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Quantitative inequality for the eigenvalue of a Schrödinger operator in the ball

Idriss Mazari*

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Abstract

The aim of this article is to prove a quantitative inequality for the first eigenvalue of a Schrödinger operator in the ball. More precisely, we optimize the first eigenvalue $\lambda(V)$ of the operator $\mathcal{L}_v := -\Delta + V$ with Dirichlet boundary conditions with respect to the potential V , under L^1 and L^∞ constraints on V . The solution has been known to be the characteristic function of a centered ball, but this article aims at proving a sharp growth rate of the following form: if V^* is a minimizer, then $\lambda(V) - \lambda(V^*) \geq C\|V - V^*\|_{L^1(\Omega)}^2$ for some $C > 0$.

The proof relies on two notions of derivatives for shape optimization: parametric derivatives and shape derivatives. We use parametric derivatives to handle radial competitors, and shape derivatives to deal with normal deformation of the ball. A dichotomy is then established to extend the result to all other potentials. We develop a new method to handle radial distributions and a comparison principle to handle second order shape derivatives at the ball. Finally, we add some remarks regarding the coercivity norm of the second order shape derivative in this context.

Keywords: Stability for eigenvalues, Schrödinger operator, mathematical biology.

AMS classification: 35J15,35Q93,47A75,49R05,49Q10 .

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*Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France (idriss.mazari@sorbonne-universite.fr)

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1 Introduction

1.1 Structure of the paper

In the first part (Section 1.2) of the introduction, we lay out the mathematical setting of this article, the main results and give the relevant definitions. In the second part of the Introduction, we give bibliographical references concerning quantitative inequalities and the biological motivations this study stems from. We then explain, in Subsection 2.1, how the proof differs from that of other quantitative inequalities.

The core of this paper is devoted to the proof of Theorem 1. In the conclusion, we give a conjecture and a final comment on a possible way to obtain an optimal exponent using parametric derivatives.

1.2 Mathematical setting

The optimization of eigenvalues of elliptic operators defined on domains with a zero-order term (i.e with a potentiel) with respect to either the domain (with a fixed potential defined on a bigger domain) or the potential (with a fixed domain) is a classical question in optimization under partial differential equations constraints. The example under scrutiny here is the operator

$$\mathcal{L}_V : u \in W_0^{2,2}(\Omega) \mapsto -\Delta u - Vu, \tag{1}$$

where Ω is a smooth domain. Under the assumption that $V \in L^\infty(\Omega)$, this operator is known to have a first, simple eigenvalue, denoted by $\lambda(V)$, and associated with an eigenfunction u_V , which solves

$$\begin{cases} -\Delta u_V - V u_V = \lambda(V) u_V & \text{in } \Omega, \\ u_V = 0 & \text{on } \partial\Omega. \\ \int_{\Omega} u_V^2 = 1. \end{cases} \quad (2)$$

Alternatively, this eigenvalue admits the following variational formulation in terms of Rayleigh quotients:

$$\lambda(V) = \inf_{u \in W_0^{1,2}(\Omega), \int_{\Omega} u^2 = 1} \left\{ \int_{\Omega} |\nabla u|^2 - \int_{\Omega} V u^2 \right\}. \quad (3)$$

We make stronger assumptions on the potential V and require that it lies in

$$\mathcal{M}(\Omega) := \left\{ V \in L^\infty(\Omega), 0 \leq V \leq 1, \frac{1}{|\Omega|} \int_{\Omega} V = V_0 \right\}, \quad (4)$$

where $V_0 > 0$ is a real parameter such that

$$V_0 < 1$$

so that $\mathcal{M}(\Omega)$ is non-empty. The optimization problem we focus on in this paper is

$$\inf_{V \in \mathcal{M}(\Omega)} \lambda(V). \quad (5)$$

This problem has drawn a lot of attention from the mathematical community over the last decades, and is quite a general one. It is particularly relevant in the context of mathematical biology, see Section 1.4.2.

1.3 Main results

1.3.1 Notations

Here and throughout, the underlying domain is $\Omega = \mathbb{B}(0; R)$.

The parameter r^* is chosen so that

$$V^* := \chi_{\mathbb{B}(0; r^*)} = \chi_{E^*} \in \mathcal{M}(\Omega).$$

The constant $c_n > 0$ is the $(n-1)$ -dimensional volume of the unit sphere in dimension n . For any $E \subset \mathbb{B}$, we define $\lambda(E) := \lambda(\chi_E)$.

1.3.2 Quantitative inequality

A classical application of Schwarz' rearrangement shows that V^* is the unique minimizer of λ in $\mathcal{M}(\Omega)$:

$$\forall V \in \mathcal{M}(\Omega), \lambda(V) \geq \lambda(V^*).$$

We refer to [32] for an introduction to the Schwarz rearrangement and to [35] for its use in this context. For the sake of completeness, we also prove this result in Annex A.

The goal of this paper is to establish the following quantitative spectral inequality:

Theorem 1. *Let $n = 2$. There exists a constant $C > 0$ such that, for any $V \in \mathcal{M}(\Omega)$, there holds*

$$\lambda(V) - \lambda(V^*) \geq C \|V - V^*\|_{L^1}^2. \quad (6)$$

We can rephrase this result in terms of Fraenkel asymmetry: as we will explain in Subsection 1.4.1, this is the natural property to expect in the context of quantitative inequalities. Indeed, if $V = \chi_E \in \mathcal{M}(\Omega)$, then

$$\|V - V^*\|_{L^1} = |E^* \Delta E|$$

where Δ stands for the symmetric difference, and, if we define the fraenkel asymmetry of E as

$$\mathcal{A}(E) := \inf_{\mathbb{B}(x;r), \chi_{\mathbb{B}(x;r)} \in \mathcal{M}(\Omega)} |E \Delta \mathbb{B}(x;r)|$$

then it follows from Theorem 1 that

$$\lambda(\chi_E) - \lambda(\chi_{E^*}) \geq C \mathcal{A}(E)^2.$$

We thus have a parametric version of a quantitative inequality with what is in fact a sharp exponent.

1.3.3 A comment on parametric and shape derivatives

The proof of Theorem 1 relies on parametric and shape derivatives, and the aim of this Section is to give possible links between the two notions and, most notably, to give a situation where this link is no longer possible. This is obviously in sharp contrast with classical shape optimization, since here we are only optimizing with respect to the potential, while it is customary to derive Faber-Krahn type inequalities, i.e to optimize with respect to the domain Ω itself, see Subsection 1.4.1.

Roughly speaking, there are two ways to tackle spectral optimization problems such as (5): the parametric approach and the shape derivative approach. By parametric approach we mean the following:

Definition 1. We define, for any $V \in \mathcal{M}(\Omega)$, the tangent cone to $\mathcal{M}(\Omega)$ at V as

$$\mathcal{T}_V := \{h : \Omega \rightarrow [-1; 1], \forall \varepsilon \leq 1, V + \varepsilon h \in \mathcal{M}(\Omega)\}$$

and define, provided it exists, the parametric derivative of λ at V in the direction $h \in \mathcal{T}_V$ as

$$\dot{\lambda}(V)[h] := \lim_{t \rightarrow 0} \frac{\lambda(V + th) - \lambda(V)}{t}.$$

In this case, the optimality condition reads

$$\forall h \in \mathcal{T}_V, \dot{\lambda}(V)[h] \geq 0.$$

In the conclusion of this article, we will explain why there holds

$$\forall h \in \mathcal{T}_{V^*}, \dot{\lambda}(V^*)[h] \geq C \|h\|_{L^1}^2.$$

Since, for any $V \in \mathcal{M}(\Omega)$, $h := V - V^* \in \mathcal{T}_{V^*}$, what we get is

$$\dot{\lambda}(V^*)[V - V^*] \geq C \|V - V^*\|_{L^1}^2$$

that is, an infinitesimal version of Estimate (6), and one might wonder whether such infinitesimal estimates might lead to global qualitative inequality of the type (6). This is customary, in the context of shape derivatives. We need to define the notion of shape derivative before going further:

Definition 2. Let $\mathcal{F} : E \mapsto \mathcal{F}(E) \in \mathbb{R}$ be a shape functional. We define

$$\mathcal{X}_1(V^*) := \{ \Phi : \mathbb{B}(0; R) \rightarrow \mathbb{R}^2, \|\Phi\|_{W^{1,\infty}} \leq 1, \forall t \in (-1; 1), \chi_{(Id+t\Phi)(\mathbb{B}^*)} \in \mathcal{M}(\Omega) \}$$

as the set of admissible perturbations at E^* . The shape derivative of first (resp. second) order of a shape function \mathcal{F} at V^* in the direction Φ is

$$\begin{aligned} \mathcal{F}'(E^*)[\Phi] &= \lim_{t \rightarrow 0} \frac{\mathcal{F}((Id + t\Phi)E^*) - \mathcal{F}(E^*)}{t} \\ (\text{resp. } \mathcal{F}''(E^*)[\Phi, \Phi] &:= \lim_{t^2 \rightarrow 0} \frac{\mathcal{F}((Id + t\Phi)E^*) - \mathcal{F}(E^*) - \mathcal{F}'(E^*)(\Phi)}{t^2}.) \end{aligned} \quad (7)$$

A customary way to derive quantitative inequality is to show that, at a given shape E , there holds

$$\mathcal{F}'(E)[\Phi] = 0, \mathcal{F}''(E)[\Phi, \Phi] > 0$$

and to lift the last inequality to a quantitative inequality of the form

$$\mathcal{F}'(E)[\Phi, \Phi] \geq C \|\Phi\|_s^2$$

where $\|\cdot\|_s$ is a suitable norm; we refer to [19] for more details but for instance one might have $\|\Phi\|_{\partial E}^2_{L^1}$ which often turns out to be the suitable exponent for a quantitative inequality. This quantitative inequality for shape deformations is usually not enough, and we refer to Section 1.4.1 for more details and bibliographical references.

1.3.4 A remark on the proof

The main innovation of this paper is the proof of Theorem 1 which, although it uses shape derivatives as is customary while proving quantitative inequalities, see Subsection 1.4.1, relies heavily on parametric derivatives. This is allowed by the fact that we are working with a potential defined on the interior of the domain.

Furthermore, we also prove that, unlike classical shape optimization, our coercivity norm for the second order shape derivative is the L^2 norm.

1.4 Bibliographical references

1.4.1 Quantitative spectral inequalities

Spectral deficit for Faber-Krahn type inequalities: Quantitative spectral inequalities have received a lot of attention for a few decades, and are usually set in a context which is more general than the one introduced here. The main goal of such inequalities were to derive quantitative versions of the Faber-Krahn inequality: for a given parameter $\beta \in (0; +\infty]$ and a bounded domain $\Omega \subset \mathbb{R}^n$, consider the first eigenvalue $\eta_\beta(\Omega)$ of the Laplacian with Robin boundary conditions:

$$\begin{cases} -\Delta u_{\beta,\Omega} = \eta_\beta(\Omega) u_{\beta,\Omega} \text{ in } \Omega, \\ \frac{\partial u_{\beta,\Omega}}{\partial \nu} + \beta u_{\beta,\Omega} = 0 \text{ on } \partial\Omega, \end{cases} \quad (8)$$

with the convention that $\eta_\infty(\Omega)$ is the first Dirichlet eigenvalue of the Laplace operator. Although it has been known since the independent works of Faber [22] and Krahn [34] that, whenever Ω^* is a ball with the same volume as Ω , there holds

$$\eta_\infty(\Omega) \geq \eta_\infty(\Omega^*),$$

the question of providing a sharp lower bound for the so-called *spectral deficit*

$$R_\infty(\Omega) = \eta_\infty(\Omega) - \eta_\infty(\Omega^*)$$

remained largely open until Nadriashvili and Hansen [27] and Melas, [40], using Bonnesen type inequalities, obtained a lower bound on the spectral deficit involving quantities related to the geometry of domain Ω through the inradius. In a later work, Brasco, De Philippis and Velichkov, [11], the sharp version of the quantitative inequality, namely:

$$R_\infty(\Omega) \geq C\mathcal{A}(\Omega)^2. \quad (9)$$

Their proof relies uses as a first step a second order shape derivative argument, a series of reduction to small asymmetry regime and, finally, a quite delicate selection principle. We comment in the next paragraph on the role of second order shape derivative for generic quantitative inequalities and the part it plays in our proof.

For a survey of the history and proofs of quantitative Faber-Krahn inequalities, we refer to the survey [10] and the references therein.

For quantitative forms of spectral inequalities with general Robin boundary conditions, the Bossel-Daners inequality, first derived by Bossel in dimension 2 in [9] and later extended by Daners in all dimensions in [21] reads:

$$R_\beta(\Omega) := \eta_\beta(\Omega) - \eta_\beta(\mathbb{B}) \geq 0, \beta > 0,$$

and a quantitative version of the Inequality was proved by Bucur, Ferone, Nitsch and Trombetti in [12]:

$$R_\beta(\Omega) \geq C\mathcal{A}(\Omega)^2.$$

Their method for $\beta < \infty$ is different from the case of Dirichlet eigenvalue and relies on a free boundary approach.

The role of second order shape derivatives: We only want to mention here the results we draw our inspiration from in Step 4 of the proof, and do not aim at giving out the rigorous mathematical setting of the results mentioned below. We refer to [19] for a thorough presentation of the link between second order shape derivatives, local shape stability and local quantitative inequalities.

As we said in the previous paragraph, most proofs of quantitative inequalities start with a local quantitative inequality for shape perturbation of the optimum E_1 : namely, if \mathcal{F} is a regular enough shape functional and E_1 is an admissible set such that

$$\mathcal{F}(E_1)[\Phi] = 0, \mathcal{F}''(E_1)[\Phi, \Phi] > 0 \quad (10)$$

for any $\Phi \in \mathcal{X}_1(E_1)$ then it is proved in [17], [20] that E_1 is a strict local minimizer in a $\mathcal{C}^{2,\alpha}$ neighbourhood of E (actually, in these two articles, the authors assume a coercivity of the second order derivative in in $H^{\frac{1}{2}}$ norm on Φ). In [19], Dambrine and Lamboley proved that the same conditions imply a local quantitative inequality under certain technical assumptions. Roughly speaking, their result implies the following result: Condition (10) implies that if, for any function $h \in \mathcal{C}^0(\partial\Omega) \cap W^{2,\infty}(\partial\Omega)$ we define E_1^h as the domain bounded by

$$\partial E_1^h := \{x + h(x)\nu_{E_1}(x), x \in \partial E_1\}$$

then there exists $\ell > 0$ such that, for any $\|h\|_{W^{2,\infty}(\partial\Omega)} \leq \ell$ there holds

$$\mathcal{F}(E_1) + C\|h\|_{L^1}^2 \leq \mathcal{F}(E_1^h)$$

for some $C > 0$.

Their result actually holds in stronger norm, but this is the version we wanted to mention here since it is the one we will adapt in our parametric setting. We note however, that the authors, in their problems, prove their inequalities for shape using a $H^{\frac{1}{2}}$ coercivity norm for the second derivative: they usually have

$$\mathcal{F}(E_1)''[h, h] \geq C \|h\|_{H^{\frac{1}{2}}}^2.$$

In this expression, we have defined $\mathcal{F}''(E_1)[h, h] := \mathcal{F}(E_1)''[x + h\nu_{E_1}(x), x + h\nu_{E_1}(x)]$. Here, in Step 4, we will show, using comparison principles, that the optimal coercivity norm is the L^2 norm.

Difference with our proofs and contribution: To the best of our knowledge, quantitative inequalities in a parametric setting have not been studied yet, and this article aims at providing a first step in that direction. As we will see while proving Theorem 1, a shape derivative approach can not be sufficient in of its own for our purposes, as is usually the case, but is needed. To tackle the second order shape derivative, we use a comparison principle. The parametric setting also enables us more freedom while dealing with other competitors and enables us to introduce a new method for dealing with such problems.

1.4.2 Mathematical biology

We briefly sketch some of the biological motivations for the problem under scrutiny here. Following the works of Fisher, [24], Kolmogoroff, Petrovsky and Piscounoff [33], a popular model for population dynamics in a bounded domain is the following so-called logistic-diffusive equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(m - u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(t = 0) = u_0 \geq 0, u_0 \neq 0. \end{cases} \quad (11)$$

In this equation, $m \in L^\infty(\Omega)$ accounts for the spatial heterogeneity and can be interpreted in terms of resources distribution: the zones $\{m \geq 0\}$ are favorable to the growth of the population, while the zones $\{m \leq 0\}$ are detrimental to this population. The particular structure of the non-linearity $-u^2$ (which accounts for the Malthusian growth of the population) makes it so that two linear steady states equations are relevant to our study: the steady-logistic diffusive equation

$$\begin{cases} \Delta\theta + \theta(m - \theta) = 0 \text{ in } \Omega, \\ \theta = 0 \text{ on } \partial\Omega, \\ \theta \geq 0, \end{cases} \quad (12)$$

and the first eigenvalue equation of the linearization of (11) around the solution $z \equiv 0$:

$$\begin{cases} -\Delta\varphi_m - m\varphi_m = \lambda(m)\varphi \text{ in } \Omega, \\ \varphi = 0 \text{ on } \partial\Omega, \end{cases} \quad (13)$$

where $\lambda(m)$ is the first eigenvalue of the operator \mathcal{L}_m defined in (1). More precisely, it is known (see [7, 13, 42]) that

1. Whenever $\lambda(m) < 0$, has a unique solution θ_m , and any solution $u = u(t, x)$ of (11) with initial datum $u_0 \geq 0, u_0 \neq 0$ converges in any L^p to θ_m as $t \rightarrow \infty$.
2. Whenever $\lambda(m) \geq 0$, any solution $u = u(t, x)$ of (11) with initial datum $u_0 \geq 0, u_0 \neq 0$ converges in any L^p to 0 as $t \rightarrow \infty$.

The eigenvalue which we seek to minimize can thus be interpreted as a measure of the survival ability given by a resources distribution, and later works investigated the problem of minimizing $\lambda(m)$ with respect to m under the constraint $m \in \mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is defined in (4). In other words, this is the problem (5).

In the case of Neumann boundary conditions, Berestycki, Hamel and Roques introduced the use of a rearrangement (due to Berestycki and Lachand-Robert,[8]) in that context, see [7] and [32] for an introduction to rearrangement, and further geometrical properties of optimizers were derived by Lou and Yanagida, [38], by Kao, Lou and Yanagida [31]. We do not wish to be exhaustive regarding the literature of this domain and refer to [35] where Lamboley, Laurain, Nadin and Privat investigate several properties of solutions of (5) under a variety of boundary conditions, and the references therein.

2 Proof of Theorem 1

2.1 Background on (5) and structure of the proof

We recall that we work in $\Omega = \mathbb{B}(0; R) \subset \mathbb{R}^2$, that

$$\mathcal{M} := \mathcal{M}(\mathbb{B}(0; R)) = \left\{ 0 \leq V \leq 1, \int_{\Omega} V = V_0 \right\}$$

and that r^* is chosen so that

$$|\mathbb{B}(0; r^*)| = V_0$$

i.e such that

$$V^* = \chi_{\mathbb{B}(0; r^*)} \in \mathcal{M}.$$

We define

$$\mathbb{S}^* := \partial\mathbb{B}^*.$$

We first recall the following simple consequence of Schwarz' rearrangement:

Lemma 1. *V^* is the unique minimizer of λ in \mathcal{M} : for any $V \in \mathcal{M}$,*

$$\lambda(V^*) < \lambda(V).$$

The associated eigenfunction u_ is decreasing and radially symmetric.*

This result is well-known, but for the sake of completeness we prove it in Annex A.

The proof of Theorem 1 relies on the study of two auxilliary problem: we introduce, for a given $\delta > 0$, the new admissible sets

$$\mathcal{M}_{\delta} := \{V \in \mathcal{M}, \|V - V^*\|_{L^1} = \delta\}, \quad (14)$$

$$\tilde{\mathcal{M}}_{\delta} := \{V \text{ radially symmetric}, V \in \mathcal{M}, \|V - V^*\|_{L^1} = \delta\} \quad (15)$$

and study the two variational problems

$$\inf_{V \in \mathcal{M}_{\delta}} \lambda(V) \quad (16)$$

and

$$\inf_{V \in \tilde{\mathcal{M}}_{\delta}} \lambda(V). \quad (17)$$

Obviously, Theorem 1 is equivalent to the existence of $C > 0$ such that

$$\forall \delta > 0, \forall V \in \mathcal{M}_{\delta}, \lambda(V) - \lambda(V^*) \geq C\delta^2. \quad (18)$$

Remark 1. This is a parametric version of the selection principle of [1], that was developed in [11]. We refer to [10] for a synthetic presentation of this selection principle. We note however that the fact that they use a perimeter constraint enables them to prove that a solution to their auxiliary problem is a normal deformation of the optimal shape. The main difficulty in the analysis of [11] is establishing \mathcal{C}^2 bounds for this normal deformation. Here, working with subsets as shape variables gives, from elliptic regularity, enough regularity to carry out this step when the solution of the auxiliary problem is a normal deformation of \mathbb{B}^* . However, we conjecture that the solutions of (16) and (17) are equal and are disconnected (see Step 3 and the Conclusion for a precise conjecture), so that the core difficulty is proving that handling the inequality for normal deformations and for radial distributions is enough to get the inequality for all other sets.

To prove (18) we follow the steps below:

1. We first show that (17) and (16) have solutions. The solutions of (17) will be denoted by \mathcal{V}_δ , the solution of (16) will be denoted by \mathcal{H}_δ .
2. We prove that it suffices to establish (18) for δ small enough.
3. For (16) we fully characterize the solutions for $\delta > 0$ small enough and prove that

$$\forall \delta > 0, \forall V \in \tilde{\mathcal{M}}_\delta, \lambda(V) - \lambda(V^*) \geq C\delta^2. \quad (19)$$

In other words, we prove that Theorem 1 holds for radially symmetric functions.

4. We compute the first and second order shape derivatives of the associated Lagrangian at the ball \mathbb{B}^* and prove a L^2 -coercivity estimate for the second order derivative. We comment upon the fact that (unlike many shape optimization problems) this is the optimal coercivity norm at the beginning of this Step. We use this information to prove that Theorem 1 holds for domains that are small normal deformations of \mathbb{B}^* with bounded mean curvature.
5. We establish a dichotomy for the behaviour of \mathcal{V}_δ and prove that (18) holds for any V by using (19) and Step 4.

2.2 Step 1: Existence of solutions to (16)-(17)

We prove the following Lemma:

Lemma 2. *The optimization problems (16) and (17) have solutions.*

Proof of Lemma 2. The proof follows from the following claim:

Claim 1. *\mathcal{M}_δ and $\tilde{\mathcal{M}}_\delta$ are compact for the weak L^∞ topology.*

We postpone the proof to the end of this Proof.

Lemma 2 follows from this claim, and we only write the details for (17). Let $\{V_k\}_{k \in \mathbb{N}}$ be a minimizing sequence for λ in \mathcal{M}_δ . From Claim 1, there exists $V_\infty \in \mathcal{M}_\delta$ such that

$$V_k \rightharpoonup V_\infty.$$

The notation \rightharpoonup stands for the weak convergence in the weak L^∞ -* sense. Let, for any $k \in \mathbb{N}$, $u_k := u_{V_k}$ and $\lambda_k := \lambda(V_k)$.

We first note that the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ is bounded. Indeed, let $\varphi \in W_0^{1,2}(\Omega)$ be such that $\int_\Omega \varphi^2 = 1$. From the formulation in terms of Rayleigh quotients and $V_k \geq 0$ there holds

$$\lambda_k \leq \int_\Omega |\nabla \varphi|^2 - \int_\Omega V_k \varphi^2 \leq \int_\Omega |\nabla \varphi|^2.$$

This gives an upper bound. For a lower bound, let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of Ω (equivalently, this is the eigenvalue associated with $V = 0$). From $V \leq 1$, $\int_{\Omega} u_k^2 = 1$ and the variational formulation for $\lambda_1(\Omega)$ there holds

$$\lambda_1(\Omega) - 1 \leq \int_{\Omega} |\nabla u_k|^2 - \int_{\Omega} V u_k^2 \leq \lambda_k$$

so that the sequence also admits a lower bound. It is straightforward to see that $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$ so that from the Rellich-Kondrachev Theorem there exists $u_{\infty} \in W_0^{1,2}(\Omega)$ such that $\{u_k\}_{k \in \mathbb{N}}$ converges strongly in L^2 and weakly in $W_0^{1,2}(\Omega)$ to u_{∞} . Passing to the limit in the weak formulation

$$\forall v \in W_0^{1,2}(\Omega), \int_{\Omega} \langle \nabla u_k, \nabla v \rangle - \int_{\Omega} V_k u_k v = \lambda(V_k) \int_{\Omega} u_k v,$$

in the normalization condition

$$\int_{\Omega} u_k^2 = 1$$

and in

$$u_k \geq 0$$

readily shows that u_{∞} is a non-trivial eigenfunction of $\mathcal{L}_{V_{\infty}}$. Furthermore, it is non-negative. Since the first eigenfunction is the only eigenfunction with a constant sign, this proves that $\lambda_{\infty} = \lambda(V_{\infty})$ and that u_{∞} is the eigenfunction associated with V_{∞} . Thus:

$$\lambda(V_{\infty}) = \inf_{V \in \mathcal{M}_{\delta}} \lambda(V).$$

It remains to prove Claim 1:

Proof of Claim 1. We only prove it for \mathcal{M}_{δ} .

Let $\{V_k\}_{k \in \mathbb{N}} \in \mathcal{M}_{\delta}^{\mathbb{N}}$. We define, for any $k \in \mathbb{N}$

$$h_k := V_k - V^*.$$

Since $V^* = \chi_{\mathbb{B}^*}$ and $0 \leq V_k \leq 1$, the following sign conditions hold on h_k :

$$h_k \geq 0 \text{ in } (\mathbb{B}^*)^c, h_k \leq 0 \text{ in } \mathbb{B}^*. \quad (20)$$

Since $\int_{\Omega} V_k = \int_{\Omega} V^*$ there holds

$$\int_{\mathbb{B}^*} h_k = - \int_{(\mathbb{B}^*)^c} h_k. \quad (21)$$

Finally from

$$\|V_k - V^*\|_{L^1} = \delta$$

there comes

$$\delta = \int_{\Omega} |V_k - V^*| \quad (22)$$

$$= \int_{\Omega} |h_k| \quad (23)$$

$$= \int_{(\mathbb{B}^*)^c} h_k - \int_{\mathbb{B}^*} h_k \text{ from (20)} \quad (24)$$

$$= -2 \int_{\mathbb{B}^*} h_k \text{ from (21)} \quad (25)$$

$$= 2 \int_{(\mathbb{B}^*)^c} h_k. \quad (26)$$

We see $h_k \chi_{(\mathbb{B}^*)^c}$ as an element of $L^\infty((\mathbb{B}^*)^c)$. Let h_∞^+ be a weak- L^∞ closure point of $\{h_k\}_{k \in \mathbb{N}}$ in $L^\infty((\mathbb{B}^*)^c)$. From (26) and (20) we have

$$1 \geq h_\infty^+ \geq 0, \int_{(\mathbb{B}^*)^c} h_\infty^+ = \frac{\delta}{2}. \quad (27)$$

For the same reason, there exists $h_\infty^- \in L^\infty(\mathbb{B}^*)$ such that

$$-1 \leq h_\infty^- \leq 0, \int_{\mathbb{B}^*} h_\infty^- = -\frac{\delta}{2}$$

and

$$h_k \rightharpoonup h_\infty \text{ weak-* in } L^\infty(\mathbb{B}^*). \quad (28)$$

We define

$$h_\infty := h_\infty^- \chi_{(\mathbb{B}^*)^c} + h_\infty^- \chi_{\mathbb{B}^*}$$

and it is clear that

$$h_k \rightharpoonup h_\infty \text{ weak-* in } L^\infty(\Omega).$$

Setting $V_\infty := V^* + h_\infty$ there holds

$$V_k \rightharpoonup V_\infty$$

and, by (27),(28), $V_\infty \in \mathcal{M}_\delta(\Omega)$. □

□

2.3 Step 2: Reduction to small L^1 neighbourhoods of V^*

We now prove the following Lemma:

Lemma 3. *To prove Theorem 1, it suffices to prove (18) for δ small enough, in other words it suffices to prove that there exists $C > 0$ such that*

$$\liminf_{\delta \rightarrow 0} \left(\inf_{V \in \mathcal{M}_\delta} \frac{\lambda(V) - \lambda(V^*)}{\delta^2} \right) \geq C.$$

Proof of Lemma 3. We define, for any $V \neq V^*$,

$$G(V) := \frac{\lambda(V) - \lambda(V^*)}{\|V - V^*\|_{L^1}^2}$$

and consider a minimizing sequence $\{V_k\}_{k \in \mathbb{N}}$ for G . Then we either have, up to subsequence,

$$\forall k \in \mathbb{N}, \|V_k - V^*\|_{L^1} \geq \varepsilon > 0$$

or

$$\|V_k - V^*\|_{L^1} \xrightarrow{k \rightarrow \infty} 0.$$

In the first case, up to a converging subsequence, $V_k \rightharpoonup V_\infty$ in a weak L^∞ -* sense and, by the same arguments as in the Proof of Claim 1,

$$\|V_\infty - V^*\|_{L^1} \geq \varepsilon > 0.$$

Furthermore, by the same arguments as in the proof of Lemma 2,

$$\lambda(V_k) \xrightarrow[k \rightarrow \infty]{} \lambda(V_\infty)$$

so that

$$G(V_k) \xrightarrow[k \rightarrow \infty]{} \frac{\lambda(V_\infty) - \lambda(V^*)}{\|V_\infty - V^*\|_{L^1}}$$

and, by Lemma 1, $G(V_\infty) > 0$. Hence we only need to study the case $\|V_k - V^*\|_{L^1} \xrightarrow[k \rightarrow \infty]{} 0$, as claimed. \square

The same arguments yield the following Lemma:

Lemma 4. (19) is equivalent to proving that there exists $C > 0$ such that

$$\liminf_{\delta \rightarrow 0} \left(\inf_{V \in \mathcal{M}_\delta} \frac{\lambda(V) - \lambda(V^*)}{\delta^2} \right) \geq C.$$

2.4 Step 3: Proof of (19)

In this Subsection we prove (19) or, in other words, we prove that Theorem 1 holds for radial distributions.

Proof of (19). We recall that, by Lemma 2, there exists a solution to (16). Let \mathcal{H}_δ be such a minimizer.

We first characterize \mathcal{H}_δ for δ small enough. Let

$$\mathbb{A}_\delta := \{|x| \leq r^* + r'_\delta\} \setminus \{x, r^* - r_\delta \leq |x| \leq r^* + r'_\delta\} \quad (29)$$

be the annular structure such that

$$\chi_{\mathbb{A}_\delta} \in \tilde{\mathcal{M}}_\delta. \quad (30)$$

We represent it below

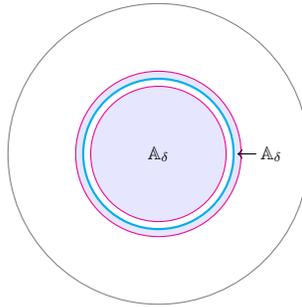


Figure 1: An example of \mathbb{A}_δ

Since r'_δ is defined by the relation

$$|\{r^* \leq |x| \leq r^* + r'_\delta\}| = \frac{\delta}{2}$$

the set \mathbb{A}_δ is uniquely defined. We claim the following:

Claim 2. *There exists $\bar{\delta} > 0$ such that, for any $\delta \leq \bar{\delta}$,*

$$\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}.$$

Proof of Claim 2. To prove this claim, we need the optimality conditions associated with (16). We first note that, if u_δ is the eigenfunction associated with \mathcal{H}_δ , there exist two real numbers η_δ, μ_δ such that

1. $\mathcal{H}_\delta = \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*}$ in \mathbb{B}^* ,
2. $\mathcal{H}_\delta = \chi_{\{\sup_{\mathbb{S}^*} u_\delta \geq u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c}$ in $(\mathbb{B}^*)^c$,
3. $|\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*| = V_0 - \frac{\delta}{2}$, $|\{\sup_{\mathbb{S}^*} u_\delta \geq u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c| = \frac{\delta}{2}$.

This is readily seen from the Rayleigh quotient formulation (3). We only prove 1: let $\mu_\delta \in \mathbb{R}$ be the only real number such that

$$|\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*| = V_0 - \frac{\delta}{2} = \int_{\mathbb{B}^*} \mathcal{H}_\delta$$

and replace \mathcal{H}_δ by

$$\tilde{\mathcal{H}}_\delta := \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*} + \mathcal{H}_\delta \chi_{(\mathbb{B}^*)^c}.$$

Since u_δ is radially symmetric (because \mathcal{H}_δ is radially symmetric), $\tilde{\mathcal{H}}_\delta$ is radially symmetric. Then, because $\int_{\mathbb{B}^*} \tilde{\mathcal{H}}_\delta = \int_{\mathbb{B}^*} \mathcal{H}_\delta$, we have, by the bathtub principle (see [30])

$$\int_{\mathbb{B}^*} u_\delta^2 \mathcal{H}_\delta \leq \int_{\mathbb{B}^*} u_\delta^2 \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*},$$

hence

$$\lambda(\mathcal{H}_\delta) \geq \int_{\mathbb{B}} |\nabla u_\delta|^2 - \int_{\mathbb{B}} \tilde{\mathcal{H}}_\delta u_\delta^2 \geq \lambda(\tilde{\mathcal{H}}_\delta).$$

This gives the required property.

We now need to exploit these optimality conditions. First of all, since u_δ is radially symmetric, we can define

$$\zeta_\delta := u_\delta|_{\mathbb{S}^*}.$$

By standard elliptic estimates, we also have

$$u_\delta \xrightarrow{\delta \rightarrow 0} u_* \text{ in } \mathcal{C}^{1,s} \ (s < 1),$$

where u_* is the eigenfunction associated with V^* . Since $\frac{\partial u_*}{\partial r} < -c$ on $\{|x| \geq \varepsilon\}$, u_δ is radially decreasing in $\{|x| > \varepsilon\}$ for $\delta > 0$ small enough. It follows that $\mu_\delta > \zeta_\delta$ for δ small enough. For the same reason, $\zeta_\delta > \eta_\delta$ for δ small enough. Hence we have

$$\mu_\delta > \eta_\delta > \eta_\delta, \mathcal{H}_\delta = \chi_{\{u_\delta > \mu_\delta\}} + \chi_{\{\zeta_\delta \geq u_\delta > \eta_\delta\}}.$$

Finally, once again because u_δ is radially decreasing on $\{|x| > \varepsilon\}$ for δ small enough, both level sets $\{u_\delta > \mu_\delta\}$ and $\{\zeta_\delta \geq u_\delta > \eta_\delta\}$ are connected, and \mathcal{H}_δ is the characteristic function of a centered ball and of an annulus, i.e

$$\mathcal{H}_\delta = \chi_{\{\|x\| \leq r^* - z_\delta\}} + \chi_{\{r^* \leq \|x\| \leq r^* + y_\delta\}}.$$

Since

$$|\mathbb{A}_\delta \Delta \mathbb{B}^*| = \int_{\mathbb{B}} |\mathcal{H}_\delta - V^*|$$

there holds

$$\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}$$

for δ small enough, as claimed. □

We now turn to the proof of (19): since \mathcal{H}_δ is the minimizer of λ in $\tilde{\mathcal{M}}_\delta$ we are going to prove that there exists a constant $C > 0$ such that

$$\lambda(\mathcal{H}_\delta) \geq \lambda(V^*) + C\delta^2 \quad (31)$$

for $\delta \leq \bar{\delta}$. Because of Lemma 3, (19) will follow. To prove (31), we use parametric derivatives. Let us fix notations:

1. For any $\delta > 0$ small enough so that $\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}$, we set

$$h_\delta := \mathcal{H}_\delta - V^*.$$

2. For any $t \in [0; 1]$ we define $V_{\delta,t}$ as

$$V_{\delta,t} := V^* + th_\delta$$

and $u_{\delta,t}$ as the eigenfunction associated with $V_{\delta,t}$:

$$\begin{cases} -\Delta u_{\delta,t} - V_{\delta,t} u_{\delta,t} = \lambda_t u_{\delta,t} & \text{in } \Omega, \\ u_{\delta,t} = 0 & \text{on } \partial\Omega. \\ \int_{\Omega} u_{\delta,t}^2 = 1. \end{cases} \quad (32)$$

3. For any such δ , \dot{u}_δ is the first order parametric derivative of u at $V_{\delta,t}$ in the direction h_δ and $\dot{\lambda}_{\delta,t}$ is the first order parametric derivative of λ at $V_{\delta,t}$ in the direction h_δ .

As is proved in Annex B, these objects are well-defined. Differentiating the equation with respect to t gives

$$\begin{cases} -\Delta \dot{u}_{\delta,t} = \lambda(V^*) \dot{u}_{\delta,t} + V^* \dot{u}_{\delta,t} + \dot{\lambda}_{\delta,t} u_{\delta,t} + h_{\delta,t} u_{\delta,t}^*, & \text{in } \Omega, \\ \dot{u}_{\delta,t} = 0 & \text{on } \partial\Omega, \\ \int_{\mathbb{B}} u_{\delta,t} \dot{u}_{\delta,t} = 0, \end{cases} \quad (33)$$

and, multiplying the first equation by $u_{\delta,t}$ and integrating by parts,

$$\dot{\lambda}_{\delta,t} = - \int_{\mathbb{B}} h_{\delta,t} u_{\delta,t}^2.$$

We apply the mean value Theorem to $f : t \mapsto \lambda_{\delta,t}$. This gives the existence of $t_1 \in [0; 1]$ such that

$$\lambda(\mathcal{H}_\delta) - \lambda(\mathbb{B}^*) = f(1) - f(0) = f'(t_1) = - \int_{\mathbb{B}} h_{\delta,t_1} u_{\delta,t_1}^2.$$

Our goal is now to prove that

$$- \int_{\mathbb{B}} h_{\delta} u_{\delta, t_1}^2 \geq C \delta^2 \quad (34)$$

for some constant $C > 0$ whenever δ is small enough. We will actually prove the existence of $\underline{\delta} > 0$ such that, for any $t \in [0; 1]$ and any $\delta \leq \underline{\delta}$, there holds

$$- \int_{\mathbb{B}} h_{\delta} u_{\delta, t}^2 \geq C \delta^2 \quad (35)$$

for some $C > 0$.

Proof of Estimate (35). We recall that r_{δ} and r'_{δ} were defined in (29)-(30). We can rewrite $\mathcal{H}_{\delta} = \chi_{\mathbb{A}_{\delta}} \in X_{\delta}$ under the form

$$\int_{r^* - r_{\delta}}^{r^*} t^{n-1} dt + \int_{r^*}^{r^* + r'_{\delta}} t^{n-1} dt = \frac{\delta}{c_n}$$

where $c_n = \mathcal{H}^{n-1}(\mathbb{S}(0; 1))$ and the condition $\int_{\Omega} h_{\delta} = 0$ implies

$$\int_{r^* - r_{\delta}}^{r^*} t^{n-1} dt = \int_{r^*}^{r^* + r'_{\delta}} t^{n-1} dt.$$

An explicit computation yields the existence of a constant $C > 0$ such that

$$r_{\delta}, r'_{\delta} \underset{\delta \rightarrow 0}{\sim} C \delta. \quad (36)$$

Let $I_{\delta}^{\pm} := \{h_{\delta} = \pm 1\}$. Since h_{δ} is radial, for any $t \in [0; 1]$ the function $u_{\delta, t}$ is radial.

First facts regarding $u_{\delta, t}$ Identifying $u_{\delta, t}$ (resp. $V_{\delta, t}$) with the unidimensional function $\tilde{u}_{\delta, t}$ (resp. $\tilde{V}_{\delta, t}$) such that

$$u_{\delta, t}(x) = \tilde{u}_{\delta, t}(|x|) \quad (\text{resp. } V_{\delta, t}(x) = \tilde{V}_{\delta, t}(|x|))$$

we have the following equation on $u_{\delta, t}$:

$$\begin{cases} -\frac{1}{r^{n-1}} \left(r^{n-1} u'_{\delta, t} \right)' = V_{\delta, t} u_{\delta, t} + \lambda_t u_{\delta, t} & \text{in } [0; R], \\ u'_{\delta, t}(0) = u_{\delta, t}(1) = 0, \\ \int_0^1 x u_{\delta, t}(x)^2 dx = \frac{1}{c_n}. \end{cases} \quad (37)$$

Since $V_{\delta, t}$ is constant in $(r^* - r_{\delta}; r^*) \cup (r^*; r^* + r'_{\delta})$ and since $u_{\delta, t}$ is uniformly bounded in L^{∞} by standard elliptic estimates, $u_{\delta, t}$ is \mathcal{C}^2 in $(r^* - r_{\delta}; r^*) \cup (r^*; r^* + r'_{\delta})$. Furthermore, Equation (37) readily gives the existence of a constant M such that, uniformly in δ and in $t \in [0; 1]$,

$$\|u_{\delta, t}\|_{W^{2, \infty}} \leq M. \quad (38)$$

Finally, it is standard to see that Equation (32) gives

$$\|u_{\delta, t} - u_{*}\|_{\mathcal{C}^1} \underset{\delta \rightarrow 0}{\rightarrow} 0 \quad (39)$$

uniformly in t . As a consequence, since

$$u'_{*}(r^*) < 0$$

there exists $\underline{\delta}_1 > 0$ such that, for any $\delta \leq \underline{\delta}_1$ and any $t \in [0; 1]$,

$$u'_{\delta, t}(r^*) \leq -C < 0. \quad (40)$$

End of the Proof For any $x \in I_\delta^\pm$ and any $t \in [0; 1]$, a Taylor expansion gives

$$u_{\delta,t}^2(x) = u_{\delta,t}^2|_{\mathbb{S}^*} \mp 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o(\text{dist}(x; \mathbb{S}^*)),$$

and $o(\text{dist}(x; \mathbb{S}^*))$ is uniform in $\delta > 0$ small enough and $t \in [0; 1]$ by Estimate (38). This Taylor expansion gives

$$\begin{aligned} \dot{\lambda}_{\delta,t} &= - \int_{\mathbb{B}} h u_{\delta,t}^2 \\ &= - u_{\delta,t}^2|_{\mathbb{S}^*} \int_{\mathbb{B}} h_\delta \left(= 0 \text{ because } \int_{\mathbb{B}} h = 0 \right) \\ &\quad + \int_{I_\delta^-} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o\left(\int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*)\right) \\ &\quad + \int_{I_\delta^+} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o\left(\int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*)\right) \end{aligned}$$

where the $o\left(\int_{I_\delta^\pm} \text{dist}(x; \mathbb{S}^*)\right)$ are uniform in $t \in [0; 1]$ and δ . Furthermore,

$$u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \geq C > 0$$

for some constant $C > 0$ independent of δ and t by Estimate (40).

Hence

$$\dot{\lambda}_{\delta,t} \underset{\delta \rightarrow 0}{\sim} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \left(\int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*) + \int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*) \right).$$

However,

$$\begin{aligned} \int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*) &= \int_{r^*}^{r^*+r'_\delta} t^{n-1}(t-r^*)dt \\ &= (r^*)^n r'_\delta \left(\frac{1}{n+1} \binom{n+1}{n} - \frac{1}{n} \binom{n}{n-1} \right) \\ &\quad + (r^*)^{n-1} (r'_\delta)^2 \left(\frac{1}{n+1} \binom{n+1}{n-1} - \frac{1}{n} \binom{n}{n-2} \right) \\ &\quad + \underset{\delta \rightarrow 0}{o}(\delta^2) \quad \text{by (36)} \\ &= (r^*)^{n-1} (r'_\delta)^2 + \underset{\delta \rightarrow 0}{o}(\delta^2) \\ &\underset{\delta \rightarrow 0}{\sim} C\delta^2 \end{aligned}$$

for some $C > 0$ by (36). In the same manner,

$$\int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*) = \int_{r^*-r_\delta}^{r^*} t^{n-1}(r^*-t)dt \underset{\delta \rightarrow 0}{\sim} C'\delta^2$$

and so, combining these estimates gives

$$\dot{\lambda}_{\delta,t} \underset{\delta \rightarrow 0}{\geq} C\delta^2 + o(\delta^2)$$

uniformly in δ and t , which concludes the proof. □

□

2.5 Step 4: shape derivatives and quantitative inequality for graphs

2.5.1 Preliminaries and notations

In this Subsection, we aim at proving Theorem 1 for $V = \chi_E$ and where E can be obtained as a normal graph over \mathbb{B}^* .

Introduction of the Lagrangian and optimality conditions We introduce the Lagrange multiplier τ associated with the volume constraint and define the Lagrangian

$$L_\tau : E \mapsto \lambda(E) - \tau \text{Vol}(E).$$

From classical results in the calculus of variations, we have the following optimality conditions.

Claim 3. *The necessary optimality conditions for a shape E to be a local minimizer (however they are not sufficient) are:*

$$\forall \Phi \in \mathcal{X}_1(E), \lambda'(E)[\Phi] = 0, \quad \forall \Phi \in \mathcal{X}_1(E), L''_\tau(E)[\Phi, \Phi] > 0.$$

Here, we use the notion of shape derivatives introduced in Definition 2. Since we only want a local quantitative inequality for shapes that can be obtained as normal graphs over the ball \mathbb{B}^* , we introduce some notations.

Notations We consider in this parts functions g belonging to

$$\mathcal{X}_0(\mathbb{B}^*) = \left\{ g \in W^{1,\infty}(\mathbb{B}^*), \|g\|_{L^\infty(\partial\mathbb{B}^*)} \leq 1, \int_{\partial\mathbb{B}^*} g = 0 \right\}.$$

Whenever $g \in \mathcal{X}_0(\mathbb{B}^*)$, there exists $\Phi_g \in \mathcal{X}_1(\mathbb{B}^*)$ such that

$$\langle \Phi_g, \nu \rangle = g \text{ on } \partial\mathbb{B}^*.$$

The set \mathcal{X}_0 corresponds to a linearization of the volume constraints for normal graphs and can be seen as a subset of the set of admissible perturbations $\mathcal{X}_1(\mathbb{B}^*)$ defined in Definition 2, in the sense that we restrict admissible perturbations to normal graphs.

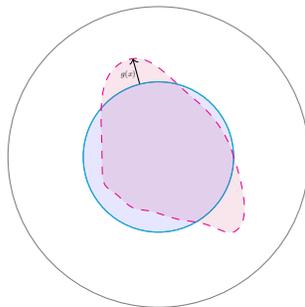


Figure 2: A normal deformation of the ball. The dotted line can be understood as the graph of g .

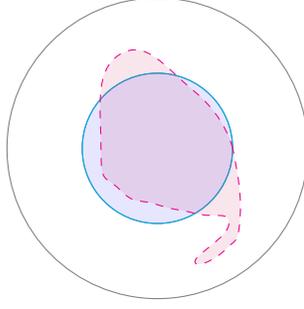


Figure 3: A perturbation of the ball which can not be seen as the graph of a function

We define, for any $g \in \mathcal{X}_0(\mathbb{B}^*)$ and any $t \in [0; T]$ (with T uniform in g because of the L^∞ constraint) the set $\mathbb{B}_{t,g}$ whose boundary is defined as

$$\partial\mathbb{B}_{t,g} := \{x + tg(x)\nu(x), x \in \partial\mathbb{B}^*\}$$

i.e a slight deformation of E^* .

We define

$$\lambda_{g,t} := \lambda(\mathbb{B}_{t,g}).$$

Recall that τ is the Lagrange multiplier associated with the volume constraint. By defining $L'_\tau(\mathbb{B}^*)[g] := L'_\tau(\mathbb{B}^*)[\Phi_g]$ and $L''_\tau(\mathbb{B}^*)[g, g] := L''_\tau(\mathbb{B}^*)[\Phi_g, \Phi_g]$, and with the same convention for other shape functionals involved, necessary optimality conditions are

$$\forall g \in \mathcal{X}_0(\mathbb{B}^*), L'_\tau(\mathbb{B}^*)[g] = 0, L''_\tau(\mathbb{B}^*)[g, g] > 0. \quad (41)$$

We first prove that these optimality conditions hold in the case of a ball and then use them to obtain Theorem 1 for normal perturbations.

2.5.2 Strategy of proof and comment on the coercivity norm

The strategy of proof is the same as the one used in many articles devoted to quantitative spectral inequalities. For example, we refer to [1, 11] for applications of these methods and to the recent [19], which presents a general framework for the study of stability and local quantitative inequalities using second order shape variations.

Although our method of proof is similar, we point out that the main thing to be careful with here is the coercivity norm for the second-order shape derivative. Indeed, let $J : E \mapsto J(E)$ be a differentiable shape function. In the context of shape spectral optimization, the "typical" coercivity norm at a local minimum E^* for J is the $H^{\frac{1}{2}}$ norm: in [19] a summary of shape functionals known to satisfy

$$J''(E^*)[\Phi, \Phi] \geq C \|\langle \Phi, \nu \rangle\|_{H^{\frac{1}{2}}(\partial E^*)}^2$$

is established and proofs of these coercivity properties are given. In this estimate, Φ is an admissible vector field at E^* .

Here, in the context of parametric shape derivatives, i.e when the shape is a subdomain, it appears (see Subsubsection 2.5.5) that the natural coercivity norm is the L^2 norm:

$$L''_\tau(E^*)[\Phi, \Phi] \geq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial A^*)}^2$$

and this coercivity norm is optimal. This makes things a bit more complicated when dealing with the terms of the second order derivative that involve the mean curvature. This lack of coercivity

might be accounted for by the fact that, while in shape optimization, it is the normal derivative of the shape derivative of the eigenfunction that is involved (see [19]) here, it is just the trace of the shape derivative of the eigenfunction on the boundary of the optimal shape that matters.

Once this L^2 coercivity is established, we will prove that there exists a constant $\xi, M, C > 0$ such that, for any g satisfying $\|g\|_{W^{1,\infty}} \leq \xi$ and such that the mean curvature of $\mathbb{B}_{g,t}$ is bounded by M for any $t \leq T$, there holds

$$|L''_\tau(\mathbb{B}_{t,g})[\Phi_g, \Phi_g] - L''_\tau(\mathbb{B}^*)[\Phi_g, \Phi_g]| \leq (C + M)\|g\|_{W^{1,\infty}}\|g\|_{L^2}^2. \quad (42)$$

We then apply the Taylor-Lagrange formula to $f : t \mapsto L_\tau(\mathbb{B}_{t,g})$ to get the desired conclusion, see Subsubsection 2.5.7.

2.5.3 Analysis of the first order shape derivative at the ball and computation of the Lagrange multiplier

The aim of this section is to prove the following Lemma:

Lemma 5. *\mathbb{B}^* is a critical shape and the Lagrange multiplier associated with the volume constraint is*

$$\tau = -u_*^2|_{\partial\mathbb{B}^*}.$$

Proof of Lemma 5. We recall (see [30]) that

$$Vol'(\mathbb{B}^*)[g] = \int_{\partial\mathbb{B}^*} g \quad (= 0 \text{ if } g \in \mathcal{X}_0(\mathbb{B}^*)).$$

We now compute the first order shape derivative of λ . The shape differentiability of λ follows from an application of the implicit function Theorem of Mignot, Murat and Puel, [41], and is proved in Appendix B.

Let, for any $g \in \mathcal{X}_0(\mathbb{B}^*)$, u'_g be the shape derivative of the $u_{\mathbb{B}_{t,g}}$ at $t = 0$. We recall that this derivative is defined as follows (see [30, Chapitre 5] for more details): we first define $v_t := u_{g,t} \circ \Phi_{g,t}$, we define \dot{u}_g as the derivative in $W_0^{1,2}(\Omega)$ of v_t with respect to t at $t = 0$, and set

$$u'_g := \dot{u}_g - \langle \Phi_g, \nabla u_0 \rangle.$$

We proceed formally to get the equation on u'_g (for rigorous computations we refer to Appendix B): we first differentiate the main equation

$$-\Delta u_{g,t} = \lambda_{g,t} u_{g,t} + V_{g,t} u_{g,t}$$

with respect to t , yielding

$$-\Delta u'_g = \lambda'_g u_* + \lambda_* u'_g + (V^*)u'_g.$$

We then differentiate the continuity equations to get the jump conditions: if we define

$$[f](x) := \lim_{y \rightarrow x, y \in (\mathbb{B}^*)^c} f(y) - \lim_{y \rightarrow x, y \in \mathbb{B}^*} f(y),$$

we have

$$[u_{g,t}]|_{\partial\mathbb{B}_{t,g}} = \left[\frac{\partial u_{g,t}}{\partial \nu} \right] \Big|_{\partial\mathbb{B}_{t,g}} = 0,$$

yielding

$$[u'_g]|_{\partial\mathbb{B}^*} = -g \left[\frac{\partial u_*}{\partial \nu} \right] \Big|_{\partial\mathbb{B}^*} = 0$$

because u_* is \mathcal{C}^1 , and

$$\left[\frac{\partial u'_g}{\partial \nu} \right] \Big|_{\partial \mathbb{B}^*} = -g \left[\frac{\partial^2 u_*}{\partial \nu^2} \right] \Big|_{\partial \mathbb{B}^*}.$$

However, from the Equation on u^* we see that

$$\left[\frac{\partial^2 u_*}{\partial \nu^2} \right] \Big|_{\partial \mathbb{B}^*} = u_* \Big|_{\partial \mathbb{B}^*},$$

so that we finally have the following equation on u'_g :

$$\begin{cases} -\Delta u'_g = \lambda'_g u_* + \lambda_* u_* + (V^*) u_* & \text{in } \mathbb{B}(0; R), \\ \left[\frac{\partial u'_g}{\partial \nu} \right] = -g u_* \Big|_{\partial \mathbb{B}^*}. \end{cases} \quad (43)$$

The weak formulation of this equation reads: for any $\varphi \in W_0^{1,2}(\mathbb{B})$,

$$\int_{\mathbb{B}} \langle \nabla u'_g, \nabla \varphi \rangle - \int_{\partial \mathbb{B}^*} g u_* \varphi = \lambda'_g \int_{\mathbb{B}} \varphi u_* + \lambda_* \int_{\mathbb{B}} \varphi u_* + \int_{\mathbb{B}} (V^*) \varphi u_*. \quad (44)$$

We finally remark that, by differentiating $\int_{\mathbb{B}} u_{g,t}^2 = 1$, we get

$$\int_{\mathbb{B}} u_* u'_g = 0.$$

Taking u_* as a test function in (44) and using the normalization condition $\int_{\mathbb{B}} u_*^2 = 1$ thus gives

$$\lambda'_g = \int_{\mathbb{B}} \langle \nabla u_*, \nabla u'_g \rangle - \lambda_* \int_{\mathbb{B}} u'_g u_* - \int_{\mathbb{B}} (V^*) u_* u'_g - \int_{\partial \mathbb{B}^*} g u_*^2. \quad (45)$$

However, since u'_g does not have a jump at $\partial \mathbb{B}^*$, we have

$$\int_{\mathbb{B}} \langle \nabla u_*, \nabla u'_g \rangle - \lambda_* \int_{\mathbb{B}} u'_g u_* - \int_{\mathbb{B}} (V^*) u_* u'_g = 0$$

by using the Equation on u_* .

In the end, we get

$$\lambda'_g = - \int_{\partial \mathbb{B}^*} g u_*^2.$$

Since $u_*^2 = \mu_*$ is constant on $\partial \mathbb{B}^*$ and $g \in \mathcal{X}_0(\mathbb{B}^*)$,

$$\lambda'_g = -\mu_* \int_{\partial \mathbb{B}^*} g = 0.$$

This also enables us to compute the Lagrange multiplier: for a function $g \in W^{1,\infty}(\partial \mathbb{B}^*)$ which is no longer assumed to satisfy $\int_{\partial \mathbb{B}^*} g = 0$, one must have

$$L'_\tau[\mathbb{B}^*](g) = 0.$$

Indeed, we know, from Lemma 1, that \mathbb{B}^* is the unique minimizer of λ under the volume constraint. However, the same computations show that

$$L'_\tau[\mathbb{B}^*](g) = -\mu^* \int_{\partial \mathbb{B}^*} g - \tau \int_{\partial \mathbb{B}^*} g,$$

and the Lagrange multiplier is thus

$$\tau = -\mu^* = -u_*^2 \Big|_{\mathbb{B}^*}. \quad (46)$$

□

We now compute the second order shape derivative of L_τ at any given shape.

2.5.4 Computation of the second order shape derivative of λ

We explained in Subsection 2.5.2 that we need to compute the second order derivative at any given shape in order to apply the Taylor-Lagrange formula. Thus, the objectif of this section is the proof of the following Lemma:

Lemma 6. *The second order derivative of the eigenvalue λ at a shape E in the direction $\Phi \in \mathcal{X}_1(E)$ is given by*

$$\lambda''(E)[\Phi, \Phi] = -2 \int_{\partial E} uu' \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} \left([\nabla^2 u[\Phi, \Phi]] - \left[\frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2 \right) + \int_{\partial E} \left(-Hu^2 - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2,$$

where u' is defined by Equation (43) and H is the mean curvature of E .

Proof of Lemma 6. To compute $\lambda''(E)[\Phi, \Phi]$, we use Hadamard's second variation formula (see [30, Chapitre 5, page 227]): let K be a \mathcal{C}^2 domain, $f(t)$ be a shape differentiable function, then

$$\frac{d^2}{dt^2} \Big|_{t=0} \int_{K_{\Phi,t}} f(t) = \int_K f''(0) + 2 \int_{\partial K} f'(0)g + \int_{\partial K} \left(Hf(0) + \frac{\partial f(0)}{\partial \nu} \right) g^2. \quad (47)$$

Let u' be the shape derivative of u_E with respect to t and u'' the second order shape derivative of u_E with respect to t . We successively apply (47) to $E_{\Phi,t}$ and $f(t) = |\nabla u_t|^2 - u_t^2$ and to $(E_{\Phi,t})^c$ and $f(t) = |\nabla u_t|^2$.

Since $\lambda(E_{\Phi,t}) = \int_{E_{\Phi,t}} (|\nabla u_t|^2 - u_t^2) + \int_{E_{\Phi,t}^c} |\nabla u_t|^2$, this gives

$$\lambda''(E)[\Phi, \Phi] = 2 \int_{\Omega} \langle \nabla u'', \nabla u \rangle - 2 \int_E u''u \quad (48)$$

$$+ 2 \int_{\Omega} |\nabla u'|^2 - 2 \int_E (u')^2 \quad (49)$$

$$+ 4 \int_{\partial E} [\langle \nabla u, \nabla u' \rangle] \langle \Phi, \nu \rangle - 4 \int_{\partial E} u'u \langle \Phi, \nu \rangle \quad (50)$$

$$+ \int_{\partial E} \left(-Hu^2 - 2 \frac{\partial u}{\partial \nu} \left[\frac{\partial^2 u}{\partial \nu^2} \right] - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2. \quad (51)$$

Let us simplify this expression: First of all the weak formulation of the equation on u' gives

$$2 \int_{\Omega} |\nabla u'|^2 - 2 \int_E (u')^2 = \underbrace{2\lambda' \int_{\Omega} u'u}_{=0 \text{ since } \int_{\Omega} u'u=0} + 2\lambda \int_{\Omega} (u')^2 - 2 \int_{\partial E} \left[u' \frac{\partial u'}{\partial \nu} \right].$$

We also note that by differentiating $\int_{\Omega} u_t^2 = 1$ twice with respect to t we get

$$\int_{\Omega} u''u + \int_{\Omega} (u')^2 = 0. \quad (52)$$

We note one last simplification to handle Line (50) in the expression for λ'' : we decompose

$$\nabla u = \frac{\partial u}{\partial \nu} \nu + \nabla^{\perp} u, \langle \nabla^{\perp} u, \nu \rangle = 0.$$

We adopt the same decomposition for u' and notice that, since u' does not have a jump at ∂E ,

$$[\nabla^{\perp} u] = \vec{0}.$$

The notation $\vec{0}$ stands for the zero vector in \mathbb{R}^n . The same holds true for u , and so, since $\frac{\partial u}{\partial \nu}$ has no jump at ∂E , we get

$$[\langle \nabla u, \nabla u' \rangle] = \frac{\partial u}{\partial \nu} \left[\frac{\partial u'}{\partial \nu} \right].$$

Finally, using the weak formulation of the equation on u (Equation (2)) we get

$$2 \int_{\Omega} \langle \nabla u'', \nabla u \rangle - 2 \int_E u'' u = 2\lambda \int_{\Omega} u'' u - \int_{\partial E} [u''] \frac{\partial u}{\partial \nu}.$$

We then only need to compute the jump $[u'']$ at ∂E . However, invoking the $W^{2,2}$ regularity of the material derivative \ddot{u} , we get for the shape derivative u'' :

$$[u''] = -2 [\nabla u'[\Phi]] - [\nabla u[D\Phi(\Phi)]] - [\nabla^2 u][[\Phi, \Phi]].$$

We now use the fact that

$$[\nabla u] = [\nabla^\perp u] = [\nabla^\perp u'] = 0 \text{ on } \partial E$$

to rewrite

$$[u''] = -2 \left[\frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle - [\nabla^2 u][[\Phi, \Phi]].$$

If we gather these expressions we get

$$\begin{aligned} \lambda''(E)[\Phi, \Phi] &= 2\lambda \int_{\Omega} u'' u + 2\lambda \int_{\Omega} (u')^2 && (= 0 \text{ because } \int_{\Omega} u'' u = 0) \\ &+ 4 \int_{\partial E} \frac{\partial u}{\partial \nu} \left[\frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} [\nabla^2 u][[\Phi, \Phi]] \\ &+ 2 \int_{\partial E} u' u \\ &- 4 \int_{\partial E} \frac{\partial u}{\partial \nu} \left[\frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle - 4 \int_{\partial E} u' u \langle \Phi, \nu \rangle \\ &+ \int_{\partial E} \left(-Hu^2 - 2 \frac{\partial u}{\partial \nu} \left[\frac{\partial^2 u}{\partial \nu^2} \right] - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 \\ &= -2 \int_{\partial E} uu' \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} \left([\nabla^2 u][\Phi, \Phi] - \left[\frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2 \right) + \int_{\partial E} \left(-Hu^2 - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 \end{aligned}$$

□

2.5.5 Analysis of the second order shape derivative at the ball

The aim of this paragraph is to prove the following Lemma:

Proposition 1. *There exists a constant $C > 0$ such that*

$$\forall g \in \mathcal{X}_0(\mathbb{B}^*), L''_\tau[B^*](g, g) \geq C \|g\|_{L^2(\partial \mathbb{B}^*)}^2.$$

We note that the proof of this Lemma relies on a monotonicity principle, which guarantees the weak L^2 coercivity. In fact, this is to be the optimal coercivity, in sharp contrast with shape optimization with respect to the boundary of the whole domain $\partial \Omega$, where the optimal coercivity usually occurs in the $H^{\frac{1}{2}}$ norm, as noted in Subsection 2.5.2.

Proof of Proposition 1. We proceed in several steps. We identify g with the normal vector field Φ_g that can be constructed from g , and, to alleviate notations, write $\Phi = \Phi_g$.

1. Computation of λ'' We use Lemma 6 and first note that, since the vector field Φ associated with g is normal,

$$[\nabla^2 u[\Phi, \Phi]] = \left[\frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2.$$

In the case of a ball, $H = \frac{1}{r^*}$. For notational simplicity, we stick to the notation

$$H^* = \frac{1}{r^*}.$$

The second derivative of λ becomes

$$\lambda''(\mathbb{B}^*)[\Phi, \Phi] = \int_{\partial \mathbb{B}^*} \left(-H^* u_*^2 - 2u_* \frac{\partial u_*}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 - 2 \int_{\partial \mathbb{B}^*} u_* u' \langle \Phi, \nu \rangle.$$

Taking into account the value of the Lagrange multiplier τ associated with the volume constraint, see Equation (46), and

$$Vol''(\mathbb{B}^*)[\Phi, \Phi] = \int_{\partial \mathbb{B}^*} H^* \langle \Phi, \nu \rangle^2$$

we get

$$L''_{\tau}(\mathbb{B}^*)[\Phi, \Phi] = 2 \int_{\partial \mathbb{B}^*} -u_* \frac{\partial u_*}{\partial \nu} \langle \Phi, \nu \rangle^2 - u_* u' \langle \Phi, \nu \rangle. \quad (53)$$

2. Separation of variables and first simplifications We identify g with a function $g : [0; 2\pi] \rightarrow \mathbb{R}$. We write the decomposition of g as a Fourier series:

$$g = \sum_{k \in \mathbb{N}} \alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot).$$

Since $g \in \mathcal{X}_0(\mathbb{B}^*)$ we have $\alpha_0 = 0$ and thus

$$g = \sum_{k=1}^{\infty} \alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot). \quad (54)$$

We define, for any $k \in \mathbb{N}^*$, u'_k (resp. w'_k) as the shape derivative of u with respect to the perturbation $g = \cos(k \cdot)$ (resp. $g = \sin(k \cdot)$). Since \mathbb{B}^* is a critical shape from Lemma 5, u'_k satisfies

$$\begin{cases} -\Delta u'_k = \lambda_* u'_k + V^* u'_k, \\ [u'_k] = -u_* \cos(k \cdot) \text{ on } \partial \mathbb{B}^*, \\ u'_k = 0 \text{ on } \partial \Omega. \end{cases}$$

Since u_* is constant on partial \mathbb{B}^* , we can write, in polar coordinates

$$u'_k(r, \theta) = \psi_k(r) \cos(k\theta)$$

where ψ_k satisfies the following equation (and we identify V^* with the one dimensional function \tilde{V}^* such that $(V^*)(x) = \tilde{V}^*(|x|) = \chi_{|x| \leq r^*}$):

$$\begin{cases} -\frac{1}{r}(r\psi'_k)' = \left(\lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_k \\ [\psi'_k](r^*) = -u_*(r^*) \\ \psi_k(R) = 0. \end{cases} \quad (55)$$

In the same way, we have

$$w'_k(r, \theta) = \psi_k(r) \sin(k\theta).$$

Whenever g admits the Fourier decomposition (54), the linearity (with respect to g) of the equation on u'_g gives

$$u'_g = \sum_{k=1}^{\infty} \alpha_k u'_k + \beta_k w'_k.$$

Plugging this in the expression of L''_τ , see Equation (53), and using the orthogonality properties of $\{\cos(k\cdot), \sin(k\cdot)\}_{k \geq 1}$ finally yields

$$L''_\tau(\mathbb{B}^*)[g, g] = L''_\tau(\mathbb{B}^*)[\Phi, \Phi] = \sum_{k=1}^{\infty} \{\alpha_k^2 + \beta_k^2\} u_*|_{\partial\mathbb{B}^*} \left(- \frac{\partial u_*}{\partial \nu} \Big|_{\partial\mathbb{B}^*} - \psi_k(r^*) \right).$$

We define the relevant sequence $\{\omega_k\}_{k \in \mathbb{N}^*}$:

$$\forall k \in \mathbb{N}^*, \omega_k := - \frac{\partial u_*}{\partial \nu} \Big|_{\partial\mathbb{B}^*} - \psi_k(r^*), \quad (56)$$

so that

$$L''_\tau(\mathbb{B}^*)[g, g] = u_*|_{\partial\mathbb{B}^*} \sum_{k=1}^{\infty} \omega_k \{\alpha_k^2 + \beta_k^2\}. \quad (57)$$

Our goal is now the following Lemma:

Lemma 7. *There exists $C > 0$ such that*

$$\forall k \in \mathbb{N}^*, \omega_k \geq C > 0.$$

In order to prove this Lemma, we use a comparison principle for one-dimensional differential equations.

3. Proof of Lemma 7: monotonicity principle We will first prove that

$$\forall k \in \mathbb{N}^*, \omega_k \geq \omega_1. \quad (58)$$

We first note that

$$\psi_1 > 0 \text{ in } (0; R). \quad (59)$$

Before we prove, let us see how (59) implies (58). Let, for any $k \geq 2$, z_k be the function defined as

$$z_k := \psi_k - \psi_1.$$

Since ψ_k can be expressed as $\psi_k = A_k J_k(\frac{r}{R})$ for $r \leq r^*$ where J_k is the k -th Bessel function of first kind, we have $z_k(0) = z_k(R) = 0$. Furthermore, z_k satisfies

$$\begin{aligned} -\frac{1}{r}(rz'_k)' &= \left(\lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_k - \left(\lambda_* + V^* - \frac{1}{r^2} \right) \psi_1 \\ &\leq \left(\lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_k - \left(\lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_1 && \text{because } \psi_1 > 0 \text{ by (59)} \\ &\leq \left(\lambda_* + V^* - \frac{k^2}{r^2} \right) z_k. \end{aligned} \quad (60)$$

We also have

$$[z'_k](r^*) = 0.$$

However, this equation and this no-jump condition imply

$$z_k \leq 0. \tag{61}$$

For k large enough, this simply follows by a contradiction argument: if $\lambda_* + V^* - \frac{k^2}{r^2} < 0$ in $(0; R)$ then, if z_k reached a positive maximum at some interior point \bar{r} , we should have

$$0 \leq -\frac{1}{\bar{r}}(\bar{r}z''_k(\bar{r})) \leq \left(\lambda_* + V^* - \frac{k^2}{\bar{r}^2}\right) z_k(\bar{r}) < 0,$$

yielding a contradiction.

A proof of (61) that is valid for all values of k reads as follows: identifying u_* with its one-dimensional counterpart (i.e with the function $\tilde{u}_* : [0; R] \rightarrow \mathbb{R}$ such that $u_*(x) = \tilde{u}_*(|x|)$) we define

$$p_k := \frac{z_k}{u_*}.$$

We notice that $p_k(0) = 0$ and that

$$[p'_k](r^*) = 0.$$

Furthermore, by straightforward computation, p_k satisfies

$$-\frac{1}{r}(rp'_k)' = \frac{-1}{u_*}(rz'_k)' + \frac{z_k}{u_*^2} \frac{1}{r}(ru'_*)' + 2\frac{u'_*}{u_*} p'_k.$$

By (60) and by non-negativity of u_* we get

$$-\frac{1}{r}(rp'_k)' \leq -\frac{k^2}{r^2} p_k.$$

From this it is straightforward to see by a contradiction argument that p_k can not reach a positive maximum at an interior point. It remains to exclude the case $p_k(R) > 0$.

We argue, once again, by contradiction, and assume that $p_k(R) > 0$. Since by l'Hospital rules we have

$$p_k(r) \underset{r \rightarrow R}{\sim} \frac{z'_k(r)}{u'_*(r)}$$

we must have $z'_k(r) \leq 0$. However once again by l'Hospital's rule,

$$p'_k(r) = \frac{u_* z'_k}{u_*^2} - \frac{u'_* z_k}{u_*^2} \underset{r \rightarrow 0}{\sim} \frac{1}{2} z'_k 2u_* < 0.$$

Hence p_k is locally decreasing at R , yielding a contradiction. Thus $p_k \leq 0$ and in turn $\psi_k - \psi_1 = z_k \leq 0$, completing the proof of (58).

The proof of (59) follows from the same arguments: we define

$$\Psi_1 := \frac{\psi_1}{u_*}$$

and observe that

$$-\frac{1}{r}(r\Psi'_1)' = -\frac{1}{r^2}\Psi_1 + 2\frac{u'_*}{u_*}\Psi'_1, [\Psi'_1](r^*) = -u_*(r^*).$$

We once again argue by contradiction and assume that Ψ_1 reaches a negative minimum. From the jump condition at r^* , if this maximum is reached at an interior point, it cannot be at $r = r^*$ and the contradiction follows from the Equation. We exclude the case of a negative minimum at R through the same reasons as for p_k .

It follows that $\psi_k \leq \psi_1$ so that

$$\omega_k - \omega_1 = \psi_1(r^*) - \psi_k(r^*) \geq 0.$$

To conclude the proof of Lemma 7, it remains to prove that

$$\omega_1 > 0. \tag{62}$$

Proof of (62). We define

$$\Psi := u'_* + \psi_1.$$

We note that

$$-\frac{1}{r}(r(u'_*)')' = \left(\lambda_* + V^* - \frac{1}{r^2}\right)u'_*, \quad [(u'_*)'](r^*) = [u''](r^*) = u_*(r^*).$$

By Hopf's Lemma, $u'_*(R) < 0$ and, since u_* is \mathcal{C}^2 in \mathbb{B}^* , $u'_*(0) = 0$.

We get the following equation on Ψ :

$$-\frac{1}{r}(r\Psi')' = \left(\lambda_* + V^* - \frac{1}{r^2}\right)\Psi, \quad [\Psi'](r^*) = 0, \quad \Psi(0) = 0, \quad \Psi(R) < 0.$$

Defining

$$\Theta := \frac{\Psi}{u_*}$$

we get

$$-\frac{1}{r}(r\Theta')' = -\frac{1}{r^2}\Theta + 2\frac{u'_*}{u_*}\Theta'$$

and Θ can thus not reach a positive maximum at an interior point. Since it is negative at $r = R$ we get $\Theta \leq 0$ in $[0; R]$. Furthermore, it is not identically zero since $\Theta(R) \neq 0$, and the strong maximum principle implies $\Theta < 0$ in $(0; R)$. This gives

$$\Theta(r^*) < 0$$

or, equivalently

$$\omega_1 = -\Theta(r^*) > 0$$

and this concludes the proof of Lemma 7. \square

4. Conclusion of the proof To prove Proposition 1, we simply write

$$\begin{aligned} L''_\tau(\mathbb{B}^*)[g, g] &= \sum_{k=1}^{\infty} \omega_k \{\alpha_k^2 + \beta_k^2\} && \text{by (57)} \\ &\geq C \sum_{k=1}^{\infty} \{\alpha_k^2 + \beta_k^2\} && \text{by Lemma 7} \\ &= C \|g\|_{L^2}^2 \\ &= C \|\langle \Phi, \nu \rangle\|_{L^2}^2. \end{aligned}$$

The proof of the Proposition is now complete. \square

2.5.6 Taylor-Lagrange formula and control of the remainder

We now state the main estimate which will enable us to apply the Taylor-Lagrange formula.

Proposition 2. *Let $M > 0$ and $\eta > 0$. There exists $s \in (0; 1)$, $\varepsilon = \varepsilon(\eta) > 0$ such that for any $\Phi \in \mathcal{X}_1(\mathbb{B}^*)$ satisfying*

$$\|\Phi\|_{\mathcal{C}^{1,1}} \leq M, \|\Phi\|_{\mathcal{C}^{1,s}} \leq \varepsilon$$

there holds

$$|L''_{\tau}(\mathbb{B}_{\Phi})[\Phi, \Phi] - L''_{\tau}(\mathbb{B}^*)[\Phi, \Phi]| \leq \eta \|\langle \Phi, \nu \rangle\|_{L^2}^2. \quad (63)$$

As mentioned earlier, this will prove a quantitative inequality for sets of bounded curvature that are in a $\mathcal{C}^{1,s}$ neighbourhood of \mathbb{B}^* . Note that working in the $\mathcal{C}^{1,s}$ norm rather than in a $\mathcal{C}^{2,s}$ norm will be enough, since elliptic regularity estimates will prove sufficient for our proofs.

The proof of this proposition is technical but not unexpected in this context. We postpone the proof to Appendix C.

2.5.7 Conclusion of the proof of Step 4

Recall that we have defined

$$f(t) := \lambda(\mathbb{B}_{t\Phi}).$$

Then:

$$\begin{aligned} \lambda(\mathbb{B}_{\Phi}) - \lambda(\mathbb{B}^*) &= f(1) - f(0) \\ &= f'(0) && (= 0 \text{ because } \mathbb{B}^* \text{ is critical}) \\ &+ \int_0^1 (1-t) f''(t) dt \\ &= \frac{1}{2} f''(0) + \int_0^1 (1-t) (f''(t) - f''(0)) dt \\ &\geq C \|\langle \Phi, \nu \rangle\|_{L^2}^2 - \eta \|\langle \Phi, \nu \rangle\|_{L^2}^2 \\ &\geq \frac{C}{2} \|\langle \Phi, \nu \rangle\|_{L^2}^2 \\ &\geq C' \|\langle \Phi, \nu \rangle\|_{L^1}^2 && \text{by the Cauchy-Schwarz Inequality} \\ &= C' \delta^2 && \text{because } |\mathbb{B}_{\Phi} \Delta \mathbb{B}^*| = \delta. \end{aligned}$$

whenever $\|\nabla \Phi\|_{L^\infty}$ is small enough. This concludes the proof of Step 4.

2.5.8 A remark on the coercivity norm

The L^2 coercivity established in Proposition 1 is not only sufficient, but also optimal. Indeed, since ψ_k is non-negative in $(0; R)$, we immediately have the bound

$$0 \leq \omega_k \leq -\frac{\partial u_*}{\partial \nu}.$$

In other words, the coercivity norm for the second derivative is the L^2 (rather than the $H^{\frac{1}{2}}$) norm of the perturbation. This is due to the fact that here, in the context of parametric optimization, it is the value of the shape derivative u' rather than the value of its normal derivative that is involved in the second order shape derivative. We note that this is in sharp contrast with classical shape optimization, where the optimization is carried out with respect to the whole domain Ω , and where the coercivity norm is the $H^{\frac{1}{2}}$ norm, see [19] and the references therein.

2.6 Step 5: Conclusion of the proof of Theorem 1

We now conclude the proof of Theorem 1: let \mathcal{V}_δ be a solution of the variational problem (17) In the same way we derived the optimality conditions for the radial version of the optimization problem, that is, for the variational problem (16), it is easy to see that V_δ is equal to 0 or 1 almost everywhere and that, furthermore, if u_δ is the associated eigenfunction, that there exists two real numbers μ_δ and η_δ such that

$$\{\mathcal{V}_\delta = 1\} = \left(\{u_\delta \geq \mu_\delta\} \cap \mathbb{B}^* \right) \cup \left(\{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c \right).$$

Remark 2. We actually expect that

$$\mathcal{V}_\delta = \mathcal{H}_\delta$$

where \mathcal{H}_δ was defined in Step 3, at least for $\delta > 0$ small enough, in which case Step 4 would prove irrelevant. We were however not able to prove this fact. Put otherwise, we expect the solution to (17) to be a radially symmetric set, given the symmetries properties involved.

We introduce one last parameter: let ζ_δ be the unique real number such that

$$\left| \{u_\delta > \zeta_\delta\} \right| = V_0. \quad (64)$$

In the two figures below, we represent the two most extreme cases we might face (note that we always represent sets that are symmetric with respect to the x -axis; this is allowed by Steiner's rearrangement but this property will not be used in what follows)

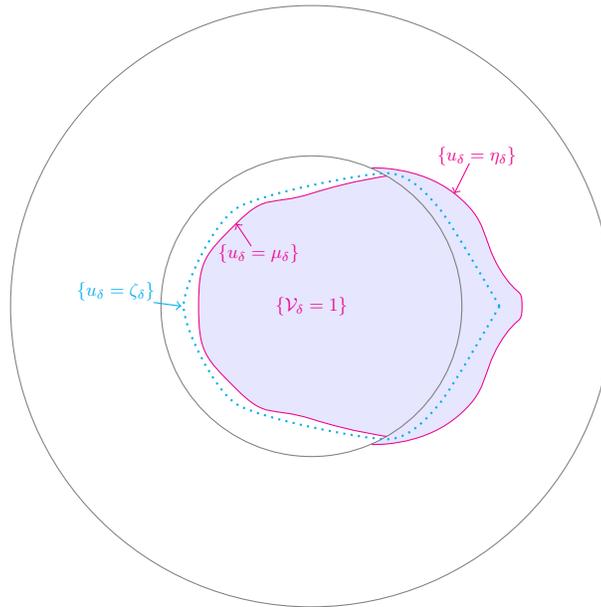


Figure 4: Here, the set \mathcal{V}_δ is connected, and we might compare it with a normal deformation of \mathbb{B}^* .

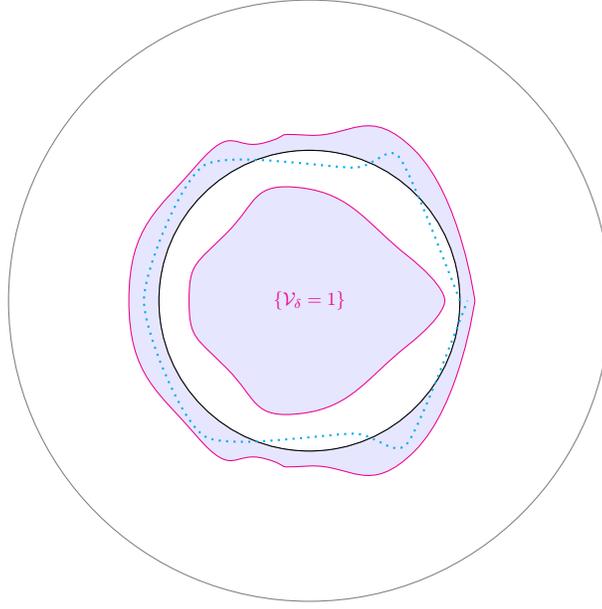


Figure 5: Here, the set \mathcal{V}_δ is disconnected, and we might compare it with a radial distribution.

To formalize this, we introduce the quantity

$$f(\delta) := |\{u_\delta \geq \zeta_\delta\} \Delta (\{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c)|.$$

Since

$$|\{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c| = \frac{\delta}{2}$$

because $\mathcal{V}_\delta \in \mathcal{M}_\delta$,

$$f(\delta) \leq \frac{\delta}{2}.$$

We now distinguish two cases:

1. First case: comparison with a radial distribution The first case is defined by

$$\frac{f(\delta)}{\delta} \xrightarrow{\delta \rightarrow 0} \ell > 0.$$

In that case, $f(\delta) \underset{\delta \rightarrow 0}{\sim} \ell \delta$. We now apply the bathtub principle: let E_δ^1 be the solution of

$$\inf_{E \subset \{u_\delta > \zeta_\delta\}, |E| = V_0 - f(\delta)} \int_{\{u_\delta \geq \zeta_\delta\} \cap E} u_\delta^2.$$

If $\zeta_{\delta,1}$ is defined through

$$|\{u_\delta > \zeta_{\delta,1}\}| = V_0 - f(\delta)$$

then $\zeta_{\delta,1} > \zeta_\delta$ and consequently

$$E_\delta^1 = \{u_\delta > \zeta_{\delta,1}\}.$$

In the same way, we define E_δ^2 as the solution of

$$E \subset \left(\{u_\delta > \zeta_\delta\} \right)^c, |E|=f(\delta) - \int_{\{u_\delta \geq \zeta_\delta\}^c \cap E} u_\delta^2$$

and, if we define $\zeta_{\delta,2}$ through the equation

$$\left| \{\zeta_\delta > u_\delta > \zeta_{\delta,2}\} \right| = f(\delta)$$

then

$$E_\delta^2 = \{\zeta_\delta > u_\delta > \zeta_{\delta,2}\}.$$

We replace V_δ by

$$\mathcal{W}_\delta := \chi_{E_\delta^1} + \chi_{E_\delta^2}.$$

From the bathtub principle,

$$- \int_{\Omega} \mathcal{V}_\delta u_\delta^2 \geq - \int_{\Omega} \mathcal{W}_\delta u_\delta^2.$$

However, \mathcal{W}_δ might not satisfy

$$\int_{\Omega} |\mathcal{W}_\delta - \chi_{\mathbb{B}^*}| = \delta.$$

We represent E_δ^i , $i = 1, 2$, below:

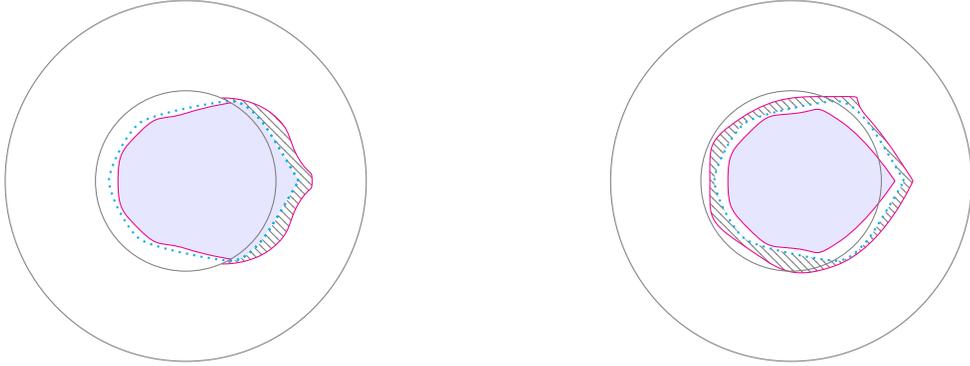
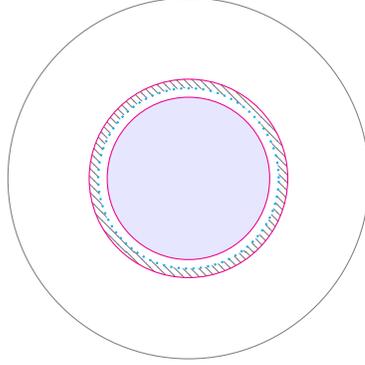


Figure 6: An illustration of the process

Finally, following the notations of step 3, we recall that $\mathbb{A}_{f(\delta)}$ is defined as

$$\mathbb{A}_{f(\delta)} := \{|x| \leq r^* - r_{f(\delta)}\} \cup \{r^* \leq |x| \leq r^* + r'_{f(\delta)}\}, \chi_{\mathbb{A}_{f(\delta)}} \in \mathcal{M}_{f(\delta)}.$$

Our competitor is $\mathbb{A}_{f(\delta)}$:



Let u_δ^* be the Schwarz rearrangement of u_δ . By equimeasurability of the Schwarz rearrangement, we have

$$\int_{\mathbb{A}_{f(\delta)}} (u_\delta^*)^2 = \int_{\Omega} \mathcal{W}_\delta u_\delta^2.$$

By the Polya-Szego Inequality (see [32]),

$$\int_{\Omega} |\nabla u_\delta^*|^2 \leq \int_{\Omega} |\nabla u_\delta|^2.$$

Finally, we have established the chain of inequalities

$$\begin{aligned} \lambda(\mathcal{V}_\delta) &\geq \int_{\Omega} |\nabla u_\delta|^2 - \int_{\Omega} \mathcal{W}_\delta u_\delta^2 \\ &\geq \int_{\Omega} |\nabla u_\delta^*|^2 - \int_{\Omega} \chi_{\mathbb{A}_{f(\delta)}} (u_\delta^*)^2 \\ &\geq \lambda(\mathbb{A}_{(f(\delta))}) \qquad \text{by the Rayleigh quotient formulation (3).} \end{aligned}$$

Now, by (19),

$$\lambda(\mathbb{A}_{(f(\delta))}) \geq \lambda_* + Cf(\delta)^2$$

and thus, since

$$f(\delta) \underset{\delta \rightarrow 0}{\sim} \ell\delta$$

we have

$$\lambda(\mathcal{V}_\delta) - \lambda_* \geq C'\ell^2\delta^2$$

which concludes the proof.

2. Second case: comparison with a normal deformation The second case is defined by

$$\frac{f(\delta)}{\delta} \underset{\delta \rightarrow 0}{\rightarrow} 0. \tag{65}$$

In this case, we use Step 4 of the proof, i.e the quantitative inequality for normal deformations of the ball.

Let us replace \mathcal{V}_δ with

$$\mathcal{W}_\delta := \chi_{\{u_\delta > \zeta_\delta\}}.$$

Recall that ζ_δ was defined in such a way that $\mathcal{W}_\delta \in \mathcal{M}(\Omega)$. By the bathtub principle,

$$\lambda(\mathcal{V}_\delta) \geq \lambda(\mathcal{W}_\delta).$$

Furthermore, Condition (65) implies

$$|\{u_\delta > \zeta_\delta\} \Delta \mathbb{B}^*| = \delta + \underset{\delta \rightarrow 0}{o}(\delta).$$

Indeed,

$$\frac{\delta}{2} = |\{u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c| = |\{u_\delta > \zeta_\delta\} \cap (\mathbb{B}^*)^c| + f(\delta).$$

Finally, standard elliptic estimates imply (in dimension 2 and 3) that

$$u_\delta \underset{\delta \rightarrow 0}{\xrightarrow{\mathcal{C}^{1,s}(\bar{\Omega})}} u_* \tag{66}$$

and, since $\frac{\partial u_*}{\partial \nu} \Big|_{\partial \mathbb{B}^*} \neq 0$ and $\zeta_\delta \underset{\delta \rightarrow 0}{\rightarrow} u_* \Big|_{\partial \mathbb{B}^*}$ it follows that $\partial \{u_\delta > \zeta_\delta\}$ is a \mathcal{C}^1 hypersurface by the implicit function Theorem.

It remains to prove that $\partial \{u_\delta > \zeta_\delta\}$ is a graph above $\partial \mathbb{B}^*$. We start by noticing that (66) implies

$$d_H(\{u_\delta > \zeta_\delta\}, \mathbb{B}^*) \underset{\delta \rightarrow 0}{\rightarrow} 0, \tag{67}$$

where d_H is the Hausdorff distance. We then argue by contradiction and assume that, for every $\delta > 0$ there exists $x_\delta \in \partial \mathbb{B}^*$, $t_1 \neq t_2 \in \mathbb{R}$ such that

$$x_\delta + t_k \nu(x_\delta) \in \partial \{u_\delta > \zeta_\delta\}, k = 1, 2.$$

It follows that

$$u_\delta(x_\delta + t_1 \nu(x_\delta)) = u_\delta(x_\delta + t_2 \nu(x_\delta))$$

and by the intermediate value Theorem and (67), there exists $t_\delta \in \mathbb{R}$ such that

$$\langle \nabla u_\delta(x_\delta + t_\delta \nu(x_\delta)), \nu(x_\delta) \rangle = 0, t_\delta \underset{\delta \rightarrow 0}{\rightarrow} 0.$$

By passing to the limit in this equation up to a subsequence, there exists a point $x_* \in \partial \mathbb{B}^*$ such that

$$\langle \nabla u_*(x_*), \nu(x_*) \rangle = 0.$$

This is a contradiction since

$$\frac{\partial u_*}{\partial \nu} \Big|_{\partial \mathbb{B}^*} \neq 0.$$

We can then say that $\partial \{u_\delta > \zeta_\delta\}$ is the graph of a function φ_δ over $\partial \mathbb{B}^*$. Besides, the Convergence result (66) implies that

$$\varphi_\delta \underset{\delta \rightarrow 0}{\xrightarrow{\mathcal{C}^1(\partial \mathbb{B}^*)}} 0.$$

Finally, since the set $\{u_\delta \geq \mu_\delta\}$ converges in the $\mathcal{C}^{1,s}$ topology to \mathbb{B}^* , there exists a uniform radius $r > 0$ such that, for any $x \in \partial \{u_\delta \geq \mu_\delta\}$, there exists y_x satisfying

$$x \in \mathbb{B}(y_x, r), \mathbb{B}(y_x, r) \subset \{u_\delta \geq \mu_\delta\}.$$

Since V_δ is constant in $\mathbb{B}(y_x, r)$, we can apply elliptic regularity results to get a uniform \mathcal{C}^2 norm on u_δ in $\{u_\delta \geq \mu_\delta\}$: there exists $M > 0$ such that, for any $x \in \{u_\delta \geq \mu_\delta\}$, $dist(x, \partial \{u_\delta, \mu_\delta\}) \leq r$, $|\nabla^2 u(x)| \leq M$. Hence, the curvature of $\{u_\delta \geq \mu_\delta\}$ is uniformly bounded by some constant M .

To prove that the curvature of $\{u_\delta > \zeta_\delta\}$ is uniformly bounded as well, we note the following fact: for any $x \in \partial\{u_\delta > \zeta_\delta\}$, let x_δ be its orthogonal projection on $\{u_\delta \geq \mu_\delta\}$. Then,

$$\mu_\delta - \zeta_\delta \underset{\delta \rightarrow 0}{\sim} |x_\delta - x| \frac{\partial u_\delta}{\partial \nu}(x_\delta)$$

and so the map $\partial\{u_\delta \geq \zeta_\delta\} \ni (x, y) \mapsto \frac{|x - x_\delta|}{|y - y_\delta|}$ converges uniformly to 1. $\{u_\delta > \zeta_\delta\}$ can thus be described, asymptotically, as $\{u_\delta \geq \mu_\delta\} + \mathbb{B}(0; t_\delta)$, so that it also has a uniformly bounded curvature.

Remark 3. We could have worked directly with $\{u_\delta \geq \mu_\delta\} + \mathbb{B}(0; t_\delta)$, by choosing a suitable t_δ but in this context, it seemed more relevant to work with level sets.

We can hence apply Step 4:

$$\begin{aligned} \lambda(\mathcal{V}_\delta) &> \lambda(\mathcal{W}_\delta) \\ &\geq \lambda_* + C |\{u_\delta > \zeta_\delta\} \Delta \mathbb{B}^*|^2 \text{ by Step 4} \\ &\geq \lambda_* + C(\delta + \underset{\delta \rightarrow 0}{o}(\delta))^2 \\ &\geq \lambda_* + C'\delta^2. \end{aligned}$$

The proof of Theorem 1 is now complete. □

3 Concluding remarks and conjecture

3.1 Extension to other domains

We do believe that this quantitative inequality is valid not only in the ball but for more general domains. Let, for any domain Ω , V_Ω be a solution of (5). Let u_Ω be the associated eigenfunction. By the bathtub principle, it is easy to see that there exists $\mu_\Omega \in \mathbb{R}$ such that

$$V_\Omega = \chi_{\{u_\Omega \geq \mu_\Omega\}} = \chi_{E_\Omega}.$$

We give the following conjecture:

Conjecture 1. *Assume that*

1. *The minimizer is regular in the sense that $\frac{\partial u_\Omega}{\partial \nu} \leq -C < 0$ on ∂E_Ω ,*
2. *E_Ω is a non-degenerate shape minimizer: for any admissible variation $\Phi \in \mathcal{X}_1(E_\Omega)$, if L_τ is the associated lagrangian, there holds*

$$L''_\tau(E_\Omega)[\Phi, \Phi] > 0.$$

Then there exists a parameter $\eta > 0$ such that, for any $V \in \mathcal{M}(\Omega)$,

$$\|V - V_\Omega\|_{L^1(\Omega)} \leq \eta \Rightarrow \lambda(V) - \lambda(V^*) \geq C\|V - V^*\|_{L^1(\Omega)}^2.$$

Here the main difficulty lies not only in the quantitative inequality for normal perturbations of the domain (Step 4 of the proof of Theorem 1) but also in the quantitative inequality for possibly disconnected competitors (Step 3 of the proof). Indeed, since the parametric derivatives u_Ω are no longer constant on the boundary of the set E_Ω , the approach used in Step 3 might fail. We note, however, that the infinitesimal quantitative bound

$$\dot{\lambda}(V_\Omega)[h] \geq C \|h\|_{L^1(\Omega)}^2$$

still holds. To see why, we notice that

$$\dot{\lambda}(V_\Omega)[h] = - \int_{\Omega} h u_\Omega^2$$

and consider, for a parameter δ , the solution h_δ of

$$\min_{h \text{ admissible at } V_\Omega, \|h\|_{L^1(\Omega)} = \delta} \dot{\lambda}(V_\Omega)[h].$$

By the bathtub principle, h_δ can be written for any δ as a level set of u_δ and, for δ small enough, one can prove that h_δ writes as follows:

$$h_\delta = \chi_{E_\delta^+} - \chi_{E_\delta^-},$$

where $E_\delta^+ \subset E_\Omega^c$ and $E_\delta^- \subset E_\Omega$ and can be described as follows: if ν is the unit normal vector to E_Ω ,

$$E_\delta^\pm := \{x \pm t\nu(x), t \in (0; t_\pm^\delta(x))\}.$$

We can then prove that

$$\frac{t_\pm^\delta}{\delta} \xrightarrow{\delta \rightarrow 0} f_\pm > 0$$

uniformly in $x \in \partial E$. It remains to apply the methods of Step 3 of the Proof of Theorem 1 and to do a Taylor expansion of $-u_\Omega^2$ at ∂E_Ω to get

$$\dot{\lambda}(V_\Omega)[h_\delta] \geq C\delta^2$$

for some constant C that depends on $\inf \frac{\partial u_\Omega}{\partial \nu}$. Thus, the infinitesimal inequality seems valid. However, it seems complicated to go further using only this information, since the parametric derivatives u are no longer constant on ∂E_Ω .

3.2 Other constraints

It would be relevant to consider perimeter constraints instead of volume constraints, but we expect the behaviour of the sequences of solutions to the auxiliary problems to be quite different. We nonetheless believe that the free boundary techniques used in [11] might apply directly to get regularity.

3.3 A conjecture

Our conjecture is that the solution of the optimization problem (17) is, for any $\delta > 0$ small enough, equal to \mathcal{H}_δ . This would make Step 4 of the proof irrelevant. Moreover, this problem seems interesting in of itself. Je sais comment faire pour une des deux parties: pour tout x , $h_-(x) = h_+(x)$ en termes de longueur. Donc ça passe. Il faut également noter que $h_- > h_+$

A Proof of Lemma 1

We briefly recall that the Schwarz rearrangement of a function $u \in W_0^{1,2}(\mathbb{B})$, $u \geq 0$ is defined as the only radially symmetric non-increasing function u^* such that, for any $t \in \mathbb{R}$,

$$|\{u \geq t\}| = |\{u^* \geq t\}|.$$

Proof of Lemma 1. We use the Polya-Szego Inequality for the Schwarz rearrangement: for any $u \in W_0^{1,2}(\mathbb{B})$, $u \geq 0$,

$$\int_{\mathbb{B}} |\nabla u^*|^2 \leq \int_{\mathbb{B}} |\nabla u|^2.$$

We also use the Hardy-Littlewood Inequality: for any $u, v \in L^2(\Omega)$,

$$\int_{\mathbb{B}} u^* v^* \geq \int_{\mathbb{B}} uv$$

and the equimeasurability of the rearrangement:

$$\int_{\mathbb{B}} u^2 = \int_{\mathbb{B}} (u^*)^2.$$

We refer to [32] for proofs. Using the Rayleigh quotient formulation (3), for any $V \in \mathcal{M}(\mathbb{B})$,

$$\begin{aligned} \lambda(V) &= \frac{\int_{\mathbb{B}} |\nabla u_V|^2 - \int_{\mathbb{B}} V u_V^2}{\int_{\mathbb{B}} u_V^2} \\ &\geq \frac{\int_{\mathbb{B}} |\nabla u_V^*|^2 - \int_{\mathbb{B}} V^* (u_V^*)^2}{\int_{\mathbb{B}} (u_V^*)^2} \\ &\geq \lambda(V^*). \end{aligned}$$

This also proves that $u_{V^*} = u_{V^*}^*$. Since the eigenvalue is simple, the eigenfunction is radially symmetric. The fact that it is decreasing follows from the Equation satisfied by u_{V^*} in polar coordinates. □

B Proof of the shape differentiability of λ

B.1 Proof of the shape differentiability

Proof of the shape differentiability. Let E be a regular subdomain of \mathbb{B} , (u_0, λ_0) be the eigenpair associated with $V := \chi_E$, and let Φ be an admissible vector field at E . Let $T_\Phi := (Id + \Phi)$ and $E_\Phi^* := T_\Phi(E)$. Let u_Φ be the eigenvalue associated with $V_\Phi := \chi_{E_\Phi}$ and λ_Φ be the associated eigenvalue. If we introduce

$$J_\Omega(\Phi) := \det(\nabla T_\Phi), A_\Phi := J_\Omega(\Phi) DT_\Phi^{-1} (DT_\Phi^{-1})^t$$

then the weak formulation of the equation on u_Φ is: for any $v \in W_0^{1,2}(\Omega)$,

$$\int_{\mathbb{B}} \langle A_\Phi \nabla u_\Phi, \nabla v \rangle = \lambda_\Phi \int_{\mathbb{B}} u_\Phi v J_\Omega(\Phi) + \int_{\mathbb{B}} V_\Phi u_\Phi v J_\Omega(\Phi).$$

We define the map F in the following way:

$$F : \begin{cases} W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times W_0^{1,2}(\mathbb{B}) \times \mathbb{R} \rightarrow W^{-1,2}(\mathbb{B}) \times \mathbb{R}, \\ (\Phi, v, \lambda) \mapsto (-\nabla \cdot (-\nabla \cdot (A_\Phi \nabla v) - \lambda v J_\Omega(\Phi) - V_\Phi v J_\Omega(\Phi)), \int_{\mathbb{B}} v^2 J_\Omega(\Phi) - 1). \end{cases}$$

It is clear from the definition of the eigenvalue that

$$F(0, u_0, \lambda_0) = 0.$$

Furthermore, the same arguments as in [18, Lemma 2.3] show that F is \mathcal{C}^∞ in $B \times W_0^{1,2}(\Omega) \times \mathbb{R}$, where B is an open ball centered at $\vec{0}$.

The differential of F at $(0, u_0, \lambda_0)$ is given by

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = \left(-\Delta w - \mu u_0 - \lambda_0 w - V w, \int_{\mathbb{B}} 2u_0 w \right).$$

To prove that this differential is invertible, it suffices to show that, if $(z, k) \in W^{-1,2}(\Omega) \times \mathbb{R}$, then there exists a unique couple (w, μ) such that

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = (z, k).$$

By the Fredholm alternative, we know that we must have

$$\mu = -\langle z, u_0 \rangle.$$

There exists a solution w_1 to the equation

$$-\Delta w - \mu u_0 - \lambda_0 w - m^* w = z.$$

We fix such a solution. Any other solution is of the form $w = w_1 + t u_0$ for a real parameter t . We look for such a t . From the equation

$$2 \int_{\Omega} u_0 w = k$$

there comes

$$t = \frac{k}{2} - \int_{\Omega} w_1 u_0.$$

hence the couple (w, μ) is uniquely determined. From the implicit function theorem, the map $\Phi \mapsto (u_\Phi, \lambda_\Phi)$ is \mathcal{C}^∞ in a neighbourhood of $\vec{0}$. \square

B.2 Computation of the first order shape derivative

Proof of Lemma 6. Let Φ be a smooth vector field at E and $E_t = T_t(E)$ where $T_t := Id + t\Phi$. We define $J_t := J_\Omega(t\Phi)$ and $A_t := A_{t\Phi}$. The other notations are the same as in the previous paragraph. Let (λ_t, u_t) be the eigencouple associated with $V_t := \chi_{E_t}$. We first define

$$v_t := u_t \circ T_t : \mathbb{B} \rightarrow \mathbb{R}.$$

The derivative of v_t with respect to t will be denoted \dot{u} . This is the material derivative, while we aim at computing the shape derivative u' defined as

$$u' = \dot{u} + \langle \Phi, \nabla u_0 \rangle.$$

For more on these notions, we refer to [30].

Obviously $v_t \in W_0^{1,2}(\Omega)$. The weak formulation on u_t writes: for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_{\mathbb{B}} \langle \nabla u_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}} u_t \varphi + \int_{E_t} u_t \varphi.$$

We do the change of variables

$$x = T_t(y),$$

so that, for any test function φ ,

$$\int_{\mathbb{B}} \langle A_t \nabla v_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}} v_t \varphi J_t + \int_E v_t \varphi J_t. \quad (68)$$

It is known that

$$\mathcal{J}(x) := \left. \frac{\partial J_t}{\partial t} \right|_{t=0} (t, x) = \nabla \cdot \Phi,$$

and that

$$\mathcal{A}(x) := \left. \frac{\partial A_t}{\partial t} \right|_{t=0} (t, x) = (\nabla \cdot \Phi) I_n - (\nabla \Phi + (\nabla \Phi)^T).$$

We recall that \mathcal{A} has the following property: if Φ_1 and Φ_2 are two vector fields, there holds

$$\langle \mathcal{A} \Phi_1, \Phi_2 \rangle = \nabla \cdot (\langle \Phi_1, \Phi_2 \rangle \Phi) - \langle \nabla (\Phi \cdot \Phi_1), \Phi_2 \rangle - \langle \nabla (\Phi \cdot \Phi_2), \Phi_1 \rangle. \quad (69)$$

We differentiate Equation (68) with respect to t to get the following equation on \dot{u} :

$$\int_{\mathbb{B}} \langle \nabla \varphi, \nabla \dot{u} + \mathcal{A} \nabla u_0 \rangle = \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \mathcal{J} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi. \quad (70)$$

Through Property (69) we get

$$\langle \mathcal{A} \nabla u_0, \nabla \varphi \rangle = \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) - \langle \nabla (\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle - \langle \nabla (\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle.$$

We deal with these three terms separately: from the divergence Formula

$$\int_{\mathbb{B}} \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) = - \int_{\partial E} [\langle \nabla u_0, \nabla \varphi \rangle] \langle \Phi, \nu \rangle.$$

We do not touch the second term.

The third term is dealt with using the weak equation on u_0 :

$$\int_{\mathbb{B}} \langle \nabla (\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle = \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 + \int_E u_0 \langle \Phi, \nabla \varphi \rangle - \int_{\partial E} [\langle \nabla u_0, \nabla \varphi \rangle] \langle \Phi, \nu \rangle.$$

Hence

$$\begin{aligned} \int_{\mathbb{B}} \langle \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}} \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) - \langle \nabla (\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle - \langle \nabla (\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle \\ &= - \int_{\mathbb{B}} \langle \nabla (\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \nabla \varphi \rangle. \end{aligned}$$

The left hand term of (70) becomes

$$\begin{aligned} \int_{\mathbb{B}} \langle \nabla \dot{u} + \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}} \langle \nabla (\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \varphi \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\mathbb{B}} \langle \nabla (\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \nabla \varphi \rangle \\ &= \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi. \end{aligned}$$

By rearranging the terms, we get

$$\begin{aligned} &\int_{\mathbb{B}} \langle \nabla (\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle \\ &= + \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \left(\int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 \right) \\ &+ \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi + \int_E u_0 \langle \Phi, \nabla \varphi \rangle. \end{aligned}$$

However, since $\mathcal{J}(x) = \nabla \cdot \Phi(x)$, we have

$$\mathcal{J}(x) \varphi + \langle \Phi, \nabla \varphi(x) \rangle = \nabla \cdot (\varphi \Phi).$$

Hence

$$\begin{aligned} \int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 &= \int_{\mathbb{B}} \nabla \cdot (\Phi \varphi) u_0 \\ &= - \int_{\mathbb{B}} \varphi \langle \Phi, \nabla u_0 \rangle, \end{aligned}$$

because u_0 satisfies homogeneous Dirichlet boundary conditions. In the same way

$$\begin{aligned} \int_E \mathcal{J}(x) u_0 \varphi + \int_E \langle \Phi, \nabla \varphi \rangle u_0 &= \int_{\mathbb{B}} \nabla \cdot (\Phi \varphi) u_0 \\ &= - \int_E \varphi \langle \Phi, \nabla u_0 \rangle + \int_{\partial E} \langle \Phi, \nu \rangle u_0 \varphi. \end{aligned}$$

We turn back to the shape derivative; recall that it is defined as

$$u' := \dot{u} - \langle \Phi, \nabla u \rangle.$$

The previous equation rewrites

$$\begin{aligned} \int_{\mathbb{B}} \langle \nabla u', \nabla \varphi \rangle &= \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} u' \varphi + \int_E u' \varphi \\ &\quad + \int_{\partial E} \langle \Phi, \nu \rangle u_0 \varphi \end{aligned}$$

Thus there appears that u' solves

$$-\Delta u' = \lambda' u_0 + \lambda_0 u_1 + V u'$$

along with Dirichlet boundary conditions and

$$\left[\frac{\partial u'}{\partial r} \right] = -\langle \Phi, \nu \rangle u_0.$$

Obtaining the jump condition on u'' is done in the same way as in [18]. \square

B.3 Gâteaux-differentiability of the eigenvalue

The parametric differentiability is also proved using the implicit function theorem applied to the following map:

$$G : \begin{cases} L^\infty(\Omega) \times W_0^{1,2}(\Omega) \times \mathbb{R} \rightarrow W^{-1,2}(\Omega) \times \mathbb{R}, \\ (h, v, \lambda) \mapsto (-\Delta v - \lambda v - (V + h)v, \int_\Omega v^2 - 1). \end{cases}$$

The invertibility of the differential follows from the same arguments as the ones used to prove the invertibility of DF in the previous section.

C Proof of Proposition 2

Proof of Proposition 2. We can not apply in a straightforward manner the methods of [19], which are well-suited for the proof of a convergence in the $H^{\frac{1}{2}}$ topology. Some minor adjustments are in order.

Let us define $T_\Phi := (Id + \Phi)$ and, for any function $f : \Omega \rightarrow \mathbb{R}$,

$$\hat{f} := f \circ T_\Phi.$$

We define the surface Jacobian

$$J_\Sigma(\Phi) := \det(\nabla T_\Phi) \left| \left({}^t \nabla T_\Phi^{-1} \right) \nu \right|,$$

the volume Jacobian

$$J_\Omega(\Phi) := \det(\nabla \Phi)$$

and, finally

$$A_\Phi := J_\Omega(\Phi) (Id + \nabla \Phi)^{-1} (Id + {}^t \nabla \Phi)^{-1}.$$

It is known (see [19, Lemma 4.8]) that

$$\|J_{\Omega/\Sigma}(\Phi) - 1\|_{L^\infty} \leq C \|\Phi\|_{W^{1,\infty}}, \|A_\Phi - 1\|_{L^\infty} \leq C \|\Phi\|_{W^{1,\infty}}. \quad (71)$$

We define u_0 as the eigenfunction associated with \mathbb{B}^* and u'_0 the shape derivative of u_0 in the direction Φ .

Finally let u'_Φ be the shape derivative in the direction Φ and $\hat{u}'_\Phi := u'_\Phi \circ T_\Phi$. Let H_Φ be the mean curvature of \mathbb{B}_Φ . Using the change of variable $y = T_\Phi(x)$, the fact that Φ is normal to \mathbb{B}^* and the value of the Lagrange multiplier τ given by (46), we get

$$\begin{aligned} L''_\tau(\mathbb{B}_\Phi)[\Phi, \Phi] &= -2 \int_{\partial \mathbb{B}^*} J_\Sigma(\Phi) \hat{u}_\Phi \hat{u}'_\Phi \langle \Phi, \nu \rangle + \int_{\partial \mathbb{B}^*} J_\Sigma(\Phi) \left(-\hat{H} \hat{u}_\Phi^2 - 2 \hat{u}_\Phi \frac{\partial \hat{u}_\Phi}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 \\ &\quad - \tau \int_{\partial \mathbb{B}^*} \hat{H} \langle \Phi, \nu \rangle^2 \\ &= -2 \int_{\partial \mathbb{B}^*} J_\Sigma(\Phi) \hat{u}_\Phi \hat{u}'_\Phi \langle \Phi, \nu \rangle + \int_{\partial \mathbb{B}^*} J_\Sigma(\Phi) \left(\hat{H} (\hat{u}_\Phi^2 - u_0^2) - 2 \hat{u}_\Phi \frac{\partial \hat{u}_\Phi}{\partial \nu} \right) \langle \Phi, \nu \rangle^2. \end{aligned}$$

Hence we have

$$\begin{aligned}
L''_\tau(\mathbb{B}_\Phi)[\Phi, \Phi] - L_\tau(\mathbb{B}^*)[\Phi, \Phi] &= -2 \int_{\partial\mathbb{B}^*} (J_\Sigma(\Phi)\hat{u}_\Phi\hat{u}'_\Phi - u_0u'_0)\langle\Phi, \nu\rangle \\
&\quad + \int_{\partial\mathbb{B}^*} \left(\hat{H}(J_\Sigma(\Phi)\hat{u}_\Phi^2 - u_0^2) \right) \langle\Phi, \nu\rangle^2 \\
&\quad + \int_{\partial\mathbb{B}^*} \left(2u_0 \frac{\partial u_0}{\partial\nu} - 2J_\Sigma(\Phi)\hat{u}_\Phi \frac{\partial\hat{u}_\Phi}{\partial\nu} \right) \langle\Phi, \nu\rangle^2.
\end{aligned} \tag{72}$$

We will prove the Proposition using the following estimates

Claim 4. *For any $\eta > 0$ there exists $\varepsilon > 0$ such that, for any Φ satisfying*

$$\|\Phi\|_{\mathcal{C}^1} \leq \varepsilon$$

there holds

1.

$$\|\hat{u}_\Phi - u_*\|_{\mathcal{C}^1(\Omega)} \leq \eta, \tag{73}$$

2.

$$\|\hat{u}'_\Phi - u'_0\|_{W_0^{1,2}} \leq \eta \|\langle\Phi, \nu\rangle\|_{L^2(\Sigma)}. \tag{74}$$

Proof of Claim 4. Estimate (73) follows from a simple contradiction argument and by using the fact that, if a sequence $\{\Phi_k\}_{k \in \mathbb{N}}$ converges in the \mathcal{C}^1 norm to 0, then u_{Φ_k} converges, in every $\mathcal{C}^{1,s}(\Omega)$ ($s < 1$) to u_0 . To prove (74), we first prove that there exists a constant M such that

$$\|\hat{u}'_\Phi\|_{W_0^{1,2}} \leq M \|\langle\Phi, \nu\rangle\|_{L^2(\Sigma)}. \tag{75}$$

By the change of variable $y := T_\Phi(x)$, we see that \hat{u}'_Φ satisfies

$$-\nabla \cdot \left(A_\Phi \nabla \hat{u}'_\Phi \right) = J_\Omega(\Phi) (\lambda_\Phi \hat{u}'_\Phi + (V^*)\hat{u}'_\Phi + \lambda'_\Phi \hat{u}_\Phi), [A_\Phi \partial_\nu \hat{u}'_\Phi] = -J_\Sigma \langle\Phi, \nu\rangle \hat{u}_\Phi \tag{76}$$

with homogeneous Dirichlet boundary conditions. The orthogonality conditions gives

$$\int_\Omega J_\Omega(\Phi) \hat{u}_\Phi \hat{u}'_\Phi = 0$$

and we will use a Spectral Gap Estimate (81) combined with a bootstrap argument.

Spectral gap estimate For any $V \in \mathcal{M}(\mathbb{B})$, $\lambda(V)$ was defined as the first eigenvalue of the operator \mathcal{L}_V defined in (1). We recalled in the Introduction that this eigenvalue is simple. Let, for any $V \in \mathcal{M}(\Omega)$, $\lambda_2(V) > \lambda(V)$ and $u_{2,V}$ be the second eigenvalue and an associated eigenfunction (we choose a L^2 normalization). We claim there exists $\omega > 0$ such that, for any $V \in \mathcal{M}(\Omega)$,

$$\omega \leq \lambda_2(V) - \lambda(V). \tag{77}$$

To prove this, we use a direct argument. Let $S(V) := \lambda_2(V) - \lambda(V)$ be the spectral gap associated with V . We consider a minimizing sequence $\{V_k\}_{k \in \mathbb{N}} \in \tilde{\mathcal{M}}(\Omega)$ (the radially assumption is not necessary here) which, up to a subsequence, converges weakly in $L^\infty - *$ to some $V_\infty \in \tilde{\mathcal{M}}(\Omega)$. It is standard to see that

$$\lambda(V_k) \xrightarrow{k \rightarrow \infty} \lambda(V_\infty), u_{V_k} \xrightarrow{k \rightarrow \infty} u_{V_\infty} \text{ strongly in } L^2(\mathbb{B}), \text{ weakly in } W_0^{1,2}(\mathbb{B}).$$

The only part which is not completely classical is to prove that

$$\lambda_2(V_k) \xrightarrow{k \rightarrow \infty} \lambda_2(V_\infty). \quad (78)$$

However, for any $k \in \mathbb{N}$, $\lambda_2(V_k)$ is defined as

$$\lambda_2(V_k) = \min_{u \in W_0^{1,2}(\mathbb{B}), \int_{\mathbb{B}} u^2 = 1, u \in \langle u_{V_k} \rangle^\perp} R_{V_k}[u], \quad (79)$$

where $\langle u \rangle^\perp$ is the subspace of functions that are L^2 -orthogonal to u , and u_{2,V_k} is defined as a minimizer for this problem (there a possibly multiple eigenfunctions). In the same way we proved that $\lambda(V)$ is uniformly bounded in V , one proves that $\lambda_2(V)$ is uniformly bounded in V . Let $\lambda_{2,\infty}$ be such that

$$\lambda_2(V_k) \xrightarrow{k \rightarrow \infty} \lambda_{2,\infty}.$$

Standard elliptic estimates prove that there exists a function $u_{2,\infty} \in W_0^{1,2}(\Omega)$ such that $u_{2,V_k} \xrightarrow{k \rightarrow \infty} u_{2,\infty}$ strongly in $L^2(\mathbb{B})$ and weakly in $W_0^{1,2}(\mathbb{B})$. Passing to the limit in

$$\int_{\mathbb{B}} u_{2,V_k} u_{V_k} = 0$$

gives

$$\int_{\mathbb{B}} u_{2,\infty} u_{V_\infty} = 0. \quad (80)$$

Passing to the limit in the weak formulation of the equation on u_{2,V_k} proves that $u_{2,\infty}$ is an eigenfunction of \mathcal{L}_{V_∞} associated with $\lambda_{2,\infty}$. It follows from the orthogonality relation (80) that

$$\lambda_{2,\infty} \geq \lambda_2(V_\infty).$$

Hence

$$\liminf_{k \rightarrow \infty} S(V_k) \geq \lambda_2(V_\infty) - \lambda(V_\infty) \geq \omega_1 > 0$$

because $\lambda(V_\infty)$ is a simple eigenvalue.

As a consequence of the spectral gap estimate (77), we get the following estimate:

$$\forall V \in \mathcal{M}(\Omega), \forall u \in \langle u_V \rangle^\perp, \omega \int_{\Omega} u^2 \leq \int_{\Omega} |\nabla u|^2 - \int_{\Omega} V u^2 - \lambda(V) \int_{\mathbb{B}} u^2. \quad (81)$$

Indeed, let $V \in \mathcal{M}(\Omega)$ and $u \in \langle u_V \rangle^\perp, u \neq 0$. Then, by the Rayleigh quotient formulation on $\lambda_2(V)$, see Equation (79),

$$\int_{\mathbb{B}} |\nabla u|^2 - \int_{\mathbb{B}} V u^2 \geq \lambda_2(V) \int_{\mathbb{B}} u^2 \geq \omega \int_{\mathbb{B}} u^2 + \lambda(V) \int_{\mathbb{B}} u^2 \text{ by (77),}$$

which is exactly the desired conclusion.

Proof of (75) First of all, multiplying (76) by \hat{u}'_Φ and integrating by parts gives

$$\int_{\Omega} A_\Phi |\nabla \hat{u}'_\Phi|^2 - \int_{\Omega} V^* J_\Omega(\Phi) (\hat{u}'_\Phi)^2 = \int_{\partial \mathbb{B}^*} J_\Sigma \hat{u}_\Phi \hat{u}'_\Phi \langle V, \nu \rangle.$$

By the Spectral gap estimate, using the fact that eigenfunctions are uniformly bounded and by continuity of the trace operator we get the existence of a constant M such that

$$\int_{\Omega} (\hat{u}'_{\Phi})^2 \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\mathbb{B}^*)} \|\hat{u}'_{\Phi}\|_{W_0^{1,2}(\Omega)}. \quad (82)$$

We rewrite

$$\|\hat{u}'_{\Phi}\|_{W_0^{1,2}(\Omega)} = \|\hat{u}'_{\Phi}\|_{L^2(\Omega)} + \|\nabla \hat{u}'_{\Phi}\|_{L^2(\Omega)}.$$

By the shape differentiability of $E \mapsto (\lambda(E), u_E)$, there exists C such that

$$\|\nabla \hat{u}'_{\Phi}\|_{L^2(\Omega)} \leq C$$

for any Φ such that $\|\Phi\|_{W^{1,\infty}} \leq 1$.

We then let $X := \|\hat{u}'_{\Phi}\|_{L^2(\Omega)}$. The estimate (82) rewrites

$$X^2 \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} X + MC \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)},$$

from where it follows that there exists $M > 0$ such that

$$\|\hat{u}'_{\Phi}\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}}.$$

We now multiply (75) by \hat{u}'_{Φ} and integrate by part, giving, for some constant M ,

$$\int_{\Omega} |\nabla \hat{u}'_{\Phi}|^2 \leq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\nabla \hat{u}'_{\Phi}\|_{L^2(\Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}^{\frac{3}{2}}$$

which in turn yields, using the same arguments,

$$\|\nabla \hat{u}'_{\Phi}\|_{L^2(\Omega)} \leq \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}}.$$

We use this in (82), giving

$$\|\hat{u}'_{\Phi}\|_{L^2(\Omega)}^2 \leq M \left(\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{u}'_{\Phi}\|_{L^2(\Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}^{\frac{3}{2}} \right).$$

This yields

$$\|\hat{u}'_{\Phi}\|_{L^2(\Omega)} \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}$$

and, finally, from the weak formulation of the equation,

$$\|\nabla \hat{u}'_{\Phi}\|_{L^2(\Omega)} \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}.$$

Proof of (74) We now turn to the proof of the continuity estimate (74), for which we will apply the same kind of bootstrap arguments, combined with a version of the splitting method, see [19, Lemma 4.10].

Let us define H_{Φ} as the solution of

$$\begin{cases} -\Delta H_{\Phi} = V_{\Phi} H_{\Phi}, \\ \left[\frac{\partial H_{\Phi}}{\partial \nu} \right] = u_{\Phi} \langle \Phi, \nu_{\Phi} \rangle \text{ on } \partial E_t, \\ H_{\Phi} = 0 \text{ on } \partial\Omega. \end{cases} \quad (83)$$

Then it appears that

$$\lambda'_{\Phi} = - \int_{\partial E_t} u_{\Phi}^2 \langle \Phi, \nu_{\Phi} \rangle = \lambda_{\Phi} \int_{\Omega} H_{\Phi} u_{\Phi}.$$

We can prove using the same bootstrap arguments used to prove (75) that

$$\|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)} \leq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}.$$

Indeed, multiplying (84) by H_Φ , doing a change of variables and integrating by parts gives

$$\int_\Omega A_t |\nabla \hat{H}_\Phi|^2 - \int_\Omega J_t \hat{H}_\Phi^2 \leq \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)}$$

and, by the variational formulation of the eigenvalue,

$$\int_\Omega \hat{H}_\Phi^2 \leq \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)}.$$

We then use the same bootstrap argument: we first prove that this implies $\|\hat{H}_\Phi\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}}$ and plug this estimate in the weak formulation of the equation. The conclusion follows.

We turn back to (74).

Let π_Φ be the orthogonal projection on $\langle u_\Phi \rangle^\perp$. We decompose u'_Φ as

$$u'_\Phi = -\pi_\Phi H_\Phi + \xi_\Phi$$

where ξ_Φ solves

$$\begin{cases} -\Delta \xi_\Phi = \lambda_\Phi \xi_\Phi + V_\Phi \xi_\Phi - \lambda_\Phi \pi_\Phi H_\Phi, \\ \xi_\Phi = 0 \text{ on } \partial\Omega. \\ \int_\Omega \xi_\Phi u_\Phi = 0. \end{cases} \quad (84)$$

Thanks to the Fredholm alternative, such a ξ_Φ exists and is uniquely defined.

We now prove that

$$\left\| \hat{H}_\Phi - H_0 \right\|_{W_0^{1,2}(\Omega)} \leq M \eta \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \quad (85)$$

for $\|\Phi\|_{\mathcal{C}^1}$ small enough. To that end, we define

$$\mathfrak{H}_\Phi := \hat{H}_\Phi - H_0.$$

Direct computation shows that

$$-\Delta \mathfrak{H}_\Phi = (V^*) \mathfrak{H}_\Phi + (V^*) \hat{H}_\Phi (J_t - 1) + \nabla \cdot \left((A_t - Id) \nabla \hat{H}_\Phi \right)$$

along with Dirichlet boundary conditions and

$$\left[\frac{\partial \mathfrak{H}_\Phi}{\partial \nu} \right] = (u_0 - J_\Sigma \hat{u}_\Phi) \langle \Phi, \nu \rangle + \left[\langle (Id - A_t) \nabla \hat{H}_\Phi, \nu \rangle \right].$$

We proceed in the same fashion: we first multiply the equation on \mathfrak{H}_Φ by \mathfrak{H}_Φ , integrate by parts and use the variational formulation of the eigenvalue to get

$$\begin{aligned} \|\mathfrak{H}_\Phi\|_{L^2(\Omega)}^2 &\leq \|J_t - 1\|_{L^\infty} \|\hat{H}_\Phi\|_{L^2(\Omega)} + \|A_t - Id\|_{L^\infty(\Omega)} \|\nabla \hat{H}_\Phi\|_{L^2(\Omega)} \|\nabla \mathfrak{H}_\Phi\|_{L^2(\Omega)} \\ &\quad + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\mathfrak{H}_\Phi\|_{W_0^{1,2}(\Omega)} \|u_0 - J_\Sigma \hat{u}_\Phi\|_{L^\infty(\partial\Omega)} \end{aligned}$$

up to a multiplicative constant. This first gives, using (73),

$$\|\mathfrak{H}_\Phi\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} (\|\nabla \Phi\|_{L^\infty} + \eta)}.$$

We then apply the same bootstrap method to get the desired conclusion.
 Finally, we need to show the following estimate, which will conclude the proof:

$$\|\hat{\xi}_{\Phi} - \xi_0\|_{W_0^{1,2}(\Omega)} \leq \eta \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \quad (86)$$

for $\|\Phi\|_{\mathcal{C}^1}$ small enough. However, this follows from the same arguments as in [19, Lemma 4.10, Paragraph 3 of the proof] and from the same bootstraps arguments. \square

Finally, going back to (72), it suffices to use the continuity of the trace to control the terms involving u'_{Φ} and Estimates (73)-(74) to conclude the proof of Proposition 2. \square

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