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To cite this version:

Jimmy Lamboley, Pieralberto Sicbaldi. Existence and regularity of Faber-Krahn minimizers in a Riemannian manifold. 2020. hal-02283353v2

HAL Id: hal-02283353

https://hal.science/hal-02283353v2

Preprint submitted on 19 Mar 2020

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Existence and regularity of Faber-Krahn minimizers in a Riemannian manifold

Jimmy Lamboley[1], Pieralberto Sicbaldi[2]

Abstract

In this paper, we study the minimization of $\lambda_1(\Omega)$, the first Dirichlet eigenvalue of the Laplace-Beltrami operator, within the class of open sets Ω of fixed volume in a Riemmanian manifold (M, g). In the Euclidian setting (when $(M, g) = (\mathbb{R}^n, e)$), the well-known Faber-Krahn inequality asserts that the solution of such problem is any ball of suitable volume. Even if similar results are known or may be expected for Riemannian manifolds with symmetries, we cannot expect to find explicit solutions for general manifolds (M, g). In this paper we study existence and regularity properties for this spectral shape optimization problem in a Riemannian setting, in a similar fashion as for the isoperimetric problem. We first give an existence result in the context of compact Riemannian manifolds, and we discuss the case of non-compact manifolds by giving a counter-example to existence. We then focus on the regularity theory for this problem, and using the tools coming from the theory of free boundary problems, we show that solutions are smooth up to a possible residual set of co-dimension 5 or higher.

Résumé

Dans cet article on étudie le problème de minimiser $\lambda_1(\Omega)$, la première valeur propre de Dirichlet de l'opérateur de Laplace-Beltrami, parmi les ensembles ouverts Ω de volume fixé dans une variété riemannienne (M, g). Dans le cadre euclidien (quand $(M, g) = (\mathbb{R}^n, e)$), l'inégalité de Faber-Krahn afirme que les solutions de ce problème sont des boules. Même si des résultats similaires ont été démontrés, ou sont susceptibles d'être prouvés pour des variétés riemanniennes ayant des symétries, on ne peut pas s'attendre à pouvoir déterminer explicitement les solutions du problème pour des variétés générales (M, g). Dans cet article, on étudie des propriétés d'existence et de régularité pour ce problème spectral d'optimisation de forme dans le cadre riemannien, avec une approche similaire au problème isopérimétrique. En premier lieu, on donne un résultat d'existence dans le contexte d'une variété riemannienne compacte, et on discute le cas des variétés non compactes avec la construction d'un contre-exemple à l'existence. Ensuite, on se concentre sur la théorie de la régularité pour ce problème, et en utilisant des techniques de la théorie des problèmes à frontière libre, on prouve que les solutions sont régulières sauf dans un ensemble résiduel de codimension supérieure ou égale à 5.

Keywords: Shape optimization, Laplace-Beltrami operator, first eigenvalue, regularity of free boundaries, Riemannian manifold, Faber-Krahn profile, isoperimetric problems.

MCS codes: 35P05, 35R35, 49Q10, 49R50, 53C21, 74P20.

1. Introduction and main results

Let (M, g) be a smooth n-dimensional Riemannian manifold (without boundary), where $n \geq 2$. For all open subset Ω of M, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace-Beltrami operator Δ_g in Ω, with zero Dirichlet boundary conditions on $\partial\Omega$, that is

$$\lambda_1(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega \|\nabla^g u\|_g^2 \, \mathrm{dvol}_g}{\displaystyle\int_\Omega u^2 \, \mathrm{dvol}_g}, \quad u \in H_0^1(\Omega) \right\}, \tag{1}$$

where dvol_g, $\|\cdot\|_g$ and ∇^g represent respectively the volume form, the norm and the gradient, all with respect to the metric g. The Sobolev space $H_0^1(\Omega)$ also refers to the metric g. When Ω is smooth enough, we can characterize $\lambda_1(\Omega)$ by the existence of u_Ω such that

$$\begin{cases} \Delta_g u_\Omega + \lambda_1(\Omega)\, u_\Omega = 0 & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{with} \quad u_\Omega \geq 0 \quad \text{and} \quad \int_\Omega u_\Omega^2 \, \mathrm{dvol}_g = 1, \tag{2}$$

[1]Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, F-75005, Paris, France. E-mail: jimmy.lamboley@imj-prg.fr

[2]Universidad de Granada, Departamento de Geometría y Topología, Facultad de Ciencias, Campus Fuentenueva, 18071 Granada, Spain & Aix Marseille Université, CNRS, Centrale Marseille, I2M, Marseille, France. E-mail: pieralberto@ugr.es

and u_Ω is called the first normalized eigenfunction of the Laplace-Beltrami operator with Dirichlet boundary condition on $\partial\Omega$.

Let $\mathrm{Vol}_g(M)$ denote the volume of the Riemannian manifold M, that can be infinite. We are interested in the existence and the regularity of optimal sets for the following shape optimization problem: for any $m \in (0, \mathrm{Vol}_g(M))$, find an open subset $\Omega^* \subset M$ of volume m such that

$$\lambda_1(\Omega^*) = \min\{\lambda_1(\Omega);\ \Omega \text{ open subset of } M, \mathrm{Vol}_g(\Omega) = m\}. \tag{3}$$

The solutions Ω^* of such optimization problem are called *Faber-Krahn minimizers*, and the function

$$FK : m \in (0, \mathrm{Vol}_g(M)) \mapsto FK(m)$$

associating to m the value of the infimum in (3), is called the *Faber-Krahn profile* of the manifold (M, g).

This problem is inspired by the classical *isoperimetric problem*: for any $m \in (0, \mathrm{Vol}_g(M))$, find an open domain $\Omega \subset M$ whose boundary $\Sigma = \partial\Omega$ minimizes area among regions of volume m. The region $\overline{\Omega}$ and its boundary Σ are called *isoperimetric region* and *isoperimetric hypersurface* respectively. In the Euclidean space, Ω must be a ball by the standard isoperimetric inequality, and Σ is then a sphere. For a general Riemannian manifold this fact fails, and the shape of the optimal region can be very difficult to understand. Nevertheless, the following fundamental results about the existence and the regularity of isoperimetric regions are now very well-known: by the seminal papers of Almgren [2], Grüter [32], and Gonzalez, Massari, Tamanini [31], if M is a compact n-dimensional Riemannian manifold, then, for any positive $m < \mathrm{Vol}_g(M)$, there exists an open set $\Omega \subset M$ whose boundary Σ minimizes area among regions of volume m, and, except for a closed singular set of Hausdorff dimension at most $n-8$, Σ is a smooth embedded hypersurface with constant mean curvature. In particular, for dimensions of the ambient manifold less or equal to 7, the isoperimetric hypersurface Σ (that is an objet of dimension $n-1$, then less or equal to 6) is an embedded hypersurface. In fact, an isoperimetric hypersurface Σ has an area-minimizing tangent cone at each point, and if a tangent cone at $p \in \Sigma$ is an hyperplane, then p is a regular point of Σ. The value of the critical dimension (i.e. 8 if we consider the dimension of the ambient manifold, and 7 if we consider the dimension of the isoperimetric hypersurface) relies on the existence of the Simons cone in $\mathbb{R}^8$, i.e.

$$C = \{(x_1, ..., x_8) \in \mathbb{R}^8 \ \mid\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

which is a global minimizer of the area functional.

Coming back to the problem of finding Faber-Krahn minimizers, when the manifold is the Euclidean space it is well known that a ball Ω^* of volume $m > 0$ is a solution to the problem (3) (as for the isoperimetric problem), and in particular it exists and it is smooth. This fact follows from the Faber-Krahn inequality : for all open subset Ω of $\mathbb{R}^n$ whose volume is m, we have

$$\lambda_1(\Omega) \ \geq\ \lambda_1(B_m) \tag{4}$$

where B_m is a round ball in $\mathbb{R}^n$ with volume m; moreover equality holds in (4) if and only if $\Omega = B^m$ up to translation and to sets of 0 capacity. We point out that the proof of the Faber-Krahn inequality relies on the isoperimetric inequality.

When (M, g) is a general Riemannian manifold with no symmetry, we cannot expect to explicitely identify minimizers for problem (3), and very few results are known. Nevertheless, the Faber-Krahn profile and the isoperimetric profile are linked (see for example [18]), and starting from the analogy with the isoperimetric problem, in the following papers there is a construction of examples of domains that are critical for $\Omega \mapsto \lambda_1(\Omega)$ under volume constraint in some Riemannian manifolds, but it is not known a priori if such critical domains are or not Faber-Krahn minimizers: [44, 46, 47, 25, 42, 40]. Notice that when the manifold has no symmetry, the constructions of such examples is limited to small or big volumes. Moreover, all such examples have regular boundary.

In this paper, we are inspired by the existence and regularity results for the isoperimetric problem recalled before, and we plan to obtain similar results for the Faber-Krahn problem (3). Nevertheless, for the existence, it is now classical (see for example [12]) that one cannot expect to prove directly that there exists an open set solution of (3). Indeed, such a result (first part of Theorem 1.2, which will be proven in Section 4.3) is already a regularity result for an optimal set; the reason is that the class of open sets does not satisfy any suitable compactness property for our problem. The usual way to overcome this difficulty is to relax our minimization problem in the class of *quasi-open sets*, relying on the notion of capacity. Basically, quasi-open sets are level sets of functions of $H^1(M)$, which happen to not be necessarily continuous, so their level sets aren't necessarily open (however, any open set is quasi-open); for more details on the study of capacity and quasi-open sets, see for example [28, 34]. If $\Omega \subset M$ is a quasi-open set, we can define

$$H_0^1(\Omega) = \{u \in H^1(M),\ u = 0 \ q.e. \text{ on } M \setminus \Omega\}. \tag{5}$$

where $q.e.$ means quasi-everywhere, which means everywhere except on a set of capacity 0 (see Section 2 for the definition of the capacity in the Riemannian setting). This definition retrieves the usual definition of $H_0^1(\Omega)$ when Ω is an open set, namely the closure of $C_c^\infty(\Omega)$ for the H^1-norm. Once we have a definition of the space $H_0^1(\Omega)$, definition

(1) can be applied to define $\lambda_1(\Omega)$ for any quasi-open set Ω, and it is classical that equation (2) has a meaning in the weak sense, in particular $u_\Omega \in H^1_0(\Omega)$ solution to (2) exists and is unique.

For the regularity of Faber-Krahn minimizers, as it happens for the isoperimetric hypersurfaces, we will prove that there exists a critical dimension k^* for which Faber-Krahn minimizers are regular if $n < k^*$, and singularities can appear starting from the dimension $n = k^*$. In order to define this critical dimension k^*, we need to recall notion of homogeneous global minimizer of the Alt-Caffarelli functional in $\mathbb{R}^n$, that is a homogeneous function $u_0 \in H^1_{loc}(\mathbb{R}^n)$ such that:

$$\int_{B_R(0)} |\nabla^e u_0|^2 + \mathrm{Vol}_e(\{u_0 > 0\} \cap B_R(0)) \leq \int_{B_R(0)} |\nabla^e w|^2 + \mathrm{Vol}_e(\{w > 0\} \cap B_R(0)). \tag{6}$$

for every $R > 0$ and $w \in H^1_{loc}(\mathbb{R}^n)$ such that $w = u_0$ outside $B_R(0)$ (the Euclidean ball of radius R and center 0), where ∇^e is the Euclidean gradient and Vol_e is the Lebesgue measure in $\mathbb{R}^n$. We will see that the Alf-Caffarelli functional plays in our problem the same role as the area functional in the isoperimetric problem, and k^* can be defined as the smallest integer such that there exists a non-trivial homogeneous global minimizer of the Alt-Caffarelli function. Unfortunately, finding the exact value of k^* is still a difficult open problem. Nevertheless, thanks to the important results by Caffarelli, Jerison, Kenig [17], De Silva, Jerison [24], and Jerison, Savin [38] we know such dimension belongs to the set $\{5, 6, 7\}$.

We are now in position to state our main results. The first one is the following:

Theorem 1.1. *(Existence) If M is compact and $m \in (0, \mathrm{Vol}_g(M))$, then there exists a quasi-open set Ω^* solution of*

$$\lambda_1(\Omega^*) = \min\{\lambda_1(\Omega);\ \Omega \text{ quasi-open subset of } M, \mathrm{Vol}_g(\Omega) = m\}. \tag{7}$$

Notice that in this result we need the compactness of the manifold M. In Section 3.2, we discuss the compactness hypothesis, and we exhibit a non-compact manifold M so that for any parameter $m > 0$, problems (3) and (7) do not have solutions. Compactness in not required to state the regularity result about Faber-Krahn minimizers. Nevertheless we need an other important topological assumption that is the connectedness (the discussion of the connectedness hypothesis of the manifold M is done in Remark 1.4 that follows). Our second main result is then the following:

Theorem 1.2. *(Regularity) Let Ω^* be a solution of (7) for $m \in (0, \mathrm{Vol}_g(M))$, and assume M is connected. Let k^* be the lowest dimension k such that there exists a non-trivial homogeneous global minimizer of the Alt-Caffarelli functional in $\mathbb{R}^k$ (it is known that $k^* \in [\![5, 7]\!]$). Then:*

1. *Ω^* is open (and therefore solves (3)) and has finite perimeter in M.*
2. *We can decompose $\partial\Omega^* = \Sigma_{reg} \cup \Sigma_{sing}$ in two disjoint sets such that:*
   (a) *Σ_{reg} is relatively open in $\partial\Omega$ and is a smooth hypersurface in M (C^∞ if M is C^∞, analytic if M is analytic),*
   (b) *we have:*
   - *if $n < k^*$, then $\Sigma_{sing} = \emptyset$,*
   - *if $n = k^*$, then Σ_{sing} is made of isolated points,*
   - *if $n > k^*$, then $dim_{\mathcal{H}}(\Sigma_{sing}) \leq n - k^*$, i.e.*

$$\forall s > n - k^*, \mathcal{H}^s(\Sigma_{sing}) = 0. \tag{8}$$

Combining these two results, we get that for any n-dimensional connected and compact Riemannian manifold M, one can find an open set Ω^* solution of (3), which is C^∞, up to a singular set of dimension less than $n - 5$.

Remark 1.3. *Notice that the first eigenvalue of a quasi-open Ω does not change if one modifies the set Ω with a set of zero-capacity (see Section 2 for definitions); as usual, the topological boundary $\partial\Omega$ here refers to the measure theoretic boundary, which is equal in this context to the set*

$$\partial\Omega = \{x \in M,\ \ \forall r > 0 \text{ small enough},\ 0 < \mathrm{Vol}(\Omega \cap B_r(x)) < \mathrm{Vol}(B_r(x))\}$$

where $B_r(x)$ denote the euclidian ball centered in x and with radius $r > 0$ smaller than the injectivity radius at x.

Remark 1.4. Without connectedness assumption for the manifold M, regularity of a minimizer may fail. Consider M to be the union of two disjoint copies of unit spheres $\mathbb{S}^n_1 \cup \mathbb{S}^n_2$ endowed with its usual metric g; for

$$m \in (\mathrm{Vol}_g(\mathbb{S}^n), 2\mathrm{Vol}_g(\mathbb{S}^n) = \mathrm{Vol}_g(M)),$$

any set of the form $\Omega = \mathbb{S}^n_1 \cup \omega$ where ω is any quasi-open subset of $\mathbb{S}^n_2$ of volume $m - \mathrm{Vol}_g(\mathbb{S}^n)$ is a solution to (7) because $\lambda_1(\mathbb{S}^n_1 \cup \omega) = \lambda_1(\mathbb{S}^n_1) < \lambda_1(\omega)$, and to (3) if ω is open, so one cannot expect any regularity property. Nevertheless, by replacing ω by any other smooth set of same volume, it is not hard to see that there still exists a smooth solution to (3), see also [11, Appendix].

Let us discuss the strategy for proving these results, and their relation to the state of the art.

About Theorem 1.1. In the Euclidian setting, while the ball is known to be a solution, the problem retrieves its interest if one consider an extra "box constraint" of the form $\Omega \subset D$ where D is an open and bounded subset of $\mathbb{R}^n$. In this setting, existence results were obtain for problem (3) with two different strategies in [33] and [10]. The main difficulty for these results is to obtain a solution that is an open set. We focus first only on proving that there exists a quasi-open set solution to (7); the fact that solutions are open will be dealt with in Theorem 1.2.

Similarly to [10], we use the variational formulation of $\lambda_1(\Omega)$ to show that problem (7) is equivalent to solving a free boundary problem (namely (12)), which is a calculus of variation problem consisting in the minimization of an energy $J(u)$ involving the level set $\Omega_u = \{u \neq 0\}$, among functions $u \in H^1(M)$. Considering a solution u to this free boundary problem, the set Ω_u will be a solution to (7). Once we obtain a free boundary formulation, one can use the same strategy as in the seminal paper [3] of Alt and Caffarelli (see below for more details) to prove existence of a solution, which relies on classical tools of calculus of variation.

This strategy may fail if the manifold M is not compact, and, as we said before, we give in Section 3 an explicit example of manifold M so that (3) has no solution. In order to exhibit such a manifold, we will need two essential properties:

- the Faber-Krahn profile of the manifold (M, g) is strictly bounded from below by the Faber-Krahn profile of the euclidian space $(\mathbb{R}^n, e)$; this happens to be true if the same is valid for the isoperimetric profile, see Proposition 3.6;
- M is asymptotically Euclidian in the sense that a geodesic ball in M converging to infinity is a smooth perturbation of a euclidian ball of same volume.

We show that both properties are valid when M is the usual catenoid in $\mathbb{R}^3$, which provides the expected counter-example to existence of Faber-Krahn minimizers (Theorem 3.4).

About Theorem 1.2. Regularity results for such kind of shape optimization problems are quite involved and will rely on many steps that we will describe in the introduction of Section 4. Similarly to the existence result, we start with formulation (11) and the definition of the functional J in (12) introduced in Section 3, and we are naturally led to the field of "regularity of free boundaries". In order to understand our strategy, let us start by commenting on the extensive litterature on this topic: the regularity theory for such problems was initiated in [3], where the authors study the regularity of the free boundary for

$$\min \left\{ \int_D |\nabla^e v|^2 + \gamma \mathrm{Vol}_e(\{v > 0\}) \; ; \; v \in H^1(D), v = u_0 \in \partial D \right\}, \tag{9}$$

where $u_0 \geq 0$ is given and D is an open bounded set in $\mathbb{R}^n$. In particular, the authors show that an optimal solution v is locally Lipschitz continuous inside D, which is the optimal regularity one can expect for v, and implies in particular that $\{v > 0\}$ is an open set. They show then that the free boundary $\partial\{v > 0\} \cap D$ can be decomposed into a smooth (analytic) part Σ_{reg} where one can write the classical optimality condition $|\nabla^e v|^2_{|\Sigma_{reg}} = \gamma$, and a (possibly) singular part Σ_{sing} which is small in the sense that $\mathcal{H}^{n-1}(\Sigma_{sing}) = 0$ (where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure); they also show that in fact $\Sigma_{sing} = \emptyset$ if $n = 2$. These results have been improved by Weiss in [53] who introduced a monotonicity formula to study blow-up limits, which combined to the study of global homogeneous minimizer of the Alt-Caffarelli functional lead to the estimate $\dim_{\mathcal{H}}(\Sigma_{sing}) \leq n - 5$ if $n \geq 5$, and $\Sigma_{sing} = \emptyset$ if $n < 5$.

We will apply a similar strategy for solutions to (11)-(12) (which are the free boundary formulations of our intial problem (7)), with three main differences that we need to take into account:

- deal with the term $\int_M w^2$ in $J(w)$, coming from the fact that we are dealing with an eigenvalue problem,
- deal with the Riemannian metric g; if $g_{ij}(x)$ is the matrix of the coefficients of the metric g in some suitable local coordinates system, from the PDE point of view to deal with g replaces the Euclidian Laplace operator by an operator of the form
$$h \, \mathrm{div}(A \, \nabla^e \cdot)$$
where $h(x) = \frac{1}{\sqrt{|g_{ij}(x)|}}$ and $A = g^{ij}(x) \sqrt{|g_{ij}(x)|}$, being $|g_{ij}(x)|$ and $g^{ij}(x)$ respectively the determinant of the matrix $g_{ij}(x)$ and the inverse matrix of the matrix $g_{ij}(x)$.
- handle the volume constraint instead of a penalization of the volume as in (9).

Let us mention a few important contributions in similar developements. In [11], T. Briançon and the first author of the present paper managed to overcome the first and third difficulties in the Euclidian setting. Note however that they only adapted the result by Alt and Caffarelli and did not adapt the improvement given by Weiss, therefore the estimate of the singular set they obtain was not optimal. In other words, even in the Euclidian setting, Theorem 1.2 improves the results in [11] (of course, one could argue that solutions are Euclidian balls in this context, but as in [11], one can consider a

box constraint of the type $\Omega \subset D$ where $D \subset M$, so that it may happen that balls are not admissible sets; even if we did not take into account this constraint in the current paper, when we are concerned with the regularity inside the box (far from ∂D), since all argument are local, Theorem 1.2 remains valid in this case).

In [51], A. Wagner did study the first steps of the strategy from [3] for a problem similar to (12) (in the Euclidian setting but with an operator of the form $\mathrm{div}(A\nabla\cdot)$). He studies a penalized version of the problem, in a similar fashion to [1], which leads to the existence of a solution to (3) (in particular, it is an open set), enjoying some density estimates and a weak formulation of the optimality condition (named "weak solutions" in [3]). Again, our result is an improvment in the sense that we show that every solution is an open set, and we improve their regularity properties.

More recently, in [21, 20], the authors develop a regularity theory for quasi-minimizers of the Alt-Caffarelli functional, including the improvment given by the Weiss-monotonicity formula (therefore, leading to a similar regularity as in Theorem 1.2). However, it is not true that minimizers we are interested in are quasi-minimizers in the sense of [20], as one cannot see the Laplace-Beltrami operator as a small deformation of the Euclidian Laplacian. A similar regularity theory for $\mathrm{div}(A\nabla\cdot)$ operators has been started in [22], though they only deal with the first step of the strategy in proving that optimal solutions are Hölder-continuous in the general case. But even if a similar regularity theory was valid for such elliptic operator, it would still remain the difficulty to prove that a solution to (12) is a quasi-minimizer in the sense of [20], the main difficulty here being to handle the volume constraint. This seems to be a significant and important open problem.

The last contributions we would like to mention are [41, 45] which were a strong inspiration for our work. The results in [45] are similar to ours in the sense that they extend results for the Euclidean Laplace operator to a more general class, though the author deal with a drifted operator of the form $-\Delta + \nabla\Phi \cdot \nabla$. Nevertheless, as we have to deal with a Laplace-Beltrami operator, several steps and ideas differ from the current paper.

Description of the paper. In the following section, we introduce the Riemannian setting of our problem. In the third section we introduce the main free boundary formulation which is in some sense equivalent to our shape optimization problem (7), and we use this formulation to prove Theorem 1.1. We also exhibit a non-compact manifold leading to a non-existence phenomenon. Finally, in Section 4, we prove Theorem 1.2.

2. Basic Riemannian notations

Since one of the goals of this paper is to bring the regularity theory for free boundary problems to Riemannian manifold, and then to mix together the geometric and the analytic language, we think it can be convenient for the reader to fix the basic Riemannian notation. For a more complete presentation see [18]. Let M be a Riemannian manifold with metric g. If $p \in M$ and f is a C^1 real function defined in a neighborhood of p, we will denote by $\nabla^g f(p)$ the gradient of f at p, i.e. the only vector of the tangent space T_pM such that

$$g(\nabla^g f(p), \nu_p) = \nu_p f$$

for every vector $\nu_p \in T_pM$, where $\nu_p f$ is the directional derivative of f at p in the direction ν_p; $\nabla^g f$ will be the gradient vector field, i.e. $\nabla^g f \in TM$, where TM is the tangent bundle of M. If $\nabla_\nu X$ is the covariant derivative of a vector field X on the manifold M with respect to $\nu \in TM$, the divergence of X is defined as

$$\mathrm{div}_g X = \mathrm{trace}(\nu \to \nabla_\nu X),$$

and the generalization of the Laplacian on a manifold, known as the Laplace-Beltrami operator, is defined by

$$\Delta_g f = \mathrm{div}_g(\nabla^g f),$$

where f is supposed to be of class C^2. Let U be an open set of M and $\phi : U \to \mathbb{R}^n$ a chart on M, i.e. a diffeomorphism of U into $\mathbb{R}^n$. If $(x^1, ..., x^n)$ are the local coordinates and $\frac{\partial}{\partial x^i}$ are the coordinate vector fields, $i = 1, ..., n$, we can define the matrix

$$G = (g_{ij})_{i,j=1,\dots,n}$$

where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are the coefficients of the metric g, which can be written as

$$g = g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j$$

using the Einstein summation convention. We denote by $|g|$ the determinant of G and by g^{ij} the coefficients of G^{-1}, the inverse matrix of G. Straightforward computations show:

$$\nabla^g f = g^{ij}\, \partial_j f$$

and

$$\Delta_g f = \frac{1}{\sqrt{|g|}}\, \partial_i \left(g^{ij}\, \sqrt{|g|}\, \partial_j f\right) = g^{ij}\, \partial_i \partial_j f + \partial_i\, g^{ij}\, \partial_j f + \frac{1}{2}\, g^{ij}\, \partial_i (\log |g|)\, \partial_j f$$

where ∂_j denotes the standard derivation with respect to x^j. We will denote dvol_g the Riemannian measure associate to g, that in U is

$$\mathrm{dvol}_g = \sqrt{|g|}\,\mathrm{d}x$$

where $\mathrm{d}x$ is the Lebesgue measure on $\phi(U)$. More generally, if $\{\phi_i : U_i \to \mathbb{R}^n\}$ is a chart covering of M, with an associate partition of unity $\{b_i\}$, we have

$$\mathrm{dvol}_g = \sum_i b_i \sqrt{|g|}\,\mathrm{d}x$$

in every chart. A function $f : M \to \mathbb{R}$ is measurable if it is measurable in every chart, and the definition of dvol_g makes possible the integration of such functions. A set $K \subset M$ is measurable if it is measurable in every chart, and its volume is given by

$$\mathrm{Vol}_g(K) = \int_K \mathrm{dvol}_g\,.$$

The L^p-space on M, denoted as $L^p(M)$, the Sobolev spaces on M (in particular $H^1(M)$) and the distribution space on M, $\mathcal{D}'(M)$, are defined in the same way as in the Euclidean space, but using the Riemannian measure, and g as the scalar product on the tangent bundle. If $\Omega \subset M$, the *capacity* of Ω is given by

$$\mathrm{cap}_g(\Omega) = \inf\ \{\|u\|^2_{H^1}\ \mid\ u \in H^1(M)\ ,\ \ u \geq 1\ \ a.e.\ \text{ in a neighborhood of }\ \Omega\}$$

where $a.e.$ means *almost everywhere* with respect to the measure dvol_g. The notion of capacity plays an important role in the definition of the space $H^1_0(\Omega)$, see definition (5). We say that a property holds *quasi-everywhere (q.e.)* if it holds on the complementary of a set of zero capacity. A set $\Omega \subset M$ is *quasi-open* if there exists a decreasing sequence ω_i of open sets such that, for every $\Omega \cup \omega_i$ is open $\lim_{i\to\infty} \mathrm{cap}_g(\omega_i) = 0$. A function $u : M \to \mathbb{R}$ is *quasi-continuous* if there exists a decreasing sequence ω_i of open sets such that the restriction of u to the complement of ω_i is continuous, and $\lim_{i\to\infty} \mathrm{cap}_g(\omega_i) = 0$. Every function in $H^1(M)$ has a quasi-continuous representative which is unique up to a set of zero capacity, and in this paper we will identify every function in $H^1(M)$ with one of its quasi-continuous representative (see Section 4.1 for more details about the choice of the right representative). If $u \in H^1(M)$, the set $\{u > 0\}$ is quasi-open and for a quasi-open set Ω, there exists a function $u \in H^1(M)$ such that $\Omega = \{u > 0\}$ up to a set of zero capacity. We refer to [14, 28, 13, 34] for more details about capacity, quasi-open sets and quasi-continuous functions, and remark that the Riemannian metric does not change the basic theory about such topics.

We will denote by $d_g(p, q)$ the distance between two points $p, q \in M$, i.e. the infimum of

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\,\mathrm{d}t$$

taken over all continuous and piecewise C^1 curve $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. For every $p \in M$ and every vector $\nu \in T_pM$ there exists a unique maximal geodesic $\gamma_{p,\nu}$ satisfying $\gamma_{p,\nu}(0) = p$ and $\gamma'_{p,\nu}(0) = \nu$ (here I is an open interval containing the origin, and is maximal with respect to the existence of $\gamma_{p,\nu}$). The injectivity radius at a point p will be

$$i_p = \inf_{\nu\in\mathcal{S}} c(\nu)$$

where $\mathcal{S}$ denotes the unit sphere in T_pM, and

$$c(\nu) = \sup\{t \in I\ \mid\ d(p, \gamma_{p,\nu}(t)) = t\}$$

The injectivity radius of the manifold is $i_M = \inf_p i_p$, that is positive if the manifold is compact. We will denote by $\exp_p : T_pM \to M$ the exponential map, that is given by

$$\exp_p(\nu) = \gamma_{p,\nu}(1)$$

if $\gamma_{p,\nu}(1)$ exists. We notice that $\gamma_{p,\nu}(t) = \exp_p(t\,\nu)$. If r is less than the injectivity radius at p, then we can have a geodesic ball of radius r and center p, that in the following will be denoted by

$$B^g_r(p) = \{q \in M\ \mid\ d(p, q) \leq r\} = \{\exp_p(\nu)\ \mid\ \|\nu\|_g < r\}\,.$$

For vector fields X, Y, Z on M we denote by

$$R(X, Y)Z = \nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) - \nabla_{[Y,X]}Z$$

the Riemann curvature tensor, where $[Y, X]$ is the vector field

$$[Y, X] = \nabla_Y X - \nabla_X Y\,.$$

If $\nu_1, \nu_2 \in T_pM$, we will denote by $\mathrm{Rm}_p(\nu_1, \nu_2)$ the sectional curvature at p of the 2-plane determined by ν_1, ν_2, i.e.

$$\mathrm{Rm}_p(\nu_1, \nu_2) = \frac{g(R(\nu_1, \nu_2)\nu_1, \nu_2)}{\|\nu_1\|_g\,\|\nu_2\|_g - g(\nu_1, \nu_2)}\,.$$

On the other hand, we will denote by Ric_p the Ricci tensor at p, i.e.

$$\mathrm{Ric}_p(\nu_1,\nu_2)=\sum_{i=1}^n g(R(\nu_1,e_1)\nu_2,e_2)$$

where $\nu_1,\nu_2\in T_pM$ and $\{e_1,e_2,...,e_n\}$ is an orthonormal basis of T_pM. The function $\mathrm{Ric}_p(\nu,\nu)$ on the set of tangent vectors ν of length 1, that we will write $\mathrm{Ric}_p(\nu)$, is the Ricci curvature at p. If S is an $(n-1)$-submanifold of M and ν is the normal vector to S, we recall that the second fundamental form on S is given by

$$A(\xi_1,\xi_2)=(\nabla_{\xi_1}\xi_2)^\nu$$

where $\xi_1,\xi_2\in TS$ and the superscript ν indicated that we consider only the component on the direction ν. We will denote by H_S the mean curvature of S, i.e. the trace of the second fundamental form.

3. Existence

3.1. The compact case

In this section, (M,g) is a Riemannian manifold. We start by giving some free boundary formulations of a relaxed version of (3), where we replace the volume constraint by an inequality constraint:

Proposition 3.1. *If Ω^* is a solution to*

$$\lambda_1(\Omega^*)=\min\Big\{\lambda_1(\Omega);\ \Omega\textit{ quasi-open subset of }M,\mathrm{Vol}_g(\Omega)\le m\Big\},\tag{10}$$

then u_{Ω^*} (solution of (1)*) is solution of*

$$\int_M\|\nabla^g u_{\Omega^*}\|_g^2\,\mathrm{dvol}_g=\min\left\{\int_M\|\nabla^g w\|_g^2\,\mathrm{dvol}_g\,,\ w\in H^1(M),\ \int_M w^2\,\mathrm{dvol}_g=1,\ \mathrm{Vol}_g(\Omega_w)\le m\right\},\tag{11}$$

and is also solution of

$$J(u_{\Omega^*})=\min\left\{J(w),\ w\in H^1(M),\ \text{with }\mathrm{Vol}_g(\Omega_w)\le m\right\},\tag{12}$$

where we denote

$$J(w):=\int_M\|\nabla^g w\|_g^2\,\mathrm{dvol}_g-\lambda(m)\int_M w^2\,\mathrm{dvol}_g,\quad \lambda(m):=\inf\Big\{\lambda_1(\Omega);\ \Omega\textit{ quasi-open subset of }M,\mathrm{Vol}_g(\Omega)\le m\Big\}$$

$$\text{and}\qquad \Omega_w=\{w\neq 0\}.$$

Reciprocally, if u solves (11) *or* (12)*, then the set $\Omega_u=\{u\neq 0\}$ is a solution to* (10)*.*

Proof. Let Ω^* be a solution to (10). The fact that u_{Ω^*} solves (11) comes easily using the variational formulation (1). In order to see that u_{Ω^*} is also solution of (12), we apply (11) to $\frac{w}{\|w\|_2}$, leading to $J(w)\ge 0$ for all $w\in H^1(M)$ such that $\mathrm{Vol}_g(\Omega_w)\le m$, and we conclude noticing that $J(u_{\Omega^*})=0$.

Let now u be a solution to (11) (in the case where u is assumed to solve (12), it is clear that u then also solves (11)). Then given Ω a quasi-open subset of M with $\mathrm{Vol}_g(\Omega)\le m$, optimizing (11) in $w\in H_0^1(\Omega)$ such that $\|w\|_{L^2(\Omega)}=1$ gives that

$$\lambda_1(\Omega_u)=\int_M\|\nabla^g u\|_g^2\,\mathrm{dvol}_g\le\lambda_1(\Omega)$$

which concludes the proof. □

The next result deals with the question of saturation of the constraint, and explains how one can link solutions of (12) with solutions of (7). This will be used in Section 4.

Proposition 3.2 (Saturation of the constraint). *We assume M to be connected. If u is a solution to* (12)*, then $\mathrm{Vol}_g(\Omega_u)=m$. Equivalently, if Ω^* is a solution to* (10)*, then $\mathrm{Vol}_g(\Omega^*)=m$ and $\Omega^*=\{u_{\Omega^*}\neq 0\}$.*

Proof. Let u be a solution of (12). Assume to the contrary that $\mathrm{Vol}_g(\Omega_u)<m$; then $w=u+t\varphi$ is admissible in (12) for any φ smooth with small support, and $|t|$ small. It implies

$$0=\frac{dJ(u+t\varphi)}{dt}_{|t=0}=2\int_M g(\nabla^g u,\nabla^g\varphi)\,\mathrm{dvol}_g-2\lambda(m)\int_M u\varphi\,\mathrm{dvol}_g$$

which means $-\Delta_g u=\lambda(m)u$ in M in the sense of distribution. Since u has constant sign (see for example [10, Remark 2.10]), say nonnegative, from strong maximum principle and connectedness of M, we get $u>0$ on M, which contradicts $m<\mathrm{Vol}_g(M)$ (in fact, in this case u is constant because $\lambda(m)$ would be the first eigenvalue of M, which is simple).

If now Ω^* solves (10), then u_{Ω^*} solves (12) and has constant sign, say nonnegative. From the strong maximum principle, $u_{\Omega^*}>0$ on Ω^*, and from the previous point, $\mathrm{Vol}(u_{\Omega^*})=m$, which concludes the proof. □

Using Proposition 3.1, one can deduce existence of a solution for (7):

Proof of Theorem 1.1: The most convenient formulation for existence is (11): it is clear that the set of admissible functions is nonempty and that the infimum in (11) is nonnegative. Taking any minimizing sequence $(u_k)_{k\in\mathbb{N}}$, and using that it is bounded in $H^1(M)$, we get that there exists $u^* \in H^1(M)$ such that, up to a subsequence:

$$\nabla^g u_k \rightharpoonup \nabla^g u^* \quad \text{weakly in } L^2(M), \quad u_k \to u^* \text{ a.e. and strongly in } L^2(M),$$

(we use the weak-compactness of closed bounded convex sets in the Hilbert space $H^1(M)$ and the Rellich-Kondrachov Theorem about the compact embedding $H^1(M) \hookrightarrow L^2(M)$ when M is compact, see Theorem 2.34 in [5]). We therefore get

$$\mathrm{Vol}_g(\Omega_{u^*}) \leq \liminf_{k\to\infty} \mathrm{Vol}_g(\Omega_{u_k}), \qquad \int_M {u^*}^2 \,\mathrm{dvol}_g = 1$$

$$\text{and} \qquad \int_M \|\nabla^g u^*\|_g^2 \,\mathrm{dvol}_g \leq \liminf_{k\to\infty} \int_M \|\nabla^g u_k\|_g^2 \,\mathrm{dvol}_g$$

so u^* solves (11), and from Proposition 3.1, the set $\Omega^* = \{u^* \neq 0\}$ solves (10). In order to build a solution to (7), we simply check that there exists a quasi-open set $\widetilde{\Omega^*} \supset \Omega^*$ such that $\mathrm{Vol}_g(\widetilde{\Omega^*}) = m$. From monotonicity, $\lambda_1(\widetilde{\Omega^*}) \leq \lambda_1(\Omega^*)$ and therefore $\widetilde{\Omega^*}$ solves (7). □

Remark 3.3. Let us notice that it is possible to adapt to the Riemannian setting some more general existence result. Using the method in [36, Chapter 2, Theorem 2.1] one can prove existence for

$$\min\{f(\lambda_1(\Omega), \ldots, \lambda_k(\Omega)), \ \ \Omega \text{ quasi-open } \subset M, \ \mathrm{Vol}_g(\Omega) = m\},$$

where f is lower semi-continuous and non-decreasing in each variable, M is compact, and $\lambda_1(\Omega) \leq \ldots \leq \lambda_k(\Omega)$ are the k first eigenvalues of the Laplace-Beltrami operator with Dirichlet boundary conditions.

Even more generally, it is also possible to adapt the theory of Buttazzo and Dal Maso (see [14]) to the case of a compact Riemannian manifold: one obtains that there exists a solution of the minimization problem:

$$\min\{F(\Omega), \ \ \Omega \text{ quasi-open } \subset M, \ \mathrm{Vol}_g(\Omega) = m\}$$

if M is a compact Riemannian manifold, F is a shape functional defined on quasi-open subsets of M, decreasing with respect to set inclusion, and lower-semicontinuous for a suitable topology (namely γ-convergence, see [14]). For more details we refer to [15].

3.2. The non compact case: counter-example to existence

Through all this subsection, M will be the minimal catenoid in $\mathbb{R}^3$, i.e. the surface given by the equation

$$x^2 + y^2 = \lambda^2 \cosh^2\left(\frac{z}{\lambda}\right) \tag{13}$$

for some $\lambda > 0$, where (x, y, z) are the coordinates of $\mathbb{R}^3$. It is the surface of revolution generated by a catenary, i.e. the curve

$$s \to \lambda\left(\cosh\left(\frac{s}{\lambda}\right), s\right) .$$

We consider the natural metric g on M, i.e. the metric induced by the immersion of M in $\mathbb{R}^3$. It is useful to express the metric of this surface in the form

$$\mathrm{d}s^2 = (f(t))^2 \mathrm{d}\theta^2 + \mathrm{d}t^2$$

for some function f. For this we need the length parameterization of the catenary, i.e.

$$t \to \left(\sqrt{t^2 + \lambda^2}, \lambda \,\mathrm{arcsinh}\left(\frac{t}{\lambda}\right)\right) .$$

The catenoid is then the image of the annulus $S^1 \times \mathbb{R}$ in $\mathbb{R}^3$ by the application

$$F(\theta, t) \to \left(\sqrt{t^2 + \lambda^2} \cos\theta \,,\, \sqrt{t^2 + \lambda^2} \sin\theta \,,\, \lambda \,\mathrm{arcsinh}\left(\frac{t}{\lambda}\right)\right)$$

and the metric is given by

$$\mathrm{d}s^2 = \left\langle \frac{\partial F}{\partial\theta}, \frac{\partial F}{\partial\theta} \right\rangle \mathrm{d}\theta^2 + \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle \mathrm{d}t^2 + 2\left\langle \frac{\partial F}{\partial\theta}, \frac{\partial F}{\partial t} \right\rangle \mathrm{d}\theta\,\mathrm{d}t = (t^2 + \lambda^2)\,\mathrm{d}\theta^2 + \mathrm{d}t^2$$

In this section we prove the following result:

Theorem 3.4. *For any volume $m > 0$, problems* (3) *or* (7) *have no solution when M is the minimal catenoid defined in* (13).

The proof of Theorem 3.4 follows from the following Propositions 3.6 and 3.7. We start by recalling a well known fact about the isoperimetric problem in the minimal catenoid.

Proposition 3.5. *([16]). Let $m > 0$. Then, for all regular open set Ω in M whose volume is equal to m we have*

$$P(\Omega) > P(B_m)$$

where B_m is a Euclidean ball of volume m, and P represents the perimeter, i.e. the area of the boundary of the open set.

The Faber-Krahn inequality allows to obtain the same result for the Faber-Krahn profile.

Proposition 3.6. *Let $m > 0$. Then, for all regular open set Ω in M whose volume is m we have*

$$\lambda_1(\Omega) > \lambda_1(B_m)$$

where B_m is a Euclidean ball of volume m.

Proof. By the Faber-Krahn Theorem (see [18], chapter IV, section 2, Theorem 2 and Remark 1), if for all $m > 0$ the inequality

$$P(\Omega) \geq P(B_m)$$

holds for every regular open set Ω with equality if and only if Ω is isometric to B_m, then inequality

$$\lambda_1(\Omega) \geq \lambda_1(B_m) \tag{14}$$

holds for every open set (non necessarily regular) Ω with equality if and only if Ω is isometric to B_m. The result follows then from Proposition 3.5 because if there were in M a regular open set Ω isometric to B_m then we would have $P(\Omega) = P(B_m)$ □

Note that in the previous proof, the fact that there does not exist in M a regular open set Ω isometric to B_m follows also from the Egregium Theorem because the Gauss curvature of M is negative).

Now we want to show that in M there does not exist any minimizer of $\Omega \mapsto \lambda_1(\Omega)$ under volume constraint. In order to do that, we will show that there exists a minimizing sequence of domains with the same volume whose first eigenvalue converges to the first eigenvalue of a Euclidean ball. We denote by B_r the ball of radius r in $\mathbb{R}^3$, and e will represents the Euclidean metric as usual. We will need the following:

Proposition 3.7. *For any $m > 0$, there exists a sequence $B^g_{r_j}(p_j)$, $j \in \mathbb{N}$ such that*

1. $\mathrm{Vol}_g(B^g_{r_j}(p_j)) = m$ *for all $j \in \mathbb{N}$, and*
2. $\lambda_1(B^g_{r_j}(p_j)) \to \lambda_1(B_1)$ *for $j \to +\infty$.*

Proof. It is convenient to consider just the superior part M^+ of the catenoid (i.e. the part with $t > 0$). By changing the coordinates as

$$t^2 = x^2 + y^2 \qquad \theta = \arctan \frac{y}{x}$$

(remember that $t > 0$), M^+ can be seen as $\mathbb{R}^2$ with the metric

$$g = dx^2 + dy^2 + \frac{\lambda^2}{(x^2+y^2)^2}\,(xdy - ydx)^2$$

If $p = (x_p, y_p) \in M^+$ and we take a chart centered at p, the metric writes

$$g = dx^2 + dy^2 + \frac{\lambda^2}{\left[(x+x_p)^2 + (y+y_p)^2\right]^2}\left[(x+x_p)dy - (y+y_p)dx\right]^2$$

and it is clear that if $\|p\|_e = \varepsilon^{-1}$, being $\|\cdot\|_e$ the Euclidean norm in $\mathbb{R}^3$, we have

$$g = dx^2 + dy^2 + \mathcal{O}(\varepsilon^2)(dx^2 + dxdy + dy^2) \tag{15}$$

for ε small enough, uniformly in a ball B_r for a fixed value r.

Now let $m > 0$. Let R be the radius of the Euclidean ball of $\mathbb{R}^2$ of volume m. We claim that M^+ contains a geodesic ball $B^g_{r_q}(q)$ of volume m, and for every point $p \in M^+$ with $\|p\|_e \geq \|q\|_e$ there exists $r_p > 0$ such that the geodesic ball $B^g_{r_p}(p)$ is contained in M^+ and has volume m. Moreover we have that $r_p \to R$ when $\|p\|_e \to +\infty$.

Proof of the claim. If we take a point $q = (x_q, y_q, z_q) \in M^+$ (considering the catenoid in the form (13)), the intersection of M^+ with the vertical cylinder of radius $\sqrt{x_q^2 + y_q^2} - 1$ and axis $(x, y) = (x_q, y_q)$ is a graph on a 2-dimensional ball of radius $\sqrt{x_q^2 + y_q^2} - 1$, and its volume tends to ∞ when $\|q\|_e \to \infty$. In particular, if $\|q\|_e$ is big enough, it contains a geodesic ball $B^g_{r_q}(q)$ of volume m, and this is true also if we replace q by a point p such that $\|p\|_e \geq \|q\|_e$. Now, if we take $\|p\|_e = \varepsilon^{-1}$, it is clear that on a chart centered at p with radius $2R$ we have the estimation (15) uniformly in ε, i.e. g converges to the Euclidean metric uniformly in a chart of radius $2R$, and then $r_p \to R$.

Now, fix $\varepsilon > 0$ small, take $\|p\|_e \in M^+$ such that $\|p\| = \varepsilon^{-1}$ and consider the geodesic ball $B_{r_p}(p)$ of volume m. We want to show that the first eigenvalue of the Laplace-Beltrami operator on this ball converges to the first eigenvalue of the Laplacian on B_R when $\varepsilon \to 0$. To proceed, it will be more convenient to work on the fixed domain B_R, endowed with the metric (depending on p)

$$\hat{g} = \left(\frac{r_p}{R}\right)^2 g\,.$$

Let ϕ and $\hat{\phi}$ the first eigenfunction of the Laplace-Beltrami operator on $B^g_{r_p}(p)$ with respect to g and $\hat{g}$ (normalized to 1 in the L^2 norm), and let λ and $\hat{\lambda}$ the associated eigenvalue. We have

$$\left\{\begin{array}{rcll} \Delta_{\hat{g}}\,\hat{\phi} + \hat{\lambda}\,\hat{\phi} & = & 0 & \text{in} \quad B_R \\ \hat{\phi} & = & 0 & \text{on} \quad \partial B_R \end{array}\right. \tag{16}$$

being $\mathrm{Vol}_{\hat{g}}(B_R) = m$. When $\varepsilon = 0$ the metric $\hat{g}$ is the Euclidean metric and the solution of (16) is therefore given by $\hat{\phi} = \phi_1$, $\hat{\lambda} = \lambda_1$, the first eigenfunction and eigenvalue on the Euclidean ball B_R.

Let us doing now a formal reasoning. In B_R, for some constant μ_ε and some (small) function w_p defined in B_R with 0 Dirichlet boundary condition we have

$$0 = \Delta_{\hat{g}}\,(\phi_1 + w_p) + (\lambda_1 + \mu_p)\,(\phi_1 + w_p) = (\Delta_e + \lambda_1)\,w_p + (\Delta_{\hat{g}} - \Delta_e)\,(\phi_1 + w_p) + \mu_p\,(\phi_1 + w_p) \tag{17}$$

The kernel of $\Delta_e + \lambda_1$ is spanned by ϕ_1. Then if we write

$$w_p = w_p^{\|} + w_p^{\perp}$$

where $w_p^{\|} \in \ker(\Delta_e + \lambda_1)$ (say $w_p^{\|} = a_p\,\phi_1$, $a_p \in \mathbb{R}$) and $w_p^{\perp} \in (\ker(\Delta_e + \lambda_1))^{\perp}$, when we project (17) onto $\ker(\Delta_e + \lambda_1)$ we obtain

$$(1 + a_p)\,\mu_p = \int_{B_R} \phi_1\,(\Delta_e - \Delta_{\hat{g}})\,(\phi_1 + w_p)\,\mathrm{dvol}_e$$

It is then natural to ask the function w_p to be in $(\ker(\Delta_e + \lambda_1))^{\perp}$, in order that $a_p = 0$.

According to the previous formal reasoning, for all $w \in \mathcal{C}^{2,\alpha}(B_1)$ orthogonal to the kernel of $\Delta_e + \lambda_1$, we define the operator

$$N(\varepsilon, w) := (\Delta_e + \lambda_1)\,w + (\Delta_{\hat{g}} - \Delta_e + \mu)\,(\phi_1 + w)$$

where μ is given by

$$\mu = \mu(\varepsilon, w) = -\int_{B_R} \phi_1\,\left[(\Delta_{\hat{g}} - \Delta_e)\,(\phi_1 + w)\right]\,\mathrm{dvol}_e\,.$$

N is $L^2(B_1)$-orthogonal to ϕ_1 (with respect to the Euclidean metric). Our aim is to find, for all ε small enough, a function $w = w(\varepsilon)$ smooth such that $N(\varepsilon, w) = 0$, that is 0 when $\varepsilon = 0$. In fact, this condition suffices to prove the proposition, because in this case we have

$$\lambda_1(B_{r_p}(p)) = \hat{\lambda} = \lambda_1 + \mu$$

that is a smooth function with respect to ε and μ vanishes when $\varepsilon = 0$.

We have

$$N(0, 0) = 0.$$

The mapping N is a smooth map from a neighborhood of $(0, 0)$ in $[0, E) \times \mathcal{C}^{2,\alpha}_{\perp,0}(B_1)$ (for some $E > 0$) into a neighborhood of 0 in $\mathcal{C}^{0,\alpha}_{\perp}(B_1)$ (here the subscript $\perp$ indicates that functions in the corresponding space are $L^2(B_1)$-orthogonal to ϕ_1 (for the Euclidean metric) and the subscript 0 indicates that functions vanish on ∂B_1). The differential of N computed at $(0, 0)$, is given by $\Delta_e + \lambda_1$ since $\hat{g}$ is the Euclidean metric when $\varepsilon = 0$. Hence the partial differential of N computed at $(0, 0)$ is invertible from $\mathcal{C}^{2,\alpha}_{\perp,0}(B_1)$ into $\mathcal{C}^{0,\alpha}_{\perp}(B_1)$ and the implicit function theorem ensures, for all ε small enough (say $\varepsilon < \varepsilon_0$) the existence of a unique solution $w = w(\varepsilon) \in \mathcal{C}^{2,\alpha}_{\perp,0}(B_1)$ depending smoothly on ε such that $N(\varepsilon, w) = 0$. □

4. Regularity

As annouced in the introduction, we follow the strategy intitiated in [3] to study the regularity of free boundaries, and adapt it to solutions of (12). We also adapt the contribution given by G. Weiss in [52] to improve the estimate on the singular set. One major difficulty is due to the volume constraint, for which we need to show that one can consider a penalized version of the problem, similarly to [11].

Our main goal is to be able to prove existence of blow-up limits and to study them: this will be achieved in Section 4.10, only after several preleminary steps:

- choose an appropriate representative of u solution to (12) using properties of subharmonic functions in Riemannian manifolds (Section 4.1),
- prove Lipschitz regularity of this representative (Corollary 4.9) which requires a first penalization of the volume constraint (Section 4.2). This allows to prove existence of blow-up limits for u, see the beginning of the proof of Proposition 4.25,
- prove the nondegeneracy of u near the boundary (Corollary 4.19), so that blow-up limits are nontrivial. This requires a deeper analysis of the penalization of the volume constraint, in particular to show that the penalization parameter can be chosen positive for inner perturbations (Sections 4.4 and 4.5),
- deduce from the previous steps that Ω_u has finite perimeter (which allows to consider blow-ups near points of the reduced boundary, which have a normal vector in a weak sense) and density estimates (to improve the convergence of blow-up limits), see Sections 4.7 and 4.8,
- prove a Weiss type almost monotonicity formula (Proposition 4.22), to show that the blow-up will be a global *homogeneous* minimizers of the Alt-Cafarelli functional, see (61).

After all these preliminary steps, we can analyse blow-ups and then conclude, using the classification of homogenous minimizers in dimensions less or equal to 5, and the approach by De Silva in [23] to study the regularity of flat points, see Section 4.11. The first step is fundamental to prove the Lipschitz regularity and nondegenerancy of the solution u, and here our strategy is different and simpler with respect to other more recent proofs for problems involving general operators with variable coefficients, see for example [50, 19].

In this section, M denotes a smooth, connected and compact Riemannian manifold. The compactness hypothesis is made only for convenience, all arguments can be localized, which explains that we drop this hypothesis in the statement of Theorem 1.2. The function u will denote a solution to (12), as we know from Proposition 3.2 that Ω^* solution to (7) can be seen as $\Omega^* = \Omega_u = \{u \neq 0\}$ where u solves (12). It is also well-known that u can be assumed to be nonnegative (see Proposition 4.1), so that $\Omega_u = \{u > 0\}$.

Let $x_0 \in M$, let $E_1, \dots, E_n$ be an orthonormal basis of the tangent space $T_{x_0}M$. We denote by $\exp_{x_0}$ the exponential map on the manifold M at the point x_0. The coordinates we want to use are the classical geodesic normal coordinates at x_0,

$$x := (x^1, \dots, x^n) \in \mathbb{R}^n \,,$$

defined by the chart

$$X(x) := \exp_{x_0}(\Theta(x))$$

where

$$\Theta(x) := \sum_{i=1}^{n} x^i \, E_i \in T_{x_0}M \,. \tag{18}$$

In order to study the regularity of our optimal set Ω^*, we fix an arbitrary point $x_0 \in \partial\Omega^*$, and we just consider a neighborhood of x_0 in M (we do not need the rest of the manifold). If we fix r_0 as a positive constant less than the cut locus at x_0 and we define the geodesic ball of radius r_0 and center x_0 as

$$B^g_{r_0}(x_0) := \big\{\mathrm{Exp}_{x_0}(\Theta(x)) \qquad : \quad x \in \mathbb{R}^n \qquad 0 \le |x| < r_0\big\} \,,$$

we just need to understand the behavior of $\partial\Omega^* \cap B^g_{r_0}(x_0)$. If R is the Riemann curvature tensor on the manifold M, we have a Taylor expansion of the coefficients $g_{ij}(x)$ of the metric in these geodesic normal coordinates given by

$$g_{ij} = \delta_{ij} + \frac{1}{3} \, R_{ikj\ell} \, x^k \, x^\ell + \mathcal{O}(|x|^3), \tag{19}$$

where δ_{ij} is the Kronecker symbol and $R_{ikj\ell} = g\big(R(E_i, E_k)\, E_j, E_\ell\big)$ at the point x_0 (here the Einstein summation convention is understood). Hence we can choose r_0 small enough such that we have

$$\frac{id}{2} \le g \le 2id \,,$$

in $B^g_{r_0}(x_0)$, and so the Laplace-Beltrami operator is uniformly elliptic in $B^g_{r_0}(x_0)$. Moreover, a straightforward computation shows that at the point of coodinates x

$$\begin{aligned} g^{ij}(x) &= \delta_{ij} - \frac{1}{3}\, R_{ikj\ell}\, x^k\, x^\ell + \mathcal{O}(|x|^3) \\ \log|g|(x) &= \frac{1}{3}\, R_{k\ell}\, x^k\, x^\ell + \mathcal{O}(|x|^3) \end{aligned}$$

where

$$R_{k\ell} = \sum_{i=1}^n R_{iki\ell}\,,$$

and then for a function f with bounded fist derivatives we have

$$\|\nabla^g f - \nabla^e f\|_\infty \le C\, r_0\,,$$

in $B^g_{r_0}(x_0)$, where the constant C depends on the bound of the first derivatives of the function f in $B^g_{r_0}(x_0)$, and for a function f with bounded first and second derivatives we have

$$\|\Delta_g f - \Delta_e f\|_\infty \le C\, r_0 \tag{20}$$

in $B^g_{r_0}(x_0)$, where the constant C depends on the bound of first and second derivatives of the function f in $B^g_{r_0}(x_0)$.

In some parts of the proof of the regularity result, it will be necessary to study the behavior of a function in a very small geodesic ball $B^g_\varepsilon(x_0)$, and for that aim we will do a *scaling*. In the Euclidean framework, this means that given a function $w(x)$ for $x \in B_\varepsilon$ we define the function $\tilde{w}(y) = w(\varepsilon\, y)$ for $y \in B_1$, where B_ε and B_1 are the Euclidean balls of radius ε and 1. This means that the metric on B_1 (i.e. the Euclidean metric) is not the metric g induced on B_1 by the parameterization $x = \varepsilon y$ of B_ε, but $\varepsilon^{-2}\, g$. We do the same in the Riemannian framework. Given $\varepsilon > 0$ small enough, we define on the manifold M the metric $\bar{g} := \varepsilon^{-2}\, g$ and the parameterization given by

$$Y(y) := \exp^g_{x_0}\,(\varepsilon\, \Theta(y))$$

with $y = (y_1, ..., y_n) \in B_R$, being B_R the Euclidean ball of radius R, and Θ defined in (18). If x are the normal geodesic coordinates around x_0 of the same point as y, it is clear that

$$\bar{g}_{ij}(y)\, \mathrm{d}y^i\, \mathrm{d}y^j = \bar{g} = \varepsilon^{-2}\, g = \varepsilon^{-2}\, g_{ij}(x)\, \mathrm{d}x^i\, \mathrm{d}x^j = \varepsilon^{-2}\, g_{ij}(\varepsilon\, y)\, \varepsilon^2\, \mathrm{d}y^i\, \mathrm{d}y^j = g_{ij}(\varepsilon\, y)\, \mathrm{d}y^i\, \mathrm{d}y^j$$

Hence in the coordinates y, the metric $\bar{g}$ is given by

$$\bar{g}_{ij}(y) = g_{ij}(\varepsilon\, y)$$

i.e.

$$\bar{g}_{ij}(y) = \delta_{ij} + \frac{1}{3}\, \varepsilon^2\, R_{ikj\ell}\, y^k\, y^\ell + \mathcal{O}(\varepsilon^3)\,.$$

4.1. A priori regularity

In this first paragraph, we recall standard properties of eigenfunctions in a domain, without using the optimality of the shape itself, but only the optimality of u in (1). Therefore in this section, $\Omega \subset M$ denotes a bounded quasi-open set.

Proposition 4.1. *Let u a solution to* (1)*. Then*

- *u has constant sign (and we will always assume $u \ge 0$)*
- *$\Delta_g u + \lambda_1(\Omega) u \ge 0$ in $\mathcal{D}'(M)$, and in particular $\Delta_g u$ is a signed Radon measure on M,*
- *u is bounded in M.*

Proof. We know that

$$\forall v \in H^1_0(\Omega), \int_M g(\nabla^g u, \nabla^g v)\, \mathrm{dvol}_g = \lambda_1(\Omega) \int_M uv\, \mathrm{dvol}_g\,. \tag{21}$$

The fact that u has constant sign is a classical result, see for example [35, Theorem 1.3.2]. Let $\varphi \in C^\infty_c(M)$ such that $\varphi \ge 0$. We introduce $p_n(s) = \inf\{ns, 1\}_+$ for $s \in \mathbb{R}$ and apply (21) to $v = \varphi p_n(u)$ which belongs to $H^1_0(\Omega)$. This gives

$$\int_M g\,(\nabla^g u, \nabla^g \varphi)\, p_n(u)\, \mathrm{dvol}_g + \underbrace{\int_M \|\nabla^g u\|^2_g\, \varphi\, p_n'(u)\, \mathrm{dvol}_g}_{\ge 0} = \lambda_1(\Omega) \int_M (u\varphi)\, p_n(u)\, \mathrm{dvol}_g\,.$$

We make n go to ∞ and use that $p_n(u)$ converges to $\mathbb{1}_{\{u>0\}}$; using that $u \ge 0$ in M, we obtain

$$\forall \varphi \in \mathcal{C}^\infty_c(M) \text{ such that } \varphi \ge 0, \quad \int_M g(\nabla^g u, \nabla^g \varphi)\, \mathrm{dvol}_g - \lambda_1(\Omega) \int_M u\varphi\, \mathrm{dvol}_g \le 0,$$

which leads to $\Delta_g u + \lambda_1(\Omega) u \ge 0$ in the sense of distribution on M. The second point follows classically from the fact that $[-\Delta_g - \lambda_1(\Omega)]\, u \le 0$ and $u \in L^2(M)$, see for example [29, Theorem 8.15 and p. 214]. □

Remark 4.2. *We recall [6, Th 1.1] (see also [7]) that generalizes the mean value property in the Riemannian context: for every $x \in M$, there exists a family $\{D_r(x)\}_{0<r<r_0}$ of open sets, monotone with respect to the inclusion, and converging to the point x when $r \to 0$ such that for any v subharmonic in M (meaning that $\Delta_g v \geq 0$), we have that $r \in (0, r_0) \mapsto \fint_{D_r(x)} v \,\mathrm{dvol}_g$ is increasing. Moreover, for almost every $x \in M$ we have*

$$v(x) = \lim_{r \to 0} \fint_{D_r(x)} v \,\mathrm{dvol}_g,$$

and this representative is defined everywhere on M and is upper-semi-continuous.

We add the following lemma[3]:

Lemma 4.3. *Let $v : M \to \mathbb{R}$, $x_0 \in M$ and $r_0 > 0$ smaller than the injectivity radius.*

- *In $B^g_{r_0}(x_0)$, if $\Delta_g v \leq 0$, $v \geq 0$ and the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)k$ for some constant $k \in \mathbb{R}$, then $r \in (0, r_0) \mapsto \dfrac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \displaystyle\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}$ is decreasing, where $S^g_r(x_0) = \partial B^g_r(x_0)$ and $S^{g_k}_r$ is a geodesic sphere of radius r in the space form with metric g_k with constant sectional curvature k.*
- *In $B^g_{r_0}(x_0)$, if $\Delta_g v \geq 0$, $v \geq 0$ and the sectional curvature satisfies $\mathrm{Rm} \leq k$ for some constant k, then $r \in (0, r_0) \mapsto \dfrac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \displaystyle\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}$ is increasing.*

Here $\mathrm{Vol}_{g_k}(S^{g_k}_r)$ represent the area of $S^{g_k}_r$ in the metric induced by g_k.

Proof. For the first case, we assume v smooth and apply Hadamard formula for derivation of integrals defined in a domain with variable boundary:

$$\begin{aligned} 0 \geq \int_{B^g_r(x_0)} \Delta_g v \,\mathrm{dvol}_g &= \int_{S^g_r(x_0)} g(\nabla^g v, \nu) \,\mathrm{dvol}_g|_{S^g_r(x_0)} \\ &= \frac{d}{dr}\left(\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}\right) - \int_{S^g_r(x_0)} v H_{S^g_r(x_0)} \,\mathrm{dvol}_g|_{S^g_r(x_0)} \end{aligned}$$

where $H_{S^g_r(x_0)}$ represents the mean curvature at points of $S^g_r(x_0)$. Let $H_{S^{g_k}_r}$ be the mean curvature of a geodesic sphere of radius r in the space form of constant curvature k, i.e.

$$H_{S^{g_k}_r} = (n-1)\frac{c_k(r)}{s_k(r)}$$

where

$$s_k(r) = \begin{cases} r & \text{if } k = 0 \\ \sin(\sqrt{k}r) & \text{if } k > 0 \\ \sinh(\sqrt{-k}r) & \text{if } k < 0 \end{cases} \quad \text{and} \quad c_k(r) = \begin{cases} 1 & \text{if } k = 0 \\ \cos(\sqrt{k}r) & \text{if } k > 0 \\ \cosh(\sqrt{-k}r) & \text{if } k < 0 \end{cases}.$$

If $\mathrm{Ric} \geq (n-1)k$, by the mean curvature comparison theorem (see [54]) we have $H \leq H_{S^{g_k}_r}$, and then

$$\begin{aligned} 0 \geq\ & \frac{d}{dr}\left(\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}\right) - \int_{S^g_r(x_0)} v H_{S^{g_k}_r} \,\mathrm{dvol}_g|_{S^g_r(x_0)} \\ =\ & \frac{d}{dr}\left(\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}\right) - (n-1)\frac{c_k(r)}{s_k(r)} \int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}. \end{aligned}$$

If w_n is the volume of the unit n-dimensional ball, we obtain

$$0 \geq w_n s_k(r)^{n-1} \frac{d}{dr}\left(\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}\right) - (n-1) w_n s_k(r)^{n-2} c_k(r) \int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)},$$

which implies

$$0 \geq \frac{d}{dr}\left(\frac{\displaystyle\int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}}{w_n s_k(r)^{n-1}}\right) = \frac{d}{dr}\left(\frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \int_{S^g_r(x_0)} v \,\mathrm{dvol}_g|_{S^g_r(x_0)}\right)$$

[3] We didn't find a reference for this result, but it can be found in `https://cuhkmath.wordpress.com/2015/08/14/mean-value-theorems-for-harmonic-functions-on-riemannian-manifolds/`

and leads to the result. The case of non-smooth function v is obtained by approximation, because smooth subharmonic functions are dense in the space of subharmonic functions (see for example [8]).

The proof for the subharmonic case is similar except that we require the sectional curvature to be less or equal to k in order to use the Hessian comparison theorem. □

Corollary 4.4. *Let u be a solution of* (1)*. Then u can be pointwise defined by an upper-semi-continuous representative (still denoted u) with the following formula (see Remark 4.2 for the definition of $D_r(x)$):*

$$u(x) = \lim_{r\to 0} \fint_{D_r(x)} u \,\mathrm{dvol}_g. \tag{22}$$

Moreover, there exist $c, c' > 0$ and $r_0 > 0$ such that for any $r < r_0$, one has

$$u(x) \le c\left(\fint_{B^g_r(x)} u\,\mathrm{dvol}_g + r^2\right) \le c'\left(\fint_{S^g_r(x)} u\,\mathrm{dvol}_g|_{S^g_r(x_0)} + r^2\right). \tag{23}$$

Proof. From Proposition 4.1, we know that $\Delta_g u \ge -\lambda\|u\|_\infty$. Let fix $x_0 \in M$ and r_0 such that $B^g_{r_0}(x_0)$ can be represented by a unique chart, and define $w(x) = |x|^2$ in local geodesic coordinates. Then in $B^g_r(x_0)$ for $r \le r_0$, one has using (20),

$$\Delta_g w \ge \Delta w - Cr$$

for some constant C independant on r. Therefore $\widetilde{u} := (u+\lambda\|u\|_\infty w)$ is sub-harmonic in $B^g_{r_1}(x_0)$ if r_1 is small enough. Applying Remark 4.2, we obtain first that

$$\lim_{r\to 0}\fint_{D_r(x_0)} (u+\lambda\|u\|_\infty w)\,\mathrm{dvol}_g = \lim_{r\to 0}\fint_{D_r(x_0)} u\,\mathrm{dvol}_g$$

exists. As g is uniformly elliptic inside $B^g_{r_1}(x_0)$, from [7, Theorem 6.3], one has

$$B^g_{c_0 r}(x_0) \subset D_r(x_0) \subset B^g_{c_1 r}(x_0)$$

for some constants $0 < c_0 < c_1$, and $r < \frac{r_1}{c_1}$. From Lebesgue differentiation Theorem, (22) is valid almost everywhere, and this representative is upper semi-continuous, see also [7]. Using the representative of u given in (22) and the monotonicity of

$$r \in (0, r_1) \mapsto \int_{D_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g,$$

we obtain

$$\begin{aligned} u(x_0) &\le \fint_{D_{r/c_1}(x_0)} \widetilde{u}\,\mathrm{dvol}_g \le \frac{1}{\mathrm{Vol}_g(B^g_{c_0 r/c_1}(x_0))}\int_{B^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g = \frac{\mathrm{Vol}_g(B^g_r(x_0))}{\mathrm{Vol}_g(B^g_{c_0 r/c_1}(x_0))}\fint_{B^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g \\ &\le 2\left(\frac{c_1}{c_0}\right)^n \fint_{B^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g \le c\left(\fint_{B^g_r(x_0)} u\,\mathrm{dvol}_g + r^2\right), \end{aligned}$$

where $c = 2\max\{1, \lambda\|u\|_\infty\}\left(\frac{c_1}{c_0}\right)^n$. For the last property, we show there exists c_2 such that

$$\fint_{B^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g \le c_2 \fint_{S^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g|_{S^g_r(x_0)}.$$

First, we choose k positive such that $\mathrm{Rm} \le k$ in the ball $B^g_r(x_0)$. Therefore, from Lemma 4.3, we know that

$$r \in (0, r_1) \mapsto \frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)}\int_{S^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g|_{S^g_r(x_0)}$$

is increasing. Moreover

$$\begin{aligned} \fint_{B^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g &= \frac{1}{\mathrm{Vol}_g(B^g_r(x_0))}\int_0^r \left(\int_{S^g_s(x_0)} \widetilde{u}\,\mathrm{dvol}_g|_{S^g_s(x_0)}\right) ds \\ &= \frac{1}{\mathrm{Vol}_g(B^g_r(x_0))}\int_0^r \left(\frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_s(x_0))}\int_{S^g_s(x_0)} \widetilde{u}\,\mathrm{dvol}_g|_{S^g_s(x_0)}\right)\mathrm{Vol}_{g_k}(S^{g_k}_s(x_0))ds \\ &\le \frac{\mathrm{Vol}_{g_k}(B^{g_k}_r(x_0))}{\mathrm{Vol}_g(B^g_r(x_0))}\left(\frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r(x_0))}\int_{S^g_r(x_0)} \widetilde{u}\,\mathrm{dvol}_g|_{S^g_r(x_0)}\right) \end{aligned}$$

and hence the result, as $\dfrac{\mathrm{Vol}_{g_k}(B^{g_k}_r(x_0))}{\mathrm{Vol}_g(B^g_r(x_0))}$ is bounded from above for $r \le r_1$. □

4.2. First penalization

In the first steps of the analysis in [3], which will lead to the optimal regularity for u (see Section 4.3), the authors use a test function whose support is larger than $\Omega_u = \{u \neq 0\}$. Therefore, this test function is a priori not allowed because of the volume constraint. We therefore prove that (12) is equivalent to a penalized version. Note that one could skip this step as in Section 4.4 we will prove a more precise penalization result (see [45] where this strategy is applied), but we decided to keep the result in this section as they are more straightforward and global. We use in this section that M is compact. Nevetheless this hypothesis is not required for Theorem 1.2 since this section can be skipped.

Proposition 4.5 (Global Penalization from above). *If u is a solution of* (12)*, then there exists $\mu^* \in \mathbb{R}_+$ such that for all $v \in H^1(M)$*

$$\int_M \|\nabla^g u\|_g^2 \,\mathrm{dvol}_g \leq \int_M \|\nabla^g v\|_g^2 \,\mathrm{dvol}_g + \lambda(m)\left[1 - \int_M v^2 \,\mathrm{dvol}_g\right]^+ + \mu^*\Big[\mathrm{Vol}_g(\Omega_v) - m\Big]^+. \tag{24}$$

We will argue as in [10, Theorem 2.4] where they deal with the Euclidian case. Only the last part of the proof requires some new developments in the Riemannian case, though we will recall the whole proof.

Remark 4.6. *This implies in particular that if Ω^* solves* (7) *and M is compact, then for μ large enough,*

$$\forall \Omega \subset M, \quad \lambda_1(\Omega^*) \leq \lambda_1(\Omega) + \mu\left[\mathrm{Vol}_g(\Omega) - m\right]^+$$

Proof. Let us introduce u_μ a solution of

$$\min\left\{G_\mu(v) := \int_M \|\nabla^g v\|_g^2 \,\mathrm{dvol}_g + \lambda_m\left[1 - \int_M v^2 \,\mathrm{dvol}_g\right]^+ + \mu\Big[\mathrm{Vol}_g(\Omega_v) - m\Big]^+, v \in H^1(M)\right\}.$$

The existence of such u_μ follows from standard compactness arguments: observing indeed that $G_\mu(|w|) = G_\mu(w)$, and that $G_\mu(w/\|w\|_{L^2}) \leq G_\mu(w)$ if $\|w\|_{L^2} \geq 1$, one can consider a minimizing sequence in the set

$$\{w \in H^1(M) \ / \ w \geq 0, \|w\|_{L^2} \leq 1\}.$$

Such minimizing sequence has a gradient uniformly bounded in L^2 and is therefore bounded in $H^1(M)$; similarly than in the proof of Theorem 1.1, this provides a solution u_μ, which is moreover non-negative and such that $\|u_\mu\|_{L^2} \leq 1$. Then, it only remains to show that for μ large enough, we necessarily have $\mathrm{Vol}_g(\Omega_{u_\mu}) \leq m$. Indeed, in that case we have, using optimality of u_μ, $G_\mu(u_\mu) \leq G_\mu(u)$ and one hand, and on the other hand using (12):

$$\int_M \|\nabla^g u\|_g^2 \,\mathrm{dvol}_g \leq \int_M \|\nabla^g u_\mu\|_g^2 \,\mathrm{dvol}_g + \lambda(m)\left[1 - \int_M u_\mu^2 \,\mathrm{dvol}_g\right] \leq G_\mu(u_\mu)$$

so u is also a minimizer for (24).

We therefore assume that $\mathrm{Vol}_g(\Omega_{u_\mu}) > m$; then we can write, $G_\mu(u_\mu) \leq G_\mu((u_\mu - t)^+)$ for $t > 0$ small enough and get, using in particular that $\|(u-t)^+\|_{L^2} \leq \|u\|_{L^2} \leq 1$:

$$\begin{aligned}\int_{\{0<u_\mu<t\}} \|\nabla^g u_\mu\|_g^2 \,\mathrm{dvol}_g + \mu\,\mathrm{Vol}_g(\{0 < u_\mu < t\}) &\leq \lambda_m\left[\int_M u_\mu^2 \,\mathrm{dvol}_g - \int_{\{u_\mu \geq t\}} (u_\mu - t)^2 \,\mathrm{dvol}_g\right] \\ &\leq \lambda_m\left[\int_{\{0<u_\mu<t\}} u_\mu^2 \,\mathrm{dvol}_g + \int_{\{u_\mu \geq t\}} 2tu_\mu \,\mathrm{dvol}_g\right]\end{aligned}$$

and using the Cauchy-Schwarz inequality and the fact that $\|u_\mu\|_{L^2} \leq 1$, we get:

$$\int_{\{0<u_\mu<t\}} \left(\|\nabla^g u_\mu\|_g^2 - \lambda_m u_\mu^2\right) \,\mathrm{dvol}_g + \mu\,\mathrm{Vol}_g(\{0 < u_\mu < t\}) \leq 2\lambda_m t\,\mathrm{Vol}_g(\Omega_{u_\mu})^{1/2}.$$

With the co-area formula, we obtain

$$\int_0^t \int_{\{u_\mu = s\}} \underbrace{\left[\|\nabla^g u_\mu\|_g + \frac{\mu - \lambda_m u_\mu^2}{\|\nabla^g u_\mu\|_g}\right]}_{\geq \sqrt{2\mu} \text{ for } s \text{ such that } \lambda_m s^2 \leq \mu/2} \mathrm{dvol}_{g|_{\{u_\mu = s\}}} dt \leq 2\lambda_m t\,\mathrm{Vol}_g(\Omega_{u_\mu})^{1/2}, \quad \text{for } t \text{ small enough}$$

which gives, dividing by t and letting $t \to 0$:

$$\sqrt{2\mu}\,P(\Omega_{u_\mu}) \leq 2\lambda_m \mathrm{Vol}_g(\Omega_{u_\mu})^{1/2}, \quad \text{and then} \quad \sqrt{2\mu} \leq \frac{2\lambda(m)\mathrm{Vol}_g(M)^{1/2}}{I_M(\mathrm{Vol}_g(\Omega_{u_\mu}))},$$

where $I_M : [0, \mathrm{Vol}_g(M)] \to \mathbb{R}_+$ is the isoperimetric profile of M, which is positive on $(0, \mathrm{Vol}_g(M))$ as M is compact and connected. From the estimate

$$G_\mu(u_\mu) \le G_\mu(u) = \int_M \|\nabla^g u\|_g^2 \,\mathrm{dvol}_g\,,$$

we know that $\mu\big[\mathrm{Vol}(\Omega_{u_\mu}) - m\big]$ is bounded uniformly in μ, so there exists μ_0 such that forall $\mu \ge \mu_0$, $\mathrm{Vol}_g(\Omega_{u_\mu}) \in \left[m, \frac{\mathrm{Vol}_g(M)+m}{2}\right]$. As a conclusion, if $\mu > \max\left\{\mu_0, \dfrac{2\lambda(m)\mathrm{Vol}_g(M)}{\inf\left\{I_M(s), s \in \left[m, \frac{\mathrm{Vol}_g(M)+m}{2}\right]\right\}}\right\}$, then $\mathrm{Vol}_g(\Omega_\mu) \le m$, which concludes the proof. □

4.3. Lipschitz continuity of the first eigenfunction

A first step in the regularity theory is to study the regularity of the state function, seen as a function defined on M. The regularity is obvious inside or outside Ω_u (so far though, we do not know yet that the interior of Ω_u is not empty), but the regularity across the free boundary is non trivial, especially as we don't know anything about the regularity of this free boundary. It is clear that one cannot expect more than Lipschitz continuity: even if we already knew that Ω_u is smooth, then the eigenfunction vanishes outside Ω_u and has a linear growth from the boundary, inside Ω_u. The purpose of this section is to prove the Lipschitz continuity of u despite the lack of knowledge about $\partial\Omega_u$; this is often referred to as the *optimal regularity of the state function*. This will have some consequences about weak regularity of the free boundary, and will be useful to prove existence of blow-ups. We will use here the penalization result of the previous section. But we could also use the refined penalization result of Section 4.4 (for which the compactness of the manifold is not required), see also [45].

As in [3], we express the Lipschitz regularity as an uniform bound of the mean value of u on spheres crossing $\partial\Omega_u$, so the main tool in this section is the following lemma:

Lemma 4.7 (Upper bound to the growth of u near the boundary). *Let u be a solution of* (12)*. There exist $C > 0$ and $r_0 > 0$ such that, for all geodesic ball $B_r^g(x_0) \subset M$ with $r \le r_0$,*

$$\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \,\mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \ge C \quad \Longrightarrow \quad u > 0 \text{ on } B_r^g(x_0). \tag{25}$$

Proof. Let $B_r^g(x_0) \subset M$ and v satisfying

$$\begin{cases} -\Delta_g v &= \lambda(m) u & \text{in } B_r^g(x_0), \\ v &= u & \text{outside } B_r^g(x_0). \end{cases} \tag{26}$$

We first notice that from maximum principle $v \ge u \ge 0$, so $\Omega_u \subset \Omega_v$.

Step 1: Using optimality of u for the penalized version given in Proposition 4.5, let us first prove

$$\int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g \le \mu^* \,\mathrm{Vol}_g(\{u=0\} \cap B_r^g(x_0)). \tag{27}$$

We compare the energies of u and v in (24), which gives,

$$J(u) - J(v) \le \mu^* \Big[\mathrm{Vol}_g(\Omega_v) - \mathrm{Vol}_g(\Omega_u)\Big]\,.$$

We now notice, using that $u - v$ vanishes on $\partial B_r^g(x_0)$,

$$\begin{aligned} J(u) - J(v) &= \int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g + 2\int_{B_r^g(x_0)} g(\nabla^g(u-v), \nabla^g v)\,\mathrm{dvol}_g - \lambda(m)\int_{B_r^g(x_0)} (u^2 - v^2)\,\mathrm{dvol}_g \\ &= \int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g - 2\int_{B_r^g(x_0)} \Delta_g v\,(u-v)\,\mathrm{dvol}_g - \lambda(m)\int_{B_r^g(x_0)} (u^2 - v^2)\,\mathrm{dvol}_g \\ &= \int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g + \lambda(m)\int_{B_r^g(x_0)} (u-v)^2\,\mathrm{dvol}_g \end{aligned} \tag{28}$$

and therefore,

$$\int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g \le J(u) - J(v) \le \mu^* \,\mathrm{Vol}_g(\{u=0\} \cap B_r^g(x_0))\,. \tag{29}$$

Step 2: Using classical elliptic tools (here we do not use the optimality of u anymore), we obtain in this step the reverse estimate[4]

$$\int_{B_r^g(x_0)} \|\nabla^g(u-v)\|_g^2 \,\mathrm{dvol}_g \ge c\,\mathrm{Vol}_g(\{u=0\} \cap B_r^g(x_0)) \left(\frac{1}{r}\fint_{\partial B_r^g(x_0)} u \,\mathrm{dvol}_{g|_{\partial B_r^g(x_0)}}\right)^2, \tag{30}$$

[4]One can find a different proof in [3, Lemma 3.2] in the harmonic case; the proof we use here is an adaptation of an argument that has been communicated to us by Antoine Mellet, see `https://vimeo.com/118498464`

for some constant $c > 0$.
For r small enough and for any x_0 in a compact set of M the exponential map $\exp_{x_0}$ is a bi-Lipschitz continuous diffeomorphism from $B^g_r(x_0)$ to the ball B_r of radius r in the tangent space, with a Lipschitz constant independent of r and of x_0. If we denote by u^* and v^* the pull back of the functions u and v in the tangent space, by the Hardy inequality we have

$$\int_{B_r} \|\nabla(u^* - v^*)\|_e^2 \,\mathrm{dvol}_e \geq \frac{1}{4}\int_{B_r} \left(\frac{v^* - u^*}{r - |x|}\right)^2 \mathrm{dvol}_e \,.$$

Now, using the bi-Lipschitz diffeomorphism given by the exponential map we get the same estimate with a smaller constant $c_1 > 0$:

$$\int_{B^g_r(x_0)} \|\nabla^g(u - v)\|_g^2 \,\mathrm{dvol}_g \geq c_1 \int_{B^g_r(x_0)} \left(\frac{v - u}{\delta_{r,x_0}}\right)^2 \mathrm{dvol}_g$$

where $\delta_{r,x_0}(x) = d_g(x, \partial B^g_r(x_0))$ and then

$$\int_{B^g_r(x_0)} \|\nabla^g(u - v)\|_g^2 \,\mathrm{dvol}_g \geq c_1 \int_{B^g_r(x_0)\cap\{u=0\}} \left(\frac{v}{\delta_{r,x_0}}\right)^2 \mathrm{dvol}_g \,.$$

To estimate the last term, it remains to understand the behavior of v in the ball. We introduce ϕ such that $u - \phi \in H^1_0(B^g_r(x_0))$ and ϕ harmonic in $B^g_r(x_0)$, i.e. ϕ is the harmonic replacement of u in that geodesic ball. Take a negative constant k such that the Ricci curvature of the manifold M satisfies, in $B^g_{r_0}(x_0)$, $\mathrm{Ric} \geq (n-1)k$. Then by the first part of Lemma 4.3 we have

$$\phi(x_0) \geq \frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \int_{S^g_r(x_0)} \phi \,\mathrm{dvol}_g|_{S^g_r(x_0)} \,.$$

Similarly to the proof of Corollary 4.4, we write

$$\begin{aligned}
\frac{1}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \int_{S^g_r(x_0)} \phi \,\mathrm{dvol}_g|_{S^g_r(x_0)} &= \frac{1}{\mathrm{Vol}_g(S^g_r(x_0))} \frac{\mathrm{Vol}_g(S^g_r(x_0))}{\mathrm{Vol}_{g_k}(S^{g_k}_r)} \int_{S^g_r(x_0)} \phi \,\mathrm{dvol}_g|_{S^g_r(x_0)} \\
&\geq c_2 \frac{1}{\mathrm{Vol}_g(S^g_r(x_0))} \int_{S^g_r(x_0)} \phi \,\mathrm{dvol}_g|_{S^g_r(x_0)}
\end{aligned}$$

for some constant c_2 depending only on k, that is to say uniformly in $r < r_0$ and x_0. Then

$$\phi(x_0) \geq c_2 \fint_{S^g_r(x_0)} \phi \,\mathrm{dvol}_g|_{S^g_r(x_0)}$$

Using now Harnack's inequality (see for example [29, Theorem 8.20]), there is a constant $c_3 > 0$ independant on x_0 and $r \leq r_0$ such that $\phi \geq c_3\phi(x_0)$ in $B^g_{\frac{r}{2}}(x_0)$. Denoting $c_4 = c_2c_3$ we finally get

$$v \geq \phi \geq c_4 \fint_{\partial B^g_r(x_0)} u \,\mathrm{dvol}_g|_{\partial B^g_r(x_0)} \quad \text{in } B^g_{\frac{r}{2}}(x_0). \tag{31}$$

In $B^g_r(x_0) \setminus B^g_{\frac{r}{2}}(x_0)$ we argue as in the proof of the classical Hopf's lemma (see for example [27, Section 6.4.2]; see also [51, Lemma 4.1]). In normal geodesic coordinates we consider the function $w(x) = e^{-\gamma\frac{|x|^2}{r^2}} - e^{-\gamma}$. After computation we have that in the chart representing $B^g_r(x_0) \setminus B^g_{\frac{r}{2}}(x_0)$, the Euclidean Laplacian of w is

$$\Delta_e w(x) = \frac{2\gamma}{r^2}\left[2\gamma\frac{|x|^2}{r^2} - n\right] e^{-\gamma\frac{|x|^2}{r^2}} \geq \frac{2\gamma}{r^2}\left[\frac{\gamma}{2} - n\right] e^{-\frac{\gamma}{4}}$$

so that, if γ is large enough, $-\Delta_e w(x) \leq -k'$, for some positive constant k', independant on x_0 and $r \leq r_0$. Then, choosing r small enough and using (20), $-\Delta_g w \leq 0$ in $B^g_r(x_0) \setminus B^g_{\frac{r}{2}}(x_0)$. On the other hand, we define

$$\varphi := \phi - \left[\frac{c_4 \fint_{S^g_r(x_0)} u \,\mathrm{dvol}_g|_{S^g_r(x_0)}}{w\left(\frac{r}{2}\right)}\right] w$$

which is such that $\varphi \geq 0$ on $\partial B^g_{\frac{r}{2}}(x_0)$, $\varphi = u \geq 0$ on $\partial B^g_r(x_0)$ and $-\Delta_g\varphi \geq 0$. We obtain from maximum principle that

$$v \geq \phi \geq \left[c_4 \fint_{\partial B^g_r(x_0)} u \,\mathrm{dvol}_g|_{\partial B^g_r(x_0)}\right] \frac{w}{w\left(\frac{r}{2}\right)} \tag{32}$$

in $B_r^g(x_0) \setminus B_{\frac{r}{2}}^g(x_0)$. We finally remark that there is a constant $c_5 = c_5(\gamma)$ such that $\dfrac{e^{-\gamma y^2} - e^{-\gamma}}{e^{-\gamma/4} - e^{-\gamma}} \geq c_5(1-y)$ for $y \in [0,1]$, so that $\dfrac{w}{w(r/2)} \geq c_5 \left(1 - \dfrac{|x|}{r}\right)$. Combining (31) and (32) we get that there exists a constant c_6 independant on x_0 and $r \leq r_0$ such that

$$v(x) \geq c_6 \left(\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \right) (r - d_g(x,x_0)) = c_6 \left(\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \right) \delta_{r_0,r_0}(x)$$

which leads to the estimate (30).

Conclusion: Combining (27) and (30), we obtain that if $\dfrac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} > \sqrt{\dfrac{\mu^*}{c}}$ then $u = v$ almost everywhere on $B_r^g(x_0)$. Using the representation (22) for both u and v, we deduce that $u = v > 0$ everywhere on $B_r^g(x_0)$, which concludes the proof. □

Remark 4.8. *Note that from* (27) *and using the representation* (22)*, we deduce that if $u > 0$ almost everywhere in $B_r^g(x_0)$, then $u = v$ everywhere in $B_r^g(x_0)$, and therefore $u > 0$ everywhere in $B_r^g(x_0)$. This fact will be used in Sections 4.4 and 4.5.*

Corollary 4.9. *Let M be a compact Riemannian manifold, $m \in (0, \mathrm{Vol}_g(M))$ and u a solution of* (12)*. Then u is Lipschitz continuous in M.*

Remark 4.10. *If M is noncompact, one deduces that u is locally Lipschitz in M. Also, combined with Proposition 3.2, one deduces that If M is connected and Ω^* solves* (7)*, then Ω^* is an open set and hence solves* (3)*. See Remark 1.4 for a counterexample in the disconnected case.*

Remark 4.11. *In order to prove that Ω_u is open (where u solves* (12)*), we only need to prove that the eigenfunction u is continuous on M (rather than Lipschitz continuous), which can be obtained in several ways, see for example the elementary proof in [10, Lemma 3.8], or also [10, Remark 3.10] and [51, Section 3] based on a classical method from Morrey [43]; this last method gives Hölder regularity for any order $\alpha < 1$. Nevertheless, we will need Lipschitz continuity in the following sections.*

Proof of Corollary 4.9: The fact that Ω_u is open is a consequence of Corollary 4.4 and Lemma 4.7. Indeed, let $x_0 \in M$ such that $u(x_0) > 0$: then from (23) we get that (c' and r_0 coming from Corollary 4.4)

$$\fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} + r^2 \geq \frac{u(x_0)}{c'}$$

for every $r \leq r_0$. So

$$\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \geq \frac{u(x_0)}{c' r} - r$$

and with Lemma 4.7, we obtain $u > 0$ in $B_r^g(x_0)$ for r small enough.

We are now in position to prove the Lipschitz continuity of u. Since Ω_u is an open set, we can choose the maximal radius r such that the geodesic ball $B_r^g(x_0)$ is included in Ω_u. As u is smooth inside Ω_u, we may assume $r < r_0$. Thanks to Lemma 4.7,

$$\frac{1}{r+\delta} \fint_{\partial B_{r+\delta}^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_{r+\delta}^g(x_0)}} \leq C \, ,$$

where C is introduced in Lemma 4.7, and for all small $\delta > 0$. Therefore

$$\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \leq C.$$

From this estimate, we deduce now that the gradient of u is bounded: we introduce w defined by

$$\begin{cases} -\Delta_g w &= \lambda(m) u & \text{in } B_r^g(x_0), \\ w &= 0 & \text{outside } B_r^g(x_0). \end{cases}$$

On one hand, as $u - w$ is harmonic, introducing G the Green function with Dirichlet boundary condition on $\partial B_r^g(x_0)$ (see [5, Theorem 4.17]), we have

$$\begin{aligned} \|\nabla^g (u-w)(x)\|_g &\leq \int_{\partial B_r^g(x_0)} \underbrace{\|(\nabla_y^g)^2 G(x,y)\|_g}_{\leq k d(x,y)^{-n}} u(y) \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}}(y) \leq \\ &\leq k \left(\frac{r}{2}\right)^{-n} \int_{\partial B_r(x_0)} u(y) \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}}(y) \leq kC \, , \end{aligned} \tag{33}$$

for $x \in B^g_r(x_0)$, where k is a constant depending on the distance of x to $\partial B^g_r(x_0)$. This constant k is not depending on x, if we take $x \in B^g_{\frac{r}{2}}(x_0)$ (according to [5, 4.10], k is not depending on x if the injectivity radius i_x at x with respect to the manifold $B^g_r(x_0)$ is bigger than a positive constant ρ, and this is the case for $x \in B^g_{\frac{r}{2}}(x_0)$).
We use now the scaling argument introduced at the beginning of Section 4 to prove:

$$\|\nabla^g w(x)\|_g \leq Cr\lambda(m)\|u\|_{L^\infty(B_r(x_0))} \tag{34}$$

for $x \in B^g_{\frac{r}{2}}(x_0)$. Indeed, considering $y \in B_{\frac{1}{2}}$, the Euclidean ball of radius $\frac{1}{2}$, and endowing on $B_{\frac{1}{2}}$ the metric $\bar{g}$, let us define $\widetilde{w}(y) = w(\exp^g_x(r\Theta(y)))$, with Θ defined in (18). We have

$$-\Delta_{\bar{g}}\widetilde{w}(y) = -r^2\,\Delta_g w(\exp^g_x(ry)) = r^2\lambda(m)u(\exp^g_x(r\Theta(y)))$$

where $\bar{g} = r^{-2}g$, so by interior gradient estimates, there is a constant C such that

$$|\nabla^e \widetilde{w}(0)| \leq Cr^2\lambda(m)\|u\|_{L^\infty(B_r(x_0))}$$

and then also

$$\|\nabla^{\bar{g}}\widetilde{w}(0)\|_{\bar{g}} \leq Cr^2\lambda(m)\|u\|_{L^\infty(B_r(x_0))}\,.$$

Now, using that $\nabla^g w(x) = \frac{1}{r}\nabla^{\bar{g}}\widetilde{w}(0)$, we obtain (34). Combining with (33), we obtain a uniform bound on $\nabla^g u(x)$ for $x \in B^g_{\frac{r}{2}}(x_0)$, and therefore u is Lipschitz continuous.

□

4.4. Refined penalization of the volume constraint

In order to investigate further regularity properties of the free boundary, we need to prove a more involved version (though localized) of the penalization property stated in Proposition 4.5: let u be a solution of (12), and $B^g_{r_0}(x_0)$ be a geodesic ball centered at $x_0 \in \partial\Omega_u$. We define

$$\mathcal{F} = \mathcal{F}(u, x_0, r_0) = \{v \in H^1(M), u - v \in H^1_0(B^g_{r_0}(x_0))\}. \tag{35}$$

For $h > 0$, we denote by $\mu_-(x_0, r_0, h)$ the largest $\mu_- \geq 0$ such that,

$$\forall\, v \in \mathcal{F} \text{ such that } m - h \leq \mathrm{Vol}_g(\Omega_v) \leq m,\ J(u) + \mu_-\mathrm{Vol}_g(\Omega_u) \leq J(v) + \mu_-\mathrm{Vol}_g(\Omega_v). \tag{36}$$

We also define $\mu_+(x_0, r_0, h)$ as the smallest $\mu_+ \geq 0$ such that,

$$\forall\, v \in \mathcal{F} \text{ such that } m \leq \mathrm{Vol}_g(\Omega_v) \leq m + h,\ J(u) + \mu_+\mathrm{Vol}_g(\Omega_u) \leq J(v) + \mu_+\mathrm{Vol}_g(\Omega_v). \tag{37}$$

The following result deals with the aymptotic behaviour of these penalization coefficients:

Proposition 4.12. *Let u be a solution of* (12)*. There exists $\Lambda = \Lambda_u \geq 0$ such that, for every $x_0 \in \partial\Omega_u$ and r_0 small enough, there exists $h_0 > 0$ such that,*

$$\forall\, h \in (0, h_0),\ \mu_-(x_0, r_0, h) \leq \Lambda \leq \mu_+(x_0, r_0, h) < +\infty,$$
$$\text{and, moreover, } \lim_{h\to 0}\mu_+(x_0, r_0, h) = \lim_{h\to 0}\mu_-(x_0, r_0, h) = \Lambda. \tag{38}$$

We start writing a weak optimality condition for the constrained problem (12); in this way, we can define Λ as a Lagrange multiplier.

Lemma 4.13 (Euler-Lagrange equation). *If u is a solution of* (12)*, then there exists $\Lambda = \Lambda_u \geq 0$ such that,*

$$\forall \Phi \in C^\infty(M, TM),\quad \int_M \Big[2\,g(D\Phi\nabla^g u, \nabla^g u) + (\lambda(m)u^2 - \|\nabla^g u\|^2_g)\,\mathrm{div}_g\Phi\Big]\,\mathrm{dvol}_g = \Lambda\int_{\Omega_u}\mathrm{div}_g\Phi\ \mathrm{dvol}_g\,. \tag{39}$$

Proof. For $\Phi \in C^\infty(M, TM)$ and $t \in \mathbb{R}$, we consider $u_t(x) = u(\exp_x(t\Phi(x)) \in H^1(M)$. If t is small enough, $x \mapsto \exp_x(t\Phi(x))$ is a C^1-diffeomorphism of M, so with a change of variable we get:

$$\mathrm{Vol}_g(\Omega_{u_t}) = \mathrm{Vol}_g(\Omega_u) - t\int_{\Omega_u}\mathrm{div}_g\Phi\,\mathrm{dvol}_g + o(t),$$

$$J(u_t) = J(u) + t\int_M\Big[2g(D\Phi\nabla^g u, \nabla^g u) - \|\nabla^g u\|^2_g\mathrm{div}_g\Phi + \lambda(m)u^2\mathrm{div}_g\Phi\Big]\,\mathrm{dvol}_g + o(t).$$

Moreover, the linear form $\Phi \mapsto \int_{\Omega_u}\mathrm{div}_g\Phi\,\mathrm{dvol}_g$ does not vanish, so we can apply the Karush-Kuhn-Tucker condition for the minimization of J among $v \in H^1(M)$ with the constraint $\mathrm{Vol}_g(\Omega_v) \leq m$: we get the existence of $\Lambda \in \mathbb{R}_+$ such that

$$\frac{d}{dt}J(u_t)_{|t=0} = -\Lambda\frac{d}{dt}\mathrm{Vol}_g(\Omega_{u_t})_{|t=0},\quad \forall\Phi \in C^\infty(M, TM),$$

which concludes the proof. □

Remark 4.14. *We can rewrite this Euler-Lagrange equation in the following way:*

$$\forall \Phi \in C^\infty(M, TM), \ \lim_{\varepsilon \to 0} \int_{\partial\{u>\varepsilon\}} g((\|\nabla^g u\|_g^2 - \Lambda)\Phi, \nu_\varepsilon)\, \mathrm{dvol}_g = 0,$$

where ν_ε denotes the outward unit normal to $\partial\{u > \varepsilon\}$, so this property can be seen as a very weak formulation of the extremality condition for λ_1, without regularity assumption. Indeed, we first notice that inside Ω_u,

$$2g(D\Phi\nabla^g u, \nabla^g u) - g(\nabla^g(\lambda(m)u^2 - \|\nabla^g u\|_g^2), \Phi) = 2\mathrm{div}_g[g(\Phi, \nabla^g u)\nabla^g u] - 2g((\underbrace{\Delta_g u + \lambda(m)u}_{=0})\Phi, \nabla^g u),$$

so using Gauss-Green formula, we obtain

$$\begin{aligned}
&\int_M \Big[2g(D\Phi\nabla^g u, \nabla^g u) + (\lambda(m)u^2 - \|\nabla^g u\|_g^2)\mathrm{div}_g\Phi\Big]\, \mathrm{dvol}_g \\
&\quad = \lim_{\varepsilon\to 0} \int_{\partial\{u>\varepsilon\}} g(2g(\Phi, \nabla^g u)\nabla^g u + (\lambda(m)u^2 - \|\nabla^g u\|_g^2)\Phi, \nu_\varepsilon)\, \mathrm{dvol}_g = \lim_{\varepsilon\to 0} \int_{\partial\{u>\varepsilon\}} \|\nabla^g u\|_g^2 g(\Phi, \nu_\varepsilon)\, \mathrm{dvol}_g
\end{aligned}$$

since $\nabla^g u = \|\nabla^g u\|_g \nu_\varepsilon$ on $\{u = \varepsilon\}$.

We are now in position to prove Proposition 4.12: the proof is very similar to [11, Theorem 1.5], and the adaptation to the framework of a manifold requires very little change. Therefore, we only sketch the argument; see also [36, Section 3.2.3] for a heuristical description of the argument and [45] in a different framework. Until the end of this section, u denotes a solution of (12), $x_0 \in \partial\Omega_u$, $r_0 > 0$, and $\mathcal{F}$ is defined in (35); we denote $\mu_\pm(h)$ instead of $\mu_\pm(x_0, r_0, h)$ to simplify the presentation and as no confusion is possible, and we denote B_r^g for $B_r^g(x_0)$. We first need the following lemma which helps obtaining existence results:

Lemma 4.15. *There exists a constant $C = C(u)$ such that for r_0 small enough,*

$$\forall v \in \mathcal{F}, \ \ J(v) \geq \frac{1}{2}\int_{B_{r_0}^g} \|\nabla^g v\|_g^2\, \mathrm{dvol}_g - C.$$

Proof. Let $v \in \mathcal{F}$; since $u - v \in H_0^1(B_{r_0}^g)$

$$\begin{aligned}
J(v) &\geq \int_M \|\nabla^g v\|_g^2\, \mathrm{dvol}_g - \lambda(m)\|v\|_{L^2(M)}^2 \geq \int_M \|\nabla^g v\|_g^2\, \mathrm{dvol}_g - 2\lambda(m)\left(\|u - v\|_{L^2(B_{r_0}^g)}^2 + \|u\|_{L^2(M)}^2\right) \\
&\geq \int_M \|\nabla^g v\|_g^2\, \mathrm{dvol}_g - 2\lambda(m)\left(\frac{\|\nabla^g(u-v)\|_{L^2(B_{r_0}^g)}^2}{\lambda_1(B_{r_0}^g)} + \frac{\|\nabla^g u\|_{L^2(M)}^2}{\lambda_m}\right) \\
&\geq \int_M \|\nabla^g v\|_g^2\, \mathrm{dvol}_g - 4\lambda(m)\left(\frac{\|\nabla^g u\|_{L^2(B_{r_0}^g)}^2 + \|\nabla^g v\|_{L^2(B_{r_0}^g)}^2}{\lambda_1(B_{r_0}^g)}\right) - 2\|\nabla^g u\|_{L^2(M)}^2 \\
&\geq \left(\int_{B_{r_0}^g} \|\nabla^g v\|_g^2\, \mathrm{dvol}_g\right)\left(1 - 4\frac{\lambda_m}{\lambda_1(B_{r_0}^g)}\right) - 2\|\nabla^g u\|_{L^2(M)}^2\left(1 + 2\frac{\lambda_m}{\lambda_1(B_{r_0}^g)}\right)
\end{aligned}$$

which gives the result if $\lambda_1(B_{r_0}^g)$ is large enough, which is true if r_0 is small enough. □

Remark 4.16. *This lemma implies that J is bounded from below on $\mathcal{F}$, and moreover that if $v_n \in \mathcal{F}$ is a sequence such that $J(v_n)$ is bounded, then $\|\nabla^g v_n\|_{L^2(B_{r_0}^g)}$ is also bounded. Since $v_n = u$ outside $B_{r_0}^g$ we deduce that v_n weakly converges up to a sub-sequence.*

Sketch of proof of Proposition 4.12: Let $\Lambda \geq 0$ be as in Lemma 4.13. The proof is divided in three steps.

- First step: $\Lambda \leq \mu_+(h) < +\infty$. To prove that $\mu_+(h)$ is finite, we first notice that $0 < \mathrm{Vol}_g(\Omega_u \cap B_{r_0}^g) < \mathrm{Vol}_g(B_{r_0}^g)$, see [11, Lemma 2.5] and Remark 4.8. We consider then $h \in (0, \mathrm{Vol}_g(B_{r_0}^g) - \mathrm{Vol}_g(\Omega_u \cap B_{r_0}^g))$ (and so, if $v \in \mathcal{F}$ with $\mathrm{Vol}_g(\Omega_v) \leq m + h$, then $\mathrm{Vol}_g(\Omega_v \cap B_{r_0}^g) < \mathrm{Vol}_g(B_{r_0}^g)$), and the optimization problem

$$\min\left\{J(v) + \mu\mathrm{Vol}_g(\Omega_v), v \in \mathcal{F}, m \leq \mathrm{Vol}_g(\Omega_v) \leq m + h\right\}.$$

Using Remark 4.16, we have existence of a solution v_μ. If $\mathrm{Vol}_g(\Omega_{v_{\mu_0}}) = m$ for some μ_0 then u is a solution to (37) with μ_0 and therefore $\mu_+(h)$ is finite: we will suppose to the contrary that $\mathrm{Vol}_g(\Omega_{v_\mu}) > m$ for all μ. As v_μ is solution to

$$J(v_\mu) = \min\Big\{J(w)\ ,\ w \in \mathcal{F}, m \leq \mathrm{Vol}_g(\Omega_w) \leq \mathrm{Vol}_g(\Omega_{v_\mu})\Big\},$$

we can write an Euler-Lagrange equation for v_μ with a similar proof to Lemma 4.13 and there exists $\Lambda_\mu \geq 0$ such that (39) is true for v_μ in place of u and for $\Phi \in C_c^\infty(B^g_{r_0}, TM)$. On one hand, using the optimality condition for v_μ we obtain that $\Lambda_\mu \geq \mu$, while on the other hand, using compactness and extracting from (39) formulas for Λ and Λ_μ we get $\lim_{\mu_n \to \infty} \Lambda_{\mu_n} = \Lambda$ for a sequence μ_n going to $+\infty$. This leads to a contradiction, so $\mu_+(h)$ is finite. To conclude this first step, we show that $\Lambda \leq \mu_+(h)$, again using the optimality condition for u.

- Second step: $\lim_{h\to 0} \mu_+(h) = \Lambda$**.** We first see that $\mu_+(h) > 0$ for $h > 0$. Indeed, if $\mu_+(h) = 0$ we can use

$$\text{for every } \varphi \in C_0^\infty(B^g_{r_0}) \text{ with } \mathrm{Vol}_g(\{\varphi \neq 0\}) < h,\ J(u) \leq J(u + t\varphi),$$

which leads to $-\Delta_g u = \lambda(m)u$ in $B^g_{r_0}$, and contradicts $\mathrm{Vol}_g(\Omega_u \cap B^g_{r_0}) < \mathrm{Vol}_g(B^g_{r_0})$. Let $\varepsilon > 0$. Since $h \mapsto \mu_+(h)$ is non-decreasing, we just have to see that $\mu_+(h) \leq \Lambda + \varepsilon$ for some $h > 0$. If $\Lambda > 0$, we restrict to $\varepsilon < \Lambda$ and define $\mu_\varepsilon(h) := \mu_+(h) - \varepsilon > 0$; if $\Lambda = 0$, we define $\mu_\varepsilon(h) = \mu_+(h)/2 > 0$. We apply the same strategy as in the first step to v_h solution to

$$\min\left\{J(v) + \mu_\varepsilon(h)\mathrm{Vol}_g(\Omega_v),\ \ v \in \mathcal{F}, \mathrm{Vol}_g(\Omega_v) \leq m + h\right\}.$$

We notive first that by definition of $\mu_+(h)$ we have $\mathrm{Vol}_g(\Omega_{v_h}) > m$. Denoting then Λ_h the Lagrange multiplier associated to v_h, we prove as before that $\Lambda_h \geq \mu_\varepsilon(h)$ and also that $\lim \Lambda_{h_n} = \Lambda$ for some sequence h_n going to 0. This leads to $\mu_+(h) \leq \Lambda + 2\varepsilon$ for some h and concludes this step.

- Third step: $\lim_{h\to 0} \mu_-(h) = \Lambda$**.** As in the first step, we first see that $\mu_-(h) \leq \Lambda$. As in the previous step, we study v_h solutions of the following minimization problem

$$\min\left\{J(w) + (\mu_-(h) + \varepsilon)\mathrm{Vol}_g(\Omega_w),\ \ w \in \mathcal{F},\ m - h \leq \mathrm{Vol}_g(\Omega_w) \leq m\right\}.$$

to deduce that $\lim_{h\to 0} \mu_-(h) = \Lambda$ and this concludes the proof. □

4.5. Positivity of the Lagrange multiplier

Proposition 4.17. *Let u a solution of* (12)*, and Λ_u given in Proposition 4.12. Then $\Lambda_u > 0$.*

We follow [9, Proof of Proposition 6.1], though we slightly simplify the presentation. For another argument, see [45, Appendix A] which relies only on the use of (39).

Proof. We argue by contradiction and suppose that $\Lambda = 0$. Our aim is to prove that under such assumption

$$-\Delta_g u = \lambda(m)u \quad \text{in the sense of distribution in } M$$

which asserts that the measure $\Delta_g u$ does not charge $\partial\Omega_u$. We introduce $(\omega_n)_{n\in\mathbb{N}}$ an increasing sequence of smooth open sets such that

$$\left\{x \in \Omega_u, d_g(x, \partial\Omega_u) \geq \frac{1}{n}\right\} \subset \omega_n \subset \overline{\omega_n} \subset \Omega_u,$$

so that $\bigcup_n \omega_n = \Omega_u$. Take $B = B^g_{r_0}(x)$ for some $x \in \partial\Omega_u$, and $r_0 > 0$.

Step 1: the gradient of u vanishes near $\partial\Omega_u$. Let $x_0 \in (\Omega_u \cap B) \setminus \omega_n$. Denote r the largest number such that $B^g_r(x_0) \subset \Omega_u$. We use the function v defined in (26); similarly to the proof in Lemma 4.7, we obtain from the definition of μ_+:

$$\frac{1}{\rho} \fint_{\partial B^g_\rho(x_0)} u \,\mathrm{dvol}_{g|_{\partial B^g_\rho(x_0)}} \leq C\sqrt{\mu_+(\mathrm{Vol}_g(B^g_\rho(x_0)))} \quad \forall B^g_\rho(x_0) \subset B \text{ such that } \mathrm{Vol}_g(\{u = 0\} \cap B^g_\rho(x_0)) > 0 \quad (40)$$

and from the fact that $\Lambda = 0$ and using Proposition 4.12, the right hand side converges to 0 when $\rho \to 0$. By definition of r, for every $\delta > 0$, $\{u = 0\} \cap B^g_{r+\delta}(x_0) \neq \emptyset$, so as in Remark 4.8 this implies that $\mathrm{Vol}_g(\{u = 0\} \cap B^g_{r+\delta}(x_0)) > 0$, and therefore we can use (40) and the monotonicity of μ_+:

$$\frac{1}{r+\delta} \fint_{\partial B^g_{r+\delta}(x_0)} u \,\mathrm{dvol}_{g|_{\partial B^g_{r+\delta}(x_0)}} \leq C\sqrt{\mu_+(\mathrm{Vol}_g(B^g_{r+\delta}(x_0)))} \leq C\sqrt{\mu_+(\mathrm{Vol}_g(B^g_{2r}(x_0)))} \quad \forall \delta < r.$$

We can let δ go to 0. Then, as in the proof of Corollary 4.9, we obtain

$$\|\nabla^g u(x_0)\|_g \leq C\left(\frac{1}{r} \fint_{\partial B^g_r(x_0)} u \,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} + r\right) \leq C\left(\sqrt{\mu_+(\mathrm{Vol}_g(B^g_{2/n}(x_0)))} + \frac{1}{n}\right)$$

and therefore $\|\nabla^g u\|_{\infty, B\setminus\omega_n}$ converges to 0 when n goes to ∞.

Step 2. Thanks to Proposition 4.1, we know that $\Delta_g u$ is a Radon measure. Let $\varphi \in C_c^\infty(B)$. We can write

$$\int_B \varphi \,\mathrm{d}(\Delta_g u) = \int_{B\cap\omega_n} \varphi \,\mathrm{d}(\Delta_g u) + \int_{B\setminus\omega_n} \varphi \,\mathrm{d}(\Delta_g u) \quad (41)$$

The first term is equal to $\lambda(m)\int_{B\cap\omega_n} u\varphi$ and converges to $\lambda(m)\int_B u\varphi$ when n goes to ∞. We are therefore aiming at proving that the second term in (41) converges to 0. We introduce the vector valued function H_n defined by

$$H_n \in H^1(M,\mathbb{R}^n), \qquad \left\{ \begin{array}{rcll} \Delta_g H_n & = & 0 & \text{in } \omega_n \\ H_n & = & \nabla^g u & \text{in } \omega_n^c \end{array} \right.$$

Then

$$\int_{B\setminus\omega_n} \varphi\, \mathrm{d}(\Delta_g u) = \int_{B\setminus\omega_n} \varphi\, \mathrm{d}(\mathrm{div}_g H_n) = \int_B \varphi\, \mathrm{d}(\mathrm{div}_g H_n) - \int_{B\cap\omega_n} \varphi\, \mathrm{d}(\mathrm{div}_g H_n) \tag{42}$$

The first term is equal to $-\int_B H_n\varphi\, \mathrm{dvol}_g$. As H_n uniformly converges to 0 on $B\cap\omega_n^c$ and is harmonic on ω_n, by maximum principle H_n converges to 0 uniformly on any compact subset of B and therefore this first term converges to 0. In order to deal with the second term, we couple the two following statements:

- first, we know that $\mathrm{div}_g(H_n)$ is harmonic in ω_n, and its trace on $\partial\omega_n$ is $\Delta_g u = -\lambda(m)u$, so it converges uniformly to 0 on any compact set of $\Omega_u \cap B$;
- second, as we have
$$\left\{ \begin{array}{rcll} \Delta_g(H_n - \nabla^g u) & = & \lambda(m)u & \text{in } \omega_n \\ H_n - \nabla^g u & = & 0 & \text{on } \partial\omega_n \end{array} \right.$$
we get the classical bound $\|\nabla^g(H_n - \nabla^g u)\|_{L^2(B\cap\omega_n)} \leq C\lambda(m)\|u\|_{L^2(\omega_n\cap B)} \leq \lambda(m)$, for some constant C, which leads to
$$\|\mathrm{div}_g(H_n)\|_{L^2(\omega_n\cap B)} \leq \|\mathrm{div}_g(H_n - \nabla^g u)\|_{L^2(\omega_n\cap B)} + \|\Delta_g u\|_{L^2(\omega_n\cap B)} \leq (1+C)\lambda(m).$$

Combing these two statements, we deduce with the Lebesgue convergence Theorem that the last term in (42) converges to 0, which concludes the proof. □

4.6. Non-degeneracy of the state function

When we proved the Lipschitz-continuity of u, we obtained an estimate from above of the gradient of u near the boundary of Ω_u. We are going to prove a similar estimate from below, which says that in a weak sense the gradient of u cannot vanish near $\partial\Omega_u$. To that end we use the penalization from below proven in Sections 4.4 and 4.5. We use a strategy from [3, Lemma 3.4].

Let u, x_0 and r_0 as in Section 4.4.

Lemma 4.18. *There exist c such that, for every ball $B_r^g(x) \subset B_{r_0}^g(x_0)$ with r small enough,*

$$\|u\|_{\infty, B_r^g(x)} \leq cr \quad \Longrightarrow \quad u \equiv 0 \text{ on } B_{r/2}^g(x)\,. \tag{43}$$

The strategy here is the opposite from Section 4.3, in the sense that we will consider a test function who is vanishing in a small ball. We will therefore use the penalization from below that has been proven in Sections 4.4 and 4.5. Compare to [3] and [11], the statement deals with the L^∞-norm instead of the L^1-average as in Section 4.3. This looks like a weaker result, but before proving Lemma 4.18, we show in the next statement that this other version is a classical corollary. This was noticed already in [4] where they state a result with L^p-average; even though it is likely possible to directly obtain the following statement in our framework, we felt that the construction of the test function to obtain Lemma 4.18 is slightly easier, see also [45].

Corollary 4.19. *There exists c such that, for every geodesic ball $B_r^g(x) \subset B_{r_0}^g(x_0)$ with r small enough,*

$$\frac{1}{r} \fint_{\partial B_r^g(x)} u\, \mathrm{dvol}_{g|_{\partial B_r^g(x)}} \leq c \quad \Longrightarrow \quad u \equiv 0 \text{ on } B_{r/4}^g(x) \tag{44}$$

Proof. Let $B_r^g(x) \subset B_{r_0}^g(x_0)$. As in the proof of Corollary 4.4, we introduce $w(y) = |y|^2$ in local coordinates centered at x, and $\widetilde{u} = u + \lambda(m)\|u\|_\infty w$ is subharmonic with respect to the metric g. Denoting ϕ the harmonic replacement of $\widetilde{u}$ with respect to the metric g, we have:

- from maximum principle, $0 \leq u \leq \widetilde{u} \leq \phi$,
- from Lemma 4.3 (choosing k such that the condition on the curvature is satisfied in $B_{r_0}^g(x_0)$, and r_0 small enough such that the area of $S_r^{g_k}(x)$ is close enough to the area of $S_r^g(x)$ for $r < r_0$), there exists a constant c_1 independant of x, r such that $\phi(x) \leq c_1 \fint_{\partial B_r^g(x)} \widetilde{u}\, \mathrm{dvol}_{g|_{\partial B_r^g(x)}}$
- from Harnack inequality ([29, Theorem 8.20], there exists c_2 also independant of x, r such that $\phi \leq c_2\phi(x)$ in $B_{r/2}(x)$.

Therefore there exists c' such that

$$\|u\|_{\infty, B_{r/2}^g(x)} \leq c' \left[\fint_{\partial B_r^g(x)} u\, \mathrm{dvol}_{g|_{\partial B_r^g(x)}} + r^2 \right].$$

We can therefore apply Lemma 4.18 in $B_{r/2}^g(x)$ and obtain that u vanishes on $B_{r/4}^g(x)$. □

Proof of Lemma 4.18. We introduce $w \in H^1(B^g_r(x))$ such that:

$$\left\{\begin{array}{rcll} -\Delta_g w & = & \beta & \text{in } B^g_r(x) \setminus \overline{B^g_{r/2}(x)} \\ w & = & \|u\|_{\infty, B^g_r(x)} & \text{in } \partial B^g_r(x) \\ w & = & 0 & \text{in } B^g_{r/2}(x). \end{array}\right.$$

where $\beta \in (0, \infty)$ will be chosen later. Then we define

$$v = \left\{\begin{array}{l} \min\{u, w\} \text{ in } B^g_r(x) \\ u \text{ in } M \setminus B^g_r(x) \end{array}\right. .$$

As $u \leq w$ on $\partial B^g_r(x)$, v has no discontinuity across $\partial B^g_r(x)$ and therefore $v \in H^1(M)$ can be used as a test function. Moreover, by construction $\Omega_v = \Omega_u \setminus B^g_{r/2}(x)$, so we can use the lower penalization given by Proposition 4.12 with $h = \mathrm{Vol}_g(B^g_{r/2}(x))$ and obtain, since $u = v$ outside $B^g_r(x)$,

$$\int_{B^g_r(x)} \|\nabla^g u\|^2_g \,\mathrm{dvol}_g - \lambda(m) \int_{B^g_r(x)} u^2 \,\mathrm{dvol}_g + \mu_-(h)\mathrm{Vol}_g(\Omega_u \cap B^g_r(x)) \leq$$

$$\leq \int_{B^g_r(x)} \|\nabla^g v\|^2_g \,\mathrm{dvol}_g - \lambda(m) \int_{B^g_r(x)} v^2 \,\mathrm{dvol}_g + \mu_-(h)\mathrm{Vol}_g(\Omega_v \cap B^g_r(x)),$$

which can be written:

$$\int_{B^g_{r/2}(x)} \|\nabla^g u\|^2_g \,\mathrm{dvol}_g + \mu_-(h)\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x)) \leq$$

$$\leq \int_{B^g_r(x)\setminus B^g_{r/2}(x)} \left(\|\nabla^g v\|^2_g - \|\nabla^g u\|^2_g\right) + \lambda(m) \int_{B^g_r(x)\setminus B^g_{r/2}(x)} (u^2 - v^2) \,\mathrm{dvol}_g + \lambda(m) \int_{B^g_{r/2}(x)} u^2 \,\mathrm{dvol}_g \tag{45}$$

The first term in the right hand side can be estimated with

$$\begin{aligned} \int_{B^g_r(x)\setminus B^g_{r/2}(x)} \left(\|\nabla^g v\|^2_g - \|\nabla^g u\|^2_g\right) \,\mathrm{dvol}_g &\leq \int_{B^g_r(x)\setminus B^g_{r/2}(x)} 2g(\nabla^g v, \nabla^g(v-u)) \,\mathrm{dvol}_g && (46) \\ &= 2\int_{(B^g_r(x)\setminus B^g_{r/2}(x))\cap\{u>w\}} g(\nabla^g(w-u), \nabla^g v) \,\mathrm{dvol}_g \\ &\leq -2\beta \int_{(B^g_r(x)\setminus B^g_{r/2}(x))\cap\{u>w\}} (w-u) \,\mathrm{dvol}_g && (47) \\ &\quad + \int_{\partial B^g_{r/2}(x)\cap\{u>w\}} \partial_n w (w-u) \,\mathrm{dvol}_g|_{\partial B^g_{r/2}(x)} && (48) \end{aligned}$$

where ∂_n denotes the normal derivative about $\partial B^g_{r/2}(x)$ (with respect to the metric g). We deal with the two other terms with

$$\int_{B^g_r(x)\setminus B^g_{r/2}(x)} (u^2 - v^2) \,\mathrm{dvol}_g \leq 2\|u\|_{\infty, B^g_r(x)} \int_{(B^g_r(x)\setminus B^g_{r/2}(x))\cap\{u>w\}} (u-w) \,\mathrm{dvol}_g$$

and

$$\int_{B^g_{r/2}(x)} u^2 \,\mathrm{dvol}_g \leq \|u\|^2_{\infty, B^g_r(x)} \,\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x)).$$

Choosing $\beta = \lambda(m)\|u\|_{\infty, B^g_r(x)}$ so that two terms cancel, and denoting

$$E(u, r) = \int_{B^g_{r/2}(x)} \|\nabla^g u\|^2_g \,\mathrm{dvol}_g + \mu_-(h)\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x))$$

we obtain

$$E(u, r) \leq \int_{\partial B^g_{r/2}(x)} (\partial_n w) u \,\mathrm{dvol}_{g|_{\partial B^g_{r/2}(x)}} + \lambda(m)\|u\|^2_{\infty, B^g_r(x)} \,\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x)). \tag{49}$$

Using classical elliptic regularity results ([29, Theorem 9.11 and 9.15]) and a scaling argument as in the proof of 4.9, we obtain the existence of C independant of x, r such that

$$\|\nabla^g w\|_{\infty, B^g_r(x)\setminus B^g_{r/2}(x)} \leq C\left[\frac{\|w\|_{\infty, B^g_r(x)}}{r} + \beta r\right] = C\|u\|_{\infty, B^g_r(x)}\left[\frac{1}{r} + \lambda(m) r\right].$$

On the other hand, working in the normal geodesic coordinates y centered at x such that $B^g_{r/2}(x_0)$ is parametrized by $B_{r/2}(0)$, using the test function $\varphi(y) := |y|^2/r$ we get that there exist C, C', C'' independant on x, r such that

$$\begin{aligned}
\int_{\partial B^g_{r/2}(x_0)} u \,\mathrm{dvol}_{g|_{\partial B^g_{r/2}(x_0)}} &= C\left[\int_{B^g_{r/2}(x_0)} g(\nabla^g u, \nabla^g \varphi)\,\mathrm{dvol}_g + \int_{B^g_{r/2}(x_0)} (\Delta_g \varphi) u \,\mathrm{dvol}_g\right] \\
&\leq C'\left(\int_{B^g_{r/2}(x_0)} \|\nabla^g u\|_g \,\mathrm{dvol}_g + \frac{1}{r}\int_{B^g_{r/2}(x_0)} u \,\mathrm{dvol}_g\right) \\
&\leq C'\left[\int_{B^g_{r/2}(x_0)\cap\Omega_u} \left(\frac{1}{2}\|\nabla^g u\|_g^2 + \frac{1}{2}\right)\mathrm{dvol}_g + \frac{\|u\|_{\infty, B^g_r(x_0)}}{r}\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x_0))\right] \\
&\leq C''\left[\int_{B^g_{r/2}(x_0)} \|\nabla^g u\|_g^2 \,\mathrm{dvol}_g + \left(\frac{1}{2} + \frac{\|u\|_{\infty, B^g_r(x_0)}}{r}\right)\mathrm{Vol}_g(\Omega_u \cap B^g_{r/2}(x_0))\right] \quad (50)
\end{aligned}$$

From Proposition 4.12, we can consider r small enough such that $h = \mathrm{Vol}_g(B^g_{r/2}(x_0))$ satisfies $\mu_-(h) \in [\Lambda/2, \Lambda]$ and therefore, using Proposition 4.17 asserting that $\Lambda > 0$, (49) and (50) leads to

$$E(u,r) \leq \left(\frac{\|u\|_{\infty, B^g_r(x_0)}}{r}\right)\left[C''(1+\lambda(m)r^2)\left[1 + \frac{2}{\Lambda}\left(\frac{1}{2} + \frac{\|u\|_{\infty, B^g_r(x_0)}}{r}\right)\right] + \lambda(m)\frac{2}{\Lambda}\frac{\|u\|_{\infty, B^g_r(x_0)}}{r}r^2\right]E(u,r)$$

Knowing that $r \leq r_0$, if $\frac{\|u\|_{\infty, B^g_r(x_0)}}{r}$ is small enough then $E(u,r)$ must vanish, and so does u in $B^g_{r/2}(x_0)$. □

4.7. Finite perimeter

As in the previous section we will use the penalization from below to prove the following result, whose proof is inspired by [45, Lemma 5.21]:

Proposition 4.20. *Let M be compact and u solution of* (12)*. Then Ω_u has finite perimeter.*

As in Remark 4.10, if M is noncompact then one obtains that Ω_u has locally finite perimeter.

Proof. As the manifold M is compact, given $x_0 \in \partial\Omega_u$, it is enough to prove that Ω_u has a finite perimeter inside a ball $B^g_r(x_0)$ for which, applying Popositions 4.12 and 4.17, $\mu_- := \mu_-(x_0, r_0, \mathrm{Vol}_g(B^g_r(x_0))) \geq \frac{\Lambda}{2} > 0$. We want to apply the same test function as in the proof of Proposition 4.5, but one needs to localize the argument. We introduce a cut-off function $\eta : M \to \mathbb{R}$ such that $\eta = 1$ in $B^g_{r/2}(x_0)$, $\eta = 0$ outside $B^g_r(x_0)$, $0 \leq \eta \leq 1$, and $\|\nabla^g \eta\|_g \leq C/r$. Then one can consider $u_t = \eta(u-t)_+ + (1-\eta)u$ for $t > 0$, which is such that $\Omega_{u_t} \subset \Omega_u$ and $\mathrm{Vol}_g(\Omega_{u_t}) \geq \mathrm{Vol}_g(\Omega_u) - \mathrm{Vol}_g(B^g_r(x_0))$. Therefore from (36) one has

$$\int_M \|\nabla^g u\|_g^2 - \lambda(m)\int_M u^2 + \mu_-\mathrm{Vol}_g(\Omega_u) \leq \int_M \|\nabla^g u_t\|_g^2 - \lambda(m)\int_M u_t^2 + \mu_-\mathrm{Vol}_g(\Omega_{u_t}). \quad (51)$$

We easily obtain the following estimates: there exists a constant C depending on u and r_0 such that for every $t \in (0,1]$,

- $\displaystyle\int_M (u^2 - u_t^2) \leq Ct$
- $\displaystyle\int_{\{u>t\}} \left(\|\nabla_g u\|_g^2 - \|\nabla_g u_t\|_g^2\right) \geq -Ct$
- $\displaystyle\int_{[B^g_r(x_0)\setminus B^g_r(x_0)]\cap\{u<t\}} \left(\|\nabla_g u\|_g^2 - \|\nabla_g u_t\|_g^2\right) \geq -Ct$

Also, one has $\mathrm{Vol}_g(\Omega_u) - \mathrm{Vol}_g(\Omega_{u_t}) = \mathrm{Vol}_g(\{0 < u < t\} \cap B^g_{r/2}(x_0))$, therefore, (51) now leads to

$$Ct \geq \int_{\{0<u<t\}\cap B^g_{r/2}(x_0)} \left[\|\nabla_g u\|_g^2 + \mu_-\right] \geq 2\sqrt{\mu_-}\int_{\{0<u<t\}\cap B^g_{r/2}(x_0)} \|\nabla_g u\|_g.$$

Applying the co-area formula, we obtain as in the proof of Proposition 4.5 that $P(\Omega_u, B^g_{r/2}(x_0)) \leq \frac{C}{2\sqrt{\mu_-}}$, which ends the proof by compactness. □

4.8. Density estimates

From the previous results, we can obtain a first weak regularity result:

Proposition 4.21. *Let u be a solution of* (12)*. Then there exist $\delta > 0$ and $r_0 > 0$ such that*

$$\delta \leq \frac{\mathrm{Vol}_g(\Omega_u \cap B_r^g(x_0))}{\mathrm{Vol}_g(B_r^g(x_0))} \leq 1 - \delta, \quad \textit{for any } x_0 \in \partial\Omega_u \textit{ and any } r < r_0.$$

Proof. The proof is classical, see also [3, 41]: by Lemma 4.18 asserting a non-degeneracy property for u, there exists $x_r \in B_{r/2}^g(x_0)$ such that $u(x_r) \geq \frac{c}{4} r$ where c is given in Lemma 4.18. Using now Lipschitz continuity of u (Corollary 4.9), we get that $u > 0$ on $B_{\theta r}^g(x_r)$ for some θ which does not depend on x_0; this leads to

$$\frac{\mathrm{Vol}_g(\Omega_u \cap B_r^g(x_0))}{\mathrm{Vol}_g(B_r^g(x_0))} \geq \frac{\mathrm{Vol}_g(B_{\theta r}^g(x_r))}{\mathrm{Vol}_g(B_r^g(x_0))}.$$

hence the lower estimate. For the upper bound, we go back to the proof of Lemma 4.7, where we obtained the following estimate (27) (v being defined by (26), and μ^* introduced in Proposition 4.5):

$$\mathrm{Vol}_g(\{u = 0\} \cap B_r^g(x_0)) \geq \frac{1}{\mu^*} \int_{B_r^g(x_0)} \|\nabla^g (u - v)\|_g^2 \, \mathrm{dvol}_g \, .$$

As $u = v$ on $\partial B_r^g(x_0)$, we also have:

$$\int_{B_r^g(x_0)} \|\nabla^g (u - v)\|_g^2 \, \mathrm{dvol}_g \geq \lambda_1(B_r^g(x_0)) \int_{B_r^g(x_0)} (u - v)^2 \, \mathrm{dvol}_g \, .$$

We want to obtain a lower bound for this last term, which will rely on the fact that u is small near x_0, while v is large. By Lipschitz continuity, we have $u(x) \leq L\kappa r$ in $B_{\kappa r}^g(x_0)$, where L is Lipschitz constant for u. On the other hand, we had shown that

$$v \geq c_6 \left(\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \right) (r - d_g(x, x_0)).$$

From Corollary 4.19 one has $\left(\frac{1}{r} \fint_{\partial B_r^g(x_0)} u \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \right) \geq c$ and therefore $v \geq c_7(1 - \kappa) r$ on $B_{\kappa r}^g(x_0)$ for some constant $c_7 > 0$. Choosing κ small enough, this leads to $v - u \geq c_8 r$ in $B_{\kappa r}^g(x_0)$ with $c_8 > 0$, which leads to

$$\mathrm{Vol}_g(\{u = 0\} \cap B_r^g(x_0)) \geq c r^2 \lambda_1(B_r^g(x_0)) \mathrm{Vol}_g(B_{\kappa r}^g(x_0))$$

and allows to conclude from the facts that $r^2 \lambda_1(B_r^g(x_0))$ is uniformly bounded from below for r small (see [26]), and $\mathrm{Vol}_g(B_{\kappa r}^g(x_0)) \geq c \mathrm{Vol}_g(B_r^g(x_0))$. □

4.9. Weiss-monotonicity formula in a manifold

Let u be a local minimum of (12) and $x_0 \in \partial\Omega_u$. Following [52], we define, for r small enough, the function

$$\phi_{u,x_0}^g(r) := \frac{1}{r^n} \int_{B_r^g(x_0)} \left(\|\nabla^g u\|_g^2 + \Lambda \mathbb{1}_{u>0} \right) \mathrm{dvol}_g - \frac{1}{r^{n+1}} \int_{\partial B_r^g(x_0)} u^2 \, \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} \tag{52}$$

We have the following:

Proposition 4.22. *There exists $C > 0$ and $r_0 > 0$ such that for all $r < r_0$,*

$$(\phi_{u,x_0}^g)'(r) \geq \frac{2}{r^n} \int_{\partial B_r^g(x_0)} \left(\partial_\nu u - \frac{u}{r} \right)^2 \mathrm{dvol}_{g|_{\partial B_r^g(x_0)}} - Cr \, . \tag{53}$$

We give immediately the following corollary of Proposition 4.22, which will be fundamental in the blow-up procedure of the following subsection:

Corollary 4.23. *The limit $\lim_{r \to 0^+} \phi_{u,x_0}^g(r)$ exists and is finite.*

Proof. It suffices to observe firstly that the function $\phi_{u,x_0}^g(r)$ is bounded for r small because u is Lipschitz continuous (4.9), and secondly that the function

$$r \to \phi_{u,x_0}^g(r) + \frac{C}{2} r^2 \, ,$$

is monotone nondecreasing by Proposition 4.22. □

We turn now to the proof of Proposition 4.22. We cannot proceed as in [41], as if we use optimality, one cannot control the penalization term with enough precision. Therefore, we use the approach from [52] using only the Euler-Lagrange equation from Lemma 4.13, which is available with volume constraint. See also [45] for a similar strategy.

Remark 4.24. *In [52], it is proven, when M is replaced by D a bounded domain in $\mathbb{R}^n$ and when g is the euclidian metric, that*

$$(\phi^e_{u,0})'(r) \;=\; \frac{2}{r^n}\int_{\partial B_r(0)}\Big(\partial_r u - \frac{u}{r}\Big)^2 \,\mathrm{dvol}_e\,, \tag{54}$$

if u is such that

$$\forall \Phi \in C^\infty_c(D,\mathbb{R}^n),\quad \int_D \Big[2\,(D\Phi\nabla^e u)\cdot\nabla^e u - |\nabla^e u|^2\,\mathrm{div}\Phi\Big]\mathrm{dvol}_e = \Lambda\int_{\Omega_u\cap D}\mathrm{div}\Phi\;\mathrm{dvol}_e. \tag{55}$$

which is the Euler-Lagrange equation (similarly to Lemma 4.13) for the minimization of the Alt-Caffarelli functional $u\mapsto \int_D |\nabla^e u|^2\mathrm{dvol}_e + \Lambda\mathrm{Vol}_e(\Omega_u\cap D)$. We will use this result when studying blow-up limits in the next section.

Proof of Proposition 4.22. We proceed in several steps.

First step. In this first step we just compute formally the derivative of the function ϕ^g_{u,x_0} with respect to r, without using the optimality of u. If we denote

$$A(r) = \int_{B^g_r(x_0)}\big(\|\nabla^g u\|^2_g + \Lambda\mathbb{1}_{u>0}\big)\;\mathrm{dvol}_g\,,$$

we classically have

$$A'(r) = \int_{\partial B^g_r(x_0)}\big(\|\nabla^g u\|^2_g + \Lambda\mathbb{1}_{u>0}\big)\;\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\,,$$

and then we can write:

$$\phi'_{u,x_0}(r) = -\frac{n}{r^{n+1}}A(r) + \frac{1}{r^n}A'(r) - \frac{d}{dr}\left[\frac{1}{r^{n+1}}\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right]. \tag{56}$$

Let us compute explicitly the third terms of (56):

$$\frac{d}{dr}\left[\frac{1}{r^{n+1}}\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right] = -\frac{n+1}{r^{n+2}}\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} + \frac{1}{r^{n+1}}\frac{d}{dr}\left[\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right]$$

and we have

$$\begin{aligned}\frac{d}{dr}\left[\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right] &= \int_{\partial B^g_r(x_0)} g(\nabla(u^2),\nu)\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} + \int_{\partial B^g_r(x_0)} u^2\,H\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\\ &= \int_{\partial B^g_r(x_0)} 2u\partial_\nu u\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} + \int_{\partial B^g_r(x_0)} u^2\,H\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\end{aligned}$$

where ν and H denote the outer normal vector to $\partial B^g_r(x_0)$ and the mean curvature (sum of the principal curvatures) respectively, and ∂_ν denotes the normal derivative with respect to the metric g. In conclusion, the third term of the second member of (56) is given by

$$\frac{d}{dr}\left[\frac{1}{r^{n+1}}\int_{\partial B^g_r(x_0)} u^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right] = \frac{1}{r^{n+1}}\int_{\partial B^g_r(x_0)}\left[\left(H-\frac{n+1}{r}\right)u^2 + 2u\partial_\nu u\right]\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\,.$$

Finally we obtain:

$$\phi'_{u,x_0}(r) = \frac{1}{r^{n+1}}\left\{-nA(r) + rA'(r) - \int_{\partial B^g_r(x_0)}\left[\left(H-\frac{n+1}{r}\right)u^2 + 2u\partial_\nu u\right]\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right\}. \tag{57}$$

Using the fact that on $\partial B^g_r(x_0)$ the mean curvature is

$$H = \frac{n-1}{r} + \mathcal{O}(r)$$

(where the error term $\mathcal{O}(r)$ is smooth in r) we obtain:

$$\phi'_{u,x_0}(r) = \frac{1}{r^{n+1}}\left\{-nA(r) + rA'(r) - \int_{\partial B^g_r(x_0)}\left[2u\partial_\nu u + \left(-\frac{2}{r}+\mathcal{O}(r)\right)u^2\right]\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right\} \tag{58}$$

Second step: We want now to compute the main order of the term

$$-nA(r) + rA'(r)\,,$$

and in this computation we will use the optimality condition for the function u. We claim that

$$\begin{aligned} -nA(r) + rA'(r) = (1+\mathcal{O}(r))(n+2)\lambda(m)\int_{B^g_r(x_0)} u^2 \mathrm{dvol}_g + \\ + (1+\mathcal{O}(r))\int_{\partial B^g_r(x_0)} \left[2u\partial_\nu u - 2r(\partial_\nu u)^2 - \lambda(m) r u^2\right] \mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\,, \end{aligned} \tag{59}$$

where the error term $\mathcal{O}(r)$ is smooth in r. In order to prove the claim, let $x = (x_1, ..., x_n)$ be normal geodesic coordinates aroung x_0 as in (18). Let Φ_ε be an approximation of

$$\Phi(\exp_{x_0}\Theta(x)) = \Theta(x)\mathbb{1}_{B^g_r(x_0)}\,.$$

More precisely, given $\varepsilon > 0$, we consider

$$\Phi_\varepsilon(\exp_{x_0}\Theta(x)) = \rho_\varepsilon(\exp_{x_0}\Theta(x))\,\Theta(x)$$

where $\rho_\varepsilon(\exp_{x_0}\Theta(x)) = \varphi_\varepsilon(d(\exp_{x_0}\Theta(x), x_0))$ is smooth and equal to 1 in $B^g_r(x_0)$, vanishes outside $B^g_{r+\varepsilon}(x_0)$, and such that φ_ε is decreasing on $[r, r+\varepsilon]$. Using Lemma 4.13 we obtain

$$\begin{aligned} \int_M \left[2\rho_\varepsilon\|\nabla^g u\|^2_g + \left(\lambda(m)u^2 - \|\nabla^g u\|^2_g\right)\mathrm{div}_g\Theta(x)\,\rho_\varepsilon + 2g((\Theta(x)\otimes\nabla^g\rho_\varepsilon)\nabla^g u, \nabla^g u) + \right. \\ \left. + \left(\lambda(m)u^2 - \|\nabla^g u\|^2_g\right) g(\Theta(x), \nabla^g\rho_\varepsilon)\right] \mathrm{dvol}_g = \Lambda\int_{\Omega_u}\left(\mathrm{div}_g\Theta(x)\,\rho_\varepsilon + g(\Theta(x), \nabla^g\rho_\varepsilon)\right)\mathrm{dvol}_g \end{aligned}$$

Passing to the limit $\varepsilon \to 0$, we obtain:

$$\begin{aligned} \int_{B^g_r(x_0)} \left[2\|\nabla^g u\|^2_g + \mathrm{div}_g\Theta(x)\left(\lambda(m)u^2 - \|\nabla^g u\|^2_g\right)\right] \mathrm{dvol}_g + \\ - (r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)} \left[2(\partial_\nu u)^2 + \lambda(m)u^2 - \|\nabla^g u\|^2_g\right] \mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} = \\ = \Lambda\left(\int_{B^g_r(x_0)} \mathrm{div}_g\Theta(x)\,\mathbb{1}_{u>0}\,\mathrm{dvol}_g - (r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)} \mathbb{1}_{u>0}\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\right)\,. \end{aligned} \tag{60}$$

In fact, in order to justify the previous passage, we use the fact that, since r is small, the metric g can be approximated with the Euclidean one. Then, first we have

$$\begin{aligned} \int_M g((\Theta(x)\otimes\nabla^g\rho_\varepsilon)\nabla^g u, \nabla^g u)\,\mathrm{dvol}_g = (1+\mathcal{O}(r))\int_{B_{r+\varepsilon}\setminus B_r} \varphi'_\varepsilon(|x|)\frac{x_i x_j \partial_i u \partial_j u}{|x|}\,dx \\ \underset{\varepsilon\to 0}{\longrightarrow} -(r+\mathcal{O}(r^2))\int_{\partial B_r}\frac{x_i x_j\partial_i u\partial_j u}{|x|^2}\,d\sigma_x = -(r+\mathcal{O}(r^2))\int_{\partial B_r}(\partial_r u)^2\,d\sigma_x = -(r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)}(\partial_\nu u)^2\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} \end{aligned}$$

where B_r represents the Euclidean ball of radius r, $|x|$ the Euclidean lengh of x, $dx, d\theta, ds$ the Euclidean Lebesgue measure with respect to the variables x, θ, s, and $d\sigma_x$ the Euclidean Lebesgue measure induced on ∂B_r. Secondly, for a general function f

$$\begin{aligned} \int_M f(x)g(\Theta(x), \nabla^g\rho_\varepsilon)\,\mathrm{dvol}_g = (1+\mathcal{O}(r))\int_{B_{r+\varepsilon}\setminus B_r} f(x)\varphi'_\varepsilon(|x|)|x|\,dx = (1+\mathcal{O}(r))\int_r^{r+\varepsilon} s^n\varphi'_\varepsilon(s)\int_{\mathbb{S}^{n-1}} f(r\theta)d\theta ds \\ \underset{\varepsilon\to 0}{\longrightarrow} -(1+\mathcal{O}(r))r^n\int_{\mathbb{S}^{n-1}} f(r\theta)d\theta = -(r+\mathcal{O}(r^2))\int_{\partial B_r} f\,d\sigma_x = -(r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)} f\,\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} \end{aligned}$$

where we used the same notation as before, and $d\theta, ds$ are the Euclidean Lebesgue measures with respect to the spherical variables θ, s. We observe that

$$\mathrm{div}_g\Theta(x) = \frac{1}{\sqrt{\det g}}\frac{\partial_i\left(\frac{\sqrt{\det g}}{g_{ii}}x_i\right)}{\partial x_i} = n + \mathcal{O}(r^2)$$

and then (60) can be rewritten as

$$(1+\mathcal{O}(r^2))\,(n\,A(r)-r\,A'(r)) = 2\int_{B^g_r(x_0)} \|\nabla^g u\|_g^2\,\mathrm{dvol}_g - 2(r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)} (\partial_\nu u)^2 \mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}$$
$$+\lambda(m)(n+\mathcal{O}(r^2))\int_{B^g_r(x_0)} u^2\mathrm{dvol}_g - \lambda(m)(r+\mathcal{O}(r^2))\int_{\partial B^g_r(x_0)} u^2\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}.$$

Using the fact the following easy consequence of the Green formula

$$\int_{B^g_r(x_0)} \|\nabla^g u\|_g^2 = \lambda(m)\int_{B^g_r(x_0)} u^2\mathrm{dvol}_g + \int_{\partial B^g_r(x_0)} u\partial_\nu u,$$

we finally obtain (59), and the claim is proved.

Third step: In this last step we use the boundary condition $u(x_0)=0$ and the Lipschitz continuous regularity of the optimal function u, which has been proven in Corollary 4.9.
Replacing (59) in (56) we obtain:

$$\begin{aligned}
\phi'_{u,x_0}(r) &= \frac{(1+\mathcal{O}(r))}{r^{n+1}}\Bigg[-(n+2)\lambda(m)\int_{B^g_r(x_0)} u^2\mathrm{dvol}_g + \\
&\qquad + \int_{\partial B^g_r(x_0)}\left(2r(\partial_\nu u)^2+\lambda(m)ru^2-4u\partial_\nu u+2\frac{u^2}{r}\right)\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} \\
&= (1+\mathcal{O}(r))\Bigg[\frac{2}{r^n}\int_{\partial B^g_r(x_0)}\left(\partial_\nu u-\frac{u}{r}\right)^2\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}} + \\
&\qquad -\frac{(n+2)\lambda(m)}{r^{n+1}}\int_{B^g_r(x_0)} u^2\mathrm{dvol}_g + \frac{\lambda(m)}{r^n}\int_{\partial B^g_r(x_0)} u^2\mathrm{dvol}_{g|_{\partial B^g_r(x_0)}}\Bigg]
\end{aligned}$$

Proposition 4.22 follows then from the facts that $u(x_0)=0$ and that u is Lipschitz continuous. □

4.10. Blow-up procedure

Let u be a solution of (12), $x_0\in\partial\Omega_u$. In this section we want to build and study the blow up of the solution u around x_0. To that end, given $\varepsilon>0$ small enough, we define on the manifold M the metric $\bar g := \varepsilon^{-2}\,g$ as we did at the beginning of this section, and the parameterization of $B^g_{\sqrt{\varepsilon}}(x_0)$ given by

$$Y(y) := \exp^g_{x_0}(\varepsilon\,\Theta(y))$$

with $y=(y_1,...,y_n)\in B_{\frac{1}{\sqrt{\varepsilon}}} := \left\{y\in\mathbb{R}^n\,/\,|y|<\frac{1}{\sqrt{\varepsilon}}\right\}$ and Θ defined in (18). In $B^g_{\sqrt{\varepsilon}}(x_0)$ the metric $\bar g$ is the Euclidean one up to ε terms. Let us consider $B_{\frac{1}{\sqrt{\varepsilon}}}$ with the metric $\bar g$ (induced by the parameterization Y). In the coordinates y beloning to $B_{\frac{1}{\sqrt{\varepsilon}}}$ we define

$$u_\varepsilon(y) := \frac{1}{\varepsilon}\,u(\exp^g_{x_0}(\varepsilon\,\Theta(y)))\,.$$

We have the following result:

Proposition 4.25. *Let u be a solution of* (12)*, $x_0\in\partial\Omega_u$. Then there exists a sequence ε_k going to 0 such that u_{ε_k} converges to a function $u_0:\mathbb{R}^n\to\mathbb{R}$ uniformly on any compact set. Moreover u_0 is nonnegative and Lipschitz continuous.*

We call such u_0 a blow-up limit of u at x_0.

Proof. Let $R<\frac{1}{\sqrt{\varepsilon}}$. With the notation we introduced above, we have

$$\nabla^{\bar g}u_\varepsilon(y) = \nabla^g u(\exp^g_{x_0}(\varepsilon\,\Theta(y)))$$

and

$$\|\nabla^{\bar g}u_\varepsilon\|_{\infty,B_R(0)} \le \|\nabla^g u\|_{\infty,B^g_{\varepsilon R}(x_0)}$$

and so the Euclidean gradient of u_ε is uniformly bounded in $B_R(0)$. Therefore, as $u(x_0)=0$, we also get for any $y\in B_R$,

$$|u_\varepsilon(y)| \le C\,\|\nabla^{\bar g}u_\varepsilon(y)\|_{\bar g}\,\|y\|_{\bar g} \le C'\,\|\nabla^g u\|_{\infty,B^g_{\varepsilon\,R}(x_0)}$$

which is also bounded uniformly in ε. From Arzelà-Ascoli Theorem, we deduce that up to a subsequence, u_ε converges uniformly to a function u_0 on $B_R(0)$. Using a diagonalization argument, we prove that up to a subsequence, u_ε converges to $u_0:\mathbb{R}^n\to\mathbb{R}$ uniformly on every ball $B_R(0)$ and therefore on any compact set. The properties of u_0 follow easily. □

Using the previous subsections, we are also in position to prove, similarly to [52], the following properties of blow ups:

Proposition 4.26. *Let u be a solution of* (12)*, $x_0 \in \partial\Omega_u$, and u_0 a blow-up limits of u at x_0. Then u_0 is (positively)* 1*-homogeneous and is a non-trivial global solution of the Alt-Caffarelli functional, which means:*

$$\int_{B_R(0)} |\nabla^e u_0|^2 + \Lambda \operatorname{Vol}_e(\Omega_{u_0} \cap B_R(0)) \leq \int_{B_R(0)} |\nabla^e w|^2 + \Lambda \operatorname{Vol}_e(\Omega_w \cap B_R(0)). \tag{61}$$

for every $R > 0$ and $w \in H^1_{loc}(\mathbb{R}^n)$ such that $w = u_0$ outside $B_R(0)$.

Proof. By definition, there exists ε_k going to 0 such that u_{ε_k} converges to u_0 on any compact set in $\mathbb{R}^n$. For the sake of clarity, we drop reference to the subsequence and denote ε instead of ε_k. We proceed by steps.

First step: we start improving the convergence. Following the proof of [41, Proposition 4.5(a)] we obtain that for any $R > 0$, u_ε converges to u_0 strongly in $H^1(B_R(0))$, and that $\Omega_\varepsilon := \{u_\varepsilon > 0\}$ converges to $\Omega_0 := \{u_0 > 0\}$ strongly in $L^1(B_R)$.

Second step: using the density estimate given in Proposition 4.21, it is classical that we have in fact convergence of $\overline{\Omega_\varepsilon}$ to $\overline{\Omega_0}$ and Ω_ε^c to Ω_0^c for the Hausdorff metric in B_R (see for example [37, Proof of Theorem 2]).

Third step: we now prove that u_0 is a global minimizer of the Alt-Caffarelli functional. Let $R > 0$ and $w \in H^1_{loc}(\mathbb{R}^n)$ such that $w = u_0$ outside $B_R(0)$. For any given v defined at least on $B_{\frac{1}{\sqrt{\varepsilon}}}$ of $\mathbb{R}^n$ for some ε, we define for $x < \sqrt{\varepsilon}$,

$$v^\varepsilon(\exp^g_{x_0}(\Theta(x))) = \varepsilon\, v\left(\frac{x}{\varepsilon}\right)$$

the blow-down of v centered at x_0, assuming that the injectivity radius at x_0 is bigger than $\sqrt{\varepsilon}$. Moreover, by scaling properties,

$$J(v^\varepsilon) = \int_{B^g_{\sqrt{\varepsilon}}(x_0)} \|\nabla^g v^\varepsilon\|^2_g \,\mathrm{dvol}_g + \lambda(m) \int_{B^g_{\sqrt{\varepsilon}}(x_0)} (v^\varepsilon)^2 \,\mathrm{dvol}_g = \varepsilon^n \left(\int_{B_{\frac{1}{\sqrt{\varepsilon}}}} \|\nabla^{\bar g} v\|^2_{\bar g} \,\mathrm{dvol}_{\bar g} + \lambda(m)\, \varepsilon^2 \int_{B_{\frac{1}{\sqrt{\varepsilon}}}} v^2 \,\mathrm{dvol}_{\bar g} \right)$$

and $\operatorname{Vol}_g(\Omega_{v^\varepsilon}) = \varepsilon^n \operatorname{Vol}_{\bar g}(\Omega_v)$. We introduce $\eta \in C^\infty_c(B_R(0))$ such that $0 \leq \eta \leq 1$, and define

$$w_\varepsilon := w + (1-\eta)(u_\varepsilon - u_0)$$

which is equal to u_ε outside $B_R(0)$. Therefore $(w_\varepsilon)^\varepsilon = u$ in $B^g_{\sqrt{\varepsilon}}(x_0) \setminus B^g_{\varepsilon R}(x_0)$, and therefore

$$h_\varepsilon := \left|\operatorname{Vol}_g(\Omega_{(w_\varepsilon)^\varepsilon}) - \operatorname{Vol}_g(\Omega_u)\right| \leq C\,\varepsilon^n$$

so by Proposition 4.12, we have for ε small enough

$$J(u) + \mu(h_\varepsilon)\operatorname{Vol}_g(\Omega_u) \leq J((w_\varepsilon)^\varepsilon) + \mu(h_\varepsilon)\operatorname{Vol}_g(\Omega_{(w_\varepsilon)^\varepsilon}).$$

So, using the previous scaling properties and localizing in B_R we obtain

$$\begin{aligned}\int_{B_R} \|\nabla^{\bar g} u_\varepsilon\|^2_{\bar g} \,\mathrm{dvol}_{\bar g} + \lambda(m)\,\varepsilon^2 \int_{B_R} (u_\varepsilon)^2 \,\mathrm{dvol}_{\bar g} + \mu(h_\varepsilon)\operatorname{Vol}_{\bar g}(\Omega_{u_\varepsilon} \cap B_R) \\ \leq \int_{B_R} \|\nabla^{\bar g} w_\varepsilon\|^2_{\bar g} \,\mathrm{dvol}_{\bar g} + \lambda(m)\,\varepsilon^2 \int_{B_R} (w_\varepsilon)^2 \,\mathrm{dvol}_{\bar g} + \mu(h_\varepsilon)\operatorname{Vol}_{\bar g}(\Omega_{w_\varepsilon} \cap B_R).\end{aligned}$$

We first use the inclusion $\Omega_{w_\varepsilon} \cap B_R \subset \{x \in B_R, w(x) > 0 \text{ and } \eta(x) = 1\} \cup \{x \in B_R, 0 \leq \eta(x) < 1\}$ to dominate $\operatorname{Vol}_{\bar g}(\Omega_{w_\varepsilon})$. Then using that u_ε converges to u_0 strongly in $H^1_{loc}(\mathbb{R}^n)$, and as the same goes for w_ε, we obtain as $\varepsilon \to 0$ (taking in account that the metric $\bar g$ converges uniformly to the Euclidean metric):

$$\int_{B_R} |\nabla^e u_0|^2 + \Lambda \operatorname{Vol}_e(\Omega_{u_0} \cap B_R) \leq \int_{B_R} |\nabla^e w|^2 + \Lambda(\operatorname{Vol}_e(\Omega_w \cap \{\eta = 1\}) + \operatorname{Vol}_e(\{0 \leq \eta < 1\} \cap B_R).$$

We conclude by choosing $\{\eta = 1\}$ arbitrary close to B_R.

Fourth step: Let us prove now that u_0 is 1-homogeneous. with the notations of Proposition 4.22 and by scaling properties, we have

$$\phi^{\bar g}_{u_\varepsilon,0}(r) = \phi^g_{u,x_0}(\varepsilon\, r)\,.$$

From Corollary 4.23, we know that $\phi^g_{u,x_0}(0^+)$ exists, and so

$$\lim_{\varepsilon \to 0^+} \phi^{\bar g}_{u_\varepsilon,0}(r) = \phi^g_{u,x_0}(0^+)\,.$$

From the convergence properties of the blow-up that we proved above, we have on the other hand that

$$\lim_{\varepsilon\to 0^+} \phi^{\bar{g}}_{u_\varepsilon,0}(r) = \phi^e_{u_0,0}(r)\,.$$

Combining the two previous results, we obtain that $(\phi^e_{u_0,0})'(r) = 0$, which implies from [53] (see Remark 4.24) that u_0 is 1-homogeneous.

Fifth step: Classically, from the non-degeneracy of u given in Lemma 4.18, we conclude that u_0 is non-trivial, see for example [41, Proposition 4.5].

The proof of the proposition is then complete. □

From the previous results, one can deduce a formula linking the limit at 0 of the Weiss functional $\phi^g_{u,x_0}(0^+)$ and the density of $x_0 \in \partial\Omega_u$.

Proposition 4.27. *(Density formula). Let u be a solution of* (12) *and $x_0 \in \partial\Omega_u$. For any blow-up u_0 of u at x_0, we have*

$$\theta(x_0) := \liminf_{r\to 0} \frac{\mathrm{Vol}_g(\Omega \cap B^g_r(x_0))}{\mathrm{Vol}_g(B^g_r(x_0))} = \frac{1}{\Lambda\omega_n}\phi^g_{u,x_0}(0^+) = \frac{\mathrm{Vol}_e(\{u_0 > 0\} \cap B_R)}{\mathrm{Vol}_e(B_R)}\,, \tag{62}$$

where ϕ^g_{u,x_0} is given by (52) *and $R > 0$.*

Proof. On one hand, because of the definition of blow ups, considering u_ε converging (up to a subsequence) to u_0, one has for ε small enough

$$\frac{\mathrm{Vol}_g(\Omega_u \cap B^g_\varepsilon(x_0))}{\mathrm{Vol}_g(B^g_\varepsilon(x_0))} = \frac{\mathrm{Vol}_{\bar{g}}(\Omega_{u_\varepsilon} \cap B_1)}{\mathrm{Vol}_{\bar{g}}(B_1)}$$

which gives at the limit $\varepsilon \to 0$,

$$\theta(x_0) = \frac{\mathrm{Vol}_e(\Omega_{u_0} \cap B_1)}{\mathrm{Vol}_e(B_1)}\,.$$

Moreover, u_0 being 1-homogeneous, the last term is independant on the radius of the considered ball. On the other hand, in the fourth step of the proof of Proposition 4.26, we have seen that $\phi^e_{u_0,0}(r) = \phi^g_{u,x_0}(0^+)$. As u_0 is 1-homogeneous, we also have

$$\phi^e_{u_0,0}(r) = \Lambda \frac{\mathrm{Vol}_e(\{u_0 > 0\} \cap B_r)}{r^d}\,,$$

which concludes the proof. □

We conclude this section with the following corollary about the possible values of the density:

Corollary 4.28. *(Density bound-density gap). Let u be a solution of* (12) *and $x_0 \in \partial\Omega_u$. Then $\theta(x_0) \geq \frac{1}{2}$, and there exists $\eta > 0$ independant on x_0 such that if $\theta(x_0) \neq \frac{1}{2}$, then $\theta(x_0) \geq \frac{1}{2} + \eta$.*

We do not reproduce the proof of this result, as it is of purely Euclidean nature, when combined with Proposition 4.27; see [41, Lemma 5.3 and 5.4] or [20, Proposition 6.1 and Lemma 6.3] for a detailed proof. For the sake of clarity, let us nevertheless recall the idea behind the first part of the result, namely that $\theta(x_0) \geq \frac{1}{2}$. Thanks to Propositions 4.26 and 4.27, this relies on the non-existence of a non-trivial 1-homogeneous harmonic function on a cone of density less that $\frac{1}{2}$. To see this, consider u_0 such a function, then its trace on the sphere $\mathbb{S}^{n-1}$ is a first eigenfunction of the Laplace-Beltrami operator on subdomain of the sphere with eigenvalue $n-1$. This cannot happen for a strict subset of the sphere, because of Faber-Krahn type results in the sphere.

The second part of the statement (the gap estimate) is more involved and relies on a "flatness imply regularity" result ([3, Theorem 8.1] or [23, Theorem 1]).

To conclude this section, we see that the optimality condition $\|\nabla^g u\|_g = \sqrt{\Lambda}$ is valid in the sense of viscosity.

Proposition 4.29. *Let u be a solution of* (12)*. The function u is a viscosity solution to $\|\nabla^g u\|_g = \sqrt{\Lambda}$ on $\partial\Omega_u$. This means that for every $x_0 \in \partial\Omega$:*

- *if $\varphi : M \to \mathbb{R}$ is differentiable at x_0 and such that $u \geq \varphi_+$ in Ω_u with equality at x_0, then $\|\nabla^g\varphi\|_g(x_0) \leq \sqrt{\Lambda}$,*
- *if $\varphi : M \to \mathbb{R}$ is differentiable at x_0 and such that $u \leq \varphi_+$ in Ω_u with equality at x_0, then $\|\nabla^g\varphi\|_g(x_0) \geq \sqrt{\Lambda}$.*

The proof of this result follows exactly the same lines as [41, Lemma 5.2] or [45, Lemmas 5.30 and 5.31] and relies again on Proposition 4.26 and the study of non-trivial 1-homogeneous global minimizers of the Alt-Caffarelli functional.

4.11. Proof of Theorem 1.2

We are now in position to prove Theorem 1.2. The proof is now very close to the proof of [41, Proposition 5.18], so we give fewer details; see also [45, Propositions 5.32 and 5.35] or [20, Corollary 7.2 and Theorem 8.1] for more detailed proofs.

Proof of Theorem 1.2. Let Ω^* be a solution of (7). Then as M is assumed to be connected, from Proposition 3.1 and 3.2, $\Omega^* = \Omega_u$ where $u = u_{\Omega^*}$ is a solution of (12).

The first point of the Theorem has been proved in Sections 4.3 and 4.7. Let us prove the second part of the statement.

(a) Let us define $\Sigma_{reg} := \{x_0 \in \partial\Omega^*, \theta(x_0) = \frac{1}{2}\}$. From the gap estimate in Corollary 4.28, it is easy to show that Σ_{reg} is relatively open in $\partial\Omega_u$. Moreover, if $x_0 \in \Sigma_{reg}$ from the convergence properties of the blow-up, one can see that the domain is flat in a neighborhood of x_0, see also [20, Proposition 6.2]. The function u satisfies

$$\begin{cases} \Delta_g u + \lambda_1(\Omega^*)\, u = 0 & \text{on } \Omega^* \\ u = 0 & \text{on } \partial\Omega^* \\ \|\nabla^g u\|_g = \sqrt{\Lambda} & \text{on } \partial\Omega^* \end{cases}$$

where the last equation is understood in the viscosity sense. We are then in position to apply [49, Appendix A] (which is an adaptation of [23]) , which implies that near x_0 the set $\partial\{u > 0\} = \partial\Omega^*$ is $C^{1,\alpha}$. Indeed the equation $-\Delta_g u = \lambda_1(\Omega^*)u$ can be written in divergence form

$$-\partial_i \left(g^{ij} \sqrt{|g|} \partial_j \right) u = \sqrt{|g|} \lambda_1(\Omega^*) u\,.$$

Moreover

$$\|\nabla^g u\|_g = \sqrt{g(\nabla^g u, \nabla^g u)} = \sqrt{(g^{ij}\partial_j u)^T\, g_{ij}\, (g^{ij}\partial_j u)} = \sqrt{(\nabla^e u)^T\, g^{ij}\, \nabla^e u} =$$
$$= \sqrt{(\nabla^e u)^T\, (g^{ij})^{\frac{1}{2}}\, (g^{ij})^{\frac{1}{2}}\, \nabla^e u} = \sqrt{((g^{ij})^{\frac{1}{2}}\, \nabla^e u)^T\, (g^{ij})^{\frac{1}{2}}\, \nabla^e u} = |(g^{ij})^{\frac{1}{2}}\, \nabla^e u|\,.$$

Then, if we define the matrix $A = A_{ij} = g^{ij}\sqrt{|g|}$, we have that u satisfies

$$\begin{cases} -\mathrm{div}(A \cdot \nabla^e u) = \sqrt{|g|}\, \lambda_1(\Omega^*)\, u & \text{on } \Omega^* \\ u = 0 & \text{on } \partial\Omega^* \\ |A^{\frac{1}{2}} \nabla^e u| = \sqrt{\Lambda}\, |g|^{\frac{1}{4}} & \text{on } \partial\Omega^* \end{cases}$$

and this allows us to use [49, Appendix A]. In order to obtain higher regularity for Σ_{reg}, we apply the classical results in [39].

(b) We use a classical reduction of dimension argument, that can be found in two forms in the literature, namely the Federer's reduction principle (see for example [48, Appendix A]), or the approach of Weiss following [30]. We follow the latter, though we only give the sketch of the proof as it is very similar to [41, Proposition 5.18], :

- if $n < k^*$, then any blow-up at $x_0 \in \partial\Omega_u$ is an 1-homogeneous global minimizer of the Alt-Caffarelli functional, and is therefore a half-plane, by definition of k^*, so it's density at the origin is $\frac{1}{2}$, which means by (62) that $\theta(x_0) = \frac{1}{2}$, so $x_0 \in \Sigma_{reg}$.
- if $n = k^*$, let assume by contradiction that there is an infinite set of points $x_n \in \Sigma_{sing} := \partial\Omega_u \setminus \Sigma_{reg}$ in M. Up to subsequence, we can assume that x_n converges to x_0, and as Σ_{sing} is closed, we still have $x_0 \in \Sigma_{sing}$. We denote $\varepsilon_n = d_g(x_n, x_0)$ and consider the blow-up around x_0 done with the functions u_{ε_n}, converging to Ω_0, a cone with singularity at 0.
  First we note that Ω_0 has only one singularity. Indeed, if it had another singularity, then by homogeneity it would have a line of singularity, which contradicts the results of Weiss ([53, Theorem 4.1]). Therefore denoting $\xi_n := \exp^g_{x_0} \Theta\left(\frac{x_n}{\varepsilon_n}\right) \in \partial\Omega_{u_{\varepsilon_n}}$, converging (up to a subsequence) to $\xi_0 \in \partial\Omega_0$, we know that ξ_0 is a regular point of Ω_0. As a consequence, for r_0 small enough, $\phi^g_{u_0,\xi_0}(r_0)$ is close to $\frac{1}{2}$. By convergence of u_{ε_n} to u_0, this means that $\phi^{\bar{g}}_{u_{\varepsilon_n},\xi_0}(r_0)$ is close to $\frac{1}{2}$ for n large enough. With similar computations as in [41], this implies that $\phi^{\bar{g}}_{u_{\varepsilon_n},\xi_n}(r_0)$ is close to $\frac{1}{2}$ for n large enough. By monotonicity, this implies $\phi^{\bar{g}}_{u_{\varepsilon_n},\xi_n}(r)$ is close to $\frac{1}{2}$ for small r, and in particular its limit when r goes to 0, which is the density of ξ_n in $\Omega_{u_{\varepsilon_n}}$. This is a contradiction as ξ_n is a singular point.
- Assume by contradiction that for some $s > n - k^*$ we have $\mathcal{H}^s(\Sigma_{sing}) > 0$. Then using again a blow-up analysis, and the density gap result, one can prove that there is $x_0 \in \Sigma_{sing}$ and Ω_0 a blow-up at x_0 whose singular set also has a positive $\mathcal{H}^s$ measure (the details follow the same lines as in [41, Proposition 5.18], itself relying on the strategy of [53]). This constitutes a contradiction and concludes the proof. □

Acknowledgements. This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR). P. Sicbaldi is partially supported by the grant "Ramón y Cajal 2015" RYC-2015-18730 and the grant "Análisis geométrico" MTM 2017-89677-P. The authors would like to thank B. Velichkov for valuable discussions on this work and for providing them an early version of [45].

[1] N. Aguilera, H. W. Alt, and L. A. Caffarelli. An optimization problem with volume constraint. *SIAM J. Control Optim.*, 24(2):191–198, 1986.

[2] F. J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Bull. Amer. Math. Soc.*, 81(1):151–154, 01 1975.

[3] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[4] H. W. Alt, L. A. Caffarelli, and A. Friedman. A free boundary problem for quasilinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 11(1):1–44, 1984.

[5] T. Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[6] B. Benson, I. Blank, and J. LeCrone. Mean value theorems for riemannian manifolds via the obstacle problem. *J. Geom. Anal.*, 2017.

[7] I. Blank and Z. Hao. The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators. *Comm. Anal. Geom.*, 23(1):129–158, 2015.

[8] A. Bonfiglioli and E. Laconelli. Subharmonic functions in sub-riemannian settings. *J. Eur. Math. Soc.*, 15:387–441, 2013.

[9] T. Briançon. Regularity of optimal shapes for the Dirichlet's energy with volume constraint. *ESAIM Control Optim. Calc. Var.*, 10(1):99–122 (electronic), 2004.

[10] T. Briançon, M. Hayouni, and M. Pierre. Lipschitz continuity of state functions in some optimal shaping. *Calc. Var. PDE*, 23(1):13–32, 2005.

[11] T. Briançon and J. Lamboley. Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints. *Ann. Inst. Poincaré An. nonlin.*, 26(4):1149–1163, 2009.

[12] D. Bucur. Do optimal shapes exist? *Milan J. Math.*, 75:379–398, 2007.

[13] Dorin Bucur and Giuseppe Buttazzo. *Variational Methods in Shape Optimization Problems*, volume 65 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser Basel, 2005.

[14] G. Buttazzo and G. Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.*, 122(2):183–195, 1993.

[15] G. Buttazzo and B. Velichkov. Shape optimization problems on metric measure spaces. *J. Funct. Anal.*, 264(1):1–33, 2013.

[16] A. Cañete and M. Ritoré. The isoperimetric problem in complete annuli of revolution with increasing Gauss curvature. *Proc. Roy. Soc. Edinburgh Sect. A*, 138(5):989–1003, 2008.

[17] L. A. Caffarelli, D. Jerison, and C.E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In *Noncompact problems at the intersection of geometry, analysis, and topology*, volume 350 of *Contemp. Math.*, pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[18] I. Chavel. *Eigenvalues in Riemannian geometry*, volume 115 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.

[19] G. David, M. Engelstein, and T. Smit Vega Garcia, M. Toro. Regularity for almost-minimizers of variable coefficient bernoulli-type functionals. *Preprint*, 2019.

[20] G. David, M. Engelstein, and T. Toro. Free boundary regularity for almost-minimizers. *Adv. Math.*, 350:1109–1192, 2019.

[21] G. David and T. Toro. Regularity of almost minimizers with free boundary. *Calc. Var. PDE*, 54(1):455–524, 2015.

[22] O. S. de Queiroz and L. S. Tavares. Almost minimizers for semilinear free boundary problems with variable coefficients. *Mathematische Nachrichten*, 291(10):1486–1501, 2018.

[23] D. De Silva. Free boundary regularity for a problem with right hand side. *Interfaces Free Bound.*, 13(2):223–238, 2011.

[24] D. De Silva and D. Jerison. A singular energy minimizing free boundary. *J. Reine Angew. Math.*, 635:1–21, 2009.

[25] E. Delay and P. Sicbaldi. Extremal domains for the first eigenvalue in a general compact riemannian manifold. *Discr. Cont. Dyn. Syst., Series A*, 35(12):5799–5825, 2015.

[26] O. Druet. Asymptotic expansion of the faber-krahn profile of a compact riemannian manifold. *C. R. Acad. Sci. Paris, Ser. I*, 346:1163–1167, 2008.

[27] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[28] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[29] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[30] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[31] E. Gonzalez, U. Massari, and I. Tamanini. On the regularity of boundaries of sets minimizing perimeter with a volume constraint. *Indiana Univ. Math.*, 32:25–37, 1983.

[32] M. Grüter. Boundary regularity for solutions of a partioning problem. *Arch. Rat. Mech. Anal.*, 97:261–270, 1987.

[33] M. Hayouni. Sur la minimisation de la première valeur propre du laplacien. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(7):551–556, 2000.

[34] A. Henrot and M. Pierre. *Variation et Optimisation de Formes*, volume 16 of *Springer Series in Computational Mathematics*. Springer, 2005. Une analyse géométrique.

[35] Antoine Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[36] A. Henrot (editor). *Shape optimization and Spectral theory*. De Gruyter, 2017.

[37] J. A. Iglesias and G. Mercier. Influence of dimension on the convergence of level-sets intotal variation regularization. *To appear in ESAIM: COCV*, 2020.

[38] D. Jerison and O. Savin. Some remarks on stability of cones for the one-phase free boundary problem. *Geom. Funct. Anal.*, 25(4):1240–1257, 2015.

[39] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[40] Jimmy Lamboley and Pieralberto Sicbaldi. New Examples of Extremal Domains for the First Eigenvalue of the Laplace-Beltrami Operator in a Riemannian Manifold with Boundary. *International Mathematics Research Notices*, 2015(18):8752–8798, 11 2014.

[41] D. Mazzoleni, S. Terracini, and B. Velichkov. Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.*, 27(2):373–426, 2017.

[42] F. Morabito and P. Sicbaldi. Delaunay type domains for an overdetermined elliptic problem in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$. *ESAIM: COCV*, 22(1):1–28, 2016.

[43] C. B. Morrey, Jr. *Multiple integrals in the calculus of variations*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition [MR0202511].

[44] F. Pacard and P. Sicbaldi. Extremal domains for the first eigenvalue of the laplace-beltrami operator. *Ann. Institut Fourier*, 59(2):515–542, 2009.

[45] E. Russ, B. Trey, and B. Velichkov. Existence and regularity of optimal shapes for elliptic operators with drift. *To appear in Calc. Var. PDE*, 2020.

[46] P. Sicbaldi. New extremal domains for the first eigenvalue of the laplacian in flat tori. *Calc. Var. PDE*, 37(3-4):329–344, 2010.

[47] P. Sicbaldi. Extremal domains of big volume for the first eigenvalue of the laplace-beltrami operator in a compact manifold. *Ann. Institut Poincaré (C) An. nonlin.*, 31:1231–1265, 2014.

[48] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University Centre for Mathematical Analysis, Canberra, 1983.

[49] L. Spolaor, B. Trey, and B. Velichkov. Free boundary regularity for a multiphase shape optimization problem. *Preprint*, 2018.

[50] B. Trey. Lipschitz continuity of the eigenfunctions on optimal sets for functionals with variable coefficients. *Preprint*, 2019.

[51] A. Wagner. Optimal shape problems for eigenvalues. *Comm. PDE*, 30(7-9):1039–1063, 2005.

[52] G. S. Weiss. Partial regularity for weak solutions of an elliptic free boundary problem. *Comm. PDE*, 23(3-4):439–455, 1998.

[53] G. S. Weiss. Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.*, 9(2):317–326, 1999.

[54] S. H. Zhu. *Comparison geometry*, volume 30 of *Math. Sci. Res. Inst. Publ.* Cambridge Univ. Press, Cambridge, 1997.