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COHOMOLOGY OF LINKING SYSTEMS WITH TWISTED COEFFICIENTS BY A p -SOLVABLE ACTION

RÉMI MOLINIER

(*communicated by Nathalie Wahl*)

Abstract

In this paper we study the cohomology of the geometric realization of linking systems with twisted coefficients. More precisely, given a prime p and a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, we compare the cohomology of \mathcal{L} with twisted coefficients with the submodule of \mathcal{F}^c -stable elements in the cohomology of S . We start with the particular case of constrained fusion systems. Then, we study the case of p -solvable actions on the coefficients.

1. Introduction

The notion of saturated fusion system was introduced by Puig in the 90s in a context of modular representation theory. Since their introduction, topologists use them to study classifying spaces of finite groups or, more precisely, their p -completions. A p -local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p -group, \mathcal{F} a saturated fusion system over S and \mathcal{L} an associated linking system. For a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, $|\mathcal{L}|_p^\wedge$ is called its *classifying space*. The theory of p -local finite groups have been studied in details by Broto, Levi, Oliver and others (see [BLO2], [OV1], [5a1] and [5a2]). The linking system and its geometric realization, even without p -completion, play here a fundamental and central role. In fact, for a given saturated fusion systems, the existence and uniqueness of an associated linking system were shown more recently by Chermak [Ch] (using the theory of partial groups). The proof of this important conjecture highlights that linking systems and their geometric realizations form a deep link between fusion system theory and homotopy theory (we refer to Aschbacher, Kessar and Oliver [AKO] for more details about fusion systems in general).

An old and well-known result due to Cartan and Eilenberg (see [CE, Theorem XII.10.1]) expresses the cohomology of a finite group G in a $\mathbb{Z}_{(p)}[G]$ -module as the submodule of “stable” elements in the cohomology of a Sylow p -subgroup of G . This submodule of stable elements corresponds to the inverse limit over the “fusion” of the group cohomology functor. One important result in the theory of p -local finite groups is an analog of this theorem for p -local finite groups which tells us that the cohomology of the geometric realization of a linking system can be computed by \mathcal{F} -stable elements. More precisely, there is a natural inclusion of BS into $|\mathcal{L}|$ and it

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induces the following isomorphism. Here, \mathcal{F}^c is the full subcategory of \mathcal{F} consisting of \mathcal{F} -centric subgroups of S and, for A a finite $\mathbb{Z}_{(p)}$ -module, $H^*(\mathcal{F}^c, A) \subseteq H^*(S, A)$ is the submodule of \mathcal{F} -stable elements.

Theorem 1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and A be a finite $\mathbb{Z}_{(p)}$ -module. The inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|_p^\wedge, A) \cong H^*(|\mathcal{L}|, A) \xrightarrow{\cong} H^*(\mathcal{F}^c, A).$$

Proof. The case $A = \mathbb{F}_p$ is [BLO2, Theorem B] and the general case is proven in [5a2, Lemma 6.12]. \square

One question asked by Oliver in his book with Aschbacher and Kessar [AKO] is the understanding of the cohomology of $|\mathcal{L}|$ with twisted coefficients. Indeed, this cohomology appears for example in the study of extensions of p -local finite groups or, more directly, can give more information about the link between the fusion system and the spaces $|\mathcal{L}|$ or $|\mathcal{L}|_p^\wedge$. Recall that, if a space X has a universal covering space \tilde{X} , the cohomology of X with twisted coefficients in a $\mathbb{Z}[\pi_1(X)]$ -module M is the cohomology of the chain complex

$$C^*(X; M) = \text{Hom}_{\mathbb{Z}[\pi_1(X)]}(S_*(\tilde{X}), M),$$

where $S_*(\tilde{X})$ is the usual singular chain complex of \tilde{X} .

Levi and Ragnarsson [LR] already give some tools along these lines. In an other paper [Mo1], the author extends Theorem 1.1 to the case of nilpotent actions on the coefficients. The main ingredient is to construct, as in the trivial coefficient case, an idempotent of $H^*(S, M)$ with image $H^*(\mathcal{F}^c, M)$.

In this paper, we also want to extend Theorem 1.1 to twisted coefficients but when the action factors through a p -solvable group. The methods used here are completely different from the ones used in [Mo1] and also more direct. We first have a look at constrained fusion systems. In that case we are able to prove that, with any coefficient module, the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

Theorem A (see Theorem 3.5). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. If \mathcal{F} is constrained and M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module, then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Next we focus on p -solvable actions. The main ingredients here are p -local finite subgroups of index a power of p or prime to p and their homotopy properties. We start by looking at p -local subgroups of index prime to p (see Definition 2.6(b)).

Theorem B (see Theorem 4.3). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and denote by $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module and if the inclusion of BS in $|O^{p'}(\mathcal{L})|$ induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

This Theorem allows us to prove that if the action on the coefficients factor through a p' -group or, even better, a p -nilpotent group, then the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

It is much more complicated to work with p -local finite groups of index a power of p , especially on the level of stable elements. Indeed, for $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module, it is difficult to compare $H^*(\mathcal{F}^c, M)$ and $H^*(\mathcal{F}_0^c, M)$. The difficulty mostly comes from the fact that we are working on different p -groups: S and S_0 . But when we work with a p -local finite group realizable by a finite group G , and if G acts "consistently" on the coefficients it is possible to get some positive results (see Section 5).

Theorem C (see Corollary 5.5). *Let G be a finite group, S a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module and assume that G acts consistently on M . If both actions factor through a given p -solvable Γ and all the M -essential subgroups (see Definition 5.3) of S are p -centric, then we have natural isomorphisms,*

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

All of these results lead us to the following conjecture.

Conjecture A (see Conjecture 5.6). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p -solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

We finish this paper with an example for Conjecture 5.6 which does not follow from the other results.

Organization. In Section 2 we give some background on p -local finite groups and stable elements. Section 3 is dedicated to the case of constrained fusion systems, Section 4 to coprime actions and Section 5 to p -solvable actions for a realizable p -local finite group. Finally, we give in Section 6 an example for Conjecture 5.6.

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2. Background

We give here a very short introduction to p -local finite groups. The notion of fusion system was first introduced by Puig for modular representation theory purpose. Later, Broto, Levi and Oliver developed the notion of linking systems and p -local finite groups to study p -completed classifying spaces of finite groups and spaces which have similar homotopy properties. We refer the reader interested in more details to Aschbacher, Kessar and Oliver [AKO].

2.1. Fusion systems and linking systems

A fusion system over a p -group S is a way to abstract the action of a finite group G with $S \in \text{Syl}_p(G)$ on the subgroups of S by conjugation. For G a finite group and $g \in G$, we will denote by c_g the homomorphism $x \in G \mapsto gxg^{-1} \in G$ and for H, K two subgroups of G , $\text{Hom}_G(H, K)$ will denote the set of all group homomorphism c_g for $g \in G$ such that $c_g(H) \leq K$.

Definition 2.1. Let S be a finite p -group. A *fusion system* over S is a small category \mathcal{F} , where $\text{Ob}(\mathcal{F})$ is the set of all subgroups of S and which satisfies the following two properties for all $P, Q \leq S$:

- (a) $\text{Hom}_S(P, Q) \subseteq \text{Mor}_{\mathcal{F}}(P, Q) \subseteq \text{Inj}(P, Q)$;
- (b) each $\varphi \in \text{Mor}_{\mathcal{F}}(P, Q)$ is the composite of an \mathcal{F} -isomorphism followed by an inclusion.

A fusion system is *saturated* if it satisfy two more technical axioms called the saturation axioms (we refer the reader to [AKO, Definition I.2.1] for a proper definition).

The composition in a fusion system is given by composition of homomorphisms. We usually write $\text{Hom}_{\mathcal{F}}(P, Q) = \text{Mor}_{\mathcal{F}}(P, Q)$ to emphasize that the morphisms in \mathcal{F} are homomorphisms. For $P, Q \leq S$, we say that P is \mathcal{F} -conjugate to Q if there is an \mathcal{F} -isomorphism between P and Q . We denote by $P^{\mathcal{F}}$ the set of all subgroups of S which are \mathcal{F} conjugate to P .

The typical example of a saturated fusion system is the fusion system $\mathcal{F}_S(G)$ of a finite group G over $S \in \text{Syl}_p(G)$.

For the purpose of this paper, we need to distinguish some collections of subgroups of S .

Definition 2.2. Let \mathcal{F} be a saturated fusion system over a finite p -group S .

- (a) A subgroup $P \leq S$ is \mathcal{F} -centric if for every $Q \in P^{\mathcal{F}}$, $C_S(Q) = Z(Q)$.
- (b) A subgroup $P \leq S$ is \mathcal{F} -radical if $O_p(\text{Aut}_{\mathcal{F}}(P)) = \text{Inn}(P)$.
- (c) A subgroup $P \leq S$ is \mathcal{F} -quasicentric if for each $Q \leq PC_S(P)$ containing P , and each $\alpha \in \text{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_P = \text{Id}$, α has a p -power order.

We let $\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}$ denote the full subcategories of \mathcal{F} with objects the \mathcal{F} -centric and \mathcal{F} -radical subgroups, the \mathcal{F} -centric subgroups and the \mathcal{F} -quasicentric subgroups, respectively.

If $\mathcal{F} = \mathcal{F}_S(G)$, a subgroup $P \leq S$ is

- (a) \mathcal{F} -centric if and only if it is p -centric (i.e. $Z(P) \in \text{Syl}_p(C_G(P))$),
- (b) \mathcal{F} -radical if $P/Z(P) = O_p(N_G(P)/C_G(P))$.
- (c) \mathcal{F} -quasicentric if and only if $O^p(C_G(P))$ has order prime to p .

The notion of linking system has been introduced by Broto, Levi and Oliver [BLO2] and generalized by Broto, Castellana, Grodal and Oliver in [5a1]. We refer the reader to these papers, or [AKO, Part III], for a proper definition. We recall here some basic facts about linking systems which will be needed here.

For G a finite group, $S \in \text{Syl}_p(G)$ and \mathcal{H} a collection of subgroups of S , the *transporter category* of G over S with set of objects \mathcal{H} is the category $\mathcal{T}_H^{\mathcal{H}}(G)$ with objects

\mathcal{H} and for $P, Q \in \mathcal{H}$, $\text{Mor}_{\mathcal{L}}(P, Q) = T_G(P, Q) = \{g \in G \mid P^g \leq Q\}$. For \mathcal{F} a saturated fusion system over a p -group S , a *linking system* associated to \mathcal{F} is a certain finite category with objects a collection \mathcal{H} of subgroups of S together with two functors:

$$\mathcal{T}_S^{\mathcal{H}}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}.$$

δ is the identity on objects and injective on morphisms and π is injective on objects and surjective on morphisms. The collection \mathcal{H} has to be stable by overgroups and \mathcal{F} -conjugation and the following proposition tell you which collection you can have.

Proposition 2.3. *Let \mathcal{F} be a saturated fusion system over a p -group S . Let \mathcal{L} be a linking system associated to \mathcal{F} .*

- (a) *$\text{Ob}(\mathcal{F}^{cr}) \subseteq \text{Ob}(\mathcal{L}) \subseteq \text{Ob}(\mathcal{F}^q)$, and there exists a linking system \mathcal{L}^q associated to \mathcal{F} such that $\text{Ob}(\mathcal{L}^q) = \text{Ob}(\mathcal{F}^q)$, and \mathcal{L} is a full subcategory of \mathcal{L}^q .*
- (b) *For every subset $\text{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \text{Ob}(\mathcal{F}^q)$ stable by \mathcal{F} -conjugacy and overgroups, the full subcategory $\mathcal{L}^{\mathcal{H}}$ of \mathcal{L}^q with set of objects \mathcal{H} is also a linking system associated to \mathcal{F} .*

Proof. The first point of (a) can be found for example in [O4, Proposition 4(g)]. For the second statement of (a), you can find a proof in [AKO, Proposition III.4.8]. Finally, (b) is a consequence of the definition of linking systems. \square

If $\mathcal{H} = \text{Ob}(\mathcal{F}^q)$, \mathcal{L} is called a *quasicentric linking system* and if $\mathcal{H} = \text{Ob}(\mathcal{F}^c)$, \mathcal{L} is called a *centric linking system*.

Definition 2.4. A p -local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where \mathcal{F} is a saturated fusion system over S and \mathcal{L} is an associated linking system. If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is another p -local finite group, we will say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p -local subgroup* of $(S, \mathcal{F}, \mathcal{L})$ if $S_0 \leq S$ and $\mathcal{F}_0 \subseteq \mathcal{F}$ is a subsystem of \mathcal{F} . Notice that we do not require that \mathcal{L}_0 is a subcategory of \mathcal{L} .

The typical example you should have in mind is the following. For G a finite group and $S \in \text{Syl}_p(G)$ let $\mathcal{L}_S^q(G)$ be the category with objects the $\mathcal{F}_S(G)$ -quasicentric subgroups of G and, for $P, Q \in \text{Ob}(\mathcal{L})$,

$$\text{Mor}_{\mathcal{L}}(P, Q) = T_G(P, Q)/O^p(C_G(P)).$$

Then $(S, \mathcal{F}_S(G), \mathcal{L}_S^q(G))$ defines a p -local finite group where $\mathcal{L}_S^q(G)$ is a quasicentric linking system. We also denote by $\mathcal{L}_S^c(G)$ the full subcategory of $\mathcal{L}_S^q(G)$ with objects the p -centric subgroups of S and it is a centric linking system.

We finish with some basic homotopy properties about linking systems which will be needed in this paper. We refer the reader interested in more details to [AKO, Part III]. For $(S, \mathcal{F}, \mathcal{L})$ a p -local finite group, we write $|\mathcal{L}|$ for the geometric realization of \mathcal{L} and $\pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|)$ for its fundamental group. The following Theorem will allow us to change the set of objects of \mathcal{L} without changing the homotopy type of $|\mathcal{L}|$.

Theorem 2.5 ([5a1, Theorem 3.5]). *Let \mathcal{F} be a saturated fusion system over a p -group S . Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two linking systems associated to \mathcal{F} with a different set of objects. Then the inclusion induces a homotopy equivalence of space $|\mathcal{L}_0| \simeq |\mathcal{L}|$.*

2.2. p -local finite subgroups of index a power of p or prime to p

The notions p -local subgroups of index a power of p or prime to p have been introduced and studied by Broto, Castellana, Grodal, Levi and Oliver [5a2]. Here we just give the definitions what we need about these p -local subgroups and we refer the reader to [5a2] for more details.

Definition 2.6. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$. Set $\mathfrak{hnp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in O^p(\text{Aut}_{\mathcal{F}}(P)) \rangle \trianglelefteq S$.

- (a) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a p -local subgroup of index a power of p if $S_0 \geq \mathfrak{hnp}(\mathcal{F})$ and, for every $P \leq S_0$, $O^p(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$.
- (b) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a p -local subgroup of index prime to p if $S_0 = S$ and, for every $P \leq S$, $O^{p'}(\text{Aut}_{\mathcal{F}}(P)) \leq \text{Aut}_{\mathcal{F}_0}(P)$.

Notice that $\mathfrak{hnp}(\mathcal{F})$ is denoted $O_{\mathcal{F}}^p(S)$ in [5a2, Definition 2.1]. These particular p -local subgroups satisfy the following properties.

Proposition 2.7 ([5a2, Proposition 3.8]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$.*

- (a) *If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index a power of p , then $P \leq S_0$ is \mathcal{F}_0 -quasicentric if, and only if, P is \mathcal{F} -quasicentric.*
- (b) *If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index prime to p , then $P \leq S$ is \mathcal{F}_0 -centric if, and only if, P is \mathcal{F} -centric.*

For an infinite group G , we denote by $O^{p'}(G)$ the intersection of all normal subgroups in G of finite index prime to p . For \mathcal{F} a fusion system over a p -group S , let $O_*^{p'}(\mathcal{F})$ be the fusion system generated by $O^{p'}(\text{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$ and define

$$\text{Out}_{\mathcal{F}}^0(S) = \langle \alpha \in \text{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \text{Hom}_{O_*^{p'}(\mathcal{F})}(P, S), \text{ for some } P \leq S \rangle.$$

Since $\text{Aut}_{\mathcal{F}}(S)$ normalizes $O_*^{p'}(\mathcal{F})$, $\text{Out}_{\mathcal{F}}^0(S) \trianglelefteq \text{Out}_{\mathcal{F}}(S)$.

Proposition 2.8. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group.*

- (a) $\mathcal{F} = \langle \text{Aut}_{\mathcal{F}}(S), O_*^{p'}(\mathcal{F}) \rangle$.
- (b) π and the inclusion of $B\text{Aut}_{\mathcal{F}}(S)$ in $|\mathcal{F}^c|$ induce isomorphisms,

$$\theta: \pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \xrightarrow{\cong} \pi_1(|\mathcal{F}^c|) \xrightarrow{\cong} \text{Out}_{\mathcal{F}}(S)/\text{Out}_{\mathcal{F}}^0(S).$$

Proof. The point (a) is proved in [5a2, Lemma 3.4]. For (b), the second isomorphism is given in [5a2, Proposition 5.2] and the first one in [5a2, Theorem 5.5] and the comment which follows. \square

According to Proposition 2.7, when dealing with p -local subgroups of index prime to p , we will work with centric linking systems.

Theorem 2.9 ([5a2, Theorem 5.5]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. For each subgroup $H \leq \text{Out}_{\mathcal{F}}(S)$ containing $\text{Out}_{\mathcal{F}}^0(S)$, there is a unique p -local finite subgroup $(S, \mathcal{F}_H, \mathcal{L}_H)$ of index prime to p such that $\text{Out}_{\mathcal{F}_H}(S) = H$ and $\mathcal{L}_H = \pi^{-1}(\mathcal{F}_H^c)$.*

Moreover, $|\mathcal{L}_H|$ is homotopy equivalent, via its inclusion in $|\mathcal{L}|$, to the covering space of $|\mathcal{L}|$ with fundamental group $\tilde{H} \geq O^{p'}(\pi_{\mathcal{L}})$ such that $\theta(\tilde{H}/O^{p'}(\pi_{\mathcal{L}})) = H/\text{Out}_{\mathcal{F}}^0(S)$ (where θ is the isomorphism given in Proposition 2.8(b)).

Thus, for a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, with \mathcal{L} a centric linking system, we can define the *minimal p -local subgroup of index prime to p* , $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ corresponding to $(S, \mathcal{F}_H, \mathcal{L}_H)$ with $H = \text{Out}_{\mathcal{F}}^0(S)$ in Theorem 2.9.

2.3. Cohomology and stable elements

The first result about stable elements is due to Cartan and Eilenberg ([CE, Chap XII, Theorem 10.1]). It also served as a guideline in the establishment of Theorem 1.1 by Broto, Levi and Oliver. Here we recall the definition of \mathcal{F}^c -stable elements in a context of twisted coefficients. We refer the reader to [Mo1] for more details. As in [Mo1], we will denote by $\omega: \mathcal{L} \rightarrow \pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|, S)$ the functor which maps each object to the unique object in the target and sends each morphism $\varphi \in \text{Mor}_{\mathcal{L}}(P, Q)$ to the class of the loop $\iota_Q \cdot \varphi \cdot \overline{\iota_P}$ where $\iota_P = \delta(1) \in \text{Mor}_{\mathcal{L}}(P, S)$, $\iota_Q = \delta(1) \in \text{Mor}_{\mathcal{L}}(Q, S)$ and $\overline{\iota_P}$ is the edge ι_P followed in the opposite direction.

Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Recall first that $\delta: \mathcal{T}_S^{\text{Ob}(\mathcal{L})}(S) \rightarrow \mathcal{L}$ induces an inclusion $\delta_S: BS \rightarrow |\mathcal{L}|$. In particular, it induces a natural map $S \rightarrow \pi_{\mathcal{L}}$ and thus, for every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module M , we have a natural action of S , or any subgroup of S , on M . Now, let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, the group cohomology bifunctor $H^*(-, -)$ induces a functor

$$H^*(-, M): \mathcal{F}^c \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$$

(a priori, $H^*(g, M)$ is defined for $g \in \text{Mor}(\mathcal{L})$ but [Mo1, Proposition 2.2] proves that $H^*(-, M)$ is well defined on \mathcal{F}^c).

Definition 2.10. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. An element $x \in H^*(S, M)$ is called *\mathcal{F} -centric stable*, or *\mathcal{F}^c -stable*, if for all $P \in \text{Ob}(\mathcal{F}^c)$ and all $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$,

$$\varphi^*(x) = \text{Res}_P^S(x).$$

We denote by $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ the submodule of all \mathcal{F}^c -stable elements.

Notice that

$$H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{F}^c} H^*(-, M) = \varprojlim_{\mathcal{L}} H^*(-, M)$$

where the last equality holds if \mathcal{L} is a centric linking system.

3. Constrained fusion systems

Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Here, we assume that \mathcal{F} is a constrained fusion system.

Definition 3.1. Let \mathcal{F} be a fusion system over a p -group S . A subgroup $Q \leq S$ is *normal in \mathcal{F}* if

- (i) $Q \trianglelefteq S$, and
- (ii) for all $P, R \leq S$ and every $\varphi \in \text{Hom}_{\mathcal{F}}(P, R)$, φ extends to a morphism $\overline{\varphi} \in \text{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\overline{\varphi}(Q) = Q$.

We write $O_p(\mathcal{F})$ for the maximal subgroup of S which is normal in \mathcal{F} . We say that \mathcal{F} is *constrained* if $O_p(\mathcal{F})$ is \mathcal{F} -centric.

An important and classical result about constrained fusion systems is the following.

Proposition 3.2 ([5a1, Proposition 4.3]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. If \mathcal{F} is constrained, there exists a finite group G such that*

- (a) S is a Sylow p -subgroup of G ,
- (b) $C_G(O_p(G)) \leq O_p(G)$,
- (c) $\mathcal{F}_S(G) = \mathcal{F}$.

Moreover, $G \cong \text{Aut}_{\mathcal{L}}(O_p(\mathcal{F}))$ and $\mathcal{L} \cong \mathcal{L}_S^c(G)$.

This group G is called a *model* of \mathcal{F} and it is unique in a precise way (see [AKO, Theorem III.5.10]). This model can also be recovered from the homotopy type of the geometric realization of a linking system associated to \mathcal{F} .

Lemma 3.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group with \mathcal{L} a centric linking system. If \mathcal{F} is constrained, then $|\mathcal{L}|$ is a classifying space of a model G of \mathcal{F} .*

Proof. By Proposition 3.2, we can assume that $\mathcal{L} = \mathcal{L}_S^c(G)$. Set

$$\mathcal{H} = \{P \in \text{Ob}(\mathcal{L}) \mid P \geq O_p(G)\}$$

and let $\mathcal{L}^{\mathcal{H}}$ be the full subcategory of \mathcal{L} with set of objects \mathcal{H} . By [5a1, Proposition 1.6], \mathcal{H} contains all \mathcal{F} -centric and \mathcal{F} -radical subgroups. Thus, by Proposition 2.3, $\mathcal{L}^{\mathcal{H}}$ is a linking system associated to \mathcal{F} and, by Theorem 2.5, $|\mathcal{L}^{\mathcal{H}}| \cong |\mathcal{L}|$.

It remains to prove that that $|\mathcal{L}^{\mathcal{H}}| \cong BG$. For that purpose, consider, the following functor.

$$F: \begin{array}{ccc} \mathcal{L}^{\mathcal{H}} & \longrightarrow & \mathcal{L}^{\{O_p(G)\}} \\ P \in \mathcal{L}^{\mathcal{H}} & \longmapsto & O_p(G) \\ g \in T_G(P, Q) & \longmapsto & g \in N_G(O_p(G)) = G \end{array}$$

It gives us a retraction by deformation of $|\mathcal{L}^{\mathcal{H}}|$ on the geometric realization of the full subcategory of \mathcal{L} with unique object $O_p(G) \leq S$. As $\text{Aut}_{\mathcal{L}}(O_p(G)) = N_G(O_p(G)) = G$, this last category is $\mathcal{B}(G)$. in particular, its geometric realization is a classifying space of G . \square

Proposition 3.4. *Let G be a finite group and S a Sylow p -subgroup of G . If we have $C_G(O_p(G)) \leq O_p(G)$, then, for every $\mathbb{Z}_{(p)}[G]$ -module M , the inclusion of S in G induces a natural isomorphism*

$$H^*(G, M) \cong H^*(\mathcal{F}_S^c(G), M).$$

Proof. Let $(S, \mathcal{F}, \mathcal{L}) = (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$. By assumption, $\mathcal{F}_S(G)$ is constrained and

G is a model of $\mathcal{F}_S(G)$. From Cartan-Eilenberg Theorem, we know that

$$\text{Res}_S^G: H^*(G, M) \longrightarrow H^*(S, M)$$

is injective and that $\text{Im}(\text{Res}_S^G) = \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. Moreover,

$$H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{F}^c} H^*(-, M) = \varprojlim_{\mathcal{L}} H^*(-, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M) \geq \varprojlim_{\mathcal{T}_S(G)} H^*(-, M).$$

Thus, it remains to prove that $\varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M) \leq \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$.

Let then $x \in H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. For $P \leq S$ and $g \in N_G(P, S)$ we have, in $\mathcal{T}_S(G)$, the following commutative diagram

$$\begin{array}{ccc} PO_p(G) & \xrightarrow{g} & gPg^{-1}O_p(G) \\ e \uparrow & & \uparrow e \\ P & \xrightarrow{g} & gPg^{-1} \end{array}$$

where e is the trivial element of G . Hence, as the top part of the diagram is in $\mathcal{T}_S^c(G)$ and $x \in \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$,

$$\begin{aligned} c_g^* \circ \text{Res}_{gPg^{-1}}^S(x) &= \text{Res}_P^{PO_p(G)} \circ c_g^* \circ \text{Res}_{gPg^{-1}O_p(G)}^S(x) \\ &= \text{Res}_P^{PO_p(G)} \circ \text{Res}_{PO_p(G)}^S(x) \\ &= \text{Res}_P^S(x). \end{aligned}$$

Thus $x \in \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$ and this complete the proof. \square

Corollary 3.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. If \mathcal{F} is constrained and M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then δ_S induces a natural isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

4. Actions factoring through a p' -group

In this section, for each p -local finite group $(S, \mathcal{F}, \mathcal{L})$ we will assume that \mathcal{L} is a **centric linking system**.

Lemma 4.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then the inclusion $O^{p'}(\mathcal{L}) \subseteq \mathcal{L}$ induces the following isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Proof. By Theorem 2.9, $|O^{p'}(\mathcal{L})|$ is, up to homotopy, a covering space of $|\mathcal{L}|$ with

fundamental group $O^{p'}(\pi_{\mathcal{L}}) \trianglelefteq \pi_{\mathcal{L}}$. It gives us a fibration sequence

$$|O^{p'}(\mathcal{L})| \rightarrow |\mathcal{L}| \rightarrow B(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})).$$

Consider then the Serre spectral sequence associated

$$H^{s+t}(|\mathcal{L}|, M) \Leftarrow H^s(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}), H^t(|O^{p'}(\mathcal{L})|, M)).$$

M is a $\mathbb{Z}_{(p)}$ -module, thus $H^q(|O^{p'}(\mathcal{L})|, M)$ is also a $\mathbb{Z}_{(p)}$ -module. As $\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})$ is a p' -group, the E_2 -page is concentrated in the first column with terms

$$H^t(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Thus the spectral sequence collapses on the E_2 -page and the Lemma follows. \square

Lemma 4.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then*

$$H^*(\mathcal{F}^c, M) = H^*(O^{p'}(\mathcal{F})^c, M)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)}.$$

Proof. Notice first that, by Proposition 2.7, $\text{Ob}(O^{p'}(\mathcal{F})^c) = \text{Ob}(\mathcal{F}^c)$. Hence, we are working with the same underlying set of objects. Thus, by definition, $H^*(\mathcal{F}^c, M) \subseteq H^*(O^{p'}(\mathcal{F})^c, M)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)}$. On the other hand, by Proposition 2.8, we have $\mathcal{F} = \langle O^{p'}(\mathcal{F}), \text{Aut}_{\mathcal{F}}(S) \rangle$ which gives the converse inclusion. \square

Theorem 4.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p -local subgroup of index prime to p . If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and if the inclusion δ_S induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. Recall that, by 2.9, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$. Then we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{B}(S) & \xrightarrow{\delta_S} & \mathcal{L} & \xrightarrow{\omega} & \mathcal{B}\pi_{\mathcal{L}} & \longrightarrow & \mathcal{B}(\text{Aut}(M)) \\ & \searrow \delta_S & \uparrow & & \uparrow & & \nearrow \\ & & O^{p'}(\mathcal{L}) & \xrightarrow{\omega} & \mathcal{B}(O^{p'}(\pi_{\mathcal{L}})) & & \end{array}$$

Moreover, by Proposition 2.8 and Theorem 2.9, the projection $\pi: \mathcal{L} \longrightarrow \mathcal{F}$ induces an isomorphism

$$\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \cong \pi_1(|\mathcal{F}^c|) \cong \text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S).$$

Then, by the two previous Lemmas, we obtain

$$\begin{aligned} H^*(|\mathcal{L}|, M) &\cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})} \\ &\cong \left(\varprojlim_{O^{p'}(\mathcal{F})^c} H^*(-, M) \right)^{\text{Aut}_{\mathcal{F}}(S)/\text{Aut}_{O^{p'}(\mathcal{F})}(S)} \\ &\cong H^*(\mathcal{F}^c, M). \end{aligned}$$

For the second isomorphism, we have to be careful with respect to the action of $\pi_{\mathcal{L}}$ on the left side of the isomorphism and $\text{Aut}_{\mathcal{F}}(S)$ on the right side. In fact here, by Definition 2.10 of \mathcal{F}^c -stable elements, we can see it on the chain level. The map $\delta_S^*: H^*(|O^{p'}(\mathcal{L})|, M) \longrightarrow H^*(S, M)$, induced by $\delta_S: BS \longrightarrow |O^{p'}(\mathcal{L})|$, gives on the chain level,

$$\begin{aligned} \text{Hom}_{\mathbb{Z}_{(p)}[S]} \left(C_* \left(\widetilde{|O^{p'}(\mathcal{L})|} \right), M \right) &\longrightarrow \text{Hom}_{\mathbb{Z}_{(p)}[\pi_{O^{p'}(\mathcal{L})}]} (C_* (|\mathcal{E}(S)|), M) \\ f &\longmapsto f|_{C_* (|\mathcal{E}(S)|)} \end{aligned}$$

where $\mathcal{E}(S)$ is defined as the category with set of object S and for each $(s, s') \in S$, $\text{Mor}_{\mathcal{E}(S)}(s, s') = \{\varphi_{s, s'}\}$ (in particular $|\mathcal{E}(S)|$ is a universal covering space of BS). Then, for $\varphi \in \text{Aut}_S(\mathcal{F})$, if we choose a lift $\tilde{\varphi} \in \text{Aut}_{\mathcal{L}}(S)$, φ acts on the left side by

$$f \longmapsto \omega(\tilde{\varphi}^{-1})f\omega(\tilde{\varphi}),$$

and on the right side by,

$$f \longmapsto \omega(\tilde{\varphi})^{-1}f \circ \varphi^*.$$

Finally, the action of φ on $\mathcal{E}(S)$ corresponds to the action of $\omega(\tilde{\varphi})$ on $|\mathcal{E}(S)|$ (indeed, a lift of $\omega(\tilde{\varphi})$ in $\widetilde{|O^{p'}(\mathcal{L})|}$ joins every vertex $s \in S$ of $|\mathcal{E}(S)|$ to the vertex $\varphi(s)$ and similarly for higher simplices). Hence, the two actions coincide. \square

Corollary 4.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module. If the action of $\pi_{\mathcal{L}}$ on M factors through a p' -group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. By Theorem 4.3, it is enough to prove that δ_S induces an isomorphism

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M).$$

But, as the action on M factor through a p' -group, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$ acts trivially on M and Theorem 1.1 gives the wanted isomorphism. \square

We already know, from a previous article ([**Mo1**, Theorem 4.3]) that, if M is a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and the action of $\pi_{\mathcal{L}}$ on M factor through a p -group, then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$$

(it is a direct corollary of [**Mo1**, Theorem 4.3] because, any action of a p -group on an abelian p -group is nilpotent). Hence, with the same arguments, we get another corollary of Theorem 4.3.

Corollary 4.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and M be a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module. If the action of $\pi_{\mathcal{L}}$ on M factors through an extension of a normal p -group by a p' -group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

5. Realizable fusion systems and actions factoring through a p -solvable group

Consider here a finite group G , S a Sylow p -subgroup of G and let $(S, \mathcal{F}, \mathcal{L})$ be the associated p -local finite group with $\mathcal{L} = \mathcal{L}_S^c(G)$. Set $\mathcal{T} = \mathcal{T}_S^c(G)$ be the centric transporter category of G , $\mathcal{L}^q = \mathcal{L}_S^q(G)$ be the quasicentric linking system associated to G and $\mathcal{T}^q = \mathcal{T}_S^q(G)$ be the associated quasicentric transporter category. We also write $\pi_{\mathcal{T}} = \pi_1(|\mathcal{T}|)$.

We have a functor

$$\rho: \mathcal{T}_S(G) \longrightarrow \mathcal{B}(G)$$

which sends each object in the source to the unique object o_G in the target and sends, for every $P, Q \leq S$, $g \in T_G(P, Q)$ to $g \in G = \text{Mor}_{\mathcal{B}(G)}(o_G)$. As $|\mathcal{B}(G)| = BG$, this induces an homomorphism

$$\rho_*: \pi_{\mathcal{T}} \longrightarrow G.$$

Here for M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, with action $\varphi: \pi_{\mathcal{L}} \rightarrow \text{Aut}(M)$ we will suppose that we have the following commutative diagram for some homomorphism $\bar{\varphi}: G \rightarrow \text{Aut}(M)$.

$$\begin{array}{ccc} & \pi_{\mathcal{L}} & \\ \delta_* \nearrow & & \searrow \varphi \\ \pi_{\mathcal{T}} & & \text{Aut}(M) \\ \rho_* \searrow & & \nearrow \bar{\varphi} \\ & G & \end{array}$$

Then, we can compare the cohomology of $|\mathcal{L}|$ and the cohomology of G when the action factors through a p -solvable group. The main ingredients that we will use are p -local subgroups of index a power of p or prime to p .

The following lemma allows us to compare $H^*(|\mathcal{L}|, M)$ and $H^*(|\mathcal{T}|, M)$.

Lemma 5.1. *Let G be a finite group and $(S, \mathcal{F}, \mathcal{L})$ be an associated p -local finite group. Let $\mathcal{T} = \mathcal{T}_S^{\text{Ob}(\mathcal{L})}(G) \subseteq \mathcal{T}^q$ be the transporter category associated to G with set of objects $\text{Ob}(\mathcal{L})$. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module, then the canonical functor $\delta: \mathcal{T} \rightarrow \mathcal{L}$ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(|\mathcal{L}|, M)$.*

Proof. This is a consequence of [BLO1, Lemma 1.3] with $\mathcal{C} = \mathcal{T}$, $\mathcal{C}' = \mathcal{L}$ and the functor $T: \mathcal{L}^{\text{op}} \rightarrow \mathbb{Z}_{(p)}\text{-Mod}$ which sends each object to M , and each morphism to its

action on M . Then δ induces a natural isomorphism $\varprojlim_{\mathcal{T}}^*(M) \cong \varprojlim_{\mathcal{L}}^*(M)$. Then

$$H^*(|\mathcal{T}|, M) = \varprojlim_{\mathcal{T}}^*(M) \cong \varprojlim_{\mathcal{L}}^*(M) = H^*(|\mathcal{L}|, M)$$

Where the first and last equality is just an interpretation in terms of functor cohomology and can be found in [LR, Proposition 3.9]. \square

Theorem 5.2. *Let G be a finite group, S a Sylow p -subgroup of G , $\mathcal{L} = \mathcal{L}_S^c(G)$ and $\mathcal{T} = \mathcal{T}_S^c(G)$. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and assume that we have the following commutative diagram.*

$$\begin{array}{ccc} & \pi_{\mathcal{L}} & \\ \delta_* \nearrow & & \searrow \varphi \\ \pi_{\mathcal{T}} & & \text{Aut}(M) \\ \rho_* \searrow & & \nearrow \bar{\varphi} \\ & G & \end{array}$$

If ρ_* is surjective and $\Gamma = \text{Im}(\varphi) = \text{Im}(\bar{\varphi})$ is p -solvable, then δ and ρ induce natural isomorphisms

$$H^*(|\mathcal{L}|, M) \cong H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

Proof. By Lemma 5.1, we just have to show that ρ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(G, M)$. We prove this by induction on the minimal number n of extensions by p -groups or p' -groups we need to obtain Γ .

If $n = 0$, $\Gamma = 1$ and the action of $\pi_{\mathcal{T}}$ on M is trivial, then it follows from [OV1, Proposition 4.5]. Assume that, if Γ is obtained by n extensions, the result is true and suppose that Γ is obtained with $n + 1$ extensions. Consider then the last one

$$0 \rightarrow \Gamma_n \rightarrow \Gamma \rightarrow Q \rightarrow 0.$$

Denote $H = \bar{\varphi}_*^{-1}(\Gamma_n)$. Thus $(T, \mathcal{F}_H, \mathcal{L}_H) = (S \cap H, \mathcal{F}_{S \cap H}(H), \mathcal{L}_{S \cap H}^c(H))$ is a p -local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p or prime to p .

If Q is a p' -group. In that case, $(T, \mathcal{F}_H, \mathcal{L}_H)$ is a p -local finite subgroup of index prime to p (defined in 2.6). Then $\text{Ob}(\mathcal{F}^c) = \text{Ob}(\mathcal{F}_H^c)$, $\mathcal{T}_H = \mathcal{T}_{S \cap H}^c(H) \subset \mathcal{T}$ and, by [OV1, Proposition 4.1(d)], this inclusion of categories induces, up to homotopy, a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. We then have the following commutative diagram with exact rows (here, \twoheadrightarrow means onto)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{\mathcal{T}_H} & \longrightarrow & \pi_{\mathcal{T}} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\ 0 & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q \longrightarrow 0 \end{array}$$

and the following fibration sequences

$$|\mathcal{T}_H| \twoheadrightarrow |\mathcal{T}| \twoheadrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

If Q is a p -group. In that case, we have to be more careful on the collection of subgroups of S we are working with. As in the case when Q is a p' -group we want to apply [OV1, Proposition 4.1(d)]. This forces us to use the following collection. Let

$$\mathcal{H} = \{P \in \text{Ob}(\mathcal{F}^q) \mid P \cap T \in \text{Ob}(\mathcal{F}_H^q)\} .$$

Since $H \trianglelefteq G$, no element of $T = S \cap H$ is G -conjugate to any element of $S \setminus T$. Thus, by [5a2, Lemma 3.5], for every $P \in \text{Ob}(\mathcal{F}^{cr})$, $P \cap T \in \text{Ob}(\mathcal{F}_H^c) \subseteq \text{Ob}(\mathcal{F}_H^q)$. In particular, $\text{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \text{Ob}(\mathcal{F}^q)$. Hence if $\mathcal{L}^{\mathcal{H}} \subseteq \mathcal{L}^q$ is the full subcategory of \mathcal{L}^q with set of objects \mathcal{H} , by Proposition 2.3(b), $\mathcal{L}^{\mathcal{H}}$ defines a linking system associated to \mathcal{F} . On the level of transporter systems, the inclusions $\mathcal{T} \subseteq \mathcal{T}^q \supseteq \mathcal{T}^{\mathcal{H}}$ induce natural isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Indeed, we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{T} & \longrightarrow & \mathcal{T}^q & \longleftarrow & \mathcal{T}^{\mathcal{H}} \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \mathcal{L} & \longrightarrow & \mathcal{L}^q & \longleftarrow & \mathcal{L}^{\mathcal{H}} \end{array}$$

The vertical arrows induce isomorphisms in cohomology by Lemma 5.1 and the lower horizontal one induces an isomorphism since, by Theorem 2.5, the inclusions of categories $\mathcal{L} \subseteq \mathcal{L}^q \supseteq \mathcal{L}^{\mathcal{H}}$ induces $|\mathcal{L}| \simeq |\mathcal{L}^q| \simeq |\mathcal{L}^{\mathcal{H}}|$. Hence the upper arrows induce isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Finally, By Proposition 2.7, $P \in \text{Ob}(\mathcal{F}_H^q)$ if and only if $P \leq T$ and $P \in \mathcal{H}$. In particular, $\mathcal{T}_H^q \subseteq \mathcal{T}^{\mathcal{H}}$. Thus we can assume for this part that $\mathcal{T} = \mathcal{T}^{\mathcal{H}}$ and $\mathcal{T}_H = \mathcal{T}_H^q$.

We have $\mathcal{T}_H \subseteq \mathcal{T}$ is a transporter system associated to \mathcal{F}_H and, by definition of \mathcal{H} , the hypotheses of [OV1, Proposition 4.1(d)], are satisfied. Thus this inclusion induces a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. Therefore, We have the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{\mathcal{T}_H} & \longrightarrow & \pi_{\mathcal{T}} & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & \searrow & \parallel \\ 0 & \longrightarrow & \Gamma_n & \longrightarrow & \Gamma & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & \nearrow & \downarrow & \nearrow & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q \longrightarrow 0 \end{array}$$

and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

Hence, in both cases, we have the following Serre spectral sequences

$$H^{s+t}(|\mathcal{T}|, M) \Leftarrow H^s(Q, H^t(|\mathcal{T}_H|, M)),$$

$$H^{s+t}(G, M) \Leftarrow H^s(Q, H^t(H, M)),$$

and ρ induces a morphism ρ^* of spectral sequences between these two. By induction, ρ^* gives an isomorphism on the E_2 page and then induces an isomorphism of spectral sequences. In particular, ρ induces a natural isomorphism

$$H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

The result follows by induction. \square

Assume the hypotheses of Theorem 5.2. It remains to compare $H^*(G, M)$ with the \mathcal{F}^c -stable elements. This is also not obvious and they are not isomorphic in all cases. On one hand, by Cartan-Eilenberg Theorem, we have $H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. On the other hand, we have $H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{L}_S^c(G)} H^*(-, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. Hence, it remains to compare $\varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$ and $\varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. For that we can use a result of Grodal [Gr].

Definition 5.3. Let G be a finite group, $S \in \text{Syl}_p(G)$ and M be a $\mathbb{Z}_{(p)}[G]$ -module. Let K be the kernel of $G \rightarrow \text{Aut}(M)$. A subgroup $P \leq S$ is called M -essential if

- (i) the poset of non trivial p -subgroup of $N_G(P)/P$ is empty or disconnected,
- (ii) $Z(P) \cap K \in \text{Syl}_p(C_G(P) \cap K)$,
- (iii) $O_p(N_G(P)/(P(C_G(P) \cap K))) = 1$.

The property (ii) looks like the definition of p -centric and (iii) looks like the definition of \mathcal{F} -radical. For the property (i), if P is \mathcal{F} -centric and fully normalized in \mathcal{F} , it is equivalent to $P = S$ or P is \mathcal{F} -essential ([AKO, Definition I.3.2]).

Theorem 5.4 ([Gr, Corollary 10.4]). *Let G be a finite group, S a Sylow p -subgroup of G and M a $\mathbb{Z}_{(p)}[G]$ -module.*

Let \mathcal{H} be a family of subgroup of S containing S and all the subgroups which are M -essential.

Then, the inclusion of S in G induce a natural isomorphism,

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^{\mathcal{H}}(G)} H^*(-, M).$$

From this Theorem and Theorem 5.2, we get the following Corollary.

Corollary 5.5. *Let G be a finite group, S a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module and assume that we have*

the following commutative diagram,

$$\begin{array}{ccc}
 & \pi_{\mathcal{L}} & \\
 \delta_* \nearrow & & \searrow \varphi \\
 \pi_{\mathcal{T}} & & \text{Aut}(M) \\
 \rho_* \searrow & & \nearrow \overline{\varphi} \\
 & G &
 \end{array}$$

that ρ_* is surjective and that $\Gamma := \text{Im}(\varphi) = \text{Im}(\overline{\varphi})$. If Γ is p -solvable and all the M -essential subgroups of S are p -centric, then δ and ρ induce natural isomorphisms,

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

We also conjecture that it can be generalized to any abstract p -local finite group and any $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$ -module with a p -solvable action.

Conjecture 5.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$ -module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p -solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

Corollary 4.5 and Corollary 5.5 give good evidence for Conjecture 5.6 to be true.

The next section, which is a bit technical, is dedicated to give an example of Conjecture 5.6 where Corollary 5.5 doesn't apply (see Remark 6.7).

6. The p -local structure of wreath products by C_p : an example for Conjecture 5.6

Let G_0 be a finite group, S_0 a Sylow p -subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. We are interested in the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ and the associated p -local finite group $(S, \mathcal{F}, \mathcal{L})$. By [CL, Theorem 5.2 and Remark 5.3], we have that $|\mathcal{L}| \simeq |\mathcal{L}_0| \wr BC_p := |\mathcal{L}_0|^p \times_{C_p} EC_p$ and an extension $(\pi_{\mathcal{L}_0})^p \rightarrow \pi_{\mathcal{L}} \rightarrow C_p$. In addition we have a section $C_p \rightarrow \pi_{\mathcal{L}}$ coming from $* \wr BC_p \rightarrow |\mathcal{L}_0| \wr BC_p$ and thus $\pi_{\mathcal{L}} = \pi_{\mathcal{L}_0} \wr C_p$.

We first give a lemma on strongly p -embedded subgroups. For a finite group G , a subgroup $H < G$ is *strongly p -embedded*, if $p \mid |H|$ and for each $x \in G \setminus H$, $H \cap xHx^{-1}$ has order prime to p .

Lemma 6.1. *Let G be a finite group, $G_0 \leq G$ a subgroup of index a power of p . If G contains a strongly p -embedded subgroup and $p \mid |G_0|$, then G_0 contains a strongly p -embedded subgroup.*

Proof. Let H be a strongly p -embedded subgroup of G . By [AKO, Proposition A.7], H contains a Sylow p -subgroup of G so, up to conjugacy, we can choose H such that H contains a Sylow p -subgroup of G_0 . Hence $G_0 \cap H$ contains a Sylow p -subgroup of G_0 and $p \mid |G_0 \cap H|$. We will show that $G_0 \cap H$ is a strongly p -embedded subgroup of G_0 .

As $[G : H]$ is prime to p and $[G : G_0]$ is a power of p , $G_0 \cap H$ is a proper subgroup of G_0 .

It remains to show that, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p . But $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1} \leq H \cap xHx^{-1}$, thus, as H is a strongly p -embedded subgroup of G , this last subgroup has order prime to p for every $x \in G \setminus H$. In particular, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p and $G_0 \cap H$ is a strongly p -embedded subgroup of G_0 . \square

We give also a lemma on \mathcal{F}_1 -essential subgroups for $\mathcal{F}_1 \subseteq \mathcal{F}$ a subsystem of index a power of p . A proper subgroup $P < S$ is \mathcal{F} -essential if P is \mathcal{F} -centric and fully normalized in \mathcal{F} , and if $\text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup.

Lemma 6.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ a p -local subgroup of index a power of p . If $P < S_1$ is \mathcal{F} -essential, then P is \mathcal{F}_1 -conjugate to an \mathcal{F}_1 -essential subgroup and P is \mathcal{F}_1 -essential if and only if P is fully normalized in \mathcal{F}_1 .*

Proof. Let $P < S_1$ be an \mathcal{F} -essential subgroup. Since \mathcal{F}_1 is saturated, P is \mathcal{F}_1 -conjugate to a subgroup of S_1 fully normalized in \mathcal{F}_1 . If P is \mathcal{F}_1 -essential, it is in particular fully normalized in \mathcal{F}_1 . Thus, it remains to prove that if P is fully normalized in \mathcal{F}_1 , then P is \mathcal{F}_1 -essential. For the remaining, we assume that P is fully normalized in \mathcal{F}_1 and \mathcal{F} -essential.

P is \mathcal{F}_1 -centric: As P is \mathcal{F} -centric, $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. In particular, for all $Q \in P^{\mathcal{F}_1} \subseteq P^{\mathcal{F}}$, $C_{S_1}(Q) = Z(Q)$ and P is \mathcal{F}_1 -centric.

$\text{Out}_{\mathcal{F}_1}(P)$ contains a strongly p -embedded subgroup: Since P is \mathcal{F} -essential, the group $\text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup. As \mathcal{F}_1 is a subsystem of \mathcal{F} of index a power of p , $\text{Out}_{\mathcal{F}_1}(P)$ is a subgroup of $\text{Out}_{\mathcal{F}}(P)$ of index a power of p . Moreover, as P is a proper subgroup of S_1 , $P < N_{S_1}(P)$ and, as P is \mathcal{F}_1 -centric, every element of $N_{S_1}(P) \setminus Z(P)$ induces a non trivial element in $\text{Out}_{\mathcal{F}_1}(P)$. Hence $p \mid |\text{Out}_{\mathcal{F}_1}(P)|$ and, by Lemma 6.1, $\text{Out}_{\mathcal{F}_1}(P)$ contains a strongly p -embedded subgroup. \square

We can easily describe the essential subgroups of a product of fusion systems.

Lemma 6.3. *Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be p -local finite groups and set $S = S_1 \times S_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. The \mathcal{F} -essential subgroups of S are of the form $Q_1 \times S_2$ with $Q_1 < S_1$ \mathcal{F}_1 -essential or $S_1 \times Q_2$ with $Q_2 > S_2$ \mathcal{F}_2 -essential.*

Proof. Let $P \leq S$ be a \mathcal{F} -essential subgroup. By [AKO, Proposition I.3.3], P is \mathcal{F} -centric and \mathcal{F} -radical. Thus, by [AOV, Lemma 3.1], $P = P_1 \times P_2$ with $P_i \leq S_i$ and P_i \mathcal{F}_i -centric.

Remark also that, if we have two groups G_1 and G_2 such that p divide $|G_1|$ and $|G_2|$ then $G_1 \times G_2$ cannot contain a strongly p -embedded subgroup. To see that let S_i be a Sylow p -subgroup of G_i and set $H = \langle x \in G \mid x(S_1 \times S_2)x^{-1} \cap S_1 \times S_2 \neq 1 \rangle$. H contains $G_1 \times \{0\}$ and $\{0\} \times G_2$ so that $H = G$. Thus, by [AKO, Proposition A.7], this implies that G has no strongly p -embedded subgroups.

We also have that $\text{Out}_{\mathcal{F}}(P) = \text{Out}_{\mathcal{F}_1}(P_1) \times \text{Out}_{\mathcal{F}_2}(P_2)$. Hence, the only possibility for P to be \mathcal{F} -essential is that $P_1 = S_1$ and P_2 is \mathcal{F}_2 -essential or the contrary. \square

Let G_0 be a finite group, S_0 a Sylow p -subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. We consider the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$

and the associated p -local finite group $(S, \mathcal{F}, \mathcal{L})$. Here, for the direct computation, we will take the notation of Alperin and Fong [AF]: an element of G will be represented by permutation matrix corresponding to the powers of $(1, 2, \dots, p)$ with entries in G_0 and the composition will follow the matrix product with the composition in G_0 . Denote by $e \in G$ the element

$$e \otimes P_{(1,2,\dots,p)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & e \\ e & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & e & 0 \end{pmatrix}$$

where e is the trivial element of G_0 . Here, we are interested in the \mathcal{F} -essential subgroups.

Lemma 6.4. *Let $P \leq S$ be an \mathcal{F} -essential subgroup.*

- (E₁) *If $P \leq S_0^p$, then either $P = S_0^p$ and $N_G(P) = N_{G_0}(S_0) \wr C_p$ or P is \mathcal{F}_0^p -essential and $N_G(P) = N_{G_0^p}(P)$.*
- (E₂) *If $P \not\leq S_0^p$, then $P \cong_{\mathcal{F}} Q \wr C_p$ where Q is \mathcal{F}_0 -essential and we have $N_G(P)/P \cong N_{G_0}(Q)/Q$ through the diagonal map $G_0 \hookrightarrow G_0^p$.*

Proof. Let $P \leq S$ be an \mathcal{F} -essential subgroup.

Assume first that $P \leq S_0^p$. If $P = S_0^p$ a direct calculation gives $N_G(P) = N_{G_0}(S_0) \wr C_p$. Else, by Lemma 6.2, we know that P is \mathcal{F}_0^p -conjugate to an \mathcal{F}_0^p -essential subgroup $Q \leq S_0^p$. By Lemma 6.3 we have $N_G(Q) \leq G_0^p$ and, in particular, $N_G(Q) = N_{G_0^p}(Q)$. Thus, since P is \mathcal{F}_0^p -conjugate to Q , we also have $N_G(P) = N_{G_0^p}(P)$ and, since P is fully normalized in \mathcal{F} , it is fully normalized in \mathcal{F}_0^p . Hence, by Lemma 6.2, P is \mathcal{F}_0 -essential.

Secondly, assume that $P \not\leq S_0^p$. As all choices of a splitting $C_p \rightarrow G$ are conjugate in G , we can assume that $P = \langle P_0, x \rangle$ where $P_0 = P \cap S_0^p$ and $x = ((x_1, x_2, \dots, x_p), c)$ is such that $x^p \in P_0$. Up to conjugation in $S_0 \wr C_p$ we can assume that x is of the form $((a, 1, 1, \dots, 1), c)$ where $a \in N_{S_0}(Q)$ where Q is the projection of P_0 on the first factor. If we write $P_0^{(i)}$ the projection of P_0 on its i th factor, as x normalizes P_0 , we have that $P_0^{(i)} = P_0^{(j)}$ for all i, j and then $P_0 \leq (P_0^{(1)})^p = Q^p$.

Notice also that $N_G(P) = \langle N_{G_0^p}(P), x \rangle$. If $g = (g_1, \dots, g_p) \in N_{G_0^p}(P)$, as g normalizes $P \cap G_0^p = P_0$, we have, for all i , $g_i \in N_{G_0}(Q)$. Moreover, if we denote $h = (h_1, \dots, h_p) = gxg^{-1}x^{-1} \in P_0$, we have, for all i , $g_i h_i = g_{i-1}$ (with $g_0 = g_p$). Therefore, there is $h' \in Q^p$ such that $g = (g_1, g_1, \dots, g_1) \cdot h' \in \langle N_{G_0}(Q) \otimes \text{Id}, Q^p \rangle \leq N_G(Q^p)$. Hence, every automorphism $c_g \in \text{Aut}_{\mathcal{F}}(P)$ can be extended to an automorphism of $\langle Q^p, x \rangle$. As P is \mathcal{F} essential, by [AKO, Proposition I.3.3], $P = \langle Q^p, x \rangle$. Now, $x^p \in Q^p$ implies that $a \in Q$ so $P = \langle Q^p, x \rangle = \langle Q^p, c \rangle = Q \wr C_p$.

Finally, direct computations give that

$$C_G(P) \cong C_{G_0}(Q) \otimes \text{Id} = \left\{ \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g \end{pmatrix} ; g \in C_{G_0}(Q) \right\}$$

and

$$N_G(P)/P \cong N_{G_0}(Q)/Q \otimes \text{Id} \cong N_{G_0}(Q)/Q.$$

In particular, as P is p -centric, Q is G_0 -centric. Moreover, as $N_G(P)/P = \text{Out}_{\mathcal{F}}(P)$ contains a strongly p -embedded subgroup, $\text{Out}_{\mathcal{F}_0}(Q) = N_{G_0}(Q)/Q$ does as well. Up to conjugacy, we can also assume that Q is fully normalized in \mathcal{F}_0 and thus Q is \mathcal{F}_0 -essential. \square

Let us now look at some cohomological results. Recall that for a group G , a subgroup $H \leq G$, and M a $\mathbb{F}_p[H]$ -module, we define the *induced* and *coinduced* $\mathbb{F}_p[G]$ -module by,

$$\text{Ind}_H^G(M) = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M \quad \text{coInd}_H^G(M) = \text{Hom}_{\mathbb{F}_p[H]}(\mathbb{F}_p[G], M).$$

Recall also that, when the index of H in G is finite, these two $\mathbb{F}_p[G]$ -modules are isomorphic (by [We, Lemma 6.3.4]).

Lemma 6.5. *Let X be a CW complex and denote by G its fundamental group. If X_0 is a covering space of X with fundamental group $G_0 \triangleleft G$ of finite index, then, for every $\mathbb{F}_p[G_0]$ -module M , we have a natural isomorphism of $\mathbb{F}_p[G/G_0]$ -modules,*

$$H^*(X_0, \text{Ind}_{G_0}^G(M)) \cong H^*(X_0, M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

Where, on the right side, G/G_0 is only acting by translation on $\mathbb{F}_p[G/G_0]$.

Proof. This can be easily seen on the chain level. Let \tilde{X} be the universal covering space of X . As $\mathbb{F}_p[G/G_0]$ -modules, we have the following

$$\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \text{Ind}_{G_0}^G(M)) = \bigoplus_{g \in [G/G_0]} \text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), g.M)$$

where the action of G/G_0 is permuting the terms in the sum. But, each terms in the sum is isomorphic, as (trivial) $\mathbb{F}_p[G/G_0]$ -modules, to $\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M)$. Thus

$$\text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \text{Ind}_{G_0}^G(M)) \cong \text{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

This induces the wanted isomorphism in cohomology. \square

Proposition 6.6. *Let G_0 be a finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p -local finite group. Consider $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ a Sylow p -subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p -local finite group. Let M be a $\mathbb{F}_p[\pi_{\mathcal{L}_0}]$ -module.*

If δ_{S_0} induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M),$$

and

$$H^*(|\mathcal{L}_0|^p, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}_0}}(M^{\otimes p})) \cong H^*((\mathcal{F}_0^p)^c, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}_0}}(M^{\otimes p}))$$

then δ_S induces a natural isomorphism

$$H^*(|\mathcal{L}|, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})) \cong H^*(\mathcal{F}^c, \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})).$$

Proof. Write $N = \text{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} (M^{\otimes p})$ and, for $i \in \{1, 2\}$, denote by $H^*(\mathcal{F}^{E_i}, N)$ the stable elements of $H^*(S, N)$ under the full subcategory of \mathcal{F} with objects S and all the subgroups of S of type (E_i) defined in Lemma 6.4.

By the Mackey Formula,

$$\text{Res}_{Q \wr C_p}^{\pi_{\mathcal{L}}} \text{Ind}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} = \text{Ind}_{Q^p}^{Q \wr C_p} \text{Res}_{Q^p}^{\pi_{\mathcal{L}_0}^p}.$$

Thus by Shapiro's Lemma (see for example [Ev, Proposition 4.1.3]) and the Kunnetth Formula, for every $P = Q \wr C_p$ of type (E_2) , we have a natural isomorphism $H^*(Q \wr C_p, N) \cong H^*(Q^p, M^{\otimes p}) \cong H^*(Q, M)^{\otimes p}$ and, by the computation of normalizers in Lemma 6.4,

$$H^*(Q \wr C_p, N)^{\text{Aut}_{\mathcal{F}}(Q \wr C_p)} \cong (H^*(Q, M)^{\text{Aut}_{\mathcal{F}_0}(Q)})^{\otimes p}.$$

Hence, applying this to all the subgroups of type (E_2) and, by naturality of the Shapiro isomorphisms, we have that,

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p}.$$

On the other hand, by [CL, Theorem 5.2 and Remark 5.3], $|\mathcal{L}_0|^p$ has the homotopy type of a covering space of $|\mathcal{L}|$ with covering group C_p . Then, if we denote by X the universal covering space of $|\mathcal{L}|$ (which is also the universal covering space of $|\mathcal{L}_0|^p$), we have the following isomorphism on the chain level (because Res and coInd are adjoint functors)

$$\text{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}_0}^p]}(C_*(X), M^{\otimes p}) \cong \text{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]}(C_*(X), N)$$

which is analogue to the Shapiro isomorphism (see [Ev, Proposition 4.1.3]). By the Kunnetth Formula, it gives us the following isomorphism on cohomology

$$H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N)$$

and give the following commutative diagram

$$\begin{array}{ccc} H^*(S_0, M)^{\otimes p} & \xrightarrow{\cong} & H^*(S, N) \\ (\delta_{S_0})^* \downarrow & & \downarrow \delta_S^* \\ H^*(|\mathcal{L}_0|, M)^{\otimes p} & \xrightarrow{\cong} & H^*(|\mathcal{L}|, N) \end{array}$$

Thus δ_S induces an isomorphism

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p} \cong H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N).$$

Secondly, by factoring the Shapiro isomorphism (see [Ev, Proposition 4.1.3]), the inclusion of S_0^p in S induces an injection $H^*(S, N) \hookrightarrow H^*(S_0^p, N)$. Hence

$$H^*(\mathcal{F}^{E_1}, N) \cong H^*((\mathcal{F}_0^p)^c, N)^{C_p} \leq H^*(S_0^p, N).$$

By assumption, $\delta_{S_0^p}$ induces an isomorphism

$$H^*((\mathcal{F}_0^p)^c, N) \cong H^*(|\mathcal{L}_0|^p, N).$$

Moreover, by Lemma 6.5, this last term is isomorphic to $H^*(|\mathcal{L}_0|^p, M^{\otimes p}) \otimes \mathbb{F}_p[C_p]$ and, in particular, it is a projective $\mathbb{F}_p[C_p]$ -module.

Consider now the Serre spectral sequence associated to the fibration sequence

$$|\mathcal{L}_0|^p \longrightarrow |\mathcal{L}| \longrightarrow BC_p$$

with coefficients in N . The E_2 page is the following,

$$E_2^{s,t} = H^s(C_p, H^t(|\mathcal{L}_0|^p, N))$$

and, by projectivity of $H^t(|\mathcal{L}_0|^p, N)$, the E_2 page is concentrated in the 0th column. Hence, we have that, $H^*(|\mathcal{L}_0|^p, N)^{C_p} = E_2^{0,*} \cong H^*(|\mathcal{L}|, N)$.

In conclusion,

$$H^*(\mathcal{F}^c, N) = H^*(\mathcal{F}^{E_1}, N) \cap H^*(\mathcal{F}^{E_2}, N) \cong H^*(|\mathcal{L}|, N)$$

and the theorem follows. \square

This proposition is a bit technical but we will use it in a specific case. Consider $p = 5$, the group $G_0 = GL_{20}(F_2)$, the wreath product $G = G_0 \wr C_5$ and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ and $(S, \mathcal{F}, \mathcal{L})$ the associated 5-local finite groups. By [Ru, Theorem 6.3], we know that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ admits a 5-local subgroup of index 4 which is exotic $(S_e, \mathcal{F}_e, \mathcal{L}_e)$ and that we have a fibration sequence

$$|\mathcal{L}_e| \longrightarrow |\mathcal{L}_0| \longrightarrow BC_4.$$

In particular, we have $\pi_{\mathcal{L}}/\pi_{\mathcal{L}_e}^5 = C_4 \wr C_5$ and we can be interested in comparing $H^*(|\mathcal{L}|, N)$ and $H^*(\mathcal{F}^c, N)$ for

$$N = \mathbb{F}_5[C_4 \wr C_5] = \text{Ind}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5}) \cong \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})$$

(the action factors through a finite group) with $M = \mathbb{F}_5[C_4]$.

By Corollary 4.4, we have that δ_{S_0} and $\delta_{S_0^p}$ induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M) \text{ and}$$

$$H^*(|\mathcal{L}_0|^5, \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})) \cong H^*((\mathcal{F}_0^5)^c, \text{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}} (M^{\otimes 5})),$$

(for the second isomorphism, notice that $|\mathcal{L}_0|^5$ has the homotopy type of a linking system associated to \mathcal{F}_0^5 by [CL, Proposition 2.17]). Hence, all the hypothesis of Proposition 6.6 are satisfied and

$$H^*(|\mathcal{L}|, N) \cong H^*(\mathcal{F}^c, N).$$

Remark 6.7. This gives us an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the stable elements when the action factors through a p -solvable group which cannot be recovered by a previous result. Notice that, even if the fusion system \mathcal{F} is realizable, as \mathcal{F}_e is exotic, we cannot find a group G with $S \in \text{Syl}_p(G)$ such that G acts on M in the same way as asked in Section 5. This example gives us some additional evidence for Conjecture 5.6.

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