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On the coupling of large deformations and elastic-plasticity in the mechanics of a simple system

Alain Cimetière, Alain Léger, Elaine Pratt

Abstract The buckling and the post buckling of a simple model involving both large deformations and elastic-plasticity are studied. Plasticity is described by the classical incremental constitutive law involving in particular non regularized transition between plastic loading and elastic unloading. The kinematics is chosen in order to take into account any rotation, however large, without having to resort to approximations or asymptotic expansions. The work focusses essentially on the specificity of the coupling between these two types of nonlinearity. At small strains the global behavior is correctly given by Hutchinson's simple model involving the same elastic-plastic constitutive law and linearized deformations. But at very large strains, the response notably differs from that of Hutchinson's model. Indeed plasticity seems to have no effect and the behavior is that of a system involving the same nonlinear deformations but with a constant modulus. From the point of view of bifurcation analysis this result is qualitatively interesting due to the fact that the elastic-plastic constitutive law can be regarded as a strong, or non-smooth, nonlinearity, whereas large deformations is a smooth, we could say weaker, non linearity, so that the result contradicts the intuition that the strongest non linearity should be associated with the strongest bifurcation effect. The analysis is carried out using two different geometrical nonlinearities.

Keywords critical load, incremental plasticity, large deformations, nonsmooth non linearity, plastic buckling, simple model, tangent modulus.

1 Introduction

It is well known in mechanics of structures that the buckling phenomenon results in particular from the nonlinearities that are involved when taking into account the kinematics at large strains. In the case of elastic structures the bifurcation diagram has been widely studied, it consists in general of the discrete spectrum of an operator obtained by the linearization of the kinematics on a trivial branch, together with the bifurcated branches at large, or very occasionally at very large deformations.

Alain Cimetière
Institut Pprime, Physique et Mécanique des Matériaux, ISAE-ENSMA et Université de Poitiers, 86360 Chasseneuil du Poitou, France, E-mail: alain.cimetiere@univ-poitiers.fr
Alain Léger
Institut de Mécanique et d'Ingénierie (I2M), Université Bordeaux 1, 33405 Talence cedex, France, E-mail: alain.leger.1@u-bordeaux.fr
Elaine Pratt
Laboratoire de Mécanique et d'Acoustique at Marseille, AMU - CNRS - Centrale Marseille 4 impasse Nikola Tesla, CS 40006, 13453 Marseille Cedex 13, France, E-mail: pratt@lma.cnrs-mrs.fr

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In the present paper we are interested in the elastic-plastic buckling of a class of mechanical systems, with the aim of not only determining the set of bifurcation points but also understanding their asymptotic behavior at very large post-buckling deformations, which, as far as the authors are aware, had never been studied. Simple mechanical models, made of springs and bars, are built for that purpose. The bars are rigid, possibly rigidly connected together or articulated, the springs are elastic-plastic.

For each mechanical model we first give the solution of the elastic buckling: plasticity is removed and we recall the effects of the nonlinear strains within linear elasticity. The systems are investigated within linearized strains in a first case then within two different models of geometrical non linearity allowing very large deformations. These elastic buckling problems are elementary exercises, but they will be useful for the asymptotic analysis. The same examples with the same kinematic choices are then carefully calculated within elastic-plasticity.

The essential results expected from this study concern the coupling between the two types of non linearity involved, one due to the kinematics, which is described by smooth functions, and the other arising from the constitutive law, which is nonsmooth and described by graphs or inequalities.
All the systems at hand possess a trivial solution for any positive real value of the parameter which is the external loading, the expected behavior then deals with bifurcated branches. The qualitative results are such that close to the trivial solution, that means within the so-called incipient buckling or close post-buckling, the set of solutions results exclusively from plasticity, and in particular seems to be unchanged by the different cases of kinematics. But following the bifurcated branches involves a boundary between plastic loading and elastic unloading, and the evolution of this free boundary may be rather complicated even though the discrete system is very simple.

Let's now outline the main steps of the paper.

- After this short introduction, the first part of this paper deals with the basic statements of elastic-plastic buckling and post buckling which are concerned by the present work, and recalls the complete analysis of the case where the elastic-plastic constitutive law is coupled with geometrically linear kinematics.
- The next part is completely new: studying the same type of mechanical device but just changing the mechanical links, the elastic-plastic constitutive law is now coupled with the geometrically non linear kinematics which had been introduced in [2].
- The third part aims at the same kind of analysis as the previous one but using another example of mechanical link leading to another type of geometrical non linearity. This is done in order to be closer to the kind of behavior that can be observed in the investigation of real mechanical structures.
- The last section is a conclusion: at first it gathers and comments the results that arise from the three previous parts. Here the main conclusion consists in the discussion of the qualitative behavior either in the range of incipient buckling and close post buckling or in the range of far post buckling. Then remaining problems are suggested from the point of view of the role of the chosen geometrical non linearities and from the point of view of the extensions to continuous structures.

The three main parts contain several repetitions, in particular when presenting the constitutive law, the functions which appear in the equilibrium equations, etc... This just aims at facilitating the use of the results since the different parts could be considered independently.

2 Motivation

2.1 Some reminders

From a historical point of view, while the buckling load of an elastic structure had been given in some generic cases since the eighteenth century, the first correct result supported by a correct justification for elastic-plastic buckling dates only from the middle of the twentieth century and states that the first buckling load is given by the so-called critical load of the tangent modulus and occurs always under an increasing load, which ruled out the

possibility of calculations by the perturbation method called adjacent equilibrium [13]. In the case of continuous media, this qualitative difference with respect to elastic buckling was stated in a complete general way ten years later in [7], but the second qualitative difference with respect to elastic buckling had been given only forty years later when it was established that the buckling diagram of an elastic-plastic structure generically involves not only a critical bifurcation load but a whole interval completely filled with incipient bifurcation directions [1].

In the present work attention is restricted to a two-degree of freedom system. The simple model which is tackled all along this work, referred to as the Shanley-Hutchinson model, had been suggested by J. Hutchinson [8], and aimed at providing an analysis of the post buckling by the study of the extension of an unloading zone inside the structure. This system had been extended in [14] by the addition of a supplementary degree of freedom in order to take into account general pre-bifurcation conditions, leading to the so-called tangent bifurcations. First obtained in the case of simple models, the results of the two or three degree of freedom systems were generalized to any structure [9], although lots of open problems remain for this generalization.

2.2 Shanley-Hutchinson

Shanley-Hutchinson's system is made of two rigid bars, rigidly connected to each other, making an inverted T shape as represented on Figure 1. Let u be the vertical displacement of the whole structure, and θ a possible rotation around the junction of the two bars. Without external force the reference configuration of the system is such that one bar is horizontal and connected to a rigid horizontal support by a continuous distribution of vertical springs, the other bar is vertical and designed to receive an external vertical load at the top. The horizontal bar is on the segment $[-1, +1]$ of the horizontal Ox axis, the vertical bar has a length L, the behavior of the springs is given by a simple model of elastic-plasticity, with Young modulus E and tangent modulus E_T. The external vertical force is denoted by λ. It is worth recalling that this structure was a continuous version of Shanley's model which involved only two springs, respectively at the ends $x = \pm 1$ of the horizontal bar, which complete bifurcation diagram had been given in [12].

The results provided in [8] confirmed that the buckled solution appears at the critical load of the tangent modulus, namely $\lambda_T = \dfrac{2E_T}{3L}$, and that a nontrivial branch actually exists given locally by a power expansion of the following form, in which τ stands for a measure of an unloading zone :

$$\lambda = \lambda_T + \mu, \text{ with } \mu = \text{Coef. } \tau^n + \text{ t.o.s.}$$

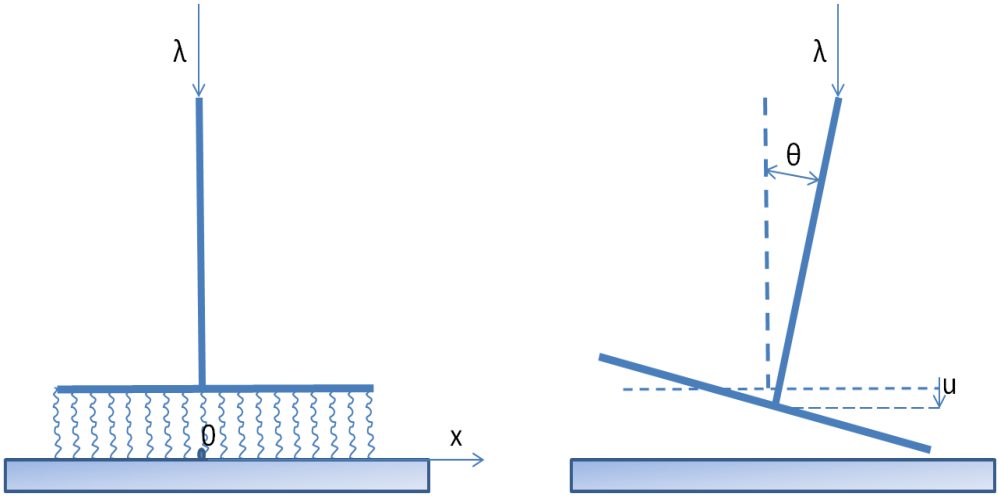
$$\lambda = \lambda_T + \lambda_1\theta + \lambda_2\theta^{1+\beta} + ...$$

$$\text{with} \quad \begin{cases} \lambda_1 > 0, \\ \lambda_2 < 0, \\ \beta \text{ rational}, \ \ 0 < \beta < 1. \end{cases} \tag{1}$$

Due to the sign of λ_2 and to the value of β, it was claimed, and presented as a general characterization of plastic bifurcations, that the nontrivial branch, which started at increasing load due to the strict positivity of λ_1, was non monotonic, with a maximal load close to the buckling point, and involved fractional exponents.

2.3 The case of linear kinematics

We now recall the analysis of the mechanical system at small strains, that is when the deformation of each spring is linearized. In fact, whether the compression of each spring and the rotation of the system is really small or not, we first look at the behavior of a model where the kinematics is linear and abstract (in fact it can be seen as a correct approximation of a physical realization of the system only at very small strains, which means here very



The reference configuration. The buckling shape.

Fig. 1 Shanley-Hutchinson's model.

small rotations), and in particular it involves $\theta \in [0, +\infty[$ and not $\theta \in [0, +\frac{\pi}{2}[$.
$\varepsilon(x)$ stands for the deformation of each spring and $\sigma(x)$ is the corresponding stress. Then the quasi-static elastic-plastic equilibrium problem is built in the following way:

1. equilibrium equations:
   - equilibrium of the resultant: $\lambda = \displaystyle\int_{-1}^{+1} \sigma(x)dx$;
   - equilibrium of the moment: $\lambda L\theta = \displaystyle\int_{-1}^{+1} x\sigma(x)dx$;
2. elastic-plastic constitutive law, referred to as incremental plasticity in the following:
   - $\dot{\sigma} = E_T\dot{\varepsilon}$ for plastic loading,
   - $\dot{\sigma} = E\dot{\varepsilon}$ for elastic loading or unloading, where the notation $(\dot{\cdot})$ results from the definition of quasi-static plasticity and denotes a derivative with respect to the load λ;
3. linearized kinematics: $\varepsilon(x, \lambda) = u(\lambda) + x\theta(\lambda)$. The associated convention is $\varepsilon(0, \lambda) > 0$ when λ is positive, the positive vertical axis for the load λ being oriented downward.

Let us first assume that buckling occurs in the elastic range, that is $\sigma = E\varepsilon$ for any x and any λ, then the result, which is here elementary, is that the complete bifurcated diagram is made of two branches

$$\begin{cases} \theta = 0, \ \lambda \in \ \mathbb{R}^+, \\ \\ \theta \in \ \mathbb{R}^+, \ \ \lambda = \dfrac{2E}{3L} := \lambda_E. \end{cases} \tag{2}$$

In order to take into account the elastic-plastic equilibrium problem we now assume that, given the constitutive parameters, that is the Young modulus E, the tangent modulus E_T and the limit of elasticity σ_y, the structure is such that the springs are all in plastic loading before buckling[1]. Then Shanley's criterion [13] establishes that, although buckling occurs under increasing load, the stress no longer increases everywhere in the structure, but stops increasing at one point. This has been stated as a necessary condition for buckling, and Hutchinson's result was precisely based upon the fact that post-buckling involves an unloading zone which spreads out inside the structure, so that any nontrivial state becomes characterized by a boundary, say $Y(\lambda)$ in the segment $[-1, +1]$, between plastic loading and elastic unloading, this boundary being, as in any free boundary problem, an unknown

[1] In the case of compressed beams, this assumption simply means that the beam is short enough not to buckle in the elastic range; in the present case it means that L is small enough.

of the problem. The very clever contribution of this model due to J. Hutchinson was that it allowed a continuous displacement of the boundary $Y(\lambda)$ while it has only two degrees of freedom, namely $u(\lambda)$ and $\theta(\lambda)$. The kinematics which is built with a rigid horizontal bar and the constitutive law given above imply the following continuity for the characterization of the boundary $Y(\lambda)$:

$$\dot{\varepsilon}(Y,\lambda) \equiv \dot{u} + Y(\lambda)\dot{\theta} = 0. \tag{3}$$

The starting point of the analysis was to consider the boundary $Y(\lambda)$ as the main unknown of the equilibrium problem: introducing the constitutive law, including Equation (3) for the boundary $Y\lambda)$, into the equations of the resultant and the moment, we obtain:

$$(\mathcal{P}) \begin{cases} i)\ \dot{\lambda} - \beta f(Y)\dot{\theta} = 0, \\ \\ ii)\ \dot{\lambda}L\theta + [L\lambda - \beta g(Y)]\dot{\theta} = 0, \end{cases} \tag{4}$$

where $f(Y)$ and $g(Y)$ are polynomials which read

$$\begin{cases} f(Y) := -\dfrac{1}{2}Y^2 - \alpha Y - \dfrac{1}{2}, \\ \\ g(Y) := -\dfrac{1}{6}Y^3 + \dfrac{1}{2}Y + \dfrac{\alpha}{3}, \end{cases} \tag{5}$$

and where $\alpha := \dfrac{E+E_T}{E-E_T}$ and $\beta := E - E_T$ are introduced for simplicity's sake.

Then, by showing that Equations (4) give, through an implicit function argument, the boundary as a function of the load λ and of the angle θ, the quasi-static equilibrium problem turned out to be an ordinary differential equation written as

$$(\mathcal{P}) : \begin{cases} \dfrac{d\lambda}{d\theta} = \beta f(Y(\theta,\lambda)), \\ \\ \lambda(0) = \lambda_0. \end{cases} \tag{6}$$

Dealing with the case of the linearized kinematics the results of this study had been given in [4] and [5], and are essentially gathered in two theorems the proof of which is not recalled here as it can be found in [4] or with more detail in [11]. The first one reads:

Theorem 1 *Problem $(\mathcal{P})$ does not have any transverse solution if $\lambda_0 \notin [\lambda_T, \lambda_E]$.*
On the other hand for each $\lambda_0 \in]\lambda_T, \lambda_E[$:
1) there exists a transverse solution, with initial unloading on $[-1, Y_0]$, where Y_0 is the unique solution of the equation $\lambda_0 = \dfrac{\beta}{L}g(Y_0)$;
2) there exists a unique function $\lambda(\theta)$, defined on $[0, +\infty[$, monotonic,
i) strictly increasing if $Y_0 \in]-1, Y_R[$, that is if $\lambda_0 \in]\lambda_T, \lambda_R[$,
ii) strictly decreasing if $Y_0 \in]Y_R, +1[$, that is if $\lambda_0 \in]\lambda_R, \lambda_E[$,
iii) constant if $Y_0 = Y_R$, that is $\lambda_0 = \lambda_R$.
iv) Moreover, for any $\lambda_0 \in [\lambda_T, \lambda_E]$ the transverse solutions satisfy the following asymptotic behavior:
$\lim\limits_{\theta\to+\infty} Y(\theta) = Y_R$,
$\lim\limits_{\theta\to+\infty} \lambda(\theta) = \lambda_R$.

The point Y_R appeared from the very beginning of the discussion of plastic buckling (see [6], [3], [15]). It is defined here as the unique zero in $[-1, 1]$ of polynomial $f(Y)$. The value λ_R is then given by Equation (4-*ii*) at $\theta = 0$ that is $\lambda_R = \dfrac{\beta}{L}g(Y_R)$ and is referred to as the critical load of the reduced modulus [10]. It can be seen as a kind of average of λ_T and λ_E, the main property of which being associated with point *iii* of Theorem 1. The second result is the following:

Theorem 2 *All branches issued from* $\lambda_0 \in]\lambda_T, \lambda_E[$ *are analytical on* $[0, +\infty[$. *Moreover, Problem* $(\mathcal{P})$ *possesses a single bifurcated branch* $\theta(\lambda)$ *verifying* $\theta(\lambda_T) = 0$. *This branch involves an initial unloading zone restricted to the single point* $Y = -1$, *and is analytical with respect to* $\sqrt{\lambda - \lambda_T}$ *for* λ *close enough to* λ_T.

The bifurcation diagram of this simple model is plotted on Figure 2.

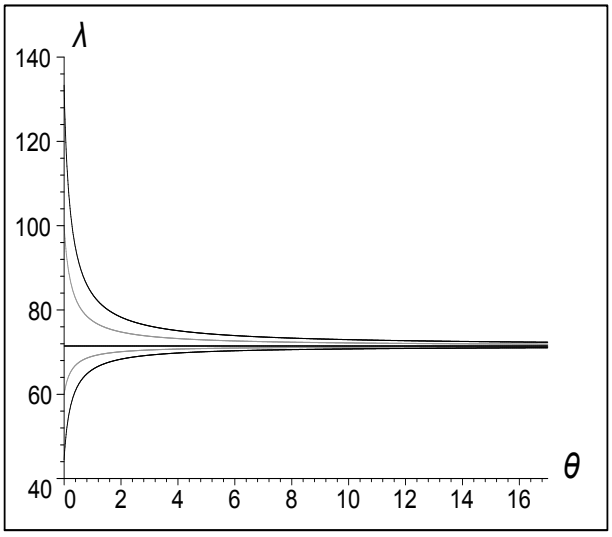


Fig. 2 The bifurcation diagram of Shanley-Hutchinson's model.

2.4 Conclusion: observing the coupling of incremental plasticity and linearized kinematics.

1. The fractional power expansion, obtained in [8] and recalled in Equation (1), does not concern the whole continuous set of bifurcated branches issued from any real number λ in $[\lambda_T, \lambda_E]$ at $\theta = 0$, but only the branches starting from the ends of this interval and simply results from the loss of regularity at $\theta = 0$. All the other branches, i.e., those starting everywhere in $]\lambda_T, \lambda_E[$ are analytical.
2. We observe in addition, and in view of the objective of the present study it is the most important point, that all the branches are converging towards a "horizontal" asymptote which is precisely the branch starting from λ_R. This branch is given by $\{[0, +\infty[\times \{\lambda_R\}\}$. According to Theorem 1 the position of the boundary is fixed and does not depend on θ, so that it is natural to say that it can be obtained by the same elementary calculation as that used for the result given by Equation (2), only using the adequate modulus.

3 Coupling incremental plasticity with nonlinear deformations: a first case

3.1 Model

This model is similar to Shanley-Hutchinson's one, except for the fastening of the springs. The structure is still the T-shape system made of two rigid bars, rigidly connected at a right angle and supported by a continuous distribution of elastic-plastic springs, but as it is supposed to describe large strains by a physically realistic mechanism, it is necessary to add some information concerning compression or traction deformations of the springs. In order to do this we now assume that the springs are free to slide on the horizontal support, while

they are tied at fixed points on the initially horizontal bar of the T-shape structure, these fixed points being uniformly distributed on the segment $[-1, +1]$ of this bar and sufficiently close to each others to be considered as an approximation of a continuous distribution. This new system is qualitatively represented on Figure 3, and we stress the fact that, contrary to the previous system which was abstract due to the assumption of linearity whatever the deformation, large or not, the present one can be manufactured in a laboratory[2]. We can see that the points where the springs are sliding are closer to each other on the horizontal support, which would mean in a continuous model that the density of springs becomes larger, as the structure is rotating, while the momentum of the reaction of each spring becomes smaller.

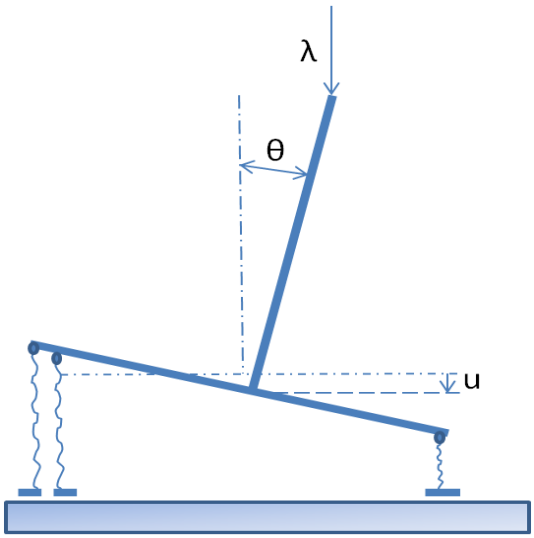


Fig. 3 A geometrically non linear version of Shanley-Hutchinson's model.

Taking the full description of this system into account, and using the same notations as those of the previous section, elementary calculations lead to the following set of equilibrium equations, giving respectively the equilibrium of the resultant, the equilibrium of the moment and the kinematic assumption:

$$\begin{cases} \lambda = \displaystyle\int_{-1}^{1} \sigma(x)dx, \\ \\ \lambda L \sin\theta = \displaystyle\int_{-1}^{1} x\sigma(x)\cos\theta dx, \\ \\ \varepsilon(x) = u + x\sin\theta. \end{cases} \tag{7}$$

In the same way as for the geometrically linearized case, let us first assume that buckling occurs in the elastic range. The constitutive relation is then assumed to be $\sigma = E\varepsilon$ everywhere on $[-1, +1]$ and through elementary calculations the complete bifurcation diagram consists of the two following branches:

$$\begin{cases} \theta = 0, \ \lambda \in \ \mathbb{R}^+, \\ \\ \theta \in \ [0, \dfrac{\pi}{2}[, \quad \lambda = \dfrac{2E}{3L}\cos\theta, \end{cases} \tag{8}$$

[2] Of course additional devices, not represented on Figure 3, would be necessary for an experimental realization in a laboratory: One device should prevent the structure from any global horizontal displacement; this could be a rigid axis normal to the plane of the structure at the intersection of the two bars and sliding without friction along a vertical guide. Another device should force the springs to remain in the initial vertical direction, for example by inserting the springs inside vertical tubes in which they could slide without friction while the structure is rotating.

which means, as opposed to the geometrically linear case, that function $\lambda(\theta)$ is no longer constant on the non trivial branch and that the load goes to zero as the rotation goes to $\pi/2$.

Let us now assume that the external load λ is smoothly varying in such a way that the load itself or any given nonzero smooth and monotonous function of the load can be taken as a kinematic parameter. As the foundation of quasi-static plasticity this function can be seen as a fictitious time, let τ be this fictitious time and $\dot{()}$ be the derivative with respect to τ. Then the elastic plastic constitutive law, in the so-called simple model of elastic-plasticity with constant strain hardening, reads:

$$\begin{cases} i)\ \dot{\sigma} = E\dot{\varepsilon} \ \text{ if } \sigma < \sigma_y \text{ or } \dot{\sigma} < 0, \ \text{ i.e. for elastic loading or for unloading,} \\ \\ ii)\ \dot{\sigma} = E_T\dot{\varepsilon} \ \text{ if } \sigma = \sigma_y \text{ and } \dot{\sigma} \geq 0, \text{ i.e. for plastic loading,} \end{cases} \tag{9}$$

where $0 < E_T < E$[3], and σ_y denotes the actual limit of elasticity of the springs which increases with the actual state.

3.2 the quasi-static equilibrium problem

After a differentiation with respect to τ, equations (7) lead to a quasi-static equilibrium problem which is now:

$$\begin{cases} i)\ \dot{\lambda} = \displaystyle\int_{-1}^{1} \dot{\sigma}(x)dx, \\ \\ ii)\ L\dot{\lambda}\sin\theta + L\lambda\cos\theta\ \dot{\theta} = \cos\theta \displaystyle\int_{-1}^{1} \dot{\sigma}(x)xdx - \sin\theta\ \dot{\theta} \displaystyle\int_{-1}^{1} \sigma(x)xdx, \\ \\ iii)\ \dot{\varepsilon}(x,\lambda) = \dot{u} + x\dot{\theta}\cos\theta. \end{cases} \tag{10}$$

From the reference configuration the system may symmetrically rotate with a positive or with a negative angle θ. Let us choose the positive rotation. From an initial state where the springs are uniformly plastically loaded (i.e., assuming that condition 9-*ii)* holds everywhere in $[-1,+1]$), unloading appears at the point $x = -1$ since the horizontal bar is rigid. Let us then assume that for some strictly positive angle θ, there exists a boundary Y depending on λ, which seperates the domain $[-1,+1]$ into an elastic part $[-1,Y]$ and a plastic part $[Y,+1]$. This specificity changes problem (7) from a classical equilibrium problem into a so-called free boundary problem, which means in the present case that in addition to the components $u(\lambda)$ and $\theta(\lambda)$ of the displacement the problem involves a new unknown, namely the boundary Y.

Due to the constitutive law (10-*iii*) a continuity relation should be satisfied at the boundary Y, which gives the complementary equation that closes system (10). It reads

$$\dot{\varepsilon}(Y,\lambda) = 0 \quad \text{that is} \quad \dot{u} = -Y\dot{\theta}\cos\theta. \tag{11}$$

Inserting Equations (9) and (11) into Equation (10), the quasi-static equation of the resultant is now

$$\dot{\lambda} = (E - E_T)\dot{\theta}\cos\theta\Big(-\frac{Y^2}{2} - Y\frac{E+E_T}{E-E_T} - \frac{1}{2}\Big). \tag{12}$$

The equation of the moment is a bit more complicated. Step by step it reads:

$$L\dot{\lambda}\sin\theta + L\lambda\dot{\theta}\cos\theta \ = \int_{-1}^{1} x\dot{\sigma}(x)\cos\theta dx - \int_{-1}^{1} x\dot{\theta}\sigma(x)\sin\theta dx, \tag{13}$$

[3] Here the tangent modulus E_T is strictly positive, which is the mechanical meaning of hardening, it is assumed to be constant in the present study. Recall that $E_T = 0$ would specify the case of the so-called perfect plasticity, and $E_T < 0$, the mechanical meaning of which is softening, is usually associated with damage analysis

where the second integral of the right hand side can be calculated through the static equation of the moment, which gives

$$L\dot{\lambda}\sin\theta + L\lambda\dot{\theta}\cos\theta \;= \int_{-1}^{1} x\dot{\sigma}(x)\cos\theta dx - L\lambda\dot{\theta}\frac{\sin^2\theta}{\cos\theta},$$

or

$$L\dot{\lambda}\sin\theta \;= \int_{-1}^{1} x\dot{\sigma}(x)\cos\theta dx - L\lambda\dot{\theta}\frac{1}{\cos\theta}. \tag{14}$$

Now the right hand side is classically calculated through the constitutive relation, respectively in $[-1, Y]$ and in $[Y, +1]$. We get :

$$L\dot{\lambda}\sin\theta \;= (E - E_T)\dot{\theta}\cos^2\theta\Big(-\frac{Y^3}{6} + \frac{Y}{2} + \frac{E + E_T}{3(E - E_T)}\Big) - L\lambda\dot{\theta}\frac{1}{\cos\theta}. \tag{15}$$

The quasi-static equilibrium problem described by Equations (12) and (15) involves the same constants α and β and the same functions $f(Y)$ and $g(Y)$ of the boundary as those of the geometrically linear case, and finally reads:

$$\begin{cases} i)\ \dot{\lambda} - \beta f(Y)\cos\theta\ \dot{\theta} \;= 0, \\ ii)\ L\dot{\lambda}\sin\theta - \Big(\beta g(Y)\cos^2\theta - \lambda L\dfrac{1}{\cos\theta}\Big)\dot{\theta} \;= 0, \end{cases} \tag{16}$$

where we recall that

$$\beta = E - E_T, \qquad \alpha = \frac{E + E_T}{E - E_T}, \qquad f(Y) = -\frac{Y^2}{2} - \alpha Y - \frac{1}{2}, \qquad g(Y) = -\frac{Y^3}{6} + \frac{Y}{2} + \frac{\alpha}{3}.$$

It is now useful to recall some properties of functions f and g.

1. The function $f : Y \in [-1, +1] \longrightarrow f(Y) = -\dfrac{Y^2}{2} - \alpha Y - \dfrac{1}{2}$:
   - is monotonically strictly decreasing in the whole interval $[-1, 1]$ between $f(-1) = \dfrac{2E_T}{E - E_T} > 0$ and $f(1) = \dfrac{-2E}{E - E_T} < 0$,
   - has consequently a single zero in $]-1, 1[$ denoted by Y_R with $Y_R = -\dfrac{1 - \sqrt{\dfrac{E_T}{E}}}{1 + \sqrt{\dfrac{E_T}{E}}}$, $\;-1 < Y_R < 0$,
   - is strictly positive in $[-1, Y_R[$ and negative in $]Y_R, 1]$.
2. The function $g : Y \in [-1, 1] \longrightarrow g(Y) = -\dfrac{Y^3}{6} + \dfrac{Y}{2} + \dfrac{\alpha}{3}$:
   - is strictly increasing in $]-1, 1[$ between $\dfrac{2E_T}{3(E - E_T)}$ and $\dfrac{2E}{3(E - E_T)}$.
   - is strictly positive in $[-1, 1]$, with a zero derivative at $Y = \pm 1$.

We first observe that Problem (10) always has the solution $\theta \equiv 0$ for any $\lambda \in \mathbb{R}^+$. Let us now come back to Equations (7). It has just been stated that if the springs are all loaded in the elastic range the equilibrium problem is satisfied by two branches of solutions given by Equation (8) . It seems then natural that the same result would be obtained if the springs were all loaded in the plastic range, by just changing the notation and by replacing E by E_T in Equation (8), but Shanley's argument states that this is impossible. Indeed a solution branch $\{\lambda = \lambda_0, \theta \in [0, \pi/2[\}$ cannot exist if the springs are all in plastic loading due to the kinematic relation (10-iii), which implies that the springs will be elastically unloaded at the point $x = -1$ of the horizontal bar as soon as $\theta \neq 0$. Consequently, only the solution branch $\{\lambda \in \mathbb{R}^+,\ \theta = 0\}$ exists if plastic loading is assumed to hold everywhere[4], or in other words, a possible bifurcated branch involving $\theta \neq 0$ from an initial state where all the springs were plastically loaded would necessarily imply elastic unloading in a part of the system. For the moment

[4] Actually, if the springs are all assumed to remain in plastic loading for any λ, then $\theta = 0, \dot{\theta} = 0$ satisfies Problem (10) where the equation of the resultant gives $\dot{\lambda} = 2E_T\dot{u}$; the equation of the moment reduces to $0 = 0$, and the kinematics is simply $\dot{\varepsilon} = \dot{u}$

we keep in mind that $\theta \equiv 0$ is solution for any λ, which means that looking for bifurcated solutions amounts to looking for solutions such that $\dot{\theta} \neq 0$ for $\theta = 0$.

Let us now look at Equation (16-i):
The boundary Y is a bounded function since it belongs to $[-1, +1]$ by definition, so that $f(Y)$ is bounded and $\dfrac{d\lambda}{d\theta}$ is also bounded on any interval of $[0, \pi/2[$ in which the free boundary Y exists, which means that the equation of the quasi-static equilibrium is a differential equation with a bounded right hand side at least on a compact sub-interval of $[0, \pi/2[$. Using this result in Equation (16-ii) and recalling that $g(Y)$ is also bounded we get that function $\lambda(\theta)$ is also bounded on the same interval of $[0, \pi/2[$. Moreover, since $\dot{\theta} \neq 0$ Problem (16) leads to the following equation:

$$0 = \beta f(Y) \cos\theta \sin\theta - \frac{\beta}{L} g(Y) \cos^2\theta + \frac{\lambda}{\cos\theta}, \tag{17}$$

which is rewritten as:

$$\Psi(Y, \theta, \lambda) \equiv \beta f(Y) \sin\theta - \frac{\beta}{L} g(Y) \cos\theta + \frac{\lambda}{\cos^2\theta} = 0. \tag{18}$$

This algebraic equation gives

$$\frac{\partial \Psi}{\partial Y} = \beta f'(Y) \sin\theta - \frac{\beta}{L} g'(Y) \cos\theta = -\beta(Y + \alpha) \sin\theta - \frac{\beta}{2}(1 - Y^2) \cos\theta.$$

where $Y + \alpha$ is always positive and as long as a boundary Y exists $(1 - Y^2)$ is also positive since the boundary must be such that $|Y| \leq 1$ by definition. It is consequently readily seen that the derivative $\dfrac{\partial \Psi}{\partial Y}$ is negative or equal to zero for $\theta \in [0, \pi/2[$ and $\lambda \in \mathbb{R}^+$, strictly negative for $\theta \in]0, \pi/2[$, and equal to zero for $\theta = 0$ at $\lambda = \dfrac{2E_T}{3L} := \lambda_T$ and $\lambda = \dfrac{2E}{3L} := \lambda_E$. The implicit function theorem then asserts that Equation (18) can be inverted giving Y as a single function $\widetilde{Y}(\lambda, \theta)$, for which we keep the notation $Y(\lambda, \theta)$ for simplicity, wherever $\dfrac{\partial \Psi}{\partial Y}$ is strictly negative. Moreover, for the free boundary to separate the interval $[-1, +1]$ into two nonzero measure sub-intervals, Y should be strictly larger than -1 and strictly smaller than $+1$.

Only one root Y of the third degree algebraic equation (18) belongs to the interval $[-1, +1]$ for $\theta = 0$ so that non trivial branches are the solutions to the following differential equation:

$$\begin{cases} \dfrac{d\lambda}{d\theta} = \beta f(Y(\theta, \lambda)) \cos\theta, \\ \\ \lambda(0) = \lambda_0, \quad \lambda_0 \in [\lambda_T, \lambda_E], \end{cases} \tag{19}$$

and now the right-hand side of Equation (19) is not only bounded but is as smooth as function $Y(\theta, \lambda)$.

3.3 Incipient buckling and close post-buckling

The first step consists in the structure of the set of solutions close to the trivial branch. At $\theta = 0$, Equation (17) becomes:

$$0 = \frac{\beta}{L} g(Y) - \lambda,$$

so that the buckling load and the boundary $Y(\lambda, 0)$ are connected by the same relation as in the geometrically linear case[5]. In particular we have the following properties which follow immediately from the above study of polynomials $f(Y)$ an $g(Y)$:

- The relation between Y and λ is such that λ_T is associated with $Y = -1$ and λ_E is associated with $Y = +1$.
- When $\lambda = \lambda_T$, there exists a strictly positive non trivial initial velocity $\dfrac{d\lambda}{d\theta}$, while a strictly negative non trivial initial velocity exists when $\lambda = \lambda_E$.

[5] This result is quite natural, since the case with linear deformations can be seen precisely as a linearization for small θ of the present one.

- The value $\lambda = \lambda_R$ corresponds to the so called adjacent equilibrium, which in other words means bifurcation point with initial velocity equal to zero, or bifurcation at constant load.
- Since Y must belong to the interval $[-1, +1]$, there do not exist non trivial initial velocities for $\lambda \notin [\lambda_T, \lambda_E]$.

Let us introduce a small enough value θ_0 of the angle and study the solution branches for $\theta \in [0, \theta_0[$, which may depend on λ_0. Then:

Lemma 1 *There exist bifurcated solutions to Problem (16), transverse to the trivial one, for any $\lambda_0 \in [\lambda_T, \lambda_E]$, analytical if $\lambda_0 \in]\lambda_T, \lambda_E[$, singular at $\theta = 0$ if $\lambda_0 = \lambda_T$* or *$\lambda_0 = \lambda_E]$.*

Proof: The proof of this Lemma involves two steps:

1. We first remark that due to implicit function theorem θ_0 is strictly positive. Let then $\lambda \in]\lambda_T, \lambda_E[$ and $\theta \in]0, \theta_0[$. Equation (17) can be inverted giving Y as an analytical function of θ and λ in this open domain. Moreover, Equation (17) can also be inverted in the open interval $]\lambda_T, \lambda_E[$ for $\theta = 0$. This implies that for any $\lambda_0 \in]\lambda_T, \lambda_E[$, Equation (19) has a single solution $\lambda(\theta)$ which is analytical on $\theta \in [0, \theta_0[$.
2. Equation (17) cannot be inverted at the points $\{\lambda = \lambda_T, \theta = 0\}$ and $\{\lambda = \lambda_E, \theta = 0\}$. Let us study one of these points and the argument will be adapted to the other. As presented in [4] for the geometrically linear case, assume that for a given natural integer n, the boundary of the initial unloading zone at $\theta = 0$ is given by $Y_{0n} = -1 + \dfrac{1}{n}$. Using this unloading zone, Equation (17) gives for any n a value $\lambda_n(0)$, which, taken as initial data to Problem (19), leads to a branch $\lambda_n(\theta)$ in $[0, \theta_0[$. Let us consider compact sets included in $[0, \theta_0[\times[\lambda_T, \lambda_E]$. Elementary techniques then give the existence of a solution branch $\lambda_\infty(\theta)$ starting from $\lambda_\infty(0) \equiv \lambda_T$, as the uniform limit of the $\lambda_n(\theta)$ when $n \longrightarrow \infty$.

□

3.4 Intermediate and asymptotic post-buckling

The main qualitative feature of the evolution on the nontrivial branches is that the free boundary, which enters the domain $[-1, +1]$ at $\theta = 0$ according to the value of the initial data λ_0, may be a non monotonic function of θ moving back to $x = +1$ for some $\theta > 0$. In fact the first step consists in observing that all solution branches associated with any initial data $\lambda_0 \in [\lambda_T, \lambda_E]$ attain the boundary $x = +1$ for some positive value of the angle θ strictly smaller than $\pi/2$. More precisely, the numerical investigation of the behavior of functions $Y(\theta)$ gives the following:

Lemma 2 *i) Let the initial data λ_0 of equation (19) belong to the interval $[\lambda_T, \lambda_R[$ of the trivial branch. Then the functions $Y(\theta)$ are monotonic, strictly increasing from an initial value Y_0, associated with λ_0 by Lemma 1, to $Y = +1$.*
ii) Let the initial data λ_0 of equation (19) belong to the interval $[\lambda_R, \lambda_E]$ of the trivial branch. Then the functions $Y(\theta)$ are non monotonic with a minimum larger than Y_R, after which they increase up to $Y = +1$ for some angle θ strictly smaller than $\pi/2$.
iii) Let Y_{01} and Y_{02} with $Y_{01} < Y_{02}$ be two initial positions of the free boundary and Y_1 and Y_2 the curves $Y(\theta)$ on the corresponding solutions branches. Then the curves Y_1 and Y_2 do not intersect and reach $x = +1$ respectively for some θ_1 and θ_2 with $\theta_1 > \theta_2$.

The free boundaries as functions of θ are plotted on Figure 4

As noticed in Figure 4, the admissible domains of functions $Y(\theta)$ are strictly included in $[0, \pi/2[$. In order to study the branches on the whole domain $[0, \pi/2[$, it is then necessary to understand the continuation of the branches up to $\pi/2$.

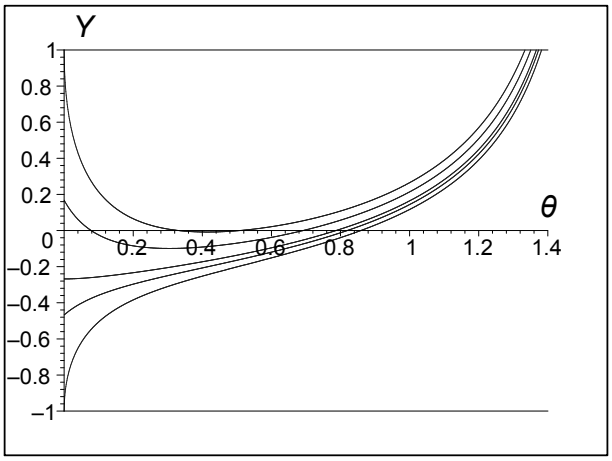


Fig. 4 Functions $Y(\theta)$ are defined for $Y \in [-1, +1]$ and we observe that $Y = +1$ for $1.335 \leq \theta \leq 1.383$. The parameters of this computation are chosen as $L = 1,\ E = 3E_T = 200.$, i.e., $Y_R = -0.268$.

Theorem 3 *The global bifurcation diagram satisfies the following statements:*
i) the incipient buckling is identical to the one of the geometrically linear case;
ii) there exist bifurcated branches on $[0, \pi/2[$*, non monotonous for* $\lambda_0 \in [\lambda_T, \lambda_R[$*, decreasing for* $\lambda_0 \in [\lambda_R, \lambda_E]$*, strictly decreasing for* $\lambda_0 > \lambda_R$*;*
iii) all these branches attain zero for a rotation θ *depending on* λ_0 *but smaller than* $\pi/2$*.*

Proof
Point *i)* simply recalls the result of the geometrically linear case given in Lemma 1.

The proof of points *ii)* and *iii)* need another lemma which completes what has been described in Lemma 2. Let λ_0 be the initial data of Problem (19), and $\lambda_{\lambda_0}(\theta)$ its corresponding solution. Let also θ_{λ_0} be the rotation at which the free boundary $Y(\theta)$ reaches $x = +1$, and denote $\lambda_{\theta_0} := \lambda(\theta_{\lambda_0})$.

Lemma 3 *: The free boundaries associated with the solution branches starting from any initial data* $\lambda_0 \in [\lambda_T, \lambda_E]$ *actually attain the border of the domain* $x = +1$ *for some sufficiently large rotation* θ*. Moreover, when* $Y(\lambda)$ *reaches the point* $x = +1$*, the springs are all elastically unloaded and remain in elastic unloading for* $\theta > \theta_{\lambda_0}$*.*

Proof
Let δ be the part due to the rotation θ of the vertical component of the displacement at a point $x > 0$ of the initially horizontal bar, and recall that u stands for the vertical displacement of the middle point $x = 0$ of the same bar. Since the bar is rigid, the geometry gives $\delta = x \sin\theta$. Under a variation $\dot{\lambda}$ of the external vertical load, the constitutive relation then leads to $\dot{u} - \dot{\delta} = 0$ at $Y = +1$.

Observe that $\beta f(+1) = 2E$ and $\dfrac{\beta}{L} g(+1) = \dfrac{2E}{3L}$; recall in addition that the quasi static equilibrium equation of the resultant (16-i) reads:

$$\dot{\lambda} = \beta f(Y) \cos\theta\ \dot{\theta},$$

which becomes, for $Y = +1$

$$\dot{\lambda} = 2E \cos\theta\ \dot{\theta}\ = 2E\ \dot{\delta}.$$

On the other hand the constitutive law inserted into the equilibrium equation (10-i) gives the increment of the vertical displacement of the point $x = 0$ of the structure:

$$\dot{\lambda} = 2E\ \dot{u}. \qquad \text{QED}$$

Remark 1 This simple result is interesting: it means that the global motion down to the horizontal support is equal to the part of the vertical displacement of the right end of the initially horizontal bar due to the rotation. The free boundary being at this end, the whole interval $[-1, +1]$ is either elastically loaded or in elastic unloading since the deformation $\varepsilon = u + x \sin\theta$ is maximum for $x = +1$. Although the external force associated with the quasi-static equilibrium is decreasing, the T-shape structure does not move back to its original position, as it would be the case in elasticity, because a part of the interval $[-1, +1]$, i.e., a part of the springs, has been deformed in the plastic range.

The part of Point *ii)* concerning the branches where $Y < +1$ is a simple corollary of Lemma 2. As a matter of fact, assume $\lambda_0 \in [\lambda_T, \lambda_R[$, then Lemma 2 states that function $Y(\theta)$ is monotonically strictly increasing from $Y_0 < Y_R$ to $+1$. Since it is a smooth function, Y_R is its single zero. Recalling Equation (19) and the properties of function $f(Y)$, we get immediately that as long as $Y(\theta) \in]Y_0, Y_R[$, function $f(Y)$ is positive so that function $\lambda(\theta)$ is strictly increasing, while when $Y(\theta) \in]Y_R, +1[$, function $f(Y)$ is negative and function $\lambda(\theta)$ is strictly decreasing. Since $\lambda(\theta)$ is smooth, it has a maximum at a point θ_M interior to $]0, \theta_{\lambda_0}[$.

Let us now consider the case where $\lambda_0 \in]\lambda_R, \lambda_E]$, then Lemma 2 states that function $Y(\theta)$ is non monotonic, at first decreasing from $Y_0 > Y_R$ at $\theta = 0$ to some value larger than Y_R after which $Y(\theta)$ is increasing up to $Y = +1$, so that $Y(\theta)$ is greater than Y_R everywhere. Function $f(Y)$ is consequently negative everywhere in $]0, \theta_{\lambda_0}[$ and function $\lambda(\theta)$ is strictly decreasing in this domain.

Then for each $\lambda_0 \in]\lambda_T, \lambda_E]$, only the continuation for $\theta > \theta_{\lambda_0}$ is needed to complete Point *ii)*. Since each free boundary reaches the border of the domain at some rotation θ_{λ_0}, we may assume that a free boundary no longer exists for any larger rotation. Actually, such a fully elastic continuation satisfies all conditions of the system, and, taking into account the match with the left limit of the elastic-plastic solution at θ_{λ_0}, it reads:

$$\lambda(\theta) = \frac{2E}{3L}(\cos\theta - \cos\theta_{\lambda_0}) + \lambda_{\theta_0}, \qquad \text{in }]\theta_{\lambda_0}, \pi/2[, \tag{20}$$

where θ_{λ_0} is defined as the value of the rotation depending on λ_0 for which $Y = +1$, and λ_{θ_0}, also depending on λ_0, is given by Equation (17)[6]. □

The global bifurcation diagram of the quasi static evolution is represented on Figure 5. According with Remark 1, the load tends to zero whereas the rotation tends to $\pi/2$.

4 Coupling incremental plasticity with nonlinear deformations: a second case

4.1 The model

In the same way as in the elastic-plastic and geometrically non linear system represented on Figure 3, the structure which will be studied now differs from Shanley-Hutchinson's model only by the fastening of the springs. The structure is still the T-shape system made of two rigid bars, rigidly connected with a right angle and supported by a continuous distribution of elastic-plastic springs but in the present case the springs are tied at their bottom, at fixed points uniformly distributed on the interval $[-1, +1]$ of the horizontal rigid support, and their upper end is assumed to slide without friction on the initially horizontal bar of the structure. In the same way as in the case represented on Figure 3 or in Shanley-Hutchinson's original model, the external load is assumed to remain strictly in the vertical direction while the structure buckles or undergoes vertical displacements. This new system is qualitatively represented on Figure 6. Again it is supposed to describe large strains by a physically realistic mechanism which could be manufactured in a laboratory[7].

Using the same notations as those of the previous section, the above description of the system leads, through elementary calculations to the following set of equilibrium equations, giving respectively the equilibrium of the

[6] It is important to stress that the free boundary does not exist in $]\theta_{\lambda_0}, \pi/2[$, the force in each spring being strictly lower than the yield value of plasticity, which means that the effective elastic continuation is completely different from the one

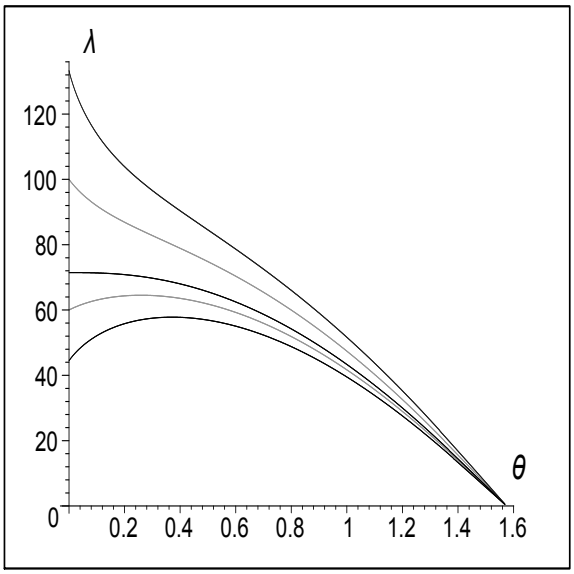


Fig. 5 The global bifurcation diagram in the $\{\theta, \lambda\}$-plane of the geometrically non linear version of Shanley-Hutchinson's model represented on Figure 3. Three branches are plotted, issued respectively from $\lambda_0 = \lambda_T$, $\lambda_0 = \lambda_R$, $\lambda_0 = \lambda_E$. The match between the left part and the right part of these curves depends on the initial data λ_0, ranging from $\theta = 1.335$ for $\lambda_0 = \lambda_E$ to $\theta = 1.383$ for $\lambda_0 = \lambda_T$.

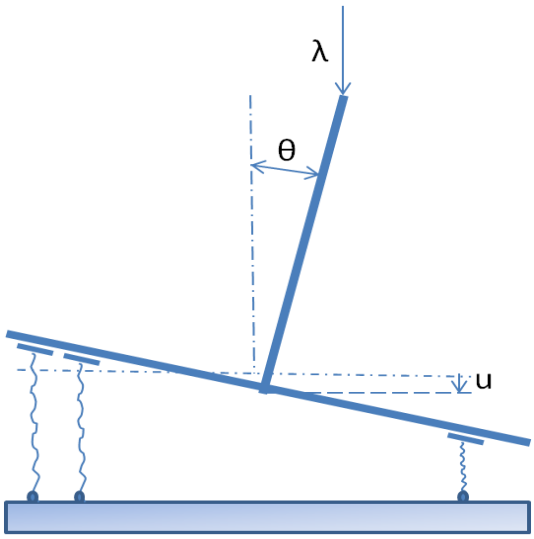


Fig. 6 A generalization of Shanley-Hutchinson's model.

that would have been obtained by fixing $Y(\theta)$ at $Y = +1$ after θ_{λ_0}, although such a fixed position of the free boundary would also mean that elastic behavior holds everywhere.

[7] Again, an additional device, not represented on Figure 6, should be added in an experimental realization. This time the structure cannot undergo global horizontal displacements contrary to the previous case, but it is crucial here again to force the springs to remain vertical by some mechanism while the system is rotating. This could be done by sufficiently many tubes clamped at their lower end in the upright position in which the springs could slide without friction.

resultant, the equilibrium of the moment and the kinematic assumption:

$$\begin{cases} i)\ \lambda = \displaystyle\int_{-1}^{+1} \sigma(x)dx, \\ \\ ii)\ \lambda L \sin\theta = \displaystyle\int_{-1}^{+1} x\sigma(x)\frac{1}{\cos^2\theta}dx, \\ \\ iii)\ \varepsilon(x) = u + x\tan\theta. \end{cases} \tag{21}$$

In the same way as what has been given in the geometrically linearized case and in the previous geometrically non linear case, let us first assume that buckling occurs in the elastic range. The constitutive relation is then assumed to be $\sigma = E\varepsilon$ everywhere on $[-1, +1]$ and the complete bifurcation diagram is made of the two following branches

$$\begin{cases} \theta = 0,\ \lambda \in\ \mathbb{R}^+, \\ \\ \theta \in\ [0, \dfrac{\pi}{2}[,\quad \lambda = \dfrac{2E}{3L\cos^3\theta}, \end{cases} \tag{22}$$

which again means that function $\lambda(\theta)$ is no longer constant on the non trivial branch, but in the present case tends to infinity as θ tends to $\pi/2$.

The elastic plastic constitutive law has been given in Equation (9) and reads:

$$\begin{cases} i)\ \dot\sigma = E\dot\varepsilon\ \text{ if } \sigma < \sigma_y \text{ or } \dot\sigma < 0,\ \text{ i.e. for elastic loading or for unloading,} \\ \\ ii)\ \dot\sigma = E_T\dot\varepsilon\ \text{ if } \sigma = \sigma_y \text{ and } \dot\sigma \geq 0,\ \text{ i.e. for plastic loading,} \end{cases} \tag{23}$$

where σ_y denotes the actual limit of elasticity of the springs and $\dot{()}$ stands for the derivative with respect to the fictitious time τ introduced in the same way as in the previous section.

4.2 the quasi-static equilibrium problem

Obtained by a differentiation of problem (21) with respect to τ, the quasi-static equilibrium problem reads:

$$\begin{cases} i)\ \dot\lambda = \displaystyle\int_{-1}^{1} \dot\sigma(x)dx, \\ \\ ii)\ L\dot\lambda\sin\theta + L\lambda\cos\theta\ \dot\theta = \dfrac{1}{\cos^2\theta}\displaystyle\int_{-1}^{1} \dot\sigma(x)xdx + \dfrac{2\sin\theta}{\cos^3\theta}\ \dot\theta\displaystyle\int_{-1}^{1} \sigma(x)xdx, \\ \\ iii)\ \dot\varepsilon(x,\lambda) = \dot u + x\dfrac{\dot\theta}{\cos^2\theta}. \end{cases} \tag{24}$$

Again we choose without restriction the case of the positive rotation and Problem (24) is closed by a complementary equation which connects a free boundary $Y(\lambda, \theta)$ with the components u and θ of the displacement. Due to the constitutive law (24-*iii*) the boundary Y is given by:

$$\dot\varepsilon(x,\lambda) = 0 \quad \text{that is} \quad \dot u = -Y\frac{\dot\theta}{\cos^2\theta}. \tag{25}$$

Recalling Equation (21-*ii*), Problem (24) together with relation (25) gives, after elementary integrations:

$$\begin{cases} \dot\lambda = (E - E_T)\dfrac{1}{\cos^2\theta}\Big(-\dfrac{1}{2}Y^2 - \dfrac{E+E_T}{E-E_T}Y - \dfrac{1}{2}\Big)\dot\theta, \\ \\ \dot\lambda\sin\theta = \Big[\dfrac{E-E_T}{L}\Big(-\dfrac{1}{6}Y^3 + \dfrac{1}{2}Y + \dfrac{1}{3}\dfrac{E+E_T}{(E-E_T)}\Big)\dfrac{1}{\cos^4\theta} - \lambda\big(\cos\theta - 2\tan\theta\sin\theta\big)\Big]\dot\theta, \end{cases} \tag{26}$$

Using notations already defined, and in particular introducing the polynomials $f(Y)$ and $g(Y)$ and the constitutive parameter β which are the same as in the previous cases, problem (26) is rewritten as:

$$\begin{cases} i)\ \dot{\lambda} = \beta f(Y)\dfrac{1}{\cos^2\theta}\dot{\theta}, \\ \\ ii)\ \dot{\lambda}\sin\theta = \Big[\dfrac{\beta}{L}g(Y)\dfrac{1}{\cos^4\theta} - \lambda\big(\cos\theta - 2\tan\theta\sin\theta\big)\Big]\dot{\theta}. \end{cases} \tag{27}$$

Again it is clear that Problem (21) always has the solution $\theta \equiv 0$ for any $\lambda \in \mathbb{R}^+$. The study of solutions such that $\dot{\theta} \neq 0$ at $\theta = 0$ starts from Problem (27): As a matter of fact, Y is bounded since it belongs to $[-1, +1]$ by definition, then $f(Y)$ is bounded, so that $\beta f(Y)\dfrac{1}{\cos^2\theta}$ is bounded on any compact of $[0, \pi/2[$ in which the free boundary $Y(\lambda)$ exists, and so is consequently $\dfrac{d\lambda}{d\theta}$ from Equation (27-i). Introducing this result in Equation (27-ii) implies that function $\lambda(\theta)$ is also bounded in the same interval.

Moreover, since we are looking for solutions such that $\dot{\theta} \neq 0$ at $\theta = 0$, Problem (27) leads to the following equation:

$$0 = -\beta f(Y)\frac{\sin\theta}{\cos^2\theta} + \frac{\beta}{L}g(Y)\frac{1}{\cos^4\theta} - \lambda\big(\cos\theta - 2\tan\theta\sin\theta\big). \tag{28}$$

Let $\Phi(Y, \lambda, \theta)$ be the right hand side of Equation (28). The derivative of Φ with respect to Y is easily obtained as

$$\frac{\partial\Phi}{\partial Y} = -\beta f'(Y)\frac{\sin\theta}{\cos^2\theta} + \frac{\beta}{L}g'(Y)\frac{1}{\cos^4\theta} = \beta(Y+\alpha)\frac{\sin\theta}{\cos^2\theta} + \frac{\beta}{2L}(1-Y^2)\frac{1}{\cos^4\theta}.$$

Then, function Φ can be inverted, giving Y as a function of λ and θ, in the same domain as in the previous case that is in:

$$\mathcal{V} = \big\{\theta \in]0, \frac{\pi}{2}[\ ,\ \lambda \in \mathbb{R}^+\big\} \cup \big\{\theta = 0\ ,\ \lambda \in]\lambda_T, \lambda_E[\big\}.$$

Only one root $\widetilde{Y}$ of the third degree algebraic equation (28) belongs to the interval $[-1, +1]$ for $\theta = 0$ so that the non trivial branches are the solutions to the following differential equation:

$$\begin{cases} \dfrac{d\lambda}{d\theta} = \beta f(\widetilde{Y}(\lambda, \theta))\dfrac{1}{\cos^2\theta}, \\ \\ \lambda(0) = \lambda_0, \quad \lambda_0 \in]\lambda_T, \lambda_E[, \end{cases} \tag{29}$$

where again the right hand side is not only bounded but is analytical in the domain where Equation (28) can be inverted. We shall now investigate the buckling and post-buckling behavior on the whole domain $[0, \frac{\pi}{2}[$.

4.3 Incipient buckling and close post-buckling

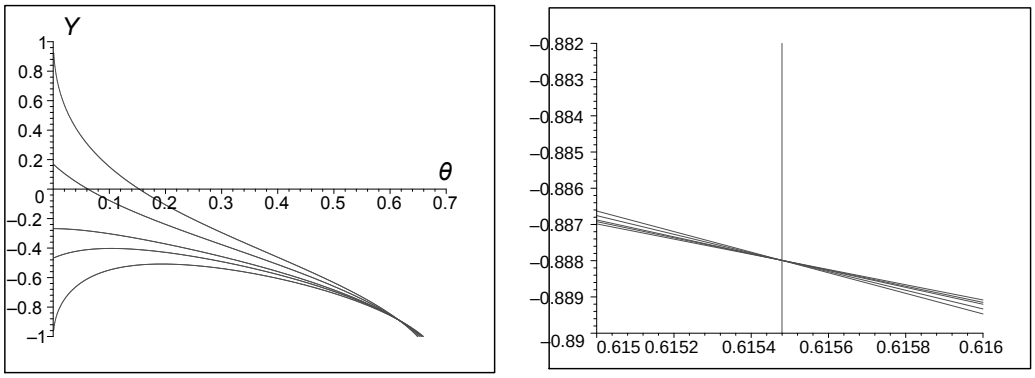
The first part of the bifurcation analysis is exactly the same as the one of the previous case studied in Section 3, itself identical to the case of the geometrically linear case recalled at Section 2. The reason is that Equation (28) of the present geometrically non linear case taken at $\theta = 0$ is the same as Equation (17) of the previous case also taken at $\theta = 0$, both being $0 = \dfrac{\beta}{L}g(Y) - \lambda$, which is also the result of the geometrically linear case given in Section 2. Then Subsection 3.3 gives a good approximation of the bifurcation diagram for the present case as long as θ is small enough.

4.4 Intermediate and asymptotic post-buckling

The evolution of the non trivial branches is such that the free boundary may be a non monotonic function of θ and may move back to $x = -1$ for some $\theta > 0$. More precisely, the numerical investigation of the solutions of equation (28) gives:

Lemma 4 *i) All branches are such that the free boundaries move down to $x = -1$.*
ii) Let the initial data λ_0 of equation (29) belong to the interval $[\lambda_T, \lambda_E]$ of the trivial branch. Then the functions $Y(\theta)$ are defined on some interval $[0, \theta_{\lambda_0}]$ with $0 < \theta_{\lambda_0} < \pi/2$. Moreover, the root $Y(\theta, \lambda)$ of equation (28) reaches $x = -1$ at θ_{λ_0}[8].
iii) Fonctions $Y(\theta)$ are non monotonic, with a maximum strictly smaller than Y_R if $\lambda_0 \in [\lambda_T, \lambda_R[$, monotonic strictly decreasing from $Y(0) > Y_R$ if $\lambda_0 \in]\lambda_R, \lambda_E]$

This result is close to the one obtained in the previous section, but there is nevertheless a qualitative difference since the curves $Y(\theta)$ intersect at some point θ_s, with $0 < \theta_s < \theta_{\lambda_0}$, as opposed to point *iii)* of Lemma 2. As a matter of fact, it is easily seen from Equation (28), that for all $\lambda_0 \in]\lambda_T, \lambda_E]$, the curves $Y(\theta)$ intersect for θ satisfying $\cos\theta - 2\tan\theta\sin\theta = 0$ that is $\theta = 0.61548$ whatever the value of λ. From symbolic calculations we get that $Y = -0.8878$ at this point[9], which is consequently a point where equation (28) may be satisfied for any finite λ, or in other words a common position of the free boundary for the bifurcated branches. Functions $Y(\theta)$ are plotted on Figure 7.



a - Functions $Y(\theta)$ are defined on $[0, 0.67]$.

b - Functions $Y(\theta)$ intersect at $\theta = 0.61548$.

Fig. 7 From Equation (28), fonctions $Y(\theta)$ belong to $[-1, +1]$ for $\theta \in [0, 0.67]$. But the interval $[0, 0.67[$ includes a crossing point of all functions $Y(\theta)$.

From the point of view of bifurcation theory, this point is singular in a particular way: the bifurcated branches do not intersect nor do they have a limit point, but the all the free boundaries associated with the branches intersect at the same point[10]. We can now state the global result of this section:

Theorem 4 *The bifurcation diagram involves the following properties:*
i) For any λ_0, initial data of Equation (29) belonging to the interval $]\lambda_R, \lambda_E]$ of the trivial branch, there exist non monotonic bifurcated branches defined on $[0, \theta_{\lambda_0}]$ having a minimum at a point θ_m with $0 < \theta_m < \theta_s$.
ii) For any λ_0, initial data of equation (29) belonging to the interval $[\lambda_T, \lambda_R]$ of the trivial branch, there exist bifurcated branches defined on $[0, \theta_{\lambda_0}]$ monotonically increasing, strictly increasing if $\lambda_0 \in [\lambda_T, \lambda_R[$.
iii) All bifurcated solutions tend to $+\infty$ as θ tends to $\pi/2$.

Proof

[8] We keep the notation θ_{λ_0} already used in the previous section as no misunderstanding is possible.

[9] In fact we obtain from equation (28), written at $\cos\theta - 2\tan\theta\sin\theta = 0$, that the third degree algebraic equation $-f(Y)\sin\theta + g(Y)\dfrac{1}{L\cos^2\theta} = 0$ has a single root $Y = -0.8878$ in the interval $[-1, +1]$. In addition, this algebraic equation shows that this result is not generic but depends on L. But numerical calculations give that this intersection point no longer exist in $[0, \pi/2[$ for very small L, which, recalling Shanley's model, would mean extremely short beams.

[10] The question of whether this kind of singularity is a result of bifurcation theory for variational inequalities is very important and might be related with the fact that the free boundary itself results from variational inequalities.

Points *i)* and *ii)* deal with the solutions in $[0, \theta_{\lambda_0}]$, where the behavior is given by Equation (29). They are consequently direct corollaries of Lemma 4.

- Let $\lambda_0 \in]\lambda_R, \lambda_E]$, then Lemma 4 states that function $Y(\theta)$ is monotonically strictly decreasing from $Y_0 > Y_R$ at $\theta = 0$ to $Y = -1$ at some $\theta < \theta_{\lambda_0}$. Since it is a smooth function, Y_R is its single zero. Recalling the properties of function $f(Y)$ and equation (29) we get immediately that as long as $Y(\theta) \in]Y_0, Y_R[$, function $f(Y)$ is negative so that function $\lambda(\theta)$ is strictly decreasing, while when $Y(\theta) \in]Y_R, -1[$, function $f(Y)$ is positive and function $\lambda(\theta)$ is strictly increasing. Since $\lambda(\theta)$ is smooth, it has a minimum at a point θ_m strictly included in $]0, \theta_s[$. Moreover, the minimum tends to zero as Y_0 tends to Y_R.
- Let us now consider the case where $\lambda_0 \in]\lambda_T, \lambda_R]$, then Lemma 4 states that function $Y(\theta)$ is non monotonous, increasing from $Y_0 \geq -1$ to some value strictly smaller than Y_R, decreasing after, down to -1. Recalling the properties of function $f(Y)$ and equation (29) we conclude that function $f(Y)$ is always positive so that function $\lambda(\theta)$ is increasing.
- Only the continuation in $]\theta_{\lambda_0}, \pi/2[$ remains to be examined in order to complete the proof of Theorem 4. This requires the following lemma:

  Lemma 5 *Let ζ be the part due to the rotation θ of the vertical component of the displacement of the initially horizontal bar, and recall that u stands for the vertical displacement of the middle point $x = 0$ of the same bar. Since the bar is rigid, the geometry gives $\zeta = x \tan\theta$. Under a variation $\dot{\lambda}$ of the external vertical load, the constitutive relation then leads to $\dot{u} + \dot{\zeta} = 0$ at the end $x = -1$*

  In the same way as for Theorem 3 this simple result follows here from the quasi-static equations in which we insert $\beta f(-1) = 2E_T$ and $\dfrac{\beta}{L} g(-1) = \dfrac{2E_T}{3L}$. It means that when the rotation is large enough the downward vertical displacement of the whole structure exceeds the upward displacement of the left end of the initially horizontal bar. This result for large rotations is interesting since the buckling phenomenon gave $\dot{u} + \dot{\zeta} < 0$ for any smaller rotation. It implies in turn that all springs, everywhere in $[-1, +1]$, are strictly compressed in the plastic range for large rotation.

According to Lemma 5 the springs are all in strict plastic loading in $]\theta_{\lambda_0}, \pi/2[$. We may consequently assume that the continuation of the solution of Equation (29) up to $\pi/2$ is given by plasticity with no unloading, that is a formally elastic behavior but involving the tangent modulus[11]. Then for any λ_0 the branches are continued by the following functions:

$$\lambda(\theta) = \frac{2E_T}{3L}\left(\frac{1}{\cos^3\theta} - \frac{1}{\cos^3\theta_{\lambda_0}}\right) + \lambda_{\theta_0}, \qquad \text{in }]\theta_{\lambda_0}, \pi/2[. \tag{30}$$

QED □

The global bifurcation diagram is represented on Figure 8

5 Concluding comments

The analysis which has been presented here aimed at understanding the coupling between two types of non linearity, one non smooth such as elastic-plasticity, and the other smooth such as large deformations in classical theories of the mechanics of structures.

The results can be summarized as follows:

- At small strains, that is in the case of incipient buckling or close post-buckling, only elastic-plasticity is involved in the bifurcation analysis. This result is emphasized by the fact that either linear strains or two different types of nonlinear strains all with the same model of plasticity lead to the same bifurcation result. We recall that contrary to elasticity of simple structures, the main qualitative property of elastic-plastic buckling

[11] We have observed that such a fully plastic behavior implies $\dot{\theta} = 0$ at $\theta = 0$, but of course $\dot{\theta} \neq 0$ is possible, and compatible with all conditions of the quasi-static problem, when $\theta \neq 0$.

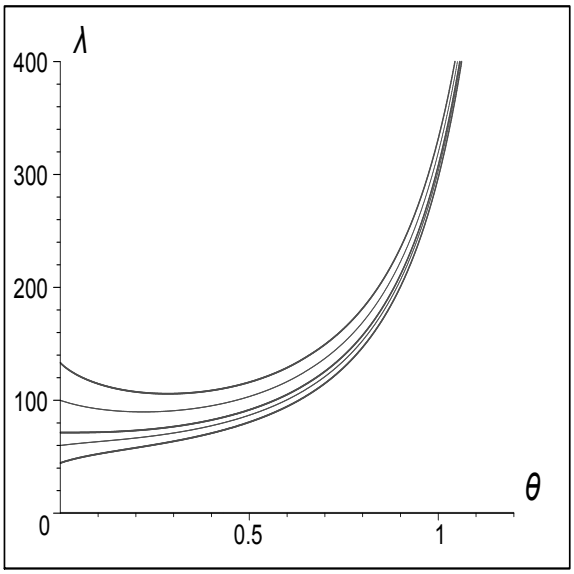


Fig. 8 The global bifurcation diagram in the second geometrically nonlinear case. Each branch is made of two parts, the left one from $\theta = 0$ to θ such that $0.65 < \theta < 0.66$ where $Y = -1$, which involves the free boundary $Y(\theta, \lambda)$, and the right part from this rotation to $\pi/2$ where the behavior is plastic loading everywhere.

is that the set of bifurcation points fills completely an interval, ranging on the real axis from the buckling load of the tangent modulus to the buckling load of the Young modulus. This interval does not depend on the geometrical non linearity. In the same way, the set of incipient buckling deflections only depends on the elastic-plastic behavior and does not depend on the geometrical non linearity.

– Quite the opposite, for very large rotations the asymptotic behavior is formally elastic, but in a specific sense which must be taken carefully: Using basic properties of elastic-plasticity, the whole analysis has been based upon the motion of a free boundary which separates the domain into a plastic loading part and an elastic unloading part, but asymptotically the free boundary is no longer involved. As a matter of fact, for sufficiently large rotations the free boundary attains the border of the domain either at one end, in which case the behavior is elastic everywhere with stresses strictly lower than the yield value for plasticity, or at the other end, in which case the behavior is plastic in strict loading everywhere. Obviously this comment is dependent on the fact that the domain is an interval, but let us look at the result that asymptotically no free boundary remains although we are dealing with incremental elastic-plasticity. Such a result could be interpreted qualitatively by saying that in some sense there remains no memory of the non smooth non linearity at very large smooth non linearity.

The present analysis has been performed in the case of a very simple discrete system having only two degrees of freedom. Although it needed a careful understanding of the mechanics, the mathematical tools were classical and simple. But extending the results to the mechanics of structures may involve important mathematical difficulties.

In elastic buckling problems it is well known that the bifurcated branches can be either subcritical or supercritical, tending asymptotically to zero or to infinity depending on the geometrical non linearity. The idea of the introduction of the two simple models of geometrical non linearity was precisely to represent qualitatively these behaviors. But in the case of elastic-plasticity, the free boundary is no longer a point in an interval but a curve or a surface which implies huge additional complexities and the problem has been carried out only in the case of a straight beam [5] and [11], ... which corresponds in the Kirchhoff-Love kinematics to the geometrically linear case of the simple models that have been studied here !!

But although incomplete from the point of view of bifurcation in the mechanics of structures, the analysis presented here may have interesting engineering applications. As a matter of fact the important qualitative result

is that the asymptotic behavior is obtained without any free boundary analysis. This means that if a constant modulus is used, the Young modulus everywhere or the tangent modulus everywhere according to the behavior of the elastic buckling of the original T-shape structure, we obtain a qualitative information of the behavior at very large strains, which is a strong simplification at least as a first approximation.

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