



HAL
open science

New pancake series for π

Yannick Saouter

► **To cite this version:**

Yannick Saouter. New pancake series for π . *Mathematical gazette*, 2020, 10.1017/mag.2020.53 .
hal-02280806

HAL Id: hal-02280806

<https://hal.science/hal-02280806>

Submitted on 5 Jan 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NEW PANCAKE SERIES FOR π

YANNICK SAOUTER

Introduction. In [1], Dalzell proved that $\pi = \frac{22}{7} - \int_0^1 \frac{t^4(1-t)^4}{1+t^2} dt$. He then used this equation to derive a new series converging to π . In [2], Backhouse studied the general case of integrals of the form $\int_0^1 \frac{t^m(1-t)^n}{1+t^2} dt$ and derived conditions on m and n so that they could be used to evaluate π . As a sequel, he derived accurate rational approximations of π . This work was extended in [3] where new rational approximations of π are obtained. Some related integrals of the forms $\int_0^1 \frac{t^m(1-t)^n}{1+t^2} P(t) dt$ and $\int_0^1 \frac{t^m(1-t)^n}{\sqrt{1-t^2}} P(t) dt$ with $P(t)$ being of polynomial form are also investigated. In [4], the author gives more new approximations and new series for the case $m = n = 4k$. In [5], new series for π are obtained with the integral $\int_0^\alpha \frac{t^{12m}(\alpha-t)^{12m}}{1+t^2} dt$ where $\alpha = 2 - \sqrt{3}$. The general problem of improving the convergence speed of the arctan series by transformation of the argument has also been considered in [6, 7]. In this work, the author considers an alternative form for the denominators in integrals. As a result, new series are obtained for multiples of π by some algebraic numbers.

The alternative integral. In the following θ will be a real number with $0 < \theta < \pi$ and we define $I_\theta = \int_0^1 \frac{dt}{t^2 - 2t \cos \theta + 1}$. We have $t^2 - 2t \cos \theta + 1 = (t - e^{i\theta})(t - e^{-i\theta})$ and thus I_θ is a proper well-defined integral. The computation of I_θ is a classical exercise of integration. We have

$$\begin{aligned} I_\theta &= \int_0^1 \frac{dt}{(t - \cos \theta)^2 + \sin^2 \theta} = \frac{1}{\sin^2 \theta} \int_0^1 \frac{dt}{1 + \left(\frac{t - \cos \theta}{\sin \theta}\right)^2} \\ &= \frac{1}{\sin \theta} \int_{-\frac{\cos \theta}{\sin \theta}}^{\frac{1 - \cos \theta}{\sin \theta}} \frac{du}{1 + u^2} \end{aligned}$$

Moreover:

$$\begin{aligned} \frac{1 - \cos \theta}{\sin \theta} &= \frac{1 - \cos^2(\theta/2) + \sin^2(\theta/2)}{2 \sin(\theta/2) \cos(\theta/2)} = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan(\theta/2) \\ -\frac{\cos \theta}{\sin \theta} &= -\frac{1}{\tan(\theta)} = -\tan\left(\frac{\pi}{2} - \theta\right) = \tan\left(\theta - \frac{\pi}{2}\right) \end{aligned}$$

Therefore:

$$I_\theta = \frac{1}{\sin \theta} \left[\arctan u \right]_{\tan(\theta - \frac{\pi}{2})}^{\tan(\theta/2)} = \frac{\pi - \theta}{2 \sin \theta}$$

This integral can alternatively be computed by first decomposing the integrand as:

$$\frac{1}{t^2 - 2t \cos \theta + 1} = \frac{1}{2i \sin \theta} \left[\frac{1}{t - e^{i\theta}} - \frac{1}{t - e^{-i\theta}} \right]$$

and then integrating with the function \log extended to the complex domain.

Factorizing pancake functions. We let $P_\theta(x) = (x - e^{i\theta})(x - e^{-i\theta})$. Let now $A(x) = x^m(1 - x)^n$ be a function with m, n integers such that $m, n \geq 1$. Such functions have been called pancake functions because of the shape of their graph. Following the steps of previous works, we are then looking for a factorization $A(x) = P_\theta(x)Q(x) + R$ where $Q(x)$ is a polynomial with real coefficients and R is a real constant. In the following, such a decomposition will be called an acceptable factorization. We then prove the following theorem.

Theorem 1. *Acceptable factorizations exist only for θ being a rational multiple of π . Let then $\theta = r\pi/s$ with r and s positive integers with $0 < r < s$ and $(r, s) = 1$. Then an acceptable factorization exists according to the following criteria:*

- For any r and s values, the pair (m, n) with n even is a solution if and only if $m + n/2 = 0[\text{mod } s]$,
- If r and s are odd, the pair (m, n) with n odd is also a solution if and only if $2m + n = 0[\text{mod } s]$.

If an acceptable factorization exists then necessarily $A(e^{i\theta}) = A(e^{-i\theta}) = R$. Since $A(x)$ is a polynomial with real coefficients, we have $\bar{A}(x) = A(\bar{x})$ for any $x \in \mathbb{C}$. Thus we have $R = \bar{R}$ and R is a real number. Reciprocally if $A(e^{i\theta}) = R$ with R real, then by conjugacy $A(e^{-i\theta}) = R$. Then the polynomial $A(x) - R$ has $x - e^{i\theta}$ and $x - e^{-i\theta}$ as factors. Since $0 < \theta < \pi$, both factors are distinct and $A(x) - R$ admit $P_\theta(x)$ as a factor. We also have $A(e^{i\theta}) = e^{im\theta}(1 - e^{i\theta})^n = e^{i(m+n/2)\theta}(-2i \sin(\theta/2))^n$. Therefore $\text{Arg}(e^{i\theta}) = ((m + n/2)\theta - n\pi/2)[\text{mod } 2\pi]$ and $A(e^{i\theta})$ is real if and only if:

$$(1) \quad (m + n/2)\theta = n\pi/2[\text{mod } \pi]$$

It is clear that if the latter equation has solutions then θ is necessarily a rational multiple of π . Therefore let r and s integers such that $\theta = r\pi/s$, $0 < r < s$ and $(r, s) = 1$. Equation (1) becomes then successively:

$$(2) \quad \begin{aligned} (m + n/2)r\pi &= ns\pi/2[\text{mod } s\pi] \\ (2m + n)r\pi &= ns\pi[\text{mod } 2s\pi] \\ (2m + n)r &= ns[\text{mod } 2s] \end{aligned}$$

We suppose first that n is even, so that $n = 2n'$ with n' an integer. Then (2) successively becomes:

$$(3) \quad \begin{aligned} (2m + 2n')r &= 0[\text{mod } 2s] \\ (m + n')r &= 0[\text{mod } s] \\ m + n' &= 0[\text{mod } s] \end{aligned}$$

since $(r, s) = 1$. We suppose now that n is odd. Then (2) becomes:

$$(4) \quad (2m + n)r = s[\text{mod } 2s]$$

Modulo 2, we obtain then $r = s[\text{mod } 2]$. Therefore, since $(r, s) = 1$, we have necessarily $r = s = 1[\text{mod } 2]$ and both r and s are odd. Modulo s , (4) gives

$$(5) \quad \begin{aligned} (2m + n)r &= 0[\text{mod } s] \\ 2m + n &= 0[\text{mod } s] \end{aligned}$$

Reciprocally if $2m + n = 0[\text{mod } s]$ and $r = s = 1[\text{mod } 2]$ then (4) is satisfied.

If $\theta = \pi/2$, using Theorem 1, the criterion derived in [2] is recovered. Thus the latter theorem is a generalization of this work. Another remark that can be made is that, since acceptable factorizations exist only for values of θ being rational multiple of π , in these cases, corresponding values of I_θ are products of π by some real algebraic numbers.

Series development. We suppose now that this is the case and this factorization will be used to develop I_θ in series. For $x \in (0, 1)$, we have $\frac{1}{P_\theta(x)} = \frac{Q(x)}{A(x)-R}$, so that:

$$(6) \quad \begin{aligned} I_\theta &= - \int_0^1 \frac{Q(x)}{R - A(x)} \\ &= - \frac{1}{R} \int_0^1 \frac{Q(x)}{1 - \frac{A(x)}{R}} \end{aligned}$$

For any real number t such that $t \neq 1$ and any integer $K \geq 0$, we have

$$\sum_{k=0}^K t^k = \frac{1 - t^{K+1}}{1 - t}$$

We suppose then that $|A(x)/R| \leq M < 1$ for all x in $(0, 1)$. Therefore we have

$$(7) \quad RI_\theta + \sum_{k=0}^K \left[\int_0^1 \left(\frac{A(x)}{R} \right)^k Q(x) dx \right] = - \int_0^1 \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} dx$$

However, by the mean value theorem, we have

$$(8) \quad \begin{aligned} \left| \int_0^1 \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} \right| &\leq \sup_{x \in (0,1)} \left| \frac{(A(x)/R)^{K+1} Q(x)}{1 - \frac{A(x)}{R}} \right| \\ &\leq \frac{M^{K+1}}{1 - M} \sup_{x \in (0,1)} |Q(x)| \end{aligned}$$

Now it is clear that the right-hand side of (8) tends to 0 when K goes to infinity. Therefore, passing to the limit, we obtain

$$(9) \quad I_\theta = - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \int_0^1 Q(x) (A(x))^k dx$$

The same result can be obtained by using the uniform convergence of the series $\sum_{k=0}^{\infty} t^k$ to the function $\frac{1}{1-t}$ in $(0, M)$. The commutation of sum and integral is then justified and the result follows. It can also be remarked that functions $(A(x))^k$ are pancake functions and since $Q(x)$ is a polynomial, integrands of (9) are sums of pancake functions. At this point, it is useful to recall definitions and well-known properties of the Beta and Gamma Eulerian functions.

Definition 2. The function $\Gamma(x)$ is defined for $x > 0$ by $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$. If $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$. The function $B(x, y)$ is defined for $x, y > 0$ by $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$ and we have $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Therefore the integral I_θ is the sum of beta integrals and can be evaluated by the use of Definition 2. More precisely, if we set $Q(x) = \sum_{i=0}^{\deg(Q)} q_i x^i$, we have, if

convergence is assumed:

$$\begin{aligned}
I_\theta &= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \int_0^1 \left[\sum_{i=0}^{\deg(Q)} q_i x^i \right] x^{mk} (1-x)^{nk} dx \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \int_0^1 x^{mk+i} (1-x)^{nk} dx \right] \\
(10) \quad &= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i B(mk+i+1, nk+1) \right] \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \frac{\Gamma(mk+i+1)\Gamma(nk+1)}{\Gamma((m+n)k+i+2)} \right] \\
&= - \sum_{k=0}^{\infty} \frac{1}{R^{k+1}} \left[\sum_{i=0}^{\deg(Q)} q_i \frac{(mk+i)!(nk)!}{((m+n)k+i+1)!} \right]
\end{aligned}$$

Using Stirling's formula, i.e. $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, it is then possible to find the order of convergence of the final right-hand side of (10). We have, for k large:

$$\begin{aligned}
(11) \quad & \frac{(mk+i)!(nk)!}{((m+n)k+i+1)!} \sim \\
& \left(\frac{mk+i}{e}\right)^{mk+i} \left(\frac{nk}{e}\right)^{nk} \left(\frac{e}{(m+n)k+i+1}\right)^{(m+n)k+i+1} \times \\
& \sqrt{\frac{2\pi(mk+i)(nk)}{(m+n)k+i+1}} \\
& \sim e \frac{(mk+i)^{mk+i} (nk)^{nk}}{((m+n)k+i+1)^{(m+n)k+i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
& = e \frac{(mk)^{mk+i} (nk)^{nk}}{((m+n)k)^{(m+n)k+i+1}} \frac{(1+\frac{i}{mk})^{mk+i}}{(1+\frac{i+1}{(m+n)k})^{(m+n)k+i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
& \sim \frac{e}{k} \frac{(m)^{mk+i} (n)^{nk}}{((m+n)^{(m+n)k+i+1}} \frac{e^i}{e^{i+1}} \sqrt{\frac{2\pi mnk}{m+n}} \\
& \sim \frac{1}{\sqrt{k}} \left(\frac{m^m n^n}{(m+n)^{m+n}}\right)^k \frac{m^i}{(m+n)^{i+1}} \sqrt{\frac{2\pi mn}{m+n}}
\end{aligned}$$

Therefore the final right-hand side of (10) is convergent if and only if $\frac{m^m n^n}{(m+n)^{m+n}} < |R|$. In order to justify the passage to the limit in (10), we need to check that $|A(x)| < |R|$ on the entire interval $(0, 1)$. Therefore, we are now looking for an upper bound for function $A(x)$. Since the function \log is concave on the positive

real axis, for $x \in (0, 1)$, we have, successively:

$$\begin{aligned} \frac{m \log(nx) + n \log(m - mx)}{m + n} &\leq \log\left(\frac{mn}{m + n}\right) \\ (nx)^m (m - mx)^n &\leq \left(\frac{mn}{m + n}\right)^{m+n} \\ A(x) = \frac{(nx)^m (m - mx)^n}{n^m m^n} &\leq \frac{m^m n^n}{(m + n)^{m+n}} = A\left(\frac{m}{m + n}\right) \end{aligned}$$

Therefore we obtain the same condition as in the previous paragraph. Moreover, since the bound is tight, the condition cannot be relaxed.

With a little work, it is also possible to find an upper bound independent of m and n . For $m, n \geq 1$, we have

$$\left(\frac{m}{n + m}\right)^m \left(\frac{n}{n + m}\right)^n \leq \frac{mn}{(m + n)^2} = \frac{1}{2 + \frac{m}{n} + \frac{n}{m}} \leq \frac{1}{4}$$

However, we have $R = A(e^{i\theta})$ and from preceding computations, we obtain $|R| = (2 \sin(\theta/2))^n$. Therefore if we have

$$2^{n+2} \sin^n(\theta/2) > 1$$

the convergence criterion is met and the series development of (9) is valid. Alternatively, the following sensitive criterion can also be used:

$$2^n \sin^n(\theta/2) > \frac{m^m n^n}{(n + m)^{m+n}}$$

Examples. In this section, we detail computations for some values of θ in both cases of Theorem 1 if possible. The case $\theta = \pi/2$ is the classical case already exposed in [2] and is thus not addressed.

• $\theta = \pi/3$. We have then $r = 1$, $s = 3$, $I_{\pi/3} = \frac{2\pi}{3\sqrt{3}}$ and $P_{\pi/3}(x) = x^2 - x + 1$. From what precedes, acceptable pairs are of the form $(m, 2n)$ with $m+n = 0 \pmod{3}$ and of the form (m, n) with n odd and $2m + n = 0 \pmod{3}$. The simplest case for odd n is obtained with the pair $(m, n) = (1, 1)$. We have then $A(x) = x(1 - x)$, $R = 1$ and $Q(x) = -1$. We have $2^{n+2} \sin^n(\theta/2) = 4$ and the convergence criterion is met. From (10), we have then:

$$\frac{2\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k + 1)!}$$

From (11), the convergence of the series is of order $\mathcal{O}(k^{-1/2} \cdot 4^{-k})$. This series was already obtained by Euler [8] using his transformation of the arctan summation series. For even n , the simplest case is obtained by the pair $(m, n) = (2, 2)$. We have then $A(x) = (x(1 - x))^2$, $R = 1$ and $Q(x) = x^2 - x - 1$. Since $2^{n+2} \sin^n(\theta/2) = 4$, the series development is valid and (10) gives

$$\begin{aligned} \frac{2\pi}{3\sqrt{3}} &= \sum_{k=0}^{\infty} \left[\frac{(2k)!(2k)!}{(4k + 1)!} + \frac{(2k + 1)!(2k)!}{(4k + 2)!} - \frac{(2k + 2)!(2k)!}{(4k + 3)!} \right] \\ &= \sum_{k=0}^{\infty} \frac{(2k)!^2}{(4k + 3)!} (20k^2 + 24k + 7) \end{aligned}$$

The general term of this series is then of order $\mathcal{O}(k^{-1/2} \cdot 16^{-k})$.

• $\theta = \pi/4$. We have then $r = 1$, $s = 4$, $I_{\pi/4} = \frac{3\sqrt{2}\pi}{8}$ and $P_{\pi/4}(x) = x^2 - \sqrt{2}x + 1$. From Theorem 1, acceptable factorizations exist only for even values of n and the condition is then $m + n/2 = 0[\text{ mod } 4]$. The simplest case is then $(m, n) = (3, 2)$ for which $R = 2 - \sqrt{2}$. We have $2^{n+2} \sin^n(\theta/2) \sim 2.34 > 1$, so that the convergence criterion is met. After computations the following series is obtained from (10).

$$\frac{3\sqrt{2}\pi}{8} = \sum_{k=0}^{\infty} \frac{(3k)!(2k)!}{(5k+4)!} \frac{1}{(2-\sqrt{2})^{k+1}} \times \\ [(163 - 20\sqrt{2})k^3 + (298 - 46\sqrt{2})k^2 + (177 - 34\sqrt{2})k + 34 - 8\sqrt{2}]$$

Using (11), it is found that the general term of this series is of order $\mathcal{O}(k^{-1/2} \cdot (\frac{108}{3125(2-\sqrt{2})})^k)$. The second simplest case is $(m, n) = (2, 4)$ for which we have $R = -6 + 4\sqrt{2}$. We have then $2^{n+2} \sin^n(\theta/2) \sim 1.37 > 1$ and the convergence criterion is met. We obtain

$$\frac{3\sqrt{2}\pi}{8} = \sum_{k=0}^{\infty} \frac{(2k+1)!(4k)!}{(6k+5)!} \frac{1}{(-6+4\sqrt{2})^{k+1}} \times \\ [(-2576 + 1560\sqrt{2})k^3 + (-4648 + 2800\sqrt{2})k^2 + (-2592 + 1550\sqrt{2})k \\ - 424 + 250\sqrt{2}]$$

The general term of this series is $\mathcal{O}(k^{-1/2} \cdot (\frac{16}{729(6-4\sqrt{2})})^k)$.

• $\theta = \pi/6$. In this case, we have then $r = 1$, $s = 6$, $I_{\pi/6} = \frac{5\pi}{6}$ and $P_{\pi/6}(x) = x^2 - \sqrt{3}x + 1$. Again acceptable factorizations only exist for even values of n . The two simplest cases are $(m, n) = (5, 2)$ and $(m, n) = (4, 4)$. In the first case, we have $2^{n+2} \sin^n(\theta/2) \sim 1.07$ and the convergence criterion is met. In the second case, we have $2^{n+2} \sin^n(\theta/2) \sim 0.28$. However, we also have $2^n \sin^n(\theta/2) \sim 0.07$ and $m^m n^n / (m+n)^{m+n} = 1/256 \sim 0.004$ and thus the sensitive convergence criterion is met. We obtain then the following two series.

$$\frac{5\pi}{6} = \sum_{k=0}^{\infty} \frac{(2k)!(5k)!}{(7k+6)!} \frac{1}{(2-\sqrt{3})^{k+1}} \times \\ [(19149 - 2072\sqrt{3})k^5 + (55007 - 7068\sqrt{3})k^4 + (61627 - 9464\sqrt{3})k^3 \\ + (33685 - 6240\sqrt{3})k^2 + (9008 - 2036\sqrt{3})k + 948 - 264\sqrt{3}]$$

$$\frac{5\pi}{6} = \sum_{k=0}^{\infty} \frac{(4k)!(4k+3)!}{(8k+7)!} \frac{1}{(-7+4\sqrt{3})^{k+1}} \times \\ [(-13888 + 7552\sqrt{3})k^3 + (-26544 + 14448\sqrt{3})k^2 + (-16184 + 8828\sqrt{3})k \\ - 3144 + 1722\sqrt{3}]$$

General terms of these series are respectively $\mathcal{O}(k^{-1/2} \cdot (\frac{12500}{823543(2-\sqrt{3})})^k)$ and $\mathcal{O}(k^{-1/2} \cdot (\frac{1}{256(7-4\sqrt{3})})^k)$.

• $\theta = \pi/15$. In this case, we have $r = 1$ and $s = 15$. From Theorem (1), acceptable factorizations exist for (m, n) such that n is odd and $2m + n = 0[\text{ mod } 15]$. A possible case is $(m, n) = (5, 5)$. However, we have then $2^n \sin^n(\theta/2) \sim 0.0004$ while $m^m n^n / (m+n)^{m+n} = 1/1024 \sim 0.00097$. Therefore, in this case, the

sensitive convergence criterion fails. Equation (6) still holds but the right-hand side of (7) is then diverging, so that (10) is no more justified. Therefore the existence of an acceptable factorization is not a sufficient condition to obtain convergent series. However, in this case, other choices of parameters, like $(m, n) = (7, 1)$, give proper convergent series.

Acknowledgements. The author wishes to warmly thank the anonymous referee for his rigorous reading of this note and his helpful comments on the initial manuscript. Additional thanks are in order for the example of divergent series of the last paragraph which has been suggested by the referee.

REFERENCES

- [1] D.P. Dalzell. On 22/7. *Journal of London Mathematical Society*, 19:133–134, 1944. <https://doi.org/10.1112/jlms/19.75.Part.3.133>.
- [2] N. Backhouse. Pancake functions and approximation to π . *Mathematical Gazette*, 79(485):371–374, 1995. <https://www.jstor.org/stable/3618318>.
- [3] S.K. Lucas. Integral proofs that $355/113 > \pi$. *Australian Mathematical Society Gazette*, 32(4):263–266, 2005. <http://www.austms.org.au/Publ/Gazette/2005/Sep05/Lucas.pdf>.
- [4] S.K. Lucas. Approximations to π derived from integrals with nonnegative integrands. *The American Mathematical Monthly*, 116(2):166–172, 2009. https://www.researchgate.net/publication/267998655_Integral_approximations_to_pi_with_nonnegative_integrands.
- [5] C.M. Bouey, H.M. Medina, and E. Meza. A new series for π via polynomial approximations to arctangent. *Involve*, 5(4):421–430, 2012. https://projecteuclid.org/download/pdf_1/euclid.involve/1513733534.
- [6] J.A. Scott. Another series for the inverse tangent. *Mathematical Gazette*, 95(534):518–520, 2011. <https://www.jstor.org/stable/23248526>.
- [7] A. Sofo and J. Cimadevilla Villacorta. New identities for the arctan function. *Journal of Mathematical Analysis*, 3(3):1–10, 2012. <http://www.ilirias.com/jma/repository/docs/JMA3-3-1.pdf>.
- [8] L. Euler. Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae. *Nova acta academia scientiarum petropolitanae*, 11:150–154, 1798. <http://eulerarchive.maa.org/pages/E705.html>.

LAB-STICC, INSTITUT MINES TÉLÉCOM ATLANTIQUE
 Current address: 655 Avenue du Technopôle, 29200 Plouzané, France.
 Email address: Yannick.Sauter@imt-atlantique.fr