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Conditional survival probabilities under partial information: a recursive quantization approach with applications

Cheikh MBAYE * Abass SAGNA † Frédéric VRINS ‡

Abstract

We consider a structural model where the survival/default state is observed together with a noisy version of the firm value process. This assumption makes the model more realistic than most of the existing alternatives, but triggers important challenges related to the computation of conditional default probabilities. In order to deal with general diffusions as firm value process, we derive a numerical procedure based on the recursive quantization method to approximate it. Then, we investigate the error approximation induced by our procedure. Eventually, numerical tests are performed to evaluate the performance of the method, and an application is proposed to the pricing of CDS options.

Keywords: default model, structural model, noisy information, non-linear filtering, credit risk.

1 Introduction

In the recent decades, credit risk received an increasing attention from academics and practitioners. In particular, the 2008 financial crisis shed the light on the importance of having sound credit risk models to better assess the default likelihood of firms and counterparties. The structural approach is one of the two most popular frameworks. It is originated to the seminal work of Merton [14] and uses the dynamics of structural variables of a firm, such as asset and debt, to determine whether the firm defaulted before a given maturity. To better deal with the actual timing of the default event, first passage time models were then introduced. Among them is the celebrated Black and Cox model [1] which adds a time-dependent barrier, among others. Yet, the Black and Cox model has few parameters and is not easily calibrated to structural data such as CDS quotes along different maturities. To that end, extensions of the same models called AT1P and SBTV were introduced in [2] and [3] allowing exact calibration to credit spreads using efficient closed-form formulas for default probabilities.

In practice however, it is difficult for investors to perfectly assess the value of the firm's assets. In this case, modeling the firm value in a Black-Cox framework is problematic, since the model assumes that the firm's underlying assets are observable. Moreover, in such a framework, the default time is predictable, leading to vanishing credit spreads for short maturities. In order to address

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these drawbacks, Duffie and Lando [7] proposed a model where the investors have only partial information on the firm value and observe at discrete time intervals a noisy accounting report. The default time becomes totally inaccessible in the market filtration. As a result, the corresponding short term spreads are always higher compared to the complete information short term spreads. Alternatively, some extensions of this model based on noisy information in continuous time can be found among others in [6] or recently in [9].

In this paper, we both generalize and improve results derived in earlier studies. The models presented in [5] and [20] can deal with arbitrary firm-value diffusions, but are heavy. Moreover, the considered information flow is only made of a noisy version of the firm-value. This is not realistic as in practice, investors can obviously observe the default state of the firm. This larger filtration is considered in [6], but under the restrictive assumption that the firm-value process is a continuous and invertible function of a Gaussian martingale. In this work, we consider the same information set as [6] but relax the restriction regarding the firm-value dynamics. To deal with this general case, we propose a numerical scheme based on fast quantization recently introduced in [17]. This technique is faster compared to [5] and [20] as there is no need to rely on Monte-Carlo simulations to compute the conditional survival probabilities. A detailed analysis of the error induced by the approximation is provided. Eventually, we illustrated our method on the pricing of CDS option credit derivatives.

The reminder of the paper is organized as follows. In Section 2, we introduce the model. Different information flows and corresponding survival probabilities will be discussed. Section 3 presents the estimations of the survival probabilities using the recursive quantization method and the stochastic filtering theory. We then give a brief introduction to the quantization method before deriving the error analysis of these estimations. Section 4 is devoted to the results of the numerical experiments.

2 The model

Assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, modeling the uncertainty of our economy. We consider a structural default model, and represent the default time τ_X of a reference entity as the first passage time of firm value process, below a default threshold. More precisely, we consider a Black and Cox setup [1], where the stochastic process X represents the actual value of the firm and $\mathbf{a} \in \mathbb{R}$ stands for the default barrier. Assuming $\tau_X > 0$, we have:

$$\tau_X := \inf \{u \geq 0 : X_u \leq \mathbf{a}\}, \quad 0 < \mathbf{a} < X_0 \quad (1)$$

where $\inf \emptyset := +\infty$, as usual. We restrict ourselves to consider $0 \leq t \leq T$ where T is a finite time horizon.

We consider a partial information model where the true firm value X (called *signal process* hereafter) is not observable and we only observe Y (*observation process*), which is correlated with X . We suppose that the dynamics of X and Y are governed by the following stochastic differential equations (SDEs) :

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & X_0 = x_0, \\ dY_t = h(t, Y_t, X_t)dt + \nu(t, Y_t)dW_t + \delta(t, Y_t)d\widetilde{W}_t, & Y_0 = y_0, \end{cases} \quad (2)$$

where (W, \widetilde{W}) is a standard two-dimensional Brownian motion. We suppose that the functions $b, \sigma, \nu, \delta : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz in x uniformly in t and that $\sigma(t, x) > 0$ for every $(t, x) \in [0, +\infty) \times \mathbb{R}$. These conditions ensure that the above SDEs admit a unique strong solution. Moreover we assume that h is locally bounded and Lipschitz in (y, x) , uniformly in t and that $\nu(t, y) > 0$ and $\delta(t, y) > 0$ for every $(t, y) \in [0, +\infty) \times \mathbb{R}$.

2.1 Information flows

One of the major critiques of such models is that in practice, the firm value is not observable. It is therefore not realistic to consider that the information available to the investor is $\mathbb{F}^X := (\mathcal{F}_t^X)_{t \geq 0}$, $\mathcal{F}_t^X := \sigma(X_u, 0 \leq u \leq t)$. A more realistic framework has been proposed in [5] where the investor information is given by the natural filtration \mathbb{F}^Y of a noisy version Y of the process X . However, one might argue that this way of modeling the information is not realistic either since given \mathcal{F}_t^Y , the investor is unable to know whether the reference entity defaulted or not by time t . In other words, the default indicator process $H = (H_t)_{t \geq 0}$, $H_t := \mathbb{1}_{\{t \geq \tau_X\}}$, $t \geq 0$, is not adapted to \mathbb{F}^Y .

In this paper, we address this point by considering a more realistic information flow, defined as the progressive enlargement of \mathbb{F}^Y with \mathbb{F}^H , the natural filtration of the default indicator process,

$$\mathcal{F}_t^H := \sigma(H_u, 0 \leq u \leq t), \quad t \geq 0.$$

In other words, we have two investors' information flows. On the one hand we have the information available to the common investor, defined as the progressive enlargement of the natural filtration of the default indicator process with that of the noisy firm-value, noted $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ where

$$\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_t^H, \quad t \geq 0.$$

In this setup, the following relationships hold:

$$\mathbb{F}^H \subsetneq \mathbb{F}^X = \mathbb{F}^W \subsetneq \mathbb{G} \quad \text{and} \quad \mathbb{F}^Y \subsetneq \mathbb{G}.$$

where $\mathbb{G} := (\mathcal{G}_t)_{t \geq 0}$ is the full information, i.e., the information available for example to a small number of stock holders of the company, who have access to Y and X .

On the other hand, the natural filtration of the actual (i.e. noise-free) firm-value process, \mathbb{F}^X , could be seen as the information available to insiders. Economically, X would represent the value of the firm, which is unobservable to the common investors, while Y might be the market price of an asset issued by the firm, accessible to all market participants, and a would stand for the solvency capital requirement imposed by regulators.

Remark 2.1. The results in the paper can be straightforwardly extended in the case when the default barrier a is a piecewise constant function of time $a : [0, \infty) \rightarrow [0, \infty)$, with $0 < a(0) < x_0$. The other extensions are beyond the scope of this paper. Nevertheless, let us mention that in, e.g., [19] crossing probabilities for the Brownian motion are obtained in the case when the (double) barrier is a piecewise linear function on $[0, T]$ and approximations for crossing probabilities are obtained for general nonlinear bounds.

2.2 Survival probability

A fundamental output of a credit model is the survival probability of the firm up to time t conditional upon \mathcal{F}_s , $s \leq t \leq T$:

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s) = \mathbb{E} \left(\mathbb{1}_{\{\tau_X > t\}} \middle| \mathcal{F}_s \right) \quad (3)$$

Note that this probability collapse to zero whenever $\{\tau_X \leq s\}$. Recall that the specificity of our approach is that, the actual value of the firm X is not revealed in \mathcal{F}_s ; only a noisy version Y is accessible.

Using the Markov property of X , the fact that the two Brownian motions are independent and the chain rule of the conditional expectation, we show the following result.

Proposition 2.1. *We have, for $s \leq t$,*

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s) = \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau_X > s\}} F(s, t, X_s) | \mathcal{F}_s^Y]}{\mathbb{P}(\tau_X > s | \mathcal{F}_s^Y)} \quad (4)$$

where, for every $x \in \mathbb{R}$,

$$F(s, t, x) := \mathbb{P}\left(\inf_{s < u \leq t} X_u > \mathbf{a} \mid X_s = x\right) \quad (5)$$

is the conditional survival probability under full information. Furthermore, it holds on the set $\{\tau_X > s\}$ that

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y) \leq \mathbb{P}(\tau_X > t | \mathcal{F}_s). \quad (6)$$

Proof. Using a key result in the theory of conditional expectations commonly referred to as the *Key lemma* (see e.g. Lemma 3.1 in [8]) we have

$$\begin{aligned} \mathbb{P}(\tau_X > t | \mathcal{F}_s) &= \mathbb{1}_{\{\tau_X > s\}} \mathbb{E}\left(\mathbb{1}_{\{\tau_X > t\}} \mid \mathcal{F}_s^Y \vee \mathcal{F}_s^H\right) \\ &= \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{E}\left(\mathbb{1}_{\{\tau_X > t\}} \mid \mathcal{F}_s^Y\right)}{\mathbb{P}(\tau_X > s | \mathcal{F}_s^Y)} \\ &= \mathbb{1}_{\{\tau_X > s\}} \frac{\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y)}{\mathbb{P}(\tau_X > s | \mathcal{F}_s^Y)}. \end{aligned} \quad (7)$$

The proof is completed by noting that $\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y) = \mathbb{E}[\mathbb{1}_{\{\tau_X > s\}} F(s, t, X_s) | \mathcal{F}_s^Y]$ (see e.g. [20]). The particular case follows from (7) since on the event $\{\tau_X > s\}$,

$$\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y) = \mathbb{P}(\tau_X > t | \mathcal{F}_s) \mathbb{P}(\tau_X > s | \mathcal{F}_s^Y) \leq \mathbb{P}(\tau_X > t | \mathcal{F}_s)$$

or, equivalently, $\mathbb{P}(\tau_X \leq t | \mathcal{F}_s) \leq \mathbb{P}(\tau_X \leq t | \mathcal{F}_s^Y)$. \square

Remark 2.2. The inequality (6), confirmed by the numerical experiments in Section 4, means that the less we have information on the state of the system ($\mathcal{F}_s^Y \subset \mathcal{F}_s^Y \vee \mathcal{F}_s^H = \mathcal{F}_s$), the higher the default probability. This also shows the difference with [20] where the quantity of interest is just $\mathbb{P}(\tau_X > t | \mathcal{F}_s^Y)$.

2.3 The problem

Note that we have clearly stated the expression of interest, namely the survival probability of the reference entity up to time t conditional upon the investor's information up to time s , we need to actually compute it. In order to even more comply with real market practice, we further consider that we can only access to, say, *discrete time* observations of Y up to time s . To that end, let us start by fixing a time discretization grid over $[0, t]$:

$$0 = t_0 < \dots < t_m = s < t_{m+1} < \dots < t_n = t.$$

Our aim is to approximate the right hand side of (4) by recursive quantization. In some specific models (those for which (2) admits an explicit solution (X, Y) , like in the Black-Scholes framework), we will consider the discrete trajectories $(X_{t_k}, Y_{t_k})_{k=0, \dots, n}$. In more general models, we need to make a discrete time approximation of the quantity of interest. To this end, we suppose

that we have access to a trajectory of Y sampled at m times: $(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m})$, with $t_0 = 0$ and $t_m = s$ (which in practice will be approximated from the paths of the Euler scheme associated to the stochastic process Y) and will estimate (3) by

$$\frac{\mathbb{P}\left(\tau_X > t \mid \mathcal{F}_s^{\bar{Y}}\right)}{\mathbb{P}\left(\tau_X > s \mid \mathcal{F}_s^{\bar{Y}}\right)},$$

on the event $\{\tau_X > s\}$, where $\mathcal{F}_s^{\bar{Y}} = \sigma(\bar{Y}_{t_k}, t_k \leq s) = \sigma(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m})$.

2.4 Discrete time approximation

We denote by \bar{X} the continuous Euler scheme associated to the process X in Equation (2), namely:

$$\bar{X}_s = \bar{X}_{\underline{s}} + b(\underline{s}, \bar{X}_{\underline{s}})(s - \underline{s}) + \sigma(\underline{s}, \bar{X}_{\underline{s}})(W_s - W_{\underline{s}}), \quad \bar{X}_0 = x_0,$$

with $\underline{s} = t_k$ if $s \in [t_k, t_{k+1})$, for $k = 0, \dots, n$. Based on the Euler scheme, we introduce the discretized version of our state-observation processes (\bar{X}, \bar{Y})

$$\begin{cases} \bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b(t_k, \bar{X}_{t_k})\Delta_k + \sigma(t_k, \bar{X}_{t_k})(W_{t_{k+1}} - W_{t_k}) \\ \bar{Y}_{t_{k+1}} = \bar{Y}_{t_k} + h(t_k, \bar{Y}_{t_k}, \bar{X}_{t_k})\Delta_k + \nu(t_k, \bar{Y}_{t_k})(W_{t_{k+1}} - W_{t_k}) + \delta(t_k, \bar{Y}_{t_k})(\widetilde{W}_{t_{k+1}} - \widetilde{W}_{t_k}) \end{cases} \quad (8)$$

where $k \in \{0, \dots, n-1\}$ for the signal process and $k \in \{0, \dots, m-1\}$ for the observation process and where $\Delta_k := t_{k+1} - t_k$.

Supposing that we have access to a discrete trajectory of Y , $(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m})$, our first goal is to approximate (recall that $t_m = s$)

$$\frac{\mathbb{P}(\tau_X > t \mid \mathcal{F}_s^Y)}{\mathbb{P}(\tau_X > s \mid \mathcal{F}_s^Y)} \quad \text{by} \quad \frac{\mathbb{P}(\tau_{\bar{X}} > t \mid \mathcal{F}_s^{\bar{Y}})}{\mathbb{P}(\tau_{\bar{X}} > s \mid \mathcal{F}_s^{\bar{Y}})}, \quad (9)$$

where (recall Equation (1))

$$\tau_{\bar{X}} := \inf\{u \geq 0, \bar{X}_u \leq \mathbf{a}\}.$$

Using the Brownian Bridge method and the Markov property of $(\bar{X}_{t_k}, \bar{Y}_{t_k})_k$, we show that the quantity (9) can be written in a closed formula.

Theorem 2.2. *We have:*

$$\frac{\mathbb{P}(\tau_{\bar{X}} > t \mid \mathcal{F}_s^{\bar{Y}})}{\mathbb{P}(\tau_{\bar{X}} > s \mid \mathcal{F}_s^{\bar{Y}})} = \Psi(\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}), \quad (10)$$

where for $y = (y_0, \dots, y_m) \in \mathbb{R}^{m+1}$,

$$\Psi(y) = \frac{\mathbb{E}[\bar{F}(t_m, t_n, \bar{X}_{t_m})K_{\mathbf{a}}^m L_y^m]}{\mathbb{E}[K_{\mathbf{a}}^m L_y^m]}, \quad (11)$$

with

$$K_{\mathbf{a}}^m = \prod_{k=0}^{m-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}), \quad L_y^m = \prod_{k=0}^{m-1} g_k(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1})$$

and where for every $x \in \mathbb{R}$,

$$\bar{F}(t_m, t_n, x) = \mathbb{E}\left[\prod_{k=m}^{n-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}) \mid \bar{X}_{t_m} = x\right]. \quad (12)$$

The function g_k is defined by

$$\begin{aligned} g_k(x_k, y_k; x_{k+1}, y_{k+1}) &= \frac{\mathbb{P}((\bar{X}_{t_{k+1}}, \bar{Y}_{t_{k+1}}) = (x_{k+1}, y_{k+1}) | (\bar{X}_{t_k}, \bar{Y}_{t_k}) = (x_k, y_k))}{\mathbb{P}(\bar{X}_{t_{k+1}} = x_{k+1} | \bar{X}_{t_k} = x_k)} \\ &= \frac{1}{(2\pi\Delta_k)^{1/2}\delta_k} \exp\left(-\frac{\nu_k^2}{2\delta_k^2\Delta_k} \left(\frac{x_{k+1} - m_k^1}{\sigma_k} - \frac{y_{k+1} - m_k^2}{\nu_k}\right)^2\right) \end{aligned} \quad (13)$$

with $m_k^1 := x_k + b_k\Delta_k$ and $m_k^2 := y_k + h_k\Delta_k$. Finally,

$$\begin{aligned} G_{\Delta_k\sigma_k^2}^{x_k, x_{k+1}}(\mathbf{a}) &= \mathbb{P}\left(\inf_{u \in [t_k, t_{k+1}]} \bar{X}_u \geq \mathbf{a} | \bar{X}_{t_k} = x_k\right) \\ &= \left(1 - \exp\left(-\frac{2(x_k - \mathbf{a})(x_{k+1} - \mathbf{a})}{\Delta_k\sigma^2(t_k, x_k)}\right)\right) \mathbb{1}_{\{x_k \geq \mathbf{a}; x_{k+1} \geq \mathbf{a}\}}. \end{aligned} \quad (14)$$

Proof. Following Theorem 2.5. in [20], we have,

$$\mathbb{P}(\tau_{\bar{X}} > t | \mathcal{F}_s^{\bar{Y}}) = \frac{\mathbb{E}[\bar{F}(t_m, t_n, \bar{X}_{t_m}) K_{\mathbf{a}}^m L_y^m]}{\mathbb{E}[L_y^m]} \quad \text{and} \quad \mathbb{P}(\tau_{\bar{X}} > s | \mathcal{F}_s^{\bar{Y}}) = \frac{\mathbb{E}[K_{\mathbf{a}}^m L_y^m]}{\mathbb{E}[L_y^m]}.$$

□

The question of interest is now to know how to estimate efficiently $\Psi(y)$ for $y = (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m})$. Owing to the form of the random vector $K_{\mathbf{a}}^m$, we may put it together with L_y^m to be reduced to similar formula as the filter estimate in a standard nonlinear filtering problem. In other work we may write for $y := (y_0, \dots, y_m)$,

$$\Psi(y) = \frac{\mathbb{E}[\bar{F}(t_m, t_n, \bar{X}_{t_m}) L_{y, \mathbf{a}}^m]}{\mathbb{E}[L_{y, \mathbf{a}}^m]} \quad \text{where} \quad L_{y, \mathbf{a}}^m = \prod_{k=0}^{m-1} g_k(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1}) \times G_{\Delta_k\sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}).$$

3 Approximation by recursive quantization

Notice that defining the operator $\pi_{y,m}$, for every bounded measurable function f , by

$$\pi_{y,m} f := \mathbb{E}[f(\bar{X}_{t_m}) L_{y, \mathbf{a}}^m],$$

we have

$$\Psi(y) = \frac{\pi_{y,m} \bar{F}(t_m, t_n, \cdot)}{\pi_{y,m} \mathbf{1}} =: \Pi_{y,m} \bar{F}(t_m, t_n, \cdot), \quad (15)$$

where $\mathbf{1}(x) = 1$, for every real x . Then it is enough to tell how to compute the numerator

$$\pi_{y,m} \bar{F}(t_m, t_n, \cdot) = \mathbb{E}\left[\bar{F}(t_m, t_n, \bar{X}_{t_m}) \prod_{k=0}^{m-1} g_k^{\mathbf{a}}(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1})\right]$$

where

$$g_k^{\mathbf{a}}(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1}) = g_k(\bar{X}_{t_k}, y_k; \bar{X}_{t_{k+1}}, y_{k+1}) \times G_{\Delta_k\sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}).$$

At this stage, several methods involving Monte Carlo simulations as the particle method can be used to approximate $\Pi_{y,m}$. Optimal quantization is an alternative and some times as a substitute

to the Monte Carlo method to approximate such a quantity (we refer to [22] for a comparison of particle like methods and optimal quantization methods).

To use the optimal quantization methods we have to quantize the marginals of the process $(\bar{X}_{t_k})_k$, means, to represent every marginal \bar{X}_{t_k} , $k = 0, \dots, n$, by a discrete random variable $\widehat{X}_{t_k}^{\Gamma_k}$ (we will simply denote it \widehat{X}_{t_k} when there is no ambiguity) taking N_k values $\Gamma_k = \{x_1^k, \dots, x_{N_k}^k\}$. As we will see later, we have also need in our context to compute the transition probabilities $\hat{p}_k^{ij} = \mathbb{P}(\widehat{X}_{t_k} = x_j^k | \widehat{X}_{t_{k-1}} = x_i^{k-1})$, for $i = 1, \dots, N_{k-1}$; $j = 1, \dots, N_k$. To this end, we may use stochastic algorithms to get the optimal grids and the associated transition probabilities (see e.g. [16, 22]). This method works well but may be very time consuming. The so-called marginal functional quantization method (see [5, 20, 21]) is used as an alternative to the previous method. It consists to construct the marginal quantizations by considering the ordinary differential equation (ODE) resulting to the substitution of the Brownian motion appearing in the dynamics of X in (2) by a quadratic quantization of the Brownian motion (see [12]). This procedure performs the marginal quantizations quite instantaneous and works well enough from the numerical point of view even if the rate of convergence (which has not been computed yet from the theoretical point of view) seems to be poor. As an alternative to the two previous methods, we propose the recursive marginal quantization (also called fast quantization) method introduced in [17]. It consists of quantizing the process $(\bar{X}_{t_k})_{k=0, \dots, n}$, based on a recursive method involving the conditional distributions $\bar{X}_{t_{k+1}} | \bar{X}_{t_k}$, $k = 0, \dots, n-1$. For the problem of interest, this last method is more performing than the previous ones due to its computation speed and to its robustness.

On the other hand, the function \bar{F} has been estimated by Monte Carlo method in [5, 20]. For competitiveness reasons of the recursive quantization w.r.t. the previously raised methods, we propose here to approximated both quantities Π and \bar{F} by the recursive quantization method.

3.1 Approximation of $\Pi_{y,m}$ by recursive quantization

Given that the denominator in the right hand side of (15) has a similar form as the numerator, we will only show how to compute the numerator. We remark that $\pi_{y,m}$ can be computed from the following recursive formula:

$$\pi_{y,k} = \pi_{y,k-1} H_{y,k}, \quad k = 1, \dots, m, \quad (16)$$

where, for every $k = 1, \dots, m$, and for every bounded and measurable function f , the transition kernel $H_{y,k}$ is defined by

$$H_{y,k} f(z) = \mathbb{E} \left[f(\bar{X}_{t_k}) g_{k-1}^{\mathbf{a}}(\bar{X}_{t_{k-1}}, y_{k-1}; \bar{X}_{t_k}, y_k) | \bar{X}_{t_{k-1}} = z \right] \quad \text{with} \quad H_{y,0} f := \mathbb{E}[f(\bar{X}_0)].$$

In fact, for any bounded Borel function f we have

$$\begin{aligned} \pi_{y,k} f &= \mathbb{E} \left[f(\bar{X}_{t_k}) \prod_{\ell=0}^{k-1} g_{\ell}^{\mathbf{a}}(\bar{X}_{t_{\ell}}, y_{\ell}; \bar{X}_{t_{\ell+1}}, y_{\ell+1}) \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(f(\bar{X}_{t_k}) \prod_{\ell=0}^{k-1} g_{\ell}^{\mathbf{a}}(\bar{X}_{t_{\ell}}, y_{\ell}; \bar{X}_{t_{\ell+1}}, y_{\ell+1}) | \mathcal{F}_{k-1}^{\bar{X}} \right) \right]. \end{aligned}$$

Since \bar{X} still be a Markov process we deduce that

$$\begin{aligned}\pi_{y,k}f &= \mathbb{E}\left[\mathbb{E}\left(f(\bar{X}_{t_k})g_{k-1}^{\mathbf{a}}(\bar{X}_{t_{k-1}}, y_{k-1}; \bar{X}_k, y_k) \middle| \mathcal{F}_{k-1}^{\bar{X}}\right) \prod_{\ell=0}^{k-2} g_{\ell}^{\mathbf{a}}(\bar{X}_{t_{\ell}}, y_{\ell}; \bar{X}_{\ell+1}, y_{\ell+1})\right] \\ &= \mathbb{E}\left[H_{y,k}f(\bar{X}_{t_{k-1}}) \prod_{\ell=0}^{k-2} g_{\ell}^{\mathbf{a}}(\bar{X}_{t_{\ell}}, y_{\ell}; \bar{X}_{\ell+1}, y_{\ell+1})\right] \\ &= \pi_{y,k-1}H_{y,k}f.\end{aligned}$$

Then, when we have access to the quantization of the marginals of the process \bar{X} , the functional $\pi_{y,k}$ can be approximated recursively by optimal quantization as $\hat{\pi}_{y,k} = \hat{\pi}_{y,k-1}\hat{H}_{y,k}$ where for every $k \geq 1$, $\hat{H}_{y,k}$ is a matrix $N_k \times N_{k-1}$ which components $\hat{H}_{y,k}^{i,j}$ read

$$\hat{H}_{y,k}^{ij} = g_{k-1}^{\mathbf{a}}(x_{k-1}^i, y_{k-1}; x_k^j, y_k) \hat{p}_k^{ij} \delta_{x_k^j}$$

where

$$p_k^{ij} = \mathbb{P}(\hat{X}_{t_k} = x_k^j | \hat{X}_{t_{k-1}} = x_{k-1}^i)$$

and $(\hat{X}_{t_k})_k$ is the quantization of the process $(\bar{X}_t)_{t \geq 0}$ over the time steps $t_k, k = 1, \dots, m$: on the grids $\Gamma_k = \{x_k^1, \dots, x_k^{N_k}\}$, of sizes N_k .

As a consequence, the quantity of interest $\Pi_{y,m}\bar{F}(t_m, t_n, \cdot)$ is estimated by

$$\hat{\Pi}_{y,m}\bar{F}(t_m, t_n, \cdot) = \sum_{i=1}^{N_m} \hat{\Pi}_{y,m}^i \bar{F}(t_m, t_n, x_m^i). \quad (17)$$

where

$$\hat{\Pi}_{y,m}^i := \frac{\hat{\pi}_{y,m}^i}{\sum_{j=1}^{N_m} \hat{\pi}_{y,m}^j}, \quad i = 1, \dots, N_m$$

and where $\hat{\pi}_{y,m}$ is the estimation (by optimal quantization) of $\pi_{y,m}$ defined recursively by

$$\begin{cases} \hat{\pi}_{y,0} = \hat{H}_{y,0} \\ \hat{\pi}_{y,k} = \hat{\pi}_{y,k-1}\hat{H}_{y,k} := \left[\sum_{i=1}^{N_{k-1}} \hat{H}_{y,k}^{i,j} \hat{\pi}_{y,k-1}^i \right]_{j=1, \dots, N_k}, \quad k = 1, \dots, m \end{cases} \quad (18)$$

with

$$\hat{H}_{y,k}^{ij} = g_{k-1}^{\mathbf{a}}(x_{k-1}^i, y_{k-1}; x_k^j, y_k) \hat{p}_k^{ij} \delta_{x_k^j}. \quad (19)$$

Our aim is now to use the (marginal) recursive quantization method to estimate the $F(t_m, t_n, x_m^i)$'s.

3.2 Approximation of $\bar{F}(t_m, t_n, \cdot)$ by recursive quantization

Recall that for every x ,

$$\bar{F}(t_m, t_n, x) = \mathbb{E}\left(\prod_{k=m}^{n-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}) \middle| \bar{X}_{t_m} = x\right).$$

As previously, we remark that if we define the functional $\pi_{n,m}$ by

$$(\pi_{n,m}f)(x) = \mathbb{E}\left(f(\bar{X}_{t_n}) \prod_{k=m}^{n-1} G_{\Delta_k \sigma_k^2}^{\bar{X}_{t_k}, \bar{X}_{t_{k+1}}}(\mathbf{a}) \middle| \bar{X}_{t_m} = x\right),$$

for every bounded and measurable function f , then $F(t_m, t_n, x)$ reads

$$\bar{F}(t_m, t_n, x) = (\pi_{n,m} \mathbf{1})(x).$$

Now, for every bounded and measurable function f , defining as previously the transition kernel H_k as,

$$(H_k f)(z) = \mathbb{E} \left(f(\bar{X}_{t_k}) G_{\Delta_{k-1} \sigma_{k-1}^2}^{\bar{X}_{t_{k-1}}, \bar{X}_{t_k}}(\mathbf{a}) | \bar{X}_{t_{k-1}} = z \right),$$

for every $k = m + 1, \dots, n$ and setting

$$H_m f = \mathbb{E} [f(\bar{X}_{t_m})] \quad (20)$$

yields for every $k = m + 1, \dots, n$,

$$\begin{aligned} (\pi_{k,m} f)(x) &= \mathbb{E} \left(\mathbb{E} \left(f(\bar{X}_{t_k}) \prod_{i=m}^{k-1} G_{\Delta_i \sigma_i^2}^{\bar{X}_{t_i}, \bar{X}_{t_{i+1}}}(\mathbf{a}) | (\bar{X}_{t_\ell})_{\ell=m, \dots, k-1} \right) | \bar{X}_{t_m} = x \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(f(\bar{X}_{t_k}) G_{\Delta_{k-1} \sigma_{k-1}^2}^{\bar{X}_{t_{k-1}}, \bar{X}_{t_k}}(\mathbf{a}) | (\bar{X}_{t_\ell})_{\ell=m, \dots, k-1} \right) \prod_{\ell=m}^{k-2} G_{\Delta_\ell \sigma_\ell^2}^{\bar{X}_{t_\ell}, \bar{X}_{t_{\ell+1}}}(\mathbf{a}) | \bar{X}_{t_m} = x \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(f(\bar{X}_{t_k}) G_{\Delta_{k-1} \sigma_{k-1}^2}^{\bar{X}_{t_{k-1}}, \bar{X}_{t_k}}(\mathbf{a}) | \bar{X}_{t_{k-1}} \right) \prod_{\ell=m}^{k-2} G_{\Delta_\ell \sigma_\ell^2}^{\bar{X}_{t_\ell}, \bar{X}_{t_{\ell+1}}}(\mathbf{a}) | \bar{X}_{t_m} = x \right) \\ &= (\pi_{k-1,m} H_k f)(x). \end{aligned}$$

Consequently, if one has access to the recursive quantizations $(\hat{X}_{t_k})_{k=m, \dots, n}$ and the transition probabilities $\{\hat{p}_k^{ij}, k = m + 1, \dots, n\}$ of the process $(\bar{X}_{t_k})_{k=m, \dots, n}$, the quantity $F(t_m, t_n, x)$ will be estimated by

$$\hat{F}(t_m, t_n, x) = \sum_{j=1}^{N_n} \hat{\pi}_{n,m} \delta_{\{x_m^j = x\}}, \quad (21)$$

where the $\hat{\pi}_{n,m}$'s are defined from the following recursive formula

$$\begin{cases} \hat{\pi}_{m,m} = \hat{H}_m \\ \hat{\pi}_{k,m} = \hat{\pi}_{k-1,m} \hat{H}_k := \left[\sum_{i=1}^{N_{k-1}} \hat{H}_k^{i,j} \hat{\pi}_{k-1,m} \right]_{j=1, \dots, N_k}, k = m + 1, \dots, n \end{cases} \quad (22)$$

with

$$\hat{H}_k^{ij} = G_{\Delta_{k-1} \sigma_{k-1}^2}^{x_{k-1}^i, x_k^j}(\mathbf{a}) \hat{p}_k^{ij} \delta_{x_k^j}, \quad i = 1, \dots, N_{k-1}; j = 1, \dots, N_k.$$

3.3 Approximation of $\Pi_{y,m} \bar{F}(t_m, t_n, \cdot)$ by recursive quantization

Combining equations (17) and (21), the conditional survival probability $\Pi_{y,m} F(t_m, t_n, \cdot)$ will be estimated (for a fixed trajectory (y_0, \dots, y_m) of the observation process $(Y_{t_0}, \dots, Y_{t_m})$) by

$$\hat{\Pi}_{y,m} \hat{F}(t_m, t_n, \cdot) = \sum_{i=1}^{N_m} \sum_{j=1}^{N_n} \hat{\Pi}_{y,m}^i \hat{\pi}_{n,m} \delta_{x_m^j}. \quad (23)$$

Remark 3.1. In Section 4, the formula (23) will be compared to the one of interest in [20]: $\mathbb{P}(\tau_{\bar{X}} > t | \mathcal{F}_s^{\bar{Y}})$, which reads (following the previous notations)

$$\mathbb{P}(\tau_{\bar{X}} > t | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) = \frac{\mathbb{E}[\bar{F}(t_m, t_n, \bar{X}_{t_m}) L_{y, \mathbf{a}}^m]}{\mathbb{E}[L_y^m]}. \quad (24)$$

The conditional probability has been approximated in [20] via an hybrid Monte Carlo - optimal quantization method. It may be approximated following the procedure we propose using only optimal quantization method as

$$\widehat{\omega}_{y, m} \widehat{F}(t_m, t_n, \cdot) = \sum_{i=1}^{N_m} \sum_{j=1}^{N_n} \widehat{\omega}_{y, m}^i \widehat{\pi}_{n, m} \delta_{x_m^j} \quad (25)$$

where the $\widehat{\omega}_{y, m}^i$'s are obtained from (18) by replacing the function $g_k^{\mathbf{a}}$ by g_k of equation (13).

Let us say now how to quantize the signal process X from the recursive quantization method.

3.4 The recursive quantization method

Recall first that for a given \mathbb{R}^d -valued random vector X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution \mathbb{P}_X , the $L^r(\mathbb{P}_X)$ -optimal quantization problem of size N for X (or for the distribution \mathbb{P}_X) aims to approximate X by a Borel function of X taking at most N values. If $X \in L^r(\mathbb{P})$ and defining $\|X\|_r := (\mathbb{E}|X|^r)^{1/r}$ where $|\cdot|$ denotes an arbitrary norm on \mathbb{R}^d , this turns out to solve the following optimization problem (see e.g. [11]):

$$e_{N, r}(X) = \inf \{ \|X - \widehat{X}^\Gamma\|_r, \Gamma \subset \mathbb{R}^d, \text{card}(\Gamma) \leq N \} = \inf_{\substack{\Gamma \subset \mathbb{R}^d \\ \text{card}(\Gamma) \leq N}} \left(\int_{\mathbb{R}^d} d(x, \Gamma)^r d\mathbb{P}_X(x) \right)^{1/r} \quad (26)$$

where \widehat{X}^Γ , the quantization of X on the subset $\Gamma = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$ (called a codebook, an N -quantizer or a grid) is defined by

$$\widehat{X}^\Gamma = \text{Proj}_\Gamma(X) := \sum_{i=1}^N x_i \mathbb{1}_{\{X \in C_i(\Gamma)\}}$$

and where $(C_i(\Gamma))_{i=1, \dots, N}$ is a Borel partition (Voronoi partition) of \mathbb{R}^d satisfying for every $i \in \{1, \dots, N\}$,

$$C_i(\Gamma) \subset \{x \in \mathbb{R}^d : |x - x_i| = \min_{j=1, \dots, N} |x - x_j|\}.$$

Keep in mind that for every $N \geq 1$, the infimum in (26) is reached at one grid at least. Any N -quantizer realizing this infimum is called an L^r -optimal N -quantizer. Moreover, if $\text{card}(\text{supp}(\mathbb{P}_X)) \geq N$ then the optimal N -quantizer is of size N (see [11] or [15]). On the other hand, the quantization error, $e_{N, r}(X)$, decreases to zero at an $N^{-1/d}$ -rate as the grid size N goes to infinity. This convergence rate (known as Zador Theorem) has been investigated in [4] and [23] for absolutely continuous probability measures under the quadratic norm on \mathbb{R}^d . A detailed study of the convergence rate under an arbitrary norm on \mathbb{R}^d and for both absolutely continuous and singular measures may be found in [11].

The recursive quantization of the Euler scheme of an \mathbb{R}^d -valued diffusion process has been introduced in [17]. The method allows to speak of fast online quantization and consists on a sequence of quantizations $(\widehat{X}_{t_k}^{\Gamma_k})_{k=0, \dots, n}$ of the Euler scheme $(\bar{X}_{t_k})_{k=0, \dots, n}$ defined recursively as

$$\widetilde{X}_0 = \bar{X}_0, \quad \widehat{X}_{t_k}^{\Gamma_k} = \text{Proj}_{\Gamma_k}(\widetilde{X}_{t_k}) \quad \text{and} \quad \widetilde{X}_{t_{k+1}} = \mathcal{E}_k(\widehat{X}_{t_k}^{\Gamma_k}, Z_{k+1}), \quad k = 0, \dots, n-1, \quad (27)$$

where $(Z_k)_{k=1,\dots,n}$ is an i.i.d. sequence of $\mathcal{N}(0; I_q)$ -distributed random vectors, independent from \bar{X}_0 and

$$\mathcal{E}_k(y, z) = y + \Delta b(t_k, y) + \sqrt{\Delta} \sigma(t_k, y) z, \quad y \in \mathbb{R}^d, \quad z \in \mathbb{R}^q, \quad k = 0, \dots, n-1.$$

The sequence of quantizers satisfies for every $k \in \{0, \dots, n\}$,

$$\Gamma_k \in \operatorname{argmin}\{\tilde{D}_k(\Gamma), \Gamma \subset \mathbb{R}^d, \operatorname{card}(\Gamma) \leq N_k\},$$

where for every grid $\Gamma \subset \mathbb{R}^d$, $\tilde{D}_{k+1}(\Gamma) := \mathbb{E}[\operatorname{dist}(\tilde{X}_{t_{k+1}}, \Gamma)^2]$.

This recursive quantization method raises some problems among which the computation of the quadratic error bound $\|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}\|_2 := (\mathbb{E}|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}|_2)^{1/2}$, for every $k = 0, \dots, n$. It has been shown in [17, 18] that for any sequences of (quadratic) optimal quantizers Γ_k for $\tilde{X}_{t_k}^{\Gamma_k}$, for every $k = 0, \dots, n-1$, the quantization error $\|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}\|_2$ is bounded by the cumulative quantization errors $\|\tilde{X}_{t_i} - \hat{X}_{t_i}^{\Gamma_i}\|_2$, for $i = 0, \dots, k$. This result is obtained under the following assumptions and is stated in Proposition 3.1 below:

1. *L²-Lipschitz assumption.* The mappings $x \mapsto \mathcal{E}_k(x, Z_{k+1})$ from \mathbb{R}^d to $L^2(\Omega, \mathcal{A}, \mathbb{P})$, $k = 1 : n$ are Lipschitz continuous i.e.

$$(\text{Lip}) \quad \equiv \quad \forall x, x' \in \mathbb{R}^d, \quad \|\mathcal{E}_k(x, Z_{k+1}) - \mathcal{E}_k(x', Z_{k+1})\|_2 \leq [\mathcal{E}_k]_{\text{Lip}} |x - x'|, \quad k = 1 : n.$$

2. *L^p-linear growth assumption.* Let $p \in (2, 3]$.

$$(\text{SL})_p \quad \equiv \quad \forall k \in \{1, \dots, n\}, \quad \forall x \in \mathbb{R}^d, \quad \mathbb{E}|\mathcal{E}_k(x, Z_{k+1})|^p \leq \alpha_{p,k} + \beta_{p,k} |x|^p.$$

Proposition 3.1. *Let $\hat{X} = (\hat{X}_{t_k})_{k=0:n}$ be defined by (27) and suppose that all the grids Γ_k are quadratic optimal. Assume that both assumptions (Lip) and (SL)_p (for some $p \in (2, 3]$) hold and that $X_0 \in L^p(\mathbb{P})$. Then,*

$$\|\bar{X}_{t_k} - \hat{X}_{t_k}\|_2 \leq C_{d,p} \sum_{i=0}^k [\mathcal{E}_{i+1:k}]_{\text{Lip}} \left[\sum_{\ell=0}^i \alpha_{p,\ell} \beta_{p,\ell+1:i} \right]^{\frac{1}{p}} N_i^{-\frac{1}{d}} \quad (28)$$

where $C_{d,p} > 0$ and $\alpha_{p,0} = \mathbb{E}|X_0|^p = \|X_0\|_p^p$, $\beta_{p,\ell:i} = \prod_{m=\ell}^i \beta_{p,m}$ (with $\prod_{\emptyset} = 1$) and

$$[\mathcal{E}_{i:k}]_{\text{Lip}} := \prod_{\ell=i}^k [\mathcal{E}_\ell]_{\text{Lip}}, \quad 1 \leq \ell \leq k \leq n \quad \text{and} \quad [\mathcal{E}_{k+1:k}]_{\text{Lip}} = 1.$$

The associated probability weights and transition probabilities are computed from explicit formulas we recall in the following result.

Proposition 3.2. *Let Γ_{k+1} be a quadratic optimal quantizer for the marginal random variable $\tilde{X}_{t_{k+1}}$. Suppose that the quadratic optimal quantizer Γ_k for \tilde{X}_{t_k} is already computed and that we have access to its associated weights $\mathbb{P}(\tilde{X}_{t_k} \in C_i(\Gamma_k))$, $i = 1, \dots, N_k$. The transition probability $\hat{p}_k^{ij} = \mathbb{P}(\tilde{X}_{t_{k+1}} \in C_j(\Gamma_{k+1}) | \tilde{X}_{t_k} \in C_i(\Gamma_k)) = \mathbb{P}(\hat{X}_{t_{k+1}} = x_{k+1}^j | \hat{X}_{t_k} = x_k^i)$ is given by*

$$\hat{p}_k^{ij} = \Phi(x_{k+1,j+}(x_k^i)) - \Phi(x_{k+1,j-}(x_k^i)), \quad (29)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution,

$$x_{k+1,j-}(x) := \frac{x_{k+1}^{j-1/2} - m_k(x)}{v_k(x)} \quad \text{and} \quad x_{k+1,j+}(x) := \frac{x_{k+1}^{j+1/2} - m_k(x)}{v_k(x)},$$

with $m_k(x) = x + \Delta b(t_k, x)$, $v_k(x) = \sqrt{\Delta} \sigma(t_k, x)$ and, for $k = 0, \dots, n-1$ and for $j = 1, \dots, N_{k+1}$,

$$x_{k+1}^{j-1/2} = \frac{x_{k+1}^j + x_{k+1}^{j-1}}{2}, \quad x_{k+1}^{j+1/2} = \frac{x_{k+1}^j + x_{k+1}^{j+1}}{2}, \quad \text{with } x_{k+1}^{1/2} = -\infty, x_{k+1}^{N_{k+1}+1/2} = +\infty.$$

Once we have access to the marginal quantizations and to its associated transition probabilities, the right hand side of (23) can be computed explicitly.

3.5 The error analysis

Our aim in this section is to investigate the error resulting from the approximation of $\Pi_{y,m} \bar{F}(t_m, t_n, \cdot) = [\Pi_{y,m}(\pi_{n,m} \mathbf{1})](\cdot)$ by $[\hat{\Pi}_{y,m}(\hat{\pi}_{n,m} \mathbf{1})](\cdot)$. This error is an aggregation of three terms (see the proof of Theorem 3.6) involving the approximation errors $|\Pi_{y,m} \bar{F}(t_m, t_n, \cdot) - \hat{\Pi}_{y,m} \bar{F}(t_m, t_n, \cdot)|$ and $|(\pi_{n,m} \mathbf{1})(x) - (\hat{\pi}_{n,m} \mathbf{1})(x)|$, $x \in \mathbb{R}$. The two following results give the bounds associated to the two former approximation errors. Both are (carefully) adjustments of Theorem 4.1. and Lemma 4.1. in [16] to our context so that we refer to the former paper for their detailed proofs.

Theorem 3.3. *Suppose that Assumption (Lip) holds true. Then, for any bounded Lipschitz continuous function f on \mathbb{R}^d we have,*

$$|\Pi_{y,m} f - \hat{\Pi}_{y,m} f| \leq \frac{K_g^m}{\phi_m \vee \hat{\phi}_m} \sum_{k=0}^m A_k^m(f, y) \|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}\|_2,$$

where

$$\phi_m := \pi_{y,m} \mathbf{1}, \quad \hat{\phi}_m := \hat{\pi}_{y,m} \mathbf{1},$$

$$A_k^m(f, y) := 2 \frac{\|f\|_\infty}{K_g^m} [g_k^2]_{\text{Lip}}(y_{k-1}, y_k) + 2 \frac{\|f\|_\infty}{K_g^m} \sum_{j=k+1}^m [\mathcal{E}]_{\text{Lip}}^{j-k-1} \left([g_j^1]_{\text{Lip}}(y_{j-1}, y_j) + [\mathcal{E}]_{\text{Lip}} [g_j^2]_{\text{Lip}}(y_{j-1}, y_j) \right),$$

and, for every $k \in \{1, \dots, m\}$, $[g_k^1]_{\text{Lip}}(y, y')$ and $[g_k^2]_{\text{Lip}}(y, y')$ are such that for every $x, x', \hat{x}, \hat{x}' \in \mathbb{R}^d$,

$$|g_k^a(x, y; x', y') - g_k^a(\hat{x}, y; \hat{x}', y')| \leq [g_k^1]_{\text{Lip}}(y, y') |x - \hat{x}| + [g_k^2]_{\text{Lip}}(y, y') |x' - \hat{x}'|.$$

The quantities K_g and $[\mathcal{E}]_{\text{Lip}}$ are defined as

$$K_g = \max_{k=1, \dots, m} \|g_k^a\|_\infty \quad \text{and} \quad [\mathcal{E}]_{\text{Lip}} = \max_{k=1, \dots, m} [\mathcal{E}_k]_{\text{Lip}}.$$

Remark that the existence of $[g_k^1]_{\text{Lip}}(y_{k-1}, y_k)$ and $[g_k^2]_{\text{Lip}}(y_{k-1}, y_k)$ is guaranteed by the fact that the function $g_k^a(x, y; x', y')$ is Lipschitz with respect to (x, x') .

Let us give now the error bound associated to the approximation of $\pi_{n,m} \mathbf{1}$.

Proposition 3.4. *Let $y = (y_0, \dots, y_m) \in (\mathbb{R}^q)^{m+1}$. Then, we have for any $x \in \mathbb{R}$*

$$|(\pi_{n,m} \mathbf{1})(x) - (\hat{\pi}_{n,m} \mathbf{1})(x)| \leq \sum_{k=m+1}^n B_k(G) \|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}\|_2 \quad (30)$$

where

$$B_k(G) = \Lambda^{n-m-1} \left([G_m^1]_{\text{Lip}} \delta_{\{k=m\}} + ([G_k^1]_{\text{Lip}} \vee [G_{k-1}^2]_{\text{Lip}}) \delta_{\{k \in \{m+1, \dots, n-1\}\}} + [G_n^2]_{\text{Lip}} \delta_{\{k=n\}} \right)$$

with

$$\Lambda = \max_{k=m+1, \dots, n} \|G_{\Delta_k \sigma_k^2}^{(\bullet, \bullet)}(\mathbf{a})\|_{\infty}$$

Proof. Recall that

$$(\pi_{n,m} \mathbf{1})(x) = \mathbb{E} \left(\Lambda_m(\bar{X}_{t_{m:n}}) \mid \bar{X}_{t_m} = x \right)$$

where for every $k \geq m$,

$$\Lambda_m(\bar{X}_{t_{m:k}}) := \prod_{\ell=m}^{k-1} G_{\Delta_{\ell} \sigma_{\ell}^2}^{\bar{X}_{t_{\ell}}, \bar{X}_{t_{\ell+1}}}(\mathbf{a})$$

with the convention that $\Lambda_m(\bar{X}_{t_{m:m}}) = 1$. Now, we have for any $k \geq m$,

$$\begin{aligned} \Lambda_m(\bar{X}_{t_{m:k}}) - \Lambda_m(\hat{X}_{t_{m:k}}) &= \left(G_{\Delta_{k-1} \sigma_{k-1}^2}^{\bar{X}_{t_{k-1}}, \bar{X}_{t_k}}(\mathbf{a}) - G_{\Delta_{k-1} \sigma_{k-1}^2}^{\hat{X}_{t_{k-1}}, \hat{X}_{t_k}}(\mathbf{a}) \right) \Lambda_m(\bar{X}_{t_{m:k-1}}) \\ &\quad + G_{\Delta_{k-1} \sigma_{k-1}^2}^{\hat{X}_{t_{k-1}}, \hat{X}_{t_k}}(\mathbf{a}) \left(\Lambda_m(\bar{X}_{t_{m:k-1}}) - \Lambda_m(\hat{X}_{t_{m:k-1}}) \right). \end{aligned}$$

Since the function $G_{\Delta_k \sigma_k^2}^{(\bullet, \bullet)}(\mathbf{a})$ is Lipschitz and bounded and that for any $k \geq m+1$, $\Lambda_m(\bar{X}_{t_{m:k-1}}) \leq \Lambda^{k-m-1}$, we have

$$\begin{aligned} |\Lambda_m(\bar{X}_{t_{m:k}}) - \Lambda_m(\hat{X}_{t_{m:k}})| &\leq \left([G_k^1]_{\text{Lip}} |\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\text{Lip}} |\bar{X}_{t_k} - \hat{X}_{t_k}| \right) \Lambda^{k-m-1} \\ &\quad + \Lambda |\Lambda_m(\bar{X}_{t_{m:k-1}}) - \Lambda_m(\hat{X}_{t_{m:k-1}})|. \end{aligned}$$

Keeping in mind that $\Lambda_m(\bar{X}_{t_{m:m}}) = \Lambda_m(\hat{X}_{t_{m:m}}) = 1$, we deduce from an induction on k that

$$|\Lambda_m(\bar{X}_{t_{m:n}}) - \Lambda_m(\hat{X}_{t_{m:n}})| \leq \Lambda^{n-m-1} \sum_{k=m+1}^n [G_k^1]_{\text{Lip}} |\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\text{Lip}} |\bar{X}_{t_k} - \hat{X}_{t_k}|.$$

The result follows by noting that

$$\begin{aligned} |(\pi_{n,m} \mathbf{1})(x) - (\hat{\pi}_{n,m} \mathbf{1})(x)| &\leq \mathbb{E} \left(|\Lambda_m(\bar{X}_{t_{m:n}}) - \Lambda_m(\hat{X}_{t_{m:n}})| \mid \bar{X}_{t_m} = x \right) \\ &\leq \Lambda^{n-m-1} \sum_{k=m+1}^n [G_k^1]_{\text{Lip}} |\bar{X}_{t_{k-1}} - \hat{X}_{t_{k-1}}| + [G_k^2]_{\text{Lip}} |\bar{X}_{t_k} - \hat{X}_{t_k}| \end{aligned}$$

and by using the non-decreasing property of the L^p -norm. \square

We may deduce now the global error induced by our procedure, means, the error deriving from the estimation of

$$\Pi_{y,m} F(t_m, t_n, \cdot) = \mathbb{P}(\tau_X > t_n \mid (Y_{t_0}, \dots, Y_{t_m}) = y)$$

by Equation (23). To this end, we need the following additional assumptions which will be used to compute (see [10]) the convergence rate of the quantity $\mathbb{E} |\mathbb{1}_{\{\tau_{\bar{X}} > t\}} - \mathbb{1}_{\{\tau_X > t\}}|$ towards 0. We suppose that the diffusion is homogeneous and

(H1) b is a $\mathcal{C}_b^{\infty}(\mathbb{R})$ function and σ is in $\mathcal{C}_b^{\infty}(\mathbb{R})$.

(H2) There exists $\sigma_0 > 0$ such that $\forall x \in \mathbb{R}, \sigma(x)^2 \geq \sigma_0^2$ (*uniform ellipticity*).

We have the following result.

Proposition 3.5 (See [10]). *Let $t > 0$. Suppose that Assumptions **(H1)** and **(H2)** are fulfilled. Then, for every $\eta \in (0, \frac{1}{2}]$ there exists an increasing function $K(T)$ such that for every $t \in [0, T]$ and for every $x \in \mathbb{R}$,*

$$\mathbb{E}_x \left[\left| \mathbb{1}_{\{\tau_X > t\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t\}} \right| \right] \leq \frac{1}{n^{\frac{1}{2}-\eta}} \frac{K(T)}{\sqrt{t}},$$

where n is the number of discretization time steps over $[0, t]$.

Theorem 3.6. *Suppose that the coefficients b and σ of the continuous signal process X are such that Assumptions **(H1)** and **(H2)** are satisfied and let $\eta \in (0, \frac{1}{2}]$. We also suppose that Assumption **(Lip)** holds. Then, for any bounded Lipschitz continuous function $f : \mathbb{R}^d \mapsto \mathbb{R}$ and for any fixed observation $y = (y_0, \dots, y_m)$ we have*

$$\begin{aligned} \left| \mathbb{P}(\tau_X > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) - \sum_{i=1}^{N_n} \sum_{j=1}^{N_m} \hat{\Pi}_{y,m}^i \hat{\pi}_{n,m} \delta_{x_m^j} \right| &\leq \mathcal{O}\left(n^{-\frac{1}{2}+\eta}\right) \\ &+ \sum_{k=0}^n C_k^n(\bar{F}(s, t, \cdot), y) \|\bar{X}_{t_k} - \hat{X}_{t_k}^{\Gamma_k}\|_2, \end{aligned}$$

where

$$C_k^n(\bar{F}(s, t, \cdot), y) = \frac{K_g^m}{\phi_m \vee \hat{\phi}_m} A_k^m(f, y) \delta_{\{k \leq m\}} + B_k(G) \delta_{\{k \geq m+1\}}$$

and where K_g^m , ϕ_m , $\hat{\phi}_m$, $A_k^m(f, y)$ and $B_k(G)$ are defined in Theorem 3.3 and in Proposition 3.4.

Proof. We have

$$\begin{aligned} \left| \mathbb{P}(\tau_X > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) - \sum_{i=1}^{N_n} \sum_{j=1}^{N_m} \hat{\Pi}_{y,m}^i \hat{\pi}_{n,m} \delta_{x_m^j} \right| \\ \leq \left| \mathbb{P}(\tau_X > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) - \mathbb{P}(\tau_{\bar{X}} > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) \right| \\ + \left| \Pi_{y,m} \bar{F}(t_m, t_n, \cdot) - \hat{\Pi}_{y,m} \bar{F}(t_m, t_n, \cdot) \right| \\ + \left| \hat{\Pi}_{y,m} \bar{F}(t_m, t_n, \cdot) - \hat{\Pi}_{y,m} \hat{F}(t_m, t_n, \cdot) \right|. \end{aligned} \quad (31)$$

Now, we have

$$\begin{aligned} \left| \hat{\Pi}_{y,m} \bar{F}(t_m, t_n, \cdot) - \hat{\Pi}_{y,m} \hat{F}(t_m, t_n, \cdot) \right| &= \left| \sum_{i=1}^{N_m} \left(\hat{\Pi}_{y,m}^i \bar{F}(t_m, t_n, x_m^i) - \hat{\Pi}_{y,m}^i \hat{F}(t_m, t_n, x_m^i) \right) \right| \\ &\leq \sup_{x \in \mathbb{R}} \left| \bar{F}(t_m, t_n, x) - \hat{F}(t_m, t_n, x) \right| \sum_{i=1}^{N_m} \hat{\Pi}_{y,m}^i \\ &= \sup_{x \in \mathbb{R}} \left| \bar{F}(t_m, t_n, x) - \hat{F}(t_m, t_n, x) \right| \\ &= \sup_{x \in \mathbb{R}} \left| (\pi_{n,m} \mathbf{1})(x) - (\hat{\pi}_{n,m} \mathbf{1})(x) \right|. \end{aligned}$$

The error bound for $|(\pi_{n,m} \mathbf{1})(x) - (\hat{\pi}_{n,m} \mathbf{1})(x)|$ is independent from x and is given by (30). On the other hand we have

$$\begin{aligned} \left| \mathbb{P}(\tau_X > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) - \mathbb{P}(\tau_{\bar{X}} > t_n | (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y) \right| \\ \leq \mathbb{E} \left(\left| \mathbb{1}_{\{\tau_X > t_n\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t_n\}} \right| \middle| (\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y \right) \\ \leq \frac{1}{\mathbb{P}((\bar{Y}_{t_0}, \dots, \bar{Y}_{t_m}) = y)} \mathbb{E} \left| \mathbb{1}_{\{\tau_X > t_n\}} - \mathbb{1}_{\{\tau_{\bar{X}} > t_n\}} \right|. \end{aligned}$$

We conclude by Proposition 3.5. \square

4 Numerical results

We illustrate the numerical part by considering a Black-Scholes model. More specifically, the dynamics of the signal process X and observation process Y are in the form

$$\begin{cases} dX_t = X_t(\mu dt + \sigma dW_t), & X_0 = x_0, \\ dY_t = Y_t(\mu dt + \sigma dW_t + \delta d\widetilde{W}_t), & Y_0 = y_0 \end{cases} \quad (32)$$

meaning that

$$\frac{dY_t}{Y_t} = \frac{dX_t}{X_t} + \delta d\widetilde{W}_t$$

which can be interpreted as the yields of the observation process Y are noised yields of the signal process X with magnitude δ . Intuitively, in order to deal with CDS option implied volatilities below, we will play with the parameter δ .

4.1 Comparison of conditional survival probabilities

Before tackling the CDS examples, we first test the performance of our method in two setups:

- By comparing the function $F(s, t, x) = \mathbb{P}(\inf_{s \leq u \leq t} X_u > \mathbf{a} | X_s = x)$ and its quantized version $\widehat{F}(s, t, x)$ defined in (21), keeping in mind that in the model (32),

$$F(s, t, x) = \Phi(h_1(x, t - s)) - \left(\frac{\mathbf{a}}{x}\right)^{2\sigma^{-2}(\mu - \sigma^2/2)} \Phi(h_2(x, t - s)), \quad (33)$$

where

$$\begin{aligned} h_1(x, u) &= \frac{1}{\sigma\sqrt{u}} \left(\log\left(\frac{x}{\mathbf{a}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u \right), \\ h_2(x, u) &= \frac{1}{\sigma\sqrt{u}} \left(\log\left(\frac{\mathbf{a}}{x}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)u \right). \end{aligned}$$

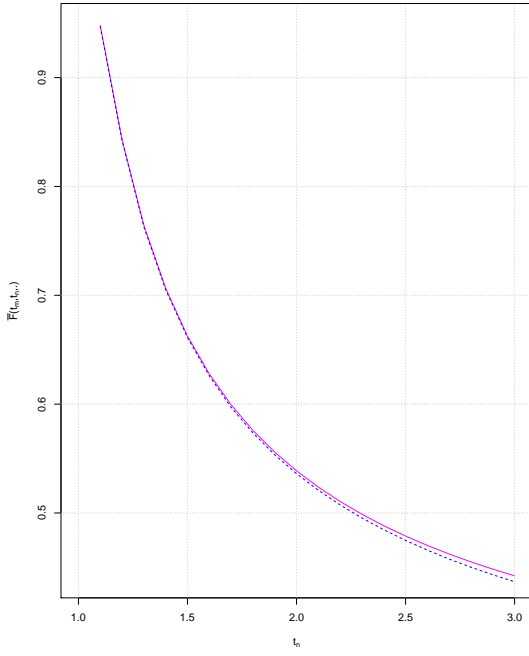
This comparison allows us to test our method given benchmark values.

- By comparing the conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ and $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$ respectively estimated by (23) and (25), where $F(s, t, \cdot)$ is computed using the exact formula (33). Our aim here is to confirm the impact of the additional information $\mathcal{F}_s^{\bar{H}}$ on the conditional probability $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$.

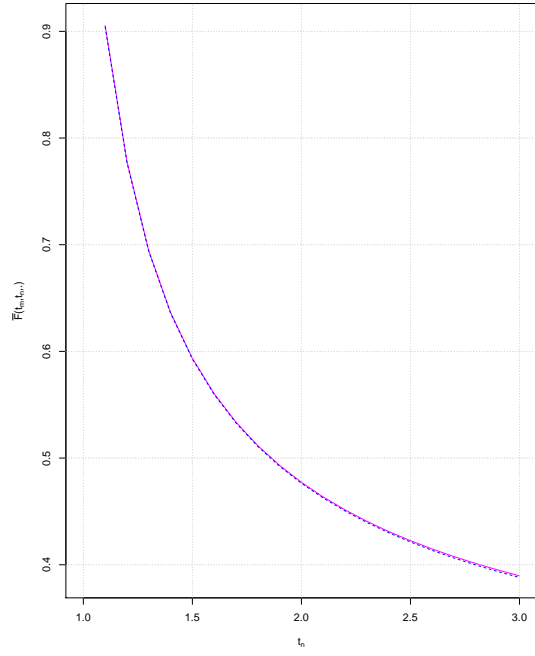
Notice that we deal in this paper with a general framework where the signal and the observation processes have no closed formula. Even if in our model both processes have explicit solutions and we may deduce a similar formula to (11) using these closed formulae (the only change will come from the function g_k which involves the conditional density of $(X_{t_{k+1}}, Y_{t_{k+1}})$ given (X_{t_k}, Y_{t_k})), we still consider their associated Euler schemes processes \bar{X} and \bar{Y} in order to stay in the scope of the proposed numerical method.

To compare the functions $F(s, t, \cdot)$ and $\widehat{F}(s, t, \cdot)$, we choose the following set of parameters (like those of [20]): $\mu = 0.03$, $\sigma = 0.09$, $\delta = 0.5$, $x_0 = y_0 = 86.3$ and $\mathbf{a} = 76$. Figure 1 shows the convergence of the quantized function $\widehat{F}(t_m, t_n, x)$ toward the exact one $F(t_m, t_n, x)$ with $t_m = 1$, $t_n \in [1.1, 3]$ and where x is one point, say x_m^* , on the grid $\{x_m^i, i = 1, \dots, N_m\}$ (see equation (21)). Once we fix t_m , $F(t_m, t_n, x)$ depends on both t_n and x . Therefore, to show the convergence, we fix x and plot both $F(t_m, t_n, x)$ and $\widehat{F}(t_m, t_n, x)$ with respect to t_n . The number of discretization points m is set to 50 and the convergence is achieved by increasing the number of quantization

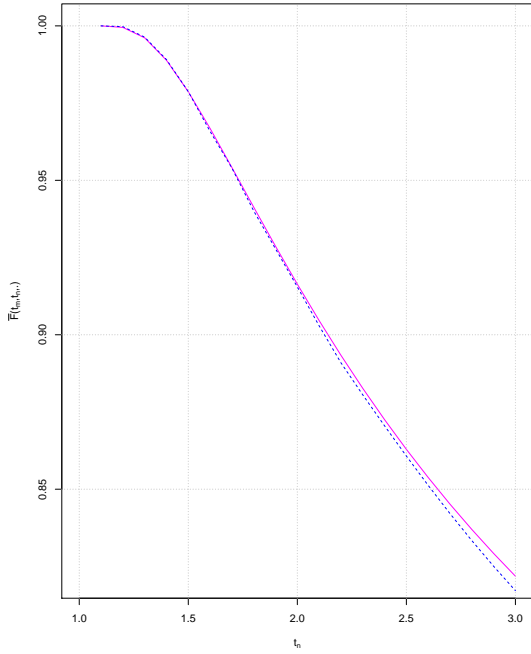
points N_n . Since the fixed quantization point x_m^* can differ when moving N_n , the corresponding figures can take different shapes but, we have only to make sure that the convergence is achieved when increasing N_n .



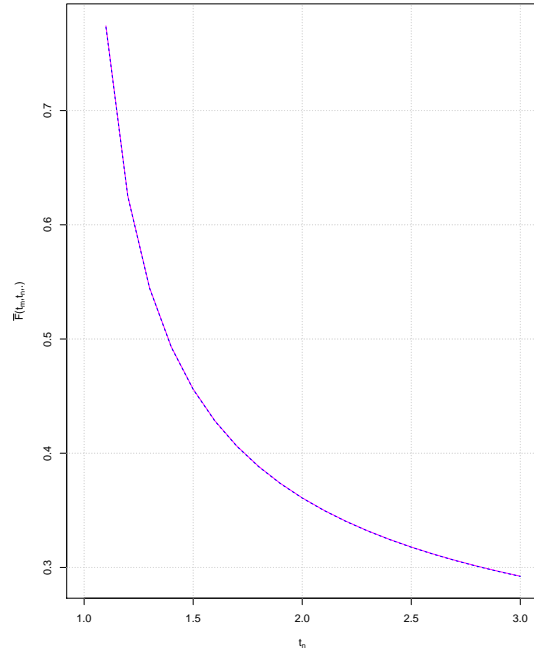
(a) $N_n = 50$



(b) $N_n = 100$



(c) $N_n = 50$

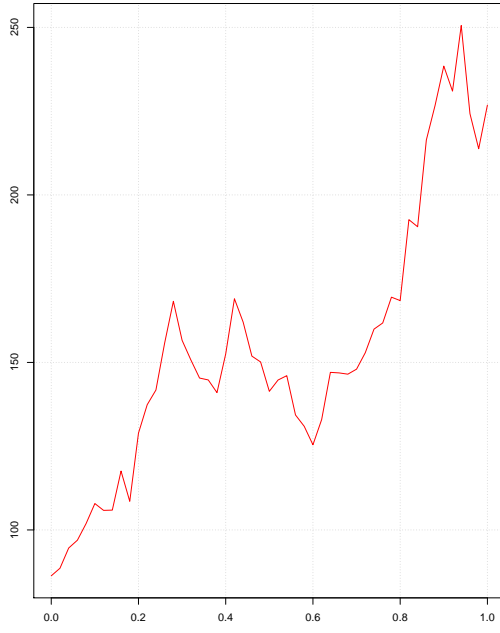


(d) $N_n = 400$

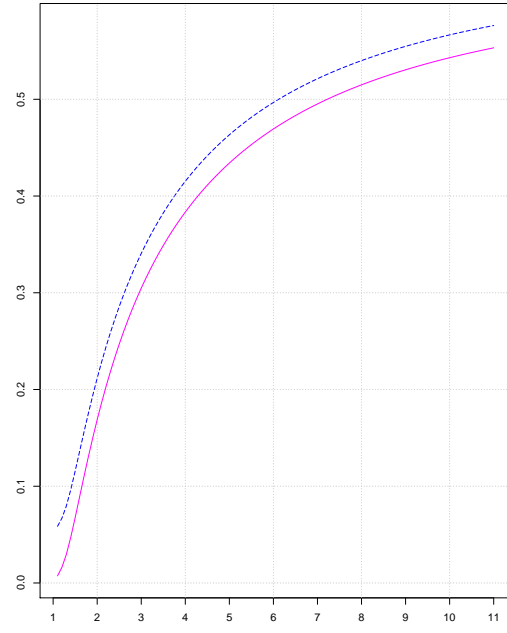
Figure 1: Convergence of $\bar{F}(t_m, t_n, \cdot)$ estimated by $\hat{F}(t_m, t_n, \cdot)$ (dotted blue) toward $F(t_m, t_n, \cdot)$ (solid magenta).

We now proceed to the numerical comparison between the conditional default probabilities $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ and $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$, respectively estimated by (23) and (25), in order to check

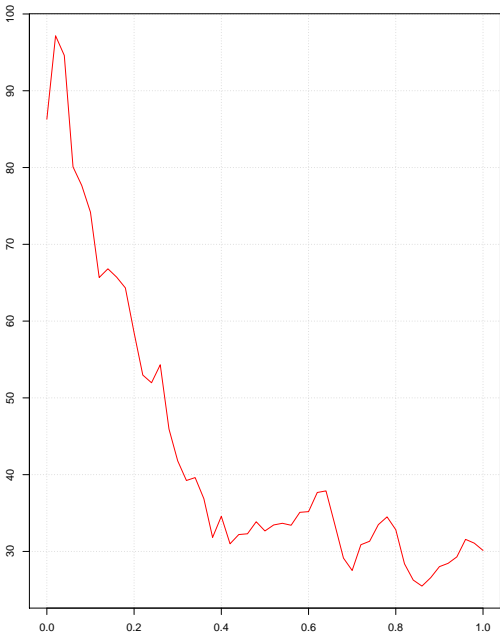
the statements of Remark 3.1. Setting $s = t_m = 1$, $t = t_n$ and considering the same parameter set as in the previous figure but with $t_n \in [1.1, 11]$, Figure 2 depicts the trajectories of the observation process \bar{Y} from 0 to t_m and the associated conditional default probabilities as a function of t_n . First, we notice that equation (6) is fulfilled as given a trajectory of the observation process \bar{Y} represented in red, $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$ lined up in dotted blue is always above $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ in magenta. Second, the gap between the two quantities is larger for an downward movement of Y compared to an upward movement for which the firm is less exposed to default. This can be understood by the fact that the more the firm is creditworthy, the less the default information is important and the less the default probability is. Then, the model preserves the memory of all the observed path of the process Y when computing default probabilities. This path-dependent future of the default probabilities has already been shown in [6] and is known to be very important as it is implicit in reduced-form models for calibration purpose to historical data.



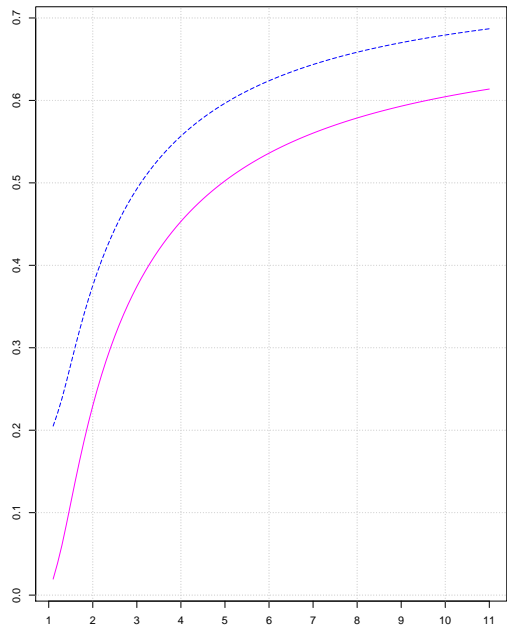
(a) \bar{Y} Up



(b) Default probability



(c) \bar{Y} Down



(d) Default probability

Figure 2: Trajectories of the observation process \bar{Y} (solid red), Y up: panel (a), \bar{Y} Down: panel (c) and the associated conditional default probabilities functions $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}} \vee \mathcal{F}_s^{\bar{H}})$ (solid magenta) and $\mathbb{P}(\tau_{\bar{X}} \leq t | \mathcal{F}_s^{\bar{Y}})$ (dotted blue): panels (b) and (d), with $t_m = 1$ and $t_n \in [1.1, 11]$ and number of quantization points $N_n = N_m = 30$.

4.2 Application to CDS option pricing

In this section, we briefly recall the concept and valuation of credit default swaps and swaptions before analyzing the quantization procedure applied to such models. This will allow to give a full pricing formula of credit swaps derivatives in a firm value approach using partial information theory and optimal quantization. In addition, the fact that we add the default filtration in the model indicating whether default has already taken place or not is very important in this case as it is pointless to price a default swap post-default.

A credit default swap (CDS) is an agreement between two counterparties to buy or sell protection against the default risk of a third party called *reference entity*. We set τ_X as the default time of the latter. In this case, if the contract is signed at time s , started at time T_a with maturity T_b , the protection buyer pays a coupon (or spread) k at payments dates T_{a+1}, \dots, T_b as long as the reference entity does not default or until τ_X . If the default occurs at time τ_X with $T_a < \tau_X \leq T_b$, the protection seller will make a single payment LGD (that we assume to be a known constant) to the protection buyer. A CDS option (CDSO) or default swaption is an option written on a default swap. From this perspective, it requires to recall the no-arbitrage pricing equation of a CDS. The time- s price of a general buyer CDS $CDS_s(a, b, k)$ with unit notional starting at time T_a with maturity T_b , $s \leq T_a < T_b$, a spread k and loss given default LGD is given by the difference of the conditional risk-neutral expectations of the protection and the premium discounted cashflows:

$$CDS_s(a, b, k) = \mathbb{E} \left[LGD \mathbb{1}_{\{T_a < \tau_X \leq T_b\}} D_s(\tau_X) | \mathcal{F}_s \right] - k \mathbb{E} \left[\sum_{i=a+1}^b \left(\mathbb{1}_{\{\tau_X \geq T_i\}} \alpha_i D_t(T_i) + \mathbb{1}_{\{T_{i-1} \leq \tau_X < T_i\}} \alpha_i \frac{\tau_X - T_{i-1}}{T_i - T_{i-1}} D_t(\tau_X) \right) \middle| \mathcal{F}_s \right]$$

with α_i the day count fraction between dates T_{i-1} and T_i which, in a standard CDS, is around 0.25 (quarterly payment dates) and $D_s(t) = e^{-\int_s^t r(u) du}$ is a time- s discount factor with maturity t and deterministic interest rates r . In a reduced-form setup, where $\mathcal{F}_s := \mathcal{F}_s^Y \vee \mathcal{F}_s^H$, this expression can be developed explicitly thanks to the Key lemma:

$$CDS_s(a, b, k) = \mathbb{1}_{\{\tau_X > s\}} \left(-LGD \int_{T_a}^{T_b} D_s(u) \partial_u P_s(u) du - k C_s(a, b) \right), \quad (34)$$

where

$$P_s(t) := \frac{S_s(t)}{S_s(s)} \quad (35)$$

and $S_s(t) = \mathbb{P}(\tau_X \geq t | \mathcal{F}_s^Y)$ is known as the *Azéma supermartingale* and $C_s(a, b)$ is the risky duration, i.e. the time- s value of the CDS premia paid during the life of the contract when the spread is 1:

$$C_s(a, b) := \sum_{i=a+1}^b \alpha_i D_s(T_i) P_s(T_i) - \int_{T_{i-1}}^{T_i} \frac{u - T_{i-1}}{T_i - T_{i-1}} \alpha_i D_s(u) \partial_u P_s(u) du.$$

The spread which, at time s , sets the forward start CDS at 0, called *par spread*, is given by:

$$\mathbb{1}_{\{\tau_X > s\}} k_s^*(a, b) := \mathbb{1}_{\{\tau_X > s\}} \frac{-LGD \int_{T_a}^{T_b} D_s(u) \partial_u P_s(u) du}{C_s(a, b)}. \quad (36)$$

The no-arbitrage price of a call option on such a contract at time $s = 0$ becomes

$$\begin{aligned} PSO(a, b, k) &= \mathbb{E} \left[(CDS_{T_a}(a, b, k))^+ D_0(T_a) \right] \\ &= D_0(T_a) \mathbb{E} \left[S_{T_a}(T_a) \left(LGD - \sum_{i=a+1}^b \int_{T_{i-1}}^{T_i} g_i(u) D_{T_a}(u) P_{T_a}(u) du \right)^+ \right] \end{aligned} \quad (37)$$

where $g_i(u) := LGD(r(u) + \delta_{T_b}(u)) + k \frac{\alpha_i}{T_i - T_{i-1}} (1 - (u - T_{i-1})r(u))$, with $\delta_s(u)$ the Dirac delta function centered at s .

The random terms inside the expectation (37) mainly the survival processes $S_{T_a}(\cdot)$ and $P_{T_a}(\cdot)$ are ready to be computed using optimal quantization. To do so, using equations (23) and (25), one only needs to set

$$P_{T_a}(u) = \widehat{\Pi}_{y,a} \widehat{F}(T_a, u, \cdot) \quad \text{and} \quad S_{T_a}(T_a) = \widehat{\omega}_{y,a} \mathbf{1}. \quad (38)$$

Hence the randomness in the expectation (37) is only from the observation process Y simulated from time 0 to T_a . This means that we should not need a lot of paths when estimating the expectation (37) using Monte Carlo simulation after computing the above mentioned survival processes using optimal quantization. This in turn motivates to fully estimate (37) using a hybrid Monte Carlo-optimal quantization procedure.

A CDS option has little liquidity but, just like usual equity options, is quoted in term of its Black implied volatility $\bar{\sigma}$ which is based on the assumption that the credit spread follows a geometric Brownian motion.¹

The Black formula for payer swaptions at time 0 with maturity T_a is

$$PSO^{Black}(a, b, k, \bar{\sigma}) = C_0(a, b) [k_0^*(a, b) \Phi(d_1) - k \Phi(d_2)]$$

where

$$d_1 = \frac{\ln \frac{k_0^*(a, b)}{k} + \frac{1}{2} \bar{\sigma}^2 T_a}{\bar{\sigma} \sqrt{T_a}}, \quad d_2 = d_1 - \bar{\sigma} \sqrt{T_a}.$$

Hence, the CDS option implied volatility $\bar{\sigma}$ can be found by solving the following equation

$$PSO(a, b, k) = PSO^{Black}(a, b, k, \bar{\sigma}).$$

We now assess the numerical results based on the model's applications to the pricing of CDS option. The model's parameter set is the same as before except here we take $\sigma = 5\%$ and δ is varying. Table 1 shows the estimated values of a European payer CDS option and the corresponding Black's volatilities with different strikes and different values of δ . First, we observe that both CDS option prices and the implied volatilities are increasing with the noise volatility, δ . This can be explained by the fact that, the higher δ , the noisier the observations are and the higher the default probability. Since δ measures the degree of transparency of the firm, this will have a positive impact on the prices of the CDS option, hence on the corresponding implied volatilities. In contrast, while the option prices are always decreasing with the strike, this is not the case with the implied volatilities except for $\delta = 2\%$ and $\delta = 3\%$. In the case where $\delta = 1\%$, the implied volatility is increasing with respect to the strike. Hence with the help of the parameter δ , one can observe different levels of skewness.

k (bps)	Payer			Implied vol (%)		
	$\delta = 1\%$	$\delta = 2\%$	$\delta = 3\%$	$\delta = 1\%$	$\delta = 2\%$	$\delta = 3\%$
52.9	0.004655	0.007214	0.009276	69.44	133.84	196.80
66.2	0.003739	0.006077	0.008032	73.36	124.16	173.70
79.4	0.003107	0.005298	0.007081	76.85	121.08	161.57

Table 1: CDS options and corresponding Black volatilities (with spread $k_0^*(a, b) = 66.20$ pbs, $T_a = 1$ and $T_b = 3$) implied by the structural model using Monte Carlo simulation ($1.5 \cdot 10^5$) paths and for various volatility parameter δ and different strikes (80%, 100%, 120%) $k_0^*(a, b)$ and $\sigma = 5\%$.

¹Recall that this does not mean in any way that the market naively believes that credit spreads exhibit log-normal dynamics. Market participants simply rely on the Black-Scholes machinery to convert a price into a quantity that is more intuitive to traders, namely implied volatilities.

Notice that in this example, we focus more on the numerical performances of the model and do not address the calibration problem. Hence, we use the model implied term structure given by the time-0 model survival probability curve as a CDS term structure. Calibration issues of the model to real market data will be investigated in a future work.

5 Conclusion

In this paper, a new structural model for credit risk has been proposed, generalizing earlier works. Our model deals with an incomplete information, where the default state and a noisy observation of the firm value are accessible to the investor. It is therefore an extension of [6], as the firm-value triggering the default is no longer restricted to be a continuous and invertible function of a Gaussian martingale, but can be any diffusion.

This more general framework benefits however from a limited analytical tractability. Therefore, we propose a numerical method that relies on nonlinear filtering theory associated with recursive quantization. Compared to earlier works such as [5] or [20], our numerical procedure is based on the fast quantization method recently introduced in [17], which avoids the use of Monte Carlo simulations. A rigorous analysis of the global error induced to the estimation of the survival processes is performed. We analyze the shapes of the default probabilities which are characterized by a path-dependent feature keeping the memory of all the path of the observed process. Eventually we quantify the impact of the volatility of the noise impacting the firm-value process on the pricing of CDS options and the corresponding implied volatilities using a hybrid Monte Carlo-optimal quantization method.

In future research, we will first investigate the calibration issues of the model which can be tackled by either using observed prices or CDS quotes. In this case, our model can be easily extended to other works dealing with exact calibration to survival probabilities such as including a specific time-dependent barrier [2] or using time change techniques [13]. Another possible research area is to deal with the price of general default sensitive securities. While we have derived a full quantization scheme to estimate the conditional default probabilities, this was not the case in the pricing of CDS option which required additional Monte Carlo simulations in order to be estimated. To derive a full quantization scheme for the pricing of defaultable claims, a possible route is to exploit the functional quantization method.

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