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► **To cite this version:**

Josephine Evans, Iván Moyano. QUANTITATIVE RATES OF CONVERGENCE TO EQUILIBRIUM FOR THE DEGENERATE LINEAR BOLTZMANN EQUATION ON THE TORUS. 2019. hal-02285239

**HAL Id: hal-02285239**

**<https://hal.science/hal-02285239>**

Preprint submitted on 12 Sep 2019

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**QUANTITATIVE RATES OF CONVERGENCE TO  
EQUILIBRIUM FOR THE DEGENERATE LINEAR BOLTZMANN  
EQUATION ON THE TORUS**

JOSEPHINE EVANS AND IVÁN MOYANO

ABSTRACT. We study the linear relaxation Boltzmann equation on the torus with a spatially varying jump rate which can be zero on large sections of the domain. In [5] Bernard and Salvarani showed that this equation converges exponentially fast to equilibrium if and only if the jump rate satisfies the geometric control condition of Bardos, Lebeau and Rauch [3]. In [22] Han-Kwan and Léautaud showed a more general result for linear Boltzmann equations under the action of potentials in different geometric contexts, including the case of unbounded velocities. In this paper we obtain quantitative rates of convergence to equilibrium when the geometric control condition is satisfied, using a probabilistic approach based on Doeblin’s theorem from Markov chains.

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1. INTRODUCTION AND MAIN RESULTS

In this article, we study the linear Boltzmann equation in the phase space  $\Omega \times V$ , i.e., the system

$$(1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x W(x) \cdot \nabla_v f = \mathcal{L}(f), & \text{in } (0, T) \times \Omega \times V, \\ f|_{t=0} = f_0, & \text{in } \Omega \times V, \end{cases}$$

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*Key words and phrases.* Convergence to equilibrium; Hypocoercivity; Linear Boltzmann Equation; Degenerate Hypocoercivity, Geometric Control Condition.

where the density function,  $f = f(t, x, v)$ , undergoes the action of the potential  $W = W(x)$  and the collision term

$$\mathcal{C}(f) := \sigma(x) \int_V (p(v, v')f(v') - p(v', v)f(v)) dv',$$

for some  $\sigma \in L^\infty(\Omega)$ , assumed to be non-negative. Physically we can think of (1) as a modeling a radiative transfer system where different parts of the space may have different transparencies, according to the scattering function  $p = p(v, v')$ . When  $\sigma = \sigma(x)$  is a positive constant, (1) is the linear relaxation equation, linear BGK equation or linear Boltzmann equation.

In this work we set  $\Omega = \mathbb{T}^d$ , the  $d$ -dimensional torus, with the usual identification

$$(2) \quad \mathbb{T}^d = \mathbb{R}^d / [0, 1]^d.$$

According to the nature of the space of velocities,  $V$ , the potential  $W$  and the scattering function  $p$ , (1) has the following measure-valued equilibrium state

$$\nu = \nu_x \otimes \nu_v,$$

where

$$\nu_x = \frac{1}{Z} e^{-W(x)} dx, \quad Z = \int_{\mathbb{T}^d} e^{-W(x)} dx,$$

and

$$\nu_v = \begin{cases} \frac{1}{|V|} & \text{if } W = 0, \quad p(v, v') = \frac{1}{|V|}, \\ \mathcal{M}(v) & \text{if } W \neq 0, \quad p(v, v') = \mathcal{M}(v), \end{cases}$$

where  $\mathcal{M}(v)$  denotes the normalised Maxwellian, i.e.,

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^d.$$

In the non-degenerate case  $\sigma > 0$ , the study of the trend to equilibrium of solutions to system (1) has been the object of many publications, using techniques as hypocoercivity (see Section 1.2 for details). In the degenerate case  $\sigma \geq 0$ , the problem of characterising the trend to equilibrium is deeply connected to the structure of the phase space  $\mathbb{T}^d \times V$  and the geometry of the set  $\{\sigma > 0\}$ , as (1) reduces to a transport equation outside this region. In [6] Bernard and Salvarani showed that exponential convergence towards equilibrium cannot hold in general. On the other hand, the same authors proved in [5] that the solutions to (1) with  $\Omega \times V = \mathbb{T}^d \times \mathbb{S}^{d-1}$  and  $W = 0$  converge to equilibrium exponentially in  $L^1$  if and only if the support of  $\sigma$  satisfies the geometric control condition (GCC for short), inspired from [3, 26] and characterized in the following way.

**Definition 1.** *The function  $\sigma$  satisfies the Geometric Control Condition (GCC) if there exists  $T = T(\sigma) > 0, \kappa > 0$  such that*

$$(3) \quad \inf_{(x, v) \in \mathbb{T}^d \times V} \int_0^T \sigma(x + vt) dt \geq \kappa.$$

The case  $W \neq 0$  and  $\sigma \geq 0$  has been analysed by Han-Kwan and Léautaud in [22], where the action of the potential may generate many different dynamics. Considering the characteristic flow

$$(4) \quad \Phi_t(x, v) = (\Phi_t^X(x, v), \Phi_t^V(x, v)), \quad t \in \mathbb{R},$$

where, for  $(x, v) \in \mathbb{T}^d \times V$  given,  $(\Phi_t^X, \Phi_t^V) = (\Phi_t^X(x, v), \Phi_t^V(x, v)) =$  solve the characteristic equations

$$(5) \quad \begin{aligned} \frac{d}{dt} \Phi_t^X &= \Phi_t^V, & \Phi_0^X &= x, \\ \frac{d}{dt} \Phi_t^V &= -\nabla_x W(\Phi_t^X), & \Phi_0^V &= v, \end{aligned}$$

the authors redefine the Geometric Control Condition in the following way.

**Definition 2.** *There exists a  $T = T(\sigma, W) > 0, \kappa > 0$  such that*

$$(6) \quad \inf_{(x, v) \in \mathbb{T}^d \times V} \int_0^T \sigma(\Phi_t^X(x, v)) dt \geq \kappa.$$

This definition is again inspired from the study of the controllability of the wave equation in [3, 26] (see Section 1.2 for more details). In this context, Han-Kwan and Léautaud give in [22] conditions linking the collision kernel and the potential which imply either convergence to a steady state or exponential convergence to a steady state. Let us mention that the results in [22] are much more general (see Section 1.2) than the setting presented here.

The methods developed in the works [5, 6, 22] do not yield constructive convergence rates for the trend to equilibrium. The goal of the present work is to obtain quantitative rates using different methods, inspired in tools from Markov chains.

**1.1. Main results.** We shall consider the following three regimes

**(R1):**  $V$  is an open nonempty set of  $\mathbb{R}^d$ , the scattering is isotropic and the potential is zero, i.e.,

$$(7) \quad p(v, v') = p(v', v) = 1, \quad \forall v, v' \in V \quad \text{and} \quad W = 0,$$

**(R2):**  $V = \mathbb{S}^{d-1}$  and (7) holds,

**(R3):**  $V = \mathbb{R}^d$ , the scattering function is given by a Maxwellian and the potential is nonzero, i.e.,

$$(8) \quad p(v, v') = \mathcal{M}(v), \quad \forall v, v' \in V \quad \text{and} \quad W \text{ is a smooth function on } \mathbb{T}^d.$$

In what follows we consider measure-valued solutions to (1) and we refer to Section 2 for details. We denote by  $\mathcal{M}(\mathbb{T}^d \times V)$  the space of measures on  $\mathbb{T}^d \times V$ , which is a Banach space endowed with the *total variation* norm, denoted  $\|\cdot\|_{TV}$  (see (19) for details). We denote  $\mathcal{P}(\mathbb{T}^d \times V)$  the space of probability measures on  $\mathbb{T}^d \times V$ . Finally,  $(T_t)_{t \geq 0}$  denotes the semigroup generated by the free transport operator on measures (see Definition 4).

Our first result corresponds to the situation described in **(R1)**.

**Theorem 1.** *Assume that  $V \subseteq \mathbb{R}^d$  is an open set satisfying that there exists  $T_* < \infty$  and  $\beta \in (0, 1)$  such that for all  $t \geq T_*$  we have*

$$(9) \quad \inf_{x_0 \in \mathbb{T}^d} \int_V T_t(\delta_{x_0} \otimes \nu_v) dv \geq \beta \nu_{x_0}.$$

*Let  $\sigma \in C^0(\mathbb{T}^d)$  such that Definition 1 holds. If  $(\mu_t)_{t \geq 0}$  is a measure solution to (1) with initial datum  $\mu_0 \in \mathcal{P}(\mathbb{T}^d \times V)$ , then*

$$(10) \quad \|\mu_t - \nu\|_{TV} \leq e^{-\lambda(t-2T-T_*)} \|\mu_0 - \nu\|_{TV}, \quad \forall t \geq t_*,$$

*with the quantitative rate*

$$(11) \quad \lambda = -\frac{1}{2T + T_*} \log \left( 1 - \beta \kappa^2 e^{-(2T+T_*)\|\sigma\|_\infty} \right).$$

The lower bound in (9) is a crucial hypothesis intimately linked to Doeblin's theorem and is key to obtain the exponential rate (11), as can be seen in Section 4.2. In order to refine the quantitative bound in (11), we give in Lemma 1 some sufficient conditions on  $V$  so that (9) holds with concrete choices of  $\beta$  and  $T_*$ .

Our next result concerns the situation described by (R2).

**Theorem 2.** *Let  $V = \mathbb{S}^{d-1}$  and assume that there exist  $T_{**}, \beta_{**} > 0$  so that if  $t, s \geq T_{**}$  then*

$$(12) \quad \int_V T_t \left( \int_{u \in \mathbb{S}^{d-1}} T_s (\delta_{x_0} \mathbf{1}_{v \in \mathbb{S}^{d-1}})(\cdot, u) du \mathbf{1}_{v \in \mathbb{S}^{d-1}} \right) dv \geq \beta \nu_x.$$

Let  $\sigma \in C^0(\mathbb{T}^d)$  such that Definition 1 holds. Then, if  $(\mu_t)_{t \geq 0}$  is a measure solution to (1) with initial datum  $\mu_0 \in \mathcal{P}(\mathbb{T}^d \times V)$ , then

$$(13) \quad \|\mu_t - \nu\|_{TV} \leq e^{-\lambda(t-2T-T_*)} \|\mu_0 - \nu\|_{TV}, \quad \forall t \geq T_*,$$

with the quantitative rate

$$(14) \quad \lambda = -\frac{1}{3T+2T_{**}} \log \left( 1 - \frac{\kappa^3 \beta_{**}}{2} e^{-(3T+2T_{**})\|\sigma\|_\infty} \right).$$

Condition (12) replaces (9) as  $T_*$  is not finite when  $V = \mathbb{S}^{d-1}$ . Loosely speaking, this set cannot be spread out to the whole of  $\mathbb{T}^d$  by the transport operator  $(T_t)_{t \geq 0}$  since its dimension is too small. The new condition (12) essentially says that we need three jumps to spread out the initial measure.

Our last result concerns the regime (R3), with non-zero potentials.

**Theorem 3.** *Let  $V = \mathbb{R}^d$  and  $W \in C^2(\mathbb{T}^d; \mathbb{R}^d)$ . Assume that there exist  $\beta_{***} \in (0, 1)$  and  $T_{***} > 0$ , depending on  $W$ , such that for all  $t \in [T_{***}, T_{***} + T]$  we have*

$$(15) \quad \int T_t (\delta_{x_0} \otimes \nu_v)(x, v) dv \geq \beta_{***} \nu_x.$$

Suppose that  $\sigma \in C^0(\mathbb{T}^d)$  satisfies the geometric control condition (2) with  $W \neq 0$ . Then, if  $(\mu_t)_{t \geq 0}$  is a measure-valued solution to (1) with initial datum  $\mu_0 \in \mathcal{P}(\mathbb{T}^d \times V)$ , then

$$(16) \quad \|\mu_t - \nu\|_{TV} \leq e^{-\lambda(t-2T-T_{***})} \|\mu_0 - \nu\|_{TV},$$

with the quantitative rate

$$(17) \quad \lambda = -\frac{1}{2T+T_{***}} \log \left( 1 - \beta_{***} \kappa^2 e^{-(2T+T_{***})\|\sigma\|_\infty} \right).$$

**Remark.** Observe that Theorems 1, 2 and 3 contain quantitative rates in terms of  $\beta$  and  $T$ . We will give in Section 3 precise results with explicit rates and assumptions in section 3.

**Remark.** Observe that we are assuming that  $\sigma \in \mathcal{C}^0(\mathbb{T}^d)$  instead of just bounded and measurable. This is a technical assumption due to the fact that we are working with measure-valued solutions. See Section 2 for details.

**1.2. Previous works: Hypocoercivity, Doeblin's theorem and the geometric control condition.**

1.2.1. *Hypoocoercivity results when  $\sigma$  is strictly positive.* Finding quantitative rates of convergence to equilibrium is a longstanding problem in kinetic theory. In the context of spatially inhomogeneous kinetic equations this is usually done using the tools of hypoocoercivity, a name given by Villani in [30] to equations exhibiting convergence like  $Ce^{-\lambda t}$  where  $C \geq 1$ . In the context of kinetic equations, hypoocoercive behaviour is typically found when considering spatially inhomogeneous equations where the dissipation of natural entropies vanishes on a large class of functions, the local equilibria, making it impossible to prove entropy-entropy production inequalities. Techniques to prove convergence for such equations based on hypoellipticity methods were developed in [24, 28, 30] as well as in many other works.

When  $\sigma$  is constant, equation (1) is a key example of a hypoocoercive equation, shown to converge faster than any power of  $t$  in  $H^1$  norm in [13] using the framework of [16]. It was then shown to converge exponentially fast to equilibrium in  $H^1$  weighted against the equilibrium in [28] and in  $L^2$  weighted against the equilibrium in [23]. The convergence in weighted  $L^2$  can also be seen as a result of the general theorem in [18]. There are several other works showing exponential convergence in various norms or for various more complex versions of this equation we mention in particular [11] since this work uses Doeblin/Harris's theorem, which is also the tool we will apply to the spatially degenerate case.

1.2.2. *Hypoocoercivity results when  $\sigma$  can vanish.* The case where  $\sigma = \sigma(x)$  is non constant and can vanish on areas of the spatial domain was first studied in [4] although it is mentioned somewhat indirectly. This paper deals with non-equilibrium steady states for scattering operators and is a pioneering example of the use of probabilistic tools in statistical physics, but without quantitative rates.

The more recent works on these spatially degenerate models was begun in [15] where the authors study a model where  $\sigma$  vanishes at a discrete set of points. In [6] Bernard and Salvarani showed that there are situations where the velocity space and form of  $\sigma$  together mean that there is no exponential convergence towards equilibrium. On the other hand, Bernard and Salvarani proved in [6] that the solutions to (1) with  $\Omega \times V = \mathbb{T}^d \times \mathbb{S}^{d-1}$  and  $W = 0$  convergence to equilibrium exponentially in  $L^1$  if and only if the support of  $\sigma$  satisfies the geometric control condition of Definition 1. This work is then extended in [27] to give a more delicate sense of when exponential convergence to equilibrium will occur. The approaches followed in [6, 27], based on semigroup theory and abstract functional analysis, do not allow one to obtain a quantitative rate of the convergence.

An equation related to (1), the  $1d$  Goldstein-Taylor type model, has been studied in [7] where the authors do get explicit rates via comparing this equation to a damped wave equation for which explicit rates were obtained by Lebeau in [26].

The case where  $V$  is unbounded is treated in [22] by Han-Kwan and Léautaud, where the authors study linear Boltzmann type equations for a general class of collision operators and external confining potential terms on a closed, smooth, connected and compact Riemannian manifold  $M$  (and in particular the torus). In this context, the authors indentify geometric control conditions in the natural phase space  $T^*M$  (similar to Definition 2 in the case  $M = \mathbb{T}^d$ ) allowing to completely characterise the convergence to equilibrium and exponentially fast convergence to equilibrium for the corresponding linear Boltzmann equation. On the other hand, the techniques developed in [22], using phase-space and microlocal tools inspired from [3, 26] do not give explicit rates of convergence.

In [17] the kinetic Fokker-Planck case is studied and here it is shown that the GCC is not equivalent to exponential convergence to equilibrium.

1.2.3. *Doebelin's theorem.* We use techniques which are inspired from Doebelin's theorem from Markov process theory (see [20] for a detailed exposition of this theorem). This theorem was used to show convergence to equilibrium for scattering equations in [4]. It has been used several times to study convergence to equilibrium for kinetic equations in the context of Non-Equilibrium Steady States [14] and is currently being used for studying the convergence to equilibrium for solutions of PDEs from mathematical biology. We mention in particular the works on the renewal equation [19], and the neuron population model [12]. This last paper contains a similar type of degeneracy to that studied in this work. In this context Doebelin's theorem and Harris's theorem have been extended to PDEs which do not conserve mass and/or have time-periodic limiting solutions rather than steady states, as in [1, 2].

1.2.4. *The geometric control condition in control theory.* . The geometric control condition mentioned in the previous section plays a fundamental role in the study of controllability and stabilisation properties of some linear PDEs, typically of hyperbolic type. The GCC condition was introduced in the seminal works [29, 3, 25] in order to prove that the linear wave equation and the Schrödinger equation in a domain  $\Omega \subset \mathbb{R}^d$ , possibly with boundary, are exactly controllable from an open subset  $\omega$  (or a subset of the boundary) as long as  $\omega$  satisfies the geometric control condition. In [9] the GCC condition is proved to be necessary for the exact controllability of the wave equation. As for the stabilisation properties, the works [3, 26, 10] prove that under the GCC condition one can expect an exponential trend to equilibrium for the wave equation with a localised damping, which is a crucial inspiration for the works [6, 22] on the linear Boltzmann equation.

1.3. **Strategy and Outline.** We first prove Theorems 1 and 3. As stated above the proof is based around Doebelin's theorem for Markov processes. The key element to executing a Doebelin argument is to find a time  $t_*$  such that we can prove a lower bound on the solution of the equation at time  $t_*$  which is independent of the initial condition. We give a detailed proof of this fact based on using Duhamel's formula. We then explain how this implies exponential convergence to equilibrium via Doebelin's theorem. We then give a more stochastic flavored proof of the same lower bound. This section is intended mainly to be illustrate the intuition behind the first proof so we are a bit briefer with the calculations. The argument written in terms of conditional probabilities is more clearly linked to the aspects of the trajectories of the stochastic process which allow us to prove convergence to equilibrium. Finally we prove Theorem 2. Here the proof is very similar to that of the other theorems. However we have to use a strategy involving 3 jumps. This means it is much simpler to write in the stochastic formulation. Lastly we have a concluding section which includes possible extensions of this project and a discussion of the rates.

*Acknowledgements.* We would like to thank many people for some useful discussion. In particularly José Cañizo for help with the deterministic version of the proof of Theorem 1. We had useful discussions with Francesco Salvarani, Havva Yoldaş, Chuqi Cao, Helge Dietert and Clément Mouhot. The first author was supported by

FSPM postdoctoral fellowship (since October 2018) and the grant ANR-17-CE40-0030. Much of this was written while the first author was visiting the Hausdorff Research Institute for Mathematics on a Junior Trimester fellowship. We would like to thank them for their hospitality. The second author was supported by the ERC grant MAFRAN.

## 2. MEASURED-VALUED SOLUTIONS TO THE LINEAR BOLTZMANN EQUATION

Let us first define some notation in order to state our results. Given  $(\mathcal{X}, \Sigma)$  a measurable space, we denote by  $\mathcal{M}(\mathcal{X})$  the set of Radon measures on  $\mathcal{X}$ . We denote by  $\mathcal{P}(\mathcal{X})$  the set of probability measures on  $\mathcal{X}$ , i.e., all measures  $\mu \in \mathcal{M}(\mathcal{X})$  satisfying  $\mu(\mathcal{X}) = 1$  and  $\mu(A) \geq 0$  for every measurable  $A$ . As usual the space  $\mathcal{P}(\mathcal{X})$  is endowed with the weak topology, denoted  $w - \mathcal{P}(\mathcal{X})$ , induced by the family of semi-norms

$$\phi \mapsto \int_{\mathcal{X}} \phi(z) d\mu(z), \quad \forall \phi \in C_b(\mathcal{X}),$$

i.e., we are using test functions which are continuous and bounded on  $\mathcal{X}$ . Recall that  $\mu \in \mathcal{M}(\mathcal{X})$  is said to be non-negative whenever

$$(18) \quad \int_{\mathcal{X}} \phi(x) \mu(dz) \geq 0, \quad \forall \phi \in C_b(\mathcal{X}; \mathbb{R}_+).$$

The *total variation* distance in  $\mathcal{M}(\mathcal{X})$  is defined as usual as

$$(19) \quad \|\mu\|_{TV} := \sup \left\{ \int_{\mathcal{X}} \phi(z) \mu(dz); \phi \in C_b(\mathcal{X}) \right\}.$$

Consider next a phase space of the form  $\mathcal{X} = \Omega \times V$ , where  $\Omega = \mathbb{T}^d$  or  $\mathbb{R}^d$ . If  $\Sigma_{\Omega \times V}$  is the Borel  $\sigma$ -algebra on  $\Omega \times V$ , we denote by  $\mathcal{L}_{\Omega \times V}$  the Lebesgue measure on  $\Omega \times V$ . If  $A \in \Sigma_{\Omega \times V}$ , we simply denote by  $|A|$  the Lebesgue measure of  $A$  if no confusion arises.

**2.1. Measure-valued solutions.** With the notation of the previous section, given  $T > 0$  and  $\mu_0 \in \mathcal{P}(\Omega \times V)$ , we consider the transport equation

$$(20) \quad \begin{cases} \partial_t \mu + v \cdot \nabla_x \mu - \nabla_x W \cdot \nabla_v \mu = 0, & \text{in } (0, T) \times \Omega \times V, \\ \mu|_{t=0} = \mu_0, & \text{in } \Omega \times V. \end{cases}$$

**Definition 3.** A *measure solution* to (20) is an element of  $C^0([0, T]; w - \mathcal{P}(\Omega \times V))$ , denoted  $\mu_t = \mu_t(dx, dv)$ , satisfying that for every  $\phi \in C_c^1([0, T] \times \Omega \times V)$ ,

$$\int_0^T \iint_{\Omega \times V} (\partial_t \phi - v \cdot \nabla_x \phi + \nabla_x W \cdot \nabla_v \phi) \mu_t(dx dv) dt = \iint_{\Omega \times V} \phi(0, x, v) \mu_0(dx dv).$$

We can write any weak solution to (20) using the transport semigroup.

**Definition 4.** The *transport semigroup* on  $\mathcal{P}(\Omega \times V)$ , noted  $(T_t)_{t \geq 0}$ , is defined by

$$(T_t \mu_0)(\phi) = \iint_{\Omega \times V} \phi(\Phi_{-t}(x, v)) d\mu_0(dx, dv), \quad \forall \phi \in C_b(\Omega \times V),$$

for any  $\mu_0 \in \mathcal{P}(\Omega \times V)$  and  $t \geq 0$ . In particular,  $\mu_t = T_t \mu_0(dx, dv)$  is a *measure solution* to (20).



In this article we work with the linear Boltzmann equation (1) in the sense of measures. Given  $\mu \in \mathcal{P}(\Omega \times V)$  we set

$$(21) \quad m_\sigma \mu(dx, dv) := \sigma(x) \mu(dx, dv), \quad L^+ \mu(dx) := \int_V p(v, v') \mu(dx, dv'),$$

which are respectively the multiplication by  $\sigma$  and the average in the variable  $v \in V$ . Given  $\mu_0 \in \mathcal{P}(\Omega \times V)$  we set

$$(22) \quad \begin{cases} \partial_t \mu + v \cdot \nabla_x \mu - \nabla_x W(x) \cdot \nabla_v \mu = m_\sigma(L^+ \mu - \mu), & \text{in } (0, T) \times \Omega \times V, \\ \mu|_{t=0} = \mu_0, & \text{in } \Omega \times V. \end{cases}$$

which is a version of (1) for measured-valued solutions.

**Definition 5.** A measure solution to (22) is an element of  $C^0([0, T]; w\text{-}\mathcal{P}(\Omega \times V))$ , denoted  $\mu_t = \mu_t(dx, dv)$ , satisfying that for every  $\phi \in C_c^1([0, T] \times \Omega \times V)$ ,

$$\begin{aligned} & \int_0^T \iint_{\Omega \times V} (\partial_t \phi - v \cdot \nabla_x \phi + \nabla_x W \cdot \nabla_v \mu + m_\sigma(\phi - L^+ \phi)) \mu_t(dx, dv) dt \\ &= \iint_{\Omega \times V} \phi(0, x, v) \mu_0(dx, dv). \end{aligned}$$

**Proposition 1.** Given  $T > 0$  and given  $\mu_0 \in \mathcal{P}(\Omega \times V)$ , there exists a unique measure-valued solution to (22), namely  $\mu_t = \mu_t(dx, dv)$ . Moreover, this solution admits the representation

$$(23) \quad \mu_t(dx, dv) = \exp\left(-\int_0^t \sigma(\Phi_s^X(x, v)) ds\right) (T_t \mu_0)(dx, dv) + S_t[\mu_t](dx, dv)$$

where  $(T_t)_{t \geq 0}$  is given by Definition 4 and

$$(24) \quad S_t[\mu_t](dx, dv) = \int_0^t \exp\left(-\int_s^t \sigma(\Phi_r^X(x, v)) dr\right) (T_{t-s} m_\sigma L^+ \mu_s)(dx, dv) ds.$$

Denoting

$$(25) \quad \mu_t(dx, dv) = \mathcal{P}_t \mu_0, \quad t \geq 0,$$

the family  $(\mathcal{P}_t)_{t \geq 0}$  is a semigroup on  $\mathcal{M}(\Omega \times V)$  enjoying the following properties

$$(26) \quad \|\mathcal{P}_t \mu_0\|_{TV} = 1, \quad \forall \mu_0 \in \mathcal{P}(\Omega \times V),$$

$$(27) \quad \|\mathcal{P}_t \mu_0 - \mathcal{P}_t \bar{\mu}_0\|_{TV} \leq \|\mu_0 - \bar{\mu}_0\|_{TV}, \quad \forall \mu_0, \bar{\mu}_0 \in \mathcal{P}(\Omega \times V).$$

We can equivalently treat (22) as a random system in the following way. We can write this equation as a stochastic process in integral form in the case  $W = 0$ ,  $V$  is compact and  $p = 1$ .

(28)

$$X_t = X_0 + \int_0^t V_s ds, \quad V_t = V_0 + \int_0^t \int_V \int_0^{\|\sigma\|_\infty} 1_{\theta \leq \sigma(X_t)} (w - V_{s-}) \Pi(ds, dw, d\theta).$$

Here  $\Pi$  is a Poisson random measure with intensity being  $\lambda \otimes \lambda \otimes U$  where  $\lambda$  is Lebesgue measure and  $U$  is the uniform measure. This is to say that the velocity process ‘jumps’ at random times  $T_1, T_2, \dots$  where  $T_{i+1} - T_i$  are exponentially distributed with rate  $A$ . Then the jump  $i$  is accepted with probability  $\sigma(X_{T_i})$  and if the jump is accepted the velocity variable will jump to a new velocity chosen

uniformly at random from  $V$  and if the jump is not accepted then the velocity remains the same. When we work in this formulation we write  $\mathbb{P}_{x_0, v_0}(A)$  for the probability of an event  $A$  happening when we begin this process with deterministic initial conditions  $X_0 = x_0, V_0 = v_0$ .

### 3. GEOMETRIC ASSUMPTIONS ON THE PHASE SPACE

In this section we introduce some hypothesis on the phase space  $\Omega \times V$  connecting the geometry of  $\Omega \times V$  with the transport operator acting on it. We essentially require that the phase space spreads out in a quantitative way any punctual mass in space after thermalisation in all directions in velocity. This property ensures that the Doeblin type argument of the next section can be applied. We also prove that some usual choices of phase spaces, such as  $V$  containing an annulus or  $V$  a sphere, satisfy the mentioned hypothesis with quantitative rates.

**Lemma 1.** *If  $W = 0$  and  $V$  contains a set like  $\{r_1 \leq |v| \leq r_2\}$  for some  $0 < r_1 < r_2$  then we can find such a  $T_*, \beta$ . In this case we can choose*

$$T_* = \max \left\{ 8\sqrt{d}d|B(0, 1)| (r_1^{d-1} + r_2^{d-1}), \sqrt{d}/r_1 \right\}$$

and

$$\beta = \frac{|\{r_1 \leq |v| \leq r_2\}|}{2|V|}.$$

We begin by proving lemma 1 for the case of theorem 1

*Proof of Lemma 1.* Assume that there exist  $0 < r_1 < r_2$  such that

$$\{v \in \mathbb{R}^d; r_1 \leq |v| \leq r_2\} \subseteq V.$$

Let us fix  $x_0 \in \mathbb{T}^d$ . Then, changing variables and using the identification (2),

$$\frac{1}{|V|} \int_{r_1 \leq |v| \leq r_2} \delta_{x_0}(x - vt) dv = \frac{1}{t^d|V|} \int_S \delta(x' = x_0 \bmod 1) dx' \geq \frac{|Q|}{t^d|V|},$$

where

$$S = \{x' \in \mathbb{R}^d; tr_1 \leq |x' - x_0| \leq tr_2\}$$

and  $Q$  is the union of all the cubes with volume one and integer vertices contained inside  $S$ .

Now, observe that the difference between  $S$  and  $Q$  is only along the perimeter of  $S$ . The parts of  $S$  that are not in  $Q$  are contained in the annuli  $\{tr_1 - \sqrt{2} \leq |v| \leq tr_1 + \sqrt{2}\}$  and  $\{tr_2 - \sqrt{2} \leq |v| \leq tr_2 + \sqrt{2}\}$ . Therefore

$$\begin{aligned} \|S\| - |Q| &\leq |B(0, 1)| \left( (tr_2 + \sqrt{2})^d + (tr_1 + \sqrt{2})^d - (tr_2 - \sqrt{2})^d - (tr_1 - \sqrt{2})^d \right) \\ &\leq |B(0, 1)| t^d 2d \left( r_2^{d-1} \sqrt{2} t^{-1} + r_1^{d-1} \sqrt{2} t^{-1} \right) = C(d, r_1, r_2) t^{d-1}. \end{aligned}$$

In this last inequality we write

$$(tr + \sqrt{2})^d - (tr - \sqrt{2})^d = (tr)^d \left( \left(1 + \frac{\sqrt{2}}{tr}\right)^d - \left(1 - \frac{\sqrt{2}}{tr}\right)^d \right).$$

Then we use the fact that for  $x < 1$  the derivatives of  $(1 + x)^d$  and  $(1 - x)^d$  are bounded by  $2d$ . With this and Taylor's theorem we conclude that if  $t > \sqrt{2}/r_1$  then we have that

$$(tr + \sqrt{2})^d - (tr - \sqrt{2})^d \leq 2d(tr)^{d-1} \sqrt{2}.$$

As a result, we get

$$\frac{1}{|V|} \int_{r_1 \leq |v| \leq r_2} \delta_{x_1}(x - vt) dv \geq (1 - \frac{1}{t} C(d, r_1, r_2)) \geq \frac{1}{2},$$

as long as  $t$  is taken large enough.

Now if the set  $V$  contains but is not equal to the annulus  $\{r_1 \leq |v| \leq r_2\}$  then the argument above yields the result with

$$\beta = \frac{|\{r_1 \leq |v| \leq r_2\}|}{2|V|}.$$

□

**Lemma 2.** *If  $V = \mathbb{S}^{d-1}$  and  $W = 0$  then we can find such a  $T_{**}$  and  $\beta_{**}$  with*

$$T_{**} = 8\sqrt{d}|B(0, 1)|(d-1)/d,$$

and  $\beta = 1/2$ .

*Proof of lemma 2.* Lets look at this against a test function.

$$\begin{aligned} & \int \int T_t \left( \int_{u \in \mathbb{S}^{d-1}} T_s(\delta_{x_0} 1_{v \in \mathbb{S}^{d-1}})(\cdot, u) du 1_{v \in \mathbb{S}^{d-1}} \right) dv \phi(x) dx \\ &= \int \int \int_{u \in \mathbb{S}^{d-1}} T_s(\delta_{x_0} 1_{v \in \mathbb{S}^{d-1}})(x, u) du 1_{v \in \mathbb{S}^{d-1}} \phi(x - vt) dv dx \\ &= \int \int_{u \in \mathbb{S}^{d-1}} \int_{v \in \mathbb{S}^{d-1}} \delta_{x_0}(x) \phi(x - vt - us) dudv dx \\ &= \int \phi(x') \left( \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \delta_{x_0}(x' + vt + us) dudv \right) dx'. \end{aligned}$$

At this point the calculations become a bit technical. Firstly since we are looking at an integral which is symmetrical in  $s, t$  we can assume w.l.o.g. that  $s \leq t$ . Lets introduce some notation. First lets write  $\lambda_d(A)$  for the  $d$ -dimensional Lebesgue measure of a set  $A$ . Lets also write  $\mathbb{S}^{d-1}(x, r)$  to be the  $d-1$  dimensional sphere of radius  $r$  around point  $x$ . Now we introduce some notations for sets depending on a set of parameters,  $s, t$  being positive numbers and  $x', x''$  points in  $\mathbb{R}^d$ .

$$(29) \quad A_{s,t,x'} = \{x \in \mathbb{R}^d \mid t - s \leq |x - x'| \leq t + s\},$$

$$(30) \quad B_{s,t,x',x''} = \{y \in \mathbb{S}^{d-1}(x', t) \mid |x'' - y| = s\}.$$

Here  $A_{s,t,x'}$  is an annulus in  $\mathbb{R}^d$  and  $B_{s,t,x',x''}$  is a  $d-2$  dimensional sphere of radius  $s$  around the point  $x''$  provided that the point  $x''$  is in  $A_{s,t,x'}$  and is empty if  $x''$  is not in  $A_{s,t,x'}$ . We then look at the integral

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \delta_{x_0}(x' + vt + us) dudv.$$

We then reparametrize in terms of  $x'' = x' + vt + us, y = x' + vt$ . Using this we can rewrite this integral as

$$\begin{aligned} & (st)^{-(d-1)} |\mathbb{S}^{d-1}|^{-2} \int_{A_{s,t,x'}} \int_{B_{s,t,x',x''}} \delta_{x_0}(x'') dy dx'' \\ &= (st)^{-(d-1)} |\mathbb{S}^{d-1}|^{-2} \int_{A_{s,t,x'}} \delta_{x_0}(x'') \lambda_{d-2}(B_{s,t,x',x''}) dx'' \end{aligned}$$

$$\begin{aligned}
&= s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} \int_{A_{s,t,x'}} \delta_{x_0}(x'') dx'' \\
&= s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} \mathbf{1}_{x_0 \in A_{s,t,x'}} \\
&= s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} \mathbf{1}_{x' \in A_{s,t,x_0}}.
\end{aligned}$$

We can see that if  $s$  and  $t$  are big enough we will cover enough of the torus several times so we can repeat the argument of lemma 1. We look at  $Q_{s,t,x_0}$  being the union of all the cubes with integral points contained inside  $A_{s,t,x_0}$ . As in lemma 1 the set difference between  $A_{s,t,x_0}$  is contained in two small annuluses containing the perimeter of  $A_{s,t,x_0}$  specifically the annuluses

$$\left\{ x \mid t - s - \sqrt{d} \leq |x - x_0| \leq t - s + \sqrt{d} \right\}, \left\{ x \mid t + s - \sqrt{d} \leq |x - x_0| \leq t + s + \sqrt{d} \right\}.$$

Emulating the calculations in lemma 1 we can bound the area of these annuluses by

$$2\sqrt{2}d \left( (t+s)^{d-1} + (t-s)^{d-1} \right).$$

Going back to our original calculation we have

$$\begin{aligned}
&\int \int T_t \left( \int_{u \in \mathbb{S}^{d-1}} T_s(\delta_{x_0} \mathbf{1}_{v \in \mathbb{S}^{d-1}})(\cdot, u) du \mathbf{1}_{v \in \mathbb{S}^{d-1}} \right) dv \phi(x) dx \\
&\geq \int \phi(x') s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} \mathbf{1}_{x' \in A_{s,t,x_0}} dx' \\
&\geq \int \phi(x') s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} \mathbf{1}_{x' \in Q_{s,t,x_0}} dx' \\
&\geq \int_{\mathbb{T}^d} \phi(x') dx' s^{-1} t^{-(d-1)} |\mathbb{S}^{d-2}| |\mathbb{S}^{d-1}|^{-2} |Q_{s,t,x_0}| \\
&= \int_{\mathbb{T}^d} \phi(x') dx' \left( \frac{|Q_{s,t,x_0}|}{|A_{s,t,x_0}|} \right).
\end{aligned}$$

Now we can compute that

$$\frac{|Q_{s,t,x_0}|}{|A_{s,t,x_0}|} \geq 1 - 2\sqrt{2}|B(0,1)| \frac{(t+s)^{d-1} + (t-s)^{d-1}}{(t+s)^d - (t-s)^d}.$$

We have that

$$(t+s)^d - (t-s)^d \geq dt^{d-1}s,$$

and

$$(t+s)^{d-1} + (t-s)^{d-1} \leq 2(d-1)t^{d-1}.$$

Therefore

$$\frac{|Q_{s,t,x_0}|}{|A_{s,t,x_0}|} \geq 1 - 2\sqrt{2}|B(0,1)| \frac{2(d-1)t^{d-1}}{dt^{d-1}s}.$$

So our lemma holds with  $\beta_{**} = 1/2$  and  $T_{**} = 8\sqrt{d}|B(0,1)|d/(d-1)$ . Therefore, if  $t, s$  are sufficiently large then this will be greater than  $1/2$ .  $\square$

**Lemma 3.** *For  $V$  smooth, periodic and positive we can find such  $(\beta_{***}, T_{***})$ . With  $T_{***} = 1/2$  and*

$$\beta = \left( \int_{\mathbb{T}^d} e^{-W(x)} dx \right) \exp \left( -(T+1) (1 + \|Hess(W)\|_\infty) \right)$$

*Proof of Lemma 3.* The first thing to note is that since the  $x$  space is compact we have that  $|\nabla_x V(x)| \leq G$  for some  $G$ . This means that we move from very high to other high velocities. We can see that we have for any  $x_0$  if  $t \leq T$  then

$$T_t(\{x_0\} \times \{R_1 \leq |v| \leq R_2\}) \supset \{R_1 + GT \leq |v| \leq R_2 - GT\} \times S_1(v),$$

with  $S_1(v) = \{x \mid x = \Phi_t^X(x_0, u) \text{ for some } u \text{ with } \phi_t^V(x_0, u) = v\}$  which is non-empty for any  $v$  in the annulus on the right hand side. This is since

$$\int T_t(\delta_{x_0} \otimes 1_{R_1 \leq |v| \leq R_2}) \phi(x, v) dx dv = \int \delta_{x_0} 1_{R_1 \leq |v| \leq R_2} \phi(T_{-t}(x, v)) dx dv,$$

and  $T_{-t}$  moves the  $v$  variable by a distance of at most  $GT$  since  $\Phi_t^V(x, v) = v - t \nabla_x W(\Phi_s^X(x, v))$  for some  $s \in [0, t]$ .

Now in a similar way we have that  $\Phi_t^X(x, v) = x + vt + \frac{1}{2}t^2 \nabla_x W(\Phi_s^X(x, v))$  for some  $s \in [0, t]$  this means we can approximate  $T_t$  by free transport for  $t$  small enough. Using this we want to show that if we take  $R$  large enough then for  $t \in [1/2, 1]$  we have

$$T_t(\{x_0\} \times \{R_1 \leq |v| \leq R_2\}) \supset \{2R_1 + G \leq |x_0 - x| \leq R_2/2 - G\} \times S_2(v),$$

with  $S_2(v) = \{v \mid v = \Phi_t^V(x_0, u) \text{ for some } u \text{ with } \Phi_t^V(x_0, u) = v\}$  which is non-empty for any  $v$  in the annulus on RHS. Therefore if  $t \in [1/2, T + 1]$  then we have

$$\int T_t(\delta_{x_0} 1_{R_1 \leq |v| \leq R_2}) dv \geq \alpha 1_{2R_1 + 2GT + G \leq |x_0 - x| \leq R_2/2 - GT/2 - G}.$$

Here  $\alpha$  is  $\min_{x,v} |\text{Jac}(T_{-t}(x, v))| \geq \exp(-t(1 + \|\text{Hess}(W)\|_\infty))$  where the last inequality follows from the lioville formula. So provided we choose  $R_1, R_2$  such that

$$R_2/2 - 2R_1 - 5GT/2 - 2G \geq 1,$$

then this annulus will contain at least one square with integral vertices. Hence, we have

$$\int T_t(\delta_{x_0} 1_{R_1 \leq |v| \leq R_2}) dv \geq \alpha.$$

Now since  $e^{-W(x)}/Z$  is bounded above we have some  $\beta = Z\alpha$  such that

$$\int T_t(\delta_{x_0} 1_{R_1 \leq |v| \leq R_2}) dv \geq \beta e^{-W(x)}/Z.$$

□

#### 4. PROOF OF THEOREMS 1 AND 3 (DUHAMEL VERSION)

**4.1. Some key lemmas.** The strategy of this section is to prove the two theorems 1 and 3 in an entirely deterministic way, based on the strategy of Doeblin's theorem. The proofs of the two theorems are identical except for the crucial lemmas 1 and 3. First we will prove both these lemmas and then write the remainder of the argument in a general framework which covers both cases.

**Lemma 4.** *Assume that  $V$  satisfies Assumption 9 and  $\sigma$  satisfies the geometric control condition 2 or we are in the situation where  $V = \mathbb{R}^d$  with the Maxwellian measure and we have a confining potential  $W \neq 0$  and  $\sigma$  satisfies the GCC. Let  $\mu_t = \mu_t(dx, dv)$  be the solution to (1) with initial datum*

$$(31) \quad \mu_0 = \delta_{x_0} \otimes \delta_{v_0},$$

for  $(x_0, v_0) \in \mathbb{T}^d \times V$  given. Let  $T_*$  be as in lemma 1 or  $T_{***}$  as in lemma 3 and  $T$  given as in (3, 2). Then, for  $t = 2T + T_*$  in the case  $W = 0$  or  $T = 2T + T_{***}$  in the case  $W \neq 0$  we have

$$(32) \quad \mu_t(dx, dv) \geq \beta \kappa^2 e^{-t\|\sigma\|_\infty} \nu \quad \text{in } \mathcal{M}(\Omega \times V).$$

*Proof.* Using Duhamel's formula (23) we have that, for every  $t \geq 0$ ,

$$(33) \quad \begin{aligned} \mu_t(dx, dv) &= \exp\left(-\int_0^t \sigma(\Phi_s^X(x, v)) ds\right) (T_t \mu_0)(dx, dv) + S_t[\mu_t](dx, dv) \\ &\geq \exp\left(-\int_0^t \sigma(\Phi_s^X(x, v)) ds\right) (T_t \mu_0)(dx, dv) \\ &\geq e^{-t\|\sigma\|_\infty} (T_t \mu_0)(dx, dv), \end{aligned}$$

as, according to (24),

$$S_t[\mu_t](dx, dv) \geq 0 \quad \text{in } \mathcal{M}(\Omega \times V).$$

Injecting (33) in (23) we get

$$\begin{aligned} \mu_t(dx, dv) &\geq \int_0^t \exp\left(-\int_s^t \sigma(\Phi_\tau^X(x, v)) d\tau\right) (T_{t-s} m_\sigma L^+ \mu_s)(dx, dv) ds \\ &\geq \int_0^t e^{-(t-s)\|\sigma\|_\infty} (T_{t-s} m_\sigma L^+ \mu_s)(dx, dv) ds \\ &\geq e^{-t\|\sigma\|_\infty} \int_0^t (T_{t-s} m_\sigma L^+ T_s \mu_0)(dx, dv) ds. \end{aligned}$$

Now we can substitute this in a second time to get

$$(34) \quad \mu_t(dx, dv) \geq e^{-t\|\sigma\|_\infty} \int_0^t \int_0^s (T_{t-s} m_\sigma L^+ T_{s-\tau} m_\sigma L^+ T_\tau \mu_0)(dx, dv) d\tau ds.$$

Now using (31) we may write

$$\begin{aligned} T_{s-\tau} m_\sigma L^+ T_\tau \mu_0 &= T_{s-\tau} m_\sigma L^+ (\delta_{\Phi_\tau^X(x_0, v_0)} \otimes \delta_{\Phi_\tau^Y(x_0, v_0)}) \\ &= T_{s-\tau} m_\sigma (\nu_v(dv) \delta_{\Phi_\tau^X(x_0, v_0)}(dx)) \\ &= T_{s-\tau} (\sigma(x) \delta_{\Phi_\tau^X(x_0, v_0)}(dx) \nu_v(dv)) \\ &= \sigma(\Phi_\tau^X(x_0, v_0)) T_{s-\tau} (\delta_{\Phi_\tau^X(x_0, v_0)}(dx) \nu_v(dv)). \end{aligned}$$

Now assuming that  $s - \tau \geq T_*$ , the definition of  $T_*$  in Assumption 9 implies

$$L^+ T_{s-\tau} m_\sigma L^+ T_\tau \mu_0 = \nu_v \sigma(\Phi_\tau^X(x_0, v_0)) \int_V T_{s-\tau} (\delta_{\Phi_\tau^X(x_0, v_0)} \nu_v) dv \geq \beta \sigma(\Phi_\tau^X(x_0, v_0)) \nu.$$

Therefore

$$T_{t-s} m_\sigma L^+ T_{s-\tau} m_\sigma L^+ T_\tau f_0 = \beta \sigma(\Phi_\tau^X(x_0, v_0)) \sigma(\Phi_{t-s}^X(x, v)) \nu.$$

Now, taking  $t = 2T + T_*$  as in the statement and integrating (34) with respect to  $\tau \in [0, T]$ ,  $s \in [T + T_*, 2T + T_*]$  we get

$$\begin{aligned} \mu_t(dx, dv) &\geq e^{-(2T+T_*)\|\sigma\|_\infty} \int_{T+T_*}^{2T+T_*} \int_0^T \sigma(\Phi_{t-s}^X(x, v)) \sigma(\Phi_\tau^X(x_0, v_0)) \beta \nu d\tau ds \\ &\geq \beta \kappa^2 e^{-(2T+T_*)\|\sigma\|_\infty} \nu, \end{aligned}$$

whence (32) follows.  $\square$

The next result is an extension of Lemma 4, valid for Dirac masses, to any initial data that is a probability measure.

**Lemma 5.** *Under the same hypothesis of Lemma 4, let  $\mu_0 \in \mathcal{P}(\Omega \times V)$  and let  $\mu_t$  be the associated solution to (22). Then, for  $t = 2T + T_*, 2T + T_{***}$  we have*

$$(35) \quad \mu_t(dx, dv) \geq \beta\kappa^2 e^{-t\|\sigma\|_\infty} \nu \quad \text{in } \mathcal{M}(\Omega \times V).$$

*Proof.* Let  $\mu_0 \in \mathcal{P}(\Omega \times V)$  and let  $\mu_t$  be given as in the statement. According to (25), we can write  $\mu_t = \mathcal{P}_t \mu_0$ . We claim that it suffices to prove that

$$(36) \quad \mu_t = \iint_{\Omega \times V} (\mathcal{P}_t \delta_{x_0, v_0}) \mu_0(dx_0, dv_0).$$

If (36) holds, Lemma 4 implies

$$\mathcal{P}_t \mu = \iint_{\Omega \times V} (\mathcal{P}_t \delta_{x_0, v_0}) \mu_0(dx_0, dv_0) \geq \beta\kappa^2 e^{-t\|\sigma\|_\infty} \iint_{\Omega \times V} \nu \mu_0(dx_0, dv_0) = \beta\kappa^2 e^{-t\|\sigma\|_\infty} \nu.$$

In order to prove (36), we observe that it is sufficient to check that

$$\nu_t := \int (\mathcal{P}_t \delta_{x_0, v_0}) \mu_0(dx_0, dv_0)$$

is indeed a measure-valued solution to (22) with initial datum  $\mu_0$ , as uniqueness of solutions (Proposition 1) would imply  $\nu_t = \mu_t$  and a fortiori (36).

According to Definition 5, let  $\phi \in C_c^1((0, T] \times \Omega \times V)$ . As  $\phi$  and  $\nabla_{t,x} \phi$  are bounded and compactly supported, then

$$P\phi = \partial_t \phi + v \cdot \nabla_x \phi - \sigma(\bar{\phi} - \phi) \in C_c^1((0, T] \times \Omega \times V).$$

Then, using Fubini's theorem,

$$\begin{aligned} & \int_0^T \iint_{\mathbb{T}^d \times V} (\partial_t \phi + v \cdot \nabla_x \phi - \sigma(\bar{\phi} - \phi)) \nu_t(dx, dv) \\ &= \int_0^T \iint_{\mathbb{T}^d \times V} P\phi \left( \iint_{\mathbb{T}^d \times V} \mathcal{P}_t \delta_{x_0, v_0} \mu_0(dx_0, dv_0) \right) (dx, dv) \\ &= \iint_{\mathbb{T}^d \times V} \left( \int_0^T \iint_{\mathbb{T}^d \times V} P\phi(\mathcal{P}_t \delta_{x_0, v_0})(dx, dv) \right) \mu_0(dx_0, dv_0) \\ &= - \iint_{\mathbb{T}^d \times V} \phi(0, x_0, v_0) \mu(dx_0, dv_0). \end{aligned}$$

This ends the proof.  $\square$

**4.2. Doeblin type argument and exponential decay.** Now we want to make a Doeblin type argument.

*Proof of Theorem 1.* Let  $t_* = 2T + T_*$  in the case  $W = 0$  and  $V$  is compact, or  $t = 2T + T_{***}$  in the case  $W \neq 0$  as in Lemma 5 and set.

$$\alpha := \beta\kappa^2 e^{-t_*\|\sigma\|_\infty}.$$

*Step 1: Estimate for positive disjoint probability measures.* Assume that are such that

$$(37) \quad \mu_1, \mu_2 \in \mathcal{P}(\Omega \times V) \quad \text{supp } \mu_1 \cap \text{supp } \mu_2 = \emptyset.$$

This implies that

$$(38) \quad \|\mu_1 - \mu_2\|_{TV} = 2.$$

Using the conservation of mass and Lemma 5, we can write

$$\mathcal{P}_{t^*}\mu_1 - \alpha\nu = (1 - \alpha)f_1, \quad \mathcal{P}_t\mu_2 - \alpha\nu = (1 - \alpha)f_2,$$

for some  $f_1, f_2 \in \mathcal{P}(\Omega \times V)$ . Hence,

$$\begin{aligned} \|\mathcal{P}_{t^*}\mu_1 - \mathcal{P}_t\mu_2\|_{TV} &\leq \|\mathcal{P}_{t^*}\mu_1 - \alpha\nu\|_{TV} + \|\mathcal{P}_t\mu_2 - \alpha\nu\|_{TV} \\ &\leq (1 - \alpha)\|f_1\|_{TV} + (1 - \alpha)\|f_2\|_{TV} \\ &\leq 2(1 - \alpha) \\ &= (1 - \alpha)\|\mu_1 - \mu_2\|_{TV}, \end{aligned}$$

as a consequence of (38). Iterating this estimate and using that  $(\mathcal{P}_t)_{t \geq 0}$  is a semi-group, we obtain

$$(39) \quad \forall \mu_1, \mu_2 \text{ satisfying (37)} \quad \forall k \in \mathbb{N}, \quad \|\mathcal{P}_{kt^*}\mu_1 - \mathcal{P}_{kt^*}\mu_2\|_{TV} \leq (1 - \alpha)^k \|\mu_1 - \mu_2\|_{TV}.$$

*Step 2: Estimate for positive measures with the same mass.* Assume now that  $\mu_1, \mu_2 \in \mathcal{M}(\Omega \times V)$  are such that

$$(40) \quad \text{supp } \mu_1 \cap \text{supp } \mu_2 = \emptyset \quad \text{and} \quad \mu_1(\Omega \times V) = \mu_2(\Omega \times V) > 0.$$

Then, setting

$$\overline{\mu}_1 := \frac{\mu_1}{\mu_1(\Omega \times V)}, \quad \overline{\mu}_2 := \frac{\mu_2}{\mu_2(\Omega \times V)},$$

we readily have that

$$\text{supp } \overline{\mu}_1 \cap \text{supp } \overline{\mu}_2 = \emptyset \quad \text{and} \quad \overline{\mu}_1, \overline{\mu}_2 \in \mathcal{P}(\Omega \times V).$$

Hence, using (39),

$$\begin{aligned} \|\mathcal{P}_{kt^*}\mu_1 - \mathcal{P}_{kt^*}\mu_2\|_{TV} &= \left\| \frac{1}{\mu_1(\Omega \times V)} \mathcal{P}_{kt^*}\overline{\mu}_1 - \frac{1}{\mu_2(\Omega \times V)} \mathcal{P}_{kt^*}\overline{\mu}_2 \right\|_{TV} \\ &= \frac{1}{\mu_1(\Omega \times V)} \|\mathcal{P}_{kt^*}\overline{\mu}_1 - \mathcal{P}_{kt^*}\overline{\mu}_2\|_{TV} \\ &\leq \frac{(1 - \alpha)^k}{\mu_1(\Omega \times V)} \|\overline{\mu}_1 - \overline{\mu}_2\|_{TV}, \end{aligned}$$

for any  $k \in \mathbb{N}$ . Hence,

$$(41) \quad \forall \mu_1, \mu_2 \text{ satisfying (40)} \quad \forall k \in \mathbb{N}, \quad \|\mathcal{P}_{kt^*}\mu_1 - \mathcal{P}_{kt^*}\mu_2\|_{TV} \leq (1 - \alpha)^k \|\mu_1 - \mu_2\|_{TV}.$$

*Step 3: Estimate for general measures probability measures.* Consider  $\mu_1, \mu_2 \in \mathcal{P}(\Omega \times V)$ . Using the Jordan's decomposition (cf. [8, Eq. (32.3), p. 421]), we can write

$$\mu_1 - \mu_2 = (\mu_1 - \mu_2)_+ - (\mu_2 - \mu_1)_+,$$

which satisfy

$$\text{supp}(\mu_1 - \mu_2)_+ \cap \text{supp}(\mu_2 - \mu_1)_+ = \emptyset \quad \text{and} \quad (\mu_1 - \mu_2)_+(\Omega \times V) = (\mu_2 - \mu_1)_+(\Omega \times V),$$

for  $(\mu_1 - \mu_2)(\Omega \times V) = 0$ . As a consequence, we can use (41) and this gives

$$(42) \quad \forall \mu_1, \mu_2 \in \mathcal{P}(\Omega \times V), \quad \forall k \in \mathbb{N}, \quad \|\mathcal{P}_{kt^*}\mu_1 - \mathcal{P}_{kt^*}\mu_2\|_{TV} \leq (1 - \alpha)^k \|\mu_1 - \mu_2\|_{TV}.$$

*Step 4: Conclusion and quantitative exponential bound.*



We observe that the equilibrium distribution satisfies

$$(43) \quad \mathcal{P}_t \nu = \nu, \quad \forall t \geq 0 \quad \text{and} \quad \nu \in \mathcal{P}(\Omega \times V).$$

Let  $t > t_*$  and set  $k \in \mathbb{N}$  be such that

$$\frac{t}{t_*} \leq k + 1.$$

Then, using (43), (42) and (27),

$$\begin{aligned} \|\mathcal{P}_t \mu_0 - \nu\|_{TV} &= \|\mathcal{P}_t \mu_0 - \mathcal{P}_t \nu\|_{TV} \\ &\leq \|\mathcal{P}_{kt_*} \mu_0 - \mathcal{P}_{kt_*} \nu\|_{TV} \\ &\leq (1 - \alpha)^k \|\mu_0 - \nu\|_{TV} \\ &\leq \exp\left(\frac{t - t_*}{t_*} \log(1 - \alpha)\right) \|\mu_0 - \nu\|_{TV}. \end{aligned}$$

where we have used that, thanks to the choice of  $k$ ,

$$(k + 1) \log(1 - \alpha) \leq \frac{t}{t_*} \log(1 - \alpha).$$

This gives (10) with the rate (11). □

## 5. PROOF OF THEOREM 1 (STOCHASTIC VERSION)

We now look at a more stochastic proof of Theorem 1 as similar proof would also work in the case of Theorem 3. We split this into two Lemmas. Here we only prove the lower bound on  $\mathcal{P}_t \mu$  as afterwards we would argue exactly as in the previous section. We want to look at paths where the velocity process jumps exactly once in  $[0, T]$ , no times in  $[T, T + T_*]$  and exactly once in  $[T + T_*, 2T + T_*]$ . Here  $T$  is from the GCC and  $T_*$  is from above.

**Lemma 6.** *Given an arbitrary set  $S$  we want to have for the following probability conditional on starting at  $(x_0, v_0)$  for our process (28)*

$$\begin{aligned} \mathbb{P}_{x_0, v_0}((X_t, V_t) \in S \mid \text{jump exactly once in } [0, T], \text{ no times in } [T, T + T_*], \\ \text{and exactly once in } [T + T_*, t]) \geq \beta \nu(S). \end{aligned}$$

Here  $\beta$  is as before and  $\nu$  is the steady state which in this case is the uniform measure on  $\mathbb{T}^d \times V$ .

**Remark.** *This is essentially the same as computing a lower bound on the integrand in the Duhamel's formula in the previous section. We repeat the computations briefly since it is technically a different quantity.*

*Proof.* Suppose the process jumps once at time  $r$  in  $[0, T]$  and again at time  $s$  in  $[T + T_*, t]$ . and we start with law  $\delta_{x_0} \otimes \delta_{v_0}$ . Immediately before the first jump we have only undergone pure transport so the law is

$$T_r(\delta_{x_0} \otimes \delta_{v_0}) = \delta_{x_0 + v_0 r} \otimes \delta_{v_0}.$$

Now after the first jump we thermalise the velocity so they become uniform over  $V$ . The operator  $L^+$  acts on measures by

$$(L^+ \mu)(A_x \times A_v) = \mu(A_x \times V) \lambda(A_v),$$

where  $\lambda$  is the uniform measure on  $V$ . Then immediately after the first jump we have the law being

$$L^+(\delta_{x_0+v_0r} \otimes \delta_{v_0}) = \delta_{x_0+v_0r} \otimes \lambda.$$

Now from our conditions on  $s$  and  $r$  we know that  $s - r > T_*$ . So this means that immediately after the second jump we have from our earlier calculation

$$L^+T_{s-r}(\delta_{x_0+v_0r} \otimes \lambda) = \frac{1}{|V|} \int_V T_{s-r} \left( \delta_{x_0+v_0r} \frac{1}{|V|} \right) dv \geq \frac{\beta}{|V|}.$$

The transport semigroup preserves the uniform measure. So we have shown that after a jump at exactly on point in  $[0, T]$ , exactly one point in  $[T + T_*, t]$  and no other jumps then we are uniformly distributed. Therefore,

$$\begin{aligned} \mathbb{P}_{x_0, v_0}((X_t, V_t) \in S | \text{jump exactly once in } [0, T], \text{ no times in } [T, T + T_*], \\ \text{and exactly once in } [T + T_*, t]) \geq \beta \nu(S). \end{aligned}$$

□

So now we need to calculate the probability of jumping exactly once in  $[0, T]$  no times in  $[T, T + T_*]$  and exactly once in  $[T + T_*, t]$ .

**Lemma 7.** *For our process (28) we have for any  $(x_0, v_0)$ .*

$$\begin{aligned} \mathbb{P}_{x_0, v_0}(\text{exactly one jump in } [0, T], \text{ none in } [T, T + T_*] \\ \text{and once in } [T + T_*, 2T + T_*]) \geq C^2 e^{-\|\sigma\|_\infty(2T + T_*)}. \end{aligned}$$

*Proof.* First lets look at the probability of jumping exactly once in  $[T + T_*, t]$  conditional on having jumped exactly once in  $[0, T]$  and zero times in  $[T, T + T_*]$ . By our earlier calculation we can see that after the process has jumped once in  $[0, T]$  and then transported to time  $T + T_*$  then the  $x$  marginal of the law will be uniform. That is to say we want to study the probability

$$\mathbb{P}_{x_0, v_0}(\text{jump exactly once in } [T + T_*, t] | \text{jumped exactly once in } [0, T], \text{ never in } [T, T + T_*]).$$

Using the GCC and the fact that the number of jumps in time  $T$  is a Poisson random variable with mean  $\int_0^T \sigma(X_t(x, v)) dt$  we have

$$\begin{aligned} \mathbb{P}_{x_*, v_*}(\text{jump exactly once in time } [0, T]) &= \int_0^T \exp\left(-\int_0^T \sigma(x_* + v_*s) ds\right) \sigma(x_* + v_*t) dt \\ &\geq e^{-\|\sigma\|_\infty T} \int_0^T \sigma(x_* + v_*t) dt \geq e^{-\|\sigma\|_\infty T} C. \end{aligned}$$

This is uniform over all starting points so for any distribution  $\mu$

$$\mathbb{P}(\text{jump exactly once in } [T + T_*, 2T + T_*] | (X_{T+T_*}, V_{T+T_*}) \sim \mu) \geq C e^{-\|\sigma\|_\infty T}.$$

So if we set  $t = 2T + T_*$  then we have

$$\begin{aligned} \mathbb{P}_{x_0, v_0}(\text{jump exactly once in } [T + T_*, t] | \text{jumped exactly once in } [0, T], \\ \text{never in } [T, T + T_*]) \geq C e^{-\|\sigma\|_\infty T}. \end{aligned}$$

Similarly, we can look at

$$\mathbb{P}_{x_0, v_0}(\text{No jumps in } [T, T + T_*] | \text{exactly one jump in } [0, T]).$$

Like before we have that

$$\mathbb{P}_{x_*, v_*}(\text{jump no times in } [0, T_*]) = \exp\left(-\int_0^{T_*} \sigma(x_* - v_* t) dt\right) \geq e^{-\|\sigma\|_\infty T_*}.$$

Again this is uniform over starting points so we have that

$$\mathbb{P}(\text{jump no times in } [T, T + T_*] | (X_T, V_T) \sim \mu) \geq e^{-\|\sigma\|_\infty T_*}.$$

Furthermore we have

$$\mathbb{P}_{x_0, v_0}(\text{exactly one jump in } [0, T]) \geq C e^{-\|\sigma\|_\infty T}.$$

Therefore, putting all these together we have

$$\begin{aligned} \mathbb{P}_{x_0, v_0}(\text{exactly one jump in } [0, T], \text{ none in } [T, T + T_*], \\ \text{and one in } [T + T_*, 2T + T_*]) \geq C^2 e^{-\|\sigma\|_\infty (2T + T_*)}. \end{aligned}$$

□

Finally we conclude by Bayes's Theorem.

**Lemma 8.** *For the stochastic process (28) we have that if  $t = 2T + T_*$  then*

$$\mathbb{P}_{x_0, v_0}((X_t, V_t) \in S) \geq \beta C^2 e^{-\|\sigma\|_\infty t} \nu(S),$$

*Proof.* We want to use the fact that

$$\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c) \geq \mathbb{P}(A|B)\mathbb{P}(B).$$

We apply this with  $A = \{(X_t, V_t) \in S\}$  and  $B$  being the event of jumping exactly once in  $[0, T]$ , no times in  $[T, T + T_*]$  and exactly once in  $[T + T_*, 2T + T_*]$ . □

**Remark.** *This gives a more stochastic flavored proof of Lemma 4. We can then pick up at that point and conclude the same way as the previous section.*

## 6. PROOF IN THE CASE THAT $V = \mathbb{S}^{d-1}$ .

Here we emulate the stochastic proof as in the previous section since it is better adapted when we have more jumps. First we need to prove our control on the transport semigroup.

*Proof of Theorem 2.* Lets fix  $t = 3T + 2T_{**}$ . We emulate the probabilistic proof we want to look at paths with one jump in  $[0, T]$  another jump in  $[T + T_{**}, 2T + T_{**}]$  another in  $[2T + 2T_{**}, 3T + 2T_{**}]$  and no other jumps. The probability of such a sequence is given by the same calculations as in lemma 7

$$C^3 e^{-\|\sigma\|_\infty (3T + 2T_{**})}.$$

Lets call  $J_1$  the event that the process jumps exactly once in  $[0, T]$ ,  $J_2$  the event that the process jumps no times in  $[T, T + T_{**}]$ ,  $J_3$  the even the process jumps exactly once in  $[T + T_{**}, 2T + T_{**}]$ ,  $J_4$  the even that the process jumps no times in  $[2T + T_{**}, 2T + 2T_{**}]$  and finally  $J_4$  the event that the process jumps exactly once in  $[2T + 2T_{**}, 3T + 2T_{**}]$ . Then by our calculations in Lemma 7 we know that

$$\mathbb{P}(\text{exactly one jump in } [0, T] | (X_0, V_0) \tilde{\mu}) \geq C e^{-\|\sigma\|_\infty T},$$

for every probability  $\mu$ . Furthermore by similar calculations to in Lemma 7 we have

$$\mathbb{P}(\text{no jumps in } [0, T_{**}] | (X_0, V_0) \tilde{\mu}) \geq e^{-\|\sigma\|_\infty T_{**}},$$

for every probability  $\mu$ . Consequently we have

$$\begin{aligned}
\mathbb{P}_{x_0, v_0}(J_1, J_2, J_3, J_4) &= \mathbb{P}_{x_0, v_0}(J_2, J_3, J_4 | J_1) \mathbb{P}_{x_0, v_0}(J_1) \\
&= \mathbb{P}_{x_0, v_0}(J_3, J_4 | J_2, J_1) \mathbb{P}_{x_0, v_0}(J_2 | J_1) \\
&= \mathbb{P}_{x_0, v_0}(J_4 | J_3, J_2, J_1) \mathbb{P}_{x_0, v_0}(J_3 | J_2, J_1) \mathbb{P}_{x_0, v_0}(J_2 | J_1) \mathbb{P}_{x_0, v_0}(J_1) \\
&= \mathbb{P}_{x_0, v_0}(J_4 | J_3) \mathbb{P}_{x_0, v_0}(J_3 | J_2) \mathbb{P}_{x_0, v_0}(J_2 | J_1) \mathbb{P}_{x_0, v_0}(J_1) \\
&\geq C^3 e^{-\|\sigma\|_\infty (3T + 2T_{**})}.
\end{aligned}$$

In this calculations we can replace conditioning by  $J_2, J_1$  by just conditioning by  $J_2$  by the Markov property and similarly in the later lines only conditioning by the most recent event.

Lets write  $J = \{J_1, J_2, J_3, J_4\}$  then we want to show that

$$\mathbb{P}_{x_0, v_0}((X_t, V_t) \in S | J) \geq \frac{1}{2} \nu(S).$$

Lets look at the law conditional on jumping at times  $q, r, s$  lying in the right sets. Immediately before the jump at time  $q$  the law will be

$$T_q(\delta_{x_0} \delta_{v_0}) = \delta_{x_0 + qv_0} \delta_{v_0}.$$

Then after the first jump the law becomes

$$\delta_{x_0 + qv_0} 1_{v \in \mathbb{S}^{d-1}}.$$

We procede like this to see that immediately before the third jump we have

$$T_{s-r} \left( \int_{\mathbb{S}^{d-1}} T_{r-q}(\delta_{x_0 + qv_0} 1_{v \in \mathbb{S}^{d-1}})(\cdot, u) du 1_{v \in \mathbb{S}^{d-1}} \right).$$

Since  $s - r, r - q \geq T_{**}$  we have that the  $x$  marginal of this is greater than

$$\frac{1}{2} \frac{1}{|\mathbb{T}^d|} = \frac{1}{2} \nu^x.$$

Now immediately after the third jump we have that the measure is bounded bellow by

$$\frac{1}{2} \nu.$$

Then since the transport part does not change the uniform measure this shows that

$$\mathbb{P}_{x_0, v_0}((X_t, V_t) \in S | \text{jumps at times } q, r, s) \geq \frac{1}{2} \nu(S).$$

Since this result is uniform over the even  $J$  we have that

$$\mathbb{P}_{x_0, v_0}((X_t, V_t) \in S | J) \geq \frac{1}{2} \nu(S).$$

Now putting this together gives that

$$\mathbb{P}_{x_0, v_0}((X_t, V_t) \in S) \geq \frac{1}{2} C^3 e^{-\|\sigma\|_\infty t} \nu(S).$$

We can now repeat the arguments for the proofs of Theorems 1 and 3 starting from the end of Lemma 4  $\square$

## 7. COMMENTS ON THE RATES

Lastly we comment on the rates we get. For the main model our rate is

$$\lambda = -\frac{\log(1 - C^2 e^{-\|\sigma\|_\infty(2T+T_*)/2})}{2T + T_*}.$$

This is almost definitely not optimal. To the best of our knowledge the rate should vary quite strongly depending on the geometry. We can give a little bit of information about a bound on the spectral gap and examples of situations where the spectral gap is well below this bound. In [21] the authors prove some results on the spectrum of this operator. Defining the constants

$$C_\infty^- = \sup_{T>0} \inf_{x,v} \frac{1}{T} \int_0^T \sigma(\Phi_t^X(x,v)) dt, \quad C_\infty^+ = \inf_{T>0} \sup_{x,v} \int_0^T \sigma(\Phi_t^X(x,v)) dt,$$

it is proven in [21] that the essential spectrum of the linear Boltzmann operator lies in the strip  $\{z : C_\infty^- \leq \operatorname{Re}(z) \leq C_\infty^+\}$ . They also show that the spectrum is contained in a strip of the form  $\{0 \leq \operatorname{Re}(z) \leq L_\infty\}$ , where  $L_\infty$  is related to the supremum of the collision kernel. We can give an upper bound on the spectral gap in total variation using a simple probabilistic argument.

**Lemma 9.** *If there exists  $\lambda > 0, A > 0$  such that for all initial data*

$$\|f(t) - \nu\|_{TV} \leq Ae^{-\lambda t},$$

*then  $\lambda \leq C_\infty^+$  using the notation above.*

*Proof.* If we initially start with a delta function we get no closer in total variation until we have jumped at least once, then we have that

$$\|f(t) - \nu\|_{TV} \geq \mathbb{P}(\text{jumped no times in time } t) = \exp\left(-\int_0^t \sigma(\Phi_s^X(x,v)) ds\right).$$

Fixing  $\epsilon$  there exists  $T(\epsilon)$  such that

$$\sup_{x,v} \int_0^{T(\epsilon)} \sigma(\Phi_s^X(x,v)) ds \leq (C_\infty^+ + \epsilon)T(\epsilon).$$

Therefore,

$$\|f(nT(\epsilon)) - \nu\|_{TV} \geq \exp\left(-\int_0^{nT(\epsilon)} \sigma(\Phi_s^X(x,v)) ds\right) \geq \exp(-nT(\epsilon)(C_\infty^+ + \epsilon)),$$

for every  $n$ . Therefore  $\lambda \leq C_\infty^+ + \epsilon$  and  $\epsilon$  is arbitrary which gives the result.  $\square$

The consideration of optimal rates raises several natural further questions. The first is to investigate the optimal rates. Secondly it would be interesting to characterize which possible choices of  $\sigma$  lead to the fastest and slowest rates. This is especially interesting since it is not obvious that having constant  $\sigma$  gives the fastest rates, particularly in the presence of a confining potential. If it is possible to choose a degenerate  $\sigma$  so that the convergence to equilibrium was much faster than the optimal choice of constant  $\sigma$  then this could have implications for Hamiltonian Markov chain Monte-Carlo simulation.

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