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# Generalized logarithmic Hardy-Littlewood-Sobolev inequality

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This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, that allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

## 1 Main result and motivation

On  $\mathbb{R}^2$ , let us define the *density of probability*  $\mu = e^{-V}$  and the *external potential*  $V$  by

$$\mu(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \text{and} \quad V(x) := -\log \mu(x) = 2 \log(1+|x|^2) + \log \pi \quad \forall x \in \mathbb{R}^2.$$

We shall denote by  $L_+^1(\mathbb{R}^2)$  the set of a.e. nonnegative functions in  $L^1(\mathbb{R}^2)$ . Our main result is the following *generalized logarithmic Hardy-Littlewood-Sobolev inequality*.

**Theorem 1.1.** For any  $\alpha \geq 0$ , we have that

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log \pi) \geq \frac{2}{M}(\alpha-1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \quad (1)$$

for any function  $f \in L_+^1(\mathbb{R}^2)$  with  $M = \int_{\mathbb{R}^2} f dx > 0$ . Moreover, the equality case is achieved by  $f_\star = M\mu$  and  $f_\star$  is the unique optimal function for any  $\alpha > 0$ .  $\square$

With  $\alpha = 0$ , the inequality is the classical *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy + M(1+\log \pi) \geq 0. \quad (2)$$

In that case  $f_\star$  is an optimal function as well as all functions generated by a translation and a scaling of  $f_\star$ . As long as the parameter  $\alpha$  is in the range  $0 \leq \alpha < 1$ , the coefficient of the right-hand side of (1)

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is negative and the inequality is essentially of the same nature as the one with  $\alpha = 0$ . It can indeed be written as

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + M(1-\alpha)(1+\log\pi) + \frac{2}{M}(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log|x-y| dx dy \geq 0$$

For reasons that will be made clear below, we shall call this range the *attractive range*.

If  $\alpha = 1$ , the inequality is almost trivial since

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \int_{\mathbb{R}^2} V f dx = \int_{\mathbb{R}^2} f \log\left(\frac{f}{f_\star}\right) dx \geq 0 \quad (3)$$

is a straightforward consequence of Jensen's inequality. Now it is clear that by adding (2) multiplied by  $(1-\alpha)$  and (3) multiplied by  $\alpha$ , we recover (1) for any  $\alpha \in [0, 1]$ . As a consequence (1) is a straightforward interpolation between (2) and (3) in the *attractive range*.

Now, let us consider the *repulsive range*  $\alpha > 1$ . It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient  $(\alpha - 1)$  in the right-hand side of (1) is now positive. Since

$$G(x) = -\frac{1}{2\pi} \log|x|$$

is the Green function associated with  $-\Delta$  on  $\mathbb{R}^2$ , so that we can define

$$(-\Delta)^{-1}f(x) = (G * f)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| f(y) dy,$$

it is interesting to write (1) as

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f dx + \frac{4\pi}{M}(\alpha-1) \int_{\mathbb{R}^2} f (-\Delta)^{-1}f dx \geq M(\alpha-1)(1+\log\pi). \quad (4)$$

If  $f$  has a sufficient decay as  $|x| \rightarrow +\infty$ , for instance if  $f$  is compactly supported, we know that  $(-\Delta)^{-1}f(x) \sim -\frac{M}{2\pi} \log|x|$  for large values of  $|x|$  and as a consequence,

$$\alpha V + \frac{4\pi}{M}(\alpha-1)(-\Delta)^{-1}f \sim 2(\alpha+1) \log|x| \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

In a minimization scheme, this prevents the runaway of the left-hand side in (4). On the other hand,  $\int_{\mathbb{R}^2} f \log f dx$  prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (4) indeed admits a minimizer.

Inequality (2) was proved in [8] by E. Carlen and M. Loss (also see [2]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [7]: see Section 2 for a sketch of their proof. Our proof of Theorem 1.1 relies on an extension of this approach which takes into account the presence of the external potential  $V$ . A remarkable feature of this approach is that it is insensitive to the sign of  $\alpha - 1$ .

One of the key motivations for studying (4) arises from entropy methods applied to *drift-diffusion-Poisson* models which, after scaling out all physical parameters, are given by

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \quad (5)$$

with a nonlinear coupling given by the *Poisson* equation

$$-\varepsilon \Delta \phi = f. \quad (6)$$

Here  $V = -\log \mu$  is the external *confining potential* and we choose it as in the statement of Theorem 1.1, while  $\beta \geq 0$  is a coupling parameter with  $V$ , which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient  $\varepsilon$  in (6) is either  $\varepsilon = -1$ , which corresponds to the *attractive* case, or  $\varepsilon = +1$ , which corresponds to the *repulsive* case. In terms of applications, when  $\varepsilon = -1$ , (6) is the equation for the mean field potential obtained from Newton's law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case  $\varepsilon = +1$  is used for repulsive electrostatic forces in semiconductor physics, electrolytes, plasmas and charged particle models.

In view of *entropy methods* applied to PDEs (see for instance [15]), it is natural to consider the *free energy functional*

$$\mathcal{F}_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx \quad (7)$$

because, if  $f > 0$  solves (5)-(6) and is smooth enough, with sufficient decay properties at infinity, then

$$\frac{d}{dt} \mathcal{F}_\beta[f(t, \cdot)] = - \int_{\mathbb{R}^2} f |\nabla \log f + \beta \nabla V + \nabla \phi|^2 \, dx \quad (8)$$

so that  $\mathcal{F}_\beta$  is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions  $\mathcal{F}_\beta$  is bounded from below. The answer is given by the following result.

**Corollary 1.2.** Let  $M > 0$ . The functional  $\mathcal{F}_\beta$  is bounded from below and admits a minimizer on the set of the functions  $f \in L^1_+(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f \, dx = M$  if and only if either  $\varepsilon = -1$  and  $\beta \geq 1 + \frac{M}{8\pi}$ , or  $\varepsilon = +1$ ,  $\beta \geq 0$  and  $M \leq 8\pi$ . If  $\varepsilon = -1$ , the minimizer is unique.  $\square$

As we shall see in Section 3.1, Corollary 1.2 is a simple consequence of Theorem 1.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with  $\varepsilon = -1$  and  $\beta = 0$ , this has been used in [12, 4] to provide a sharp range of existence of the solutions to the evolution problem. In [6], the case  $\varepsilon = -1$  with a potential  $V$  with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the *drift-diffusion-Poisson* model (5)-(6) and considerations on the *free energy*, in the electrostatic case, we can quote, among many others, [14, 13] and subsequent papers. In the Euclidean space with confining potentials, we shall refer to [10, 11, 3, 1]. However, as far as we know, these papers are primarily devoted to dimensions  $d \geq 3$  and the sharp growth condition on  $V$  when  $d = 2$  has not been studied so far. The goal of this paper is to fill this gap. The specific choice of  $V$  has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 3.3, we shall give a result for general potentials on  $\mathbb{R}^2$ : see Theorem 3.4 for a statement.

## 2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (2) given by E. Carlen, J. Carrillo and M. Loss in [7]. The main idea is to use the nonlinear diffusion equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f}$$

with a nonnegative initial datum  $f_0$ . The equation preserves the mass  $M = \int_{\mathbb{R}^2} f dx$  and is such that

$$\frac{d}{dt} \left( \int_{\mathbb{R}^2} f \log f dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f ((-\Delta)^{-1} f) dx \right) = -\frac{8}{M} \left( \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx \int_{\mathbb{R}^2} f dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right).$$

According to [9], the Gagliardo-Nirenberg inequality

$$\|\nabla g\|_2^2 \|g\|_4^4 \geq \pi \|g\|_6^6 \quad (9)$$

applied to  $g = f^{1/4}$  guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [17]), we know that the solution behaves for large values of  $t$  like a self-similar solution, the so-called Barenblatt solution, which is given by  $B(t, x) := t^{-2} f_\star(x/t)$ . As a consequence, we find that

$$\begin{aligned} & \int_{\mathbb{R}^2} f_0 \log f_0 dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_0 ((-\Delta)^{-1} f_0) dx \\ & \geq \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^2} B \log B dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} B ((-\Delta)^{-1} B) dx = \int_{\mathbb{R}^2} f_\star \log f_\star dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_\star ((-\Delta)^{-1} f_\star) dx \end{aligned}$$

After an elementary computation, we observe that the above inequality is exactly (2) written for  $f = f_0$ .

The point is now to adapt this strategy to the case with an external potential. This justifies why we have to introduce a nonlinear diffusion equation with a drift. As we shall see below, the method is insensitive to  $\alpha$  and applies when  $\alpha > 1$  exactly as in the case  $\alpha \in (0, 1)$ . A natural question is whether solutions are regular enough to perform the computations below and in particular if they have a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum  $f_0$  which is as smooth and decaying as  $|x| \rightarrow +\infty$  as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [17] for regularity issues and to [15] for the truncation method. In the proof, we will therefore leave these issues apart, as they are purely technical.

**Proof of Theorem 1.1.** By homogeneity, we can assume that  $M = 1$  without loss of generality and consider the evolution equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (x f).$$

1) Using simple integrations by parts, we compute

$$\int_{\mathbb{R}^2} (1 + \log f) \Delta \sqrt{f} dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx$$

and

$$\int_{\mathbb{R}^2} (1 + \log f) \nabla \cdot (x f) dx = - \int_{\mathbb{R}^2} \frac{\nabla f}{f} \cdot (x f) dx = - \int_{\mathbb{R}^2} x \cdot \nabla f dx = 2 \int_{\mathbb{R}^2} f dx = 2.$$

As a consequence, we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f \log f dx = -8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx + 8\pi \int_{\mathbb{R}^2} \mu^{3/2} dx \quad (10)$$

using

$$\int_{\mathbb{R}^2} \mu^{3/2} dx = \frac{1}{2\sqrt{\pi}}.$$

2) By elementary considerations again, we find that

$$4\pi \int_{\mathbb{R}^2} f(-\Delta)^{-1} (\Delta \sqrt{f}) dx = -4\pi \int_{\mathbb{R}^2} f^{3/2} dx$$

and

$$\begin{aligned} 4\pi \int_{\mathbb{R}^2} \nabla \cdot (xf) (-\Delta)^{-1} f dx &= -4\pi \int_{\mathbb{R}^2} xf \cdot \nabla (-\Delta)^{-1} f dx \\ &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) x \cdot \frac{x-y}{|x-y|^2} dx dy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) (x-y) \cdot \frac{x-y}{|x-y|^2} dx dy = 1 \end{aligned}$$

where, in the last line, we exchanged the variables  $x$  and  $y$  and took the half sum of the two expressions. This proves that

$$\frac{d}{dt} \left( 4\pi \int_{\mathbb{R}^2} f ((-\Delta)^{-1} f) dx \right) = -8\pi \int_{\mathbb{R}^2} (f^{3/2} - \mu^{3/2}) dx. \quad (11)$$

3) We observe that

$$\mu(x) = \frac{1}{\pi(1+|x|^2)^2} = e^{-V(x)}$$

solves

$$\Delta V = -\Delta \log \mu = 8\pi \mu \quad (12)$$

and, as a consequence,

$$\int_{\mathbb{R}^2} V \Delta \sqrt{f} dx = \int_{\mathbb{R}^2} \Delta V \sqrt{f} dx = 8\pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx.$$

Since

$$\begin{aligned} 2\sqrt{\pi} \int_{\mathbb{R}^2} V \nabla \cdot (xf) dx &= -2\sqrt{\pi} \int_{\mathbb{R}^2} fx \cdot \nabla V dx = -8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{|x|^2}{1+|x|^2} f dx \\ &= -8\sqrt{\pi} + 8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{f}{1+|x|^2} dx = -8\sqrt{\pi} + 8\pi \int_{\mathbb{R}^2} \sqrt{\mu} f dx, \end{aligned}$$

we conclude that

$$\frac{d}{dt} \int_{\mathbb{R}^2} f V dx = 8\pi \int_{\mathbb{R}^2} (\mu \sqrt{f} + \sqrt{\mu} f - 2\mu^{3/2}) dx. \quad (13)$$

Let us define

$$\mathcal{F}[f] := \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1-\alpha)(1+\log \pi) + 2(1-\alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x-y| dx dy.$$

Collecting (10), (11) and (13), we find that

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -8 \left( \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \right) - 8\pi \alpha \int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx.$$

Notice that

$$\int_{\mathbb{R}^2} (f^{3/2} - \mu \sqrt{f} - \sqrt{\mu} f + \mu^{3/2}) dx = \int_{\mathbb{R}^2} \varphi \left( \frac{f}{\mu} \right) \mu^{3/2} dx \quad \text{with} \quad \varphi(t) := t^{3/2} - t - \sqrt{t} + 1$$

and that  $\varphi$  is a strictly convex function on  $\mathbb{R}^+$  such that  $\varphi(1) = \varphi'(1) = 0$ , so that  $\varphi$  is nonnegative. On the other hand, by (9), we know that

$$\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \geq 0$$

as in the proof of [7]. Altogether, this proves that  $t \mapsto \mathcal{F}[f(t, \cdot)]$  is monotone nonincreasing. Hence

$$\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \rightarrow +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_\star] = 0.$$

This completes the proof of (1). ■

### 3 Consequences

#### 3.1 Proof of Corollary 1.2

To prove the result of Corollary 1.2, we have to establish first that the *free energy* functional  $\mathcal{F}_\beta$  is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5)-(6).

• **Repulsive case.** Let us consider the *free energy* functional defined in (7) where  $\phi$  is given by (6) with  $\varepsilon = +1$ , i.e.,  $\phi = -\frac{1}{2\pi} \log|\cdot| * f$ .

**Lemma 3.1.** Let  $M > 0$  and  $\varepsilon = +1$ . Then  $\mathcal{F}_\beta$  is bounded from below on the set of the functions  $f \in L^1_+(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f dx = M$  if and only if  $\beta \geq 1 + \frac{M}{8\pi}$ . □

**Proof.** With  $g = \frac{f}{M}$  and  $\alpha = 1 + \frac{M}{8\pi}$ , this means that

$$\begin{aligned} \frac{1}{M} \mathcal{F}_\beta[f] - \log M &= \int_{\mathbb{R}^2} g \log g dx + \beta \int_{\mathbb{R}^2} V g dx - \frac{M}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x-y| dx dy \\ &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx - 2(\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x-y| dx dy \\ &\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi) \end{aligned}$$

according to Theorem 1.1: the condition  $\beta \geq \alpha$  is enough to prove that  $\mathcal{F}_\beta[f]$  is bounded from below.

Reciprocally, let us assume that  $\beta < 1 + \frac{M}{8\pi}$  and let  $f_\varepsilon(x) := \varepsilon^2 f_\star(\varepsilon x)$ . It is then straightforward to check that  $\mathcal{F}_\beta$  is not bounded from below because

$$\mathcal{F}_\beta[f_\varepsilon] \sim -4M(\beta - \alpha) \log \varepsilon \rightarrow -\infty \quad \text{as } \varepsilon \rightarrow 0_+.$$
■

**Proof of Corollary 1.2 with  $\varepsilon = +1$ .** Let us consider a smooth solution of (5)-(6). We refer to [16] for details and to [1] for similar arguments in dimension  $d \geq 3$ . According to (8),  $f$  converges as  $t \rightarrow +\infty$  to a solution of

$$\nabla \log f + \beta \nabla V + \nabla \phi = 0.$$

Notice that this already proves the existence of a stationary solution. The equation can be solved as

$$f = M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} dx}$$

after taking into account the conservation of the mass. With (6), the problem is reduced to solving

$$-\Delta\psi = M\left(\frac{e^{-\gamma V-\psi}}{\int_{\mathbb{R}^2} e^{-\gamma V-\psi} dx} - \mu\right), \quad \psi = (\beta - \gamma)V + \phi, \quad \gamma = \beta - \frac{M}{8\pi}$$

using (12). It is a critical point of the functional  $\psi \mapsto \mathcal{J}_{M,\gamma}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla\psi|^2 dx + M \int_{\mathbb{R}^2} \psi \mu dx + M \log\left(\int_{\mathbb{R}^2} e^{-\gamma V-\psi} dx\right)$ . Such a functional is strictly convex as, for instance, in [10, 11]. We conclude that  $\psi$  is unique up to an additional constant. ■

• **Attractive case.** Let us consider the *free energy* functional (7)  $\mathcal{F}_\beta$  where  $\phi$  is given by (6) with  $\varepsilon = -1$ , i.e.,  $\phi = \frac{1}{2\pi} \log|\cdot| * f$ . Inspired by [12], we have the following estimate.

**Lemma 3.2.** Let  $\beta \geq 0$  and  $\varepsilon = -1$ . Then  $\mathcal{F}_\beta$  is bounded from below on the set of the functions  $f \in L^1_+(\mathbb{R}^2)$  such that  $\int_{\mathbb{R}^2} f dx = M$  if and only if  $M \leq 8\pi$ . □

**Proof.** With  $g = \frac{f}{M}$  and  $\alpha = 1 - \frac{M}{8\pi}$ , Theorem 1.1 applied to

$$\begin{aligned} & \frac{1}{M} \mathcal{F}_\beta[f] - \log M \\ &= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx + 2(1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log|x - y| dx dy \\ & \geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log\pi) \end{aligned}$$

proves that the free energy is bounded from below if  $M \leq 8\pi$ . On the other hand, if  $f_\lambda(x) := \lambda^2 f(\lambda x)$  and  $M > 8\pi$ , then

$$\mathcal{F}_\beta[f_\lambda] \sim -2M \left(\frac{M}{8\pi} - 1\right) \log \lambda \rightarrow -\infty \quad \text{as } \lambda \rightarrow +\infty,$$

which proves that  $\mathcal{F}_\beta$  is not bounded from below. ■

**Proof of Corollary 1.2 with  $\varepsilon = -1$ .** The proof goes as in the case  $\beta = 0$ . We refer to [4] and leave details to the reader. ■

**Remark 3.3.** Let us notice that  $\mathcal{F}_\beta$  is unbounded from below if  $\beta < 0$ . This follows from the observation that  $\lim_{|y| \rightarrow \infty} \mathcal{F}_\beta[f_y] = -\infty$  where  $f_y(x) = f(x + y)$  for any admissible  $f$ . □

### 3.2 Duality

When  $\alpha > 1$ , we can write a first inequality by considering the *repulsive case* in the proof of Corollary 1.2 and observing that

$$\mathcal{J}_{M,\gamma}[\psi] \geq \min \mathcal{J}_{M,\gamma}$$

where  $\psi \in W_{\text{loc}}^{2,1}(\mathbb{R}^2)$  is such that  $\int_{\mathbb{R}^2} (\Delta\psi) dx = 0$  and the minimum is taken on the same set of functions.

When  $\alpha \in [0, 1)$ , it is possible to argue by duality as in [5, Section 2]. Since  $f_\star$  realizes the equality case in (1), we know that

$$\int_{\mathbb{R}^2} f_\star \log\left(\frac{f_\star}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f_\star dx + M(1 - \alpha) (1 + \log\pi) = \frac{2}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_\star(x) f_\star(y) \log|x - y| dx dy$$

and, using the fact that  $f_\star$  is a critical point of the difference of the two sides of (1), we also have that

$$\int_{\mathbb{R}^2} \log\left(\frac{f}{f_\star}\right) (f - f_\star) dx + \alpha \int_{\mathbb{R}^2} V (f - f_\star) dx = \frac{4}{M} (\alpha - 1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (f(x) - f_\star(x)) f_\star(y) \log|x - y| dx dy.$$



By subtracting the first identity to (1) and adding the second identity, we can rephrase (1) as

$$\mathcal{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log \left( \frac{f}{f_\star} \right) dx \geq \frac{4\pi}{M} (1-\alpha) \int_{\mathbb{R}^2} (f - f_\star) (-\Delta)^{-1} (f - f_\star) dx := \mathcal{F}_{(2)}[f].$$

Let us consider the Legendre transform

$$\mathcal{F}_{(i)}^*[g] := \sup_f \left( \int_{\mathbb{R}^2} g f dx - \mathcal{F}_{(i)}[f] \right)$$

where the supremum is restricted to the set of the functions  $f \in L_+^1(\mathbb{R}^2)$  such that  $M = \int_{\mathbb{R}^2} f dx$ . After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that

$$M \log \left( \int_{\mathbb{R}^2} e^{g-V} dx \right) = \mathcal{F}_{(1)}^*[g] \leq \frac{M}{16\pi(1-\alpha)} \int_{\mathbb{R}^2} |\nabla g|^2 dx + M \int_{\mathbb{R}^2} g e^{-V} dx = \mathcal{F}_{(2)}^*[g].$$

We can get rid of  $M$  by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as  $\alpha \rightarrow 0_+$ , which is clearly the sharpest one for all possible  $\alpha \in [0, 1)$ .

### 3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential  $W$  on  $\mathbb{R}^2$  such that

$$W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{W(x)}{V(x)} = \beta \quad (\mathcal{H}_W)$$

and the associated *free energy functional*

$$\mathcal{F}_{\beta,W}[f] := \int_{\mathbb{R}^2} f \log f dx + \beta \int_{\mathbb{R}^2} W f dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f dx$$

where  $\phi$  is given in terms of  $f > 0$  by (6). With previous notations,  $\mathcal{F}_\beta = \mathcal{F}_{\beta,V}$ . Our last result is that the asymptotic behaviour obtained from  $(\mathcal{H}_W)$  is enough to decide whether  $\mathcal{F}_{\beta,W}$  is bounded from below or not. The precise result goes as follows.

**Theorem 3.4.** Under Assumption  $(\mathcal{H}_W)$ ,  $\mathcal{F}_{\beta,W}$  defined as above is bounded from below if either  $\varepsilon = -1$  and  $\beta > 1 + \frac{M}{8\pi}$ , or  $\varepsilon = +1$ ,  $\beta > 0$  and  $M \leq 8\pi$ . The result is also true in the limit case if  $(W - \beta V) \in L^\infty(\mathbb{R}^2)$  and either  $\varepsilon = -1$  and  $\beta = 1 + \frac{M}{8\pi}$ , or  $\varepsilon = +1$ ,  $\beta = 0$  and  $M \leq 8\pi$ .  $\square$

**Proof.** If  $(W - \beta V) \in L^\infty(\mathbb{R}^2)$ , we can write that

$$\mathcal{F}_{\beta,W}[f] \geq \mathcal{F}_\beta[f] - M \|W - \beta V\|_{L^\infty(\mathbb{R}^2)}.$$

This completes the proof in the limit case. Otherwise, we redo the argument using  $\tilde{\beta} V - (\tilde{\beta} V - W)_+$  for some  $\tilde{\beta} \in (0, \beta)$  if  $\varepsilon = +1$ , and for some  $\tilde{\beta} \in (1 + \frac{M}{8\pi}, \beta)$  if  $\varepsilon = -1$ .  $\blacksquare$

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## References

- [1] Arnold, A., Markowich, P., Toscani, G., and Unterreiter, A. (2001). On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100.
- [2] Beckner, W. (1993). Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242.
- [3] Biler, P. and Dolbeault, J. (2000). Long time behavior of solutions of Nernst-Planck and Debye-Hückel drift-diffusion systems. *Ann. Henri Poincaré*, 1(3):461–472.
- [4] Blanchet, A., Dolbeault, J., and Perthame, B. (2006). Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations*, 44:1–33.
- [5] Campos, J. and Dolbeault, J. (2012). A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities. *C. R. Math. Acad. Sci. Paris*, 350(21-22):949–954.
- [6] Campos, J. F. and Dolbeault, J. (2014). Asymptotic Estimates for the Parabolic-Elliptic Keller-Segel Model in the Plane. *Comm. Partial Differential Equations*, 39(5):806–841.
- [7] Carlen, E. A., Carrillo, J. A., and Loss, M. (2010). Hardy-Littlewood-Sobolev inequalities via fast diffusion flows. *Proc. Natl. Acad. Sci. USA*, 107(46):19696–19701.
- [8] Carlen, E. A. and Loss, M. (1992). Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on  $\mathbb{S}^n$ . *Geom. Funct. Anal.*, 2(1):90–104.
- [9] Del Pino, M. and Dolbeault, J. (2002). Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875.
- [10] Dolbeault, J. (1991). Stationary states in plasma physics: Maxwellian solutions of the Vlasov-Poisson system. *Math. Models Methods Appl. Sci.*, 1(2):183–208.
- [11] Dolbeault, J. (1999). Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states). *J. Math. Pures Appl. (9)*, 78(2):121–157.
- [12] Dolbeault, J. and Perthame, B. (2004). Optimal critical mass in the two-dimensional Keller-Segel model in  $\mathbb{R}^2$ . *C. R. Math. Acad. Sci. Paris*, 339(9):611–616.
- [13] Dressler, K. (1990). Steady states in plasma physics – the Vlasov-Fokker-Planck equation. *Math. Methods Appl. Sci.*, 12(6):471–487.
- [14] Gogny, D. and Lions, P.-L. (1989). Sur les états d’équilibre pour les densités électroniques dans les plasmas. *RAIRO Modél. Math. Anal. Numér.*, 23(1):137–153.
- [15] Jüngel, A. (2016). *Entropy methods for diffusive partial differential equations*. SpringerBriefs in Mathematics. Springer, [Cham].
- [16] Li, X. (2019). Asymptotic behavior of Nernst-Planck equation. In preparation.
- [17] Vázquez, J. L. (2006). *Smoothing and decay estimates for nonlinear diffusion equations*, volume 33 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford.