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# OPTIMAL LIPSCHITZ MAPS ON ONE-HOLED TORI AND THE THURSTON METRIC THEORY OF TEICHMÜLLER SPACE

YI HUANG AND ATHANASE PAPADOPOULOS

**ABSTRACT.** We study Thurston's Lipschitz and curve metrics, as well as the arc metric on the Teichmüller space of the torus equipped with hyperbolic metrics with one boundary component of fixed length. We construct natural Lipschitz maps between two such hyperbolic surfaces that generalize Thurston's stretch maps. The construction is based on maps between ideal Saccheri quadrilaterals. We prove the following: (1) On the Teichmüller space of the torus with one boundary component, the Lipschitz metric and the curve metric coincide and give a geodesic metric. (2) On the same Teichmüller space, the arc metric and the curve metrics coincide when the length of the boundary component is  $\leq 4 \operatorname{arcsinh}(1)$ , but differ when the boundary length is large.

We obtain several applications of this construction, including results on the Teichmüller spaces of closed hyperbolic surfaces: we construct novel Thurston geodesics and use them in particular to show that the sum-symmetrization of the Thurston metric is not Gromov hyperbolic.

*Keywords.*— Teichmüller space, hyperbolic surface, Lipschitz metric, curve metric, Thurston metric, arc metric, stretch map, stretch path, partial stretch path, geodesic, Gromov hyperbolicity.

*AMS classification.*— 32G15, 30F60, 30F10, 53C23, 53C70.

## 1. INTRODUCTION

Let  $S = S_{g,n}$  be a finite-type topological surface of negative Euler characteristic with genus  $g \geq 0$  and  $n \geq 0$  (open) borders labeled from 1 to  $n$ . We consider the following variants of Teichmüller space:

- $\mathcal{T}(S)$  is the space of homotopy classes of complete hyperbolic structures on  $S$ , where both cuspidal and hyperbolic boundary monodromy are admitted;
- $\mathcal{T}(S, \vec{b} = b_1, \dots, b_n)$  is the subset of  $\mathcal{T}(S)$  where the geodesic representative of the  $k$ -th boundary has length  $b_k \in [0, \infty)$ , here  $b_k = 0$  means a cusp;
- $\mathcal{T} := \mathcal{T}(S, \vec{0})$  is the subset of  $\mathcal{T}(S)$  consisting of homotopy classes of complete finite-area (cusps are admitted) hyperbolic structures on  $S$ .

We will, at times, need to consider the convex core of a complete hyperbolic surface  $(S, h)$  and we denote its convex core by  $(\bar{S}, \bar{h})$ , where  $\bar{S}$  is a closed bordered surface in  $S$  and  $\bar{h}$  is obtained from  $h$  by restricting to  $\bar{S}$  a homotopy representative of  $h$  which is geodesic on the boundary of  $\bar{S} \subset S$ . In this context, we adopt the notation  $\mathcal{T}(\bar{S})$  and  $\mathcal{T}(\bar{S}, \vec{b})$  to refer to Teichmüller spaces of geodesic-bordered (finite-area) hyperbolic surfaces. Note that  $\mathcal{T}(S)$  is naturally identified with  $\mathcal{T}(\bar{S})$ , and  $\mathcal{T}(S, \vec{b})$  with  $\mathcal{T}(\bar{S}, \vec{b})$  via the map which takes a complete marked hyperbolic surface to its convex core.

**1.1. Thurston's asymmetric metric.** Thurston defined in [14] two asymmetric metrics on the Teichmüller space  $\mathcal{T} = \mathcal{T}(S, \vec{0})$  of  $S$ . We recall their definitions:

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**Definition 1.1** (Thurston's Lipschitz metric). Let  $h_0$  and  $h_1$  be two hyperbolic structures on  $S$  and let  $\varphi : (S, h_0) \rightarrow (S, h_1)$  be a homeomorphism homotopic to the identity map on  $S$ . The *Lipschitz constant*  $\text{Lip}(\varphi)$  of  $\varphi$  is the quantity

$$\text{Lip}(\varphi) := \sup_{x \neq y \in S} \frac{d_{h_1}(\varphi(x), \varphi(y))}{d_{h_0}(x, y)}.$$

We denote by  $L(h_0, h_1)$  the infimum of the Lipschitz constant over all homeomorphisms  $\varphi : (S, h_0) \rightarrow (S, h_1)$  which are homotopic to the identity:

$$(1) \quad L(h_0, h_1) := \inf_{\varphi \sim \text{id}_S} \log \text{Lip}(\varphi).$$

The quantity  $L(h_0, h_1)$  depends only on the homotopy classes of  $h_0$  and  $h_1$ , and therefore descends to a real function on  $\mathcal{T} \times \mathcal{T}$ . Thurston showed in [14, §2] that  $L$  satisfies all (distance) metric axioms except for symmetry; that is, there exist hyperbolic structures  $h_0$  and  $h_1$  on  $S$  such that  $L(h_0, h_1) \neq L(h_1, h_0)$ . In this paper, we refer to asymmetric metrics as metrics for simplicity and, as such, we refer to  $L(\cdot, \cdot)$  as the *Lipschitz metric*.

**Definition 1.2** (Thurston's curve metric). Denote by  $\mathcal{S}$  the set of homotopy classes of essential simple closed curves on  $S$  (i.e. curves neither null-homotopic nor homotopic to a puncture). Consider the quantity

$$(2) \quad K(h_0, h_1) := \sup_{\gamma \in \mathcal{S}} \log \frac{l_{h_1}(\gamma)}{l_{h_0}(\gamma)}.$$

Thurston showed [14, §2] that  $K$  also defines an asymmetric metric on  $\mathcal{T}$ . We refer to  $K(\cdot, \cdot)$  as the *curve metric*.

In the same paper [14, Theorem 8.5], Thurston proved that  $K \equiv L$  and that it gives a geodesic metric on the Teichmüller space  $\mathcal{T}$  of complete finite-area metrics on  $S$ .

**1.2.  $k$ -Lipschitz maps and Thurston geodesics.** Thurston constructed a class of distinguished geodesics for the metric  $L$  (and  $K$ ). His construction is based on certain Lipschitz maps between ideal triangles that we now define. Consider the most symmetric foliation by horocycles of the ideal triangle. This is a foliation of the three cusps of the triangle by horocycle segments perpendicularly interpolating between boundaries, with a central unfoliated region bounded by horocyclic segments which meet tangentially at their ends (see Figure 1). We refer to the three boundary points where two distinct horocycle leaves meet as *anchor points*.

**Definition 1.3** ( $k$ -expansion map). For any given  $k \geq 1$ , the  $k$ -expansion map between two ideal triangles is defined to be the identity on the central unfoliated region and defined to send each horocycle at distance  $d \geq 0$  from this central region onto the horocycle (at the same cusp) at distance  $kd$  from this region, where each horocycle is mapped linearly with respect to its parametrization by arclength.

Thurston utilized  $k$ -expansion maps to construct rays (that is, one-parameter families parametrized by  $[0, \infty)$ ) of complete finite-area hyperbolic metrics on  $S$

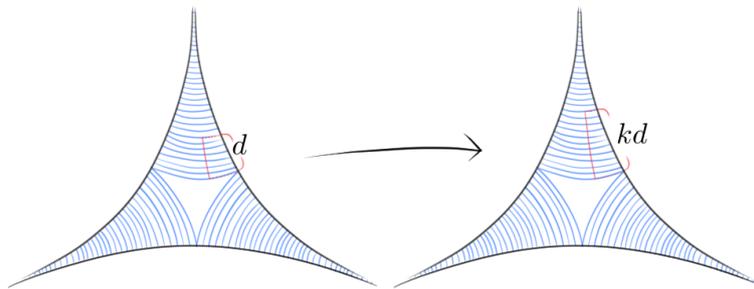
$$h_t := \text{stretch}(h_0, \lambda, t), \text{ for } t \geq 0,$$

where an initial hyperbolic metric  $h_0$  is stretched along a maximal geodesic lamination  $\lambda$ . In particular, the identity map

$$\text{id}_S : (S, h_0) \rightarrow (S, h_t)$$

is an  $e^t$ -Lipschitz map for every  $t \geq 0$ . We refer to these maps as *stretch maps*.

Thurston's stretch map construction produces hyperbolic metrics  $h_t$  by expanding distances along the maximal lamination  $\lambda$  by a factor of  $e^t$  and replacing the

FIGURE 1. The  $k$ -expansion map of an ideal hyperbolic triangle.

metric on the complementary triangles with the pullback (hyperbolic) metric with respect to  $e^t$ -expansion maps on ideal triangles. It is not difficult to see that this gives a well-defined construction when  $\lambda$  consists of finitely many leaves, but careful analysis is required when there are infinitely many. This is done by Thurston in [14, §4]. Stretch maps yield geodesic rays  $(S, h_t)$  for the Thurston metric. We refer to geodesic segments of  $(S, h_t)$  as *stretch paths*; they are at the heart of the results developed in [14].

**1.3. Thurston metric for bordered surfaces.** Neither the Lipschitz metric nor the curve metric, as respectively expressed by (1) and (2), form (asymmetric) metrics on  $\mathcal{T}(S)$  because they assume negative values (see [4, Theorem 1.8] and [10, Theorem 2.4]). However, (1) and (2) *do* give positive (asymmetric) metrics when restricted to Teichmüller spaces  $\mathcal{T}(S, \vec{b})$  with prescribed boundary monodromy [6, Theorem 7.9]. Combined with the result [4, Corollary 1.12] due to Guéritaud and Kassel that  $L$  and  $K$  are equal whenever either is positive, we see that these two metrics agree on  $\mathcal{T}(S, \vec{b})$  and naturally generalize the Thurston metric. Guéritaud-Kassel did not investigate the positivity of the naïve curve ratio metric (or Lipschitz metric) on  $\mathcal{T}(S, \vec{b})$ , because they instead achieve positivity by adjusting the Thurston metric with critical exponent renormalization factors [4, (1.6)]. The benefit of their approach is that they obtained a metric for a very general class of representations (or rather, characters) which in turn encode very different geometric objects. Guéritaud has communicated to us a succinct alternative argument for how to obtain the positivity of the naïve Thurston metric on  $\mathcal{T}(S, \vec{L})$  via their work with Danciger [2].

Guéritaud and Kassel prove that  $L \equiv K$  by establishing an equivariant form of the Kirszbraun-Valentine theorem as a machine for producing Lipschitz maps. There is no guarantee that the maps so-produced are injective. It is tempting, therefore, to wonder if one might be able to reutilize Thurston's stretch map construction to build optimal Lipschitz homeomorphisms and hence Thurston geodesics. Unfortunately, stretch maps with respect to maximal geodesic laminations (i.e. those whose complementary regions are ideal triangles) generally distort boundary monodromy and hence do not lie in  $\mathcal{T}(S, \vec{b})$ . There are, however, special cases where one is able to reuse Thurston's construction in a clever way (see, for example, Lenzhen-Rafi-Tao's [7, §6]).

**1.4. The arc metric.** The *arc metric* [9], based on lengths of simple orthogeodesics is a way of defining a nonnegative asymmetric metric on  $\mathcal{T}(S)$  in the guise of the Teichmüller space  $\mathcal{T}(\vec{S})$  of geodesically bordered hyperbolic surfaces.

**Definition 1.4** (The arc metric). Let  $\mathcal{A}$  denote the set of boundary-relative homotopy classes of arcs on  $\vec{S}$  (i.e. essential simple paths with endpoints on the

boundary  $\partial\bar{S}$  of  $\bar{S}$ , “essential” meaning that the curve is not homotopic to a piece of a boundary component) and consider the quantity

$$(3) \quad A(h_0, h_1) := \sup_{\alpha \in \mathcal{A} \cup \mathcal{S}} \log \frac{l_{h_1}(\alpha)}{l_{h_0}(\alpha)} = \sup_{\alpha \in \mathcal{A}} \log \frac{l_{h_1}(\alpha)}{l_{h_0}(\alpha)},$$

where  $l_h(\alpha)$  denotes the length of the unique orthogeodesic representative of  $\alpha$  in its boundary-relative homotopy class. We refer to  $A(\cdot, \cdot)$  as the *arc metric*.

The arc metric is more than superficially similar to the Thurston metric, it is in fact equal to the pullback of the Thurston metric with respect to the doubling embedding  $\mathcal{T}(\bar{S}) \hookrightarrow \mathcal{T}(dS)$ , where  $dS$  is the closed surface obtained by doubling  $\bar{S}$  along its boundary [9, Corollary 2.9].

**Remark 1.5.** In (3), the fact that the supremum of length ratios over the collection of arcs and curve  $\mathcal{A} \cup \mathcal{S}$  is equal to the supremum taken only over the collection of arcs. This is established in [9, Proposition 2.13], and comes from the fact that any sequence of arcs Dehn-twisted a high number of times about a curve  $\gamma$  (roughly) detects the length ratio for  $\gamma$ . In any case, we see definitionally that  $A \geq K$ .

The hitherto study of geodesics for this metric has been based on explicit constructions of Lipschitz maps between right angled hexagons (see [11] and its generalization in [12]).

**1.5. The one-holed torus.** Our work in the present paper centers on the one-holed torus case. We set  $S = S_{1,1}$  in notation such as  $\mathcal{T}(S, b)$ , where  $b = b_1 > 0$ , to denote the Teichmüller space of one-holed hyperbolic tori with prescribed boundary monodromy.

In §2, we construct natural generalizations of Thurston’s stretch maps for one-holed tori. Namely, we give an elementary construction of piece-wise smooth homeomorphisms between complete hyperbolic metrics in  $\mathcal{T}(S, b)$  with optimal Lipschitz constant.

Using these stretch map generalizations, we show in

- §3.1 that  $L \equiv K$  for  $\mathcal{T}(S, b)$  and that  $\mathcal{T}(S, b)$  is a geodesic space;
- §3.2 that the equality  $K \equiv A$  holds, when restricted to  $\mathcal{T}(S_{1,1}, b)$ , if the boundary length  $b \leq 4 \operatorname{arcsinh}(1)$ . However, this fails when  $b \gg 0$ ;
- §3.3 various constructions of Thurston geodesics on  $\mathcal{T}(S_{g,n}, \vec{b})$ ;
- §3.4 that for any closed surface of genus  $g \geq 2$ ,  $\mathcal{T}(S)$  cannot be Gromov hyperbolic with respect to the Thurston metric (or, more precisely, its symmetrization).

We develop many of these ideas further in an upcoming paper [5] for surfaces of general (finite) topological type, as well as the claim that the arc metric on  $\mathcal{T}(\bar{S})$  is in fact equal to the Lipschitz metric on  $\mathcal{T}(\bar{S})$  defined by considering optimal Lipschitz constants for geodesic-bordered hyperbolic surfaces. This is not easily obtainable as a consequence of doubling arguments, and we use Guéritaud and Kassel’s equivariant Kirszbraun-Valentine theory.

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## 2. LIPSCHITZ MAPS ON ONE-HOLED TORI

**2.1. Saccheri quadrilaterals.** Classically, a *Saccheri quadrilateral* in the hyperbolic plane is a geodesic quadrilateral with two opposite sides of equal length perpendicular to a third. See the quadrilateral  $ABCD$  in the left hand side of Figure 2,

where the two equal sides  $AD$  and  $BC$  are perpendicular to  $AB$ . The angles at  $C$  and  $D$  are then equal and necessarily acute. On the right hand side of this figure, we have represented an *ideal Saccheri quadrilateral*, by which we mean that the vertices  $C$  and  $D$  are ideal points at infinity and (hence) the sides  $AD$  and  $BC$  have infinite length. In this case, the angles at  $C$  and  $D$  are 0.

The isometry type of the ideal Saccheri quadrilateral  $ABCD$  is determined by the length of  $AB$ . Note that an ideal triangle may be regarded as a limit of ideal Saccheri quadrilaterals where the length of the side  $AB$  tends to 0.

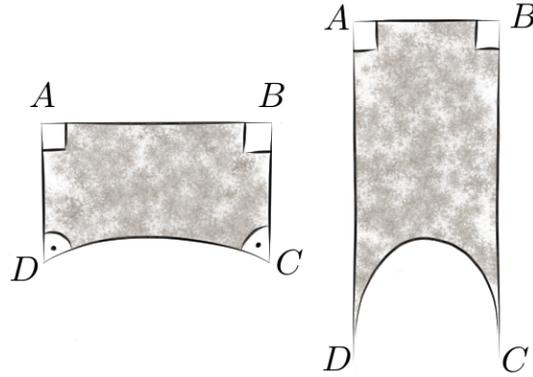


FIGURE 2. A Saccheri quadrilateral (left) and an ideal Saccheri quadrilateral (right)

**2.2. Partial horocyclic foliations.** An ideal Saccheri quadrilateral has two ideal vertices, and we foliate the neighborhood of each cusp with horocycles centered at these ideal vertices. We extend this foliation reflection-symmetrically until the two foliations meet tangentially in the middle (see Figure 3) and refer to this intersection point as an *anchor point*. This reflection-symmetric partial foliation of the quadrilateral is uniquely determined, and in the special case that the length of  $AB$  is 0, corresponds to “two out of three sectors” of the horocyclic foliation of the ideal triangle employed by Thurston. Figure 3 (a), (b) and (c) illustrate each of the three (mutually disjoint) situations that may arise with regards to the partial horocyclic foliation:

- (a) occurs when the length of  $AB$  is smaller than  $2 \operatorname{arcsinh}(1)$ ;
- (b) occurs when equal to  $2 \operatorname{arcsinh}(1)$ , and
- (c) occurs when greater than  $2 \operatorname{arcsinh}(1)$ .

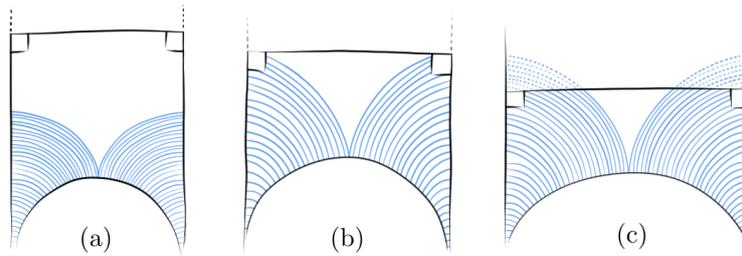


FIGURE 3. Partial horocyclic foliations of various ideal Saccheri quadrilaterals and their extensions (dotted lines).

In cases (a) and (b), we define  $k$ -*expansion maps* of an ideal Saccheri quadrilateral in much the same way as Thurston did for ideal triangles (see Definition 1.3). Specifically, the map is equal to the identity map on the unfoliated region and sends any leaf of the horocyclic foliation situated at distance  $d \geq 0$  from the unfoliated region to the one at distance  $kd$  from that region, mapping linearly with respect to arclength on each horocyclic leaf (see Figure 4). For case (c), however, this construction is hampered by  $AB$  excising some of the horocyclic segments. We resolve this issue by working instead with *extended ideal Saccheri quadrilaterals*: extend each ideal Saccheri quadrilateral (regarded as being embedded in the hyperbolic plane  $\mathbb{H}^2$ ) to the complete infinite-area convex domain bordered by  $CD$  and the two bi-infinite geodesics respectively containing  $AD$  and  $BC$  (see dotted lines in Figure 3) and define the  $k$ -expansion map on the extended ideal Saccheri quadrilateral.

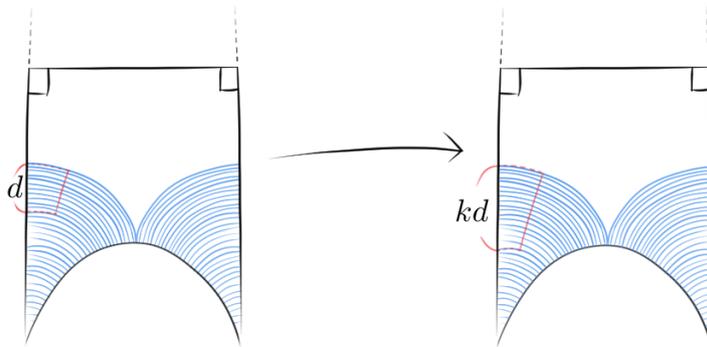


FIGURE 4. The  $k$ -expansion map of an ideal Saccheri quadrilateral

**Proposition 2.1.** *The  $k$ -expansion map of the (extended) ideal Saccheri quadrilateral has Lipschitz constant precisely equal to  $k$ .*

*Proof.* The proof is the same as Thurston’s proof for the  $k$ -expansion map on ideal triangles; proof details are omitted in [11, Proposition 2.2] but are easily recovered from (for example) [12, Lemma 5.2]. We may, in essence, ignore the unfoliated region. On the horocyclically foliated regions, the  $k$ -expansion map expands the orthogonal geodesic foliation by  $k$  and contracts the horocyclic foliation. The orthogonality of the two invariant foliations ensures  $k$ -Lipschitz-ness.  $\square$

**2.3. Partial stretch maps on one-holed tori.** We now construct  $k$ -Lipschitz maps between hyperbolic one-holed tori using  $k$ -expansion maps between ideal Saccheri quadrilaterals.

**Lemma 2.2** (Chain recurrence characterization). *A one-holed torus admits in the interior of its convex core three types of geodesic laminations which are chain recurrent (that is, which are limits in the Hausdorff topology of simple closed geodesics); see Figure 5:*

- (1) simple closed geodesics  $\gamma$ ;
- (2) the union of a simple closed geodesic  $\gamma$  and a bi-infinite geodesic  $\ell$  which spirals to  $\gamma$  from one side of  $\gamma$  and to  $\gamma^{-1}$  from the other side;
- (3) geodesic laminations  $\mu$  with uncountably many leaves corresponding to the support of some measured laminations.

*Proof.* We first note that all three of these types of geodesic laminations are in fact chain recurrent. Conversely, let  $\lambda$  be a chain recurrent geodesic lamination. We know that every (compactly supported) lamination contains a sublamination which supports a transverse measure. Any such sublamination of  $\lambda$  falls either into cases (1) or (3), which respectively correspond to the support of rational and irrational measured laminations on  $S$ . Since the complement of an irrational measured lamination on a one-holed torus is an annulus homotopy equivalent to the boundary of the convex core of  $S$  (one may see this by taking a train track approximating this lamination), case (3) is already maximal among chain recurrent laminations. On the other hand, simple closed geodesics can be extended in precisely two ways, both of which fall into class (2). This covers all possibilities for  $\lambda$ .  $\square$

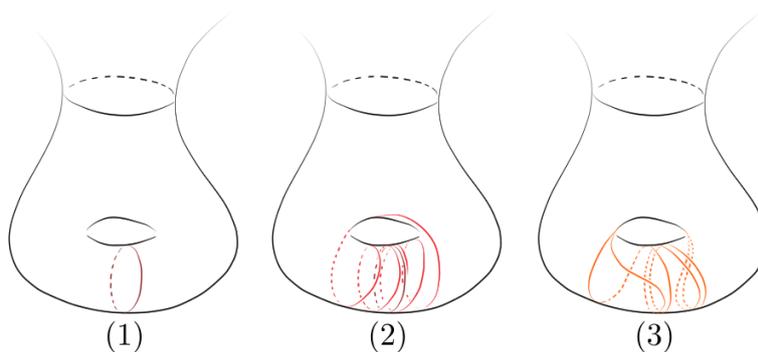


FIGURE 5. The three types of chain recurrent laminations classified in Lemma 2.2, the rightmost figure depicts a train-track carrying an irrational lamination.

**Theorem 2.3** (Partial stretch maps). *For any complete hyperbolic metric  $h_0$  in  $\mathcal{T}(S, b)$  and a chain recurrent lamination  $\lambda$ , there is a ray*

$$(4) \quad h_t := \text{pstretch}(h_0, \lambda, t), \text{ for } t \geq 0,$$

*of complete hyperbolic metrics in  $\mathcal{T}(S, b)$  such that*

- *for  $0 \leq t$ , the identity map  $\text{id}_S : (S, h_0) \rightarrow (S, h_t)$  is an  $e^t$ -Lipschitz homeomorphism which is an isometry outside of a compact set contained in the convex core of  $S$ ;*
- *the identity map expands arclength along  $\lambda$  by a factor of  $e^t$ .*

*In addition, the path  $(S, h_t)$  in  $\mathcal{T}(S, b)$  defines a geodesic ray for both the Lipschitz metric  $L$  and the curve metric  $K$ , with  $L \equiv K$  along  $(S, h_t)$ . We refer to subsegments of  $(S, h_t)$  as partial stretch paths.*

**Remark 2.4.** Note that we do not define partial stretch maps for arbitrary maximal (compactly supported) laminations in this paper, and require the stretched lamination to be invariant under the hyperelliptic involution  $\iota$  on  $S$ . This property is satisfied by all chain recurrent laminations as they are Hausdorff limits of simple closed geodesics, which are always invariant under  $\iota$ .

*Proof.* Lemma 2.2 lets us deal with this construction on a case-by-case basis.

**When  $\lambda$  is irrational.** This corresponds to case (3) of Lemma 2.2. The irrational lamination  $\lambda = \mu$  is fixed under the hyperelliptic involution on  $S$  because it is the limit of simple closed geodesics, which we know are fixed under the hyperelliptic involution  $\iota$  on  $S$ . There are precisely two (simple) orthogeodesic rays  $\sigma_1, \sigma_2$  on the convex core of  $S$  which spiral toward  $\mu$ , and the set  $\sigma_1 \cup \sigma_2$  is therefore also fixed

by  $\iota$  — in fact,  $\iota$  permutes  $\sigma_1$  and  $\sigma_2$ . Cutting the convex core of  $S$  along  $\mu$ ,  $\sigma_1$  and  $\sigma_2$  produces two ideal Saccheri quadrilaterals  $Q_1, Q_2$ . In particular, due to the  $\iota$ -invariance of  $\sigma_1 \cup \sigma_2$ , the involution  $\iota$  define an isometry between these two Saccheri quadrilaterals. This in turn means that the extended ideal Saccheri quadrilaterals  $\widehat{Q}_i$ , obtained by cutting  $S - \mu$  along the bi-infinite geodesics extending  $\sigma_1$  and  $\sigma_2$ , are isometric via  $\iota$ . Hence, the endpoints of the partial horocyclic foliations (see Figure 3) on  $\widehat{Q}_1$  and  $\widehat{Q}_2$  completely match up on the crowned hyperbolic surface  $S - \mu$  (see Figure 6). The upshot is that the  $k$ -expansion maps on  $\widehat{Q}_i$  glue together to give a  $k$ -expansion map on  $S - \mu$  with Lipschitz constant  $k$ .

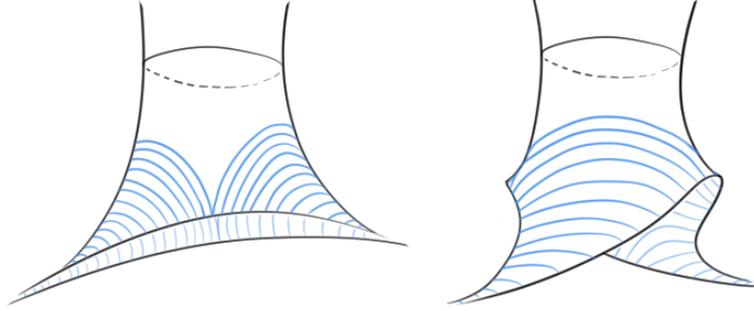


FIGURE 6. The partial horocyclic foliation on  $S - \mu$ , viewed from two different perspectives.

We now produce the family  $h_t$  of complete metrics on  $S$ . First, we observe that we can employ [14, Proposition 4.1] essentially without alteration by doubling the convex core of  $S$ , noting that any sufficiently small neighborhood  $N_\epsilon(\mu)$  of  $\mu$  is also doubled as  $\mu$  is compact and supported on the interior of the convex core. We use the previously constructed  $k$ -expansion map on  $S - \mu$ , with  $k = e^t$ , to redefine the metric outside of  $\mu$ . Since the  $k$ -expansion map takes points of distance  $d$  from the unfoliated region to points of distance  $e^t d$  away, the new “sharpness functions” (see the proof of [14, Corollary 4.2]) are rescaled by  $e^{-t}$  and we invoke Thurston’s Proposition 4.1 to extend the new metrics over  $N_\epsilon(\mu)$  and hence produce a family  $h_t$  of hyperbolic metrics on  $S$ . Since the  $k$ -expansion map is equal to the identity outside of a compact set, the metrics  $h_t$  are all isometric outside of a compact set. The developing map then tells us that the boundary monodromy remains constant.

**When  $\lambda$  is a simple closed geodesic.** This corresponds to case (1) of Lemma 2.2. In this scenario, we extend  $\lambda = \gamma$  to a chain recurrent lamination comprised of  $\gamma$  and a bi-infinite simple closed geodesic spiraling to  $\gamma$  from one side and  $\gamma^{-1}$  from the other. Thus, we have reduced this case to:

**Remaining case.** This corresponds to case (2) of Lemma 2.2, where  $\lambda$  is the union  $\gamma \cup \ell$  comprised of a simple closed geodesic  $\gamma$  and a bi-infinite geodesic  $\ell$  spiraling to  $\gamma$  on one side of  $\gamma$  and to  $\gamma^{-1}$  on the other side. We first observe that both  $\gamma$  and  $\ell$  are preserved under the hyperelliptic involution  $\iota$  on  $S$ . And just as with the case when  $\lambda$  is irrational, the crowned hyperbolic surface  $C := S - (\gamma \cup \ell)$  is obtained by gluing together two isometric extended ideal Saccheri quadrilaterals (see again Figure 6) whose partial horocyclic foliations perfectly align. Again, the upshot is that we obtain a  $k$ -expansion map on  $C$  which expands along its boundaries by a factor of  $K$ . As with the last step of the proof for the irrational  $\lambda$  case, we define  $h_t$  on  $C$  by pullback with respect to the  $e^t$ -expansion map, and extend  $h_t$  over  $\lambda = \gamma \cup \ell$  either by invoking [14, Proposition 4.1] or more naïvely by observing that the pullback metric on  $C$  glues continuously on the tangent spaces over  $\gamma$  and  $\ell$ .

**Geodesic ray**  $(S, h_t)$ . To complete the proof, we show that  $(S, h_t)$  defines a geodesic ray for both  $L$  and  $K$ . By construction, the stretch map  $\text{id}_S : (S, h_0) \rightarrow (S, h_t)$ , for  $t \geq 0$ , is an  $e^t$ -Lipschitz map which stretches  $\lambda$  by a factor of  $e^t$ . This means that (the measured lamination support in)  $\lambda$  realizes the maximum curve ratio between  $(S, h_0)$  and  $(S, h_t)$  for all time and  $K(h_0, h_t) = t$ . Moreover, observe that the composition of the  $e^s$ -expansion map and the  $e^t$ -expansion map on the ideal Saccheri quadrilateral is precisely the  $e^{s+t}$ -expansion map, and the respective sharpness function rescalings by  $e^{-s}$  and  $e^{-t}$  for the first two maps gives the requisite rescaling by  $e^{-s-t}$  needed for the third. This tells us that the stretch map  $\text{id}_S : (S, h_0) \rightarrow (S, h_s)$  composed with the stretch map  $\text{id}_S : (S, h_s) \rightarrow (S, h_{s+t})$  is precisely equal to the stretch map  $\text{id}_S : (S, h_0) \rightarrow (S, h_{s+t})$ . Therefore, we see that

$$(5) \quad K(h_s, h_{s+t}) = t \text{ for all } s, t \geq 0,$$

and hence  $(S, h_t)$  defines a geodesic ray for the curve metric  $K$ . Finally, the Lipschitz metric is at least the curve metric. Thus, we have

$$(6) \quad t \geq L(h_s, h_{s+t}) \geq K(h_s, h_{s+t}) = t,$$

where the first inequality follows from the stretch map  $\text{id}_S : (S, h_s) \rightarrow (S, h_{s+t})$  being  $e^t$ -Lipschitz. This in turn tells us that  $(S, h_t)$  is a geodesic ray for  $L$ .  $\square$

The partial measured foliations on the two ideal Saccheri quadrilaterals, when they are glued together, define a measured foliation class  $F$  on the torus with one boundary component. When the Teichmüller space of this surface with boundary is equipped with its Thurston boundary, we have the following:

**Theorem 2.5.** *The geodesic ray  $(S, h_t)$  defined in Theorem 2.3 converges, as  $t \rightarrow \infty$ , to the projective class  $[F]$  of  $F$ , considered as an element of Thurston's boundary of the Teichmüller space of the torus with one boundary component.*

*Proof.* The proof mimics the proof of Theorem 3.7 in [8] which asserts a similar property in the case of the Thurston metric of a non-bordered surface. It is based on a double inequality [8, Proposition 3.1] comparing hyperbolic length and intersection number.  $\square$

### 3. APPLICATIONS

**3.1. The Lipschitz metric versus the curve metric on  $\mathcal{T}(S = S_{1,1}, b)$ .** We now establish a generalization of Thurston's [14, Corollary 8.5]. The following result can be obtained as a corollary of a combiantion of either [2] or [6, Theorem 7.9] along with [4, Corollary 1.12]:

**Theorem 3.1.** *The Lipschitz metric  $L$  and the curve metric  $K$  on  $\mathcal{T}(S, b)$ , respectively defined by (1) and (2), are equal.*

We adapt Thurston's proof, which relies fundamentally upon [14, Theorem 8.2] and [14, Theorem 8.4].

**Definition 3.2** (ratio-maximizing laminations). Given a pair of marked hyperbolic metrics  $h_0, h_1$  in  $\mathcal{T}(S, b)$ , we say that a geodesic lamination  $\lambda$  is *ratio-maximizing* if there is a  $k$ -Lipschitz homeomorphism, where  $k := e^{K(h_0, h_1)}$ , from a neighborhood of  $\lambda$  in  $(S, h_0)$  to a neighborhood of  $\lambda$  in  $(S, h_1)$ , in the correct homotopy class.

We already know from [14, Proposition 4.1 or Theorem 8.1] that such a neighborhood map exists if  $\lambda$  is contained in the support of the projective measured lamination maximizing  $K$ . Thurston's insightful observation is that the notion of a "(length-)ratio-maximizing" lamination can be extended to (non-necessarily measured) geodesic laminations containing isolated bi-infinite geodesic leaves provided that one requires also that an immediate neighborhood of the lamination be

mapped across so as to preserve the optimal Lipschitz constant. Fortunately, the proofs for [14, Theorems 8.2 and 8.4] hold true in our context.

**Theorem 3.3** ([14, Theorems 8.2 and 8.4]). *There is a unique chain recurrent geodesic lamination which is ratio-maximizing and contains all other ratio-maximizing chain recurrent geodesic laminations for the pair  $h_0, h_1$ . We refer to it as the maximal ratio-maximizing lamination and denote it by  $\mu(h_0, h_1)$ . This lamination has the following property: if  $\{h_0^{(i)}\}$  and  $\{h_1^{(i)}\}$  are sequences of complete hyperbolic structures in  $\mathcal{T}(S, b)$  which respectively converge to  $h_0$  and  $h_1$ , then  $\mu(h_0, h_1)$  contains any lamination in the limit set of  $\mu(h_0^{(i)}, h_1^{(i)})$  with respect to the Hausdorff topology on the set of geodesic laminations on  $(S, h)$ .*

*Proof of Theorem 3.1.* We first note that  $\mu(h_0, h_1)$  cannot contain the boundary of the convex core as that geodesic is unstretched. Indeed, since the boundary monodromy is always fixed, every ratio-maximizing chain recurrent lamination we encounter during the course of this proof must lie within the interior of the convex core. Since  $\mu(h_0, h_1)$  is chain recurrent and supported on the interior of the convex core, it lies in one of the three classes described in Lemma 2.2. If  $\mu(h_0, h_1)$  is a maximal chain recurrent lamination (i.e. cases (2) and (3)), there is a unique partial stretch path

$$h_t := \text{pstretch}(h_0, \mu(h_0, h_1), tK(h_0, h_1))$$

which stretches along  $\mu(h_0, h_1)$ . We claim that this geodesic ray must reach  $h_1$  at time 1. If not, then there exists a first time  $0 < s < 1$  such that the topology of the maximal ratio-maximizing lamination  $\mu(h_t, h_1)$  changes. However, Theorem 3.3 tells us that  $\mu(h_t, h_1)$  necessarily contains  $\mu(h_{t-\epsilon}, h_1) = \mu(h_0, h_1)$ , which is impossible due to the maximality of  $\mu(h_0, h_1)$  among all chain recurrent laminations (supported on the interior of the convex core).

The only case that remains is when  $\mu(h_0, h_1)$  is a simple closed geodesic, in which event there are precisely two maximal chain recurrent laminations  $\mu_{\pm}$ , supported on the convex core interior, containing  $\mu(h_0, h_1)$ . We stretch along  $\mu_+$  without loss of generality. One of two things can happen: either we reach  $h_1$  or the maximal ratio-maximizing lamination changes at some point to a new lamination  $\mu'$ . Theorem 3.3 then tells us that  $\mu'$  is either  $\mu_+$  or  $\mu_-$ . However, it cannot be  $\mu_+$  as then  $\mu(h_0, h_1)$  would have been  $\mu_+$  in the first place, and hence must be  $\mu_-$ . Since  $\mu_-$  is maximal, we necessarily reach  $h_1$ . In particular, since  $\mu(h_0, h_1)$  is a part of the maximally stretched locus all throughout, the path must end precisely at time  $t = 1$ .

We have constructed geodesics to both  $L$  and  $K$  which join arbitrary points  $(S, h_0)$  and  $(S, h_1)$  in  $\mathcal{T}(S, b)$ . Moreover, by construction, the Lipschitz constant for the partial stretch map at  $t = 1$  is  $e^{K(h_0, h_1)}$ . Therefore, the two metrics must therefore be equal, as desired.  $\square$

The following corollary is immediate from the proof of Theorem 3.1.

**Corollary 3.4.** *Any two points in  $\mathcal{T}(S, b)$  are joined by a Thurston geodesic which is the concatenation of at most two partial stretch paths.*

### 3.2. The curve metric versus the arc metric on $\mathcal{T}(S, b)$ .

**Theorem 3.5.** *The curve metric  $K$  and the arc metric  $A$  on  $\mathcal{T}(S, b)$  are equal if  $b \leq 4 \operatorname{arcsinh}(1)$ .*

*Proof.* When  $b \leq 4 \operatorname{arcsinh}(1)$ , the partial horocyclic foliation is completely contained in the convex core of  $S$  and hence any partial stretch map simply evaluates to being the identity map on the convex core boundary. In particular, this says that the Lipschitz constant for the optimal Lipschitz map  $\text{id}_S : (S, h_0) \rightarrow (S, h_1)$  is

the same as the constant for the optimal Lipschitz map  $\text{id}_{\bar{S}} : (\bar{S}, \bar{h}_0) \rightarrow (\bar{S}, \bar{h}_1)$ , and hence we have the following chain of inequalities:

$$L(h_0, h_1) = K(h_0, h_1) \leq A(\bar{h}_0, \bar{h}_1) \leq L(\bar{h}_0, \bar{h}_1) = L(h_0, h_1),$$

where the first inequality is explained in Remark 1.5 and the second inequality is a general consequence of Lipschitz metrics being at least as great as length-ratio-type metrics.  $\square$

In contrast:

**Proposition 3.6.** *The arc metric is strictly greater than the Thurston metric on  $\mathcal{T}(S, b)$  for all sufficiently large  $b$ . That is to say, there exist  $(S, h_0)$  and  $(S, h_1)$  such that*

$$K(h_0, h_1) < A(h_0, h_1).$$

*Proof.* We sketch the construction required for this proof. Consider a right-angled hexagon  $H_0$  with alternating sidelengths

$$2 \operatorname{arcosh}(x^4), \operatorname{arcsinh}(x^2) \text{ and } \operatorname{arcsinh}(x^2);$$

and consider another right-angled hexagon  $H_1$  with alternating sidelengths

$$2 \operatorname{arcosh}(x^4), \operatorname{arcsinh}(x^3) \text{ and } \operatorname{arcsinh}(x^3),$$

where  $x$  is understood to be a very large number (see Figure 7).

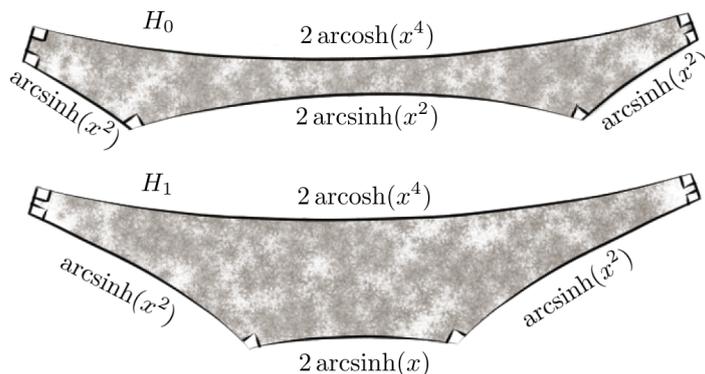


FIGURE 7. Two thin hexagons  $H_0$  and  $H_1$ .

We double the  $H_i$  to obtain pairs of pants  $P_i$  in such a way that the above listed sides remain unglued and form the boundaries of the  $P_i$ . We then glue the two  $P_i$  boundaries of equal length with no twisting to form two geodesic-bordered one-holed tori  $\bar{T}_0$  and  $\bar{T}_1$  in  $\mathcal{T}(\bar{S}, 4 \operatorname{arcosh}(x^4))$ . The two shortest interior simple closed geodesics  $\alpha, \beta$  (see Figure 8) on  $\bar{T}_0$  are both of length  $2 \operatorname{arcsinh}(x^2)$ , whereas on  $\bar{T}_1$  they are respectively of lengths  $2 \operatorname{arcsinh}(x)$  and  $2 \operatorname{arcsinh}(x^3)$ .

We regard  $\alpha$  and  $\beta$  as standard  $\mathbb{Z}$ -basis vectors  $(1, 0)$  and  $(0, 1)$  for  $H_1(\bar{T}_i; \mathbb{Z}) = \mathbb{Z}^2$ . The primitive elements of  $H_1(\bar{T}_i; \mathbb{Z})$  naturally biject with the collection of simple closed geodesics on  $\bar{T}_i$ . In particular, since these tori are so thin, the hyperbolic length of a geodesic  $\gamma$  whose homology class is  $(p, q)$  is approximately

$$2 \operatorname{arcsinh}(x^2) \cdot |p| + 2 \operatorname{arcsinh}(x^2)|q| \text{ on } \bar{T}_0, \text{ and}$$

$$2 \operatorname{arcsinh}(x) \cdot |p| + 2 \operatorname{arcsinh}(x^3)|q| \text{ on } \bar{T}_1.$$

Let  $T_i$  denote the infinite area complete hyperbolic extension of  $\bar{T}_i$ . For large  $x$ , the inverse hyperbolic sine function is close to a logarithm, and one sees that

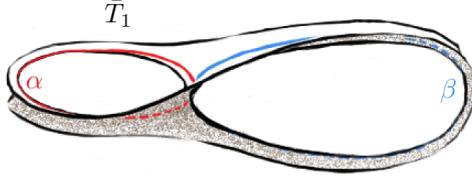


FIGURE 8. The thin torus  $\bar{T}_1$  and its two shortest geodesics of lengths  $l_\alpha = 2 \operatorname{arcsinh}(x)$  and  $l_\beta = 2 \operatorname{arcsinh}(x^3)$ .

$K(T_0, T_1) \approx \frac{3}{2}$ . On the other hand, there is an arc on  $\bar{S}$  whose lengths with respect to  $\bar{T}_0$  and  $\bar{T}_1$  are

$$\operatorname{arcsinh}\left(\frac{1}{\sqrt{x^4-1}}\right) \approx \frac{1}{x^2} \text{ and } \operatorname{arcsinh}\left(\frac{\sqrt{x^6+1}}{\sqrt{x^8-1}}\right) \approx \frac{1}{x}.$$

This means that the arc metric  $A(\bar{T}_0, \bar{T}_1)$  is at least (approximately)  $\log(x)$ , which far eclipses  $\frac{3}{2}$  for large  $x$ .  $\square$

**3.3. Novel geodesics on  $\mathcal{J}(S_{g,n}, \vec{b})$ .** There are strong and deep analogies between Teichmüller mappings and Thurston’s stretch maps. The former generate geodesics for the Teichmüller metric — a curve metric for extremal lengths, whilst the latter generate geodesics of the Thurston metric — a curve metric for hyperbolic lengths.

The analogy extends farther: Thurston’s stretch maps constitute specialized examples of optimal Lipschitz maps between hyperbolic surfaces  $(S, h_1)$  and  $(S, h_2)$ , stretching the leaves of a geodesic lamination and contracting the leaves of a (singular) measured foliation which intersects orthogonally the leaves of the lamination, whereas Teichmüller mappings yields optimal Lipschitz maps between collections  $\{(S, \hat{h}_1)\}$  and  $\{(S, \hat{h}_2)\}$  of specialized singular Euclidean metrics, stretching the leaves of a measured foliation (whose leaves are geodesics for the underlying Euclidean structure) and contracting the leaves of a transverse measured foliation which intersects it orthogonally. The (a priori) flexibility in the domain for the singular Euclidean metric Lipschitz optimization problem contrasts with the uniqueness of its solution: there is a unique Teichmüller mapping which optimizes the Lipschitz constant between certain (unique) domain and codomain metrics among the  $\{(S, \hat{h}_i)\}$ . On the other hand, for the hyperbolic metric Lipschitz optimization problem, the domain and codomain are predetermined, but there is flexibility in the optimal map. This apparent difference is due to differences in the “minimal stretch locus” (as denoted by  $E(j, \rho)$  in [4, Theorem 1.3]): the minimal stretch locus for the Teichmüller map is all of  $S$ , and corresponds to the support of a measured foliation; the minimal stretch locus for Thurston’s stretch map is a measure 0 subset of  $S$ , and corresponds to the support of a measured geodesic lamination.

In general, describing and establishing that a map is optimal Lipschitz even for simple examples is unexpectedly difficult. The partial stretch maps we describe in §2 resolve this in a concrete way for complete hyperbolic one-holed tori with fixed boundary monodromy. Moreover, the fact that they are isometric outside of a compact set easily enables additional gluing-map-based constructions, allowing for a rich family of novel Thurston geodesics. We give the following examples:

- (1) When the boundary geodesic representative satisfies  $b \leq 4 \operatorname{arcsinh}(1)$ , the stretch locus (i.e. the set of points on  $S$  where there is *any* metric distortion) lies within the convex core of the surface and we may double this convex along its boundary to yield partial stretch maps on closed genus 2 surfaces. Generally speaking, this is a new class of Thurston geodesics for  $\mathcal{J}(S_2)$ ,

as (classical) stretch maps (generically speaking<sup>1</sup>) distort the length of the central separating geodesic.

- (2) Whenever the stretch locus may be isometrically embedded in another (not necessarily hyperbolic) surface  $\Sigma$  of greater topological complexity, we obtain partial stretch maps on  $\Sigma$  by setting the new Lipschitz map to be the identity on the complement of the embedded stretch locus. One simple, but potentially useful instance of this arises (again) when  $b \leq 4 \operatorname{arcsinh}(1)$  and we extend  $S$  to  $\Sigma = S_{g,n}$  by gluing on a genus  $g - 1$  surface with  $n + 1$  holes. This construction produces new geodesics for the Thurston metric in  $\mathcal{T}(S_{g,n}, \vec{b})$ .
- (3) We may adapt Example (2) to work in far greater generality by performing small metric deformations with Lipschitz constant smaller than or equal to the Lipschitz constant on  $S$  (such as small Fenchel-Nielsen twists or earthquakes) in such a way as to not disturb the metric expansion along the embedded stretch locus of  $S \subset \Sigma$ . One example of this comes from adapting Example (1) when  $4 \operatorname{arcsinh}(1) < b \leq 4 \operatorname{arcosh}(\frac{3}{2})$ , where we double the surface (including its extruding stretch locus) and reglue to a genus 2 surface with a  $\frac{b}{4}$  twist. This enables the extruding stretch loci to fit on top of (originally) unstretched domain. The  $b = 4 \operatorname{arcosh}(\frac{3}{2})$  instance of this particular construction is used by Lenzhen-Rafi-Tao in their proof of [7, Theorem 1.1]. Another example of this is simultaneously performing stretching along  $\lambda$  as well as small earthquakes along measured laminations which do not transversely intersect  $\lambda$ .
- (4) All of our stretch maps (in the present paper) on one-holed tori  $S$  are invariant under the hyperelliptic involution  $\iota$  and hence descend to  $S/\iota$  — a hyperbolic sphere with three  $\pi$ -cone angles and one hole with geodesic representative of length  $\frac{b}{2}$  (see Figure 9(2)). This is also the quotient, with respect to a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, of a four-holed sphere  $S_{0,4}$  with boundary geodesic representatives of length  $\frac{b}{2}$  (see Figure 9(3)). This new building block immediately affords new flexibility in gluing-map based constructions.
- (5) As a final example, we observe that in Example (4), in instances where the stretching lamination contains a simple closed curve  $\gamma$  (see Figure 9(3)), this  $\gamma$  then descends to an interval  $\gamma'$  of length  $\frac{\ell_\gamma}{2}$  joining cone points in  $S/\iota$  and lifts to a separating geodesic  $\gamma''$  in  $S_{0,4}$  of length  $2\ell_\gamma$ . Since  $\gamma''$  lies on the stretching lamination, it remains geodesic under stretching, and so we may cut  $S_{0,4}$  along  $\gamma''$  to obtain two isometric pairs of pants  $P$  of boundary lengths  $2\ell_\gamma, \frac{b}{2}, \frac{b}{2}$ . In particular, the partial stretching map on  $P$  increases the length of just one of its boundaries. Moreover, we may glue the unstretched boundaries of  $P$  to then yield a stretching map on a one-holed torus which increases the boundary length but preserves the length of (at least) one interior simple closed geodesic. For  $b \leq 4 \operatorname{arcsinh}(1)$ , some of these examples may be used to construct completely novel arc metric geodesics.

**Remark 3.7.** The construction for Example (4) suffices to show that

$$(\mathcal{T}(S_{1,1}, 2b), L \equiv K) \text{ is isometric to } (\mathcal{T}(S_{0,4}, b, b, b, b), L \equiv K).$$

This positively answers a question posed by Walsh in [15, Paragraph after Theorem 7.9]. It is exceedingly unlikely for this isometry to extend to the other two cases posed:  $\mathcal{T}(S_{1,2}, \vec{0})$  versus  $\mathcal{T}(S_{0,5}, \vec{0})$  and  $\mathcal{T}(S_2)$  versus  $\mathcal{T}(S_{0,6}, \vec{0})$ .

<sup>1</sup>This is an intuitive statement, and we state it without proof.

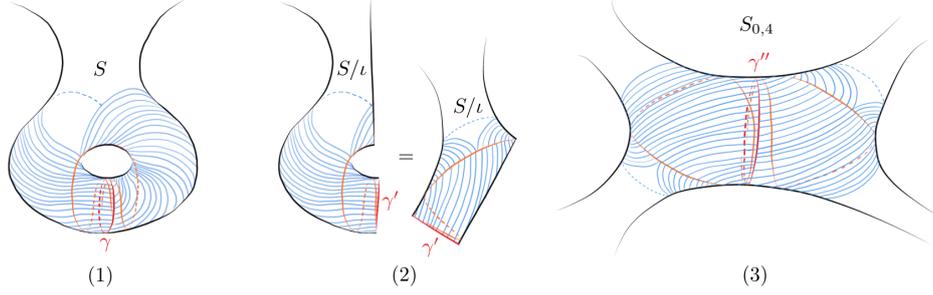


FIGURE 9. Partial horocycle foliations (blue) on (1) the one-holed torus of boundary length  $L$ ; (2) the quotient one-holed sphere with three  $\pi$ -cone points with boundary length  $\frac{L}{2}$ ; (3) the four-holed sphere with all boundaries of length  $\frac{L}{2}$ . The maximally stretched lamination consists of two geodesics: the support of the transverse measure (red) and the geodesic which spirals towards the former (orange).

**3.4. The metric geometry of the Thurston metric.** We further illustrate the versatility of partial stretch maps by proving the following claim inspired by [7, Theorem 1.1].

**Theorem 3.8** (Arbitrarily non-thin geodesic triangles). *For every  $g \geq 2$  and for every  $D > 0$ , there are points  $W, X, Y \in \mathcal{T}(S_g)$  joined by two-way Thurston geodesic segments  $G_{WX}$ ,  $G_{WY}$  and  $G_{XY}$  such that there is a point  $Z \in G_{XY}$  which is distance at least  $D$  away from  $G := G_{WX} \cup G_{WY}$  (i.e.: the distances  $K(Z, G)$  and  $K(G, Z)$  are both greater than  $D$ ). Moreover,  $G$  is also a two-way Thurston geodesic between  $X$  and  $Y$ .*

A *two-way geodesic* is a geodesic which is also a geodesic when parameterized in the reverse direction. Note that every geodesic for a symmetric metric is two-way.

*Proof.* We first give the proof for  $g = 2$ . For any metric  $h \in \mathcal{T}(S, b < 4 \operatorname{arcsinh}(1))$ , we consider the double  $(dS, dh)$  of the convex hull of  $(S, h)$  and denote by  $\gamma$  the separating simple closed geodesic we glued along to get  $dS$ . Thanks to  $b$  being small, there is a small collar neighborhood  $C$  around  $\gamma$  outside of the stretch locus and hence is unaffected by partial stretch maps on either of the components of  $dS - \gamma$ . Construct a smooth family of Lipschitz maps  $\phi_t : C \rightarrow C_t$  where  $C_t$  is the Fenchel-Nielsen twist of  $C$  by  $t$ .

For sufficiently large  $\tau$  (e.g.  $\tau \gg 2D \operatorname{arcsinh}(1)/b$ ) we can ensure that any metric on  $dS$  that comes from gluing (possibly with twisting) the convex cores of  $(S, h_0)$  and  $(S, h_1)$ , when Fenchel-Nielsen twisted by  $\phi_\tau$  on  $C$ , will give a map with Lipschitz constant  $\operatorname{Lip}(\phi_\tau) \gg D$ . We reparametrize and rescale such a family of Fenchel-Nielsen twist deformations  $\phi_{t \in [0, \tau]}$  on  $C$  so that the Lipschitz constant for  $\phi_t$  increases linearly with respect to  $t$  and finishes at  $\tau = 1$ .

We construct  $X, Y \in \mathcal{T}(S_2)$  as follows: choose arbitrary points  $h_0, h_1 \in \mathcal{T}(S, b)$  such that, without loss of generality,

$$K(h_0, h_1) \geq K(h_1, h_0) > 2 \operatorname{Lip}(\phi_\tau) \gg 2D,$$

and set  $X$  to be the convex hull of  $S$  with  $h_0$  on the left of  $\gamma$  and  $h_1$  (with the opposite orientation) on the right. Set  $Y$  to be  $h_1$  on the left of  $\gamma$  and  $h_0$  (with the opposite orientation) on the right. Performing simultaneous partial stretch maps on the two sides of  $\gamma$ , we produce a Thurston geodesic  $G$  (see, for example, Figure 10). In fact, this is necessarily a geodesic in both directions: from  $X$  to  $Y$  the stretching

on the left of  $\gamma$  dominates all other behavior and from  $Y$  to  $X$ , the stretching on the right of  $\gamma$  dominates all other behavior. Set the midpoint of this geodesic as  $W$  and define  $G_{WX}$  as the two-way Thurston geodesic between  $X$  and  $W$  and  $G_{WY}$  as the two-way Thurston geodesic between  $W$  and  $Y$ .

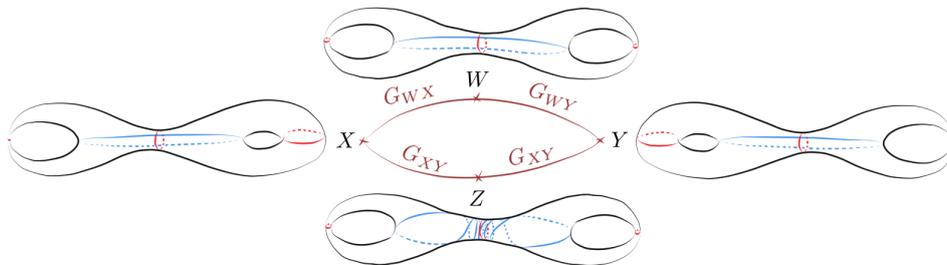


FIGURE 10. A depiction of the closed genus 2 hyperbolic surfaces  $W, X, Y, Z$  on the geodesics  $G_{WX}, G_{WY}$  and  $G_{XY}$ . Going from  $X$  to  $Y$ , the leftmost (red) curve is maximally stretched; from  $Y$  to  $X$  the rightmost (red) curve is maximally stretched;  $Z$  is distance  $D$  away from/to  $G = G_{WX} \cup G_{WY}$  because of the twisted (blue) curve.

Next, we produce another Thurston geodesic from  $X$  to  $Y$  by augmenting  $G$  with  $\phi_t$  on  $C$  until half-way  $t = \frac{1}{2}$  and then unwinding the Fenchel-Nielsen twist with  $\phi_{-t}$  until  $t = 1$ . Since  $K(h_0, h_1) \gg 2L$  and both are geodesics are uniformly parametrized (and we're producing actual Lipschitz maps for  $t \in [0, 1]$ ), the dominant behavior from  $X$  to  $Y$  is still dominated by the partial stretching on the left side of  $\gamma$  and from  $Y$  to  $X$  by the right side of  $\gamma$ . Therefore, this is once again a two-way Thurston geodesic. We label it as  $G_{XY}$  and denote the  $t = \frac{1}{2}$  midpoint by  $Z$ . Since  $Z$  has a lot of built in twisting, it is necessarily at least distance  $D$  from every single point in  $G$  (with distances measured in either direction).

For general  $g$ , the proof is essentially the same: one simply needs to glue in an additional "unstretched"  $S_{g-1,2}$  in between the convex hulls of  $(S, h_0), (S, h_1)$  which cap off the two ends.  $\square$

Roughly speaking, the above result says that there is no naïve sense in which the Thurston metric can be Gromov hyperbolic. However, without wanting to clarify what  $\delta$ -hyperbolicity might mean for an asymmetric metric, we instead make the following concrete statement:

**Corollary 3.9.** *The sum-symmetrization  $d_{\text{sum}}(h_0, h_1) := K(h_0, h_1) + K(h_1, h_0)$  of the Thurston metric on  $\mathcal{T}(S_g)$  is not Gromov hyperbolic:*

**Remarks 3.10.** We use the classical notion of Gromov hyperbolicity defined by the Gromov product, and make no assumptions about the metric being geodesic.

*Proof.* Two-way geodesics for the Thurston metric are geodesics for  $d_{\text{sum}}$ . Hence, the edges  $G_{WX}, G_{WY}, G_{XY}$  constitute the edges of a geodesic triangle  $\Delta_{WXY}$  for  $d_{\text{sum}}$ . It is evident that  $d_{\text{sum}} \geq K$ , and hence  $\Delta_{WXY}$  is not  $D$ -thin. By choosing  $D > 0$  to be arbitrarily large, this contradicts the condition that all geodesic triangles (if any exist) be  $\delta$ -thin for some  $\delta > 0$ , which in turn is a necessary (but not necessarily sufficient, due to the potential sparsity of geodesic triangles in metric spaces which might not be geodesic) condition for Gromov's  $\delta$ -hyperbolicity.  $\square$

**Remark 3.11.** Our proofs for Theorem 3.8 and Corollary 3.9 are fairly flexible, and one extends easily to  $\mathcal{T}(S_{g,n}, \vec{b})$  by gluing in undeformed surfaces or stretched  $S_{0,4}$  (Example (4) of §3.3). In particular, we can show (using this construction) that:

- $g \geq 2$  with arbitrary  $n \geq 0$ ;
- $g = 1$  and  $n \geq 2$ , with the condition that at least  $4 - n$  of the boundaries must have geodesic representatives of the same length  $b < 2 \operatorname{arcsinh}(1)$ ;
- $g = 0$  and  $n \geq 4$ , with the condition that at least  $8 - n$  of the boundaries must have geodesic representatives of the same length  $b < 2 \operatorname{arcsinh}(1)$ .

We leave the construction of these cases to interested readers.

This (informal) non-hyperbolicity of the Thurston metric (Theorem 3.8) juxtaposes with the fact that it shares an ideal boundary with an infinite dimensional “hyperbolic space” containing the Teichmüller space (endowed with the Weil-Petersson metric) as a subset [1, Proposition 15 and Corollary 16]. The crux of this paradox is that the “envelope” from  $X$  to  $Y$  [3], i.e. the union of all the geodesics from  $X$  to  $Y$ , becomes very “fat” as the distance between  $X$  and  $Y$  increases. Indeed, the geodesics  $G$  and  $G_{XY}$  constructed in the proof of Theorem 3.8 fail to fellow-travel. Moreover, due to the generality of our construction, either:

- (1) there are no arbitrarily long arc metric geodesic segments in the thick part of Teichmüller space, or
- (2) it is impossible to have a quasi-“thin triangles”-type claim of the same form as [13, Theorem E], whereby if a side of a geodesic triangle is in the thick part of Teichmüller space, then it lies near to one of the other two sides.

Nonetheless, it seems plausible that one might recover a weaker notion of hyperbolicity such as: there exists some  $\delta > 0$  so that any three points  $X, Y, Z$  form the vertices of some  $\delta$ -thin triangle. Or, perhaps the stronger statement that the envelope from  $X$  to  $Y$  lies within the union of the  $\delta$ -neighborhoods of the envelopes from  $X$  to  $Z$  and from  $Z$  to  $Y$ .

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