weber, coulomb, and ohm. The volt/metre and tesla, both square roots of energy densities proportional to $\sqrt{\text{J}/\text{m}^3}$, are obtained in s^{-2} .

Extending these ideas to the kelvin and the mole, we get a unified system embedding the new SI within a framework where quantities previously considered as “fundamental constants of nature” return to their natural status of being simply 1, i.e.

$$c = \hbar = \underline{\mu_0} = \underline{\epsilon_0} = Z_0 = k = N_A = 1 \quad (113)$$

and no longer appear within the expressions of the fundamental laws. This is achieved through the set of equalities,

$$c_0\ \text{m/s} = \hbar_0\ \text{J s} = z_0\ \Omega = k_0\ \text{J/K} = N_{A_0}\ \text{mol}^{-1} = 1 \quad (114)$$

Once the second is chosen as the unit of time, c_0 fixes the size of the metre, \hbar_0 the joule and kilogram, k_0 the kelvin, and N_{A_0} the mole. $Z_0 = 1$ (or $\underline{\mu_0}$ or $\underline{\epsilon_0} = 1$) fixes the vacuum impedance, the magnetic permeability, and the electric permittivity to 1, and the size of the electrical units from μ_0 (or the exact size of μ_0 if the coulomb is obtained from the fixing of e).

We have chosen in (114) to determine the ohm, a dimensionless unit, to be $1/z_0 = 1/\mu_0 c_0 = 1/376.730\dots$, so that the impedance of vacuum is $376.730\dots\ \Omega = 1$. This also determines $R_K = 376.730\dots\ \Omega/2\alpha = 25\ 812.807\dots\ \Omega$. We could choose as well, to fix the electrical units, one of the equivalent conditions

$$z_0\ \text{H/s} = z_0^{-1}\ \text{F/s} = \mu_0\ \text{N/A}^2 = \epsilon_0\ \text{C}^2/(\text{J s} \times \text{m/s}) = 1 \quad (115)$$

It is often said that the new SI provides a system of units by fixing the values of fundamental constants of nature such as the speed of light and the Planck constant. Including it within the proposed system leads to transcend this point of view. We take advantage of the natural set of units provided by quantum mechanics for the quantum of action $\hbar = 1$, by relativity for the speed of light $c = 1$, and by electromagnetism for the vacuum magnetic permeability $\underline{\mu_0} = 1$, its electric permittivity $\underline{\epsilon_0} = 1$, and the vacuum impedance $Z_0 = (\underline{\mu_0}/\underline{\epsilon_0})^{1/2} = 1$. We then rescale these natural units, all related to the second or simply equal to 1, to get the properly normalized units we are used to.

Deciding a choice of values for the numerical constants \hbar_0 , c_0 , and μ_0 (with related ϵ_0 and z_0) does not fix the Planck constant, the speed of light nor the permeability or the permittivity or the impedance of vacuum, which all remain identical to 1. But it fixes instead the sizes of the various units (with or without dimensions) that we have chosen to use, derived from the second. This is also the case for the Boltzmann constant, providing the usual kelvin, and for the Avogadro constant, turned back into a fixed Avogadro number N_{A_0} providing the definition of the mole.

It has been agreed to fix numerically \hbar_0 and c_0 to define the metre, the joule, and the kilogram, but decided not to fix $\mu_0 = 4\pi \times 10^{-7}\ (\text{N/A}^2)$, to let the coulomb and ampere free to adjust to the chosen value of the elementary charge. This may be practically convenient, but remains conceptually questionable. While the vacuum magnetic permeability, its electric permittivity and impedance are all equal to 1, as the speed of light, their numerical values in SI units (N/A^2 or H/m , F/m , and Ω , respectively) get now dependent on the elementary charge e , i.e. on future experimental measurements of α . This is a somewhat unpleasant consequence of the fixing of e_0 in the new SI.

This is well managed by embedding the new SI within the proposed system, where the floating character of electrical units through their dependence on an unfixed μ_0 proportional to α is made explicit and automatically taken care of. $\hbar = c = \underline{\mu_0} = \underline{\epsilon_0} = Z_0 = k = N_A = 1$ guarantees that the vacuum magnetic permeability, electric permittivity and impedance all remain constant and identical to 1. The construction combines the advantages of both systems, with a simplified description of the fundamental laws, dimensionless charges, impedances, and fluxes, an elementary charge $e = \sqrt{4\pi\alpha}$ and flux quantum π/e , the kelvin as a unit of energy, and the mole identified as a very large Avogadro number.

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