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# Structural accessibility and structural observability of nonlinear networked systems

Marco Tulio Angulo\*, Andrea Aparicio, and Claude H. Moog

**Abstract**—The classical notions of structural controllability and structural observability are receiving increasing attention in Network Science, as they provide a mathematical basis to understand how the network structure of a dynamic system affects its controllability and observability properties. However, these two notions are formulated assuming systems with linear dynamics, which significantly limits their applicability. To overcome this limitation, here we introduce and fully characterize the notions of “structural accessibility” and “structural observability” for the broad class of nonlinear systems with meromorphic dynamics. We show how nonlinearities render easier the problem of controlling and observing networked systems, in the sense that meromorphic nonlinearities reduce the number of variables that are necessary to directly control and directly measure. Our results contribute to clarify the role that the network structure and the nonlinearities play in our ability to control and observe complex dynamic systems.

**Index Terms**—network science; nonlinear systems; accessibility; observability; controllability.

## I. INTRODUCTION

In a world where complex networks underlie most biological, social and technological systems that shape the human experience [1, 2], one central research program is finding principles that help us to control and observe complex networked systems. When all that is known about a dynamical system is its network structure (i.e., a graph of the interactions between its variables), the theoretical basis for this research program has been the classical notions of “structural controllability” and “structural observability” of linear systems [3]. These two notions characterize the conditions under which almost all linear dynamical systems whose structure matches a given network are controllable or observable, respectively [4]. Thus, linear structural controllability and structural observability provide a mathematical formalism to predict how changes in the network structure of a system impact its controllability and observability properties. For example, linear structural controllability was applied to predict and then experimentally validate how removing different neurons (i.e., removing nodes in the network) affects the locomotion of the roundworm *C. elegans* [5]. Additionally, over the last few years, a central line of research has been characterizing minimal sets of “driver nodes” and “sensor nodes” that render a complex networked system linearly controllable or linearly observable, respectively [3].

The conditions of linear structural controllability and linear structural observability can be stronger than necessary when

applied to systems with nonlinear dynamics, resulting in over-conservative predictions. Such over-conservative predictions happen because, for nonlinear systems, their lack of linear controllability (resp. linear observability) does not imply their lack of controllability (resp. observability). Nonlinear systems can be controllable or observable thanks to higher-order terms. An everyday example of this fact is a car, which is controllable but not linearly controllable because it cannot move in the direction of the axis defined by its rear wheels. Despite the ubiquity of nonlinear systems in nature and technology, the effects of nonlinearities in our ability to efficiently control and observe complex networked systems remain poorly understood [6–11].

Given that most systems in nature contain nonlinearities, in this paper we introduce and fully characterize the notions of nonlinear “structural accessibility” and nonlinear “structural observability” as counterparts of linear structural controllability and linear structural observability. These two notions characterize the conditions under which almost all nonlinear systems whose structure matches a given network are locally accessible almost everywhere or locally weakly observable almost everywhere, respectively. Local accessibility and local weak observability are nonlinear generalizations of linear controllability and linear observability, which have played a central role in nonlinear control theory [12]. Our graph characterization provides, for the first time, necessary and sufficient conditions for structural observability or structural accessibility for the broad class of nonlinear systems with meromorphic dynamics. A meromorphic function is the quotient of analytic functions, such as  $x_1 x_2 / (1 + x_2 \sin(x_1))$  in the variables  $x_1$  and  $x_2$ . Thus, the class of meromorphic systems is general enough to include the dynamic models used in most scientific disciplines [13], including ecology (e.g., the Generalized Lotka-Volterra equations), biochemistry (e.g., the Michaelis-Menten equations), neuroscience (e.g., the Wilson-Cowan model), and engineering (e.g., the Euler-Lagrange equations for robots). The results presented here extend our previous work that only considered ecological dynamics [14, 15].

Somewhat counter-intuitively, we find that nonlinearities render the problem of controlling or observing a networked system significantly easier. Specifically, for the class of nonlinear systems with meromorphic dynamics, we prove that the conditions for nonlinear structural accessibility and observability are essential in the sense that they are contained in the conditions for linear structural controllability and observability. In other words, the necessary and sufficient conditions for nonlinear meromorphic dynamics are only necessary and no longer sufficient for linear dynamics. The additional condition required for linear dynamics is rather “accidental” because it applies to the special subclass of linear dynamics. A direct consequence of this analysis is that the sets of driver and sensor nodes are smaller when compared to those necessary for linear structural controllability and linear structural observability.

The organization of this paper is as follows. We start in Section II summarizing the network characterization of structural controllability and structural observability for linear

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systems, serving as a comparison point to our results. Section III contains our problem statement and main results for nonlinear systems, as well as a case of study applying our structural accessibility theory to the locomotion of *C. elegans*. We collect the proofs of our claims in Sections IV and V. We end discussing some limitations of our approach in Section VI.

## II. PRELIMINARIES: RECALL OF RESULTS FOR LINEAR SYSTEMS

The network or *graph* of a system with  $N$  state variables,  $M$  inputs, and  $P$  outputs is a directed graph  $G = (V, E)$ . The vertex set  $V = X \cup Y \cup U$  contains state nodes  $X = \{x_1, \dots, x_N\}$ , output nodes  $Y = \{y_1, \dots, y_P\}$ , and input nodes  $U = \{u_1, \dots, u_M\}$ , see Fig. 1a. The edge set  $E = A \cup B \cup C$  contains edges  $(x_j \rightarrow x_i) \in A$  to denote that the  $i$ -th state variable directly depends on the  $j$ -th one, edges  $(x_j \rightarrow y_i) \in C$  to denote that the  $i$ -th measured output directly depends on the  $j$ -th state variable, and edges  $(u_j \rightarrow x_i) \in B$  to denote that the  $i$ -th state variable directly depends on the  $j$ -th input. Graphs with empty output or input node sets represent systems without outputs or inputs, respectively.

### A. Linear structural controllability and linear structural observability.

In the framework of linear structural controllability and linear structural observability the system dynamics is of course assumed linear. Then the controllability and observability of the set of all linear systems whose structure matches a given graph  $G$  is analyzed. More precisely, the system dynamics is assumed to have the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad (1)$$

where  $x(t) \in \mathbb{R}^N$ ,  $u(t) \in \mathbb{R}^M$  and  $y(t) \in \mathbb{R}^P$  are the state, input, and output of the system at time  $t$ , respectively. Here  $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ ,  $B = (b_{ij}) \in \mathbb{R}^{N \times M}$  and  $C = (c_{ij}) \in \mathbb{R}^{P \times N}$  are matrices of parameters.

The *structure* of Eq. (1) is determined by the zero/non-zero pattern of these three matrices. Thus, given a graph  $G$ , the class  $\mathcal{D}_L(G)$  of all linear systems whose structure matches  $G$  is defined as all systems of the form (1) such that:  $a_{ij} \neq 0$  if  $(x_j \rightarrow x_i) \in A$ ,  $b_{ij} \neq 0$  if  $(u_j \rightarrow x_i) \in B$ , and  $c_{ij} \neq 0$  if  $(x_j \rightarrow y_i) \in C$ . Note that the edges  $(x_j \rightarrow x_i)$  and  $(u_j \rightarrow x_i)$  are encoded by differential equations. By contrast, the edges  $(x_j \rightarrow y_i)$  are encoded by algebraic equations. These output edges have direction because the output map is not necessarily one-to-one (e.g., the output  $y_1 = x_1 + x_2$  correspond to the edges  $x_1 \rightarrow y_1$  and  $x_2 \rightarrow y_1$ ). In other words, the class  $\mathcal{D}_L(G)$  describes the set of all linear dynamics that a system can take if its structure matches  $G$ .

The class  $\mathcal{D}_L(G)$  is said linearly *structurally controllable* (resp. linearly *structurally observable*) if almost all systems it contains are controllable (resp. observable) [4]. This means that any of its systems is either controllable (resp. observable), or becomes controllable (resp. observable) by an infinitesimal change in the nonzero entries of the matrices  $A$ ,  $B$  or  $C$ . Remarkably, it turns out that when one system in  $\mathcal{D}_L(G)$  is

linearly controllable (resp. linearly observable), then almost all other systems in  $\mathcal{D}_L(G)$  are linearly controllable as well (resp. linearly observable as well) [4, 16].

A central result, which can be traced back to the pioneering work of Lin in the 70's [16], is the following graph characterization of linear structural controllability and linear structural observability:

**Theorem 1.** (see, e.g., [4]) *The class  $\mathcal{D}_L(G)$  of linear systems is:*

- a. linearly structurally controllable if and only if: (i) each state node is the end-node of a path that starts in  $U$ ; and (ii) there is a disjoint union of cycles and paths starting in  $U$  that covers  $X$ .
- b. linearly structurally observable if and only if: (i) each state node is the start-node of a path that ends in  $Y$ ; and (ii) there is a disjoint union of cycles and paths ending in  $Y$  that covers  $X$ .

A *path* is a sequence of nodes  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_n$  where  $v_i \in X \cup Y \cup U$ . The *start-node* of this path is  $v_1$  and its *end-node* is  $v_n$ . A *cycle* is a path that starts and ends in the same node (i.e.,  $v_n = v_1$ ). Two paths are *disjoint* if they have disjoint sets of nodes. A set of edges  $E'$  covers  $X$  if each node  $x_i \in X$  is either the tip or the tail of some edge  $e \in E'$ .

Theorem 1 shows that, excepting a zero-measure set of singularities, the graph  $G$  of a linear system determines its controllability and observability properties. Note that, for linear structural controllability, it is not sufficient that the control inputs can propagate their “influence” to all state nodes (i.e., that there is a path from some input node to each state node). Similarly, for linear structural observability, it is not sufficient that each state node can propagate its state to some output (i.e., that there is a path from each state node to some output node). Both notions require that the graph  $G$  contains enough “independent paths” that propagates these effects, encoded as the disjoint union of cycles and paths that covers all state nodes.

**Example 1.** *For the graph  $G$  of Fig. 1a, the class  $\mathcal{D}_L(G)$  contains all linear systems of the form*

$$\begin{cases} \dot{x}_1(t) = b_{11}u_1(t), \\ \dot{x}_2(t) = b_{21}u_1(t), \\ y_1(t) = c_{11}x_1(t) + c_{12}x_2(t), \end{cases} \quad (2)$$

with nonzero constants  $b_{11}, b_{21}, c_{11}$  and  $c_{12}$ . Recall that:

1. Together with isolated nodes in  $G$ , the main obstacle for linear structural controllability is the presence of so-called “dilations” [16]. An elementary example of a dilation is when two nodes with identical dynamics are controlled by the same input, as in the top in Fig. 1a. A dilation makes  $G$  not structurally controllable because it is impossible to obtain a disjoint union of paths and cycles that covers  $X$ . For Fig. 1a, all systems in  $\mathcal{D}_L(G)$  are uncontrollable because their state is constrained to the plane  $b_{11}x_2(t) - b_{21}x_1(t) = b_{11}x_2(0) - b_{21}x_1(0)$  for all inputs  $u_1(t)$  and time  $t$  (Fig. 1b).
2. Analogously, the so-called “contractions” in  $G$  are the main obstacle for linear structural observability. An ele-

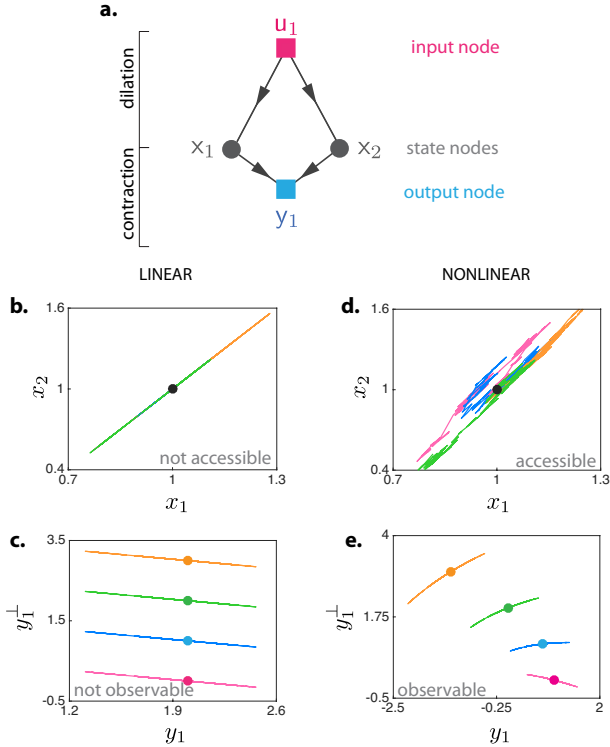


Fig. 1. (Color online) **a.** Graph of a system. Here the input, state and output nodes are  $U = \{u_1\}$ ,  $X = \{x_1, x_2\}$  and  $Y = \{y_1\}$ , respectively. **b.** The graph of panel a is not linearly structurally controllable, meaning that no linear system with this graph is controllable. We illustrate this with five trajectories (colors) of the linear dynamics of Eq. (2) with initial condition  $x(0) = (1, 1)^\top$  (black dot), parameters  $b_{11} = 0.5$  and  $b_{12} = 1$ , and random inputs  $u_1(t)$ . The lack of controllability constrains the system to the plane  $\{x \in \mathbb{R}^2 | b_{21}x_1 - b_{11}x_2 = b_{21}x_1(0) - b_{11}x_2(0)\}$  for all time and inputs, representing the autonomous element of this system. Consequently, the system is not accessible and not controllable. **c.** The graph of panel a is not linearly structurally observable, meaning that no linear system with this graph is observable. We illustrate this using the linear dynamics of Eq. (2), where five different trajectories (colors) with different initial conditions (dots) give exactly the same projection in the output  $y_1$  because they are all vertically aligned. This is characterized by the hidden element  $\zeta = c_{12}x_1 - c_{11}x_2$  which is orthogonal to the output and its derivatives. Consequently, the system is not linearly observable and not locally weakly observable. **d.** The nonlinear dynamics of Eq. (4) is accessible because it lacks autonomous elements, illustrated here by five trajectories (colors) that are not constrained to any low-dimensional manifold. Parameters are as in panel b and  $\varepsilon = 0.5$ . This shows that the graph in panel a is structurally accessible. **e.** The nonlinear dynamics of Eq. (4) is locally weakly observable since different trajectories (colors) corresponding to different initial conditions (dots) give different projections in the output  $y_1$ . Parameters are as in panel d and  $\varepsilon = 0.5$ . This shows that the graph in panel a is structurally observable.

*mentary example of a contraction is when two state nodes are measured using a single output, as in the bottom in Fig. 1a. In this case, all systems  $\mathcal{D}_L(G)$  are unobservable because, by using  $y_1 = c_{11}x_1 + c_{12}x_2$  and  $k \geq 1$  of its derivatives  $y_1^{(k)} = (c_{11}b_{11} + c_{12}b_{12})u_1^{(k)}$ , it is impossible to infer the value of  $x_1$  and  $x_2$  (Fig. 1c). Indeed, there is only one equation ( $y_1$ ) and two unknowns ( $x_1$  and  $x_2$ ).*

### B. Driver and sensor nodes for linear systems.

Theorem 1 provides a theoretical basis for the very active research line aiming to identify the “driver” and “sensor” nodes that render a system linearly structurally controllable and linearly structurally observable, respectively (see, e.g.,

[2, 3]). More precisely, consider a graph  $G(X, A)$  with only state nodes  $X$  and edges  $(x_i \rightarrow x_j) \in A$ . Then define:

#### Definition 1.

- (i)  $X_D \subseteq X$  is a set of driver nodes if there exists a set  $U$  of input nodes and a set  $B$  of edges of the form  $(u_i \rightarrow x_j)$  such that: (i) the graph  $G(X \cup U, A \cup B)$  is linearly structurally controllable; and (ii) all and only the driver nodes have incoming edges from the input nodes (i.e.,  $(u_i \rightarrow x_j) \in B$  if  $x_j \in X_D$ ).
- (ii)  $X_S \subseteq X$  is a set of sensor nodes if there exists a set  $Y$  of output nodes and a set  $C$  of edges of the form  $(x_i \rightarrow y_j)$  such that: (i) the graph  $G(X \cup Y, A \cup C)$  is linearly structurally observable; and (ii) all and only the sensor nodes have outgoing edges to the output nodes (i.e.,  $(x_i \rightarrow y_j) \in C$  if  $x_i \in X_S$ ).

A set of driver nodes or sensor nodes is called *minimal* if it has the minimal cardinality among all sets of driver nodes or sensor nodes, respectively. The conditions in Theorem 1 allow finding a minimal set of driver nodes (resp. a minimal set of sensor nodes) by mapping the satisfaction of these conditions into solving a maximum matching problem on the graph  $G$  (resp.  $G^\top$  obtained from  $G$  by reversing the direction of all its edges), see [3].

### III. PROBLEM STATEMENT AND MAIN RESULTS FOR NONLINEAR SYSTEMS

Here we generalize the analysis of Section II by enlarging the class of dynamics that the system can take to include arbitrary meromorphic nonlinearities. The main result is that the conditions (i) and (ii) in Theorem 1 do not have the same level of importance: one is essential and must be preserved, while the other is not essential and can be dropped.

Specifically, we now consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \quad (3)$$

where  $f : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$  and  $h : \mathbb{R}^N \rightarrow \mathbb{R}^P$  are arbitrary meromorphic functions of their arguments (i.e., each of their entries is the quotient of analytic functions). An instrumental property of meromorphic functions is that they are either identically zero (written as “ $\equiv 0$ ”), or they are different from zero in an open dense subset of their domain (written as “ $\neq 0$ ”), see [12, Chapter 1]. This property allows us to define:

**Definition 2.** Given a meromorphic pair  $\{f, h\}$ , its graph  $G_{f,h} = (X \cup Y \cup U, A_f \cup B_f \cup C_h)$  has the edge-set defined as:  $(x_j \rightarrow x_i) \in A_f$  if  $\partial f_i / \partial x_j \neq 0$ ;  $(u_j \rightarrow x_i) \in B_f$  if  $\partial f_i / \partial u_j \neq 0$ ; and  $(x_j \rightarrow y_i) \in C_h$  if  $\partial h_i / \partial x_j \neq 0$ .

We say that two pairs  $\{f, h\}$  and  $\{\tilde{f}, \tilde{h}\}$  are graph-equivalent if  $G_{f,h} = G_{\tilde{f},\tilde{h}}$ . Since any pair is graph-equivalent to itself, graph-equivalence is an equivalence relation. Thus, given a graph  $G$ , we can define the equivalence class

$$\mathcal{D}(G) := \{\text{all meromorphic } \{f, h\} \text{ such that } G_{f,h} = G\}.$$

The class  $\mathcal{D}(G)$  represents all meromorphic nonlinear dynamics that a system can have given that its graph is  $G$ . Note that  $\mathcal{D}_L(G) \subset \mathcal{D}(G)$ .

As the nonlinear counterparts of linear controllability and linear observability, we consider the concepts of local *accessibility* and local weak *observability*. We introduce these concepts through a mathematical formalism based on differential algebra (we refer the reader to [12] for technical details). Consider the field of meromorphic functions  $\mathcal{K}$  in the variables  $\{x, u, \dot{u}, \ddot{u}, \dots, y, \dot{y}, \ddot{y}, \dots\}$ , and the sets of differential symbols  $dx = (dx_1, \dots, dx_N)^\top$ ,  $du^{(\ell)} = (du_1^{(\ell)}, \dots, du_M^{(\ell)})^\top$ ,  $\ell \geq 0$ , and  $dy^{(k)} = (dy_1^{(k)}, \dots, dy_P^{(k)})^\top$ ,  $k \geq 0$ . For a function  $\varphi \in \mathcal{K}$ , its differential is defined as  $d\varphi = (\partial\varphi/\partial x)^\top dx + (\partial\varphi/\partial u)^\top du$ . More generally, functions in the vector space spanned over  $\mathcal{K}$  by the symbols of  $\{dx, du, \dots, du^{(\ell)}, dy, \dots, dy^{(k)}\}$  are called *one-forms*. Intuitively, these vector spaces represent “all functions” that can be written in terms of the symbols that they contain. For example, one may consider that  $\mathcal{X} = \text{span}_{\mathcal{K}}\{dx\}$  represents “all functions of the state variables”, that  $\mathcal{Y}^{(k)} = \text{span}_{\mathcal{K}}\{dy, \dots, dy^{(k)}\}$  represents “all function of the output and its first  $k$  derivatives”, and analogously for  $\mathcal{U}^{(\ell)} = \text{span}_{\mathcal{K}}\{du, \dots, du^{(\ell)}\}$ .

Based on the above framework, we introduce the following notions:

**Definition 3.**

- (i) An autonomous element of a system is a non-constant meromorphic function  $\xi(x)$  such that its  $k$ -th time derivative  $\xi^{(k)}$  is independent of  $u$  for all  $k \geq 0$ , i.e.,

$$\partial\xi^{(k)}/\partial u \equiv 0, \quad \forall k \geq 0.$$

- (ii) A hidden element of a system is a non-constant meromorphic function  $\zeta(x)$  that is independent of  $\{y, \dots, y^{(k)}\}$  and  $\{u, \dots, u^{(\ell)}\}$  for all  $k, \ell \geq 0$ , i.e.,

$$d\zeta \notin \mathcal{Y}^{(k)} \cup \mathcal{U}^{(\ell)}, \quad \forall k, \ell \geq 0.$$

An autonomous element constrains the state of the system to a low-dimensional manifold for all control inputs [12, pp. 47], just as in an uncontrollable linear system its state is constrained to a hyperplane. In this paper, we introduce the notion of a “hidden element” as an internal variable of the system whose value cannot be inferred from measured information. In other words, a hidden element cannot be rewritten as a function of the outputs and inputs and their derivatives. A non-constant function that is not a hidden element is called observable.

With these notions a system is called locally *accessible* if it does not have autonomous elements [12]. By analogy, we call a system locally weakly *observable* if it does not have hidden elements. Note that this definition is equivalent to the standard definition of local weak observability [12, Definition 4.7, pp. 57]. In terms of the systems’ trajectories, local accessibility means that, for almost all initial states, the set of reachable states has dimension  $N$ . Local weak observability means that, in an open and dense subset of the state space, any state can be distinguished from its neighbors by measuring the output and input only.

Local accessibility and local weak observability are fundamental properties of nonlinear systems as they characterize the solution to fundamental problems like exact linearization by feedback control, calculating controllability and observability

indices for nonlinear systems, among others [12]. For simplicity, in the rest of this paper we write “accessible” for locally accessible, and “observable” for locally weakly observable.

For linear systems, the lack of autonomous elements is equivalent to linear controllability, and the lack of hidden elements is equivalent to linear observability [12]. For example, all linear systems of Eq. (2) are not controllable because  $\xi(x) = b_{21}x_1 - b_{11}x_2$  is an autonomous element for all of them. Indeed  $\xi^{(k)} \equiv 0$ , which is independent of  $u$  for all  $k \geq 1$ . Similarly,  $\zeta(x) = c_{12}x_1 - c_{11}x_2$  is a hidden element for all those linear systems because it cannot be written as a function of the output  $y_1 = c_{11}x_1 + c_{12}x_2$  and/or the input  $u_1$  and their derivatives. Actually, since  $y_1^{(k)} = (c_{11}b_{11} + c_{12}b_{12})u_1^{(k-1)}$ , the output derivatives contain no information of the state at all. Hence, we conclude that  $d\zeta \notin \mathcal{Y}_1^{(k)} \cup \mathcal{U}_1^{(\ell)}$  for all  $k, \ell \geq 0$ . In this sense, the above definitions of accessibility and observability provide nonlinear generalizations of linear controllability and linear observability.

*A. Structural accessibility and structural observability for nonlinear systems.*

In analogy to the case of linear dynamics [4], for nonlinear systems we say that “almost all” systems in the class  $\mathfrak{D}(G)$  have certain property (e.g., accessibility or observability) if the subset of systems having that property is open and dense everywhere. We use this concept to define:

**Definition 4.** The class  $\mathfrak{D}(G)$  is:

- (i) structurally accessible if almost all systems it contains are accessible.  
(ii) structurally observable if almost all systems it contains are observable.

Structural accessibility (resp. structural observability) implies that accessible systems (resp. observable systems) are *generic* in the class  $\mathfrak{D}(G)$ , while non-accessible systems (resp. non-observable systems) are not and thus are “hard to find”. Consequently, if  $\mathfrak{D}(G)$  is structurally accessible (resp. structurally observable), any of its systems is either accessible (resp. observable) or becomes accessible (resp. observable) by some arbitrarily small change of its dynamics (see Example 2 below). In what follows, when  $\mathfrak{D}(G)$  is structurally accessible (resp. structurally observable), we also call the graph  $G$  structurally accessible (resp. structurally observable).

Our first main result proves that, as in the case of linear systems, if at least one system in  $\mathfrak{D}(G)$  is accessible (resp. observable) then almost all other systems in  $\mathfrak{D}(G)$  are also accessible (resp. observable), see Lemma 1 in Section IV and Lemma 4 in Section V. This first result is instrumental because it allows us to prove that certain  $\mathfrak{D}(G)$  is structurally accessible (or structurally observable) by showing that one of its systems is accessible (or observable).

Based on the above first result, our second main result provides necessary and sufficient conditions for structural accessibility and structural observability for the general class of meromorphic nonlinear systems:

**Theorem 2.** The class  $\mathfrak{D}(G)$  is:

- a. *structurally accessible if and only if each state node is the end-node of a path that starts in  $\mathcal{U}$ .*
- b. *structurally observable if and only if each state node is the start-node of a path that ends in  $\mathcal{Y}$ .*

*Proof.* See Proposition 2 in Section IV for item a, and Proposition 3 in Section V for item b.  $\square$

Although the observability of a nonlinear system may depend on the applied inputs, Theorem 2 shows that its structural observability is independent of such inputs. This situation happens because removing all edges that connect the inputs to the state variables will not change if the condition of Theorem 2b is satisfied or not. More generally, note that including edges in a graph cannot damage its structural accessibility or structural observability. Note also that a “duality” similar to the case of linear systems remains: a graph is structurally accessible if and only if its “dual graph” (with reversed edges and the labels of input and output nodes interchanged) is structurally observable.

Additionally and somewhat counterintuitively, Theorem 2 shows that meromorphic nonlinearities make it easier to “control” and “observe” networked systems because Theorem 1 contains the conditions of Theorem 2. We illustrate this point by revisiting Example 1 now considering nonlinear dynamics:

**Example 2.** *For the graph in Fig. 1a, the class  $\mathcal{D}(G)$  contains all nonlinear systems of the form*

$$\begin{cases} \dot{x}_1(t) = b_{11}u_1(t) + \varepsilon, \\ \dot{x}_2(t) = b_{21}u_1(t) + \varepsilon u_1^3(t), \\ y_1(t) = c_{11}x_1 + c_{12}x_2 + \varepsilon x_1(t)x_2(t), \end{cases} \quad (4)$$

with nonzero  $b_{11}, b_{21}$ , and  $\varepsilon$ . Note that Eq. (4) is an “ $\varepsilon$ -change” of Eq. (2) because making  $\varepsilon = 0$  renders Eq. (4) equal to Eq. (2). Note also:

1. *In the dilation of Fig. 1a, the nonlinearities in  $\mathcal{D}(G)$  eliminate the autonomous element that was present in  $\mathcal{D}_L(G)$ . That is, the function  $\xi(x) = b_{21}x_1 - b_{11}x_2$  that was an autonomous element for all linear dynamics of Eq. (2) is no longer an autonomous element for Eq. (4) because  $\dot{\xi} = -\varepsilon b_{11}u_1^3$  depends on  $u_1$ . This proves that  $\mathcal{D}(G)$  is structurally accessible. Actually, the trajectories of Eq. (4) are no longer constrained to a low-dimensional manifold (Fig. 1d).*
2. *In the contraction of Fig. 1a, the nonlinearities in  $\mathcal{D}(G)$  also eliminate the hidden element that was present in  $\mathcal{D}_L(G)$ . To see this, compute  $\dot{y}_1 = \alpha_0(u_1) + \varepsilon\alpha_1(u_1)x_1 + \varepsilon\alpha_2(u_1)x_2$ , where  $\alpha_0(u_1) = c_{11}[b_{11}u_1 + \varepsilon p_1] + c_{12}[b_{21}u_1 + \varepsilon u_1^3]$ ,  $\alpha_1(u_1) = b_{21}u_1 + \varepsilon u_1^3$  and  $\alpha_2(u_1) = b_{11}u_1 + \varepsilon p_1$ . Crucially, note that  $\alpha_1 \neq 0$  and  $\alpha_2 \neq 0$  for almost all  $u_1$ , implying that  $\dot{y}_1$  now depends on the state variables  $x_1$  and  $x_2$  (compare to the case of linear dynamics, where  $\dot{y}_1$  was independent of  $x$ ). The Jacobian*

$$\begin{pmatrix} \partial y_1 / \partial x \\ \partial \dot{y}_1 / \partial x \end{pmatrix} = \begin{pmatrix} c_{11} + \varepsilon x_2 & c_{12} + \varepsilon x_1 \\ \varepsilon \alpha_1 & \varepsilon \alpha_2 \end{pmatrix},$$

*is nonsingular almost everywhere in  $\mathbb{R}^2$ . The Implicit Function Theorem thus implies that we can locally write*

*$x_1$  and  $x_2$  in terms of  $y_1$  and  $\dot{y}_1$ . Therefore, the function  $\zeta(x) = c_{12}x_1 - c_{11}x_2$  that was a hidden element for the linear system of Eq. (2) is no longer a hidden element for Eq. (4). This proves that Eq. (4) is observable (Fig. 1e), and that  $\mathcal{D}(G)$  is structurally observable.*

**B. Minimal sets of driver/sensor nodes and input/output nodes for nonlinear systems.**

As in Section II-B, consider a graph  $G(X, A)$  consisting of state nodes  $X$  and edges  $(x_i \rightarrow x_j) \in A$ . We can extend the notion of driver nodes of Definition 1 to nonlinear systems by requiring that a set of driver nodes  $X_D \subseteq X$  renders  $G(X \cup \mathcal{U}, A \cup B)$  structurally accessible. Similarly, a set of sensor nodes  $X_S \subseteq X$  must render  $G(X \cup \mathcal{U}, A \cup C)$  structurally observable. Then, Theorem 2 has the following implication:

**Proposition 1.**

- (i) *A minimal set of driver nodes is given by arbitrarily choosing one node in each root strongly-connected-component of  $G(X, A)$ .*
- (ii) *A minimal set of sensor nodes is given by arbitrarily choosing one node in each top strongly-connected-component of  $G(X, A)$ .*

A strongly connected component (SCC) of  $G$  is a maximal subgraph such that there is a directed path in both directions between any two of its nodes [17, pp. 552-557]. A root SCC is an SCC without incoming edges, and a top SCC is an SCC without outgoing edges. Recall that any directed graph can be decomposed into an acyclic graph between its SCCs, with root and top SCCs at the start and end of this graph, respectively [17]. Let  $m$  be the number of root SCCs and  $p$  the number of top SCCs of  $G(X, A)$ . Then, Proposition 1-(i) is obtained because if a single input node  $u$  is connected to one arbitrary node  $x_j$  of each root SCC (i.e.,  $u \rightarrow x_j, j = 1, \dots, m$ ), the decomposition into the acyclic graph of SCC implies that the graph satisfies the condition of Theorem 2a. Analogously, Proposition 1-(ii) is obtained from the fact that if a single output node  $y$  is connected with one arbitrary node  $x_j$  of each top SCC (i.e.,  $x_j \rightarrow y, j = 1, \dots, p$ ), this will yield a graph that satisfies the condition of Theorem 2b. An additional consequence of this argument is the following:

**Corollary 1.**

- (i) *The minimal number of driver nodes of any graph is its number of root SCCs, and the minimal number of sensor nodes is its number of top SCCs.*
- (ii) *The minimal number of input nodes that renders any graph structurally accessible is always one, and the minimal number of output nodes that renders any graph structurally observable is also one.*

The second statement in the above Corollary generalizes the result of Ref. [18] to structural systems and to the case of observability.

All minimal sets of driver or sensor nodes of arbitrary graphs can be found in linear time because the SCCs can be computed in linear time [17, pp. 35]. By contrast, in the case of linear structural accessibility (resp. linear structural

observability), solving the maximum-matching problem to find one set of driver nodes (resp. sensor nodes) takes polynomial time. Actually, for linear dynamics, identifying all sets of driver nodes (resp. sensor nodes) is intractable for large graphs. Note that the decomposition of a graph into its SCCs has also been leveraged when assuming linear dynamics (see, e.g., [19] or [20]).

### C. Application to the locomotion of *C. elegans*.

In this final subsection, we contrast the predictions of linear structural controllability and (nonlinear) structural accessibility using the locomotion of *C. elegans* as a case of study.

In a recent pioneering work [5], Gan et al. analyzed the control properties of the *C. elegans* nervous system from a network perspective. For this aim, the authors modeled the connectome underlying the nervous system of *C. elegans* as a directed graph  $G$ . In this graph, state nodes  $X$  correspond to neurons and muscles, input nodes  $U$  represent the external stimuli applied to the touch neurons, and the output nodes  $Y$  are muscle cells. Edges in  $G$  correspond to electrical and chemical junctions between the neurons, and between neurons and muscles. The analysis aims to understand how the structure of this network impacts the ability to control the muscle cells (i.e., outputs) from external stimuli (i.e., inputs). A key finding was that ablating (i.e., deleting) the neuron marked as “PDB” generates a dilation in  $G$ . Importantly, the ablation of PDB does not disconnect any neuron or muscle from the network. However, assuming that the nervous system is approximately linear as in Eq. (1), linear structural controllability predicts that ablating PDB damages the linear controllability of the nervous system because it generates a dilation [5].

Neural systems, however, can have strong nonlinear behaviors [21]. For example, in computational neuroscience, the elementary Wilson-Cowan (WC) model [22] describes the input activity  $x_i$  of the  $i$ -th neurons as [23]:

$$\dot{x}_i = -x_i + \sum_{j=1}^N a_{ij} \frac{1}{1 + \exp(-\tau(x_j - \mu))}.$$

Here,  $\tau > 0$  and  $\mu > 0$  characterize the steepness of the activation function and the firing-rate threshold, respectively. The parameters  $A = (a_{ij}) \in \mathbb{R}^{N \times N}$  characterize the interaction strength between neurons (e.g.,  $x_j \rightarrow x_i$  if  $a_{ij} \neq 0$ ). Note that the WC model has meromorphic dynamics. However, presumably, the nervous system of *C. elegans* may be governed by more complicated dynamics than the WC model.

In particular, by applying our structural accessibility theory, we can assume that an arbitrary nonlinear system with meromorphic dynamics as in Eq. (3) governs the nervous system of the *C. elegans*. In such a case, our theory implies that the dilation caused by ablating PDB cannot damage the structural accessibility of the *C. elegans* connectome. Hence, by using the definition of accessibility in terms of trajectories of the system [12, pp. 47], our theory predicts that the nervous system of an ablated worm can reach the same set of states as those of normal worms. In other words, our structural accessibility theory predicts that PDB ablated worms can still adopt each

body pose that a non-ablated worm can adopt. More generally, we predict that the ability of a worm to adopt a body pose is preserved as long as the ablated interneurons do not fully disconnect an input (i.e., a sensory neuron) or an output (i.e., a muscle cell). However, the loss of linear controllability implies that ablated worms may use different trajectories to reach desired states (e.g., by using different paths in the connectome). This prediction agrees with the experimental results [5], as ablated worms are not reported to have reduced mobility but rather a different pattern of locomotion characterized by significantly larger negative values of the first “eigenworm”.

The following two Sections build the proofs for our main results.

## IV. PROOF OF THE STRUCTURAL ACCESSIBILITY THEOREM

Given a graph  $G = (X \cup U, A \cup B)$ , here we consider the class  $\mathcal{D}(G)$  of all controlled systems

$$\dot{x}(t) = f(x(t), u(t)), \quad (5)$$

such that  $G_f = G$ .

Our first result shows that once  $\mathcal{D}(G)$  has at least one accessible system, then  $\mathcal{D}(G)$  is structural accessible (i.e., accessible systems are generic). To prove this first result, we show that once  $\mathcal{D}(G)$  contains one accessible system, the “line” connecting this accessible system to any other system in  $\mathcal{D}(G)$  consists of accessible systems and eventually some isolated non-accessible systems (Fig. 2a).

Specifically, we construct the argument to prove this result using the notion of the  $k$ -jets  $f_k$  of a meromorphic function  $f$ —informally defined as taking the first  $k$ -terms of its Taylor expansion—and the resulting topology—the so-called “Whitney  $C^k$  topologies”. We refer the reader to [24, Section 2.1] and [25, Chapter II.3] for further details. Specifically, the topology we use is defined from the notion of an open ball of radius  $\varepsilon \geq 0$  centered at a meromorphic  $f_0$ . This ball consists of all meromorphic  $f$ ’s such that the Euclidean distance between the first  $\ell$  Taylor coefficients of  $f_0$  and  $f$  is less than  $\varepsilon$  for all  $\ell \geq 0$ .

**Lemma 1.** *If  $\mathcal{D}(G)$  contains at least one accessible system, then this class is structurally accessible. That is:*

- (i) *the subset of accessible systems is dense everywhere;*
- (ii) *the subset of accessible systems is open; and*
- (iii) *the subset of non-accessible systems is not dense.*

*Proof.* Although (iii)  $\Rightarrow$  (i) because  $\mathcal{D}(G)$  is the disjoint union of accessible and non-accessible systems, an independent proof of each statement is provided:

- (i) Let  $f^* \in \mathcal{D}(G)$  be the accessible system and choose arbitrary any other  $f \in \mathcal{D}(G)$ . Define the convex combination  $f_\lambda = \lambda f^* + (1 - \lambda)f$ , which parametrizes the “line” passing between  $f^*$  and  $f$ . Note that  $f_\lambda \in \mathcal{D}(G)$  for almost all  $\lambda \in [0, 1]$ . To prove the claim, our objective is to show that almost all systems in  $\{f_\lambda\}$ ,  $\lambda \in [0, 1]$ , are accessible. Note also that for  $\lambda = 1$  we have  $f_1 = f^*$ , implying that  $f_1$  is accessible. Consequently, due to the generic properties of meromorphic functions and the



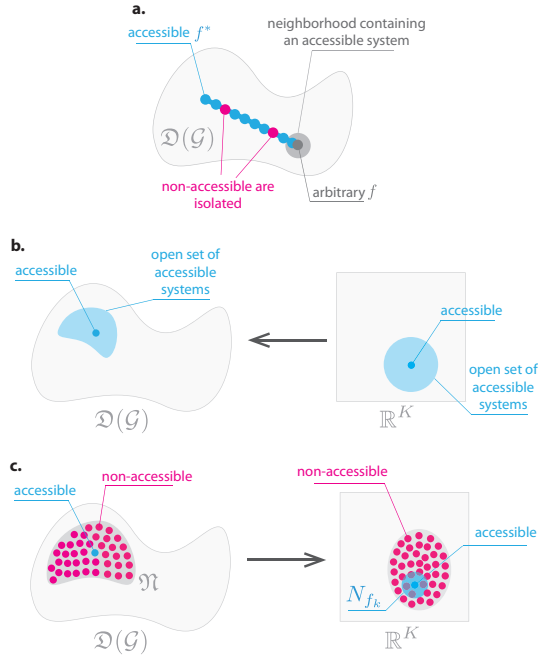


Fig. 2. (Color online) Figures for the proof of Lemma 1. **a.** Accessible systems are dense. **b.** The subset of accessible systems is open. **c.** Contradiction obtained by assuming that the subset of non-accessible systems is dense, proving that non-accessible systems are not dense.

Accessibility rank condition [12], the family of systems  $\{f_\lambda\}$  are accessible for almost all  $\lambda \in [0, 1]$ , completing the proof.

- (ii) We prove that any accessible  $f \in \mathcal{D}(G)$  has a neighborhood consisting only of accessible systems. Since  $f$  is meromorphic, we can rewrite this function as the Taylor-expansion  $f(x, u) = \alpha_0(x) + \sum_{i=1}^{\infty} \alpha_i(x)u^i$  with  $\alpha_i \in \mathcal{K}$ . Note that the accessibility of  $f$  implies there exists a  $k \in \mathbb{N}$  such that the  $k$ -jet  $f_k(x, u) := \alpha_0(x) + \sum_{i=1}^k \alpha_i(x)u^i$  is accessible. Indeed, since  $f$  is accessible there cannot be autonomous elements  $\xi \in \mathcal{K}$ , implying that  $d\xi$  is not orthogonal to at least some  $\alpha_k$ ,  $k \in \mathbb{N}$ . This implies that no (non-constant)  $\xi \in \mathcal{K}$  can be an autonomous element for  $f_k$ , making the  $k$ -jet  $f_k$  accessible.

Recall that this  $k$ -jet represents the first  $k$  terms of the Taylor expansion of  $f$ , implying we can associate  $f_k$  to a point in  $\mathbb{R}^K$  for some  $K$  that depends on  $k$  (right in Fig. 2b). Next we regard  $f_k$  as a polynomial function of its Taylor coefficients, so that the generic properties of meromorphic functions imply that  $f_k$  has a neighborhood  $N_{f_k} \subseteq \mathbb{R}^K$  of accessible systems. All  $\tilde{f} \in \mathcal{D}(G)$  such that their  $k$ -jets  $\tilde{f}_k$  satisfy  $\tilde{f}_k \in N_{f_k}$  will form the open neighborhood of  $f$  of accessible systems.

- (iii) We prove by contradiction, assuming that  $\mathcal{D}(G)$  is structurally accessible but that it contains an open set  $\mathfrak{N}$  such that non-accessible systems are dense on it (pink in Fig. 2c). Since  $\mathcal{D}(G)$  is structurally accessible and accessible systems are dense due to Lemma 1-(i), then  $\mathfrak{N}$  contains at least one accessible system  $f$  (blue in Fig. 2c). Now choose  $k \geq 0$  large enough such that the  $k$ -jet  $f_k$  of the accessible system  $f$  is accessible. The  $k$ -jets  $\tilde{f}_k$  of all non-accessible systems  $\tilde{f}$ 's remain non-accessible. Since

the  $f_k$  and the  $\tilde{f}_k$ 's represent the first  $k$  terms of the Taylor expansion of  $f$  and the  $\tilde{f}$ 's, we can associate each of them to a point in  $\mathbb{R}^K$  corresponding to the value of the first  $k$  coefficients of their Taylor expansion (here again  $K$  is some constant that depends on  $k$ ). Since  $\mathfrak{N}$  is a neighborhood of  $f$ , all its elements are mapped to a corresponding neighborhood of  $f_k$  in  $\mathbb{R}^K$  such that the points corresponding to non-accessible systems are dense (Fig. 2c). Considering now that  $f_k$  is accessible and that it is a polynomial function of its Taylor coefficients, the generic properties of meromorphic functions imply that there exists a neighborhood  $N_{f_k} \subseteq \mathbb{R}^K$  of  $f_k$  such that all its corresponding elements are accessible (blue neighborhood in Fig. 2c). This gives the desired contradiction, since it contradicts the fact that the non-accessible systems were dense.  $\square$

The next result allows us to analyze the structural accessibility of a graph from its *spanning* subgraphs, which will be instrumental for the proof of the main result. Recall that a subgraph  $\tilde{G}$  of  $G$  is spanning when  $\tilde{G}$  includes all nodes of  $G$ .

**Lemma 2.** *Let  $\tilde{G} \subseteq G$  be any spanning subgraph of  $G$ . If  $\mathcal{D}(\tilde{G})$  is structurally accessible then  $\mathcal{D}(G)$  is also structurally accessible.*

*Proof.* Since  $\mathcal{D}(\tilde{G})$  is structurally accessible, it contains one system  $\dot{x} = f(x, u)$  which is accessible. Notice that starting from  $\tilde{G}$ , we can recover  $G$  by adding some edges. Suppose that the edge  $x_j \rightarrow x_i$  is added to  $\tilde{G}$  to obtain  $G$ . Then  $\mathcal{D}(G)$  contains the systems

$$\dot{x}_i = f_i(x, u) + \alpha x_j, \quad (6)$$

for any constant  $\alpha \neq 0$ . Similarly, if the edge  $u_j \rightarrow x_i$  is added  $\mathcal{D}(G)$  contains the systems

$$\dot{x}_i = f_i(x, u) + \alpha u_j. \quad (7)$$

For  $\alpha = 0$  the systems of Eqs. (6) or (7) are accessible. Additionally, their right-hand side is a meromorphic function of  $\alpha$ . Thus, due to the generic properties of meromorphic functions [12], both systems are accessible for almost all  $\alpha \in \mathbb{R}$ . Therefore, the class  $\mathcal{D}(G)$  is structurally accessible. Repeating the same argument for all other edges completes the proof.  $\square$

Now consider a meromorphic function  $\varphi(x, u) : \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N$  and a subset of nodes  $V \subseteq X \cup U$ . We write  $\varphi \in S$  if  $\varphi(x)$  depends on *all* variables  $v_i$  for all  $v_i \in V$ . With this notation, an autonomous element of Eq. (5) is a non-constant meromorphic function  $\varphi(x)$  such that  $\varphi^{(k)} \notin U$  for all  $k \geq 0$ .

**Example 3.** *For the graph of Fig. 1a with the linear dynamics of Eq. (2) we have that  $\xi = b_{11}x_2 - b_{21}x_1$  satisfies  $\xi^{(k)} = 0$  for all  $k \geq 1$ . Thus we have that  $\xi^{(k)} \notin U$  for all  $k$  and hence  $\xi$  is an autonomous element.*

Next, for a set  $S$  of state nodes, define its ‘‘tail-set’’  $T(S) \subseteq V \cup U$  as all nodes which point to  $S$  (Fig. 3a). We



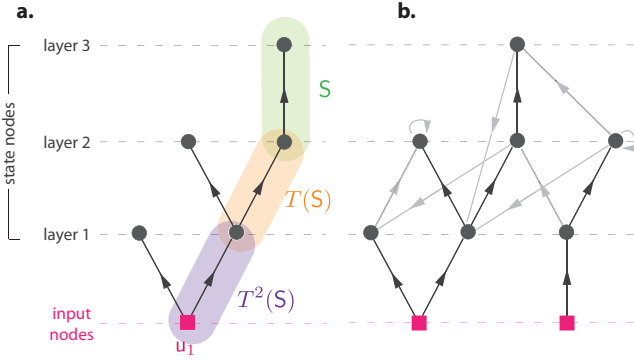


Fig. 3. (Color online) **a.** A tree graph  $G$  where each state node has only one incoming edge and its root is an input node. A set  $S$  and its tail-sets  $T(S)$  and  $T^2(S)$  are marked in green, orange and purple, respectively. **b.** From the graph  $G$  with  $M = 2$  input nodes (dark and light edges), a spanning subgraph  $\tilde{G}$  (dark edges) is obtained that has  $M$  disjoint trees, one incoming edge per state node, and roots at the input nodes.

denote  $T^k(S) := T(T^{k-1}(S))$ . In the next result we prove the structural accessibility of a special class of graphs. We prove this result by leveraging Lemma 1 and exhibiting one accessible system in this class of graphs.

**Lemma 3.** *Assume that the graph  $G$  is a (connected) directed tree with each state node  $x_i$  having a single incoming edge, and rooted at a single input node  $u_1$ . Then  $\mathcal{D}(G)$  is structurally accessible.*

*Proof.* For this graph, note that the state nodes can be organized into  $L$  layers according to their distance to the input node, with the first layer being all state nodes with distance one (Fig. 3a). To prove the claim, we show that the following system with polynomial dynamics is accessible:

$$\dot{x}_i = (f_{T(i)})^{p_i}. \quad (8)$$

Above,  $f_{T(i)} = x_{T(i)}$  if  $x_i$  is in layer  $\geq 2$ , and  $f_{T(i)} = u_1$  otherwise. The vector  $p = (p_1, \dots, p_N) \in \mathbb{N}_+^N$  contains  $N$  different integers with  $\min_k p_k$  large enough.

Next, note that for this graph  $G$  and the dynamics of Eq. (8), any non-constant meromorphic function  $\varphi \in \mathcal{S}$  satisfies  $\dot{\varphi} \in T(S)$  for any  $S \subseteq X$ . Namely, if  $\varphi$  depends on  $\{x_i, \dots, x_k\}$ , then  $\dot{\varphi}$  depends on all variables  $\{f_{T(i)}, \dots, f_{T(k)}\}$ . To show this, just note that

$$\dot{\varphi} = \sum_{i \in S} \frac{\partial \varphi}{\partial x_i} \dot{x}_i = \sum_{i \in S} \frac{\partial \varphi}{\partial x_i} f_{T(i)}^{p_i} \in T(S),$$

and that no term can cancel out in the sum because they have different exponents  $p_i$ .

The above observation actually implies that this system is accessible. To see this, take any  $S \subseteq X$  and any non-constant meromorphic function  $\varphi \in \mathcal{S}$ . The structure of  $G$  implies that all state nodes are the end-node of a U-rooted path, so there exists a finite  $k$  such that  $u_1 \in T^k(S)$ . Since  $\varphi^{(k)} \in T^k(S)$ , this implies that  $\varphi$  cannot be an autonomous element.  $\square$

We now have all the ingredients for proving our main result:

**Proposition 2.**  *$\mathcal{D}(G)$  is structurally accessible if and only if each state node is the end-node of a path that starts in U.*

*Proof.*

( $\Leftarrow$ ) By contradiction. If there is a state node  $x_i$  that is not the end-node of any U-rooted path, then  $x_i$  itself is an autonomous element.

( $\Rightarrow$ ) Since each state node is the end-node of a U-rooted path, note we can always obtain a spanning subgraph  $\tilde{G}$  of  $G$  such that: (i) it is a disjoint union of (connected) directed trees rooted at U; (ii) each state node has a single incoming edge (Fig. 2b). By applying Lemma 3 to each tree, we conclude that the class  $\mathcal{D}(G)$  is structurally accessible.  $\square$

**Remark 1.** *Note that in the trivial cases of an empty graph (i.e., a graph without nodes) or a graph without state nodes (i.e., the underlying system has no dynamics), applying Definition 4 yields that both graphs are structurally accessible because the set of autonomous element is empty.*

**Remark 2.** *Note that restricting the system dynamics of Eq. (5) to be affine in the control input changes the graph conditions for structural accessibility. In such case, graphs that contains “pure dilations of the control input” as in Fig. 1a are not structurally accessible because those subgraphs only admit linear dynamics*

**Remark 3.** *Note that the conditions for linear structural controllability and for structural accessibility are identical when each node of the graph has a self-loop. Indeed, in such a case, one can use all those self-loops to construct a cycle family that covers all state nodes.*

## V. PROOF OF THE STRUCTURAL OBSERVABILITY THEOREM

We start with the following observation, which is a direct analogy of Lemma 1 and Lemma 2:

**Lemma 4.**

- (i) *If  $\mathcal{D}(G)$  contains at least one observable system, then this class is structurally observable.*
- (ii) *Let  $\tilde{G} \subseteq G$  be any spanning subgraph of  $G$ . If  $\mathcal{D}(\tilde{G})$  is structurally observable then  $\mathcal{D}(G)$  is also structurally observable.*

*Proof.* A proof for item (i) follows using the exact same argument as in the proof of Lemma 1. Similarly, item (ii) follows using the same argument as in the proof of Lemma 2.  $\square$

We next prove the structural observability of a special class of graphs. Similar to the proof of Lemma 3, we prove this result by leveraging Lemma 4-(i) and exhibiting one observable system in this class.

**Lemma 5.** *Suppose that  $G(X \cup Y, A \cup C)$  is a (connected) directed tree topped at a single output node  $y$ , with each state node having a single outgoing edge. Then  $\mathcal{D}(G)$  is structurally observable.*

*Proof.* From the structure of the graph we can order its nodes by layers, where nodes with distance  $k$  to the output  $y$  belong

to the  $k$ -th layer (Fig. 4a). We will prove the claim by induction in the number of layers:

- (i) For one layer, denote its nodes by  $\{x_1, \dots, x_{d_1}\}$  where  $d_1$  is the number of nodes. One particular dynamics admissible for this graph is

$$\dot{x}_i = c_i, \quad i = 1, \dots, d_1; \quad y = x_1 x_2 \cdots x_{d_1}, \quad (9)$$

with  $c_i$  some non-zero constants. In the following we show that Eq. (9) is observable by proving that the span of  $dy$  and its derivatives  $dy^{(k)}$  equals  $\text{span}_{\mathcal{K}} dx$ . If  $d_1 = 1$  the claim follows directly, because there is only one state variable  $x_1$  and  $y = x_1$  renders it observable. Consider now that  $d_1 > 1$ . By direct calculation we obtain

$$\dot{y} = \frac{d}{dt} \left( \prod_{i=1}^{d_1} x_i \right) = \left( \prod_{i=1}^{d_1} x_i \right) \left( \sum_{i=1}^{d_1} \frac{c_i}{x_i} \right) = (y) \left( \sum_{i=1}^{d_1} \frac{c_i}{x_i} \right)$$

The variable  $z := \dot{y}/y$  is observable from  $y$ . Therefore, the system of Eq. (9) will be observable if the span of  $dz$  and its derivatives  $dz^{(k)}$  equals  $\text{span}_{\mathcal{K}} dx$ . Note that

$$z^{(k)} = \sum_{i=1}^{d_1} (-1)^k k! \frac{c_i^{k+1}}{x_i^{k+1}},$$

so its differential is

$$dz^{(k)} = \sum_{i=1}^{d_1} (-1)^{k+1} (k+1)! \frac{c_i^{k+1}}{x_i^{k+2}} dx_i.$$

Taking  $k = d_1 + 1$ , the set  $\{dz, \dots, dz^{(d_1+1)}\}$  will contain the functions  $\{1/x_i^2, \dots, 1/x_i^{d_i+3}\}$ ,  $i = 1, \dots, d_1$ , whose span is  $\text{span}_{\mathcal{K}} dx$ . This proves that the system of Eq. (9) is observable, and thus that a graph  $G$  with one layer is locally observable.

- (ii) For the induction step, we show that if a graph  $G$  with  $L$  layers is structurally observable, then a graph with  $L+1$  layers is also structurally observable. By definition, the nodes in the  $(L+1)$ -th layer are only connected to nodes in the  $L$ -th layer. Furthermore, they are connected in the same way as nodes in the first layer are connected to the output node (Fig. 4a). Therefore, the argument in point (i) with  $y$  replaced by the corresponding node in the  $L$ -th layer implies that the nodes in the  $(L+1)$ -th layer are observable. This completes the proof.  $\square$

The final result follows by decomposing the graph into disjoint trees topped at the output nodes:

**Proposition 3.**  $\mathfrak{D}(G)$  is structurally observable if and only if each state node is the start-node of a path that ends in  $Y$ .

*Proof.*

- ( $\Leftarrow$ ) By contradiction. If there is a state node  $x_i$  that is not the start-node of any  $Y$ -topped path, then  $x_i$  itself is a hidden element.
- ( $\Rightarrow$ ) Since each state node is the start-node of a  $Y$ -topped path, note we can always obtain a spanning subgraph  $\tilde{G}$  of  $G$  such that: (i) it is a disjoint union of (connected) directed trees topped at  $Y$ ; (ii) each state node has a

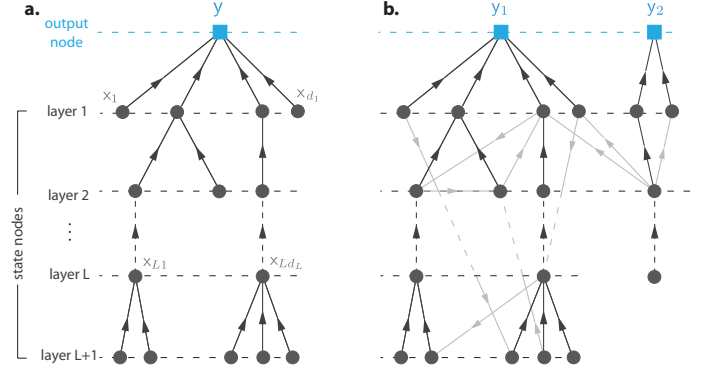


Fig. 4. (Color online) **a.** A tree graph  $G$  topped at  $y$  with a single outgoing edge per state node. **b.** From any graph  $G$  such that each node has a path to  $y$  (dark and light edges), a subgraph  $\tilde{G}$  (dark edges) can be obtained such that it is a tree topped at  $y$  and each state node has a single outgoing edge.

single outgoing edge (Fig. 4b). By Lemma 5,  $\mathfrak{D}(\tilde{G})$  is structurally observable. Since  $\tilde{G} \subseteq G$  is a spanning subgraph, Lemma 4-(ii) implies that  $\mathfrak{D}(G)$  is structurally observable.  $\square$

**Remark 4.** In analogy to Remark 1, in the trivial cases of an empty graph (i.e., a graph without nodes) or a graph without state nodes (i.e., the underlying system has no dynamics), applying Definition 4 yields that both graphs are structurally observable because the set of hidden elements is empty.

## VI. DISCUSSION AND CONCLUDING REMARKS

The notions of structural accessibility and structural observability that we have introduced and characterized are nonlinear counterparts of the notions of linear structural controllability and linear structural observability.

A limitation of our analysis is that it assumes that the system can be any arbitrary meromorphic dynamics, which may be unreasonable in some cases. More detailed predictions for the impact of the network structure on the controllability or observability properties can be obtained when the class of dynamics that the system can take is better known—such as neuronal, ecological, gene regulatory, or epidemic systems, see, e.g., [13]. Such an analysis will provide graph conditions for structural accessibility and structural observability for each specific class of system dynamics, which are potentially different from those of Theorem 1 and Theorem 2. Indeed, note that the conditions of Theorem 2 are always necessary, but they may not be sufficient when we restrict the system dynamics to belong to a special class. For example, when we restrict the class to linear dynamics, Theorem 1 shows that those conditions are not sufficient. Additionally, in [14] and [15], we analyzed the structural accessibility and structural observability for a class of nonlinear dynamics found in ecosystems. The considered class of ecosystem dynamics does not include linear dynamics, as it is assumed that ecological dynamics are always written in terms of the per-capita growth rate  $\dot{x}_i/x_i$ . Our analysis shows that the condition of Theorem 2 is again necessary but not sufficient for ecological dynamics. We emphasize that the class of ecosystem dynamics considered

in our previous work [14, 15] is significantly smaller than the class of meromorphic dynamics considered in this paper. Specifically, the considered class of ecological dynamics do not include linear, neuronal (i.e., Wilson-Cowan dynamics), and gene regulatory (i.e., Michelis-Menten) dynamics. We also point out that a more detailed topological analysis could better precise in which sense accessible (resp. observable) systems are “generic” in a structurally accessible class (resp. structurally observable class). This analysis would help us understand the type of “deformations” that a system dynamics needs to make it accessible (or observable). Such a detailed topological analysis is out of the scope for this paper, but it merits further study.

Finally, our results provide a broader perspective of what we can deduce about the controllability or observability properties of a system from knowing only its interconnection network. We have shown that if the control inputs can reach all state nodes through a path in the network, then almost all admissible system dynamics are accessible. Similarly, if all state nodes can reach an output through a path in the network, then almost all admissible system dynamics are locally observable. These two facts suggest that the interconnection network by itself only encodes the essential information of the controllability and observability properties of complex systems.

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