



HAL
open science

Cohomology of linking systems with twisted coefficients by a p -solvable action

Rémi Molinier

► **To cite this version:**

Rémi Molinier. Cohomology of linking systems with twisted coefficients by a p -solvable action. Homology, Homotopy and Applications, 2017, 19 (2), pp.61-82. 10.4310/HHA.2017.v19.n2.a4 . hal-02280278

HAL Id: hal-02280278

<https://hal.science/hal-02280278>

Submitted on 20 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COHOMOLOGY OF LINKING SYSTEMS WITH TWISTED COEFFICIENTS BY A p -SOLVABLE ACTION

RÉMI MOLINIER

(*communicated by Nathalie Wahl*)

Abstract

In this paper we study the cohomology of the geometric realization of linking systems with twisted coefficients. More precisely, given a prime p and a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, we compare the cohomology of \mathcal{L} with twisted coefficients with the submodule of \mathcal{F}^c -stable elements in the cohomology of S . We start with the particular case of constrained fusion systems. Then, we study the case of p -solvable actions on the coefficients.

1. Introduction

The notion of saturated fusion system was introduced by Puig in the 90s in a context of modular representation theory. Since their introduction, topologists use them to study classifying spaces of finite groups or, more precisely, their p -completions. A p -local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p -group, \mathcal{F} a saturated fusion system over S and \mathcal{L} an associated linking system. For a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, $|\mathcal{L}|_p^\wedge$ is called its *classifying space*. The theory of p -local finite groups have been studied in details by Broto, Levi, Oliver and others (see [BLO2], [OV1], [5a1] and [5a2]). The linking system and its geometric realization, even without p -completion, play here a fundamental and central role. In fact, for a given saturated fusion systems, the existence and uniqueness of an associated linking system were shown more recently by Chermak [Ch] (using the theory of partial groups). The proof of this important conjecture highlights that linking systems and their geometric realizations form a deep link between fusion system theory and homotopy theory (we refer to Aschbacher, Kessar and Oliver [AKO] for more details about fusion systems in general).

An old and well-known result due to Cartan and Eilenberg (see [CE, Theorem XII.10.1]) expresses the cohomology of a finite group G in a $\mathbb{Z}_{(p)}[G]$ -module as the submodule of “stable” elements in the cohomology of a Sylow p -subgroup of G . This submodule of stable elements corresponds to the inverse limit over the “fusion” of the group cohomology functor. One important result in the theory of p -local finite groups is an analog of this theorem for p -local finite groups which tells us that the cohomology of the geometric realization of a linking system can be computed by \mathcal{F} -stable elements. More precisely, there is a natural inclusion of BS into $|\mathcal{L}|$ and it

Received Month Day, Year, revised Month Day, Year; published on Month Day, Year.

2010 Mathematics Subject Classification: 55R40, 55N25, 55R35, 20J06, 20D20, 20J15.

Key words and phrases: fusion system, p -local finite group, Cohomology with twisted coefficients, group cohomology.

Article available at <http://dx.doi.org/10.4310/HHA.v.n.a>

Copyright © , International Press. Permission to copy for private use granted.

induces the following isomorphism. Here, \mathcal{F}^c is the full subcategory of \mathcal{F} consisting of \mathcal{F} -centric subgroups of S and, for A a finite $\mathbb{Z}_{(p)}$ -module, $H^*(\mathcal{F}^c, A) \subseteq H^*(S, A)$ is the submodule of \mathcal{F} -stable elements.

Theorem 1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and A be a finite $\mathbb{Z}_{(p)}$ -module. The inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|_p^\wedge, A) \cong H^*(|\mathcal{L}|, A) \xrightarrow{\cong} H^*(\mathcal{F}^c, A).$$

Proof. The case $A = \mathbb{F}_p$ is [BLO2, Theorem B] and the general case is proven in [5a2, Lemma 6.12]. \square

One question asked by Oliver in his book with Aschbacher and Kessar [AKO] is the understanding of the cohomology of $|\mathcal{L}|$ with twisted coefficients. Indeed, this cohomology appears for example in the study of extensions of p -local finite groups or, more directly, can give more information about the link between the fusion system and the spaces $|\mathcal{L}|$ or $|\mathcal{L}|_p^\wedge$. Recall that, if a space X has a universal covering space \tilde{X} , the cohomology of X with twisted coefficients in a $\mathbb{Z}[\pi_1(X)]$ -module M is the cohomology of the chain complex

$$C^*(X; M) = \text{Hom}_{\mathbb{Z}[\pi_1(X)]}(S_*(\tilde{X}), M),$$

where $S_*(\tilde{X})$ is the usual singular chain complex of \tilde{X} .

Levi and Ragnarsson [LR] already give some tools along these lines. In an other paper [Mo1], the author extends Theorem 1.1 to the case of nilpotent actions on the coefficients. The main ingredient is to construct, as in the trivial coefficient case, an idempotent of $H^*(S, M)$ with image $H^*(\mathcal{F}^c, M)$.

In this paper, we also want to extend Theorem 1.1 to twisted coefficients but when the action factors through a p -solvable group. The methods used here are completely different from the ones used in [Mo1] and also more direct. We first have a look at constrained fusion systems. In that case we are able to prove that, with any coefficient module, the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

Theorem A (see Theorem 3.5). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. If \mathcal{F} is constrained and M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module, then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Next we focus on p -solvable actions. The main ingredients here are p -local finite subgroups of index a power of p or prime to p and their homotopy properties. We start by looking at p -local subgroups of index prime to p (see Definition 2.6(b)).

Theorem B (see Theorem 4.3). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and denote by $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module and if the inclusion of BS in $|O^{p'}(\mathcal{L})|$ induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

This Theorem allows us to prove that if the action on the coefficients factor through a p' -group or, even better, a p -nilpotent group, then the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

It is much more complicated to work with p -local finite groups of index a power of p , especially on the level of stable elements. Indeed, for $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module, it is difficult to compare $H^*(\mathcal{F}^c, M)$ and $H^*(\mathcal{F}_0^c, M)$. The difficulty mostly comes from the fact that we are working on different p -groups: S and S_0 . But when we work with a p -local finite group realizable by a finite group G , and if G acts "consistently" on the coefficients it is possible to get some positive results (see Section 5).

Theorem C (see Corollary 5.5). *Let G be a finite group, S a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module and assume that G acts consistently on M . If both actions factor through a given p -solvable Γ and all the M -essential subgroups (see Definition 5.3) of S are p -centric, then we have natural isomorphisms,*

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

All of these results lead us to the following conjecture.

Conjecture A (see Conjecture 5.6). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p -solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

We finish this paper with an example for Conjecture 5.6 which does not follow from the other results.

Organization. In Section 2 we give some background on p -local finite groups and stable elements. Section 3 is dedicated to the case of constrained fusion systems, Section 4 to coprime actions and Section 5 to p -solvable actions for a realizable p -local finite group. Finally, we give in Section 6 an example for Conjecture 5.6.

Acknowledgments

I would like to thanks Bob Oliver, my PhD adviser, for his help and support all along this work. I am also grateful to Jesper Grodal for his hospitality at the Center for Symmetry and Deformation and many fruitful conversations. Finally, I would like to thanks the referee for his very careful reading of the paper and all his useful and accurate comments.

2. Background

We give here a very short introduction to p -local finite groups. The notion of fusion system was first introduced by Puig for modular representation theory purpose. Later, Broto, Levi and Oliver developed the notion of linking systems and p -local finite groups to study p -completed classifying spaces of finite groups and spaces which have similar homotopy properties. We refer the reader interested in more details to Aschbacher, Kessar and Oliver [AKO].

2.1. Fusion systems and linking systems

A fusion system over a p -group S is a way to abstract the action of a finite group G with $S \in \text{Syl}_p(G)$ on the subgroups of S by conjugation. For G a finite group and $g \in G$, we will denote by c_g the homomorphism $x \in G \mapsto gxg^{-1} \in G$ and for H, K two subgroups of G , $\text{Hom}_G(H, K)$ will denote the set of all group homomorphism c_g for $g \in G$ such that $c_g(H) \leq K$.

Definition 2.1. Let S be a finite p -group. A *fusion system* over S is a small category \mathcal{F} , where $\text{Ob}(\mathcal{F})$ is the set of all subgroups of S and which satisfies the following two properties for all $P, Q \leq S$:

- (a) $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$;
- (b) each $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ is the composite of an \mathcal{F} -isomorphism followed by an inclusion.

A fusion system is *saturated* if it satisfy two more technical axioms called the saturation axioms (we refer the reader to [AKO, Definition I.2.1] for a proper definition).

The composition in a fusion system is given by composition of homomorphisms. We usually write $\text{Hom}_{\mathcal{F}}(P, Q) = \text{Mor}_{\mathcal{F}}(P, Q)$ to emphasize that the morphisms in \mathcal{F} are homomorphisms. For $P, Q \leq S$, we say that P is \mathcal{F} -conjugate to Q if there is an \mathcal{F} -isomorphism between P and Q . We denote by $P^{\mathcal{F}}$ the set of all subgroups of S which are \mathcal{F} conjugate to P .

The typical example of a saturated fusion system is the fusion system $\mathcal{F}_S(G)$ of a finite group G over $S \in \text{Syl}_p(G)$.

For the purpose of this paper, we need to distinguish some collections of subgroups of S .

Definition 2.2. Let \mathcal{F} be a saturated fusion system over a finite p -group S .

- (a) A subgroup $P \leq S$ is \mathcal{F} -centric if for every $Q \in P^{\mathcal{F}}$, $C_S(Q) = Z(Q)$.
- (b) A subgroup $P \leq S$ is \mathcal{F} -radical if $O_p(\text{Aut}_{\mathcal{F}}(P)) = \text{Inn}(P)$.
- (c) A subgroup $P \leq S$ is \mathcal{F} -quasicentric if for each $Q \leq PC_S(P)$ containing P , and each $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_P = \text{Id}$, α has a p -power order.

We let $\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}$ denote the full subcategories of \mathcal{F} with objects the \mathcal{F} -centric and \mathcal{F} -radical subgroups, the \mathcal{F} -centric subgroups and the \mathcal{F} -quasicentric subgroups, respectively.

If $\mathcal{F} = \mathcal{F}_S(G)$, a subgroup $P \leq S$ is

- (a) \mathcal{F} -centric if and only if it is p -centric (i.e. $Z(P) \in \text{Syl}_p(C_G(P))$),
- (b) \mathcal{F} -radical if $P/Z(P) = O_p(N_G(P)/C_G(P))$.
- (c) \mathcal{F} -quasicentric if and only if $O^p(C_G(P))$ has order prime to p .

The notion of linking system has been introduced by Broto, Levi and Oliver [BLO2] and generalized by Broto, Castellana, Grodal and Oliver in [5a1]. We refer the reader to these papers, or [AKO, Part III], for a proper definition. We recall here some basic facts about linking systems which will be needed here.

For G a finite group, $S \in \text{Syl}_p(G)$ and \mathcal{H} a collection of subgroups of S , the *transporter category* of G over S with set of objects \mathcal{H} is the category $\mathcal{T}_H^{\mathcal{H}}(G)$ with objects

\mathcal{H} and for $P, Q \in \mathcal{H}$, $\text{Mor}_{\mathcal{L}}(P, Q) = T_G(P, Q) = \{g \in G \mid P^g \leq Q\}$. For \mathcal{F} a saturated fusion system over a p -group S , a *linking system* associated to \mathcal{F} is a certain finite category with objects a collection \mathcal{H} of subgroups of S together with two functors:

$$\mathcal{T}_S^{\mathcal{H}}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}.$$

δ is the identity on objects and injective on morphisms and π is injective on objects and surjective on morphisms. The collection \mathcal{H} has to be stable by overgroups and \mathcal{F} -conjugation and the following proposition tell you which collection you can have.

Proposition 2.3. *Let \mathcal{F} be a saturated fusion system over a p -group S . Let \mathcal{L} be a linking system associated to \mathcal{F} .*

- (a) *$\text{Ob}(\mathcal{F}^{cr}) \subseteq \text{Ob}(\mathcal{L}) \subseteq \text{Ob}(\mathcal{F}^q)$, and there exists a linking system \mathcal{L}^q associated to \mathcal{F} such that $\text{Ob}(\mathcal{L}^q) = \text{Ob}(\mathcal{F}^q)$, and \mathcal{L} is a full subcategory of \mathcal{L}^q .*
- (b) *For every subset $\text{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \text{Ob}(\mathcal{F}^q)$ stable by \mathcal{F} -conjugacy and overgroups, the full subcategory $\mathcal{L}^{\mathcal{H}}$ of \mathcal{L}^q with set of objects \mathcal{H} is also a linking system associated to \mathcal{F} .*

Proof. The first point of (a) can be found for example in [O4, Proposition 4(g)]. For the second statement of (a), you can find a proof in [AKO, Proposition III.4.8]. Finally, (b) is a consequence of the definition of linking systems. \square

If $\mathcal{H} = \text{Ob}(\mathcal{F}^q)$, \mathcal{L} is called a *quasicentric linking system* and if $\mathcal{H} = \text{Ob}(\mathcal{F}^c)$, \mathcal{L} is called a *centric linking system*.

Definition 2.4. A p -local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where \mathcal{F} is a saturated fusion system over S and \mathcal{L} is an associated linking system. If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is another p -local finite group, we will say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p -local subgroup* of $(S, \mathcal{F}, \mathcal{L})$ if $S_0 \leq S$ and $\mathcal{F}_0 \subseteq \mathcal{F}$ is a subsystem of \mathcal{F} . Notice that we do not require that \mathcal{L}_0 is a subcategory of \mathcal{L} .

The typical example you should have in mind is the following. For G a finite group and $S \in \text{Syl}_p(G)$ let $\mathcal{L}_S^q(G)$ be the category with objects the $\mathcal{F}_S(G)$ -quasicentric subgroups of G and, for $P, Q \in \text{Ob}(\mathcal{L})$,

$$\text{Mor}_{\mathcal{L}}(P, Q) = T_G(P, Q)/O^p(C_G(P)).$$

Then $(S, \mathcal{F}_S(G), \mathcal{L}_S^q(G))$ defines a p -local finite group where $\mathcal{L}_S^q(G)$ is a quasicentric linking system. We also denote by $\mathcal{L}_S^c(G)$ the full subcategory of $\mathcal{L}_S^q(G)$ with objects the p -centric subgroups of S and it is a centric linking system.

We finish with some basic homotopy properties about linking systems which will be needed in this paper. We refer the reader interested in more details to [AKO, Part III]. For $(S, \mathcal{F}, \mathcal{L})$ a p -local finite group, we write $|\mathcal{L}|$ for the geometric realization of \mathcal{L} and $\pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|)$ for its fundamental group. The following Theorem will allow us to change the set of objects of \mathcal{L} without changing the homotopy type of $|\mathcal{L}|$.

Theorem 2.5 ([5a1, Theorem 3.5]). *Let \mathcal{F} be a saturated fusion system over a p -group S . Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two linking systems associated to \mathcal{F} with a different set of objects. Then the inclusion induces a homotopy equivalence of space $|\mathcal{L}_0| \simeq |\mathcal{L}|$.*

2.2. p -local finite subgroups of index a power of p or prime to p

The notions p -local subgroups of index a power of p or prime to p have been introduced and studied by Broto, Castellana, Grodal, Levi and Oliver [5a2]. Here we just give the definitions what we need about these p -local subgroups and we refer the reader to [5a2] for more details.

Definition 2.6. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$. Set $\mathfrak{hnp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in O^p(\text{Aut}_{\mathcal{F}}(P)) \rangle \trianglelefteq S$.

- (a) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a p -local subgroup of index a power of p if $S_0 \geq \mathfrak{hnp}(\mathcal{F})$ and, for every $P \leq S_0$, $O^p(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$.
- (b) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a p -local subgroup of index prime to p if $S_0 = S$ and, for every $P \leq S$, $O^{p'}(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$.

Notice that $\mathfrak{hnp}(\mathcal{F})$ is denoted $O_{\mathcal{F}}^p(S)$ in [5a2, Definition 2.1]. These particular p -local subgroups satisfy the following properties.

Proposition 2.7 ([5a2, Proposition 3.8]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$.*

- (a) *If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index a power of p , then $P \leq S_0$ is \mathcal{F}_0 -quasicentric if, and only if, P is \mathcal{F} -quasicentric.*
- (b) *If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index prime to p , then $P \leq S$ is \mathcal{F}_0 -centric if, and only if, P is \mathcal{F} -centric.*

For an infinite group G , we denote by $O^{p'}(G)$ the intersection of all normal subgroups in G of finite index prime to p . For \mathcal{F} a fusion system over a p -group S , let $O_*^{p'}(\mathcal{F})$ be the fusion system generated by $O^{p'}(\text{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$ and define

$$\text{Out}_{\mathcal{F}}^0(S) = \langle \alpha \in \text{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \text{Hom}_{O_*^{p'}(\mathcal{F})}(P, S), \text{ for some } P \leq S \rangle.$$

Since $\text{Aut}_{\mathcal{F}}(S)$ normalizes $O_*^{p'}(\mathcal{F})$, $\text{Out}_{\mathcal{F}}^0(S) \trianglelefteq \text{Out}_{\mathcal{F}}(S)$.

Proposition 2.8. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group.*

- (a) $\mathcal{F} = \langle \text{Aut}_{\mathcal{F}}(S), O_*^{p'}(\mathcal{F}) \rangle$.
- (b) π and the inclusion of $B\text{Aut}_{\mathcal{F}}(S)$ in $|\mathcal{F}^c|$ induce isomorphisms,

$$\theta: \pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \xrightarrow{\cong} \pi_1(|\mathcal{F}^c|) \xrightarrow{\cong} \text{Out}_{\mathcal{F}}(S)/\text{Out}_{\mathcal{F}}^0(S).$$

Proof. The point (a) is proved in [5a2, Lemma 3.4]. For (b), the second isomorphism is given in [5a2, Proposition 5.2] and the first one in [5a2, Theorem 5.5] and the comment which follows. \square

According to Proposition 2.7, when dealing with p -local subgroups of index prime to p , we will work with centric linking systems.

Theorem 2.9 ([5a2, Theorem 5.5]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. For each subgroup $H \leq \text{Out}_{\mathcal{F}}(S)$ containing $\text{Out}_{\mathcal{F}}^0(S)$, there is a unique p -local finite subgroup $(S, \mathcal{F}_H, \mathcal{L}_H)$ of index prime to p such that $\text{Out}_{\mathcal{F}_H}(S) = H$ and $\mathcal{L}_H = \pi^{-1}(\mathcal{F}_H^c)$.*

Moreover, $|\mathcal{L}_H|$ is homotopy equivalent, via its inclusion in $|\mathcal{L}|$, to the covering space of $|\mathcal{L}|$ with fundamental group $\tilde{H} \geqslant OP'(\pi_{\mathcal{L}})$ such that $\theta(\tilde{H}/OP'(\pi_{\mathcal{L}})) = H/\text{Out}_{\mathcal{F}}^0(S)$ (where θ is the isomorphism given in Proposition 2.8(b)).

Thus, for a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, with \mathcal{L} a centric linking system, we can define the *minimal p -local subgroup of index prime to p* , $(S, OP'(\mathcal{F}), OP'(\mathcal{L}))$ corresponding to $(S, \mathcal{F}_H, \mathcal{L}_H)$ with $H = \text{Out}_{\mathcal{F}}^0(S)$ in Theorem 2.9.

2.3. Cohomology and stable elements

The first result about stable elements is due to Cartan and Eilenberg ([CE, Chap XII, Theorem 10.1]). It also served as a guideline in the establishment of Theorem 1.1 by Broto, Levi and Oliver. Here we recall the definition of \mathcal{F}^c -stable elements in a context of twisted coefficients. We refer the reader to [Mo1] for more details. As in [Mo1], we will denote by $\omega: \mathcal{L} \rightarrow \pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|, S)$ the functor which maps each object to the unique object in the target and sends each morphism $\varphi \in \text{Mor}_{\mathcal{L}}(P, Q)$ to the class of the loop $\iota_Q \cdot \varphi \cdot \overline{\iota_P}$ where $\iota_P = \delta(1) \in \text{Mor}_{\mathcal{L}}(P, S)$, $\iota_Q = \delta(1) \in \text{Mor}_{\mathcal{L}}(Q, S)$ and $\overline{\iota_P}$ is the edge ι_P followed in the opposite direction.

Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Recall first that $\delta: \mathcal{T}_S^{\text{Ob}(\mathcal{L})}(S) \rightarrow \mathcal{L}$ induces an inclusion $\delta_S: BS \rightarrow |\mathcal{L}|$. In particular, it induces a natural map $S \rightarrow \pi_{\mathcal{L}}$ and thus, for every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module M , we have a natural action of S , or any subgroup of S , on M . Now, let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, the group cohomology bifunctor $H^*(-, -)$ induces a functor

$$H^*(-, M): \mathcal{F}^c \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$$

(a priori, $H^*(g, M)$ is defined for $g \in \text{Mor}(\mathcal{L})$ but [Mo1, Proposition 2.2] proves that $H^*(-, M)$ is well defined on \mathcal{F}^c).

Definition 2.10. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. An element $x \in H^*(S, M)$ is called *\mathcal{F} -centric stable*, or *\mathcal{F}^c -stable*, if for all $P \in \text{Ob}(\mathcal{F}^c)$ and all $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$,

$$\varphi^*(x) = \text{Res}_P^S(x).$$

We denote by $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ the submodule of all \mathcal{F}^c -stable elements.

Notice that

$$H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{F}^c} H^*(-, M) = \varprojlim_{\mathcal{L}} H^*(-, M)$$

where the last equality holds if \mathcal{L} is a centric linking system.

3. Constrained fusion systems

Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Here, we assume that \mathcal{F} is a constrained fusion system.

Definition 3.1. Let \mathcal{F} be a fusion system over a p -group S . A subgroup $Q \leqslant S$ is *normal in \mathcal{F}* if

- (i) $Q \trianglelefteq S$, and
- (ii) for all $P, R \leqslant S$ and every $\varphi \in \text{Hom}_{\mathcal{F}}(P, R)$, φ extends to a morphism $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\overline{\varphi}(Q) = Q$.

We write $O_p(\mathcal{F})$ for the maximal subgroup of S which is normal in \mathcal{F} . We say that \mathcal{F} is *constrained* if $O_p(\mathcal{F})$ is \mathcal{F} -centric.

An important and classical result about constrained fusion systems is the following.

Proposition 3.2 ([5a1, Proposition 4.3]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. If \mathcal{F} is constrained, there exists a finite group G such that*

- (a) S is a Sylow p -subgroup of G ,
- (b) $C_G(O_p(G)) \leq O_p(G)$,
- (c) $\mathcal{F}_S(G) = \mathcal{F}$.

Moreover, $G \cong \text{Aut}_{\mathcal{L}}(O_p(\mathcal{F}))$ and $\mathcal{L} \cong \mathcal{L}_S^c(G)$.

This group G is called a *model* of \mathcal{F} and it is unique in a precise way (see [AKO, Theorem III.5.10]). This model can also be recovered from the homotopy type of the geometric realization of a linking system associated to \mathcal{F} .

Lemma 3.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. If \mathcal{F} is constrained, then $|\mathcal{L}|$ is a classifying space of a model G of \mathcal{F} .*

Proof. By Proposition 3.2, we can assume that $\mathcal{L} = \mathcal{L}_S^c(G)$. Set

$$\mathcal{H} = \{P \in \text{Ob}(\mathcal{L}) \mid P \geq O_p(G)\}$$

and let $\mathcal{L}^{\mathcal{H}}$ be the full subcategory of \mathcal{L} with set of objects \mathcal{H} . By [5a1, Proposition 1.6], \mathcal{H} contains all \mathcal{F} -centric and \mathcal{F} -radical subgroups. Thus, by Proposition 2.3, $\mathcal{L}^{\mathcal{H}}$ is a linking system associated to \mathcal{F} and, by Theorem 2.5, $|\mathcal{L}^{\mathcal{H}}| \cong |\mathcal{L}|$.

It remains to prove that that $|\mathcal{L}^{\mathcal{H}}| \cong BG$. For that purpose, consider, the following functor.

$$F: \begin{array}{ccc} \mathcal{L}^{\mathcal{H}} & \longrightarrow & \mathcal{L}^{\{O_p(G)\}} \\ P \in \mathcal{L}^{\mathcal{H}} & \longmapsto & O_p(G) \\ g \in T_G(P, Q) & \longmapsto & g \in N_G(O_p(G)) = G \end{array}$$

It gives us a retraction by deformation of $|\mathcal{L}^{\mathcal{H}}|$ on the geometric realization of the full subcategory of \mathcal{L} with unique object $O_p(G) \leq S$. As $\text{Aut}_{\mathcal{L}}(O_p(G)) = N_G(O_p(G)) = G$, this last category is $\mathcal{B}(G)$. in particular, its geometric realization is a classifying space of G . \square

Proposition 3.4. *Let G be a finite group and S a Sylow p -subgroup of G . If we have $C_G(O_p(G)) \leq O_p(G)$, then, for every $\mathbb{Z}_{(p)}[G]$ -module M , the inclusion of S in G induces a natural isomorphism*

$$H^*(G, M) \cong H^*(\mathcal{F}_S^c(G), M).$$

Proof. Let $(S, \mathcal{F}, \mathcal{L}) = (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$. By assumption, $\mathcal{F}_S(G)$ is constrained and

G is a model of $\mathcal{F}_S(G)$. From Cartan-Eilenberg Theorem, we know that

$$\text{Res}_S^G: H^*(G, M) \longrightarrow H^*(S, M)$$

is injective and that $\text{Im}(\text{Res}_S^G) = \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. Moreover,

$$H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{F}^c} H^*(-, M) = \varprojlim_{\mathcal{L}} H^*(-, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M) \geq \varprojlim_{\mathcal{T}_S(G)} H^*(-, M).$$

Thus, it remains to prove that $\varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M) \leq \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$.

Let then $x \in H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. For $P \leq S$ and $g \in N_G(P, S)$ we have, in $\mathcal{T}_S(G)$, the following commutative diagram

$$\begin{array}{ccc} PO_p(G) & \xrightarrow{g} & gPg^{-1}O_p(G) \\ e \uparrow & & \uparrow e \\ P & \xrightarrow{g} & gPg^{-1} \end{array}$$

where e is the trivial element of G . Hence, as the top part of the diagram is in $\mathcal{T}_S^c(G)$ and $x \in \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$,

$$\begin{aligned} c_g^* \circ \text{Res}_{gPg^{-1}}^S(x) &= \text{Res}_P^{PO_p(G)} \circ c_g^* \circ \text{Res}_{gPg^{-1}O_p(G)}^S(x) \\ &= \text{Res}_P^{PO_p(G)} \circ \text{Res}_{PO_p(G)}^S(x) \\ &= \text{Res}_P^S(x). \end{aligned}$$

Thus $x \in \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$ and this complete the proof. \square

Corollary 3.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. If \mathcal{F} is constrained and M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then δ_S induces a natural isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

4. Actions factoring through a p' -group

In this section, for each p -local finite group $(S, \mathcal{F}, \mathcal{L})$ we will assume that \mathcal{L} is a **centric linking system**.

Lemma 4.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then the inclusion $O^{p'}(\mathcal{L}) \subseteq \mathcal{L}$ induces the following isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Proof. By Theorem 2.9, $|O^{p'}(\mathcal{L})|$ is, up to homotopy, a covering space of $|\mathcal{L}|$ with

fundamental group $O^{p'}(\pi_{\mathcal{L}}) \trianglelefteq \pi_{\mathcal{L}}$. It gives us a fibration sequence

$$|O^{p'}(\mathcal{L})| \rightarrow |\mathcal{L}| \rightarrow B(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})).$$

Consider then the Serre spectral sequence associated

$$H^{s+t}(|\mathcal{L}|, M) \Leftarrow H^s(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}), H^t(|O^{p'}(\mathcal{L})|, M)).$$

M is a $\mathbb{Z}_{(p)}$ -module, thus $H^q(|O^{p'}(\mathcal{L})|, M)$ is also a $\mathbb{Z}_{(p)}$ -module. As $\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})$ is a p' -group, the E_2 -page is concentrated in the first column with terms

$$H^t(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Thus the spectral sequence collapses on the E_2 -page and the Lemma follows. \square

Lemma 4.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then*

$$H^*(\mathcal{F}^c, M) = H^*(O^{p'}(\mathcal{F})^c, M)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)}.$$

Proof. Notice first that, by Proposition 2.7, $\text{Ob}(O^{p'}(\mathcal{F})^c) = \text{Ob}(\mathcal{F}^c)$. Hence, we are working with the same underlying set of objects. Thus, by definition, $H^*(\mathcal{F}^c, M) \subseteq H^*(O^{p'}(\mathcal{F})^c, M)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)}$. On the other hand, by Proposition 2.8, we have $\mathcal{F} = \langle O^{p'}(\mathcal{F}), \text{Aut}_{\mathcal{F}}(S) \rangle$ which gives the converse inclusion. \square

Theorem 4.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and if the inclusion δ_S induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. Recall that, by 2.9, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$. Then we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{B}(S) & \xrightarrow{\delta_S} & \mathcal{L} & \xrightarrow{\omega} & \mathcal{B}\pi_{\mathcal{L}} & \longrightarrow & \mathcal{B}(\text{Aut}(M)) \\ & \searrow \delta_S & \uparrow & & \uparrow & & \nearrow \\ & & O^{p'}(\mathcal{L}) & \xrightarrow{\omega} & \mathcal{B}(O^{p'}(\pi_{\mathcal{L}})) & & \end{array}$$

Moreover, by Proposition 2.8 and Theorem 2.9, the projection $\pi: \mathcal{L} \longrightarrow \mathcal{F}$ induces an isomorphism

$$\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \cong \pi_1(|\mathcal{F}^c|) \cong \text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S).$$

Then, by the two previous Lemmas, we obtain

$$\begin{aligned} H^*(|\mathcal{L}|, M) &\cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})} \\ &\cong \left(\varprojlim_{O^{p'}(\mathcal{F})^c} H^*(-, M) \right)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)} \\ &\cong H^*(\mathcal{F}^c, M). \end{aligned}$$

For the second isomorphism, we have to be careful with respect to the action of $\pi_{\mathcal{L}}$ on the left side of the isomorphism and $\text{Aut}_{\mathcal{F}}(S)$ on the right side. In fact here, by Definition 2.10 of \mathcal{F}^c -stable elements, we can see it on the chain level. The map $\delta_S^*: H^*(|O^{p'}(\mathcal{L})|, M) \longrightarrow H^*(S, M)$, induced by $\delta_S: BS \longrightarrow |O^{p'}(\mathcal{L})|$, gives on the chain level,

$$\begin{aligned} \text{Hom}_{\mathbb{Z}_{(p)}[S]} \left(C_* \left(\widetilde{|O^{p'}(\mathcal{L})|} \right), M \right) &\longrightarrow \text{Hom}_{\mathbb{Z}_{(p)}[\pi_{O^{p'}(\mathcal{L})}]} (C_* (|\mathcal{E}(S)|), M) \\ f &\longmapsto f|_{C_* (|\mathcal{E}(S)|)} \end{aligned}$$

where $\mathcal{E}(S)$ is defined as the category with set of object S and for each $(s, s') \in S$, $\text{Mor}_{\mathcal{E}(S)}(s, s') = \{\varphi_{s, s'}\}$ (in particular $|\mathcal{E}(S)|$ is a universal covering space of BS). Then, for $\varphi \in \text{Aut}_S(\mathcal{F})$, if we choose a lift $\tilde{\varphi} \in \text{Aut}_{\mathcal{L}}(S)$, φ acts on the left side by

$$f \longmapsto \omega(\tilde{\varphi}^{-1})f\omega(\tilde{\varphi}),$$

and on the right side by,

$$f \longmapsto \omega(\tilde{\varphi})^{-1}f \circ \varphi^*.$$

Finally, the action of φ on $\mathcal{E}(S)$ corresponds to the action of $\omega(\tilde{\varphi})$ on $|\mathcal{E}(S)|$ (indeed, a lift of $\omega(\tilde{\varphi})$ in $\widetilde{|O^{p'}(\mathcal{L})|}$ joins every vertex $s \in S$ of $|\mathcal{E}(S)|$ to the vertex $\varphi(s)$ and similarly for higher simplices). Hence, the two actions coincide. \square

Corollary 4.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module. If the action of $\pi_{\mathcal{L}}$ on M factors through a p' -group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. By Theorem 4.3, it is enough to prove that δ_S induces an isomorphism

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M).$$

But, as the action on M factor through a p' -group, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$ acts trivially on M and Theorem 1.1 gives the wanted isomorphism. \square

We already know, from a previous article ([**Mo1**, Theorem 4.3]) that, if M is a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and the action of $\pi_{\mathcal{L}}$ on M factor through a p -group, then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$$

(it is a direct corollary of [**Mo1**, Theorem 4.3] because, any action of a p -group on an abelian p -group is nilpotent). Hence, with the same arguments, we get another corollary of Theorem 4.3.

Corollary 4.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M be a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module. If the action of $\pi_{\mathcal{L}}$ on M factors through an extension of a normal p -group by a p' -group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

5. Realizable fusion systems and actions factoring through a p -solvable group

Consider here a finite group G , S a Sylow p -subgroup of G and let $(S, \mathcal{F}, \mathcal{L})$ be the associated p -local finite group with $\mathcal{L} = \mathcal{L}_S^c(G)$. Set $\mathcal{T} = \mathcal{T}_S^c(G)$ be the centric transporter category of G , $\mathcal{L}^q = \mathcal{L}_S^q(G)$ be the quasicentric linking system associated to G and $\mathcal{T}^q = \mathcal{T}_S^q(G)$ be the associated quasicentric transporter category. We also write $\pi_{\mathcal{T}} = \pi_1(|\mathcal{T}|)$.

We have a functor

$$\rho: \mathcal{T}_S(G) \longrightarrow \mathcal{B}(G)$$

which sends each object in the source to the unique object o_G in the target and sends, for every $P, Q \leq S$, $g \in T_G(P, Q)$ to $g \in G = \text{Mor}_{\mathcal{B}(G)}(o_G)$. As $|\mathcal{B}(G)| = BG$, this induces an homomorphism

$$\rho_*: \pi_{\mathcal{T}} \longrightarrow G.$$

Here for M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, with action $\varphi: \pi_{\mathcal{L}} \rightarrow \text{Aut}(M)$ we will suppose that we have the following commutative diagram for some homomorphism $\bar{\varphi}: G \rightarrow \text{Aut}(M)$.

$$\begin{array}{ccc} & \pi_{\mathcal{L}} & \\ \delta_* \nearrow & & \searrow \varphi \\ \pi_{\mathcal{T}} & & \text{Aut}(M) \\ \rho_* \searrow & & \nearrow \bar{\varphi} \\ & G & \end{array}$$

Then, we can compare the cohomology of $|\mathcal{L}|$ and the cohomology of G when the action factors through a p -solvable group. The main ingredients that we will use are p -local subgroups of index a power of p or prime to p .

The following lemma allows us to compare $H^*(|\mathcal{L}|, M)$ and $H^*(|\mathcal{T}|, M)$.

Lemma 5.1. *Let G be a finite group and $(S, \mathcal{F}, \mathcal{L})$ be an associated p -local finite group. Let $\mathcal{T} = \mathcal{T}_S^{\text{Ob}(\mathcal{L})}(G) \subseteq \mathcal{T}^q$ be the transporter category associated to G with set of objects $\text{Ob}(\mathcal{L})$. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then the canonical functor $\delta: \mathcal{T} \rightarrow \mathcal{L}$ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(|\mathcal{L}|, M)$.*

Proof. This is a consequence of [BLO1, Lemma 1.3] with $\mathcal{C} = \mathcal{T}$, $\mathcal{C}' = \mathcal{L}$ and the functor $T: \mathcal{L}^{\text{op}} \rightarrow \mathbb{Z}_{(p)}\text{-Mod}$ which sends each object to M , and each morphism to its

action on M . Then δ induces a natural isomorphism $\varprojlim_{\mathcal{T}}^*(M) \cong \varprojlim_{\mathcal{L}}^*(M)$. Then

$$H^*(|\mathcal{T}|, M) = \varprojlim_{\mathcal{T}}^*(M) \cong \varprojlim_{\mathcal{L}}^*(M) = H^*(|\mathcal{L}|, M)$$

Where the first and last equality is just an interpretation in terms of functor cohomology and can be found in [LR, Proposition 3.9]. \square

Theorem 5.2. *Let G be a finite group, S a Sylow p -subgroup of G , $\mathcal{L} = \mathcal{L}_S^c(G)$ and $\mathcal{T} = \mathcal{T}_S^c(G)$. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and assume that we have the following commutative diagram.*

$$\begin{array}{ccc} & \pi_{\mathcal{L}} & \\ \delta_* \nearrow & & \searrow \varphi \\ \pi_{\mathcal{T}} & & \text{Aut}(M) \\ \rho_* \searrow & & \nearrow \bar{\varphi} \\ & G & \end{array}$$

If ρ_* is surjective and $\Gamma = \text{Im}(\varphi) = \text{Im}(\bar{\varphi})$ is p -solvable, then δ and ρ induce natural isomorphisms

$$H^*(|\mathcal{L}|, M) \cong H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

Proof. By Lemma 5.1, we just have to show that ρ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(G, M)$. We prove this by induction on the minimal number n of extensions by p -groups or p' -groups we need to obtain Γ .

If $n = 0$, $\Gamma = 1$ and the action of $\pi_{\mathcal{T}}$ on M is trivial, then it follows from [OV1, Proposition 4.5]. Assume that, if Γ is obtained by n extensions, the result is true and suppose that Γ is obtained with $n + 1$ extensions. Consider then the last one

$$0 \rightarrow \Gamma_n \rightarrow \Gamma \rightarrow Q \rightarrow 0.$$

Denote $H = \bar{\varphi}_*^{-1}(\Gamma_n)$. Thus $(T, \mathcal{F}_H, \mathcal{L}_H) = (S \cap H, \mathcal{F}_{S \cap H}(H), \mathcal{L}_{S \cap H}^c(H))$ is a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p or prime to p .

If Q is a p' -group. In that case, $(T, \mathcal{F}_H, \mathcal{L}_H)$ is a p -local finite subgroup of index prime to p (defined in 2.6). Then $\text{Ob}(\mathcal{F}^c) = \text{Ob}(\mathcal{F}_H^c)$, $\mathcal{T}_H = \mathcal{T}_{S \cap H}^c(H) \subset \mathcal{T}$ and, by [OV1, Proposition 4.1(d)], this inclusion of categories induces, up to homotopy, a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. We then have the following commutative diagram with exact rows (here, \twoheadrightarrow means onto)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{\mathcal{T}_H} & \longrightarrow & \pi_{\mathcal{T}} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\ 0 & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q \longrightarrow 0 \end{array}$$

and the following fibration sequences

$$|\mathcal{T}_H| \twoheadrightarrow |\mathcal{T}| \twoheadrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

If Q is a p -group. In that case, we have to be more careful on the collection of subgroups of S we are working with. As in the case when Q is a p' -group we want to apply [OV1, Proposition 4.1(d)]. This forces us to use the following collection. Let

$$\mathcal{H} = \{P \in \text{Ob}(\mathcal{F}^q) \mid P \cap T \in \text{Ob}(\mathcal{F}_H^q)\} .$$

Since $H \trianglelefteq G$, no element of $T = S \cap H$ is G -conjugate to any element of $S \setminus T$. Thus, by [5a2, Lemma 3.5], for every $P \in \text{Ob}(\mathcal{F}^{cr})$, $P \cap T \in \text{Ob}(\mathcal{F}_H^c) \subseteq \text{Ob}(\mathcal{F}_H^q)$. In particular, $\text{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \text{Ob}(\mathcal{F}^q)$. Hence if $\mathcal{L}^{\mathcal{H}} \subseteq \mathcal{L}^q$ is the full subcategory of \mathcal{L}^q with set of objects \mathcal{H} , by Proposition 2.3(b), $\mathcal{L}^{\mathcal{H}}$ defines a linking system associated to \mathcal{F} . On the level of transporter systems, the inclusions $\mathcal{T} \subseteq \mathcal{T}^q \supseteq \mathcal{T}^{\mathcal{H}}$ induce natural isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Indeed, we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{T} & \longrightarrow & \mathcal{T}^q & \longleftarrow & \mathcal{T}^{\mathcal{H}} \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \mathcal{L} & \longrightarrow & \mathcal{L}^q & \longleftarrow & \mathcal{L}^{\mathcal{H}} \end{array}$$

The vertical arrows induce isomorphisms in cohomology by Lemma 5.1 and the lower horizontal one induces an isomorphism since, by Theorem 2.5, the inclusions of categories $\mathcal{L} \subseteq \mathcal{L}^q \supseteq \mathcal{L}^{\mathcal{H}}$ induces $|\mathcal{L}| \simeq |\mathcal{L}^q| \simeq |\mathcal{L}^{\mathcal{H}}|$. Hence the upper arrows induce isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Finally, By Proposition 2.7, $P \in \text{Ob}(\mathcal{F}_H^q)$ if and only if $P \leq T$ and $P \in \mathcal{H}$. In particular, $\mathcal{T}_H^q \subseteq \mathcal{T}^{\mathcal{H}}$. Thus we can assume for this part that $\mathcal{T} = \mathcal{T}^{\mathcal{H}}$ and $\mathcal{T}_H = \mathcal{T}_H^q$.

We have $\mathcal{T}_H \subseteq \mathcal{T}$ is a transporter system associated to \mathcal{F}_H and, by definition of \mathcal{H} , the hypotheses of [OV1, Proposition 4.1(d)], are satisfied. Thus this inclusion induces a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. Therefore, We have the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{\mathcal{T}_H} & \longrightarrow & \pi_{\mathcal{T}} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\ 0 & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q \longrightarrow 0 \end{array}$$

and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

Hence, in both cases, we have the following Serre spectral sequences

$$H^{s+t}(|\mathcal{T}|, M) \Leftarrow H^s(Q, H^t(|\mathcal{T}_H|, M)),$$

$$H^{s+t}(G, M) \Leftarrow H^s(Q, H^t(H, M)),$$

and ρ induces a morphism ρ^* of spectral sequences between these two. By induction, ρ^* gives an isomorphism on the E_2 page and then induces an isomorphism of spectral sequences. In particular, ρ induces a natural isomorphism

$$H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

The result follows by induction. \square

Assume the hypotheses of Theorem 5.2. It remains to compare $H^*(G, M)$ with the \mathcal{F}^c -stable elements. This is also not obvious and they are not isomorphic in all cases. On one hand, by Cartan-Eilenberg Theorem, we have $H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. On the other hand, we have $H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{L}_S^c(G)} H^*(-, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. Hence, it remains to compare $\varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$ and $\varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. For that we can use a result of Grodal [Gr].

Definition 5.3. Let G be a finite group, $S \in \text{Syl}_p(G)$ and M be a $\mathbb{Z}_{(p)}[G]$ -module. Let K be the kernel of $G \rightarrow \text{Aut}(M)$. A subgroup $P \leq S$ is called M -essential if

- (i) the poset of non trivial p -subgroup of $N_G(P)/P$ is empty or disconnected,
- (ii) $Z(P) \cap K \in \text{Syl}_p(C_G(P) \cap K)$,
- (iii) $O_p(N_G(P)/(P(C_G(P) \cap K))) = 1$.

The property (ii) looks like the definition of p -centric and (iii) looks like the definition of \mathcal{F} -radical. For the property (i), if P is \mathcal{F} -centric and fully normalized in \mathcal{F} , it is equivalent to $P = S$ or P is \mathcal{F} -essential ([AKO, Definition I.3.2]).

Theorem 5.4 ([Gr, Corollary 10.4]). *Let G be a finite group, S a Sylow p -subgroup of G and M a $\mathbb{Z}_{(p)}[G]$ -module.*

Let \mathcal{H} be a family of subgroup of S containing S and all the subgroups which are M -essential.

Then, the inclusion of S in G induce a natural isomorphism,

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^{\mathcal{H}}(G)} H^*(-, M).$$

From this Theorem and Theorem 5.2, we get the following Corollary.

Corollary 5.5. *Let G be a finite group, S a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and assume that we have*

the following commutative diagram,

$$\begin{array}{ccc}
 & \pi_{\mathcal{L}} & \\
 \delta_* \nearrow & & \searrow \varphi \\
 \pi_{\mathcal{T}} & & \text{Aut}(M) \\
 \rho_* \searrow & & \nearrow \overline{\varphi} \\
 & G &
 \end{array}$$

that ρ_* is surjective and that $\Gamma := \text{Im}(\varphi) = \text{Im}(\overline{\varphi})$. If Γ is p -solvable and all the M -essential subgroups of S are p -centric, then δ and ρ induce natural isomorphisms,

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

We also conjecture that it can be generalized to any abstract p -local finite group and any $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module with a p -solvable action.

Conjecture 5.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p -solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

Corollary 4.5 and Corollary 5.5 give good evidence for Conjecture 5.6 to be true.

The next section, which is a bit technical, is dedicated to give an example of Conjecture 5.6 where Corollary 5.5 doesn't apply (see Remark 6.7).

6. The p -local structure of wreath products by C_p : an example for Conjecture 5.6

Let G_0 be a finite group, S_0 a Sylow p -subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. We are interested in the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ and the associated p -local finite group $(S, \mathcal{F}, \mathcal{L})$. By [CL, Theorem 5.2 and Remark 5.3], we have that $|\mathcal{L}| \simeq |\mathcal{L}_0| \wr BC_p := |\mathcal{L}_0|^p \times_{C_p} EC_p$ and an extension $(\pi_{\mathcal{L}_0})^p \rightarrow \pi_{\mathcal{L}} \rightarrow C_p$. In addition we have a section $C_p \rightarrow \pi_{\mathcal{L}}$ coming from $* \wr BC_p \rightarrow |\mathcal{L}_0| \wr BC_p$ and thus $\pi_{\mathcal{L}} = \pi_{\mathcal{L}_0} \wr C_p$.

We first give a lemma on strongly p -embedded subgroups. For a finite group G , a subgroup $H < G$ is *strongly p -embedded*, if $p \mid |H|$ and for each $x \in G \setminus H$, $H \cap xHx^{-1}$ has order prime to p .

Lemma 6.1. *Let G be a finite group, $G_0 \leq G$ a subgroup of index a power of p . If G contains a strongly p -embedded subgroup and $p \mid |G_0|$, then G_0 contains a strongly p -embedded subgroup.*

Proof. Let H be a strongly p -embedded subgroup of G . By [AKO, Proposition A.7], H contains a Sylow p -subgroup of G so, up to conjugacy, we can choose H such that H contains a Sylow p -subgroup of G_0 . Hence $G_0 \cap H$ contains a Sylow p -subgroup of G_0 and $p \mid |G_0 \cap H|$. We will show that $G_0 \cap H$ is a strongly p -embedded subgroup of G_0 .

As $[G : H]$ is prime to p and $[G : G_0]$ is a power of p , $G_0 \cap H$ is a proper subgroup of G_0 .

It remains to show that, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p . But $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1} \leq H \cap xHx^{-1}$, thus, as H is a strongly p -embedded subgroup of G , this last subgroup has order prime to p for every $x \in G \setminus H$. In particular, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p and $G_0 \cap H$ is a strongly p -embedded subgroup of G_0 . \square

We give also a lemma on \mathcal{F}_1 -essential subgroups for $\mathcal{F}_1 \subseteq \mathcal{F}$ a subsystem of index a power of p . A proper subgroup $P < S$ is \mathcal{F} -essential if P is \mathcal{F} -centric and fully normalized in \mathcal{F} , and if $\text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup.

Lemma 6.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ a p -local subgroup of index a power of p . If $P < S_1$ is \mathcal{F} -essential, then P is \mathcal{F}_1 -conjugate to an \mathcal{F}_1 -essential subgroup and P is \mathcal{F}_1 -essential if and only if P is fully normalized in \mathcal{F}_1 .*

Proof. Let $P < S_1$ be an \mathcal{F} -essential subgroup. Since \mathcal{F}_1 is saturated, P is \mathcal{F}_1 -conjugate to a subgroup of S_1 fully normalized in \mathcal{F}_1 . If P is \mathcal{F}_1 -essential, it is in particular fully normalized in \mathcal{F}_1 . Thus, it remains to prove that if P is fully normalized in \mathcal{F}_1 , then P is \mathcal{F}_1 -essential. For the remaining, we assume that P is fully normalized in \mathcal{F}_1 and \mathcal{F} -essential.

P is \mathcal{F}_1 -centric: As P is \mathcal{F} -centric, $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. In particular, for all $Q \in P^{\mathcal{F}_1} \subseteq P^{\mathcal{F}}$, $C_{S_1}(Q) = Z(Q)$ and P is \mathcal{F}_1 -centric.

$\text{Out}_{\mathcal{F}_1}(P)$ contains a strongly p -embedded subgroup: Since P is \mathcal{F} -essential, the group $\text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup. As \mathcal{F}_1 is a subsystem of \mathcal{F} of index a power of p , $\text{Out}_{\mathcal{F}_1}(P)$ is a subgroup of $\text{Out}_{\mathcal{F}}(P)$ of index a power of p . Moreover, as P is a proper subgroup of S_1 , $P < N_{S_1}(P)$ and, as P is \mathcal{F}_1 -centric, every element of $N_{S_1}(P) \setminus Z(P)$ induces a non trivial element in $\text{Out}_{\mathcal{F}_1}(P)$. Hence $p \mid |\text{Out}_{\mathcal{F}_1}(P)|$ and, by Lemma 6.1, $\text{Out}_{\mathcal{F}_1}(P)$ contains a strongly p -embedded subgroup. \square

We can easily describe the essential subgroups of a product of fusion systems.

Lemma 6.3. *Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be p -local finite groups and set $S = S_1 \times S_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. The \mathcal{F} -essential subgroups of S are of the form $Q_1 \times S_2$ with $Q_1 < S_1$ \mathcal{F}_1 -essential or $S_1 \times Q_2$ with $Q_2 > S_2$ \mathcal{F}_2 -essential.*

Proof. Let $P \leq S$ be a \mathcal{F} -essential subgroup. By [AKO, Proposition I.3.3], P is \mathcal{F} -centric and \mathcal{F} -radical. Thus, by [AOV, Lemma 3.1], $P = P_1 \times P_2$ with $P_i \leq S_i$ and P_i \mathcal{F}_i -centric.

Remark also that, if we have two groups G_1 and G_2 such that p divide $|G_1|$ and $|G_2|$ then $G_1 \times G_2$ cannot contain a strongly p -embedded subgroup. To see that let S_i be a Sylow p -subgroup of G_i and set $H = \langle x \in G \mid x(S_1 \times S_2)x^{-1} \cap S_1 \times S_2 \neq 1 \rangle$. H contains $G_1 \times \{0\}$ and $\{0\} \times G_2$ so that $H = G$. Thus, by [AKO, Proposition A.7], this implies that G has no strongly p -embedded subgroups.

We also have that $\text{Out}_{\mathcal{F}}(P) = \text{Out}_{\mathcal{F}_1}(P_1) \times \text{Out}_{\mathcal{F}_2}(P_2)$. Hence, the only possibility for P to be \mathcal{F} -essential is that $P_1 = S_1$ and P_2 is \mathcal{F}_2 -essential or the contrary. \square

Let G_0 be a finite group, S_0 a Sylow p -subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. We consider the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$

and the associated p -local finite group $(S, \mathcal{F}, \mathcal{L})$. Here, for the direct computation, we will take the notation of Alperin and Fong [AF]: an element of G will be represented by permutation matrix corresponding to the powers of $(1, 2, \dots, p)$ with entries in G_0 and the composition will follow the matrix product with the composition in G_0 . Denote by $e \in G$ the element

$$e \otimes P_{(1,2,\dots,p)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & e \\ e & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & e & 0 \end{pmatrix}$$

where e is the trivial element of G_0 . Here, we are interested in the \mathcal{F} -essential subgroups.

Lemma 6.4. *Let $P \leq S$ be an \mathcal{F} -essential subgroup.*

- (E₁) *If $P \leq S_0^p$, then either $P = S_0^p$ and $N_G(P) = N_{G_0}(S_0) \wr C_p$ or P is \mathcal{F}_0^p -essential and $N_G(P) = N_{G_0^p}(P)$.*
- (E₂) *If $P \not\leq S_0^p$, then $P \cong_{\mathcal{F}} Q \wr C_p$ where Q is \mathcal{F}_0 -essential and we have $N_G(P)/P \cong N_{G_0}(Q)/Q$ through the diagonal map $G_0 \hookrightarrow G_0^p$.*

Proof. Let $P \leq S$ be an \mathcal{F} -essential subgroup.

Assume first that $P \leq S_0^p$. If $P = S_0^p$ a direct calculation gives $N_G(P) = N_{G_0}(S_0) \wr C_p$. Else, by Lemma 6.2, we know that P is \mathcal{F}_0^p -conjugate to an \mathcal{F}_0^p -essential subgroup $Q \leq S_0^p$. By Lemma 6.3 we have $N_G(Q) \leq G_0^p$ and, in particular, $N_G(Q) = N_{G_0^p}(Q)$. Thus, since P is \mathcal{F}_0^p -conjugate to Q , we also have $N_G(P) = N_{G_0^p}(P)$ and, since P is fully normalized in \mathcal{F} , it is fully normalized in \mathcal{F}_0^p . Hence, by Lemma 6.2, P is \mathcal{F}_0 -essential.

Secondly, assume that $P \not\leq S_0^p$. As all choices of a splitting $C_p \rightarrow G$ are conjugate in G , we can assume that $P = \langle P_0, x \rangle$ where $P_0 = P \cap S_0^p$ and $x = ((x_1, x_2, \dots, x_p), c)$ is such that $x^p \in P_0$. Up to conjugation in $S_0 \wr C_p$ we can assume that x is of the form $((a, 1, 1, \dots, 1), c)$ where $a \in N_{S_0}(Q)$ where Q is the projection of P_0 on the first factor. If we write $P_0^{(i)}$ the projection of P_0 on its i th factor, as x normalizes P_0 , we have that $P_0^{(i)} = P_0^{(j)}$ for all i, j and then $P_0 \leq (P_0^{(1)})^p = Q^p$.

Notice also that $N_G(P) = \langle N_{G_0^p}(P), x \rangle$. If $g = (g_1, \dots, g_p) \in N_{G_0^p}(P)$, as g normalizes $P \cap G_0^p = P_0$, we have, for all i , $g_i \in N_{G_0}(Q)$. Moreover, if we denote $h = (h_1, \dots, h_p) = gxg^{-1}x^{-1} \in P_0$, we have, for all i , $g_i h_i = g_{i-1}$ (with $g_0 = g_p$). Therefore, there is $h' \in Q^p$ such that $g = (g_1, g_1, \dots, g_1) \cdot h' \in \langle N_{G_0}(Q) \otimes \text{Id}, Q^p \rangle \leq N_G(Q^p)$. Hence, every automorphism $c_g \in \text{Aut}_{\mathcal{F}}(P)$ can be extended to an automorphism of $\langle Q^p, x \rangle$. As P is \mathcal{F} essential, by [AKO, Proposition I.3.3], $P = \langle Q^p, x \rangle$. Now, $x^p \in Q^p$ implies that $a \in Q$ so $P = \langle Q^p, x \rangle = \langle Q^p, c \rangle = Q \wr C_p$.

Finally, direct computations give that

$$C_G(P) \cong C_{G_0}(Q) \otimes \text{Id} = \left\{ \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g \end{pmatrix} ; g \in C_{G_0}(Q) \right\}$$

and

$$N_G(P)/P \cong N_{G_0}(Q)/Q \otimes \text{Id} \cong N_{G_0}(Q)/Q.$$

In particular, as P is p -centric, Q is G_0 -centric. Moreover, as $N_G(P)/P = \text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup, $\text{Out}_{\mathcal{F}_0}(Q) = N_{G_0}(Q)/Q$ does as well. Up to conjugacy, we can also assume that Q is fully normalized in \mathcal{F}_0 and thus Q is \mathcal{F}_0 -essential. \square

Let us now look at some cohomological results. Recall that for a group G , a subgroup $H \leq G$, and M a $\mathbb{F}_p[H]$ -module, we define the *induced* and *coinduced* $\mathbb{F}_p[G]$ -module by,

$$\text{Ind}_H^G(M) = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M \quad \text{coInd}_H^G(M) = \text{Hom}_{\mathbb{F}_p[H]}(\mathbb{F}_p[G], M).$$

Recall also that, when the index of H in G is finite, these two $\mathbb{F}_p[G]$ -modules are isomorphic (by [We, Lemma 6.3.4]).

Lemma 6.5. *Let X be a CW complex and denote by G its fundamental group. If X_0 is a covering space of X with fundamental group $G_0 \triangleleft G$ of finite index, then, for every $\mathbb{F}_p[G_0]$ -module M , we have a natural isomorphism of $\mathbb{F}_p[G/G_0]$ -modules,*

$$H^*(X_0, \text{Ind}_{G_0}^G(M)) \cong H^*(X_0, M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

Where, on the right side, G/G_0 is only acting by translation on $\mathbb{F}_p[G/G_0]$.

Proof. This can be easily seen on the chain level. Let \tilde{X} be the universal covering space of X . As $\mathbb{F}_p[G/G_0]$ -modules, we have the following

$$\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \text{Ind}_{G_0}^G(M)) = \bigoplus_{g \in [G/G_0]} \text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), g.M)$$

where the action of G/G_0 is permuting the terms in the sum. But, each terms in the sum is isomorphic, as (trivial) $\mathbb{F}_p[G/G_0]$ -modules, to $\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M)$. Thus

$$\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \text{Ind}_{G_0}^G(M)) \cong \text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

This induces the wanted isomorphism in cohomology. \square

Proposition 6.6. *Let G_0 be a finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. Consider $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{F}_p[\pi_{\mathcal{L}_0}]$ -module.*

If δ_{S_0} induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M),$$

and

$$H^*(|\mathcal{L}_0|^p, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}_0}}(M^{\otimes p})) \cong H^*((\mathcal{F}_0^p)^c, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}_0}}(M^{\otimes p}))$$

then δ_S induces a natural isomorphism

$$H^*(|\mathcal{L}|, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})) \cong H^*(\mathcal{F}^c, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})).$$

Proof. Write $N = \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})$ and, for $i \in \{1, 2\}$, denote by $H^*(\mathcal{F}^{E_i}, N)$ the stable elements of $H^*(S, N)$ under the full subcategory of \mathcal{F} with objects S and all the subgroups of S of type (E_i) defined in Lemma 6.4.

By the Mackey Formula,

$$\text{Res}_{Q \wr C_p}^{\pi_{\mathcal{L}}} \text{Ind}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} = \text{Ind}_{Q^p}^{Q \wr C_p} \text{Res}_{Q^p}^{\pi_{\mathcal{L}_0}^p}.$$

Thus by Shapiro's Lemma (see for example [Ev, Proposition 4.1.3]) and the Kunnetth Formula, for every $P = Q \wr C_p$ of type (E_2) , we have a natural isomorphism $H^*(Q \wr C_p, N) \cong H^*(Q^p, M^{\otimes p}) \cong H^*(Q, M)^{\otimes p}$ and, by the computation of normalizers in Lemma 6.4,

$$H^*(Q \wr C_p, N)^{\text{Aut}_{\mathcal{F}}(Q \wr C_p)} \cong (H^*(Q, M)^{\text{Aut}_{\mathcal{F}_0}(Q)})^{\otimes p}.$$

Hence, applying this to all the subgroups of type (E_2) and, by naturality of the Shapiro isomorphisms, we have that,

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p}.$$

On the other hand, by [CL, Theorem 5.2 and Remark 5.3], $|\mathcal{L}_0|^p$ has the homotopy type of a covering space of $|\mathcal{L}|$ with covering group C_p . Then, if we denote by X the universal covering space of $|\mathcal{L}|$ (which is also the universal covering space of $|\mathcal{L}_0|^p$), we have the following isomorphism on the chain level (because Res and coInd are adjoint functors)

$$\text{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}_0}^p]}(C_*(X), M^{\otimes p}) \cong \text{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]}(C_*(X), N)$$

which is analogue to the Shapiro isomorphism (see [Ev, Proposition 4.1.3]). By the Kunnetth Formula, it gives us the following isomorphism on cohomology

$$H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N)$$

and give the following commutative diagram

$$\begin{array}{ccc} H^*(S_0, M)^{\otimes p} & \xrightarrow{\cong} & H^*(S, N) \\ (\delta_{S_0})^* \downarrow & & \downarrow \delta_S^* \\ H^*(|\mathcal{L}_0|, M)^{\otimes p} & \xrightarrow{\cong} & H^*(|\mathcal{L}|, N) \end{array}$$

Thus δ_S induces an isomorphism

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p} \cong H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N).$$

Secondly, by factoring the Shapiro isomorphism (see [Ev, Proposition 4.1.3]), the inclusion of S_0^p in S induces an injection $H^*(S, N) \hookrightarrow H^*(S_0^p, N)$. Hence

$$H^*(\mathcal{F}^{E_1}, N) \cong H^*((\mathcal{F}_0^p)^c, N)^{C_p} \leq H^*(S_0^p, N).$$

By assumption, $\delta_{S_0^p}$ induces an isomorphism

$$H^*((\mathcal{F}_0^p)^c, N) \cong H^*(|\mathcal{L}_0|^p, N).$$

Moreover, by Lemma 6.5, this last term is isomorphic to $H^*(|\mathcal{L}_0|^p, M^{\otimes p}) \otimes \mathbb{F}_p[C_p]$ and, in particular, it is a projective $\mathbb{F}_p[C_p]$ -module.

Consider now the Serre spectral sequence associated to the fibration sequence

$$|\mathcal{L}_0|^p \longrightarrow |\mathcal{L}| \longrightarrow BC_p$$

with coefficients in N . The E_2 page is the following,

$$E_2^{s,t} = H^s(C_p, H^t(|\mathcal{L}_0|^p, N))$$

and, by projectivity of $H^t(|\mathcal{L}_0|^p, N)$, the E_2 page is concentrated in the 0th column. Hence, we have that, $H^*(|\mathcal{L}_0|^p, N)^{C_p} = E_2^{0,*} \cong H^*(|\mathcal{L}|, N)$.

In conclusion,

$$H^*(\mathcal{F}^c, N) = H^*(\mathcal{F}^{E_1}, N) \cap H^*(\mathcal{F}^{E_2}, N) \cong H^*(|\mathcal{L}|, N)$$

and the theorem follows. \square

This proposition is a bit technical but we will use it in a specific case. Consider $p = 5$, the group $G_0 = GL_{20}(F_2)$, the wreath product $G = G_0 \wr C_5$ and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ and $(S, \mathcal{F}, \mathcal{L})$ the associated 5-local finite groups. By [Ru, Theorem 6.3], we know that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ admits a 5-local subgroup of index 4 which is exotic $(S_e, \mathcal{F}_e, \mathcal{L}_e)$ and that we have a fibration sequence

$$|\mathcal{L}_e| \longrightarrow |\mathcal{L}_0| \longrightarrow BC_4.$$

In particular, we have $\pi_{\mathcal{L}}/\pi_{\mathcal{L}_e}^5 = C_4 \wr C_5$ and we can be interested in comparing $H^*(|\mathcal{L}|, N)$ and $H^*(\mathcal{F}^c, N)$ for

$$N = \mathbb{F}_5[C_4 \wr C_5] = \text{Ind}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5}) \cong \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})$$

(the action factors through a finite group) with $M = \mathbb{F}_5[C_4]$.

By Corollary 4.4, we have that δ_{S_0} and $\delta_{S_0^p}$ induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M) \text{ and}$$

$$H^*(|\mathcal{L}_0|^5, \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})) \cong H^*((\mathcal{F}_0^5)^c, \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})),$$

(for the second isomorphism, notice that $|\mathcal{L}_0|^5$ has the homotopy type of a linking system associated to \mathcal{F}_0^5 by [CL, Proposition 2.17]). Hence, all the hypothesis of Proposition 6.6 are satisfied and

$$H^*(|\mathcal{L}|, N) \cong H^*(\mathcal{F}^c, N).$$

Remark 6.7. This gives us an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the stable elements when the action factors through a p -solvable group which cannot be recovered by a previous result. Notice that, even if the fusion system \mathcal{F} is realizable, as \mathcal{F}_e is exotic, we cannot find a group G with $S \in \text{Syl}_p(G)$ such that G acts on M in the same way as asked in Section 5. This example gives us some additional evidence for Conjecture 5.6.

References

- [5a1] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver, Subgroup families controlling p -local finite groups, *Proc. London Math. Soc.* (3) **91** (2005), no. 2, 325–354.

- [5a2] C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver, Extensions of p -local finite groups, *Trans. Amer. Math. Soc.* **359** (2007), no. 7, 3791–3858.
- [AF] J.L. Alperin and P. Fong, Weights for symmetric and general linear groups, *J. Algebra* **131** (1990), no. 1, 2–22.
- [AKO] M. Aschbacher, R. Kessar and B. Oliver, *Fusion systems in algebra and topology*, volume 391 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 2011.
- [AOV] K.K.S. Andersen, B. Oliver and J. Ventura, Reduced, tame and exotic fusion systems, *Proc. Lond. Math. Soc. (3)* **105** (2012), no. 1, 87–152.
- [BLO1] C. Broto, R. Levi and B. Oliver, Homotopy equivalences of p -completed classifying spaces of finite groups, *Invent. Math.* **151** (2003), no. 3, 611–664.
- [BLO2] C. Broto, R. Levi and B. Oliver, The homotopy theory of fusion systems, *J. Amer. Math. Soc.* **16** (2003), no. 4, 779–856.
- [CE] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999.
- [CL] N. Castellana and A. Libman, Wreath products and representations of p -local finite groups, *Adv. Math.* **221** (2009), no. 4, 1302–1344.
- [Ch] A. Chermak, Fusion systems and localities, *Acta Math.* **211** (2013), no. 1, 47–139.
- [Ev] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991.
- [Gr] J. Grodal, Higher limits via subgroup complexes, *Ann. of Math. (2)* **155** (2002), no. 2, 405–457.
- [LR] R. Levi and K. Ragnarsson, p -local finite group cohomology, *Homology Homotopy Appl.* **13** (2011), no. 1, 223–257.
- [Mo1] R. Molinier, Cohomology with twisted coefficients of the classifying space of a fusion system, *Topology Appl.* **212** (2016), 1 – 18.
- [O4] B. Oliver, Extensions of linking systems and fusion systems, *Trans. Amer. Math. Soc.* **362** (2010), no. 10, 5483–5500.
- [OV1] B. Oliver and J. Ventura, Extensions of linking systems with p -group kernel, *Math. Ann.* **338** (2007), no. 4, 983–1043.
- [Ru] A. Ruiz, Exotic normal fusion subsystems of general linear groups, *J. Lond. Math. Soc. (2)* **76** (2007), no. 1, 181–196.
- [We] C.A. Weibel, *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1994.

Rémi Molinier molinier@ksu.edu

Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA