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# Asymptotic Analysis for Periodic Perforated Shells 

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#### Abstract

We consider a perforated half-cylindrical thin shell and investigate the limit behavior when the period and the thickness simultaneously go to zero. By using the decomposition of shell displacements presented in [3] we obtain a priori estimates. With the unfolding and rescaling operator we transform the problem to a reference configuration. In the end this yields a homogenized limit problem for the shell.


## Résumé

Nous étudions une coque mince demi-cylindrique perforée et donnons le comportement asymptotique lorsque la période et l'épaisseur tendent simultanément vers zéro. En utilisant la décomposi-tion des déplacements d'une coque, introduite dans [3], nous obtenons des estimations a priori. A l'aide d'un opérateur d'éclatement et de réduction de dimension, nous transposons le problème initial en un problème posé dans la configuration de référence. Finalement, le problème limite homogénéisé est obtenu.

Keywords: Homogenization; Dimension reduction; Linear Elasticity; Shell Theory; Perforated domains
2010 MSC: 35B27, 74Q05, 74K25, 74B05

## 1. Introduction

We consider a thin heterogeneous half-cylindrical shell with an in-plane periodic porous structure, whereby the periodicity $\varepsilon$ is of the same order as the shell's thickness $2 \delta$ and small compared to its in-plane surface size. This paper provides an analysis for homogenization and dimension reduction of the shell. We want to point out that both tasks are performed simultaneously, where $\lim _{(\varepsilon, \delta) \rightarrow(0,0)} \frac{\delta}{\varepsilon} \rightarrow$ $\kappa \in(0, \infty)$. This is necessary since homogenization and dimension reduction usually do not commute as it was shown e.g. in [5]. The presented approach via the rescaling-unfolding operator is closely related to the one given in [2, Chapter 11] for plates and for heterogeneous beams in [4], but new in the context of a linear elastic shell. There are various different homogenization techniques, as for example asymptotic expansions presented in [18, 20], via Gamma-convergence in [19] and the two-scale convergence introduced in [21. Although, the homogenization of plates and shells is in focus of interest of some other well-known research groups, our approach provides all the estimates and gives the limit not in terms of energy bounds, but yields a computational tool for the effective shell coefficients on its exact topology, which is important for applications.
Dimension reduction and homogenization of elastic plates via an asymptotic expansion technique can be found in [7]. Dimension reduction and homogenization of a shell for the diffusion problem in the sense of two-scale convergence was presented in [8, where it was shown that the curvature does

[^0]not enter the homogenized model. Moreover, the homogenization for piezoelectric perforated shells without dimension reduction were presented in [9]. We want to mention, that the dimension reduction of a homogeneous shell was analyzed in [13, 17, 6]. For some notions of classical results in functional analysis we refer to [11, 12].
In our analysis, we begin with a general extension technique (based on results developed in [3]) for displacements acting on a perforated shell made of a network of thin cylinders to the full shell domain (see Proposition 1.1). The result is crucial for the following analysis. We assume that the shell is fixed along the lateral boundary and continue with a decomposition approach for thin structures introduced in [3]. This decomposition technique allows to represent any $H^{1}$-displacement of the shell through the displacement of its mid-surface, the rotations of the small segments orthogonal to the mid-surface and a warping term which takes into account the deformation of these small segments. This leads to Korn inequalities and estimates for each term of the decomposition.
In Sec. 2-4, the rescaling and unfolding operators are introduced and the strain tensor is considered on a reference domain. In the subsequent analysis we decompose the shell's displacement fields in the two orthogonal complements of extensional and in-extensional displacements, introduced in Sec. 5 . Such an approach has been considered for homogeneous thin shells in [6].
Sec. 6.1 presents assumptions on forces in the right-hand side, rescaling them in a detailed manner.
At the end the limit problem is discussed. Especially Sec. 8.1 is important for applications, where the variational problem for an anisotropic homogenized shell is presented together with an expression to compute its effective coefficients via 6 auxiliary cell problems. We note, that the anisotropic coefficient tensors coincide with those obtained in the homogenization of a plate in [2, Chapter 11]. In section 9 we focus on the effects of the boundary conditions in our model, which play an important role. Especially, if we fix the shell's curved ends we obtain a membrane dominated limit equation. In that case clamping the lateral boundary does not change the model.

### 1.1. Geometrical setting

We consider a cylindrical half-shell with constant radius $a$. We assume that our shell consists of a periodic structure with a periodicity cell of size $\varepsilon$ in its mid-plane, and is of thickness $2 \delta$, with $\delta=\kappa \varepsilon \in\left(0, \delta_{0}\right], \delta_{0}=a / 3$, where $\kappa$ is a strictly positive fixed constant.
Let $Y^{\prime}$ be a bounded domain in $\mathbb{R}^{2}$ having the paving property with respect to an additive subgroup $\mathbf{G} \doteq \mathbf{p}_{1} \mathbb{Z} \oplus \mathbf{p}_{2} \mathbb{Z}$ of $\mathbb{R}^{2}$ of dimension 2 and let $T$ be an open set such that $\bar{T} \subset Y^{\prime}$ (Figure 1.1 gives an example of such a cell $Y^{\prime}$ ). We assume the boundary of $T$ to be Lipschitz and for simplicity we also assume $T$ connected. Denote

$$
\omega \doteq(0, a \pi) \times(0, l), \quad Y \doteq Y^{\prime} \times(-\kappa, \kappa), \quad Y^{*} \doteq Y^{\prime} \backslash \bar{T}, \quad Y^{*} \doteq Y^{\prime *} \times(-\kappa, \kappa)
$$



Figure 1: Cell $Y^{\prime}$ and the perforated domain $Y^{*}$
In the periodic setting a.e. $s^{\prime} \in \mathbb{R}^{2}$ can be decomposed as

$$
\begin{equation*}
s^{\prime}=\varepsilon\left[\frac{s^{\prime}}{\varepsilon}\right]_{Y^{\prime}}+\varepsilon\left\{\frac{s^{\prime}}{\varepsilon}\right\}_{Y^{\prime}} \tag{1.1}
\end{equation*}
$$

where $[\cdot]_{Y^{\prime}}$ belongs to $\mathbf{G}$ and $\{\cdot\}_{Y^{\prime}}$ to $Y^{\prime}$.
Set

$$
\Xi_{\varepsilon}=\left\{\xi \in \mathbf{G} \mid \varepsilon \xi+\varepsilon Y^{\prime} \subset \omega\right\}, \quad \widehat{\omega}_{\varepsilon}=\operatorname{interior}\left\{\bigcup_{\xi \in \Xi_{\varepsilon}}\left(\varepsilon \xi+\varepsilon \overline{Y^{\prime}}\right)\right\}, \quad \Lambda_{\varepsilon}=\omega \backslash \overline{\widehat{\omega}_{\varepsilon}} .
$$

Let us also introduce some notations for the unions of all holes

$$
T_{\varepsilon} \doteq\left\{x \in \widehat{\omega}_{\varepsilon} \left\lvert\,\left\{\frac{x}{\varepsilon}\right\}_{Y^{\prime}} \in T\right.\right\}, \quad \omega_{\varepsilon}^{*}=\omega \backslash \overline{T_{\varepsilon}}, \quad \widehat{\omega}_{\varepsilon}^{*}=\widehat{\omega}_{\varepsilon} \backslash \overline{T_{\varepsilon}}
$$

Consider the injective mapping $\phi: \bar{\omega} \rightarrow \mathbb{R}^{3}$ defined as

$$
\phi\left(s_{1}, s_{2}\right)=\left(\begin{array}{c}
s_{2}  \tag{1.2}\\
a \cos \left(\frac{s_{1}}{a}\right) \\
a \sin \left(\frac{s_{1}}{a}\right)
\end{array}\right), \quad\left(s_{1}, s_{2}\right) \in \bar{\omega}
$$

and denote by $S=\phi(\bar{\omega})$ the mid-surface of the whole shell (without the holes). Furthermore, we introduce the vectors

$$
\mathbf{t}_{1}=\left(\begin{array}{c}
0  \tag{1.3}\\
-\sin \left(\frac{s_{1}}{a}\right) \\
\cos \left(\frac{s_{1}}{a}\right)
\end{array}\right), \quad \mathbf{t}_{2}=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \quad \mathbf{n}=\frac{\mathbf{t}_{1} \wedge \mathbf{t}_{2}}{\left\|\mathbf{t}_{1} \wedge \mathbf{t}_{2}\right\|_{2}}=\left(\begin{array}{c}
0 \\
\cos \left(\frac{s_{1}}{a}\right) \\
\sin \left(\frac{s_{1}}{a}\right)
\end{array}\right)
$$

Obviously, $\mathbf{t}_{1}$ and $\mathbf{t}_{2}$ are linearly independent and are tangential vectors to the surface $S$. Denote

- $\Omega_{\varepsilon}=\omega \times(-\kappa \varepsilon, \kappa \varepsilon), \Omega_{\varepsilon}^{*}=\omega_{\varepsilon}^{*} \times(-\kappa \varepsilon, \kappa \varepsilon)$,
- $\mathcal{Q}_{\varepsilon}^{*}=\Phi\left(\Omega_{\varepsilon}^{*}\right)$ the perforated shell,
- $\mathcal{Q}_{\varepsilon}=\Phi\left(\Omega_{\varepsilon}\right)$ the shell without the holes
where $\Phi: \bar{\Omega}_{\varepsilon} \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is given by

$$
\begin{equation*}
\Phi(s)=\phi\left(s_{1}, s_{2}\right)+s_{3} \mathbf{n}\left(s_{1}, s_{2}\right), \quad s=\left(s_{1}, s_{2}, s_{3}\right) \in \bar{\Omega}_{\varepsilon} \tag{1.4}
\end{equation*}
$$



Figure 2: Periodic perforated plane domain transformed to a periodic shell
We easily check that if $\delta=\kappa \varepsilon \in\left(0, \delta_{0}\right]$ the map $\Phi$ from $\bar{\Omega}_{\varepsilon}$ onto $\overline{\mathcal{Q}}_{\varepsilon}$ is a $C^{1}$-diffeomorphism. That means we have

$$
\begin{equation*}
c_{0} \leq\left\|\nabla_{s} \Phi\right\|_{L^{\infty}\left(\Omega_{\varepsilon}\right)} \leq c_{1} \quad \text { and } \quad c_{0} \leq\left\|\nabla_{x} \Phi^{-1}\right\|_{L^{\infty}\left(\mathcal{Q}_{\varepsilon}\right)} \leq c_{1} \tag{1.5}
\end{equation*}
$$

The constants do not depend on $\varepsilon$.
We denote by $x$ the running point of the shell while $s$, s.t. $\Phi(s)=x$, is the running point in the reference domain. A function $u$ defined on $\mathcal{Q}_{\varepsilon}$ (resp. $\mathcal{Q}_{\varepsilon}^{*}$ ) can also be considered as a function defined on $\Omega_{\varepsilon}$ (resp. $\Omega_{\varepsilon}^{*}$ ) which we also denote by $u$.
Proposition 1.1. There exists an extension operator $\mathcal{P}_{\varepsilon}$ from $H^{1}\left(\mathcal{Q}_{\varepsilon}^{*}\right)^{3}$ into $H^{1}\left(\mathcal{Q}_{\varepsilon}\right)^{3}$ satisfying for all $u \in H^{1}\left(\mathcal{Q}_{\varepsilon}^{*}\right)^{3}$

$$
\begin{equation*}
\mathcal{P}_{\varepsilon}(u)_{\mid \mathcal{Q}_{\varepsilon}^{*}}=u, \quad\left\|e\left(\mathcal{P}_{\varepsilon}(u)\right)\right\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}\right)} \leq C\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{1.6}
\end{equation*}
$$

The constant does not depend on $\varepsilon$.
The proof of Proposition 1.1 has been moved to the Appendix.
Set $\gamma_{0}=\{0\} \times[0, l] \cup\{\pi\} \times[0, l] \subset \partial \omega$. The part $\Gamma_{0, \varepsilon}=\Phi\left(\gamma_{0} \times(-\kappa \varepsilon, \kappa \varepsilon)\right)$ of the lateral boundary of the shell is clamped. The complementary of $\Gamma_{0, \varepsilon}$ in the lateral boundary of the shell is a free boundary. From now on, any displacement $u$ belonging to $H^{1}\left(\mathcal{Q}_{\varepsilon}^{*}\right)^{3}$ will be extended to a displacement belonging to $H^{1}\left(\mathcal{Q}_{\varepsilon}\right)^{3}$. We will always denote by $u$ the extended displacement, which will satisfy (1.6). This displacement (still denoted $u$ ) could also be considered as an element of $H^{1}\left(\Omega_{\varepsilon}^{*}\right)^{3}$ or $H^{1}\left(\Omega_{\varepsilon}\right)^{3}$.

### 1.2. Decomposition of shell displacements

In this section we introduce a decomposition for every displacement $u$ of the shell $\mathcal{Q}_{\varepsilon}^{*}$ as it was shown in [3].
Definition 1.1. An elementary displacement $U_{e}$ associated to $u \in H^{1}\left(\Omega_{\varepsilon}\right)^{3}$ is given by

$$
\begin{equation*}
U_{e}=\mathcal{U}\left(s_{1}, s_{2}\right)+s_{3} \mathcal{R}\left(s_{1}, s_{2}\right) \tag{1.7}
\end{equation*}
$$

where $(\alpha \in\{1,2\})$

$$
\begin{equation*}
\mathcal{U}=\frac{1}{2 \kappa \varepsilon} \int_{-\kappa \varepsilon}^{\kappa \varepsilon} u\left(\cdot, s_{3}\right) d s_{3}, \quad \mathcal{R}_{\alpha}=\frac{3}{2(\kappa \varepsilon)^{3}} \int_{-\kappa \varepsilon}^{\kappa \varepsilon} s_{3} u\left(\cdot, s_{3}\right) \cdot \boldsymbol{t}_{\alpha} d s_{3}, \quad \mathcal{R}_{3}=0 \quad \text { a.e. in } \omega . \tag{1.8}
\end{equation*}
$$

Moreover, we have that $\mathcal{U}=\left(\mathcal{U}_{1}, \mathcal{U}_{2}, \mathcal{U}_{3}\right) \in H^{1}(\omega)^{3}$ and $\mathcal{R}=\left(\mathcal{R}_{1}, \mathcal{R}_{2}\right) \in H^{1}(\omega)^{2}$. Every displacement $u$ is then decomposed as

$$
\begin{equation*}
U_{e}\left(\cdot, s_{3}\right)=\sum_{\alpha=1}^{2}\left(\mathcal{U}_{\alpha}+s_{3} \mathcal{R}_{\alpha}\right) \boldsymbol{t}_{\alpha}+\mathcal{U}_{3} \boldsymbol{n}, \quad u=U_{e}+\bar{u} \tag{1.9}
\end{equation*}
$$

where $\bar{u} \in H^{1}\left(\Omega_{\varepsilon}\right)^{3}$ is a residual displacement called warping.
Denote

$$
\begin{aligned}
& V_{\varepsilon} \doteq\left\{v \in H^{1}\left(\mathcal{Q}_{\varepsilon}\right)^{3} \mid v=0 \text { on } \Gamma_{0, \varepsilon}\right\}, \quad V_{\varepsilon}^{*} \doteq\left\{v \in H^{1}\left(\mathcal{Q}_{\varepsilon}^{*}\right)^{3} \mid v=0 \text { on } \Gamma_{0, \varepsilon}\right\} \\
& H_{\Gamma_{0}}^{1}(\omega) \doteq\left\{\Phi \in H^{1}(\omega) \mid \Phi=0 \text { on } \Gamma_{0}\right\}
\end{aligned}
$$

One has

$$
\mathcal{U} \in H_{\Gamma_{0}}^{1}(\omega)^{3}, \quad \mathcal{R} \in H_{\Gamma_{0}}^{1}(\omega)^{2}, \quad \bar{u} \in V_{\varepsilon}
$$

Remark 1.1. The warping $\bar{u}$ fulfills the following properties

$$
\begin{equation*}
\int_{-\kappa \varepsilon}^{\kappa \varepsilon} \bar{u}\left(\cdot, s_{3}\right) d s_{3}=0, \quad \int_{-\kappa \varepsilon}^{\kappa \varepsilon} s_{3} \bar{u}\left(\cdot, s_{3}\right) \cdot \boldsymbol{t}_{\alpha} d s_{3}=0 \tag{1.10}
\end{equation*}
$$

For $\mathcal{U}$ and $\mathcal{R}$ holds

$$
\mathcal{U}=\mathcal{U}_{1} \mathbf{t}_{1}+\mathcal{U}_{2} \mathbf{t}_{2}+\mathcal{U}_{3} \mathbf{n}, \quad \mathcal{R}=\mathcal{R}_{1} \mathbf{t}_{1}+\mathcal{R}_{2} \mathbf{t}_{2} .
$$

In the next step we want to establish the strain tensor in the cylindrical coordinates. The derivatives of the elementary displacement $U_{e}$ are calculated using

$$
\begin{align*}
\frac{\partial \mathcal{U}}{\partial s_{1}} & =\frac{\partial \mathcal{U}_{1}}{\partial s_{1}} \mathbf{t}_{1}-\frac{1}{a} \mathcal{U}_{1} \mathbf{n}+\frac{\partial \mathcal{U}_{2}}{\partial s_{1}} \mathbf{t}_{2}+\frac{\partial \mathcal{U}_{3}}{\partial s_{1}} \mathbf{n}+\frac{1}{a} \mathcal{U}_{3} \mathbf{t}_{1}, \\
\frac{\partial \mathcal{U}}{\partial s_{2}} & =\frac{\partial \mathcal{U}_{1}}{\partial s_{2}} \mathbf{t}_{1}+\frac{\partial \mathcal{U}_{2}}{\partial s_{2}} \mathbf{t}_{2}+\frac{\partial \mathcal{U}_{3}}{\partial s_{2}} \mathbf{n} \tag{1.11}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{\partial \mathcal{R}}{\partial s_{1}}=\frac{\partial \mathcal{R}_{1}}{\partial s_{1}} \mathbf{t}_{1}-\frac{1}{a} \mathcal{R}_{1} \mathbf{n}+\frac{\partial \mathcal{R}_{2}}{\partial s_{1}} \mathbf{t}_{2}  \tag{1.12}\\
& \frac{\partial \mathcal{R}}{\partial s_{2}}=\frac{\partial \mathcal{R}_{1}}{\partial s_{2}} \mathbf{t}_{1}+\frac{\partial \mathcal{R}_{2}}{\partial s_{2}} \mathbf{t}_{2}
\end{align*}
$$

The strain tensor for a shell displacement $u \in H^{1}\left(\mathcal{Q}_{\varepsilon}\right)$ is given by

$$
\begin{equation*}
e_{x}(u)=\frac{\nabla_{x} u+\left(\nabla_{x} u\right)^{T}}{2} \tag{1.13}
\end{equation*}
$$

A small computation yields, that $\nabla_{s}$ in the coordinates of the reference domain is given by

$$
\begin{equation*}
\nabla_{s}=\nabla_{x} \nabla \Phi \tag{1.14}
\end{equation*}
$$

Furthermore, we still have that $e_{x}(u)$ is in the shell configuration. Therefore, we consider the transformation matrix $\left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right)$ and transfer our strain matrix into the reference domain by

$$
\begin{equation*}
\left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right)^{T} e_{x}(u)\left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right) \tag{1.15}
\end{equation*}
$$

Definition 1.2. We define by $e(u)$ the strain tensor in the coordinates of the reference domain by

$$
\begin{equation*}
e(u)=\left(\boldsymbol{t}_{1}\left|\boldsymbol{t}_{2}\right| \boldsymbol{n}\right)^{T} \frac{\nabla_{s} u(\nabla \Phi)^{-1}+\left(\nabla_{s} u(\nabla \Phi)^{-1}\right)^{T}}{2}\left(\boldsymbol{t}_{1}\left|\boldsymbol{t}_{2}\right| \boldsymbol{n}\right) . \tag{1.16}
\end{equation*}
$$

Hence, we obtain

$$
\begin{align*}
\left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right)^{T} \nabla_{s} u(\nabla \Phi)^{-1}\left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right)= & \left(\mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right)^{T} \nabla_{s} u\left(\begin{array}{ccc}
\frac{a}{a+s_{3}} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \\
& =\left(\begin{array}{ccc}
\frac{a}{a+s_{3}} \frac{\partial u}{\partial s_{1}} \mathbf{t}_{1} & \frac{\partial u}{\partial s_{2}} \mathbf{t}_{1} & \frac{\partial u}{\partial s_{3}} \mathbf{t}_{1} \\
\frac{a}{a+s_{3}} \frac{\partial u}{\partial s_{1}} \mathbf{t}_{2} & \frac{\partial u}{\partial s_{2}} \mathbf{t}_{2} & \frac{\partial u}{\partial s_{3}} \mathbf{t}_{2} \\
\frac{a}{a+s_{3}} \frac{\partial u}{\partial s_{1}} \mathbf{n} & \frac{\partial u}{\partial s_{2}} \mathbf{n} & \frac{\partial u}{\partial s_{3}} \mathbf{n}
\end{array}\right), \tag{1.17}
\end{align*}
$$

where

$$
\begin{aligned}
\frac{\partial u}{\partial s_{1}} & =\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+s_{3} \frac{\partial \mathcal{R}_{1}}{\partial s_{1}}+\frac{\partial \bar{u}_{1}}{\partial s_{1}}+\frac{1}{a}\left(\mathcal{U}_{3}+\bar{u}_{3}\right)\right) \mathbf{t}_{1}+\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+s_{3} \frac{\partial \mathcal{R}_{2}}{\partial s_{1}}+\frac{\partial \bar{u}_{2}}{\partial s_{1}}\right) \mathbf{t}_{2} \\
& +\left(\frac{\partial \mathcal{U}_{3}}{\partial s_{1}}+\frac{\partial \bar{u}_{3}}{\partial s_{1}}-\frac{1}{a}\left(\mathcal{U}_{1}+s_{3} \mathcal{R}_{1}+\bar{u}_{1}\right)\right) \mathbf{n}, \\
\frac{\partial u}{\partial s_{2}} & =\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+s_{3} \frac{\partial \mathcal{R}_{1}}{\partial s_{2}}+\frac{\partial \bar{u}_{1}}{\partial s_{2}}\right) \mathbf{t}_{1}+\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{2}}+s_{3} \frac{\partial \mathcal{R}_{2}}{\partial s_{2}}+\frac{\partial \bar{u}_{2}}{\partial s_{2}}\right) \mathbf{t}_{2}+\left(\frac{\partial \mathcal{U}_{3}}{\partial s_{2}}+\frac{\partial \bar{u}_{3}}{\partial s_{2}}\right) \mathbf{n}, \\
\frac{\partial u}{\partial s_{3}} & =\left(\mathcal{R}_{1}+\frac{\partial \bar{u}_{1}}{\partial s_{3}}\right) \mathbf{t}_{1}+\left(\mathcal{R}_{2}+\frac{\partial \bar{u}_{2}}{\partial s_{3}}\right) \mathbf{t}_{2}+\frac{\partial \bar{u}_{3}}{\partial s_{3}} \mathbf{n} .
\end{aligned}
$$

We get for the strain tensor $e(u)$ of a displacement $u \in V_{\varepsilon}$ the following components:

$$
\begin{aligned}
e_{11}(u) & =\frac{a}{a+s_{3}}\left[\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right)+s_{3} \frac{\partial \mathcal{R}_{1}}{\partial s_{1}}+\frac{\partial \bar{u}_{1}}{\partial s_{1}}+\frac{1}{a} \bar{u}_{3}\right] \\
e_{22}(u) & =\frac{\partial \mathcal{U}_{2}}{\partial s_{2}}+s_{3} \frac{\partial \mathcal{R}_{2}}{\partial s_{2}}+\frac{\partial \bar{u}_{2}}{\partial s_{2}}, \\
e_{12}(u) & =\frac{1}{2} \frac{a}{a+s_{3}}\left[\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right)+s_{3}\left(\frac{\partial \mathcal{R}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{R}_{1}}{\partial s_{2}}\right)+\frac{s_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+\frac{s_{3}^{2}}{a} \frac{\partial \mathcal{R}_{1}}{\partial s_{2}}\right. \\
& \left.+\frac{\partial \bar{u}_{2}}{\partial s_{1}}+\left(1+\frac{s_{3}}{a}\right) \frac{\partial \bar{u}_{1}}{\partial s_{2}}\right], \\
e_{13}(u) & =\frac{1}{2} \frac{a}{a+s_{3}}\left[\left(\frac{\partial \mathcal{U}_{3}}{\partial s_{1}}-\frac{1}{a} \mathcal{U}_{1}+\mathcal{R}_{1}\right)-\frac{1}{a} \bar{u}_{1}+\frac{\partial \bar{u}_{3}}{\partial s_{1}}+\left(1+\frac{s_{3}}{a}\right) \frac{\partial \bar{u}_{1}}{\partial s_{3}}\right] \\
e_{23}(u) & =\frac{1}{2}\left\{\left(\frac{\partial \mathcal{U}_{3}}{\partial s_{2}}+\mathcal{R}_{2}\right)+\frac{\partial \bar{u}_{3}}{\partial s_{2}}+\frac{\partial \bar{u}_{2}}{\partial s_{3}}\right\}, \quad e_{33}(u)=\frac{\partial \bar{u}_{3}}{\partial s_{3}} .
\end{aligned}
$$

Theorem 1.1. Let $u \in H^{1}\left(\mathcal{Q}_{\varepsilon}^{*}\right)^{3}$ and $(\mathcal{U}, \mathcal{R}, \bar{u})$ be the terms of its decomposition, then the following inequalities are satisfied:

$$
\begin{gather*}
\left\|e\left(U_{e}\right)\right\|_{L^{2}\left(\Omega_{\varepsilon}\right)} \leq C\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}  \tag{1.18}\\
\|\bar{u}\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}\right)} \leq C \varepsilon\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}  \tag{1.19}\\
\|\nabla \bar{u}\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}\right)} \leq C\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{1.20}
\end{gather*}
$$

Proof. The proof is given in [3, Theorem 4.1].
From [3] we also obtain the full estimates of $u$ and the components of the elementary displacement $U_{e}$.
Proposition 1.2. For every $u \in V_{\varepsilon}^{*}$

$$
\begin{equation*}
\|u\|_{H^{1}\left(\Omega_{\varepsilon}\right)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \quad\|\mathcal{R}\|_{H^{1}(\omega)}+\|\mathcal{U}\|_{H^{1}(\omega)} \leq \frac{C}{\varepsilon^{3 / 2}}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{1.21}
\end{equation*}
$$

The constants do not depend on $\varepsilon$.
From the expression of the strain tensor $e(u)$ one derives the following estimates:
Lemma 1.1. One has also the following estimates $\left((\alpha, \beta) \in\{1,2\}^{2}\right)$ :

$$
\begin{equation*}
\left\|\frac{\partial \mathcal{U}}{\partial s_{\alpha}} \cdot \boldsymbol{t}_{\beta}+\frac{\partial \mathcal{U}}{\partial s_{\beta}} \cdot \boldsymbol{t}_{\alpha}\right\|_{L^{2}(\omega)}+\left\|\frac{\partial \mathcal{U}}{\partial s_{\alpha}} \cdot \boldsymbol{n}+\mathcal{R} \cdot \boldsymbol{t}_{\alpha}\right\|_{L^{2}(\omega)} \leq \frac{C}{\varepsilon^{1 / 2}}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{1.22}
\end{equation*}
$$

The constant does not depend on $\varepsilon$.
Proof. We will only show that

$$
\begin{equation*}
\left\|\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right\|_{L^{2}(\omega)} \leq \frac{C}{\varepsilon^{1 / 2}}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{1.23}
\end{equation*}
$$

since the other inequalities follow in the same way.
First observe that $\frac{a}{a+s_{3}}$ is uniformly bounded. Then, we start with the expression of $e_{12}(u)$ given by 1.18) Due to 1.19 and 1.20 we obtain

$$
\int_{\Omega_{\varepsilon}}\left[\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right)+s_{3}\left(\frac{\partial \mathcal{R}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{R}_{1}}{\partial s_{2}}\right)+\frac{s_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+\frac{s_{3}^{2}}{a} \frac{\partial \mathcal{R}_{1}}{\partial s_{2}}\right]^{2} d s \leq C\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}^{2}
$$

Hence, using the estimates (1.21)

$$
\varepsilon \int_{\omega}\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right)^{2} d s \leq C\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}^{2}
$$

which proves the inequality 1.23 .

## 2. The rescaling operator $\boldsymbol{T}_{\varepsilon}$

From now on we consider the reference domain

$$
\begin{equation*}
\Omega=\omega \times(-\kappa, \kappa) \tag{2.1}
\end{equation*}
$$

and we rescale the shell in its $s_{3}$ direction.
Definition 2.1. Given a measurable function $\Psi$ on $\Omega_{\varepsilon}$, we define the measurable function $\mathfrak{T}_{\varepsilon}(\Psi)$ on $\Omega$ as

$$
\begin{equation*}
\mathfrak{T}_{\varepsilon}(\Psi)\left(s_{1}, s_{2}, y_{3}\right)=\Psi\left(s_{1}, s_{2}, \varepsilon y_{3}\right), \quad \text { for a.e. }\left(s_{1}, s_{2}, y_{3}\right) \in \Omega \tag{2.2}
\end{equation*}
$$

Lemma 2.1. One has for every $\Psi \in L^{2}\left(\Omega_{\varepsilon}\right)$ and for the warping $\bar{u}$

$$
\begin{array}{lr}
\left\|\mathfrak{T}_{\varepsilon}(\Psi)\right\|_{L^{2}(\Omega)} \leq C \varepsilon^{-1 / 2}\|\Psi\|_{L^{2}\left(\Omega_{\varepsilon}\right)}, & \left\|\mathfrak{T}_{\varepsilon}(\bar{u})\right\|_{L^{2}(\Omega)} \leq C \varepsilon^{1 / 2}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \\
\left\|\frac{\partial \mathfrak{T}_{\varepsilon}(\bar{u})}{\partial s_{\alpha}}\right\|_{L^{2}(\Omega)} \leq C \varepsilon^{-1 / 2}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, & \left\|\frac{\partial \mathfrak{T}_{\varepsilon}(\bar{u})}{\partial y_{3}}\right\|_{L^{2}(\Omega)} \leq C \varepsilon^{1 / 2}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{2.3}
\end{array}
$$

## 3. Asymptotic behavior of the strain tensor

Lemma 3.1. Let $\left\{u_{\varepsilon}\right\}_{\varepsilon}$ be a sequence of displacements belonging to $V_{\varepsilon}^{*}$ and satisfying

$$
\left\|e\left(u_{\varepsilon}\right)\right\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \leq C \varepsilon^{3 / 2}
$$

with a constant independent of $\varepsilon$.
There exists a subsequence (still denoted $\varepsilon$ ) and $\mathcal{U} \in H_{\Gamma_{0}}^{1}(\omega)^{3}, \mathcal{R} \in H_{\Gamma_{0}}^{1}(\omega)^{2}, \mathcal{Z}_{\alpha \beta} \in L^{2}(\omega), \mathcal{Z}_{\alpha 3} \in$ $L^{2}(\omega)$ and $\bar{u} \in L^{2}\left(\omega ; H^{1}(-\kappa, \kappa)\right)^{3}$ satisfying

$$
\begin{equation*}
\int_{-\kappa}^{\kappa} \bar{u}\left(\cdot, y_{3}\right) d y_{3}=0, \quad \int_{-\kappa}^{\kappa} y_{3} \bar{u}_{\alpha}\left(\cdot, y_{3}\right) d y_{3}=0 \quad \text { a.e. in } \omega \tag{3.1}
\end{equation*}
$$

such that

$$
\begin{align*}
& \mathcal{U}_{\varepsilon} \longrightarrow \mathcal{U} \quad \text { strongly in } H_{\Gamma_{0}}^{1}(\omega)^{3}, \quad \mathcal{R}_{\varepsilon, \alpha} \rightharpoonup \mathcal{R}_{\alpha} \quad \text { weakly in } H_{\Gamma_{0}}^{1}(\omega), \\
& \frac{1}{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}} \cdot \boldsymbol{t}_{\beta}+\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\beta}} \cdot \boldsymbol{t}_{\alpha}\right) \rightharpoonup \mathcal{Z}_{\alpha \beta} \quad \text { weakly in } L^{2}(\omega), \\
& \frac{1}{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}} \cdot \boldsymbol{n}+\mathcal{R}_{\varepsilon} \cdot \boldsymbol{t}_{\alpha}\right) \rightharpoonup \mathcal{Z}_{\alpha 3} \quad \text { weakly in } L^{2}(\omega), \\
& \frac{1}{\varepsilon^{2}} \mathfrak{T}_{\varepsilon}\left(\bar{u}_{\varepsilon}\right) \rightharpoonup \bar{u} \quad \text { weakly in } L^{2}\left(\omega ; H^{1}(-\kappa, \kappa)\right)^{3},  \tag{3.2}\\
& \frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon}}{\partial s_{\alpha}}\right)=\frac{1}{\varepsilon} \frac{\partial}{\partial s_{\alpha}} \mathfrak{T}_{\varepsilon}\left(\bar{u}_{\varepsilon}\right) \rightharpoonup 0 \quad \text { weakly in } L^{2}(\omega \times(-\kappa, \kappa))^{3}, \\
& \frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(e\left(u_{\varepsilon}\right)\right) \rightharpoonup \mathcal{E}(\mathcal{U}, \mathcal{Z}, \bar{u}) \quad \text { weakly in } L^{2}(\omega)^{3 \times 3} .
\end{align*}
$$

Moreover, one has

$$
\frac{\partial \mathcal{U}_{3}}{\partial s_{1}}-\frac{1}{a} \mathcal{U}_{1}+\mathcal{R}_{1}=0, \quad \frac{\partial \mathcal{U}_{3}}{\partial s_{2}}+\mathcal{R}_{2}=0
$$

Proof. We start with the weak limits; they are the consequences of 1.21 .

$$
\begin{equation*}
\mathcal{U}_{\varepsilon} \rightharpoonup \mathcal{U} \quad \text { weakly in } H_{\Gamma_{0}}^{1}(\omega)^{3}, \quad \mathcal{R}_{\varepsilon} \rightharpoonup \mathcal{R} \quad \text { weakly in } H_{\Gamma_{0}}^{1}(\omega)^{2} \tag{3.3}
\end{equation*}
$$

[^1]
Both convergences (3.2) ${ }_{3,4}$ follow directly from Lemma 1.1 .
Now we prove
\[

$$
\begin{equation*}
\mathcal{U}_{\varepsilon, 3} \longrightarrow \mathcal{U}_{3} \quad \text { strongly in } H_{\Gamma_{0}}^{1}(\omega) \tag{3.4}
\end{equation*}
$$

\]

By the Sobolev embedding and convergences (3.3), one has

$$
\begin{equation*}
\mathcal{U}_{\varepsilon} \longrightarrow \mathcal{U} \quad \text { strongly in } L^{2}(\omega)^{3}, \quad \mathcal{R}_{\varepsilon} \longrightarrow \mathcal{R} \quad \text { strongly in } L^{2}(\omega)^{2} \tag{3.5}
\end{equation*}
$$

Besides, from estimate $11.22_{2}$, one obtains

$$
\begin{aligned}
& \frac{\partial \mathcal{U}_{\varepsilon, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}+\mathcal{R}_{\varepsilon, 1} \longrightarrow 0 \quad \text { strongly in } L^{2}(\omega) \\
& \frac{\partial \mathcal{U}_{\varepsilon, 3}}{\partial s_{2}}+\mathcal{R}_{\varepsilon, 2} \longrightarrow 0 \quad \text { strongly in } L^{2}(\omega)
\end{aligned}
$$

Hence $\nabla \mathcal{U}_{\varepsilon, 3}$ strongly converges to its limit in $L^{2}(\omega)^{2}$, which ends the proof of (3.4). That also proves the last equalities of the Lemma.
Now, prove the strong convergences

$$
\mathcal{U}_{\varepsilon, \alpha} \longrightarrow \mathcal{U}_{\alpha} \quad \text { strongly in } H_{\Gamma_{0}}^{1}(\omega), \quad \alpha=1,2
$$

By estimate 1.22$)_{1}$ one immediately has

$$
\begin{aligned}
& \frac{\partial \mathcal{U}_{\varepsilon, 1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{\varepsilon, 3} \longrightarrow 0 \quad \text { strongly in } L^{2}(\omega) \\
& \frac{\partial \mathcal{U}_{\varepsilon, 2}}{\partial s_{2}} \longrightarrow 0 \quad \text { strongly in } L^{2}(\omega) \\
& \frac{\partial \mathcal{U}_{\varepsilon, 1}}{\partial s_{2}}+\frac{\partial \mathcal{U}_{\varepsilon, 2}}{\partial s_{1}} \longrightarrow 0 \quad \text { strongly in } L^{2}(\omega)
\end{aligned}
$$

Furthermore, from (3.5) and the above strong convergences, one obtains the strong convergence of the strain tensor of the displacement $\left(\mathcal{U}_{\varepsilon, 1}, \mathcal{U}_{\varepsilon, 2}\right)$ in $L^{2}(\omega)^{3}$. Since $\omega$ is a Lipschitz domain, this displacement strongly converges to its limit in $H_{\Gamma_{0}}^{1}(\omega)^{2}$. The elements of the limit strain tensor are then particularly given by $(\alpha \in\{1,2\})$

$$
\begin{aligned}
\frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(e_{\alpha \alpha}\right) & \rightharpoonup \mathcal{Z}_{\alpha \alpha}+y_{3} \frac{\partial \mathcal{R}_{\alpha}}{\partial s_{\alpha}} \\
\frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(e_{12}\right) & \rightharpoonup \frac{1}{2}\left\{\mathcal{Z}_{12}+\frac{y_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+y_{3} \frac{\partial \mathcal{R}_{2}}{\partial s_{1}}+y_{3} \frac{\partial \mathcal{R}_{1}}{\partial s_{2}}\right\}, \\
\frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(e_{\alpha 3}\right) & \rightharpoonup \frac{1}{2}\left\{\mathcal{Z}_{\alpha 3}+\frac{\partial \bar{u}_{\alpha}}{\partial y_{3}}\right\} \\
\frac{1}{\varepsilon} \mathfrak{T}_{\varepsilon}\left(e_{33}\right) & \rightharpoonup \frac{\partial \bar{u}_{3}}{\partial y_{3}}
\end{aligned}
$$

Putting everything together we obtain the symmetric tensor

$$
\mathcal{E}(\mathcal{U}, \mathcal{Z}, \bar{u})=\left(\begin{array}{ccc}
\mathcal{Z}_{11}+\frac{y_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{1}}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{1}^{2}} & \frac{1}{2} \mathcal{Z}_{12}+\frac{y_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{2}}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{1} \partial s_{2}} & \frac{1}{2}\left(\mathcal{Z}_{13}+\frac{\partial \bar{u}_{1}}{\partial y_{3}}\right) \\
* & \mathcal{Z}_{22}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{2}^{2}} & \frac{1}{2}\left(\mathcal{Z}_{23}+\frac{\partial \bar{u}_{2}}{\partial y_{3}}\right) \\
* & * & \frac{\partial \bar{u}_{3}}{\partial y_{3}}
\end{array}\right)
$$

which ends the proof of the Lemma.

As a consequence of the estimates in Lemma 1.1 and the above Lemma, one has a.e. in $\omega$ with $\mathcal{U}_{i} \in H_{\Gamma_{0}}^{1}(\omega)$ and $\mathcal{R}_{\alpha} \in H_{\Gamma_{0}}^{1}(\omega)$

$$
\begin{equation*}
\frac{\partial \mathcal{U}}{\partial s_{\alpha}} \cdot \mathbf{t}_{\beta}+\frac{\partial \mathcal{U}}{\partial s_{\beta}} \cdot \mathbf{t}_{\alpha}=0, \quad \frac{\partial \mathcal{U}}{\partial s_{\alpha}} \cdot \mathbf{n}+\mathcal{R} \cdot \mathbf{t}_{\alpha}=0 \tag{3.6}
\end{equation*}
$$

From the first equation in (3.6) we obtain for $(\alpha, \beta)=(2,2)$ that $\frac{\partial \mathcal{U}_{2}}{\partial s_{2}}=0$. Hence $\mathcal{U}_{2}$ does not depend on $s_{2}, \mathcal{U}_{2}=U_{2}\left(s_{1}\right)$ and due to the boundary conditions, one has $U_{2} \in H_{0}^{1}(0, a \pi)$.
With that we conclude for $(\alpha, \beta)=(1,2)$ that

$$
\mathcal{U}_{1}\left(s_{1}, s_{2}\right)=-s_{2} \frac{d U_{2}}{d s_{1}}\left(s_{1}\right)+U_{1}\left(s_{1}\right)
$$

Since $\mathcal{U}_{1}$ belongs to $H_{\Gamma_{0}}^{1}(\omega)$, we get $U_{2} \in H_{0}^{2}(0, a \pi)$ and $U_{1} \in H_{0}^{1}(0, a \pi)$.
This yields for the last case, $(\alpha, \beta)=(1,1)$,

$$
\mathcal{U}_{3}\left(s_{1}, s_{2}\right)=a s_{2} \frac{d^{2} U_{2}}{d s_{1}^{2}}-a \frac{d U_{1}}{d s_{1}}\left(s_{1}\right)
$$

Since $\mathcal{U}_{3}$ belongs to $H_{\Gamma_{0}}^{1}(\omega)$, this implies at this step

$$
\begin{align*}
& \quad U_{2} \in H_{0}^{3}(0, a \pi), \quad U_{1} \in H_{0}^{2}(0, a \pi) \\
& \text { and } \mathcal{U}\left(s_{1}, s_{2}\right)=\left(-s_{2} \frac{d U_{2}}{d s_{1}}\left(s_{1}\right)+U_{1}\left(s_{1}\right), U_{2}\left(s_{1}\right), a s_{2} \frac{d^{2} U_{2}}{d s_{1}^{2}}-a \frac{d U_{1}}{d s_{1}}\left(s_{1}\right)\right) . \tag{3.7}
\end{align*}
$$

We now focus on the second equality given in (3.6), where we obtain with our expression for $\mathcal{U}$

$$
\mathcal{R}_{1}\left(s_{1}, s_{2}\right)=-s_{2}\left(\frac{1}{a} \frac{d U_{2}}{d s_{1}}\left(s_{1}\right)+a \frac{d^{3} U_{2}}{d s_{1}^{3}}\left(s_{1}\right)\right)+\frac{1}{a} U_{1}\left(s_{1}\right)+a \frac{d^{2} U_{1}}{d s_{1}^{2}}\left(s_{1}\right), \quad \mathcal{R}_{2}\left(s_{1}, s_{2}\right)=-a \frac{d^{2} U_{2}}{d s_{1}^{2}}\left(s_{1}\right)
$$

Observe that due to the above conditions on $U_{2}, \mathcal{R}_{2}$ belongs to $H_{\Gamma_{0}}^{1}(\omega)$. Now, since $\mathcal{R}_{1}$ also belongs to $H_{\Gamma_{0}}^{1}(\omega)$, we finally obtain

$$
U_{2} \in H_{0}^{4}(0, a \pi), \quad U_{1} \in H_{0}^{3}(0, a \pi)
$$

Thus

$$
\begin{aligned}
& \mathcal{R}_{1} \in H_{\Gamma_{0}}^{1}(\omega), \quad \mathcal{R}_{2} \in H^{2}(\omega) \cap H_{\Gamma_{0}}^{1}(\omega) \\
& \mathcal{U}_{1} \in H^{2}(\omega) \cap H_{\Gamma_{0}}^{1}(\omega), \quad \mathcal{U}_{2} \in H^{4}(\omega) \cap H_{\Gamma_{0}}^{1}(\omega), \quad \mathcal{U}_{3} \in H^{2}(\omega) \cap H_{\Gamma_{0}}^{1}(\omega) .
\end{aligned}
$$

## 4. Unfolding of the rescaled shell

Definition 4.1. The unfolding $\mathcal{T}_{\varepsilon}\left(\psi^{\prime}\right)\left(\right.$ resp. $\left.\mathcal{T}_{\varepsilon}(\psi)\right)$ of a measurable function defined on $\omega$ (resp. $\Omega$ ) is measurable on $\omega \times Y^{\prime}$ (resp. $\Omega \times Y^{\prime}$ ) and given by

$$
\begin{array}{ll}
\mathcal{T}_{\varepsilon}\left(\psi^{\prime}\right)\left(s^{\prime}, y^{\prime}\right)=\psi^{\prime}\left(\varepsilon\left[\frac{s^{\prime}}{\varepsilon}\right]_{Y^{\prime}}+\varepsilon y^{\prime}\right) & \text { for a.e. }\left(s^{\prime}, y^{\prime}\right) \in \widehat{\omega}_{\varepsilon} \times Y^{\prime} \\
\mathcal{T}_{\varepsilon}(\psi)\left(s^{\prime}, y^{\prime}\right)=0 & \text { for a.e. }\left(s^{\prime}, y^{\prime}\right) \in \Lambda_{\varepsilon} \times Y^{\prime}
\end{array}
$$

and

$$
\begin{array}{ll}
\mathcal{T}_{\varepsilon}(\psi)\left(s^{\prime}, y^{\prime}, y_{3}\right)=\psi\left(\varepsilon\left[\frac{s^{\prime}}{\varepsilon}\right]_{Y^{\prime}}+\varepsilon y^{\prime}, y_{3}\right) & \text { for a.e. }\left(s^{\prime}, y^{\prime}, y_{3}\right) \in \widehat{\omega}_{\varepsilon} \times Y \\
\mathcal{T}_{\varepsilon}(\psi)\left(s^{\prime}, y^{\prime}, y_{3}\right)=0 & \text { for a.e. }\left(s^{\prime}, y^{\prime}, y_{3}\right) \in \Lambda_{\varepsilon} \times Y
\end{array}
$$

As shown in [1] , for every $\psi^{\prime} \in L^{2}(\omega)$ we have

$$
\begin{equation*}
\left\|\mathcal{T}_{\varepsilon}\left(\psi^{\prime}\right)\right\|_{L^{2}\left(\omega \times Y^{\prime}\right)} \leq\left\|\psi^{\prime}\right\|_{L^{2}(\omega)} . \tag{4.1}
\end{equation*}
$$

Definition 4.2. The rescaling-unfolding operator is defined by $\Pi_{\varepsilon}=\mathcal{T}_{\varepsilon} \circ \mathfrak{T}_{\varepsilon}$.
Lemma 4.1. We obtain the following estimate for the warping:

$$
\begin{equation*}
\left\|\Pi_{\varepsilon}(\bar{u})\right\|_{L^{2}\left(\omega ; H^{1}(Y)\right)} \leq C \varepsilon^{1 / 2}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} . \tag{4.2}
\end{equation*}
$$

Denote $H_{\text {per }}^{1}\left(Y^{\prime}\right)\left(\right.$ resp. $\left.H_{\text {per }}^{1}(Y)\right)$ the subspace of $H_{l o c}^{1}\left(\mathbb{R}^{2}\right)\left(\right.$ resp. $H_{l o c}^{1}\left(\mathbb{R}^{2} \times(-\kappa, \kappa)\right) \cap H^{1}(Y)$ ) containing the functions $\mathbf{G}$ periodic and

$$
\widehat{\mathcal{W}} \doteq\left\{\widehat{\bar{\pi}} \in H_{p e r}^{1}(Y)^{3} \mid \int_{-\kappa}^{\kappa} \widehat{\bar{\pi}}\left(\cdot, y_{3}\right) d y_{3}=0, \int_{-\kappa}^{\kappa} y_{3} \hat{\bar{\pi}}_{\alpha}\left(\cdot, y_{3}\right) d y_{3}=0 \text { a.e. in } \omega \times Y^{\prime}\right\} .
$$

Lemma 4.2. There exists a subsequence of $\{\varepsilon\}$ (still denoted $\{\varepsilon\}$ ) and $\widehat{\mathcal{U}} \in L^{2}\left(\omega ; H_{p e r}^{1}\left(Y^{\prime}\right)\right)^{3}, \widehat{\mathcal{R}} \in$ $L^{2}\left(\omega ; H_{p e r}^{1}\left(Y^{\prime}\right)\right)^{2}$ and $\widehat{\bar{u}} \in L^{2}(\omega ; \widehat{\overline{\mathcal{W}}})$ such that

$$
\begin{align*}
& \mathcal{T}_{\varepsilon}\left(\mathcal{U}_{\varepsilon}\right) \longrightarrow \mathcal{U} \text { strongly in } L^{2}\left(\omega ; H^{1}\left(Y^{\prime}\right)\right)^{3}, \\
& \mathcal{T}_{\varepsilon}\left(\mathcal{R}_{\varepsilon}\right) \longrightarrow \mathcal{R} \text { strongly in } L^{2}\left(\omega ; H^{1}\left(Y^{\prime}\right)\right)^{2}, \\
& \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}}\right) \longrightarrow \frac{\partial \mathcal{U}}{\partial s_{\alpha}} \quad \text { strongly in } L^{2}\left(\omega \times Y^{\prime}\right)^{3},  \tag{4.3}\\
& \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{R}_{\varepsilon}}{\partial s_{\alpha}}\right) \rightharpoonup \frac{\partial \mathcal{R}}{\partial s_{\alpha}}+\frac{\partial \widehat{\mathcal{R}}}{\partial y_{\alpha}} \quad \text { weakly in } L^{2}\left(\omega \times Y^{\prime}\right)^{2}, \\
& \frac{1}{\varepsilon^{2}} \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon}\right) \rightharpoonup \widehat{\bar{u}} \text { weakly in } L^{2}\left(\omega ; H^{1}(Y)\right)^{3} .
\end{align*}
$$

One has

$$
\bar{u}\left(s_{1}, s_{2}, y_{3}\right)=\frac{1}{\left|Y^{\prime}\right|} \int_{Y^{\prime}} \widehat{\bar{u}}\left(s_{1}, s_{2}, y_{1}, y_{2}, y_{3}\right) d y_{1} d y_{2} \quad \text { for a.e. }\left(s_{1}, s_{2}, y_{3}\right) \in \Omega \text {. }
$$

Moreover,

$$
\begin{array}{rlll}
\frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}} \cdot \boldsymbol{n}+\mathcal{R}_{\varepsilon} \cdot \boldsymbol{t}_{\alpha}\right) \rightharpoonup \mathcal{Z}_{\alpha 3}+\widehat{Z}_{\alpha 3} & \text { weakly in } & L^{2}\left(\omega ; H^{1}\left(Y^{\prime}\right)\right),  \tag{4.4}\\
\frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}} \cdot \boldsymbol{t}_{\beta}+\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\beta}} \cdot \boldsymbol{t}_{\alpha}\right) \rightharpoonup \mathcal{Z}_{\alpha \beta}+\widehat{Z}_{\alpha \beta} & \text { weakly in } & L^{2}\left(\omega ; H^{1}\left(Y^{\prime}\right)\right),
\end{array}
$$

where

$$
\begin{equation*}
\widehat{Z}_{13}=\frac{\partial \widehat{\mathcal{U}}_{3}}{\partial y_{1}}+\widehat{\mathcal{R}}_{1}, \quad \widehat{Z}_{23}=\frac{\partial \widehat{\mathcal{U}}_{3}}{\partial y_{2}}+\widehat{\mathcal{R}}_{2}, \quad \widehat{Z}_{\alpha \beta}=e_{y, \alpha \beta}(\widehat{\mathcal{U}}) . \tag{4.5}
\end{equation*}
$$

Proof. The strong convergences of $(4.3)_{1,2,3}$ follow from 3.2$)_{1,2}$ and [1, Propostion 3.4]. Convergence $(4.3)_{4}$ is the consequence of [1, Theorem 3.5] and $(4.3)_{5}$ is obtained with [1, Corollary 3.2].
The convergences of $4.4{ }_{1,2}$ follow from Lemma 3.1] and [1, Theorem 3.5]. With Lemma 10.2 we then obtain the expression for $\widehat{Z}_{\alpha 3}$ in (4.5) and Lemma 10.3 yields the expression $\widehat{Z}_{\alpha \beta}$.
To do that, first we need to identify the different fields appearing in Lemma 10.2 Here

$$
u_{\varepsilon} \longleftrightarrow \mathcal{U}_{\varepsilon, 3}, \quad v_{\varepsilon} \longleftrightarrow\binom{-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}+\mathcal{R}_{\varepsilon, 1}}{\mathcal{R}_{\varepsilon, 2}}
$$

From (3.2 ${ }_{1,4}$, one has

$$
\begin{aligned}
& \frac{1}{\varepsilon}\left(\nabla \mathcal{U}_{\varepsilon, 3}+\binom{-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}+\mathcal{R}_{\varepsilon, 1}}{\mathcal{R}_{\varepsilon, 2}}\right)-\binom{\mathcal{Z}_{13}}{\mathcal{Z}_{23}} \text { weakly in } L^{2}(\omega)^{2}, \\
& \mathcal{T}_{\varepsilon}\left[\nabla\binom{-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}+\mathcal{R}_{\varepsilon, 1}}{\mathcal{R}_{\varepsilon, 2}}\right] \rightarrow \nabla\binom{-\frac{1}{a} \mathcal{U}_{1}+\mathcal{R}_{1}}{\mathcal{R}_{2}}+\nabla_{y}\binom{\widehat{\mathcal{R}}_{1}}{\widehat{\mathcal{R}}_{2}} \text { weakly in } L^{2}\left(\omega \times Y^{\prime}\right)^{2} .
\end{aligned}
$$

Then, one can apply Lemma 10.2. The function $\mathfrak{u}$ is called $\widehat{\mathcal{U}}_{3}$.
Now we determine the $\widehat{Z}_{\alpha \beta}$ 's.
Let us identify

$$
u_{\varepsilon} \longleftrightarrow\binom{\mathcal{U}_{\varepsilon, 1}}{\mathcal{U}_{\varepsilon, 2}}, \quad v_{\varepsilon} \longleftrightarrow\left(\begin{array}{cc}
\frac{1}{a} \mathcal{U}_{\varepsilon, 3} & 0 \\
0 & 0
\end{array}\right)
$$

Hence, by $(3.2)_{1,3} \frac{1}{\varepsilon}\left(e\left(u_{\varepsilon}\right)+v_{\varepsilon}\right) \rightharpoonup \mathcal{X}$, and $\mathcal{T}_{\varepsilon}\left(\nabla v_{\varepsilon}\right) \rightharpoonup \nabla v+\nabla_{y} \widehat{v}$. Here, observe that $\widehat{v}=0$. The field $\left(\mathfrak{u}_{1}, \mathfrak{u}_{2}\right)$ given by Lemma 10.3 is denoted $\left(\widehat{\mathcal{U}}_{1}, \widehat{\mathcal{U}}_{2}\right)$.

### 4.1. Limit of the rescaled-unfolded strain tensor

Proposition 4.1. Under the assumptions and the results of Lemma 4.2 we obtain the following weak convergences in $L^{2}(\omega \times Y)$ :

$$
\begin{aligned}
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{\alpha \alpha}\left(u_{\varepsilon}\right)\right) \rightharpoonup \mathcal{Z}_{\alpha \alpha}+e_{y, \alpha \alpha}(\widehat{\mathcal{U}})+y_{3}\left(\frac{\partial \mathcal{R}_{\alpha}}{\partial s_{\alpha}}+\frac{\partial \widehat{\mathcal{R}}_{\alpha}}{\partial y_{\alpha}}\right)+\frac{\partial \widehat{\bar{u}}_{\alpha}}{\partial y_{\alpha}} \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{12}\left(u_{\varepsilon}\right)\right) \rightharpoonup \frac{1}{2}\left(\mathcal{Z}_{12}+2 e_{y, 22}(\widehat{\mathcal{U}})+y_{3}\left(\frac{\partial \mathcal{R}_{1}}{\partial s_{2}}+\frac{\partial \widehat{\mathcal{R}}_{1}}{\partial y_{2}}+\frac{\partial \mathcal{R}_{2}}{\partial s_{1}}+\frac{\partial \widehat{\mathcal{R}}_{2}}{\partial y_{1}}\right)+\frac{\partial \widehat{\bar{u}}_{1}}{\partial y_{2}}+\frac{\partial \widehat{\bar{u}}_{2}}{\partial y_{1}}\right) \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{\alpha 3}\left(u_{\varepsilon}\right)\right) \rightharpoonup \frac{1}{2}\left(\mathcal{Z}_{\alpha 3}+\frac{\partial \widehat{\mathcal{U}}_{3}}{\partial y_{\alpha}}+\widehat{\mathcal{R}}_{\alpha}+\frac{\partial \widehat{\bar{u}}_{\alpha}}{\partial y_{3}}+\frac{\partial \widehat{\bar{u}}_{3}}{\partial y_{\alpha}}\right) \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{33}\left(u_{\varepsilon}\right)\right) \rightharpoonup \frac{\partial \widehat{\bar{u}}_{3}}{\partial y_{3}} .
\end{aligned}
$$

Proof. First, note that the function $y_{3} \longrightarrow \frac{a}{a+\varepsilon y_{3}}$ converges uniformly to 1 in $\omega \times Y$.
Below, we give the limits for $\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{11}\left(u_{\varepsilon}\right)\right)$ and $\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{13}\left(u_{\varepsilon}\right)\right)$, since other cases follow in a similar way. For the calculation we combine the results obtained in Lemma 3.1 and Lemma 4.2. We have,

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{11}\left(u_{\varepsilon}\right)\right)=\frac{1}{\varepsilon}\left[\frac{a}{a+\varepsilon y_{3}}\left(\mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon, 1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{\varepsilon, 3}\right)+\varepsilon y_{3} \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{R}_{\varepsilon, 1}}{\partial s_{1}}\right)+\Pi_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon, 1}}{\partial s_{1}}\right)+\frac{1}{a} \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 3}\right)\right)\right] .
$$

Therefore, we get for each term in the limit

$$
\begin{aligned}
& \frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon, 1}}{\partial s_{\varepsilon, 1}}+\frac{1}{a} \mathcal{U}_{\varepsilon, 3}\right) \rightharpoonup \mathcal{Z}_{11}+e_{y, 11}(\widehat{\mathcal{U}}) \quad \text { weakly in } \quad L^{2}\left(\omega \times Y^{\prime}\right) \\
& \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{R}_{\varepsilon, 1}}{\partial s_{1}}\right) \rightharpoonup \frac{\partial \mathcal{R}_{1}}{\partial s_{1}}+\frac{\partial \widehat{\mathcal{R}}_{1}}{\partial y_{1}} \quad \text { weakly in } L^{2}\left(\omega \times Y^{\prime}\right) \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon, 1}}{\partial s_{1}}\right)=\frac{1}{\varepsilon^{2}} \frac{\partial \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 1}\right)}{\partial y_{1}} \rightharpoonup \frac{\partial \widehat{\bar{u}}_{1}}{\partial y_{1}} \quad \text { weakly in } \quad L^{2}(\omega \times Y) \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 3}\right) \rightharpoonup 0 \quad \text { weakly in } \quad L^{2}(\omega \times Y)
\end{aligned}
$$

Hence,

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{11}\left(u_{\varepsilon}\right)\right) \rightharpoonup \mathcal{Z}_{11}+e_{y, 11}(\widehat{\mathcal{U}})+y_{3}\left(\frac{\partial \mathcal{R}_{1}}{\partial s_{1}}+\frac{\partial \widehat{\mathcal{R}}_{1}}{\partial y_{1}}\right)+\frac{\partial \widehat{\bar{u}}_{1}}{\partial y_{1}} \quad \text { weakly in } \quad L^{2}(\omega \times Y) .
$$

Now we focus on

$$
\begin{aligned}
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{13}\left(u_{\varepsilon}\right)\right)=\frac{1}{2} \frac{1}{\varepsilon} \frac{a}{a+\varepsilon y_{3}}\left[\mathcal { T } _ { \varepsilon } \left(\frac{\partial \mathcal{U}_{\varepsilon, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}\right.\right. & \left.+\mathcal{R}_{\varepsilon, 1}\right) \\
& \left.-\frac{1}{a} \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 1}\right)+\Pi_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon, 3}}{\partial s_{1}}\right)+\left(1+\frac{\varepsilon y_{3}}{a}\right) \Pi_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon, 1}}{\partial s_{3}}\right)\right]
\end{aligned}
$$

Similar to the previous case we calculate the limits of each component, obtaining

$$
\begin{aligned}
& \frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(\frac{\partial \mathcal{U}_{\varepsilon, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{U}_{\varepsilon, 1}+\mathcal{R}_{\varepsilon, 1}\right) \rightharpoonup \mathcal{Z}_{13}+\frac{\partial \widehat{\mathcal{U}}_{3}}{\partial y_{1}}+\widehat{\mathcal{R}}_{1} \quad \text { weakly in } \quad L^{2}\left(\omega \times Y^{\prime}\right), \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 1}\right) \rightharpoonup 0 \quad \text { weakly in } L^{2}(\omega \times Y), \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(\frac{\partial \bar{u}_{\varepsilon, 3}}{\partial s_{1}}\right)=\frac{1}{\varepsilon^{2}} \frac{\partial \Pi_{\varepsilon}\left(\bar{u}_{3}\right)}{\partial y_{1}} \rightharpoonup \frac{\partial \widehat{\bar{u}}_{3}}{\partial y_{1}} \quad \text { weakly in } L^{2}(\omega \times Y), \\
& \frac{1}{\varepsilon^{2}} \frac{\partial \Pi_{\varepsilon}\left(\bar{u}_{\varepsilon, 1}\right)}{\partial y_{3}} \rightharpoonup \frac{\partial \widehat{\bar{u}}_{1}}{\partial y_{3}} \quad \text { weakly in } \quad L^{2}(\omega \times Y) .
\end{aligned}
$$

Hence,

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{13}\left(u_{\varepsilon}\right)\right) \rightharpoonup \frac{1}{2}\left(\mathcal{Z}_{13}+\frac{\partial \widehat{\mathcal{U}}_{3}}{\partial y_{1}}+\widehat{\mathcal{R}}_{1}+\frac{\partial \widehat{\bar{u}}_{3}}{\partial y_{1}}+\frac{\partial \widehat{\bar{u}}_{1}}{\partial y_{3}}\right)
$$

Define the displacement $\widehat{u}$ belonging to $L^{2}\left(\omega ; H_{\text {per }}^{1}(Y)\right)^{3}$ by

$$
\widehat{u}(\cdot, y)=\widehat{\mathcal{U}}\left(\cdot, y_{1}, y_{2}\right)+y_{3} \widehat{\mathcal{R}}\left(\cdot, y_{1}, y_{2}\right)+\left(y_{3}\left(\mathcal{Z}_{13} \mathbf{t}_{1}+\mathcal{Z}_{23} \mathbf{t}_{2}\right)+\widehat{\bar{u}}(\cdot, y)\right)
$$

for a.e. $y \in Y^{*}$ and a.e. in $\omega$.
Hence, one obtains

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e\left(u_{\varepsilon}\right)\right) \rightharpoonup\left(\begin{array}{ccc}
\mathcal{Z}_{11}+\frac{y_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{1}}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{1}^{2}} & \frac{1}{2} \mathcal{Z}_{12}+\frac{y_{3}}{a} \frac{\partial \mathcal{U}_{1}}{\partial s_{2}}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{1} \partial s_{2}} & 0 \\
* & \mathcal{Z}_{22}-y_{3} \frac{\partial^{2} \mathcal{U}_{3}}{\partial s_{2}^{2}} & 0 \\
* & * & 0
\end{array}\right)
$$

where $\mathcal{E}_{y}(\widehat{u})$ is the symmetric tensor whose components are the $e_{y, i j}(\widehat{u})$ 's. We want to note here that we obtain the same kind of result in [4].
The aim of the following section is to determine the $\mathcal{Z}_{\alpha \beta}$ 's.
Remark 4.1. If we compare our results with [3, Proposition 11.13], we see that

$$
\mathcal{E}_{y}(\widehat{u})=E_{w}(\stackrel{\circ}{u})+\mathcal{E}_{y}^{1}(\widehat{u})
$$

where the terms on the right hand side follow from the given definitions in [3].

## 5. Inextensional and extensional displacements

### 5.1. Inextensional displacements

Denote $\mathbb{H} \doteq\left[H_{\Gamma_{0}}^{1}(\omega)\right]^{2} \times L^{2}(\omega)$. We equip $\mathbb{H}$ with the scalar product

$$
\begin{aligned}
<\mathcal{U}, \mathcal{V}>=\int_{\omega}\left[\frac{1}{2}\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right)\left(\frac{\partial \mathcal{V}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{V}_{3}\right)+\right. & \frac{\partial \mathcal{U}_{2}}{\partial s_{2}} \frac{\partial \mathcal{V}_{2}}{\partial s_{2}} \\
& \left.+\frac{1}{2}\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}\right)\left(\frac{\partial \mathcal{V}_{1}}{\partial s_{2}}+\frac{\partial \mathcal{V}_{2}}{\partial s_{1}}\right)+\mathcal{U}_{3} \mathcal{V}_{3}\right] d s_{1} d s_{2}
\end{aligned}
$$

The associated norm is equivalent to the usual norm of $\left[H_{\Gamma_{0}}^{1}(\omega)\right]^{2} \times L^{2}(\omega)$.
Denote $D_{I}$ the space of inextensional displacements

$$
D_{I} \doteq\left\{\Phi \in \mathbb{H} \quad \left\lvert\, \quad \frac{\partial \Phi_{1}}{\partial s_{1}}+\frac{1}{a} \Phi_{3}=0\right., \quad \frac{\partial \Phi_{2}}{\partial s_{2}}=0, \quad \frac{\partial \Phi_{1}}{\partial s_{2}}+\frac{\partial \Phi_{2}}{\partial s_{1}}=0\right\}
$$

A displacement $\mathcal{V}$ belongs to $D_{I}$ if and only if there exists $\left(V_{1}, V_{2}\right) \in H_{0}^{1}(0, a \pi) \times H_{0}^{2}(0, a \pi)$ such that for a.e. $\left(s_{1}, s_{2}\right) \in \omega$

$$
\begin{align*}
& \mathcal{V}_{2}\left(s_{1}, s_{2}\right)=V_{2}\left(s_{1}\right) \\
& \mathcal{V}_{1}\left(s_{1}, s_{2}\right)=-s_{2}^{c} V_{2}^{\prime}\left(s_{1}\right)+V_{1}\left(s_{1}\right), \quad V_{1} \in H_{0}^{1}(0, a \pi), \quad V_{2} \in H_{0}^{2}(0, a \pi)  \tag{5.1}\\
& \mathcal{V}_{3}\left(s_{1}, s_{2}\right)=a\left(s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right)-V_{1}^{\prime}\left(s_{1}\right)\right)
\end{align*}
$$

The map $\mathcal{V} \in D_{I} \longmapsto\left(V_{1}, V_{2}\right) \in H_{0}^{1}(0, a \pi) \times H_{0}^{2}(0, a \pi)$ is one to one and onto.
Denote

$$
\mathbb{D}_{I}=D_{I} \cap\left(\left[H_{\Gamma_{0}}^{1}(\omega)\right]^{2} \times H_{\Gamma_{0}}^{2}(\omega)\right) .
$$

Note that the limit of the mid surface displacement of the shell $\mathcal{U}$ belongs to $\mathbb{D}_{I}$.
We equip $D_{I}\left(\right.$ resp. $\left.\mathbb{D}_{I}\right)$ with the semi-norm

$$
\|\mathcal{V}\|_{D_{I}}=\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}, \quad\left(\text { resp. } \quad\|\mathcal{V}\|_{\mathbb{D}_{I}}=\left\|\mathcal{V}_{3}\right\|_{H^{2}(\omega)}\right)
$$

Lemma 5.1. The semi-norm $\|\cdot\|_{D_{I}}$ (resp. $\|\cdot\|_{\mathbb{D}_{I}}$ ) is a norm equivalent to the norm of the product space $\left[H^{1}(\omega)\right]^{2} \times L^{2}(\omega)$ (resp. $\left[H^{1}(\omega)\right]^{2} \times H^{2}(\omega)$ ).
Moreover, there exist two constants $c, C$ such that for every $\mathcal{V} \in D_{I}$ (resp. $\mathcal{V} \in \mathbb{D}_{I}$ ) one has

$$
\begin{aligned}
& c\left(\left\|V_{1}\right\|_{H_{0}^{1}(0, a \pi)}^{2}+\left\|V_{2}\right\|_{H_{0}^{2}(0, a \pi)}^{2}\right) \leq\|\mathcal{V}\|_{D_{I}}^{2} \leq C\left(\left\|V_{1}\right\|_{H_{0}^{1}(0, a \pi)}^{2}+\left\|V_{2}\right\|_{H_{0}^{2}(0, a \pi)}^{2}\right) \\
&\left(\text { resp. } c\left(\left\|V_{1}\right\|_{H_{0}^{3}(0, a \pi)}^{2}+\left\|V_{2}\right\|_{H_{0}^{4}(0, a \pi)}^{2}\right) \leq\|\mathcal{V}\|_{\mathbb{D}_{I}}^{2} \leq C\left(\left\|V_{1}\right\|_{H_{0}^{3}(0, a \pi)}^{2}+\left\|V_{2}\right\|_{H_{0}^{4}(0, a \pi)}^{2}\right)\right)
\end{aligned}
$$

where $\left(V_{1}, V_{2}\right)$ are associated to $\mathcal{V}$ by (5.1).
Proof. see Appendix.

### 5.2. Extensional displacements

Set

$$
s_{1}^{c}=s_{1}-\frac{a \pi}{2}, \quad s_{2}^{c}=s_{2}-\frac{l}{2} .
$$

Denote $D_{E}$ the orthogonal subspace of $D_{I}$ in $\mathbb{H}$ for the scalar product of $\mathbb{H}$.
For every $\phi$ in $L^{2}(\omega)$, denote

$$
\mathcal{M}_{2}(\phi)\left(s_{1}\right)=\frac{1}{l} \int_{0}^{l} \phi\left(s_{1}, s_{2}\right) d s_{2}, \quad \mathcal{M}_{2}^{c}(\phi)\left(s_{1}\right)=\frac{1}{l} \int_{0}^{l} \phi\left(s_{1}, s_{2}\right) s_{2}^{c} d s_{2} \quad \text { for a.e. } s_{1} \in(0, a \pi)
$$

Note that for every $\mathcal{U} \in D_{E}$, one has $\mathcal{M}_{2}\left(\mathcal{U}_{\alpha}\right), \mathcal{M}_{2}^{c}\left(\mathcal{U}_{\alpha}\right) \in H_{0}^{1}(0, a \pi)$ while $\mathcal{M}_{2}\left(\mathcal{U}_{3}\right), \mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right) \in$ $L^{2}(0, a \pi)(\alpha \in\{1,2\})$.
Let $\mathcal{U}$ be in $D_{E}$, it satisfies $\left.\quad<\mathcal{U}, \mathcal{V}\right\rangle=\int_{\omega} \mathcal{U}_{3} \mathcal{V}_{3} d s_{1} d s_{2}, \quad \forall \mathcal{V} \in D_{I}$. Thus,

$$
\int_{\omega} \mathcal{U}_{3}\left(s_{1}, s_{2}\right)\left(s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right)-V_{1}^{\prime}\left(s_{1}\right)\right) d s_{1} d s_{2}=0, \quad \forall V_{1} \in H_{0}^{1}(0, a \pi), \quad \forall V_{2} \in H_{0}^{2}(0, a \pi)
$$

That gives

$$
\begin{array}{r}
\mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\left(s_{1}\right)=C_{1}, \quad \mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right)\left(s_{1}\right)=C_{2} s_{1}^{c}+C_{3} \\
\left(C_{1}, C_{2}, C_{3}\right) \in \mathbb{R}^{3} \quad \text { for a.e. } s_{1} \in(0, a \pi) \tag{5.2}
\end{array}
$$

Hence

$$
D_{E}=\left\{\mathcal{U} \in \mathbb{H} \left\lvert\, \frac{d \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)}{d s_{1}}=\frac{d^{2} \mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right)}{d s_{1}^{2}}=0 \quad\right. \text { in } \quad(0, a \pi)\right\} .
$$

We equip $D_{E}$ with the norm

$$
\|\Phi\|_{E}=\sqrt{\int_{\omega}\left[\frac{1}{2}\left|\frac{\partial \Phi_{1}}{\partial s_{1}}+\frac{1}{a} \Phi_{3}\right|^{2}+\left|\frac{\partial \Phi_{2}}{\partial s_{2}}\right|^{2}+\frac{1}{2}\left|\frac{\partial \Phi_{1}}{\partial s_{2}}+\frac{\partial \Phi_{2}}{\partial s_{1}}\right|^{2}\right] d s_{1} d s_{2}}
$$

Endowed with this norm, $D_{E}$ is not a Hilbert space. We denote with $\mathbb{D}_{E}$ the completion of $D_{E}$ for this norm.

Lemma 5.2. For every $\mathcal{U}$ in $\mathbb{D}_{E}$, one has

$$
\begin{equation*}
\left\|\mathcal{U}_{2}\right\|_{H^{1}\left(0, l ; L^{2}(0, a \pi)\right)}+\left\|\mathcal{U}_{1}\right\|_{H^{1}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)}+\left\|\mathcal{U}_{3}\right\|_{L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E} \tag{5.3}
\end{equation*}
$$

Proof. See Appendix.
Now, consider the field $\mathcal{U}_{\varepsilon}$, the mid-surface displacement associated to $u_{\varepsilon}$ the solution of the variational problem (6.5). This field belongs to $\mathbb{H}$. We decompose it as the sum of an inextensional displacement $\mathcal{U}_{I, \varepsilon}$ and an extensional one $\mathcal{U}_{E, \varepsilon}$. By the definition of $\|\cdot\|_{E}$ and Lemma 1.1 we obtain

$$
\left\|\mathcal{U}_{E, \varepsilon}\right\|_{E} \leq \sum_{\alpha, \beta=1}^{2}\left\|\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\alpha}} \cdot \mathbf{t}_{\beta}+\frac{\partial \mathcal{U}_{\varepsilon}}{\partial s_{\beta}} \cdot \mathbf{t}_{\alpha}\right\|_{L^{2}(\omega)} \leq \frac{C}{\varepsilon^{1 / 2}}\left\|e\left(u_{\varepsilon}\right)\right\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \leq C \varepsilon
$$

Lemma 5.3. There exist a subsequence (still denoted $\{\varepsilon\}$ ) and $\mathcal{U}_{E} \in \mathbb{D}_{E}$ such that

$$
\begin{aligned}
& \frac{1}{\varepsilon} \mathcal{U}_{E, \varepsilon, 1} \rightharpoonup \mathcal{U}_{E, 1} \quad \text { weakly in } H^{1}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right) \\
& \frac{1}{\varepsilon} \mathcal{U}_{E, \varepsilon, 2} \rightharpoonup \mathcal{U}_{E, 2} \quad \text { weakly in } H^{1}\left(0, l ; L^{2}(0, a \pi)\right) \\
& \frac{1}{\varepsilon} \mathcal{U}_{E, \varepsilon, 3} \rightharpoonup \mathcal{U}_{E, 3} \quad \text { weakly in } L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)
\end{aligned}
$$

Proof. From Lemma 5.2, one has

$$
\left\|\mathcal{U}_{E, \varepsilon, 1}\right\|_{H^{1}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)}+\left\|\mathcal{U}_{E, \varepsilon, 2}\right\|_{H^{1}\left(0, l ; L^{2}(0, a \pi)\right)}+\left\|\mathcal{U}_{E, \varepsilon, 3}\right\|_{L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)} \leq C \varepsilon
$$

which yields the claim.
Going back to the expressions for $\mathcal{Z}_{\alpha \beta}$ introduced Lemma 3.1 and Proposition 4.1 we get with Lemma 5.3 that

$$
\mathcal{Z}_{\alpha \beta}=\frac{1}{2}\left[\frac{\partial \mathcal{U}_{E}}{\partial s_{\alpha}} \mathbf{t}_{\beta}+\frac{\partial \mathcal{U}_{E}}{\partial s_{\beta}} \mathbf{t}_{\alpha}\right] .
$$

## 6. The linear elasticity problem

Let $a_{i j k l} \in L^{\infty}(Y), i, j, k, l \in\{1,2,3\}$ and it should satisfy both the symmetry condition

$$
\begin{equation*}
a_{i j k l}(y)=a_{j i k l}(y)=a_{k l i j}(y) \quad \text { for a.e. } y \in Y \tag{6.1}
\end{equation*}
$$

and the coercivity condition $\left(c_{0}>0\right)$

$$
\begin{equation*}
a_{i j k l}(y) \tau_{i j} \tau_{k l} \geq c_{0} \tau_{i j} \tau_{i j} \quad \text { for a.e. } y \in Y, \tag{6.2}
\end{equation*}
$$

where $\tau$ is a $3 \times 3$ symmetric real matrix.
The coefficients $a_{i j k l}^{\varepsilon}$ of the Hooke's tensor on the shell for $x=\Phi(s)$ are given by

$$
\begin{gather*}
a_{i j k l}^{\varepsilon}(x)=a_{i j k l}\left(\left\{\frac{s^{\prime}}{\varepsilon}\right\}_{Y^{\prime}}, \frac{s_{3}}{\varepsilon}\right) \quad \text { for a.e. } x \in \mathcal{Q}_{\varepsilon}^{*}  \tag{6.3}\\
\sigma_{i j}^{\varepsilon}(v)=a_{i j k l}^{\varepsilon} e_{k l}(v) \quad \forall v \in V_{\varepsilon}^{*} \tag{6.4}
\end{gather*}
$$

For a given applied force $f_{\varepsilon}$ the displacement $u_{\varepsilon}$ of a shell is the solution to the linear elasticity problem

$$
\left\{\begin{array}{l}
\text { Find } u_{\varepsilon} \in V_{\varepsilon}^{*} \text { such that }  \tag{6.5}\\
\int_{\mathcal{Q}_{\varepsilon}^{*}} \sigma^{\varepsilon}\left(u_{\varepsilon}\right): e(v) d x=\int_{\mathcal{Q}_{\varepsilon}^{*}} f_{\varepsilon} v d x, \quad \forall v \in V_{\varepsilon}^{*}
\end{array}\right.
$$

where the colon denotes the classical Frobenius scalar product.
6.1. Assumptions on the forces

We assume that the body forces are given by

$$
f_{\varepsilon}\left(s_{1}, s_{2}, s_{3}\right)=\varepsilon^{2} f\left(s_{1}, s_{2}\right)+\varepsilon F\left(s_{1}, s_{2}\right)+s_{3} g\left(s_{1}, s_{2}\right) \quad \text { for a.e. } \quad\left(s_{1}, s_{2}\right) \in \omega
$$

where $f=f_{1} \mathbf{t}_{1}+f_{2} \mathbf{t}_{2}+f_{3} \mathbf{n},\left(f_{1}, f_{2}, f_{3}\right) \in L^{2}(\omega)^{3}$ and $g=g_{1} \mathbf{t}_{1}+g_{2} \mathbf{t}_{2},\left(g_{1}, g_{2}\right) \in L^{2}(\omega)^{2}$.
Regarding $F$, we want to choose this field so that it does not act with inextensional displacements. First, in view of Lemma 5.2, we take

$$
\begin{equation*}
F_{1} \in L^{2}\left(0, l ; H^{1}(0, a \pi)\right), \quad F_{2} \in L^{2}(\omega), \quad F_{3} \in L^{2}\left(0, l ; H^{2}(0, a \pi)\right) \tag{6.6}
\end{equation*}
$$

Then

$$
\int_{\omega} F\left(s_{1}, s_{2}\right) \cdot V\left(s_{1}, s_{2}\right) d s_{1} d s_{2}, \quad V \in D_{E}
$$

will be written

$$
\begin{aligned}
&\langle F, V\rangle=\int_{0}^{l}<F_{1}, V_{1}>_{\left.H^{1}(0, a \pi),\left(H^{1}(0, a \pi)\right)^{\prime}\right)} d s_{2}+\int_{\omega} F_{2} V_{2} d s_{1} d s_{2} \\
&+\int_{0}^{l}<F_{3}, V_{3}>_{\left.H^{2}(0, a \pi),\left(H^{2}(0, a \pi)\right)^{\prime}\right)} d s_{2}
\end{aligned}
$$

for every $V \in \mathbb{D}_{E}$. Due to Lemma 5.2. one has

$$
\begin{equation*}
|\langle F, V\rangle| \leq\left(\left\|F_{1}\right\|_{L^{2}\left(0, L ; H^{1}(0, a \pi)\right)}+\left\|F_{2}\right\|_{L^{2}(\omega)}+\left\|F_{3}\right\|_{L^{2}\left(0, L ; H^{2}(0, a \pi)\right)}\right)\|V\|_{E}, \quad \forall V \in \mathbb{D}_{E} \tag{6.7}
\end{equation*}
$$

Recall that this field has to satisfy for all $\mathcal{V} \in D_{I}$ that

$$
\int_{\omega} F\left(s_{1}, s_{2}\right) \cdot \mathcal{V}\left(s_{1}, s_{2}\right) d s=0
$$

Hence, for all $\left(V_{1}, V_{2}\right) \in H_{0}^{1}(0, a \pi) \times H_{0}^{2}(0, a \pi)$

$$
\int_{\omega}\left(\begin{array}{l}
F_{1}\left(s_{1}, s_{2}\right) \\
F_{2}\left(s_{1}, s_{2}\right) \\
F_{3}\left(s_{1}, s_{2}\right)
\end{array}\right) \cdot\left(\begin{array}{c}
-s_{2}^{c} V_{2}^{\prime}\left(s_{1}\right)+V_{1}\left(s_{1}\right) \\
V_{2}\left(s_{1}\right) \\
a\left(s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right)-V_{1}^{\prime}\left(s_{1}\right)\right)
\end{array}\right) d s=0
$$

We then get with partial integration and the boundary conditions for $V_{1}$ and $V_{2}$ that

$$
\int_{\omega}\left[\left(\frac{\partial F_{1}}{\partial s_{1}} s_{2}^{c}+F_{2}+a \frac{\partial^{2} F_{3}}{\partial s_{1}^{2}} s_{2}^{c}\right) V_{2}+\left(F_{1}+a \frac{\partial F_{3}}{\partial s_{1}}\right) V_{1}\right] d s_{1} d s_{2}=0
$$

holds for all $V_{1} \in H_{0}^{1}(0, a \pi)$ and $V_{2} \in H_{0}^{2}(0, a \pi)$. Hence, the field $F \in L^{2}(\omega)^{3}$ has to satisfy

$$
\begin{equation*}
\mathcal{M}_{2}\left(F_{1}\right)+a \frac{d \mathcal{M}_{2}\left(F_{3}\right)}{d s_{1}}=0, \quad \frac{d \mathcal{M}_{2}^{c}\left(F_{1}\right)}{d s_{1}}+\mathcal{M}_{2}\left(F_{2}\right)+a \frac{d^{2} \mathcal{M}_{2}^{c}\left(F_{3}\right)}{d s_{1}^{2}}=0, t^{2} \tag{6.8}
\end{equation*}
$$

In Lemma 9.2 we show that there exists a field $\mathbf{F} \in L^{2}(\omega)^{3}$ such that

$$
\langle F, V\rangle=\int_{\omega}\left(\mathbf{F}_{11} e_{11}(V)+\mathbf{F}_{12} e_{12}(V)+\mathbf{F}_{22} e_{22}(V)\right) d s_{1} d s_{2}
$$

Taking into account the holes, we need an additional assumption on the forces $F$. We will see this in the proof of the lemma below.
From now on, we assume that $F$ satisfies (6.6) and moreover $F \in H^{1}(\omega)^{3}$.

[^2]
## Lemma 6.1. One has

$$
\begin{align*}
\left\lvert\, \frac{1}{2 \kappa} \int_{\mathcal{Q}_{\varepsilon}^{*}} f_{\varepsilon} \cdot u d x\right. & -\varepsilon^{3}\left(\int_{\omega_{\varepsilon}^{*}} f \cdot \mathcal{U} d s_{1} d s_{2}+\frac{1}{\varepsilon} \int_{\omega_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}\right. \\
& \left.+\frac{\kappa^{2}}{3 a} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{U}_{\alpha} d s_{1} d s_{2}+\frac{\kappa^{2}}{3} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{R}_{\alpha} d s_{1} d s_{2}\right) \mid  \tag{6.9}\\
\leq & C \varepsilon^{5 / 2}\left(\|f\|_{L^{2}(\omega)}+\|g\|_{L^{2}(\omega)}+\|F\|_{L^{2}(\omega)}\right)\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
\end{align*}
$$

Furthermore

$$
\begin{equation*}
\left|\int_{\mathcal{Q}_{\varepsilon}^{*}} f_{\varepsilon} \cdot u d x\right| \leq C \varepsilon^{3 / 2}\left(\|f\|_{L^{2}(\omega)}+\|g\|_{L^{2}(\omega)}+\left\|F_{3}\right\|_{L^{2}\left(0, L ; H^{2}(0, a \pi)\right)}+\|F\|_{H^{1}(\omega)}\right)\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \tag{6.10}
\end{equation*}
$$

The constants do not depend on $\varepsilon$.
Proof. Using the decomposition of $u$ we can write (see Remark 1.1)

$$
\begin{align*}
& \int_{\mathcal{Q}_{\varepsilon}^{*}} f_{\varepsilon} \cdot u d x=\int_{\Omega_{\varepsilon}^{*}} f_{\varepsilon} \cdot u \operatorname{det}\left(\mathbf{t}_{1}+\frac{s_{3}}{a} \mathbf{t}_{1}\left|\mathbf{t}_{2}\right| \mathbf{n}\right) d s \\
& =\varepsilon^{3} 2 \kappa \int_{\omega_{\varepsilon}^{*}} f \cdot \mathcal{U} d s_{1} d s_{2}+2 \kappa \varepsilon^{2} \int_{\omega_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}+\frac{2 \varepsilon^{3} \kappa^{3}}{3 a} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{U}_{\alpha} d s_{1} d s_{2}  \tag{6.11}\\
& +\frac{2 \varepsilon^{3} \kappa^{3}}{3} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{R}_{\alpha} d s_{1} d s_{2}+\frac{2 \varepsilon^{4} \kappa^{3}}{3 a} \int_{\omega_{\varepsilon}^{*}} F \cdot \mathcal{R} d s_{1} d s_{2}+\frac{2 \varepsilon^{5} \kappa^{3}}{3 a} \int_{\omega_{\varepsilon}^{*}} f_{\alpha} \mathcal{R}_{\alpha} d s_{1} d s_{2} \\
& +\int_{\Omega_{\varepsilon}^{*}} \frac{s_{3}^{2}}{a} g \cdot \bar{u} d s+\int_{\Omega_{\varepsilon}^{*}} \frac{\varepsilon}{a} s_{3} F \cdot \bar{u} d s+\int_{\Omega_{\varepsilon}^{*}} \frac{s_{3}}{a} \varepsilon^{2} f_{3} \bar{u} \cdot \mathbf{n} d s
\end{align*}
$$

First, using the estimates $1.19_{2}$ and 1.21$)_{2}$ one gets

$$
\begin{aligned}
& \left|\int_{\Omega_{\varepsilon}^{*}} \varepsilon s_{3}^{2} g \cdot \bar{u} d s\right| \leq C \varepsilon^{9 / 2}\|g\|_{L^{2}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \quad\left|\int_{\Omega_{\varepsilon}^{*}} \varepsilon s_{3} F \cdot \bar{u} d s\right| \leq C \varepsilon^{7 / 2}\|F\|_{L^{2}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \\
& \left|\int_{\Omega_{\varepsilon}^{*}} \varepsilon^{2} s_{3} f_{3} \bar{u} \cdot \mathbf{n} d s\right| \leq C \varepsilon^{9 / 2}\|f\|_{L^{2}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \quad\left|\int_{\omega_{\varepsilon}^{*}} \varepsilon^{4} F \cdot \mathcal{R} d s_{1} d s_{2}\right| \leq C \varepsilon^{5 / 2}\|F\|_{L^{2}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \\
& \left|\int_{\omega_{\varepsilon}^{*}} \varepsilon^{5} f_{\alpha} \mathcal{R}_{\alpha} d s_{1} d s_{2}\right| \leq C \varepsilon^{7 / 2}\|F\|_{L^{2}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
\end{aligned}
$$

Hence, 6.9 is proved. Now, $1.21_{2}$ also leads to

$$
\begin{aligned}
& \left|\varepsilon^{3} 2 \kappa \int_{\omega_{\varepsilon}^{*}} f \cdot \mathcal{U} d s_{1} d s_{2}+\frac{2 \varepsilon^{3} \kappa^{3}}{3 a} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{U}_{\alpha} d s_{1} d s_{2}+\frac{2 \varepsilon^{3} \kappa^{3}}{3} \int_{\omega_{\varepsilon}^{*}} g_{\alpha} \mathcal{R}_{\alpha} d s_{1} d s_{2}\right| \\
& \quad \leq C \varepsilon^{3 / 2}\left(\|f\|_{L^{2}(\omega)}+\|g\|_{L^{2}(\omega)}\right)\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
\end{aligned}
$$

Now, it remains to estimate $\int_{\omega_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}$. For every function $\phi$ in $L^{1}(\omega)$, we denote

$$
\mathcal{M}_{\varepsilon}(\phi)\left(s^{\prime}\right)=\frac{1}{\varepsilon^{2}\left|Y^{\prime}\right|} \int_{Y^{\prime}} \phi\left(\varepsilon\left[\frac{s^{\prime}}{\varepsilon}\right]_{Y^{\prime}}+\varepsilon z\right) d z_{1} d z_{2}, \quad \text { for a.e. } s^{\prime} \in \widehat{\omega}_{\varepsilon}
$$

Function $\mathcal{M}_{\varepsilon}(\phi)$ belongs to $L^{1}\left(\widehat{\omega}_{\varepsilon}\right)$ (see [1, 2] for the properties of the operator $\mathcal{M}_{\varepsilon}$.) Recall that by $1.22,1.21)_{2}$, Lemma 5.1 and 5.3 one has

$$
\|\mathcal{U}\|_{E} \leq \frac{C}{\varepsilon^{1 / 2}}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \quad\left\|\mathcal{U}_{E}\right\|_{H^{1}(\omega)} \leq \frac{C}{\varepsilon^{3 / 2}}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
$$

One has (see [2, Proposition 1.38 ])

$$
\begin{array}{r}
\left|\int_{\widehat{\omega}_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}-\int_{\widehat{\omega}_{\varepsilon}^{*}} F \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}\right| \leq C \varepsilon\left\|\nabla \mathcal{U}_{E}\right\|_{L^{2}(\omega)}\|F\|_{L^{2}(\omega)}, \\
\left|\int_{\widehat{\omega}_{\varepsilon}^{*}} F \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}-\int_{\widehat{\omega}_{\varepsilon}^{*}} \mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}\right| \leq C \varepsilon\left\|\mathcal{U}_{E}\right\|_{L^{2}(\omega)}\|\nabla F\|_{L^{2}(\omega)} .
\end{array}
$$

Hence
$\left|\int_{\widehat{\omega}_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}-\int_{\widehat{\omega}_{\varepsilon}^{*}} \mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}\right| \leq C \varepsilon\left\|\mathcal{U}_{E}\right\|_{H^{1}(\omega)}\|F\|_{H^{1}(\Omega)} \leq \frac{C}{\varepsilon^{1 / 2}}\|F\|_{H^{1}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}$.
Since $\mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right)$ is constant on every $\varepsilon$-cell, that gives

$$
\int_{\widehat{\omega}_{\varepsilon}^{*}} \mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}=\frac{\left|Y^{\prime *}\right|}{\left|Y^{\prime}\right|} \int_{\widehat{\omega}_{\varepsilon}} \mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}
$$

Proceeding as above, one shows that

$$
\left|\int_{\widehat{\omega}_{\varepsilon}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}-\int_{\widehat{\omega}_{\varepsilon}} \mathcal{M}_{\varepsilon}(F) \cdot \mathcal{M}_{\varepsilon}\left(\mathcal{U}_{E}\right) d s_{1} d s_{2}\right| \leq \frac{C}{\varepsilon^{1 / 2}}\|F\|_{H^{1}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
$$

Summarizing the above estimates and using 6.7 give (recall that there are no holes in $\Lambda_{\varepsilon}$ )

$$
\begin{aligned}
& \quad\left|\int_{\omega_{\varepsilon}^{*}} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}-\frac{\left|Y^{\prime} *\right|}{\left|Y^{\prime}\right|} \int_{\omega} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}\right| \leq \frac{C}{\varepsilon^{1 / 2}}\|F\|_{H^{1}(\omega)}\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}, \\
& \text { and }\left|\int_{\omega} F \cdot \mathcal{U}_{E} d s_{1} d s_{2}\right| \leq \frac{C}{\varepsilon^{1 / 2}}\left(\left\|F_{3}\right\|_{L^{2}\left(0, L ; H^{2}(0, a \pi)\right)}+\|F\|_{H^{1}(\omega)}\right)\|e(u)\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)}
\end{aligned}
$$

which leads to 6.10.
Using now $u=u_{\varepsilon}$ as test function in (6.5) we obtain

$$
\left\|e\left(u_{\varepsilon}\right)\right\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}^{*}\right)} \leq C \varepsilon^{3 / 2}\left(\|f\|_{L^{2}(\omega)}+\|g\|_{L^{2}(\omega)}+\left\|F_{3}\right\|_{L^{2}\left(0, L ; H^{2}(0, a \pi)\right)}+\|F\|_{H^{1}(\omega)}\right)
$$

## 7. Unfolded limit problems

For every $\left(\mathcal{V}_{E}, \mathcal{V}\right)$ in $\mathbb{D}_{E} \times \mathbb{D}_{I}$ we define the symmetric tensor $\mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)$ by

$$
\mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)=\left(\begin{array}{ccc}
\mathcal{Z}_{11}\left(\mathcal{V}_{E}\right)-y_{3} \Lambda_{11}(\mathcal{V}) & \mathcal{Z}_{12}\left(\mathcal{V}_{E}\right)-y_{3} \Lambda_{12}(\mathcal{V}) & 0 \\
\mathcal{Z}_{12}\left(\mathcal{V}_{E}\right)-y_{3} \Lambda_{12}(\mathcal{V}) & \mathcal{Z}_{22}\left(\mathcal{V}_{E}\right)-y_{3} \Lambda_{22}(\mathcal{V}) & 0 \\
0 & 0 & 0
\end{array}\right)
$$

with

$$
\mathcal{Z}_{\alpha \beta}(\mathcal{V})=\frac{1}{2}\left[\frac{\partial \mathcal{V}}{\partial s_{\alpha}} \mathbf{t}_{\beta}+\frac{\partial \mathcal{V}}{\partial s_{\beta}} \mathbf{t}_{\alpha}\right]
$$

and

$$
\begin{aligned}
& \Lambda_{11}(\mathcal{V})=\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1}^{2}}-\frac{1}{a} \frac{\partial \mathcal{V}_{1}}{\partial s_{1}}=\frac{\partial}{\partial s_{1}}\left(\frac{\partial \mathcal{V}}{\partial s_{1}} \mathbf{n}\right), \quad \Lambda_{22}(\mathcal{V})=\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{2}^{2}}=\frac{\partial}{\partial s_{2}}\left(\frac{\partial \mathcal{V}}{\partial s_{2}} \mathbf{n}\right) \\
& \Lambda_{12}(\mathcal{V})=\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}-\frac{1}{a} \frac{\partial \mathcal{V}_{1}}{\partial s_{2}}=\frac{\partial}{\partial s_{2}}\left(\frac{\partial \mathcal{V}}{\partial s_{1}} \mathbf{n}\right)
\end{aligned}
$$

Denote $H_{p e r}^{1}\left(Y^{*}\right)$ the subspace of $H^{1}\left(Y^{*}\right)$ containing the functions $\mathbf{G}$ periodic and

$$
\mathbb{D} \doteq\left\{v=\left(\mathcal{V}_{E}, \mathcal{V}, \widehat{v}\right) \in \mathbb{D}_{I} \times \mathbb{D}_{E} \times L^{2}\left(\Omega ; H_{p e r}^{1}\left(Y^{*}\right)\right)^{3}\right\}
$$

For every $v \in \mathbb{D}$ we consider the symmetric tensor

$$
\mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)+\mathcal{E}_{y}(\widehat{v})
$$

and the semi-norm

$$
\|v\|_{\mathbb{D}}=\left\|\mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)+\mathcal{E}_{y}(\widehat{v})\right\|_{L^{2}\left(\omega \times Y^{*}\right)}
$$

Lemma 7.1. Given the expressions (3.7) for $\mathcal{V} \in \mathbb{D}_{I}$, there exist $c, C \in \mathbb{R}_{+}$such that

$$
c\|\mathcal{V}\|_{\mathbb{D}_{I}}^{2} \leq \sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}(\mathcal{V})\right\|_{L^{2}(\omega)}^{2} \leq C\|\mathcal{V}\|_{\mathbb{D}_{I}}^{2}
$$

Proof. First, one has

$$
\sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}(\mathcal{V})\right\|_{L^{2}(\omega)}^{2} \leq C\left(\left\|D_{2} \mathcal{V}_{3}\right\|_{L^{2}(\omega)}+\left\|\nabla \mathcal{V}_{1}\right\|_{L^{2}(\omega)}\right)
$$

This inequality and Lemma 5.1 give the inequality in the right-hand side.
We prove the left-hand side of the inequality by contradiction. We assume that there exists a sequence $\left(\mathcal{V}_{n}\right)_{n \in \mathbb{N}}$ in $\mathbb{D}_{I}$, such that

$$
\left\|\mathcal{V}_{n}\right\|_{\mathbb{D}_{I}}=1, \quad \sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}\left(\mathcal{V}_{n}\right)\right\|_{L^{2}(\omega)}^{2} \rightarrow 0 \text { as } n \rightarrow \infty
$$

By Lemma 5.1 and the expressions in (5.1), we can also consider a sequence $\left(V_{1, n}, V_{2, n}\right)_{n \in \mathbb{N}}$ in $H_{0}^{3}(0, a \pi) \times \bar{H}_{0}^{4}(0, a \pi)$ with

$$
\left\|V_{1, n}\right\|_{H_{0}^{3}(0, a \pi)}^{2}+\left\|V_{2, n}\right\|_{H_{0}^{4}(0, a \pi)}^{2}=1
$$

and the components $\Lambda_{\alpha \beta}$ can be expressed as

$$
\begin{align*}
& \Lambda_{11}\left(\mathcal{V}_{n}\right)=\frac{1}{a}\left(s_{2}^{c} V_{2, n}^{\prime \prime}\left(s_{1}\right)-V_{1, n}^{\prime}\left(s_{1}\right)\right)+a\left(s_{2}^{c} V_{2, n}^{\prime \prime \prime \prime}\left(s_{1}\right)-V_{1, n}^{\prime \prime \prime}\right)  \tag{7.1}\\
& \Lambda_{12}\left(\mathcal{V}_{n}\right)=\frac{1}{a} V_{2, n}^{\prime}\left(s_{1}\right)+a V_{2, n}^{\prime \prime \prime}\left(s_{1}\right), \quad \Lambda_{22}\left(\mathcal{V}_{n}\right)=0
\end{align*}
$$

We have then that there exists $\left(V_{1}, V_{2}\right)$ in $H_{0}^{3}(0, a \pi) \times H_{0}^{4}(0, a \pi)$ such that

$$
\left(V_{1, n}, V_{2, n}\right) \rightharpoonup\left(V_{1}, V_{2}\right) \quad \text { weakly in } \quad H_{0}^{3}(0, a \pi) \times H_{0}^{4}(0, a \pi)
$$

By Sobolev embedding we then get

$$
\left(V_{1, n}, V_{2, n}\right) \longrightarrow\left(V_{1}, V_{2}\right) \quad \text { strongly in } \quad H_{0}^{2}(0, a \pi) \times H_{0}^{3}(0, a \pi)
$$

Moreover, since $\left\|\Lambda_{\alpha \beta}\right\| \longrightarrow 0$ for $(\alpha, \beta) \in\{(1,1),(1,2),(2,2)\}$. We have that

$$
\begin{equation*}
\frac{1}{a}\left(s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right)-V_{1}^{\prime}\left(s_{1}\right)\right)+a\left(s_{2}^{c} V_{2}^{\prime \prime \prime \prime}\left(s_{1}\right)-V_{1}^{\prime \prime \prime}\right)=0, \quad \frac{1}{a} V_{2}^{\prime}\left(s_{1}\right)+a V_{2}^{\prime \prime \prime}\left(s_{1}\right)=0 \tag{7.2}
\end{equation*}
$$

Solving the differential equations with the respective boundary conditions we obtain that $V_{2}=V_{1}=0$. Therefore, we have that $\left(V_{1, n}, V_{2, n}\right)$ converges strongly to $(0,0)$ in $H_{0}^{2}(0, a \pi) \times H_{0}^{3}(0, a \pi)$.
Considering again equation 7.1 and with our assumption that $\left\|\Lambda_{11}\left(\mathcal{V}_{n}\right)\right\|_{L^{2}(\omega)} \rightarrow 0$, we also get $\left(V_{1, n}^{\prime \prime \prime}, V_{2, n}^{\prime \prime \prime \prime}\right) \rightarrow(0,0)$ strongly in $L^{2}(0, a \pi)^{2}$ and then the convergence $\left(V_{1, n}, V_{2, n}\right) \rightarrow(0,0)$ strongly in $H_{0}^{3}(0, a \pi) \times H_{0}^{4}(0, a \pi)$, which contradicts the fact that $\left\|V_{1}\right\|_{H_{0}^{3}(0, a \pi)}^{2}+\left\|V_{2}\right\|_{H_{0}^{4}(0, a \pi)}^{2}=1$ coming from the assumption $\left\|V_{1, n}\right\|_{H_{0}^{3}(0, a \pi)}^{2}+\left\|V_{2, n}\right\|_{H_{0}^{4}(0, a \pi)}^{2}=1$ for all $n \in \mathbb{N}$.

Lemma 7.2. Consider the space $\mathbb{S} \doteq \mathbb{R}^{3} \times \mathbb{R}^{3} \times H_{\text {per }, 0}^{1}\left(Y^{*}\right)^{3}$ with the seminorm

$$
\begin{aligned}
\left\|\left(\tau_{A}, \tau_{B}, \widehat{w}\right)\right\|_{\mathbb{S}}^{2} & =\sum_{\substack{\alpha, \beta=1 \\
\alpha \leq \beta}}^{2}\left\|\tau_{A}^{\alpha \beta}+y_{3} \tau_{B}^{\alpha \beta}+e_{\alpha \beta, y}(\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}^{2} \\
& +\left\|e_{13, y}(\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}^{2}+\left\|e_{23, y}(\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}^{2}+\left\|e_{33, y}(\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}^{2}
\end{aligned}
$$

Then this expression actually defines a norm on $\mathbb{S}$ equivalent to the product-norm.
Proof. We consider the field $\Phi \in H^{1}\left(\mathbb{R}^{3}\right)^{3}$ given by

$$
\begin{aligned}
& \Phi_{1}(y)=y_{1}\left(\tau_{A}^{11}+y_{3} \tau_{B}^{11}\right)+y_{2}\left(\tau_{A}^{12}+y_{3} \tau_{B}^{12}\right) \\
& \Phi_{2}(y)=y_{2}\left(\tau_{A}^{22}+y_{3} \tau_{B}^{22}\right)+y_{1}\left(\tau_{A}^{12}+y_{3} \tau_{B}^{12}\right), \\
& \Phi_{3}(y)=-\left[\frac{\left(y_{1}\right)^{2}}{2} \tau_{B}^{11}+\frac{\left(y_{2}\right)^{2}}{2} \tau_{B}^{22}+y_{2} y_{1} \tau_{B}^{12}\right] .
\end{aligned}
$$

Hence, we have

$$
\left\|\left(\tau_{A}, \tau_{B}, \widehat{w}\right)\right\|_{\mathbb{S}}=\left\|\mathcal{E}_{y}(\Phi+\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}
$$

We will now show that $\left\|\mathcal{E}_{y}(\Phi+\widehat{w})\right\|_{L^{2}\left(Y^{*}\right)}=0 \Longrightarrow \Phi=0, \widehat{w}=0$.
Consider the case that $\mathcal{E}_{y}(\Phi+\widehat{w})=0$, which yields that $\Phi+\widehat{w}$ is a rigid displacement. Hence, there exist $a, b \in \mathbb{R}^{3}$ such that

$$
\Phi+\widehat{w}=r, \quad r(y)=\left(\begin{array}{l}
a_{1}+b_{2} y_{3}-b_{3} y_{2} \\
a_{2}+b_{3} y_{1}-b_{1} y_{3} \\
a_{3}+b_{1} y_{2}-b_{2} y_{1}
\end{array}\right)
$$

Since, $\widehat{w}$ is a periodic function with period $\mathbf{p}_{1}, \mathbf{p}_{2}$, one has $(\Phi-r)\left(y+\mathbf{p}_{i}\right)=(\Phi-r)(y)$ for a.e. $y \in\left(\mathbb{R}^{2} \backslash \bigcup_{\xi \in \mathbf{G}}(\xi+\bar{S})\right) \times(-\kappa, \kappa)$. Considering the first two components yields the equations

$$
\begin{aligned}
& \tau_{A}^{11}+y_{3} \tau_{B}^{11}=0, \\
& \tau_{A}^{22}+y_{3} \tau_{B}^{22}=0, \\
& \tau_{A}^{12}+y_{3} \tau_{B}^{12}=-y_{3} \tau_{B}^{12}=b_{3},
\end{aligned} \quad \text { for a.e. } y_{3} \in(-\kappa, \kappa)
$$

Therefore, we obtain $\tau_{A}^{11}=\tau_{B}^{11}=\tau_{A}^{22}=\tau_{B}^{22}=\tau_{A}^{12}=\tau_{B}^{12}=0$ and $b_{3}=0$. Now, the equality of the third component yields $b_{1}=b_{2}=0$. Finally, we conclude that $\Phi=0, r$ is a constant displacement and since $\widehat{w} \in H_{p e r, 0}^{1}\left(Y^{*}\right)^{3}$ the displacement $r=0$ and therefore $\widehat{w}=0$, which proves that $\|\cdot\|_{\mathbb{S}}$ is a norm. By contradiction we easily prove that there exists a constant $C>0$ such that

$$
\left.C\left(\left|\tau_{A}\right|+\left|\tau_{B}\right|+\|\widehat{w}\|_{H^{1}\left(Y^{*}\right)}\right) \leq\left\|\left(\tau_{A}, \tau_{B}, \widehat{w}\right)\right\|_{\mathbb{S}}, \quad \forall\left(\tau_{A}, \tau_{B}, \widehat{w}\right)\right) \in \mathbb{S}
$$

which ends the proof.
Lemma 7.3. The semi-norm $\|\cdot\|_{\mathbb{D}}$ is a norm equivalent to the product-norm of $\mathbb{D}_{I} \times \mathbb{D}_{E} \times L^{2}\left(\Omega ; H_{p e r}^{1}\left(Y^{*}\right)\right)^{3}$.
Proof. By the definition of $\|\cdot\|_{\mathbb{D}_{I}}$, we have that

$$
\begin{aligned}
\|v\|_{\mathbb{D}}^{2}= & \sum_{\alpha, \beta=1}^{2}\left\|\mathcal{Z}_{\alpha \beta}\left(\mathcal{V}_{E}\right)-y_{3} \Lambda_{\alpha \beta}(\mathcal{V})+e_{\alpha \beta, y}(\widehat{v})\right\|_{L^{2}\left(\omega \times Y^{*}\right)}^{2} \\
& +2\left\|e_{13, y}(\widehat{v})\right\|_{L^{2}\left(\omega \times Y^{*}\right)}^{2}+2\left\|e_{23, y}(\widehat{v})\right\|_{L^{2}\left(\omega \times Y^{*}\right)}^{2}+2\left\|e_{33, y}(\widehat{v})\right\|_{L^{2}\left(\omega \times Y^{*}\right)}^{2} .
\end{aligned}
$$

We may further note that we have

$$
\begin{aligned}
\left\|\mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)\right\|_{L^{2}(\Omega)}^{2} & =\sum_{\alpha, \beta=1}^{2} \int_{\Omega}\left(Z_{\alpha \beta}\left(\mathcal{V}_{E}\right)+y_{3} \Lambda_{\alpha \beta}\left(\mathcal{V}_{I}\right)\right)^{2} d s \\
& =2 \kappa \sum_{\alpha, \beta=1}^{2}\left\|\mathcal{Z}_{\alpha \beta}\left(\mathcal{V}_{E}\right)\right\|_{L^{2}(\omega)}^{2}+\frac{2 \kappa^{3}}{3} \sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}(\mathcal{V})\right\|_{L^{2}(\omega)}^{2}
\end{aligned}
$$

We obtain with Lemma 7.2 and the equivalence of norms that

$$
\begin{aligned}
& c\left(\sum_{\alpha, \beta=1}^{2}\left\|\mathcal{Z}_{\alpha \beta}\right\|_{L^{2}(\omega)}+\sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}\right\|_{L^{2}(\omega)}\right.\left.+\|\widehat{v}\|_{L^{2}\left(\omega \times Y^{*}\right)}\right) \leq\|\mathcal{V}\|_{\mathbb{D}_{I}} \\
& \leq C\left(\sum_{\alpha, \beta=1}^{2}\left\|\mathcal{Z}_{\alpha \beta}\right\|_{L^{2}(\omega)}+\sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}\right\|_{L^{2}(\omega)}+\|\widehat{v}\|_{L^{2}\left(\omega \times Y^{*}\right)}\right) .
\end{aligned}
$$

Further note that $\sum_{\alpha, \beta=1}^{2}\left\|\mathcal{Z}_{\alpha \beta}\left(\mathcal{V}_{E}\right)\right\|_{L^{2}(\omega)}^{2}=\left\|\mathcal{V}_{E}\right\|_{E}^{2}$.
Besides, with Lemma 7.1 we obtain that

$$
c\|\mathcal{V}\|_{\mathbb{D}_{I}}^{2} \leq \sum_{\alpha, \beta=1}^{2}\left\|\Lambda_{\alpha \beta}(\mathcal{V})\right\|_{L^{2}(\omega)}^{2} \leq C\|\mathcal{V}\|_{\mathbb{D}_{I}}^{2}
$$

Finally, we can conclude

$$
c\left(\left\|\mathcal{V}_{E}\right\|_{E}+\|\mathcal{V}\|_{\mathbb{D}_{I}}+\|\widehat{v}\|_{L^{2}\left(\omega \times Y^{*}\right)}\right) \leq\|v\|_{\mathbb{D}} \leq C\left(\left\|\mathcal{V}_{E}\right\|_{E}+\|\mathcal{V}\|_{\mathbb{D}_{I}}+\|\widehat{v}\|_{L^{2}\left(\omega \times Y^{*}\right)}\right)
$$

Theorem 7.1. Let $u_{\varepsilon}$ be the solution of the elasticity problem 6.5). Then the following convergence holds:

$$
\begin{equation*}
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e\left(u_{\varepsilon}\right)\right) \rightarrow \mathcal{E}\left(\mathcal{U}_{E}, \mathcal{U}\right)+\mathcal{E}_{y}(\widehat{u}) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)^{9} \tag{7.3}
\end{equation*}
$$

where $\left(\mathcal{U}_{E}, \mathcal{U}, \widehat{u}\right) \in \mathbb{D}$ is the unique solution of the rescaled and unfolded problem

$$
\begin{align*}
& \frac{1}{2 \kappa} \int_{\omega \times Y^{*}} a_{i j k l}\left(E_{S, i j}\left(\mathcal{U}_{E}, \mathcal{U}\right)+\mathcal{E}_{i j, y}(\widehat{u})\right)\left(E_{S, k l}\left(\mathcal{V}_{E}, \mathcal{V}\right)+\mathcal{E}_{k l, y}(\widehat{v})\right) d s^{\prime} d y \\
& \quad=\left|Y^{\prime *}\right|\left(\int_{\omega}\left(f \cdot \mathcal{V}+\frac{\kappa^{2}}{3 a} g_{\alpha} \mathcal{V}_{\alpha}-\frac{\kappa^{2}}{3} g_{\alpha} \frac{\partial \mathcal{V}}{\partial s_{\alpha}} \boldsymbol{n}\right) d s^{\prime}+\left\langle F, \mathcal{V}_{E}\right\rangle\right), \quad \forall\left(\mathcal{V}_{E}, \mathcal{V}, \widehat{v}\right) \in \mathbb{D} \tag{7.4}
\end{align*}
$$

Proof. Take $v=\left(\mathcal{V}_{E}, \mathcal{V}, \widehat{v}\right)$ such that

$$
\mathcal{V}_{E} \in \mathcal{C}^{1}(\bar{\omega})^{3} \cap \mathbb{D}_{E}, \quad \mathcal{V} \quad \in \mathcal{C}^{2}(\bar{\omega})^{3} \cap \mathbb{D}_{I}
$$

and consider the test function $v_{\varepsilon}=v_{\varepsilon, 1}+v_{\varepsilon, 2}$, where

$$
\begin{aligned}
& v_{\varepsilon, 1}(s)=\mathcal{V}\left(s^{\prime}\right)+\varepsilon \mathcal{V}_{E}\left(s^{\prime}\right)-s_{3}\left[\frac{\partial\left(\mathcal{V}+\varepsilon \mathcal{V}_{E}\right)}{\partial s_{\alpha}}\left(s^{\prime}\right) \cdot \mathbf{n}\left(s^{\prime}\right)\right] \mathbf{t}_{\alpha}\left(s^{\prime}\right) \\
& v_{\varepsilon, 2}(s)=\varepsilon^{2} \widehat{v}\left(s^{\prime},\left\{\frac{s}{\varepsilon}\right\}\right)
\end{aligned}
$$

with $\widehat{v} \in \mathcal{C}^{1}\left(\bar{\omega} ; H_{p e r}^{1}\left(Y^{*}\right)^{3}\right)$ satisfying $\widehat{v}\left(0, s_{2}, y\right)=\widehat{v}\left(a \pi, s_{2}, y\right)$ for a.e. $\left(s_{2}, y\right) \in(0, L) \times Y^{*}$.

We calculate the elements $e_{12}\left(v_{\varepsilon, 1}\right)$ and $e_{13}\left(v_{\varepsilon, 1}\right)$, since the rest follows in a similar way. We obtain

$$
\begin{aligned}
& e_{12}\left(v_{\varepsilon, 1}\right)=\frac{1}{2} \frac{a}{a+s_{3}}\left[\frac{\partial \mathcal{V}_{2}}{\partial s_{1}}+\varepsilon \frac{\partial \mathcal{V}_{E, 2}}{\partial s_{1}}-s_{3}\left(\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}+\varepsilon \frac{\partial^{2} \mathcal{V}_{E, 3}}{\partial s_{1} \partial s_{2}}\right)\right. \\
& \left.+\left(1+\frac{s_{3}}{a}\right)\left(\frac{\partial \mathcal{V}_{1}}{\partial s_{2}}+\varepsilon \frac{\partial \mathcal{V}_{E, 1}}{\partial s_{2}}\right)-\left(s_{3}+\frac{s_{3}^{2}}{a}\right)\left(\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}-\frac{1}{a} \frac{\partial \mathcal{V}_{1}}{\partial s_{2}}+\varepsilon\left(\frac{\partial^{2} \mathcal{V}_{E, 3}}{\partial s_{1} \partial s_{2}}-\frac{1}{a} \frac{\partial \mathcal{V}_{E, 1}}{\partial s_{2}}\right)\right)\right] .
\end{aligned}
$$

Applying the rescaling-unfolding operator $\Pi_{\varepsilon}$ and dividing by $\varepsilon$ yields with the properties for $\mathbb{D}_{I}$ that

$$
\begin{aligned}
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{12}\left(v_{\varepsilon, 1}\right)\right)=\frac{1}{2} \frac{a}{a+\varepsilon y_{3}}\left[\left(\frac{\partial \mathcal{V}_{E, 2}}{\partial s_{1}}+\frac{\partial \mathcal{V}_{E, 1}}{\partial s_{2}}\right)-2 y_{3} \frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}+2 \frac{y_{3}}{a} \frac{\partial \mathcal{V}_{1}}{\partial s_{2}}\right. \\
& \left.\quad+\frac{\varepsilon y_{3}}{a} \frac{\partial \mathcal{V}_{E, 1}}{\partial s_{2}}-\frac{\varepsilon y_{3}}{a} \frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}+\frac{\varepsilon y_{3}}{a} \frac{\partial \mathcal{V}_{1}}{\partial s_{2}}-\varepsilon y_{3} \frac{\partial^{2} \mathcal{V}_{E, 3}}{\partial s_{1} \partial s_{2}}-\left(\varepsilon+\frac{\varepsilon^{2}}{a}\right)\left(\frac{\partial^{2} \mathcal{V}_{E, 3}}{\partial s_{1} \partial s_{2}}-\frac{1}{a} \frac{\partial \mathcal{V}_{E, 1}}{\partial s_{2}}\right)\right] \\
& \quad \longrightarrow \frac{1}{2} \mathcal{Z}_{12}\left(\mathcal{V}_{E}\right)-y_{3}\left(\frac{\partial^{2} \mathcal{V}_{3}}{\partial s_{1} \partial s_{2}}-\frac{\partial \mathcal{V}_{1}}{\partial s_{2}}\right) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)^{9}
\end{aligned}
$$

For $e_{13}\left(v_{\varepsilon, 1}\right)$ we then obtain

$$
\begin{aligned}
& e_{13}\left(v_{\varepsilon, 1}\right)=\frac{1}{2} \frac{a}{a+s_{3}}\left[\left(\frac{\partial \mathcal{V}_{3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)+\varepsilon\left(\frac{\partial \mathcal{V}_{E, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)+\frac{s_{3}}{a}\left(\frac{\partial \mathcal{V}_{3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)\right. \\
& \left.+\frac{s_{3} \varepsilon}{a}\left(\frac{\partial \mathcal{V}_{E, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)+\left(1+\frac{s_{3}}{a}\right)\left(-\left(\frac{\partial \mathcal{V}_{3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)-\varepsilon\left(\frac{\partial \mathcal{V}_{E, 3}}{\partial s_{1}}-\frac{1}{a} \mathcal{V}_{1}\right)\right)\right]=0
\end{aligned}
$$

In conclusion we get that

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e\left(v_{\varepsilon, 1}\right)\right) \longrightarrow \mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)^{9}
$$

In the next step we focus on the calculation for $e_{i j}\left(v_{\varepsilon, 2}\right)$, where we once again just focus on $e_{12}$ and $e_{13}$. One has

$$
\begin{aligned}
& e_{12}\left(v_{\varepsilon, 2}\right)(s)=\frac{\varepsilon^{2}}{2}\left(\frac{a}{a+s_{3}} \frac{\partial \widehat{v}}{\partial s_{1}} \mathbf{t}_{2}+\frac{\partial \widehat{v}}{\partial s_{2}} \mathbf{t}_{1}\right)\left(s^{\prime},\left\{\frac{s}{\varepsilon}\right\}\right)+\frac{\varepsilon}{2}\left(\frac{a}{a+s_{3}} \frac{\partial \widehat{v}}{\partial y_{1}} \mathbf{t}_{2}+\frac{\partial \widehat{v}}{\partial y_{2}} \mathbf{t}_{1}\right)\left(s^{\prime},\left\{\frac{s}{\varepsilon}\right\}\right), \\
& e_{13}\left(v_{\varepsilon, 2}\right)(s)=\frac{\varepsilon^{2}}{2}\left(\frac{a}{a+s_{3}} \frac{\partial \widehat{v}}{\partial s_{1}} \mathbf{n}+\frac{\partial \widehat{v}}{\partial s_{3}} \mathbf{t}_{1}\right)\left(s^{\prime},\left\{\frac{s}{\varepsilon}\right\}\right)+\frac{\varepsilon}{2}\left(\frac{a}{a+s_{3}} \frac{\partial \widehat{v}}{\partial y_{1}} \mathbf{n}+\frac{\partial \widehat{v}}{\partial y_{3}} \mathbf{t}_{1}\right)\left(s^{\prime},\left\{\frac{s}{\varepsilon}\right\}\right) .
\end{aligned}
$$

Considering now $\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{12}\left(v_{\varepsilon, 2}\right)\right)$ and $\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{13}\left(v_{\varepsilon, 2}\right)\right)$, we obtain

$$
\begin{aligned}
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{12}\left(v_{\varepsilon, 2}\right)\right) \longrightarrow e_{y, 12}(\widehat{v}) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right) \\
& \frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e_{13}\left(v_{\varepsilon, 2}\right)\right) \longrightarrow e_{y, 13}(\widehat{v}) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)
\end{aligned}
$$

which then yields

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e\left(v_{\varepsilon, 2}\right)\right) \longrightarrow \mathcal{E}_{y}(\widehat{v}) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)^{9}
$$

therefore

$$
\frac{1}{\varepsilon} \Pi_{\varepsilon}\left(e\left(v_{\varepsilon}\right)\right) \longrightarrow \mathcal{E}\left(\mathcal{V}_{E}, \mathcal{V}\right)+\mathcal{E}_{y}(\widehat{v}) \quad \text { strongly in } L^{2}\left(\omega \times Y^{*}\right)^{9}
$$

Plugging in our test function $v_{\varepsilon}$ into the weak formulation (6.5), applying the rescaling-unfolding operator on both sides. Dividing by $2 \kappa \varepsilon^{3}$ and passing to the limit, we obtain (7.4) with the chosen test functions (regarding the right-hand side, we use the results from Lemma 6.1 and [2, Proposition 4.8] to get an integral over the whole domain $\omega$ at the limit). Then, by density of $\mathcal{C}^{1}(\bar{\omega})^{3} \cap \mathbb{D}_{E}$ in $\mathbb{D}_{E}$, $\mathcal{C}^{2}(\bar{\omega})^{3} \cap \mathbb{D}_{I}$ in $\mathbb{D}_{I}$ and $\mathcal{C}^{1}\left(\bar{\omega} ; H_{p e r}^{1}\left(Y^{*}\right)^{3}\right)$ in $L^{2}\left(\omega ; H_{\text {per }}^{1}\left(Y^{*}\right)^{3}\right)$, this yields 7.4 for every $\left(\mathcal{V}_{E}, \mathcal{V}, \widehat{v}\right) \in \mathbb{D}$. Due to the coercivity of $a_{i j k l}$ and Lemma 10.1 , we can apply Lax-Milgram theorem to the weak formulation 7.4 . Therefore, this problem has an unique solution.

## 8. Homogenization of the shell

In this section we want to express the warping-microscopic displacement $\widehat{u}$ with respect to the macroscopic $\mathcal{U}_{E}$ and $\mathcal{U}$. Therefore, choosing $\mathcal{V}=0$ in equation (6.5) leads to

$$
\int_{Y^{*}} a_{i j k l}\left(E_{S, i j}\left(\mathcal{U}_{E}, \mathcal{U}\right)+\mathcal{E}_{i j, y}(\widehat{u})\right) \mathcal{E}_{k l, y}(\widehat{v}) d y=0 \quad \forall \widehat{v} \in H_{p e r}^{1}\left(Y^{*}\right)^{3}
$$

Hence, we can write $\widehat{u}$ in terms of $\left(\mathcal{U}_{E}, \mathcal{U}\right)$. We define the 3 matrices

$$
\mathbf{M}^{11}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \mathbf{M}^{12}=\mathbf{M}^{21}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \quad \mathbf{M}^{22}=\left(\begin{array}{lll}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

and introduce the 6 distinct correctors $\left((\alpha, \beta) \in\{1,2\}^{2}\right)$

$$
\begin{gathered}
\widetilde{\chi_{E}^{\alpha \beta}} \in H_{p e r}^{1}\left(Y^{*}\right)^{3}, \quad \widetilde{\chi_{I}^{\alpha \beta}} \in H_{p e r}^{1}\left(Y^{*}\right)^{3}, \quad \text { where } \\
\widetilde{\chi_{E}^{12}}=\widetilde{\chi_{E}^{21}}, \quad \widetilde{\chi_{I}^{12}}=\widetilde{\chi_{I}^{21}},
\end{gathered}
$$

and which are defined by

$$
\begin{align*}
& \int_{Y^{*}} a_{i j k l}\left(\mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{E}^{\alpha \beta}}\right)\right) \mathcal{E}_{k l, y}(\widetilde{\psi}) d y=0 \\
& \int_{Y^{*}} a_{i j k l}\left(y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{I}^{\alpha \beta}}\right)\right) \mathcal{E}_{k l, y}(\widetilde{\psi}) d y=0, \quad \forall \tilde{\psi} \in H_{p e r}^{1}\left(Y^{*}\right)^{3} .
\end{align*}
$$

Hence, we can write $\widehat{u}$ as

$$
\widehat{u}\left(s^{\prime}, y\right)=e_{\alpha \beta}\left(\mathcal{U}_{E}\right)\left(s^{\prime}\right) \widetilde{\chi_{E}^{\alpha \beta}}(y)+\Lambda_{\alpha \beta}(\mathcal{U})\left(s^{\prime}\right) \widetilde{\chi_{I}^{\alpha \beta}}(y) \quad \text { for a.e. }\left(s^{\prime}, y\right) \in \omega \times Y^{*}
$$

8.1. The limit problems in the shell's mid surface

Theorem 8.1. The limit displacement $\left(\mathcal{U}_{E}, \mathcal{U}\right) \in \mathbb{D}_{E} \times \mathbb{D}_{I}$ solves the homogenized problem

$$
\begin{align*}
& \int_{\omega}\left[a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} e_{\alpha \beta}\left(\mathcal{U}_{E}\right) e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right)+b_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m}\left(e_{\alpha \beta}\left(\mathcal{U}_{E}\right) \Lambda_{\alpha^{\prime} \beta^{\prime}}(\mathcal{V})\right.\right.  \tag{8.2}\\
& \left.\left.+\Lambda_{\alpha \beta}(\mathcal{U}) e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right)\right)+c_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} \Lambda_{\alpha \beta}(\mathcal{U}) \Lambda_{\alpha \beta}(\mathcal{V})\right] d s^{\prime} \\
& =\frac{\left|Y^{\prime}\right|}{\left|Y^{\prime}\right|}\left(\int_{\omega}\left(f \cdot \mathcal{V}+\frac{\kappa^{2}}{3 a} g_{\alpha} \mathcal{V}_{\alpha}-\frac{\kappa^{2}}{3} g_{\alpha} \frac{\partial \mathcal{V}}{\partial s_{\alpha}} \boldsymbol{n}\right) d s^{\prime}+\left\langle F, \mathcal{V}_{E}\right\rangle\right), \quad \forall\left(\mathcal{V}_{E}, \mathcal{V}\right) \in \mathbb{D}_{E} \times \mathbb{D}_{I},
\end{align*}
$$

where

$$
\begin{aligned}
a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[\boldsymbol{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{E}^{\alpha \beta}}\right)\right] \boldsymbol{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} d y \\
b_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[y_{3} \boldsymbol{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\left(\chi_{I}^{\alpha \beta}\right.}\right)\right] \boldsymbol{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} d y, \\
c_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[y_{3} \boldsymbol{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\left(\chi_{I}^{\alpha \beta}\right.}\right)\right] y_{3} \boldsymbol{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} d y .
\end{aligned}
$$

Proof. Consider equation (6.5) and choose the test function such that $\left(\mathcal{V}_{E}, \mathcal{V}\right) \in \mathbb{D}_{E} \times \mathbb{D}_{I}$ and $\widehat{v}=0$. Moreover, with the expression for $\widehat{u}$ we obtain for the left hand side in 6.5

$$
\frac{1}{2 \kappa} \int_{\omega \times Y^{*}} a_{i j k l}(y)\left(E_{S, i j}\left(\mathcal{U}_{E}, \mathcal{U}\right)+\mathcal{E}_{i j, y}(\widehat{u})\right) E_{S, k l}\left(\mathcal{V}_{E}, \mathcal{V}\right) d s^{\prime} d y
$$

Hence,

$$
\begin{align*}
\int_{\omega \times Y^{*}} a_{i j k l}(y)\left[e _ { \alpha \beta } ( \mathcal { U } _ { E } ) ( s ^ { \prime } ) \left(\mathbf{M}_{i j}^{\alpha \beta}\right.\right. & \left.\left.\left.+\mathcal{E}_{i j, y} \widetilde{\left(\chi_{E}^{\alpha \beta}\right.}\right)(y)\right)+\Lambda_{\alpha \beta}(\mathcal{U})\left(s^{\prime}\right)\left(y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{I}^{\alpha \beta}}\right)(y)\right)\right] \\
\times \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} & {\left[e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{M}\right)\left(s^{\prime}\right)+y_{3} \Lambda_{\alpha^{\prime} \beta^{\prime}}(\mathcal{V})\left(s^{\prime}\right)\right] d s^{\prime} d y } \\
& =\left|Y^{*}\right|\left(\int_{\omega}\left(f \cdot \mathcal{V}+\frac{\kappa^{2}}{3 a} g_{\alpha} \mathcal{V}_{\alpha}-\frac{\kappa^{2}}{3} g_{\alpha} \frac{\partial \mathcal{V}}{\partial s_{\alpha}} \mathbf{n}\right) d s^{\prime}+\left\langle F, \mathcal{V}_{E}\right\rangle\right) \tag{8.3}
\end{align*}
$$

Computing the expression yields,

$$
\begin{aligned}
& \frac{1}{\left|Y^{*}\right|} \int_{\omega \times Y *} a_{i j k l}(y) e_{\alpha \beta}\left(\mathcal{U}_{E}\right)\left(\mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{E}^{\alpha \beta}}\right)\right) \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right) \\
&+a_{i j k l}(y) \Lambda_{\alpha \beta}(\mathcal{U})\left(y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{I}^{\alpha \beta}}\right)\right) \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right) \\
&+a_{i j k l}(y) e_{\alpha \beta}\left(\mathcal{U}_{E}\right)\left(\mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{E}^{\alpha \beta}}\right)\right) y_{3} \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} \Lambda_{\alpha^{\prime} \beta^{\prime}}(\mathcal{V}) \\
&+a_{i j k l}(y) \Lambda_{\alpha \beta}(\mathcal{U})\left(y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\chi_{I}^{\alpha \beta}}\right)\right) y_{3} \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}} \Lambda_{\alpha^{\prime} \beta^{\prime}}(\mathcal{V}) d y d s^{\prime} \\
& \quad=\frac{\left|Y^{\prime} *\right|}{\left|Y^{\prime}\right|}\left(\int_{\omega}\left(f \cdot \mathcal{V}+\frac{\kappa^{2}}{3 a} g_{\alpha} \mathcal{V}_{\alpha}-\frac{\kappa^{2}}{3} g_{\alpha} \frac{\partial \mathcal{V}}{\partial s_{\alpha}} \mathbf{n}\right) d s^{\prime}+\left\langle F, \mathcal{V}_{E}\right\rangle\right)
\end{aligned}
$$

With the expressions for the homogenized coefficients we end up with equation 8.2.
Denote $\mathbb{S}_{2}$ the set of $2 \times 2$ symmetric matrices.
Lemma 8.1. There exists a constant $C>0$ such that the homogenized coefficients satisfy

$$
\begin{aligned}
a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} \tau_{E}^{\alpha \beta} \tau_{E}^{\alpha^{\prime} \beta^{\prime}} & +b_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m}\left(\tau_{E}^{\alpha \beta} \tau_{I}^{\alpha^{\prime} \beta^{\prime}}+\tau_{I}^{\alpha \beta} \tau_{E}^{\alpha^{\prime} \beta^{\prime}}\right) \\
& +c_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} \tau_{I}^{\alpha \beta} \tau_{I}^{\alpha^{\prime} \beta^{\prime}} \geq C\left(\tau_{E}^{\alpha \beta} \tau_{E}^{\alpha \beta}+\tau_{I}^{\alpha \beta} \tau_{I}^{\alpha \beta}\right) \quad \forall\left(\tau_{E}, \tau_{I}\right) \in \mathbb{S}_{2} \times \mathbb{S}_{2}
\end{aligned}
$$

Proof. We first note that with the variational formulations 8.1) we can calculate the homogenized coefficients as

$$
\begin{aligned}
a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[\mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y} \widetilde{\left(\chi_{E}^{\alpha \beta}\right)}\right]\left[\mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}}+\mathcal{E}_{k l, y} \widetilde{\left(\widetilde{\chi_{E}^{\alpha^{\prime} \beta^{\prime}}}\right)}\right] d y \\
b_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y} \widetilde{\left(\chi_{I}^{\alpha \beta}\right)}\right]\left[\mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}}+\widetilde{\mathcal{E}_{k l, y}\left(\widetilde{\left(\chi_{E}^{\alpha^{\prime} \beta^{\prime}}\right)}\right] d y}\right. \\
& =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[\mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y} \widetilde{\left(\widetilde{\chi_{E}^{\alpha \beta}}\right)}\right]\left[y_{3} \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}}+\widetilde{\left.\mathcal{E}_{k l, y}\left(\widetilde{\left(\chi_{I}^{\alpha^{\prime} \beta^{\prime}}\right.}\right)\right] d y}\right. \\
c_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} & =\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}(y)\left[y_{3} \mathbf{M}_{i j}^{\alpha \beta}+\mathcal{E}_{i j, y}\left(\widetilde{\left(\chi_{I}^{\alpha \beta}\right.}\right)\right]\left[y_{3} \mathbf{M}_{k l}^{\alpha^{\prime} \beta^{\prime}}+\widetilde{\left.\mathcal{E}_{k l, y}\left(\widetilde{\left(\chi_{I}^{\alpha^{\prime} \beta^{\prime}}\right.}\right)\right] d y} .\right.
\end{aligned}
$$

For every $\left(\tau_{E}, \tau_{I}\right) \in \mathbb{S}_{2} \times \mathbb{S}_{2}$, one has

$$
\begin{aligned}
a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} \tau_{E}^{\alpha \beta} \tau_{E}^{\alpha^{\prime} \beta^{\prime}}+b_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m}\left(\tau_{E}^{\alpha \beta} \tau_{I}^{\alpha^{\prime} \beta^{\prime}}+\tau_{I}^{\alpha \beta} \tau_{E}^{\alpha^{\prime} \beta^{\prime}}\right)+c_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} \tau_{I}^{\alpha \beta} \tau_{I}^{\alpha^{\prime} \beta^{\prime}} \\
\quad=\frac{1}{\left|Y^{*}\right|} \int_{Y^{*}} a_{i j k l}\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right]\left[M_{k l}+\mathcal{E}_{k l, y}(\Psi)\right] d y
\end{aligned}
$$

with $M=\left(\tau_{E}^{\alpha \beta}+y_{3} \tau_{I}^{\alpha \beta}\right) \mathbf{M}^{\alpha \beta}$, and $\Psi=\tau_{E}^{\alpha \beta} \widetilde{\chi_{E}^{\alpha \beta}}+\tau_{I}^{\alpha \beta} \widetilde{\chi_{I}^{\alpha \beta}}$. By the coercivity of $a_{i j k l}$, see (6.2), we obtain

$$
\int_{Y^{*}} a_{i j k l}(y)\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right]\left[M_{k l}+\mathcal{E}_{k l, y}(\Psi)\right] d y \geq c_{0} \int_{Y^{*}}\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right]\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right] d y
$$

Here we are again in the context of Lemma 7.2 . This then yields with the equivalence of the norms that

$$
\begin{aligned}
\int_{Y^{*}}\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right]\left[M_{i j}+\mathcal{E}_{i j, y}(\Psi)\right] d y & \geq C\left(\left|\tau_{E}\right|^{2}+\left|\tau_{I}\right|^{2}+\|\Psi\|_{L^{2}\left(Y^{*}\right)}^{2}\right) \\
& \geq C\left(\tau_{E}^{\alpha \beta} \tau_{E}^{\alpha \beta}+\tau_{I}^{\alpha \beta} \tau_{I}^{\alpha \beta}\right) \quad \forall\left(\tau_{E}, \tau_{I}\right) \in \mathbb{S}_{2} \times \mathbb{S}_{2}
\end{aligned}
$$

## 9. Different Boundary condition

In this section we want to emphasize on a change of the boundary condition, such that the previously free part is clamped, i.e. $\Gamma_{0}=\phi([0, \pi] \times\{0\} \cup[0, \pi] \times\{l\})$. We may note, that all presented estimates and resulting limits are not affected by the change of boundary conditions until we consider the split of $\mathcal{U}=\mathcal{U}_{I}+\mathcal{U}_{E}$. As in (5.1), we first obtain that $\mathcal{U}_{I}$ can be presented as

$$
\mathcal{U}_{2}\left(s_{1}, s_{2}\right)=U_{2}\left(s_{1}\right), \mathcal{U}_{1}\left(s_{1}, s_{2}\right)=-s_{2}^{c} U_{2}^{\prime}\left(s_{1}\right)+U_{1}\left(s_{1}\right), \mathcal{U}_{3}\left(s_{1}, s_{2}\right)=a\left(s_{2}^{c} U_{2}^{\prime \prime}\left(s_{1}\right)-U_{1}^{\prime}\left(s_{1}\right)\right)
$$

$U_{1} \in H^{1}(0, a \pi), \quad U_{2} \in H^{2}(0, a \pi)$. With respect to our new boundary conditions we need that $\mathcal{U}_{2}\left(s_{1}, 0\right)=\mathcal{U}_{2}\left(s_{1}, l\right)=0, \quad$ for a.e. $s_{1}$. Hence, we have $U_{2}\left(s_{1}\right)=0$ for a.e. $s_{1} \in(0, a \pi)$. With the same reasoning we get $U_{1}\left(s_{1}\right)=0$ for a.e. $s_{1} \in(0, a \pi)$. Therefore $D_{I}=\mathbb{D}_{I}=\{0\}$.

Remark 9.1. In the applied forces we consider $F$ such that

$$
\begin{equation*}
F_{1} \in L^{2}\left(0, l ; H_{0}^{1}(0, a \pi)\right) \cap H^{1}(\omega), \quad F_{2} \in H^{1}(\omega), \quad F_{3} \in L^{2}\left(0, l ; H_{0}^{2}(0, a \pi)\right) \cap H^{1}(\omega) \tag{9.1}
\end{equation*}
$$

In the case of a fully clamped shell along $\partial \omega$ the assumptions on the forces do not change and we obtain $D_{I}=\mathbb{D}_{I}=0$. Hence, we immediately get equation 9.2 .

Lemma 9.1. For every $\mathcal{U}$ in $\mathbb{D}_{E}$, where $\Gamma_{0}$ is given above, one has

$$
\left\|\mathcal{U}_{2}\right\|_{H^{1}\left(0, l ; L^{2}(0, a \pi)\right)}+\left\|\mathcal{U}_{1}\right\|_{H^{1}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)}+\left\|\mathcal{U}_{3}\right\|_{L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E} .
$$

Proof. This estimate is an immediate consequence of the fact that $D_{E}=H_{0}^{1}(\omega) \times H_{0}^{1}(\omega) \times L^{2}(\omega)$ and Lemma 5.2 .

If we consider the linear elasticity problem presented in section 6 and getting to the limit, as presented earlier, we obtain that the limit homogenized equation is given by

$$
\begin{equation*}
\int_{\omega} a_{\alpha \beta \alpha^{\prime} \beta^{\prime}}^{h o m} e_{\alpha \beta}\left(\mathcal{U}_{E}\right) e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right) d s^{\prime}=\frac{\left|Y^{\prime}\right|}{\left|Y^{\prime}\right|}\left\langle F, \mathcal{V}_{E}\right\rangle, \quad \forall \mathcal{V}_{E} \in \mathbb{D}_{E} \tag{9.2}
\end{equation*}
$$

Now, we show that $\left\langle F, \mathcal{V}_{E}\right\rangle$ can be expressed in terms of $e_{\alpha^{\prime} \beta^{\prime}}\left(\mathcal{V}_{E}\right)$ for every $V \in \mathbb{D}_{E}$.
Denote $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ the fields defined by

$$
\mathcal{F}(\cdot, 0)=0, \quad \frac{\partial \mathcal{F}}{\partial s_{2}}=F, \quad \widetilde{\mathcal{F}}(\cdot, 0)=0, \quad \frac{\partial \widetilde{\mathcal{F}}}{\partial s_{2}}=\mathcal{F}
$$

Recall that the components of $F$ are given by 9.1 .
Lemma 9.2. For every $V \in \mathbb{D}_{E}$ one has

$$
\left\langle F, \mathcal{V}_{E}\right\rangle=\int_{\omega}\left(\boldsymbol{F}_{11} e_{11}(V)+\boldsymbol{F}_{12} e_{12}(V)+\boldsymbol{F}_{22} e_{22}(V)\right) d s_{1} d s_{2}
$$

where $\quad \boldsymbol{F}_{11}=a F_{3}, \quad \boldsymbol{F}_{12}=-2\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right), \quad \boldsymbol{F}_{22}=-\mathcal{F}_{2}+\partial_{1} \widetilde{\mathcal{F}}_{1}+a \partial_{11} \widetilde{\mathcal{F}}_{3}$.

Proof. Consider $V \in D_{E}$. One has

$$
\begin{aligned}
\int_{\omega} F_{3} V_{3} d s_{1} d s_{2} & =a \int_{\omega} F_{3} e_{11}(V) d s_{1} d s_{2}-a \int_{\omega} F_{3} \partial_{1} V_{1} d s_{1} d s_{2} \\
& =a \int_{\omega} F_{3} e_{11}(V) d s_{1} d s_{2}+a \int_{\omega} \partial_{1} F_{3} V_{1} d s_{1} d s_{2}
\end{aligned}
$$

Then

$$
\begin{aligned}
& \int_{\omega}\left(F_{1}+a \partial_{1} F_{3}\right) V_{1} d s_{1} d s_{2}=-\int_{\omega}\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) \partial_{2} V_{1} d s_{1} d s_{2} \\
& =-2 \int_{\omega}\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) e_{12}(V) d s_{1} d s_{2}+\int_{\omega}\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) \partial_{1} V_{2} d s_{1} d s_{2} \\
& =-2 \int_{\omega}\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) e_{12}(V) d s_{1} d s_{2}-\int_{\omega}\left(\partial_{1} \mathcal{F}_{1}+a \partial_{11} \mathcal{F}_{3}\right) V_{2} d s_{1} d s_{2}
\end{aligned}
$$

and finally

$$
\int_{\omega}\left(F_{2}-\partial_{1} \mathcal{F}_{1}-a \partial_{11} \mathcal{F}_{3}\right) V_{2} d s_{1} d s_{2}=-\int_{\omega}\left(\mathcal{F}_{2}-\partial_{1} \widetilde{\mathcal{F}}_{1}-a \partial_{11} \widetilde{\mathcal{F}}_{3}\right) \partial_{2} V_{2} d s_{1} d s_{2}
$$

With those calculations we obtain for every $V$ in $D_{E}$

$$
\begin{aligned}
& \int_{\omega} F \cdot V d s_{1} d s_{2}=\int_{\omega}\left(F_{1} V_{1}+F_{2} V_{2}+F_{3} V_{3}\right) d s_{1} d s_{2} \\
& =\int_{\omega}\left(\left(F_{1}+a \partial_{1} F_{3}\right) V_{1}+F_{2} V_{2}+a F_{3} e_{11}(V)\right) d s_{1} d s_{2} \\
& =\int_{\omega}\left(-2\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) e_{12}(V)+\left(F_{2}-\partial_{1} \mathcal{F}_{1}-a \partial_{11} \mathcal{F}_{3}\right) V_{2}+a F_{3} e_{11}(V)\right) d s_{1} d s_{2} \\
& =\int_{\omega}\left(-2\left(\mathcal{F}_{1}+a \partial_{1} \mathcal{F}_{3}\right) e_{12}(V)+\left(-\mathcal{F}_{2}+\partial_{1} \widetilde{\mathcal{F}}_{1}+a \partial_{11} \widetilde{\mathcal{F}}_{3}\right) e_{22}(V)+a F_{3} e_{11}(V)\right) d s_{1} d s_{2}
\end{aligned}
$$

We conclude the proof by the density of $D_{E}$ in $\mathbb{D}_{E}$.

## 10. Appendix



Figure 3: Cell $Y^{\prime}$ and the perforated domain $Y^{\prime *}$
10.1. Proof of Proposition 1.1

There exists $\kappa_{0}>0$ such that

$$
\mathcal{O}_{\kappa_{0}}^{\prime}=\left\{s \in \mathbb{R}^{2} \backslash \bar{T} \mid \operatorname{dist}(s, T)<\kappa_{0}\right\} \subset Y^{\prime *}
$$

Since the boundary of $T$ is Lipschitz, there exist $R^{\prime}, R_{1}^{\prime}>0$ and $N \geq 2$ open sets $\mathcal{O}_{1}^{\prime}, \ldots, \mathcal{O}_{N}^{\prime}$ such that

- $\mathcal{O}_{i}^{\prime}$ is included in a ball of radius $R^{\prime}$ and is star-shaped with respect to a ball of radius $R_{1}^{\prime}$, $i \in\{1, \ldots, N\}$,
- $\mathcal{O}_{i}^{\prime} \cap \mathcal{O}_{i+1}^{\prime} \neq \emptyset, i \in\{1, \ldots, N-1\}$, and $\mathcal{O}_{N}^{\prime} \cap \mathcal{O}_{1}^{\prime} \neq \emptyset$,
- $\mathcal{O}_{\kappa_{0}}^{\prime} \subset \bigcup_{i=1}^{N} \mathcal{O}_{i}^{\prime} \subset Y^{\prime *}$.

Set $\mathcal{O}_{\kappa_{0}}=\mathcal{O}_{\kappa_{0}}^{\prime} \times(-\kappa, \kappa), \quad \mathcal{O}_{i}=\mathcal{O}_{i}^{\prime} \times(-\kappa, \kappa), \quad i \in\{1, \ldots, N\}$. One has

- $\mathbf{P}_{1}: \mathcal{O}_{i}$ is included in a ball of radius $R=R^{\prime}+\kappa$ and is star-shaped with respect to a ball of radius $R_{1}=\inf \left\{R_{1}^{\prime}, \kappa\right\}, i \in\{1, \ldots, N\}$,
- $\mathbf{P}_{2}: \mathcal{O}_{i} \cap \mathcal{O}_{i+1} \neq \emptyset, i \in\{1, \ldots, N-1\}$, and $\mathcal{O}_{N} \cap \mathcal{O}_{1} \neq \emptyset$,
- $\mathbf{P}_{3}: \mathcal{O}_{\kappa_{0}} \subset \bigcup_{i=1}^{N} \mathcal{O}_{i} \subset Y^{*}$.

Set $\mathcal{O}_{\kappa_{0}}^{\diamond}=\mathcal{O}_{\kappa_{0}} \cup(\bar{T} \times(-\kappa, \kappa))$. Below, we will use the classical extension result
Lemma 10.1. There exists an extension operator $\mathcal{P}$ from $H^{1}\left(\mathcal{O}_{\kappa_{0}}\right)$ into $H^{1}\left(\mathcal{O}_{\kappa_{0}}^{\diamond}\right)$ satisfying for all $\phi \in H^{1}\left(\mathcal{O}_{\kappa_{0}}\right)$

$$
\mathcal{P}(\phi)_{\mid \mathcal{O}_{\kappa_{0}}}=\phi, \quad\|\nabla(\mathcal{P}(\phi))\|_{L^{2}\left(\mathcal{O}_{\kappa_{0}}^{\circ}\right)} \leq C\|\nabla \phi\|_{L^{2}\left(\mathcal{O}_{\kappa_{0}}\right)}
$$

The constant only depends on $\partial T^{3}$
Proof of Proposition 1.1. For every $\xi \in \Xi_{\varepsilon}$ and $\mathcal{O}_{i}, i \in\{1, \ldots, N\}$, if $\varepsilon \mathrm{s}$ small enough, the domain $\Phi\left(\varepsilon \xi+\varepsilon \mathcal{O}_{i}\right)$ is included in a ball of radius $2 R \varepsilon$ and is star-shaped with respect to a ball of radius $R_{1} \varepsilon / 4$ (due to property $\mathbf{P}_{1}$ and Lemma A2 in [3]).
Now, let $u$ be a displacement belonging to $H^{1}\left(\mathcal{Q}_{\varepsilon}\right)^{3}$. For every $(\xi, i) \in \Xi_{\varepsilon} \times\{1, \ldots, N\}$ there exists a rigid displacement $r_{\xi, i}$ such that

$$
\begin{equation*}
\left\|\nabla_{x}\left(u-r_{\xi, i}\right)\right\|_{L^{2}\left(\Phi\left(\varepsilon \xi+\varepsilon \mathcal{O}_{i}\right)\right)} \leq C\|e(u)\|_{L^{2}\left(\Phi\left(\varepsilon \xi+\varepsilon \mathcal{O}_{i}\right)\right)} \tag{10.1}
\end{equation*}
$$

The constant doe not depend on $\varepsilon, \xi$ and $\mathcal{O}_{i}$, it only depends on the ratio $R / R_{1}$ (see Theorem 2.3 in [3]). Then, step by step we compare the rigid displacements $r_{\xi, 1}, r_{\xi, 2}, \ldots, r_{\xi, N}$ thanks to the properties $\mathbf{P}_{2}$ and $\mathbf{P}_{3}$. To do that, observe that there exist two constants independent of $\varepsilon$ and $\xi$ such that

$$
\begin{aligned}
& c \varepsilon^{3}\left|\mathcal{O}_{i} \cap \mathcal{O}_{i+1}\right| \leq\left|\Phi\left(\varepsilon \xi+\varepsilon \mathcal{O}_{i} \cap \mathcal{O}_{i+1}\right)\right| \leq C \varepsilon^{3}\left|\mathcal{O}_{i} \cap \mathcal{O}_{i+1}\right|, \quad i \in\{1, \ldots, N-1\} \\
& c \varepsilon^{3}\left|\mathcal{O}_{N} \cap \mathcal{O}_{1}\right| \leq\left|\Phi\left(\varepsilon \xi+\varepsilon \mathcal{O}_{N} \cap \mathcal{O}_{1}\right)\right| \leq C \varepsilon^{3}\left|\mathcal{O}_{N} \cap \mathcal{O}_{1}\right|
\end{aligned}
$$

As a consequence, there exists a rigid displacement $r_{\xi}$ such that

$$
\begin{equation*}
\left\|\nabla_{x}\left(u-r_{\xi}\right)\right\|_{L^{2}\left(\Phi\left(\varepsilon \xi+\varepsilon Y^{*}\right)\right)} \leq C\|e(u)\|_{L^{2}\left(\Phi\left(\varepsilon \xi+\varepsilon Y^{*}\right)\right)} \tag{10.2}
\end{equation*}
$$

The constant doe not depend on $\varepsilon$ and $\xi$.
At this point, transform the domain $\Phi\left(\varepsilon \xi+\varepsilon Y^{*}\right)$ by the inverse map $z \in Y^{*} \longmapsto \Phi(\varepsilon \xi+\varepsilon z)$, then apply Lemma 10.1 in order to extend the function in the hole $T$ and finally transform by the map $z \in Y \longmapsto \Phi(\varepsilon \xi+\varepsilon z)$ and to the result add the displacement $r_{\xi}$. The $L^{2}$ norm of the strain tensor of the extended displacement (now defined in $\Phi(\varepsilon \xi+\varepsilon Y)$ ) is bounded by a constant (independent of $\varepsilon$ and $\xi$ ) multiply by $\|e(u)\|_{L^{2}\left(\Phi\left(\varepsilon \xi+\varepsilon Y^{*}\right)\right)}$.
We apply this process to every domain of $\varepsilon \xi+\varepsilon Y^{*}, \xi \in \Xi_{\varepsilon}$. Finally, we obtain an extension of the displacement $u$ satisfying (1.6).

[^3]
### 10.2. Two Lemmas

For the definitions and properties of the unfolding operators $\mathcal{T}_{\varepsilon}, \mathcal{M}_{\varepsilon}$ we refer to [1, 2] Lemma 10.2 is proved in [2]. Let $\Omega$ be a bounded domain in $\mathcal{R}^{N}$ with Lipschitz boundary and $Y=\Pi_{i=1}^{N}\left(0, l_{i}\right), l_{i}>0$, $i=1, \ldots, N$.

Lemma 10.2. Suppose $p \in(1,+\infty)$. Let $\left\{\left(\boldsymbol{u}_{\varepsilon, \delta}, \boldsymbol{v}_{\varepsilon, \delta}\right)\right\}_{\varepsilon, \delta}$ be a sequence in $W^{1, p}(\Omega)^{N} \times W^{1, p}(\Omega)^{N \times N}$ (with $\boldsymbol{v}_{\varepsilon, \delta}$ a symmetric matrix) converging weakly to $(\boldsymbol{u}, \boldsymbol{v})$ in $W^{1, p}(\Omega)^{N} \times W^{1, p}(\Omega)^{N \times N}$.
Assume furthermore that there exist $\mathcal{X}$ in $L^{p}(\Omega)^{N \times N}$ and $\widehat{v}$ in $L^{p}\left(\Omega ; W_{\text {per, } 0}^{1, p}(Y)\right)^{N \times N}$ such that as $(\varepsilon, \delta) \rightarrow(0,0)$

$$
\begin{align*}
& \frac{1}{\delta}\left(e\left(\boldsymbol{u}_{\varepsilon, \delta}\right)+\boldsymbol{v}_{\varepsilon, \delta}\right) \rightharpoonup \mathcal{X} \quad \text { weakly in } \quad L^{p}(\Omega)^{N \times N}  \tag{10.3}\\
& \mathcal{T}_{\varepsilon, \delta}\left(\nabla \boldsymbol{v}_{\varepsilon, \delta}\right) \rightharpoonup \nabla \boldsymbol{v}+\nabla_{y} \widehat{\boldsymbol{v}} \quad \text { weakly in } \quad L^{p}(\Omega \times Y)^{N \times N \times N}
\end{align*}
$$

Then $\boldsymbol{u}$ belongs to $W^{2, p}(\Omega)^{N}$ and there exists $\mathfrak{u} \in L^{p}\left(\Omega ; W_{\text {per }, 0}^{1, p}(Y)\right)^{N}$ such that, up to a subsequence,

$$
\begin{array}{ll}
\text { if } \frac{\varepsilon}{\delta} \rightarrow \theta \in[0,+\infty), & \frac{1}{\delta} \mathcal{T}_{\varepsilon, \delta}\left(e\left(\boldsymbol{u}_{\varepsilon, \delta}\right)+\boldsymbol{v}_{\varepsilon, \delta}\right) \rightharpoonup \mathcal{X}+e_{y}(\mathfrak{u})+\theta \widehat{\boldsymbol{v}} \text { weakly in } L^{p}(\Omega \times Y)^{N \times N}  \tag{10.4}\\
\text { if } \frac{\varepsilon}{\delta} \rightarrow+\infty, & \widehat{\boldsymbol{v}}=e_{y}(\mathfrak{u}) .
\end{array}
$$

Proof. First, from 10.3 one obtains that $e(\mathbf{u})+\mathbf{v}=0$, then since $\Omega$ is a bounded domain with Lipschitz boundary $\mathbf{u}$ belongs to $W^{2, p}(\Omega)^{N}$. We also deduce from this convergence and the Korn inequality that $\mathbf{u}_{\varepsilon, \delta}$ strongly converges to $\mathbf{u}$ in $W^{1, p}(\Omega)^{N}$.
Then, up to a subsequence, there exists $\widehat{\mathcal{X}} \in L^{p}(\Omega \times Y)^{N}$ such that

$$
\frac{1}{\delta} \mathcal{T}_{\varepsilon, \delta}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right) \rightharpoonup \widehat{\mathcal{X}} \quad \text { weakly in } L^{p}(\Omega \times Y)^{N \times N}
$$

Step 1. In this first step we assume that $\frac{\varepsilon}{\delta} \rightarrow \theta \in[0,+\infty)$.
Introduce the function $\mathbf{Z}_{\varepsilon, \delta}$ belonging to $L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N}$, defined as

$$
\begin{equation*}
\mathbf{Z}_{\varepsilon, \delta}=\frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}-\mathcal{M}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\right)-\mathcal{M}_{\varepsilon}\left(\nabla \mathbf{u}_{\varepsilon, \delta}\right) \cdot y^{c} \tag{10.5}
\end{equation*}
$$

Its gradient and symmetric gradient with respect to $y$ are

$$
\begin{align*}
\nabla_{y} \mathbf{Z}_{\varepsilon, \delta} & =\mathcal{T}_{\varepsilon}\left(\nabla \mathbf{u}_{\varepsilon, \delta}\right)-\mathcal{M}_{\varepsilon}\left(\nabla \mathbf{u}_{\varepsilon, \delta}\right) \\
e_{y}\left(\mathbf{Z}_{\varepsilon, \delta}\right) & =\mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)\right)-\mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)\right)  \tag{10.6}\\
& =\mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)-\left(\mathcal{T}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)-\mathcal{M}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)\right)-\mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)
\end{align*}
$$

Convergence $\int 10.3_{1}$ on one side together with the fact that $\left\|\nabla \mathbf{v}_{\varepsilon, \delta}\right\|_{L^{p}(\Omega)}$ and $\frac{\varepsilon}{\delta}$ are bounded, give

$$
\left\|e_{y}\left(\mathbf{Z}_{\varepsilon, \delta}\right)\right\|_{L^{p}(\Omega \times Y)^{N}} \leq C(\delta+\varepsilon) \leq C \delta
$$

The Korn inequality implies

$$
\left\|\mathbf{Z}_{\varepsilon, \delta}\right\|_{L^{p}\left(\Omega ; W^{1, p}(Y)\right)} \leq C \delta
$$

Consequently, up to a subsequence, there exists $\widehat{\mathbf{Z}}$ in $L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N}$ such that,

$$
\begin{equation*}
\frac{1}{\delta} \mathbf{Z}_{\varepsilon, \delta} \rightharpoonup \widehat{\mathbf{Z}} \quad \text { weakly in } L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N} \tag{10.7}
\end{equation*}
$$

By 10.6 one has

$$
\frac{1}{\delta} \mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)=\frac{1}{\delta} e_{y}\left(\mathbf{Z}_{\varepsilon, \delta}\right)+\frac{\varepsilon}{\delta} \frac{\mathcal{T}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)-\mathcal{M}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)}{\varepsilon}+\frac{1}{\delta} \mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)
$$

Then going to the limit using 10.7 and [2, Proposition 1.25 and Theorem 1.41]

$$
\begin{equation*}
\frac{1}{\delta} \mathcal{T}_{\varepsilon}\left(\nabla \mathbf{u}_{\varepsilon, \delta}+\mathbf{v}_{\varepsilon, \delta}\right) \rightharpoonup \widehat{\mathcal{X}}=e_{y}(\widehat{\mathbf{Z}})+\theta\left(\nabla \mathbf{v} y^{c}+\widehat{\mathbf{v}}\right)+\mathcal{X} \quad \text { weakly in } L^{p}(\Omega \times Y)^{N \times N} \tag{10.8}
\end{equation*}
$$

Now, we prove that

$$
\mathfrak{u}=\widehat{\mathbf{Z}}-\frac{\theta}{2} \sum_{j, k=1}^{N} \frac{\partial^{2} \mathbf{u}}{\partial x_{j} \partial x_{k}}\left(y_{j}^{c} y_{k}^{c}-\mathcal{M}_{Y}\left(y_{j}^{c} y_{k}^{c}\right)\right)
$$

is periodic (note that this function belongs to $L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N}$ ).
We proceed as in the proof of [2, Theorem 1.36], one first evaluates the difference of the traces of $\mathbf{Z}_{\varepsilon, \delta}$ on the faces $Y_{1}=\{0\} \times(0,1)^{N-1}$ and $Y_{1}+\mathbf{e}_{1}$. For a.e. $\left(x, y^{\prime}\right) \in \Omega \times Y_{1}$, one has

$$
\begin{aligned}
& \mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}+\mathbf{e}_{i}\right)-\mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}\right) \\
& =\frac{1}{\varepsilon}\left(\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}+\mathbf{e}_{1}\right)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right)-\mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)(x) \\
& =\frac{1}{\varepsilon}\left(\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x+\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right)-\mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)(x)
\end{aligned}
$$

Let $\Phi$ be in $\mathcal{D}\left(\Omega \times Y_{1}\right)^{N}$, one has successively

$$
\begin{aligned}
& \int_{\Omega \times Y_{1}}\left(\mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}+\mathbf{e}_{i}\right)-\mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}\right)\right) \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime} \\
= & \int_{\Omega \times Y_{1}}\left[\frac{1}{\varepsilon}\left(\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x+\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right)-\mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)(x)\right] \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime} \\
= & \int_{\Omega \times Y_{1}} \mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right) \cdot \frac{\Phi\left(x-\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\Phi\left(x, y^{\prime}\right)}{\varepsilon} d x d y^{\prime}-\int_{\Omega \times Y_{1}} \mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)(x) \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime} \\
= & \int_{\Omega \times Y_{1}}\left(\mathbf{u}_{\varepsilon, \delta}(x)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right) \cdot \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d x d y^{\prime}+\int_{\Omega \times Y_{1}}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}-\mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)\right) \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime} \\
+ & \int_{\Omega \times Y_{1}} \mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right) \cdot \frac{\Phi\left(x-\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\Phi\left(x, y^{\prime}\right)+\varepsilon \mathbf{e}_{1} \cdot \nabla_{x} \Phi\left(x, y^{\prime}\right)}{\varepsilon} d x d y^{\prime}
\end{aligned}
$$

then

$$
\begin{aligned}
& =\int_{\Omega \times Y_{1}}\left(\mathcal{M}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)(x)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right) \cdot \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d x d y^{\prime} \\
& +\int_{\Omega \times Y_{1}}\left(\mathbf{u}_{\varepsilon, \delta}(x)-\mathcal{M}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)(x) \cdot \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d x d y^{\prime}+\int_{\Omega \times Y_{1}}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}-\mathcal{M}_{\varepsilon}\left(\frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}\right)\right) \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime}\right. \\
& +\int_{\Omega \times Y_{1}} \mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right) \cdot \frac{\Phi\left(x-\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\Phi\left(x, y^{\prime}\right)+\varepsilon \mathbf{e}_{1} \cdot \nabla_{x} \Phi\left(x, y^{\prime}\right)}{\varepsilon} d x d y^{\prime}
\end{aligned}
$$

The last right-hand side is equal to (see [2, Proposition 1.24])

$$
\begin{aligned}
& \int_{\Omega \times Y_{1}}\left(\mathcal{M}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)(x)-\mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right)\right) \cdot \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d x d y^{\prime} \\
+ & \int_{\Omega} \mathbf{u}_{\varepsilon, \delta}(x) \cdot\left(\int_{Y} \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d y^{\prime}-\mathcal{M}_{\varepsilon}\left(\int_{Y} \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d y^{\prime}\right) d x\right. \\
+ & \int_{\Omega} \frac{\partial \mathbf{u}_{\varepsilon, \delta}}{\partial x_{1}}(x) \cdot\left(\int_{Y} \Phi\left(x, y^{\prime}\right) d y^{\prime}-\mathcal{M}_{\varepsilon}\left(\int_{Y} \Phi\left(x, y^{\prime}\right) d y^{\prime}\right) d x\right. \\
+ & \int_{\Omega \times Y_{1}} \mathcal{T}_{\varepsilon}\left(\mathbf{u}_{\varepsilon, \delta}\right)\left(x, y^{\prime}\right) \cdot \frac{\Phi\left(x-\varepsilon \mathbf{e}_{1}, y^{\prime}\right)-\Phi\left(x, y^{\prime}\right)+\varepsilon \mathbf{e}_{1} \cdot \nabla_{x} \Phi\left(x, y^{\prime}\right)}{\varepsilon} d x d y^{\prime}
\end{aligned}
$$

Divide by $\delta$ and then pass to the limit using [2, Propositions 1.38 and 1.39]. It yields

$$
\begin{aligned}
& \int_{\Omega \times Y_{1}} \frac{\mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}+\mathbf{e}_{i}\right)-\mathbf{Z}_{\varepsilon, \delta}\left(x, y^{\prime}\right)}{\delta} \cdot \Phi(x, y) d x d y^{\prime} \\
\longrightarrow & \int_{\Omega \times Y_{1}}-\theta\left(\nabla \mathbf{u}(x) y^{c}\right) \cdot \frac{\partial \Phi}{\partial x_{1}}\left(x, y^{\prime}\right) d x d y^{\prime}+\frac{\theta}{2} \int_{\Omega \times Y_{1}} \mathbf{u}(x) \cdot \frac{\partial^{2} \Phi}{\partial x_{1}^{2}}\left(x, y^{\prime}\right) d x d y^{\prime} \\
= & \int_{\Omega \times Y_{1}} \theta \sum_{k=2}^{N} \frac{\partial^{2} \mathbf{u}}{\partial x_{1} \partial x_{k}}(x) y_{k}^{c^{c}} \cdot \Phi\left(x, y^{\prime}\right) d x d y^{\prime} .
\end{aligned}
$$

Hence, for a.e. $\left(x, y^{\prime}\right) \in \Omega \times Y_{1}, \widehat{\mathbf{Z}}\left(x, y^{\prime}+\mathbf{e}_{i}\right)-\widehat{\mathbf{Z}}\left(x, y^{\prime}\right)=\theta \sum_{k=2}^{N} \frac{\partial^{2} \mathbf{u}}{\partial x_{1} \partial x_{k}}(x) y_{k}^{\prime c}$. We obtain similar equalities for the difference of the traces of $\widehat{\mathbf{Z}}$ over the other faces of $Y$. That proves the claim. Then, a straightforward calculation gives (using $\nabla e(\mathbf{u})+\nabla \mathbf{v}=0$ )

$$
e_{y}(\mathfrak{u})=e_{y}(\widehat{\mathbf{Z}})-\theta \sum_{k=1}^{N} \frac{\partial e(\mathbf{u})}{\partial x_{k}} y_{k}^{c}=e_{y}(\widehat{\mathbf{Z}})+\theta \sum_{k=1}^{N} \frac{\partial \mathbf{v}}{\partial x_{k}} y_{k}^{c} .
$$

With 10.8 , that gives the convergence $10.4{ }_{1}$.
Step 2. In this step we assume that $\frac{\varepsilon}{\delta} \rightarrow+\infty$.
Again we consider the function $\mathbf{Z}_{\varepsilon, \delta}$ introduced in 10.5. Now, it satisfies

$$
\left\|\mathbf{Z}_{\varepsilon, \delta}\right\|_{L^{p}\left(\Omega ; W^{1, p}(Y)\right)} \leq C \varepsilon .
$$

Hence, up to a subsequence, there exists $\widehat{\mathbf{Z}}$ in $L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N}$ such that,

$$
\begin{equation*}
\frac{1}{\varepsilon} \mathbf{Z}_{\varepsilon, \delta} \rightharpoonup \widehat{\mathbf{Z}} \quad \text { weakly in } L^{p}\left(\Omega ; W^{1, p}(Y)\right)^{N} \tag{10.9}
\end{equation*}
$$

Observe that

$$
\begin{aligned}
& \frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)=\frac{\delta}{\varepsilon} \frac{1}{\delta} \mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right) \longrightarrow 0 \quad \text { strongly in } L^{p}(\Omega \times Y)^{N \times N}, \\
& \frac{1}{\varepsilon} \mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)=\frac{\delta}{\varepsilon} \frac{1}{\delta} \mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right) \longrightarrow 0 \quad \text { strongly in } L^{p}(\Omega)^{N \times N}
\end{aligned}
$$

One has

$$
\frac{1}{\varepsilon} \mathcal{T}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)=\frac{1}{\varepsilon} e_{y}\left(\mathbf{Z}_{\varepsilon, \delta}\right)+\frac{\mathcal{T}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)-\mathcal{M}_{\varepsilon}\left(\mathbf{v}_{\varepsilon, \delta}\right)}{\varepsilon}+\frac{1}{\varepsilon} \mathcal{M}_{\varepsilon}\left(e\left(\mathbf{u}_{\varepsilon, \delta}\right)+\mathbf{v}_{\varepsilon, \delta}\right)
$$

Passing to the limit in the above equality gives

$$
e_{y}(\widehat{\mathbf{Z}})+\nabla \mathbf{v} y^{c}+\widehat{\mathbf{v}}=0
$$

Then, as in the previous step we prove that

$$
\mathfrak{v}=\widehat{\mathbf{Z}}-\frac{1}{2} \sum_{j, k=1}^{N} \frac{\partial^{2} \mathbf{u}}{\partial x_{j} \partial x_{k}}\left(y_{j}^{c} y_{k}^{c}-\mathcal{M}_{Y}\left(y_{j}^{c} y_{k}^{c}\right)\right)
$$

is periodic. Thus $10.4_{2}{ }_{2}$ is proved with $\mathfrak{u}=-\mathfrak{v}$.
As a consequence of Lemma 10.2 one has (see also [2, Lemma 11.11])

Lemma 10.3. Suppose $p \in(1,+\infty)$. Let $\left\{\left(u_{\varepsilon, \delta}, v_{\varepsilon, \delta}\right)\right\}_{\varepsilon, \delta}$ be a sequence in $W^{1, p}(\Omega) \times W^{1, p}(\Omega)^{N}$ converging weakly to $(u, v)$ in $W^{1, p}(\Omega) \times W^{1, p}(\Omega)^{N}$. Assume furthermore that there exist $\mathcal{X}$ in $L^{p}(\Omega)^{N}$ and $\widehat{v}$ in $L^{p}\left(\Omega ; W_{p e r, 0}^{1, p}(Y)\right)^{N}$ such that as $(\varepsilon, \delta) \rightarrow(0,0)$

$$
\begin{aligned}
& \frac{1}{\delta}\left(\nabla u_{\varepsilon, \delta}+v_{\varepsilon, \delta}\right) \rightharpoonup \mathcal{X} \quad \text { weakly in } L^{p}(\Omega)^{N} \\
& \mathcal{T}_{\varepsilon, \delta}\left(\nabla v_{\varepsilon, \delta}\right) \rightharpoonup \nabla v+\nabla_{y} \widehat{v} \quad \text { weakly in } L^{p}(\Omega \times Y)^{N \times N}
\end{aligned}
$$

Then $u$ belongs to $W^{2, p}(\Omega)$ and there exists $\mathfrak{u} \in L^{p}\left(\Omega ; W_{\text {per }, 0}^{1, p}(Y)\right)$ such that, up to a subsequence,

$$
\begin{array}{ll}
\text { if } \frac{\varepsilon}{\delta} \rightarrow \theta \in[0,+\infty), & \frac{1}{\delta} \mathcal{T}_{\varepsilon, \delta}\left(\nabla u_{\varepsilon, \delta}+v_{\varepsilon, \delta}\right) \rightharpoonup \mathcal{X}+\nabla_{y} \mathfrak{u}+\theta \widehat{v} \text { weakly in } L^{p}(\Omega \times Y)^{N}  \tag{10.10}\\
\text { if } \frac{\varepsilon}{\delta} \rightarrow+\infty, & \widehat{v}=\nabla_{y} \mathfrak{u} .
\end{array}
$$

Proof. Consider the field $\mathbf{u}_{\varepsilon, \delta} \in W^{1, p}(\Omega)^{N}$ and the symmetric matrix field $\mathbf{v}_{\varepsilon, \delta} \in W^{1, p}(\Omega)^{N \times N}$ defined by

$$
\begin{aligned}
& \mathbf{u}_{\varepsilon, \delta}=\left(u_{\varepsilon, \delta}, 0, \ldots, 0\right), \quad\left(\mathbf{v}_{\varepsilon, \delta}\right)_{11}=v_{1, \varepsilon} \\
& \left(\mathbf{v}_{\varepsilon, \delta}\right)_{1 i}=\left(\mathbf{v}_{\varepsilon, \delta}\right)_{i 1}=\frac{1}{2} v_{i, \varepsilon}, \quad\left(\mathbf{v}_{\varepsilon, \delta}\right)_{i j}=0 \text { if }(i, j) \in\{2, \ldots, N\}^{2}
\end{aligned}
$$

These fields satisfy the assumptions of Lemma 10.2 and the convergences 10.3 . Therefore, the results in 10.4 give 10.10 .

### 10.3. Postponed Proofs

Proof of Proposition 5.1. Step 1. We start by showing the norm equivalences.
Take $\mathcal{V} \in D_{I}\left(\right.$ resp. $\left.\mathbb{D}_{I}\right)$, one has

$$
e_{11}(\mathcal{V})=-\frac{1}{a} \mathcal{V}_{3}, \quad e_{12}(\mathcal{V})=e_{22}(\mathcal{V})=0
$$

Now, the $2 D$-Korn inequality gives (recall that $\mathcal{V}_{1}=\mathcal{V}_{2}=0$ on $\Gamma_{0}$ )

$$
\left\|\mathcal{V}_{1}\right\|_{H^{1}(\omega)}^{2}+\left\|\mathcal{V}_{2}\right\|_{H^{1}(\omega)}^{2} \leq C\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2}
$$

Then we obtain

$$
\|\mathcal{V}\|_{\left[H^{1}(\omega)\right]^{2} \times L^{2}(\omega)}^{2}=\left\|\mathcal{V}_{1}\right\|_{H^{1}(\omega)}^{2}+\left\|\mathcal{V}_{2}\right\|_{H^{1}(\omega)}^{2}+\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2} \leq C\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2}=C\|\mathcal{V}\|_{D_{I}}^{2}
$$

On the contrary, to estimate $\|\cdot\|_{D_{I}}$ by $\|\cdot\|_{\left[H^{1}(\omega)\right]^{2} \times L^{2}(\omega)}$ from above, we can use Young's inequality such that

$$
\begin{aligned}
\|\mathcal{V}\|_{D_{I}}^{2} & =\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2}=\int_{\omega} \mathcal{V}_{3}^{2} d s^{\prime} \\
& \leq \int_{\omega}\left[\left(\frac{\partial \mathcal{V}_{1}}{\partial s_{1}}\right)^{2}+\frac{1}{a^{2}} \mathcal{V}_{3}^{2}+\left(\frac{\partial \mathcal{V}_{2}}{\partial s_{2}}\right)^{2}+\left(\frac{\partial \mathcal{V}_{1}}{\partial s_{2}}\right)^{2}+\left(\frac{\partial \mathcal{V}_{2}}{\partial s_{1}}\right)+\mathcal{V}_{3}^{2}\right] d s^{\prime} \\
& \leq C\left(\left\|\mathcal{V}_{1}\right\|_{H^{1}(\omega)}^{2}+\left\|\mathcal{V}_{2}\right\|_{H^{1}(\omega)}^{2}+\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2}\right)=C\|\mathcal{V}\|_{\left[H^{1}(\omega)\right]^{2} \times L^{2}(\omega)}^{2}
\end{aligned}
$$

Step 2. We prove the inequalities.
With expression 5.1 3 we get

$$
\begin{aligned}
\|\mathcal{V}\|_{D_{I}}^{2} & =\left\|\mathcal{V}_{3}\right\|_{L^{2}(\omega)}^{2}=\int_{\omega} \mathcal{V}_{3}^{2} d s_{1} d s_{2} \\
& =a^{2} \int_{0}^{a \pi} \int_{0}^{l}\left[\left(s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right)\right)^{2}-2 s_{2}^{c} V_{2}^{\prime \prime}\left(s_{1}\right) V_{1}^{\prime}\left(s_{1}\right)+\left(V_{1}^{\prime}\left(s_{1}\right)\right)^{2}\right] d s_{2} d s_{1} \\
& \leq C\left(\left\|V_{2}^{\prime \prime}\right\|_{L^{2}(0, a \pi)}^{2}+\left\|V_{1}^{\prime}\right\|_{L^{2}(0, a \pi)}^{2}\right)
\end{aligned}
$$

First we note that $V_{2}\left(s_{1}\right)=V_{2}^{\prime}\left(s_{1}\right)=0$ for $s_{1} \in\{0, a \pi\}$, which follows by the expressions in (5.1) and since $\mathcal{V}_{1}\left(0, s_{2}\right)=\mathcal{V}_{1}\left(a \pi, s_{2}\right)=\mathcal{V}_{2}\left(0, s_{2}\right)=\mathcal{V}_{2}\left(a \pi, s_{2}\right)=0$ for a.e. $s_{2} \in(0, l)$. Moreover, we obtain with the Poincaré inequality in $H_{0}^{1}(0, a \pi)$ and $H_{0}^{2}(0, a \pi)$ that,

$$
\begin{aligned}
\left\|V_{2}\right\|_{H^{2}(0, a \pi)}^{2}+\left\|V_{1}\right\|_{H^{1}(0, a \pi)}^{2} & \leq C\left(\left\|V_{2}^{\prime \prime}\right\|_{L^{2}(0, a \pi)}^{2}+\left\|V_{1}^{\prime}\right\|_{L^{2}(0, a \pi)}^{2}\right) \\
& \leq C \int_{\omega}\left(\left(s_{2}^{c} V_{2}^{\prime \prime}\right)^{2}-2 s_{2}^{c} V_{2}^{\prime \prime} V_{1}^{\prime}+\left(V_{1}^{\prime}\right)^{2}\right) d s^{\prime} \\
& \leq C \int_{\omega} a^{2}\left(s_{2}^{c} V_{2}^{\prime \prime}-V_{1}^{\prime}\right)^{2} d s^{\prime}=C\|\mathcal{V}\|_{D_{I}}^{2} .
\end{aligned}
$$

The second inequality is again obtained in a similar way, where we need to use that

$$
\left\|V_{2}\right\|_{H^{4}(0, a \pi)}^{2}+\left\|V_{1}\right\|_{H^{3}(0, a \pi)}^{2} \leq C\left(\left\|V_{2}^{\prime \prime}\right\|_{H^{2}(0, a \pi)}^{2}+\left\|V_{1}^{\prime}\right\|_{H^{2}(0, a \pi)}^{2}\right)
$$

which follows from the Poincaré inequality.
Proof of Lemma 5.2. Since $\mathbb{D}_{E}$ is the completion of $D_{E}$ for the norm $\|\cdot\|_{E}$, if we prove the estimates of the Lemma for $\mathcal{U} \in D_{E}$, then by density they will be satisfied for every element in $\mathbb{D}_{E}$. Let $\mathcal{U}$ be in $D_{E}$, recall that

$$
\begin{equation*}
\left\|\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right\|_{L^{2}(\omega)} \leq\|\mathcal{U}\|_{E}, \quad\left\|\frac{\partial \mathcal{U}_{2}}{\partial s_{2}}\right\|_{L^{2}(\omega)} \leq\|\mathcal{U}\|_{E}, \quad\left\|\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}+\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}\right\|_{L^{2}(\omega)} \leq\|\mathcal{U}\|_{E} \tag{10.11}
\end{equation*}
$$

Recall also that there exists $\left(C_{1}, C_{2}, C_{3}\right) \in \mathbb{R}^{3}$ such that

$$
\mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\left(s_{1}\right)=C_{1}, \quad \mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right)\left(s_{1}\right)=C_{2} s_{1}^{c}+C_{3}, \quad \text { for a.e. } s_{1} \in(0, a \pi)
$$

$S$ tep 1. In this step we prove

$$
\begin{align*}
&\left\|\mathcal{M}_{2}\left(\mathcal{U}_{1}\right)\right\|_{H^{1}(0, a \pi)}+\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)\right\|_{H^{1}(0, a \pi)}+\left\|\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(0, a \pi)}  \tag{10.12}\\
&+\left|\mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right|+\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}
\end{align*}
$$

Set

$$
\begin{align*}
\widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\left(s_{1}\right) & =C_{2} \frac{s_{1}\left(s_{1}-a \pi\right)}{2}, \\
\mathcal{M}_{22}\left(\mathcal{U}_{2}\right)\left(s_{1}\right) & =\frac{1}{l} \int_{0}^{l} \mathcal{U}_{2}\left(s_{1}, s_{2}\right) \frac{s_{2}\left(s_{2}-l\right)}{2} d s_{2}, \tag{10.13}
\end{align*} \quad \text { for a.e. } s_{1} \in(0, a \pi) .
$$

One has $\widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right), \mathcal{M}_{22}\left(\mathcal{U}_{2}\right) \in H_{0}^{1}(0, a \pi)$.
We first show that

$$
\left\|\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}+\frac{1}{a} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)} \leq \frac{1}{\sqrt{l}}\|\mathcal{U}\|_{E}
$$

By plugging in the definition for $\mathcal{M}_{2}\left(\mathcal{U}_{i}\right)$ we get

$$
\left\|\frac{1}{l} \frac{d}{d s_{1}} \int_{0}^{l} \mathcal{U}_{1}\left(s_{1}, s_{2}\right) d s_{2}+\frac{1}{a l} \int_{0}^{l} \mathcal{U}_{3}\left(s_{1}, s_{2}\right) d s_{2}\right\|_{L^{2}(0, a \pi)} .
$$

We interchange differentiation and integration, s.t. with Jensen

$$
\left\|\frac{1}{l} \int_{0}^{l}\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right) d s_{2}\right\|_{L^{2}(0, a \pi)} \leq \frac{1}{\sqrt{l}}\left\|\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right\|_{L^{2}(\omega)} \leq \frac{1}{\sqrt{l}}\|\mathcal{U}\|_{E}
$$

Moreover, we have

$$
\begin{aligned}
\left\|\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}\right\|_{L^{2}(0, a \pi)}^{2}+\left\|\frac{1}{a} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2} & =\int_{0}^{a \pi}\left(\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}+\frac{1}{a} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right)^{2} d s_{1} \\
& =\left\|\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}+\frac{1}{a} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2}
\end{aligned}
$$

since by partial integration

$$
\int_{0}^{a \pi} \frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right) d s_{1}=\left[\mathcal{M}_{2}\left(\mathcal{U}_{1}\right) \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right]_{0}^{a \pi}-\int_{0}^{a \pi} \mathcal{M}_{2}\left(\mathcal{U}_{1}\right) \frac{d \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)}{d s_{1}} d s_{1}=0
$$

Therefore, we obtain

$$
\left\|\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}\right\|_{L^{2}(0, a \pi)}^{2}+\left\|\frac{1}{a} \mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2} \leq C\|\mathcal{U}\|_{E}^{2}
$$

The Poincaré inequality in $H_{0}^{1}(0, a \pi)$ and the previous results lead to

$$
\left\|\mathcal{M}_{2}\left(\mathcal{U}_{1}\right)\right\|_{H^{1}(0, a \pi)} \leq C\left\|\frac{d \mathcal{M}_{2}\left(\mathcal{U}_{1}\right)}{d s_{1}}\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}
$$

and since $\mathcal{M}_{2}\left(\mathcal{U}_{3}\right)$ is independent of $s_{1}$ we obtain $\left|\mathcal{M}_{2}\left(\mathcal{U}_{3}\right)\right| \leq C\|\mathcal{U}\|_{E}$. Below we show the inequality

$$
\begin{equation*}
\left\|\frac{d \mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)}{d s_{1}}+\frac{1}{a} \frac{d \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)}{d s_{1}}+\frac{C_{3}}{a}\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E} \tag{10.14}
\end{equation*}
$$

Plugging in the definition for $\mathcal{M}_{2}^{c}$ and $\widetilde{\mathcal{M}}_{2}^{c}$ we get again with Jensen and since $s_{2}^{c} \in(-l / 2, l / 2)$

$$
\begin{aligned}
& \left\|\frac{d \mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)}{d s_{1}}+\frac{1}{a} \frac{d \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)}{d s_{1}}+\frac{C_{3}}{a}\right\|_{L^{2}(0, a \pi)} \\
= & \left\|\frac{1}{l} \int_{0}^{l}\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right) s_{2}^{c} d s_{2}\right\|_{L^{2}(0, a \pi)} \leq\left\|\frac{1}{\sqrt{l}}\left(\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}+\frac{1}{a} \mathcal{U}_{3}\right) s_{2}^{c}\right\|_{L^{2}(\omega)} \leq C\|\mathcal{U}\|_{E}
\end{aligned}
$$

Now, we prove the inequality

$$
\begin{equation*}
\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2}+\left|C_{3}\right|^{2} \leq C\|\mathcal{U}\|_{E}^{2} \tag{10.15}
\end{equation*}
$$

With the Poincaré inequality and since $C_{3} \in \mathbb{R}$ and $\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right), \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right) \in H_{0}^{1}(0, a \pi)$ we obtain

$$
\begin{aligned}
& \left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2}+\left|C_{3}\right|^{2} \\
& \leq C\left(\left\|\frac{d}{d s_{1}} \mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \frac{d}{d s_{1}} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}^{2}+\left\|\frac{C_{3}}{a}\right\|_{L^{2}(0, a \pi)}^{2}\right) \leq C\|\mathcal{U}\|_{E}
\end{aligned}
$$

using

$$
\int_{0}^{a \pi}\left(\frac{d}{d s_{1}} \mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \frac{d}{d s_{1}} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right) \frac{C_{3}}{a} d s_{1}=0
$$

In the following we show the inequality

$$
\begin{equation*}
\left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E} \tag{10.16}
\end{equation*}
$$

With the previous result, partial integration and 10.11) we get

$$
\begin{aligned}
& \left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)} \\
& =\left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}-\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)} \\
& \leq\left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}-\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)\right\|+\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}
\end{aligned}
$$

where we have for the first term by plugging in the definition and swapping integration with differentiation

$$
\begin{aligned}
& \left\|\frac{1}{l} \int_{0}^{l} \frac{\partial \mathcal{U}_{2}}{\partial s_{1}} \frac{s_{2}\left(s_{2}-l\right)}{2} d s_{2}-\frac{1}{l} \int_{0}^{l} \mathcal{U}_{1}\left(\cdot, s_{2}\right)\left(s_{2}-\frac{l}{2}\right) d s_{2}\right\|_{L^{2}(0, a \pi)} \\
& =\left\|\frac{1}{l} \int_{0}^{l}\left(\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right) \frac{s_{2}\left(s_{2}-l\right)}{2} d s_{2}\right\|_{L^{2}(0, a \pi)} \leq C\left\|\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}+\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right\|_{L^{2}(\omega)} \leq C\|\mathcal{U}\|_{E}
\end{aligned}
$$

Integrating $\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}+\frac{1}{a} \widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)$ over $(0, a \pi)$ and due to the above estimate 10.16 , one obtains $\left|C_{2}\right| \leq C\|\mathcal{U}\|_{E}$ and then again with 10.16 and 10.14 -10.15
$\left\|\widetilde{\mathcal{M}}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}+\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{3}\right)\right\|_{L^{2}(0, a \pi)}+\left\|\mathcal{M}_{2}^{c}\left(\mathcal{U}_{1}\right)\right\|_{H^{1}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}+\left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}$.
Since $\mathcal{M}_{22}\left(\mathcal{U}_{2}\right) \in H_{0}^{1}(0, a \pi),\left\|\mathcal{M}_{22}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(0, a \pi)} \leq C\left\|\frac{d \mathcal{M}_{22}\left(\mathcal{U}_{2}\right)}{d s_{1}}\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}$. The PoincaréWirtinger inequality gives

$$
\begin{equation*}
\left\|\mathcal{U}_{2}-\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(\omega)} \leq C\|\mathcal{U}\|_{E} . \tag{10.17}
\end{equation*}
$$

Multiply $\mathcal{U}_{2}-\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)$ by $\frac{s_{2}\left(s_{2}-l\right)}{2}$ and then integrate with respect to $s_{2}$ to get

$$
\left\|\mathcal{M}_{22}\left(\mathcal{U}_{2}\right)+\frac{l^{2}}{12} \mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E}
$$

Therefore

$$
\begin{equation*}
\left\|\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(0, a \pi)} \leq C\|\mathcal{U}\|_{E} . \tag{10.18}
\end{equation*}
$$

$S$ tep 2. We show the 3 inequalities in equation (5.3) by using Poincaré-Wirtinger inequality. We start with $\left\|\mathcal{U}_{2}\right\|_{L^{2}(\omega)} \leq C\|\mathcal{U}\|_{E}$. With the inequalities in 10.17)-10.18 we get

$$
\begin{equation*}
\left\|\mathcal{U}_{2}\right\|_{L^{2}(\omega)} \leq\left\|\mathcal{U}_{2}-\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(\omega)}+\left\|\mathcal{M}_{2}\left(\mathcal{U}_{2}\right)\right\|_{L^{2}(\omega)} \leq C\|\mathcal{U}\|_{E} \tag{10.19}
\end{equation*}
$$

Recall that if $X$ is a separable Hilbert space, then the Poincaré-Wirtinger inequality is valid in $W^{1, p}(0, l ; X)(p \in[1,+\infty])$. From 10.19 and $(10.11)_{3}$ we get

$$
\left\|\frac{\partial \mathcal{U}_{2}}{\partial s_{1}}\right\|_{L^{2}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)}+\left\|\frac{\partial \mathcal{U}_{1}}{\partial s_{2}}\right\|_{L^{2}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E}
$$

Then the Poincaré-Wirtinger inequality and estimate $10.12{ }_{1}$ in $H^{1}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)$ give

$$
\left\|\mathcal{U}_{1}\right\|_{L^{2}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)} \leq\left\|\mathcal{U}_{1}-\mathcal{M}_{2}\left(\mathcal{U}_{1}\right)\right\|_{L^{2}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)}+\left\|\mathcal{M}_{2}\left(\mathcal{U}_{1}\right)\right\|_{L^{2}\left(0, l ;\left(H^{1}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E}
$$

The above inequality leads to $\left\|\frac{\partial \mathcal{U}_{1}}{\partial s_{1}}\right\|_{L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E}$, which together with 10.11$)_{1}$ yields

$$
\left\|\mathcal{U}_{3}\right\|_{L^{2}\left(0, l ;\left(H^{2}(0, a \pi)\right)^{\prime}\right)} \leq C\|\mathcal{U}\|_{E}
$$

This ends the proof of the lemma.
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[^1]:    ${ }^{1}$ or equivalently $\left\|e\left(u_{\varepsilon}\right)\right\|_{L^{2}\left(\mathcal{Q}_{\varepsilon}\right)} \leq C \varepsilon^{3 / 2}$ since the displacements are extended to the whole shell.

[^2]:    ${ }^{2}$ As example, take $\left(\mathcal{F}_{2}, \mathcal{F}_{3}\right) \in L^{2}(0, a \pi) \times H^{2}(0, a \pi)$ and set

    $$
    F\left(s_{1}, s_{2}\right)=s_{2}^{c}\left(-a \frac{d \mathcal{F}_{3}}{d s_{1}}\left(s_{1}\right) \mathbf{t}_{1}+\mathcal{F}_{2}\left(s_{1}\right) \varepsilon_{2}+\mathcal{F}_{3}\left(s_{1}\right) \mathbf{n}\right) \quad \text { for a.e. } \quad\left(s_{1}, s_{2}\right) \in \omega
    $$

[^3]:    ${ }^{3}$ Note that if we transform the domain $\mathcal{O}_{\kappa_{0}}$ by a dilation, the constant does not change.

