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Generalized logarithmic Hardy-Littlewood-Sobolev inequality

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This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, which allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

1 Main result and motivation

On \mathbb{R}^2 , let us define the *density of probability* $\mu = e^{-V}$ and the *external potential* V by

$$\mu(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \text{and} \quad V(x) := -\log \mu(x) = 2 \log(1+|x|^2) + \log \pi \quad \forall x \in \mathbb{R}^2.$$

We shall denote by $L_+^1(\mathbb{R}^2)$ the set of a.e. nonnegative functions in $L^1(\mathbb{R}^2)$. Our main result is the following *generalized logarithmic Hardy-Littlewood-Sobolev inequality*.

Theorem 1.1. For any $\alpha \geq 0$, we have that

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log \pi) \geq \frac{2}{M}(\alpha-1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \quad (1)$$

for any function $f \in L_+^1(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f dx > 0$. Moreover, the equality case is achieved by $f_\star = M\mu$ and f_\star is the unique optimal function for any $\alpha > 0$. \square

With $\alpha = 0$, the inequality is the classical *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy + M(1+\log \pi) \geq 0. \quad (2)$$

In that case f_\star is an optimal function as well as all functions generated by a translation and a scaling of f_\star . As long as the parameter α is in the range $0 \leq \alpha < 1$, the coefficient of the right-hand side of (1)

is negative and the inequality is essentially of the same nature as the one with $\alpha = 0$. It can indeed be written as

$$\int_{\mathbb{R}^2} f \log \left(\frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log \pi) + \frac{2}{M}(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \geq 0.$$

For reasons that will be made clear below, we shall call this range the *attractive range*.

If $\alpha = 1$, the inequality is almost trivial since

$$\int_{\mathbb{R}^2} f \log \left(\frac{f}{M} \right) dx + \int_{\mathbb{R}^2} V f dx = \int_{\mathbb{R}^2} f \log \left(\frac{f}{f_\star} \right) dx \geq 0 \quad (3)$$

is a straightforward consequence of Jensen's inequality. Now it is clear that by adding (2) multiplied by $(1-\alpha)$ and (3) multiplied by α , we recover (1) for any $\alpha \in [0, 1]$. As a consequence (1) is a straightforward interpolation between (2) and (3) in the *attractive range*.

Now, let us consider the *repulsive range* $\alpha > 1$. It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient $(\alpha - 1)$ in the right-hand side of (1) is now positive. Since

$$G(x) = -\frac{1}{2\pi} \log|x|$$

is the Green function associated with $-\Delta$ on \mathbb{R}^2 , so that we can define

$$(-\Delta)^{-1} f(x) = (G * f)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy,$$

it is interesting to write (1) as

$$\int_{\mathbb{R}^2} f \log \left(\frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f dx + \frac{4\pi}{M}(\alpha - 1) \int_{\mathbb{R}^2} f (-\Delta)^{-1} f dx \geq M(\alpha - 1)(1 + \log \pi). \quad (4)$$

If f has a sufficient decay as $|x| \rightarrow +\infty$, for instance if f is compactly supported, we know that $(-\Delta)^{-1} f(x) \sim -\frac{M}{2\pi} \log|x|$ for large values of $|x|$ and as a consequence,

$$\alpha V + \frac{4\pi}{M}(\alpha - 1)(-\Delta)^{-1} f \sim 2(\alpha + 1) \log|x| \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

In a minimization scheme, this prevents the runaway of the left-hand side in (4). On the other hand, $\int_{\mathbb{R}^2} f \log f dx$ prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (4) indeed admits a minimizer.

Inequality (2) was proved in [8] by E. Carlen and M. Loss (also see [2]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [7]: see Section 2 for a sketch of their proof. Our proof of Theorem 1.1 relies on an extension of this approach which takes into account the presence of the external potential V . A remarkable feature of this approach is that it is insensitive to the sign of $\alpha - 1$.

One of the key motivations for studying (4) arises from entropy methods applied to *drift-diffusion-Poisson* models which, after scaling out all physical parameters, are given by

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad (5)$$

with a nonlinear coupling given by the *Poisson* equation

$$-\varepsilon \Delta \phi = f. \quad (6)$$

Here $V = -\log \mu$ is the external *confining potential* and we choose it as in the statement of Theorem 1.1, while $\beta \geq 0$ is a coupling parameter with V , which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient ε in (6) is either $\varepsilon = -1$, which corresponds to the *attractive* case, or $\varepsilon = +1$, which corresponds to the *repulsive* case. In terms of applications, when $\varepsilon = -1$, (6) is the equation for the mean field potential obtained from Newton's law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case $\varepsilon = +1$ is used for repulsive electrostatic forces in semiconductor physics, electrolytes, plasmas and charged particle models.

In view of *entropy methods* applied to PDEs (see for instance [15]), it is natural to consider the *free energy functional*

$$\mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx \quad (7)$$

because, if $f > 0$ solves (5)-(6) and is smooth enough, with sufficient decay properties at infinity, then

$$\frac{d}{dt} \mathcal{F}_\beta[f(t, \cdot)] = - \int_{\mathbb{R}^2} f |\nabla \log f + \beta \nabla V + \nabla \phi|^2 \, dx \quad (8)$$

so that \mathcal{F}_β is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions \mathcal{F}_β is bounded from below. The answer is given by the following result.

Corollary 1.2. Let $M > 0$. If $\varepsilon = +1$, the functional \mathcal{F}_β is bounded from below and admits a minimizer on the set of the functions $f \in L^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f \, dx = M$ if $\beta \geq 1 + \frac{M}{8\pi}$. It is bounded from below if $\varepsilon = -1$, $\beta \geq 1 - \frac{M}{8\pi}$ and $M \leq 8\pi$. If $\varepsilon = +1$, the minimizer is unique. \square

As we shall see in Section 3.1, Corollary 1.2 is a simple consequence of Theorem 1.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with $\varepsilon = -1$ and $\beta = 0$, this has been used in [12, 4] to provide a sharp range of existence of the solutions to the evolution problem. In [6], the case $\varepsilon = -1$ with a potential V with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the *drift-diffusion-Poisson* model (5)-(6) and considerations on the *free energy*, in the electrostatic case, we can quote, among many others, [14, 13] and subsequent papers. In the Euclidean space with confining potentials, we shall refer to [10, 11, 3, 1]. However, as far as we know, these papers are primarily devoted to dimensions $d \geq 3$ and the sharp growth condition on V when $d = 2$ has not been studied so far. The goal of this paper is to fill this gap. The specific choice of V has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 3.3, we shall give a result for general potentials on \mathbb{R}^2 : see Theorem 3.4 for a statement.

2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (2) given by E. Carlen, J. Carrillo and M. Loss in [7]. The main idea is to use the nonlinear diffusion equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f}$$

with a nonnegative initial datum f_0 . The equation preserves the mass $M = \int_{\mathbb{R}^2} f \, dx$ and is such that

$$\frac{d}{dt} \left(\int_{\mathbb{R}^2} f \log f \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f ((-\Delta)^{-1} f) \, dx \right) = -\frac{8}{M} \left(\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx \int_{\mathbb{R}^2} f \, dx - \pi \int_{\mathbb{R}^2} f^{3/2} \, dx \right).$$

According to [9], the Gagliardo-Nirenberg inequality

$$\|\nabla g\|_2^2 \|g\|_4^4 \geq \pi \|g\|_6^6 \quad (9)$$

applied to $g = f^{1/4}$ guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [17]), we know that the solution behaves for large values of t like a self-similar solution, the so-called Barenblatt solution, which is given by $B(t, x) := t^{-2} f_\star(x/t)$. As a consequence, we find that

$$\begin{aligned} & \int_{\mathbb{R}^2} f_0 \log f_0 \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_0 ((-\Delta)^{-1} f_0) \, dx \\ & \geq \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^2} B \log B \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} B ((-\Delta)^{-1} B) \, dx = \int_{\mathbb{R}^2} f_\star \log f_\star \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_\star ((-\Delta)^{-1} f_\star) \, dx \end{aligned}$$

After an elementary computation, we observe that the above inequality is exactly (2) written for $f = f_0$.

The point is now to adapt this strategy to the case with an external potential. This justifies why we have to introduce a nonlinear diffusion equation with a drift. As we shall see below, the method is insensitive to α and applies when $\alpha > 1$ exactly as in the case $\alpha \in (0, 1)$. A natural question is whether solutions are regular enough to perform the computations below and in particular if they have a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum f_0 that is as smooth and decaying as $|x| \rightarrow +\infty$ as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [17] for regularity issues and to [15] for the truncation method. In the proof, we will therefore leave these issues aside, as they are purely technical.

Proof of Theorem 1.1. By homogeneity, we can assume that $M = 1$ without loss of generality and consider the evolution equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (x f).$$

1) Using simple integrations by parts, we compute

$$\int_{\mathbb{R}^2} (1 + \log f) \Delta \sqrt{f} \, dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx$$

and

$$\int_{\mathbb{R}^2} (1 + \log f) \nabla \cdot (x f) \, dx = - \int_{\mathbb{R}^2} \frac{\nabla f}{f} \cdot (x f) \, dx = - \int_{\mathbb{R}^2} x \cdot \nabla f \, dx = 2 \int_{\mathbb{R}^2} f \, dx = 2.$$

As a consequence, we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f \log f \, dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx + 8\pi \int_{\mathbb{R}^2} \mu^{3/2} \, dx \quad (10)$$

using

$$\int_{\mathbb{R}^2} \mu^{3/2} \, dx = \frac{1}{2\sqrt{\pi}}.$$

2) By elementary considerations again, we find that

$$4\pi \int_{\mathbb{R}^2} f(-\Delta)^{-1} (\Delta \sqrt{f}) dx = -4\pi \int_{\mathbb{R}^2} f^{3/2} dx$$

and

$$\begin{aligned} 4\pi \int_{\mathbb{R}^2} \nabla \cdot (xf) (-\Delta)^{-1} f dx &= -4\pi \int_{\mathbb{R}^2} xf \cdot \nabla (-\Delta)^{-1} f dx \\ &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) x \cdot \frac{x-y}{|x-y|^2} dx dy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) (x-y) \cdot \frac{x-y}{|x-y|^2} dx dy = 1 \end{aligned}$$

where, in the last line, we exchanged the variables x and y and took the half sum of the two expressions. This proves that

$$\frac{d}{dt} \left(4\pi \int_{\mathbb{R}^2} f (-\Delta)^{-1} f dx \right) = -8\pi \int_{\mathbb{R}^2} (f^{3/2} - \mu^{3/2}) dx. \quad (11)$$

3) We observe that

$$\mu(x) = \frac{1}{\pi(1+|x|^2)^2} = e^{-V(x)}$$

solves

$$\Delta V = -\Delta \log \mu = 8\pi \mu \quad (12)$$

and, as a consequence,

$$\int_{\mathbb{R}^2} V \Delta \sqrt{f} dx = \int_{\mathbb{R}^2} \Delta V \sqrt{f} dx = 8\pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx.$$

Since

$$\begin{aligned} 2\sqrt{\pi} \int_{\mathbb{R}^2} V \nabla \cdot (xf) dx &= -2\sqrt{\pi} \int_{\mathbb{R}^2} f x \cdot \nabla V dx = -8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{|x|^2}{1+|x|^2} f dx \\ &= -8\sqrt{\pi} + 8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{f}{1+|x|^2} dx = -8\sqrt{\pi} + 8\pi \int_{\mathbb{R}^2} \sqrt{\mu} f dx, \end{aligned}$$

we conclude that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f V dx = 8\pi \int_{\mathbb{R}^2} (\mu \sqrt{f} + \sqrt{\mu} f - 2\mu^{3/2}) dx. \quad (13)$$

Let us define

$$\mathcal{F}[f] := \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1-\alpha)(1+\log \pi) + 2(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x-y| dx dy.$$

Collecting (10), (11) and (13), we find that

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -8 \left(\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right) - 8\pi \alpha \int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx.$$

Notice that

$$\int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx = \int_{\mathbb{R}^2} \varphi \left(\frac{f}{\mu} \right) \mu^{3/2} dx \quad \text{with} \quad \varphi(t) := t^{3/2} - t - \sqrt{t} + 1$$

and that φ is a strictly convex function on \mathbb{R}^+ such that $\varphi(1) = \varphi'(1) = 0$, so that φ is nonnegative. On the other hand, by (9), we know that

$$\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \geq 0$$

as in the proof of [7]. Altogether, this proves that $t \mapsto \mathcal{F}[f(t, \cdot)]$ is monotone nonincreasing. Hence

$$\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \rightarrow +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_\star] = 0.$$

This completes the proof of (1). ■

3 Consequences

3.1 Proof of Corollary 1.2

To prove the result of Corollary 1.2, we have to establish first that the *free energy* functional \mathcal{F}_β is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5)-(6).

• **Repulsive case.** Let us consider the *free energy* functional defined in (7) where ϕ is given by (6) with $\varepsilon = +1$, i.e., $\phi = -\frac{1}{2\pi} \log |\cdot| * f$.

Lemma 3.1. Let $M > 0$ and $\varepsilon = +1$. Then \mathcal{F}_β is bounded from below on the set of the functions $f \in L^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f dx = M$ if $\beta \geq 1 + \frac{M}{8\pi}$. □

Proof. With $g = \frac{f}{M}$ and $\alpha = 1 + \frac{M}{8\pi}$, this means that

$$\begin{aligned} \frac{1}{M} \mathcal{F}_\beta[f] - \log M &= \int_{\mathbb{R}^2} g \log g dx + \beta \int_{\mathbb{R}^2} V g dx - \frac{M}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| dx dy \\ &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx - 2(\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| dx dy \\ &\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi) \end{aligned}$$

according to Theorem 1.1; the condition $\beta \geq \alpha$ is enough to prove that $\mathcal{F}_\beta[f]$ is bounded from below. ■

Proof of Corollary 1.2 with $\varepsilon = +1$. Let us consider a smooth solution of (5)-(6). We refer to [16] for details and to [1] for similar arguments in dimension $d \geq 3$. According to (8), f converges as $t \rightarrow +\infty$ to a solution of

$$\nabla \log f + \beta \nabla V + \nabla \phi = 0.$$

Notice that this already proves the existence of a stationary solution. The equation can be solved as

$$f = M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} dx}$$

after taking into account the conservation of the mass. With (6), the problem is reduced to solving

$$-\Delta \psi = M \left(\frac{e^{-\gamma V - \psi}}{\int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx} - \mu \right), \quad \psi = (\beta - \gamma) V + \phi, \quad \gamma = \beta - \frac{M}{8\pi}$$

using (12). It is a critical point of the functional $\psi \mapsto \mathcal{J}_{M,\gamma}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + M \int_{\mathbb{R}^2} \psi \mu dx + M \log \left(\int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx \right)$. Such a functional is strictly convex as, for instance, in [10, 11]. We conclude that ψ is unique up to an additional constant. ■

• **Attractive case.** Let us consider the *free energy* functional (7) \mathcal{F}_β where ϕ is given by (6) with $\varepsilon = -1$, i.e., $\phi = \frac{1}{2\pi} \log |\cdot| * f$. Inspired by [12], we have the following estimate.

Lemma 3.2. Let $\varepsilon = -1$. Then \mathcal{F}_β is bounded from below on the set of the functions $f \in L^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f dx = M$ if $M \leq 8\pi$ and $\beta \geq 1 - \frac{M}{8\pi}$. It is not bounded from below if $M > 8\pi$. □

Proof. With $g = \frac{f}{M}$ and $\alpha = 1 - \frac{M}{8\pi}$, Theorem 1.1 applied to

$$\begin{aligned} & \frac{1}{M} \mathcal{F}_\beta[f] - \log M \\ &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx + 2(1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| dx dy \\ & \geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi) \end{aligned}$$

proves that the free energy is bounded from below if $M \leq 8\pi$ and $\beta \geq \alpha$. On the other hand, if $f_\lambda(x) := \lambda^{-2} f(\lambda^{-1} x)$ and $M > 8\pi$, then

$$\mathcal{F}_\beta[f_\lambda] \sim 2M \left(\frac{M}{8\pi} - 1 \right) \log \lambda \rightarrow -\infty \quad \text{as } \lambda \rightarrow 0_+,$$

which proves that \mathcal{F}_β is not bounded from below. ■

Proof of Corollary 1.2 with $\varepsilon = -1$. The proof goes as in the case $\beta = 0$. We refer to [4] and leave details to the reader. ■

Remark 3.3. Let us notice that \mathcal{F}_β is unbounded from below if $\beta < 0$. This follows from the observation that $\lim_{|y| \rightarrow \infty} \mathcal{F}_\beta[f_y] = -\infty$ where $f_y(x) = f(x + y)$ for any admissible f . □

3.2 Duality

When $\alpha > 1$, we can write a first inequality by considering the *repulsive case* in the proof of Corollary 1.2 and observing that

$$\mathcal{J}_{M,\gamma}[\psi] \geq \min \mathcal{J}_{M,\gamma}$$

where $\psi \in W^{2,1}_{\text{loc}}(\mathbb{R}^2)$ is such that $\int_{\mathbb{R}^2} (\Delta \psi) dx = 0$ and the minimum is taken on the same set of functions.

When $\alpha \in [0, 1)$, it is possible to argue by duality as in [5, Section 2]. Since f_\star realizes the equality case in (1), we know that

$$\int_{\mathbb{R}^2} f_\star \log \left(\frac{f_\star}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f_\star dx + M(1 - \alpha) (1 + \log \pi) = \frac{2}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_\star(x) f_\star(y) \log |x - y| dx dy$$

and, using the fact that f_\star is a critical point of the difference of the two sides of (1), we also have that

$$\int_{\mathbb{R}^2} \log \left(\frac{f}{f_\star} \right) (f - f_\star) dx + \alpha \int_{\mathbb{R}^2} V (f - f_\star) dx = \frac{4}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (f(x) - f_\star(x)) f_\star(y) \log |x - y| dx dy.$$

By subtracting the first identity to (1) and adding the second identity, we can rephrase (1) as

$$\mathcal{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log \left(\frac{f}{f_\star} \right) dx \geq \frac{4\pi}{M} (1 - \alpha) \int_{\mathbb{R}^2} (f - f_\star) (-\Delta)^{-1} (f - f_\star) dx := \mathcal{F}_{(2)}[f].$$

Let us consider the Legendre transform

$$\mathcal{F}_{(i)}^*[g] := \sup_f \left(\int_{\mathbb{R}^2} g f dx - \mathcal{F}_{(i)}[f] \right)$$

where the supremum is restricted to the set of the functions $f \in L_+^1(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} f dx$. After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that

$$M \log \left(\int_{\mathbb{R}^2} e^{g-V} dx \right) = \mathcal{F}_{(1)}^*[g] \leq \frac{M}{16\pi(1-\alpha)} \int_{\mathbb{R}^2} |\nabla g|^2 dx + M \int_{\mathbb{R}^2} g e^{-V} dx = \mathcal{F}_{(2)}^*[g].$$

We can get rid of M by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as $\alpha \rightarrow 0_+$, which is clearly the sharpest one for all possible $\alpha \in [0, 1)$.

3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential W on \mathbb{R}^2 such that

$$W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{W(x)}{V(x)} = \beta \quad (\mathcal{H}_W)$$

and the associated *free energy functional*

$$\mathcal{F}_{\beta, W}[f] := \int_{\mathbb{R}^2} f \log f dx + \beta \int_{\mathbb{R}^2} W f dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f dx$$

where ϕ is given in terms of $f > 0$ by (6). With previous notations, $\mathcal{F}_\beta = \mathcal{F}_{\beta, V}$. Our last result is that the asymptotic behaviour obtained from (\mathcal{H}_W) is enough to decide whether $\mathcal{F}_{\beta, W}$ is bounded from below or not. The precise result goes as follows.

Theorem 3.4. Under Assumption (\mathcal{H}_W) , $\mathcal{F}_{\beta, W}$ defined as above is bounded from below if either $\varepsilon = +1$ and $\beta > 1 + \frac{M}{8\pi}$, or $\varepsilon = -1$, $\beta > 1 - \frac{M}{8\pi}$ and $M \leq 8\pi$. The result is also true in the limit case if $(W - \beta V) \in L^\infty(\mathbb{R}^2)$ and either $\varepsilon = +1$ and $\beta = 1 + \frac{M}{8\pi}$, or $\varepsilon = -1$, $\beta = 1 - \frac{M}{8\pi}$ and $M \leq 8\pi$. \square

Proof. If $(W - \beta V) \in L^\infty(\mathbb{R}^2)$, we can write that

$$\mathcal{F}_{\beta, W}[f] \geq \mathcal{F}_\beta[f] - M \|W - \beta V\|_{L^\infty(\mathbb{R}^2)}.$$

This completes the proof in the limit case. Otherwise, we redo the argument using $\tilde{\beta} V - (\tilde{\beta} V - W)_+$ for some $\tilde{\beta} \in (0, \beta)$ if $\varepsilon = -1$, and for some $\tilde{\beta} \in (1 + \frac{M}{8\pi}, \beta)$ if $\varepsilon = +1$. \blacksquare

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