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1 Abstracting Linear Equation Systems

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12 — Abstract —

13 We show how to compute finite abstractions of linear equations systems over the reals. We develop
14 a general method based on elementary modes, that can be applied to the various abstractions into
15 finite structures, including the sign abstraction as used in abstract interpretation based program
16 analysis, and the difference abstraction as used in change prediction algorithms for flux networks in
17 systems biology.

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21 **1** Introduction

22 Systems of linear equations serve in abstract interpretation to abstract from the concrete
23 semantics of programs with arithmetic operations [4, 10]. In systems biology, systems of
24 linear equations are used to describe the fluxes of a reaction network in a steady state [13, 14].
25 Both applications raise a very similar question on how to compute an abstraction for linear
26 equations systems. For program analysis based on abstract interpretation [15], one might ask
27 for instance, whether the value of a variable x in a program is strictly positive if the value
28 of another variable y was. This question is about the sign abstraction of the \mathbb{R} -solution set
29 of a linear equation system (that itself abstracts from the programs semantics). In systems
30 biology [11, 17, 5, 1] one might want to know for a given flux network, whether a flux x must
31 increase (resp. decrease) if some other flux y does. This question concerns the difference
32 abstraction of a pair of \mathbb{R} -solutions of the equation system, that is of an \mathbb{R}^2 -solution of the
33 system of linear equations where addition and multiplication are defined component-wise.

We therefore study the question of how to compute finite abstractions of the solution set
of a system of linear equation over the reals. Given an homomorphism $h : \mathbb{R}^k \rightarrow \Delta$ between
 Σ -structures where $\Sigma = \{+, *, 0, 1\}$ and Δ is finite, and a linear equation system ϕ with
integer coefficients, the question is how to compute the h -abstraction of the solution set of ϕ .
If V is the set of variables of ϕ , then the set to be computed is:

$$h \circ \text{sol}^{\mathbb{R}^k}(\phi) = \{h \circ \alpha \mid \alpha : V \rightarrow \mathbb{R}^k, \text{ s.t. } \mathbb{R}^k, \alpha \models \phi\}$$

34 This problem generalizes on both questions above. The instance for the first question concerns
35 the sign abstraction $h_{\mathbb{S}} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ which satisfies $h_{\mathbb{S}}(r) = 1$ if $r > 0$, $h_{\mathbb{S}}(r) = -1$ if
36 $r < 0$ and otherwise $h_{\mathbb{S}}(0) = 0$. The second instance concerns the difference abstraction
37 $h_{\Delta_3} : \mathbb{R}^2 \rightarrow \{\uparrow, \downarrow, \sim\}$, where \uparrow stands for increase, \downarrow for decrease, and \sim for no change. It
38 satisfies $h_{\Delta_3}(r, r') = \uparrow$ if $r < r'$, $h_{\Delta_3}(r, r') = \downarrow$ if $r > r'$, and otherwise $h_{\Delta_3}(r, r') = \sim$.



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39 We next illustrate the difficulty of the problem by showing how to compute $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi)$
40 from ϕ with the existing methods. Note that the set $\text{sol}^{\mathbb{R}}(\phi)$ cannot be enumerated since it
41 is infinite. Instead we can enumerate the set of sign assignments $\alpha : V \rightarrow \{-1, 0, 1\}$ which
42 is finite. Furthermore, since $h_{\mathbb{S}}(x) = -1$ is equivalent to $x < 0$ and $h_{\mathbb{S}}(x) = 1$ to $0 < x$,
43 the system $\phi \wedge \bigwedge_{x \in V} h_{\mathbb{S}}(x) = \alpha(x)$ is equivalent to a system of linear equations and strict
44 inequations. The satisfiability of such a system can be decided in at most exponential time [6].
45 However, this method is not feasible in practice given that the number of sign assignments
46 grows exponentially with the number of variables. So the question is whether there exists a
47 more efficient algorithm for computing $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi)$.

48 From John’s overapproximation theorem [11, 17], we know for any homomorphism
49 $h : S \rightarrow \Delta$ between Σ -structures and any negation-free first-order Σ -formula ϕ that $h \circ$
50 $\text{sol}^S(\phi) \subseteq \text{sol}^\Delta(\phi)$. We call ϕ h -exact if and only if $h \circ \text{sol}^S(\phi) = \text{sol}^\Delta(\phi)$. If Δ is finite, then
51 we can compute for any h -exact formula ϕ the abstraction $h \circ \text{sol}^S(\phi)$ by computing $\text{sol}^\Delta(\phi)$
52 with finite domain constraint programming [18].

53 In a first step, we show that any integer linear matrix equation $Ax=0$ can be transformed
54 into some \mathbb{R}_+ -equivalent Σ -formula that is quasi-positive and quasi-triangular and thus
55 $h_{\mathbb{B}}$ -exact. This transformation is based on the computation of elementary modes [16, 8, 9, 19]
56 – the extreme rays of the cone $\text{sol}^{\mathbb{R}^+}(Ax=0)$ – that can be done in practice by various libraries
57 from computational geometry [3]. The conversion may take exponential time in the worst
58 case, but is often well-behaved. The correctness of the conversion relies on the fact that the
59 Σ -algebras \mathbb{R}_+ and \mathbb{B} permit unique division by nonzero natural numbers.

60 In the second step, we introduce $h_{\mathbb{B}}$ -mixed systems, which generalize on systems of linear
61 equations, positive polynomial equations $p=0$ and inequations $p \neq 0$ where polynomial p has
62 no constant term. We then show that $h_{\mathbb{B}}$ -mixed systems can be converted to an $h_{\mathbb{B}}$ -exact
63 formula too. In order to do so, extend on the results from the first step by introducing
64 the notion of $h_{\mathbb{B}}$ -invariant Σ -formulas, which subsume the polynomial equations $p=0$ and
65 inequations $p \neq 0$ for all positive polynomials p without constant terms.

66 In a third step, we rewrite linear equations systems ϕ into $h_{\mathbb{B}}$ -mixed formulas ϕ' , based on
67 the two previous steps, so that sign abstraction of $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi)$ can be computed from boolean
68 abstraction $h_{\mathbb{B}} \circ \text{sol}^{\mathbb{R}}(\phi') = \text{sol}^{\mathbb{B}}(\phi')$. It is then sufficient to compute the boolean solution set
69 $\text{sol}^{\mathbb{B}}(\phi')$ by finite domain constraint programming. The rewriting approach based the results
70 for $h_{\mathbb{B}}$ -mixed systems presented here was applied recently by the authors [1] to the difference
71 abstraction $h_{\Delta_3} : \mathbb{R}_+^2 \rightarrow \{\uparrow, \downarrow, \sim\}$ and a refinement thereof into a finite Σ -structure with 6
72 elements. This procedure was implemented and applied successfully for change prediction in
73 systems biology. It illustrates that our results presented here do provide a general framework
74 enabling the computation of various finite abstractions of linear equation systems.

75 We illustrate our results by applying the sign abstraction for program analysis based on
76 abstract interpretation. We consider the Python implementation in Fig. 1 of the function
77 `integral` : $\mathbb{R}^2 \rightarrow \mathbb{R}$ with parameter `f` : $\mathbb{R} \rightarrow \mathbb{R}$. A call `integral(a, step)` computes the
78 approximation of the integral $\int_0^a \mathbf{f}(x)dx$ with stepwidth `step`. Abstract interpretation applied
79 to this program with a polyhedral abstract domain may produce the following first-order
80 formula ϕ_{integral} :

$$81 \quad (\text{throw_exception} = 1 \iff a < 0) \wedge (\text{do_recursion} = 1 \iff \text{step} \leq a) \wedge \\ a_{\text{rec}} = a - \text{step} \wedge \text{step}_{\text{rec}} = \text{step}$$

82 This formula uses the following variables: a flag `throw_exception` that is true on exception
83 throwing; a flag `do_recursion` that is true when a recursive call is made; two variables `a_rec`,
84 `step_rec` representing the parameters passed recursively to `integral`.

```

def integral(a: float, step: float):
    if a < 0: raise ValueError('This should never happen')
    if step > a:
        return 0
    else:
        return step * f(a) + integral(a - step, step)

```

■ **Figure 1** Python function approximating the integral $\int_0^a \mathbf{f}(x)dx$ for a given function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$.

85 In order to know whether an exception may be thrown, we are interested in the sign
86 abstraction for this formula $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi_{\text{integral}})$. According to John's Theorem [11], this
87 abstraction can be overapproximated by $\text{sol}^{\mathbb{S}}(\phi_{\text{integral}})$ which in turn can be computed
88 by finite domain constraint programming. However, this approximation does not rule out
89 that a_{rec} may be strictly negative when $do_recursion$ is true, although this condition is not
90 possible when running the program. Conversely, this is correctly reflected by the abstraction
91 of its abstract interpretation $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi_{\text{integral}})$, that can be computed by converting it to a
92 $h_{\mathbb{B}}$ -mixed system which is then solved with the methods presented above.

2 Preliminaries

94 *Sets.* We start with usual notation from set theory. Let \mathbb{N} be the set natural numbers and
95 \mathbb{R}_+ the set of positive real numbers, both including 0. For any set A and $n \in \mathbb{N}$, the set
96 of n -tuples of elements in A is denoted by A^n . The i -th projection function on n -tuples of
97 elements in A , where $1 \leq i \leq n$ is the function $\pi_i : A^n \rightarrow A$ such that $\pi_i(a_1, \dots, a_n) = a_i$ for
98 all $a_1, \dots, a_n \in A$. If A is finite the number of elements of A is denote by $|A|$.

99 *Projections and Pairs.* The projection $\pi_a(f)$ of a function $f : A \rightarrow B$ is its restriction $\alpha_{|A \setminus \{a\}}$.
100 The projection of a set F of functions $f : A \rightarrow B$ is $\pi_a(F) = \{\pi_a(f) \mid f \in F\}$. Furthermore,
101 we define the pair function $f^2 : A^2 \rightarrow B^2$ such that $f^2(a_1, a_2) = (f(a_1), f(a_2))$.

102 *Σ -Algebras and Σ -Structures.* We next recall the usual notions of Σ -algebras and of homomor-
103 phism between Σ -algebras. Let $\Sigma = \cup_{n \geq 0} F^{(n)} \uplus C$ be a ranked signature. The elements of
104 $f \in F^{(n)}$ are called the n -ary function symbols of Σ and the elements in $c \in C$ its constants.

105 ► **Definition 1.** A Σ -algebra $S = (dom(S), \cdot^S)$ consists of a set $dom(S)$ and an interpretation
106 \cdot^S such that $c^S \in dom(S)$ for all $c \in C$, and $f^S : dom(S)^n \rightarrow dom(S)$ for all $f \in F^{(n)}$.

107 In order to generalize Σ -algebras to Σ -structures, we consider n -ary function symbols as
108 $n+1$ -ary relation symbols.

109 ► **Definition 2.** A Σ -structure $\Delta = (dom(\Delta), \cdot^{\Delta})$ consists of a set $dom(\Delta)$ and an interpre-
110 tation \cdot^{Δ} such that $c^{\Delta} \in dom(\Delta)$ for all $c \in C$ and $f^{\Delta} \subseteq dom(\Delta)^{n+1}$ for all $f \in F^{(n)}$.

111 Clearly, any Σ -algebra is also a Σ -structure. Note also that symbols in $F^{(0)}$ are interpreted
112 as monadic relations, i.e., as subsets of the domain, in contrast to constants in C that are
113 interpreted as elements of the domain.

114 ► **Definition 3.** A homomorphism between two Σ -structures S and Δ is a function $h :$
115 $dom(S) \rightarrow dom(\Delta)$ such that for $c \in C$, $n \in \mathbb{N}$, $f \in F^{(n)}$, and $s_1, \dots, s_{n+1} \in dom(S)$:

- 116 1. $h(c^S) = c^{\Delta}$, and
- 117 2. if $(s_1, \dots, s_{n+1}) \in f^S$ then $(h(s_1), \dots, h(s_{n+1})) \in f^{\Delta}$.

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d	d'	$d +^{\mathbb{S}} d'$	$d *^{\mathbb{S}} d'$	d	d'	$d +^{\mathbb{S}} d'$	$d *^{\mathbb{S}} d'$	d	d'	$d +^{\mathbb{S}} d'$	$d *^{\mathbb{S}} d'$
-1	1	{-1, 0, 1}	-1	0	1	1	0	1	1	1	1
-1	0	-1	0	0	0	0	0	1	0	1	0
-1	-1	-1	1	0	-1	-1	0	1	-1	{-1, 0, 1}	-1

■ **Figure 2** Interpretation of Σ -structure of signs \mathbb{S} .

118 We can convert any $n + 1$ -ary relation to a n -ary set valued functions. In this way any
 119 n -function is converted to a n -ary set valued n -functions. In other words, functions of type
 120 $D^n \rightarrow D$ are converted to functions of type $D^n \rightarrow 2^D$ where $D = \text{dom}(\Delta)$. In set-valued
 121 notation, the second condition on homomorphism can then be rewritten equivalently as
 122 $h(f^S(s_1, \dots, s_n)) \subseteq f^\Delta(h(s_1), \dots, h(s_n))$. A homomorphism for Σ -algebras thus satisfies
 123 $h(c^S) = c^\Delta$ and $h(f^S(s_1, \dots, s_n)) = f^\Delta(h(s_1), \dots, h(s_n))$.

124 ► **Definition 4.** A Σ -abstraction is a homomorphism $h: S \rightarrow \Delta$ between Σ -structures such
 125 that $\text{dom}(\Delta) \subseteq \text{dom}(S)$.

126 3 The Boolean and the Sign Abstraction

127 Throughout the paper we will use the signature $\Sigma = F^{(2)} \uplus C$ with two binary function
 128 symbols in $F^{(2)} = \{+, *\}$, and two constants $C = \{0, 1\}$. In the Σ -algebras that we will
 129 consider the functions $+$ and $*$ will be associative and commutative, with neutral element 0
 130 and 1 respectively.

131 ► **Example 5.** The set of positive reals \mathbb{R}_+ can be turned into a Σ -algebra, in which the
 132 function symbols are interpreted as addition of positive reals $+^{\mathbb{R}_+}$, multiplication of positive
 133 reals $*^{\mathbb{R}_+}$. The constants are interpreted by themselves $0^{\mathbb{R}_+} = 0$ and $1^{\mathbb{R}_+} = 1$.

134 ► **Example 6.** The set of Booleans $\mathbb{B} = \{0, 1\} \subseteq \mathbb{R}_+$ equally defines a Σ -algebra. There,
 135 the function symbols are interpreted as a disjunction $+^{\mathbb{B}} = \vee^{\mathbb{B}}$ and conjunction $*^{\mathbb{B}} = \wedge^{\mathbb{B}}$ on
 136 Booleans. The constants are interpreted by themselves $0^{\mathbb{B}} = 0$ and $1^{\mathbb{B}} = 1$.

137 In order to abstract positive real numbers into booleans, we can define a function $h_{\mathbb{B}}: \mathbb{R}_+ \rightarrow \mathbb{B}$
 138 such that $h_{\mathbb{B}}(0) = 0$ and $h_{\mathbb{B}}(r) = 1$ if $r > 0$.

139 ► **Lemma 7.** The function $h_{\mathbb{B}}: \mathbb{R}_+ \rightarrow \mathbb{B}$ is a Σ -abstraction between Σ -algebras.

140 ► **Example 8.** The set of signs $\{-1, 0, 1\} \subseteq \mathbb{R}$ can be turned into a Σ -structure $\mathbb{S} =$
 141 $(\{-1, 0, 1\}, \cdot^{\mathbb{S}})$ with the interpretation $+^{\mathbb{S}}$ and $*^{\mathbb{S}}$ given in Fig. 2. The constants are interpreted
 142 by themselves $0^{\mathbb{S}} = 0$ and $1^{\mathbb{S}} = 1$. Note that all $+^{\mathbb{S}}$ contains $(-1, 1, -1)$, $(-1, 1, 1)$ and
 143 $(-1, 1, 0)$ meaning that the sum of a strictly negative and a strictly positive real has a sign in
 144 $-1 +^{\mathbb{S}} 1$, so it may either be strictly positive, strictly negative, or zero. For this reason, \mathbb{S} is
 145 a Σ -structure but not a Σ -algebra.

146 We define the sign abstraction $h_{\mathbb{S}}: \mathbb{R} \rightarrow \mathbb{S}$ such that $h_{\mathbb{S}}(0) = 0$, $h_{\mathbb{S}}(r) = -1$ for all strictly
 147 negative reals $r < 0$ and $h_{\mathbb{S}}(r) = 1$ for all strictly positive reals $r > 0$.

148 ► **Lemma 9.** $h_{\mathbb{S}}: \mathbb{R} \rightarrow \mathbb{S}$ is a Σ -abstraction into a Σ -structure (that is not a Σ -algebra).

$$\begin{aligned}
e \in \mathcal{E}_\Sigma & ::= x \mid c \mid e \odot e' && \text{where } c \in C = \{0, 1\} \text{ and } \odot \in F^{(2)} = \{+, *\} \\
\phi \in \mathcal{F}_\Sigma & ::= e \overset{\circ}{=} e' \mid \exists x. \phi \mid \phi \wedge \phi \mid \neg \phi && \text{where } x \in \mathcal{V}
\end{aligned}$$

■ **Figure 3** First-order Σ -expressions and Σ -formulas, where $\Sigma = C \cup F^{(2)}$.

4 Abstractions and First-Order Logic

We recall the first-order logic for Σ -structures and recall John's theorem [11] on how to overapproximate the Σ -abstraction first-order Σ -formulas.

We fix a set of variables \mathcal{V} (for instance $\mathcal{V} = \mathbb{N}$). The variables in \mathcal{V} will be ranged over by x and y . The abstract syntax in Fig. 3 define the set of first-order expressions $e \in \mathcal{E}_\Sigma$ which are constructed from the function symbols in the signature Σ and the variables in \mathcal{V} . A Σ -equation is a pair $e \overset{\circ}{=} e'$ where $e, e' \in \mathcal{E}_\Sigma$. A first-order formula $\phi \in \mathcal{F}_\Sigma$ is constructed from Σ -equations with the usual first-order connectives. As shortcuts, we define the formula $true =_{\text{def}} 1 \overset{\circ}{=} 1$ and for any sequence of formulas ϕ_1, \dots, ϕ_n we define $\bigwedge_{i=1}^n \phi_i$ as $\phi_1 \wedge \dots \wedge \phi_n$ which is equal to $true$ if $n = 0$. Furthermore, we define formulas $e \neq 0$ by $\neg e \overset{\circ}{=} 0$.

The semantics of first-order logic is standard. Let S be a Σ -structure and $\alpha : V \rightarrow \text{dom}(S)$ be a variable assignment. For any expressions $e \in \mathcal{E}_\Sigma$ and variable assignment $\mathcal{V}(e) \subseteq V$, the semantics defines a subset of $\llbracket e \rrbracket^{\alpha, S} \subseteq \text{dom}(S)$, and for any formula $\phi \in \mathcal{F}_\Sigma$ with $\mathcal{V}(\phi) \subseteq V$ a truth value $\llbracket \phi \rrbracket^{\alpha, S} \in \mathbb{B}$. Expressions $e, e' \in \mathcal{E}_\Sigma$ may be set valued in the case that S is not a Σ -algebra. Therefore, the equality symbol $\overset{\circ}{=}$ will be interpreted as nondisjointness, i.e., $e \overset{\circ}{=} e'$ is true if and only if $\llbracket e \rrbracket^{\alpha, S} \cap \llbracket e' \rrbracket^{\alpha, S} \neq \emptyset$. If S is a Σ -algebra, then both sets will be singletons. Therefore, the equality symbol $\overset{\circ}{=}$ is indeed interpreted as equality for Σ -algebra, but not for Σ -structures. See Fig. 4 of the appendix for the details.

The set of solutions of a formula $\phi \in \mathcal{F}_\Sigma$ over a Σ -algebra S with respect to a set of variables V that contains $\mathcal{V}(\phi)$ is defined by $\text{sol}_V^S(\phi) = \{\alpha : V \rightarrow \text{dom}(S) \mid \llbracket \phi \rrbracket^{\alpha, S} = 1\}$. If $V = \mathcal{V}(\phi)$ we omit the index V , i.e., $\text{sol}^S(\phi) = \text{sol}_V^S(\phi)$.

► **Lemma 10 Quantification is projection.** $\text{sol}^S(\exists x. \phi) = \pi_x(\text{sol}^S(\phi))$.

Let $h : S \rightarrow \Delta$ be a Σ -abstraction and $V \subseteq \mathcal{V}$. For any subset of assignments R of type $V \rightarrow \text{dom}(S)$ we define $h \circ R = \{h \circ \alpha : V \rightarrow \text{dom}(\Delta) \mid \alpha \in R\}$.

► **Theorem 11 John's Overapproximation Theorem [1, 17, 11].** For any Σ -abstraction $h : S \rightarrow \Delta$ between Σ -structures and negation-free formula $\phi \in \mathcal{F}_\Sigma$: $h \circ \text{sol}^S(\phi) \subseteq \text{sol}^\Delta(\phi)$.

We only give a sketch of the proof. Let $\alpha : V \rightarrow \text{dom}(S)$. For any expression $e \in \mathcal{E}_\Sigma$ such that $\mathcal{V}(e) \subseteq V$ we can show that $h(\llbracket e \rrbracket^{\alpha, S}) = \llbracket e \rrbracket^{h \circ \alpha, \Delta}$ by induction on the structure of ϕ . It then follows for any positive formula $\phi \in \mathcal{F}_\Sigma$ with $\mathcal{V}(\phi) \subseteq V$ that $\llbracket \phi \rrbracket^{\alpha, S} \subseteq \llbracket \phi \rrbracket^{h \circ \alpha, \Delta}$. This is equivalent to that: $\{h \circ \alpha \mid \alpha \in \text{sol}_V^S(\phi)\} \subseteq \text{sol}_V^\Delta(\phi)$ and thus $h \circ \text{sol}_V^S(\phi) \subseteq \text{sol}_V^\Delta(\phi)$.

5 Linear Equation Systems and Elementary Modes

We are interested in systems of Σ -equation where $\Sigma = \{+, *, 1, 0\}$. The base case will be homogeneous linear equations systems with natural coefficients, which capture linear matrix integer equations $\mathbf{Ax} = 0$. We will show that elementary modes [16, 8, 9, 19] can be used to make linear integer matrix equations quasi-positive and strongly-triangular.

We also need systems of polynomial equations, with natural coefficients and no constant term, that are nonlinear. For any natural n and expression $e, e_1, \dots, e_n \in \mathcal{E}_\Sigma$, we define the

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186 expression $\prod_{i=1}^n e_i = e_1 * \dots * e_n$, which is equal to 1 if $n = 0$ and to $\sum_{i=1}^n e_i = e_1 + \dots + e_n$
 187 which is equal to 0 if $n = 0$. Furthermore, let $e^n = \prod_{i=1}^n e$ and $ne \stackrel{\text{def}}{=} \sum_{i=1}^n e$.

188 ► **Definition 12.** A Σ -equation is called positive if it has the form $e \stackrel{\circ}{=} 0$ and quasi-positive if
 189 it has the form $e \stackrel{\circ}{=} ny$, where $n \in \mathbb{N}$, $y \in \mathcal{V}$, and $e \in \mathcal{E}_\Sigma$.

190 This definition makes sense, since all constants in Σ -expressions are positive and all operators
 191 of Σ -expressions preserve positivity. Note also that any positive equation is quasi-positive
 192 since the constant 0 is equal to the polynomial $0y$. A system of Σ -equations is a conjunctive
 193 formula of the form $\bigwedge_{i=1}^n e_i \stackrel{\circ}{=} e'_i$ in \mathcal{F}_Σ . We call a system of Σ -equations positive respectively
 194 quasi-positive if all its equations are.

195 A polynomial (with natural coefficients) is an expression of the form $\sum_{j=1}^l n_j \prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}}$
 196 where l and i_j are naturals, $x_{1,1}, \dots, x_{l,i_l}$ variables, all $n_j \neq 0$ naturals called the coefficients,
 197 and all $m_{j,k} \neq 0$ naturals called the exponents. The products $\prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}}$ are called the
 198 monomials of the polynomial.

199 ► **Definition 13.** A polynomial $\sum_{j=1}^l n_j \prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}}$ with natural coefficients $n_j \neq 0$ has no
 200 constant term if none of its monomials is equal to 1, i.e., $i_j \neq 0$ for all $1 \leq j \leq l$. It is linear
 201 if all its monomials are variables, i.e. $i_j = 1$ and $m_{j,1} = \dots = m_{j,i_j} = 1$ for all $1 \leq j \leq l$.

202 Note that any linear polynomial has the form $\sum_{j=1}^l n_j x_{j,1}$ where l and all $n_j \neq 0$ are naturals
 203 and all $x_{j,1}$ are variables. In particular, linear polynomials do not have a constant term.
 204 Furthermore, note that the constant 0 is equal to the linear polynomial with $l = 0$. A
 205 polynomial equation is a Σ -equation $p \stackrel{\circ}{=} p'$ between polynomials. A (homogeneous) linear
 206 equation is a polynomial equation with linear polynomials, so without constant terms. A
 207 linear equation system is a system of linear equations.

208 An (homogeneous) linear integer matrix equation has the form $A\mathbf{y} \stackrel{\circ}{=} 0$ where A is an
 209 $n \times m$ matrix of integers for some naturals m, n such that $\mathbf{y} \in \mathcal{V}^m$. Any linear integer
 210 matrix equation can be turned into a linear equation system with natural coefficients, by
 211 bringing the negative coefficients on the right-hand side. For instance, the linear integer
 212 matrix equation on the right corresponds to the linear equation system with natural coefficients $3x \stackrel{\circ}{=} 0 \wedge 2x \stackrel{\circ}{=} 5y$.

$$\begin{pmatrix} 3 & 0 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\circ}{=} 0$$

213 This system is quasi-positive, but not positive since $5y$ appears on a right-hand side.
 214 More generally, the linear equation system for an linear integer matrix equation $A\mathbf{y} \stackrel{\circ}{=} 0$ is
 215 positive if and only if all integers in A are positive, and quasi-positive, if each line of A
 216 contains at most one negative integer. Furthermore, the above linear equation system is
 217 triangular in the following sense, but not strongly triangular:

218 ► **Definition 14.** We call a quasi-positive system of Σ -equations triangular if it has the form
 219 $\bigwedge_{l=1}^n e_l \stackrel{\circ}{=} n_l y_l$ such that the variables y_l are l -fresh for all $1 \leq l \leq n$, i.e., $y_l \notin \mathcal{V}(\bigwedge_{i=1}^{l-1} e_i \stackrel{\circ}{=} e'_i)$
 220 and if $n_l \neq 0$ then $y_l \notin \mathcal{V}(e_l)$. We call the quasi-positive polynomial system strongly-triangular
 221 if it is triangular and satisfies $n_l \neq 0$ for all $1 \leq l \leq n$.

222 Consider a linear integer matrix equation $A\mathbf{y} \stackrel{\circ}{=} 0$. If A is positive and triangular, then the
 223 corresponding linear equation system is positive and triangular too. For being quasi-positive
 224 and strongly-triangular, the integers below the diagonal of A must negative, those on the
 225 diagonal must be strictly negative, and those on the right of the diagonal must be positive.

226 ► **Theorem 15 Elementary Modes.** For any system of linear equations ϕ with natural
 227 coefficients, one can compute in at most exponential time an \mathbb{R}_+ -equivalent formula $\exists \mathbf{x}. \phi'$
 228 such that ϕ' is a quasi-positive strongly-triangular system of linear equations with natural
 229 coefficients and \mathbf{x} the sequence of variables on the left hand sides of the equations.

230 The theorem applies in particular to the linear equation systems of integer matrix
 231 equations $A\mathbf{y} \stackrel{\circ}{=} 0$. It shows that there exists a matrix E of naturals, a vector of naturals \mathbf{n} ,
 232 and a vector of fresh variables \mathbf{x} , such that $A\mathbf{y} \stackrel{\circ}{=} 0$ is \mathbb{R}_+ -equivalent to $\exists \mathbf{x}. E\mathbf{x} \stackrel{\circ}{=} \mathbf{n}\mathbf{y}$.

233 **Proof** Consider a system ϕ of linear equations with natural coefficients. Geometrically,
 234 the solution space of ϕ over the reals is a linear subspace of $\mathbb{R}^{\mathcal{V}(\phi)}$. When restricted to
 235 positive reals, as we do, this linear subspace is to be intersected with the positive cone
 236 $\mathbb{R}_+^{\mathcal{V}(\phi)}$. Therefore, $\text{sol}^{\mathbb{R}_+}(\phi)$ is a rational cone obtained by intersecting finitely many half-
 237 spaces: its H-representation is defined by the inequations $A\mathbf{y} \leq 0 \wedge A\mathbf{y} \geq 0 \wedge \mathbf{y} \geq 0$. The
 238 elementary modes of ϕ are the extreme rays of this cone and allow its V-representation.
 239 Up to normalization there are finitely many such extreme rays. Moreover, since the cone
 240 is rational and ϕ is homogeneous, the elementary modes can be normalised so that the
 241 V-representation contains only integer coefficients. The normalized elementary modes will be
 242 vectors of naturals in $\mathbb{N}^{\mathcal{V}(\phi)}$. Let e_1, \dots, e_n be the set of all normalized elementary modes
 243 in some arbitrary total order. Let E be the matrix with columns e_1, \dots, e_n . Let \mathbf{y} be the
 244 sequence of all variables of $\mathcal{V}(\phi)$ in some arbitrary total order. By construction, the variable
 245 in \mathbf{y} are pairwise distinct. According to the normalized V-representation of the system, every
 246 point of the cone is a positive linear combination of the elementary modes $\exists \mathbf{x}. E\mathbf{x} \stackrel{\circ}{=} \mathbf{n}\mathbf{y}$, where
 247 \mathbf{n} contains the normalization factors. The linear system $E\mathbf{x} \stackrel{\circ}{=} \mathbf{n}\mathbf{y}$ is quasi-positive, since
 248 E and \mathbf{n} are positive. The variables in \mathbf{x} can be chosen freshly, and thus pairwise distinct
 249 with \mathbf{y} . The linear system $E\mathbf{x} \stackrel{\circ}{=} \mathbf{n}\mathbf{y}$ is strongly-triangular, since each variable of \mathbf{y} occurs in
 250 exactly one equation and never on the left. Therefore, we can define ϕ' as $E\mathbf{x} \stackrel{\circ}{=} \mathbf{n}\mathbf{y}$. The
 251 computation of the elementary modes and thus of E can be done in at most exponential
 252 time in the size of ϕ by Motzkin's double description method [16, 8, 9].

253 6 Abstraction Exactness

254 John's overapproximation Theorem 11 shows that the set of solutions over the abstract
 255 domain $\text{sol}^{\Delta}(\phi)$ is an approximation by the abstraction of the concrete solution set $h(\text{sol}^S(\phi))$
 256 for any abstraction $h : S \rightarrow \Delta$ from concrete to abstract structure and any positive first-order
 257 formula ϕ . We say ϕ is h -exact if even equality holds.

258 **► Definition 16 Exactness.** Let $h : S \rightarrow \Delta$ be a Σ -abstraction, ϕ a Σ -formula and $V \supseteq \mathcal{V}(\phi)$.
 259 We call ϕ h -exact with respect to V if $h(\text{sol}_V^S(\phi)) = \text{sol}_V^{\Delta}(\phi)$. We call ϕ h -exact if ϕ is
 260 h -exact with respect to $\mathcal{V}(\phi)$.

261 Our next objective is to study the preservation of h -exactness by logical operators. The
 262 main difficulty of this paper is the fact that h -exactness is not preserved by conjunction.
 263 Nevertheless, as we will show next, it is preserved by disjunction and existential quantification.
 264 For the case of disjunction, we need a basic property of union which fails for intersection.

265 **► Lemma 17.** Let V be a set of variables, R_1 and R_2 be subsets of assignments of type
 266 $V \rightarrow \text{dom}(S)$ and $h : S \rightarrow \Delta$ be a Σ -abstraction. $h \circ (R_1 \cup R_2) = h \circ R_1 \cup h \circ R_2$.

267 **► Proposition 18.** The disjunction of h -exact formulas is h -exact.

268 **► Lemma 19 Projection commutes with abstraction.** For any Σ -abstraction $h : S \rightarrow \Delta$,
 269 subset R of assignments of type $V \rightarrow S$, and variable $x \in \mathcal{V}$: $h \circ \pi_x(R) = \pi_x(h \circ R)$.

270 **► Proposition 20 Quantification preserves exactness.** For any surjective Σ -abstraction
 271 $h : S \rightarrow \Delta$ and formula $\exists x.\phi \in \mathcal{F}_{\Sigma}$, if ϕ is h -exact then $\exists x.\phi$ is h -exact.

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272 We next study the h -exactness for strongly-triangular systems of Σ -equations, under the
273 condition that h is an abstraction between Σ -algebras with unique division.

274 **► Lemma 21 Singleton property.** *If S is a Σ -algebra, $e \in \mathcal{E}_\Sigma$, and $\alpha : \mathcal{V}(e) \rightarrow S$ a variable
275 assignment, then the set $\llbracket e \rrbracket^{\alpha, S}$ is a singleton.*

276 A Σ -algebra is a Σ -structure with the singleton property. Let ele be the function that
277 maps any singleton to the element that it contains.

278 **► Definition 22.** *We say that a Σ -structure S has unique division, if it satisfies the first-order
279 formula $\forall x. \exists^{=1} y. ny \overset{\circ}{=} x$ for all nonzero natural number $n \in \mathbb{N}$.*

280 Clearly, the Σ -algebras \mathbb{R}_+ and \mathbb{B} have unique division. For any element s of the domain
281 of a structure S with unique division and any nonzero natural number $n \in \mathbb{N}$, we denote by
282 $\frac{s}{n}$ the unique element of $\{\alpha(y) \mid \alpha \in sol^S(ny \overset{\circ}{=} z), \alpha(z) = s\}$.

► Lemma 23. *Let $\phi \in \mathcal{F}_\Sigma$ be a Σ -formula and S a Σ -algebra with unique division. For
nonzero natural number n , variable $y \notin \mathcal{V}(\phi)$, and expression $e \in \mathcal{E}_\Sigma$ with $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$:*

$$sol^S(\phi \wedge ny \overset{\circ}{=} e) = \{\alpha[y / \frac{ele(\llbracket e \rrbracket^{\alpha, S})}{n}] \mid \alpha \in sol^S(\phi)\}$$

283 **► Proposition 24.** *Let $\phi \in \mathcal{F}_\Sigma$ a Σ -formula, $n \neq 0$ a natural number, $e \in \mathcal{E}_\Sigma$ a Σ -expression
284 with $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$ and $y \notin \mathcal{V}(\phi)$ and the Σ -abstraction $h : S \rightarrow \Delta$ with S and Δ two Σ -algebras
285 with unique division. Then if ϕ is h -exact implies that $\phi \wedge e \overset{\circ}{=} ny$ is h -exact.*

Sketch of Proof. We can show that $h(ele(\llbracket e \rrbracket^{\alpha, S})) = ele(\llbracket e \rrbracket^{h \circ \alpha, \Delta})$ and $h(\frac{s}{n}) = \frac{h(s)}{n}$. Hence:

$$\begin{aligned} h \circ sol^S(\phi \wedge e \overset{\circ}{=} ny) &= h \circ \{\alpha[y / \frac{ele(\llbracket e \rrbracket^{\alpha, S})}{n}] \mid \alpha \in sol^S(\phi)\} && \text{by Lemma 23} \\ &= \{(h \circ \alpha)[y / h(\frac{ele(\llbracket e \rrbracket^{\alpha, S})}{n})] \mid \alpha \in sol^S(\phi)\} && \text{elementary} \\ &= \{\sigma[y / h(\frac{ele(\llbracket e \rrbracket^{\sigma, S})}{n})] \mid \sigma \in sol^\Delta(\phi)\} && h\text{-exactness of } \phi \\ &= \{\sigma[y / \frac{h(ele(\llbracket e \rrbracket^{\sigma, S}))}{n}] \mid \sigma \in sol^\Delta(\phi)\} \\ &= \{\sigma[y / \frac{ele(\llbracket e \rrbracket^{h \circ \sigma, \Delta})}{n}] \mid \sigma \in sol^\Delta(\phi)\} \\ &= sol^\Delta(\phi \wedge e \overset{\circ}{=} ny) && \text{by Lemma 23} \end{aligned}$$

286 **► Proposition 25.** *Let $h : S \rightarrow \Delta$ be a Σ -abstraction between Σ -algebras with unique division.
287 Then any strongly-triangular system of Σ -equations with natural coefficients is h -exact.*

288 **Sketch of Proof** By induction on the number of equations and Proposition 24.

289 We notice that Proposition 25 remains true for triangular systems that are not strongly-
290 triangular. This will follow with further results from the next section (Theorem 37 and
291 Proposition 31) requiring a different argument.

292 **► Theorem 26 Exactness.** *Quasi-positive strongly-triangular polynomial systems are $h_{\mathbb{B}}$ -exact.*

293 **Proof** The Σ -algebras \mathbb{R}_+ and \mathbb{B} have unique division, so we can apply Proposition 25.

294 The Elementary Modes Theorem 15 shows that any integer matrix equation $A\mathbf{x} \overset{\circ}{=} 0$ is
295 \mathbb{R}_+ -equivalent to some quasi-positive strongly-triangular linear equation system. We can
296 thus apply Theorem 26 to obtain the following corollary.

297 **► Corollary 27.** *Any matrix integer equation can be converted in at most exponential time
298 to some \mathbb{R}_+ -equivalent $h_{\mathbb{B}}$ -exact Σ -formula.*

299 This corollary permits us to compute the $h_{\mathbb{B}}$ -abstraction of an integer matrix equation by
300 computing the \mathbb{B} -solutions of the \mathbb{R}_+ -equivalent $h_{\mathbb{B}}$ -exact formula. For computing abstractions
301 between structures without unique division we need to strengthen this result.

7 Abstraction Invariance

The essential problem approached by the paper is that conjunctions of two h -exact formulas may not be h -exact. The situation changes when assuming the following notion of h -invariance for at least one of the two formulas.

► **Definition 28 Invariance.** *Let $h : S \rightarrow \Delta$ be a Σ -abstraction and $V \subseteq \mathcal{V}$ a subset of variables. We call a subset R of variable assignments of type $V \rightarrow \text{dom}(S)$ h -invariant iff:*

$$\forall \alpha, \alpha' : V \rightarrow \text{dom}(S). (\alpha \in R \wedge h \circ \alpha = h \circ \alpha' \implies \alpha' \in R).$$

We call a Σ -formula ϕ h -invariant if its solution set $\text{sol}^S(\phi)$ is.

The relevance of the notion of invariance for exactness of conjunctions – that we will formalize in Proposition 31 – is due to the the following lemma:

► **Lemma 29.** *If either R_1 or R_2 are h -invariant then: $h \circ (R_1 \cap R_2) = h \circ R_1 \cap h \circ R_2$.*

Sketch of Proof. The one inclusion is straightforward without invariance. For the other inclusion, we can assume with loss of generality that R_1 is h -invariant. So let $\beta \in h \circ R_1 \cap h \circ R_2$. Then there exist $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$ such that $\beta = h \circ \alpha_1 = h \circ \alpha_2$. By h -invariance of R_1 it follows that $\alpha_1 \in R_2$. So $\alpha_1 \in R_1 \cap R_2$, and hence, $\beta \in h \circ (R_1 \cap R_2)$.

We continue with an algebraic characterization of h -invariance. Given a Σ -abstraction $h : S \rightarrow \Delta$, and a set R of variable assignments to $\text{dom}(\Delta)$, we define the left-decomposition of R with respect to h as the following set of variable assignments to $\text{dom}(S)$:

$$h \circ R \quad =_{\text{def}} \{ \alpha \mid h \circ \alpha \in R \}$$

Clearly, $R \subseteq h \circ (h \circ R)$. The inverse inclusion characterizes the h -invariance of R .

► **Lemma 30 Algebraic characterization.** *A subset of R variables assignment of type $V \rightarrow \text{dom}(S)$ is h -invariant for an Σ -abstraction $h : S \rightarrow \Delta$ iff $h \circ (h \circ R) \subseteq R$.*

► **Proposition 31 Exactness is preserved by conjunction when assuming invariance.** *Let h be a surjective Σ -abstraction. If ϕ_1 and ϕ_2 are h -exact Σ -formulas and ϕ_1 or ϕ_2 are h -invariant then the conjunction $\phi_1 \wedge \phi_2$ is h -exact.*

Proof Let ϕ_1 and ϕ_2 be h -exact Σ -formulas. We assume without loss of generality that ϕ_1 is h -invariant. Let $V = \mathcal{V}(\phi_1 \wedge \phi_2)$. Since $\mathcal{V}(\phi_2) \subseteq V$ the set $\text{sol}_V^S(\phi_2)$ is h -invariant too by Lemma 54. We can now show that $\phi_1 \wedge \phi_2$ is h -exact as follows:

$$\begin{aligned} h \circ \text{sol}^S(\phi_1 \wedge \phi_2) &= h \circ (\text{sol}_V^S(\phi_1) \cap \text{sol}_V^S(\phi_2)) \\ &= h \circ \text{sol}_V^S(\phi_1) \cap h \circ \text{sol}_V^S(\phi_2) \quad \text{by Lemma 29} \\ &= \text{sol}_V^\Delta(\phi_1) \cap \text{sol}_V^\Delta(\phi_2) \quad \text{by } h\text{-exactness of } \phi_1 \text{ and } \phi_2 \text{ wrt } V \\ &= \text{sol}^\Delta(\phi_1 \wedge \phi_2) \end{aligned}$$

Our next objective is to show that h -invariant formulas are closed under conjunction, disjunction, and existential quantification. The two former closure properties rely on the following two algebraic properties of abstraction decomposition.

► **Lemma 32.** *For any Σ -abstraction $h : S \rightarrow \Delta$, any subsets of assignments of type $V \rightarrow \text{dom}(S)$ R_1 and R_2 and V a subset of variables:*

$$\blacksquare \quad h \circ (R_1 \cap R_2) = h \circ R_1 \cap h \circ R_2.$$

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327 ■ $h \ominus (R_1 \cup R_2) = h \ominus R_1 \cup h \ominus R_2.$

328 ► **Lemma 33 Intersection and union preserve invariance.** *Let $h : S \rightarrow \Delta$ be a Σ -abstraction.*
 329 *Then the intersection and union of any two h -invariant subsets R_1 and R_2 of variables*
 330 *assignments of type $V \rightarrow \text{dom}(S)$ is h -invariant.*

331 ► **Lemma 34 Projection commutes with left-decomposition.** $h \ominus \pi_x(R) = \pi_x(h \ominus R).$

332 ► **Proposition 35 Invariance is preserved by conjunction, disjunction, and quantification.**
 333 *If h is a surjective abstraction then the class of h -invariant FO-formulas is closed under*
 334 *conjunction, disjunction, and existential quantification.*

335 We do not know whether negation preserves h -invariance in general, but for finite Δ it
 336 can be shown that if ϕ is h -exact and h -invariant, then $\neg\phi$ is h -exact and h -invariant too.

337 ► **Proposition 36.** *Let h be a surjective Σ -abstractions. Then the class of h -exact and h -*
 338 *invariant Σ -formulas is closed under conjunction, disjunction and existential quantification.*

339 ► **Theorem 37 Invariance.** *Any positive polynomial equation $p \stackrel{\circ}{=} 0$ such that p has no constant*
 340 *term is $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant.*

Sketch of Proof. Any positive polynomial equation $p \stackrel{\circ}{=} 0$ such that p has no constant
 term and only positive coefficients has the form $\sum_{j=1}^l n_j \prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}} \stackrel{\circ}{=} 0$ where $l \geq 0$, and
 $n_j, i_j, m_{j,k} > 0$. We can now show that for both algebras $S \in \{\mathbb{B}, \mathbb{R}_+\}$ that:

$$\text{sol}^S(p \stackrel{\circ}{=} 0) = \text{sol}^S\left(\bigwedge_{j=1}^l \bigvee_{k=1}^{i_j} x_{j,k} \stackrel{\circ}{=} 0\right)$$

341 Since the formulas $x_{j,k} \stackrel{\circ}{=} 0$ are $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant, the theorem follows from the closure
 342 properties of Proposition 36.

8 Boolean Abstractions of $h_{\mathbb{B}}$ -Mixed Systems

344 In this section we prove our main result stating how to compute the $h_{\mathbb{B}}$ -abstraction of the
 345 \mathbb{R}_+ -solution set of a $h_{\mathbb{B}}$ -mixed systems exactly.

346 ► **Definition 38.** *A $h_{\mathbb{B}}$ -mixed system is a formula in \mathcal{F}_{Σ} of the form $\exists \mathbf{z}. \phi \wedge \phi'$ where ϕ is a*
 347 *linear equation system and ϕ' a $h_{\mathbb{B}}$ -invariant and $h_{\mathbb{B}}$ -exact first-order formula.*

348 Note that linear equation systems $A\mathbf{y} \stackrel{\circ}{=} 0$, with A an integer matrix and \mathbf{y} a sequence of
 349 pairwise distinct variables, need not to be $h_{\mathbb{B}}$ -exact, if A is not positive. However, any linear
 350 equation systems of this shape is \mathbb{R}_+ -equivalent to some quasi-positive strongly-triangular
 351 polynomial system, as shown by the Elementary Modes Theorem 15. And quasi-positive
 352 strongly-triangular polynomial system were shown to be $h_{\mathbb{B}}$ -exact by Exactness Theorem 26.

353 ► **Theorem 39 Main.** *Any $h_{\mathbb{B}}$ -mixed system can be converted in exponential time to an*
 354 *\mathbb{R}_+ -equivalent Σ -formula that is $h_{\mathbb{B}}$ -exact.*

355 **Proof** Consider a $h_{\mathbb{B}}$ -mixed system $\exists \mathbf{x}. (\phi \wedge \phi')$ where ϕ is a linear equation system and
 356 ϕ' a first-order formula that is $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant. Based on the Elementary Modes
 357 Theorem 15, the linear equation system ϕ can be transformed in exponential time to the
 358 form $\exists \mathbf{z}. \phi''$ where ϕ'' is a quasi-positive strongly-triangular system of linear equations. Such
 359 polynomial equation systems are $h_{\mathbb{B}}$ -exact by Theorem 26, and so is ϕ'' . The Invariance
 360 Proposition 31 shows that the conjunction $\phi'' \wedge \phi'$ is $h_{\mathbb{B}}$ -exact too, since ϕ' was assumed to
 361 be $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant. Finally, $h_{\mathbb{B}}$ -exactness is preserved by existential quantification
 362 by Proposition 20, so the formula $\exists \mathbf{x}. \exists \mathbf{z}. \phi'' \wedge \phi'$ is $h_{\mathbb{B}}$ -exact too.

363 ► **Corollary 40.** *The $h_{\mathbb{B}}$ -abstraction of the \mathbb{R}_+ -solution set of an $h_{\mathbb{B}}$ -mixed system ϕ , that is*
 364 *$h_{\mathbb{B}} \circ \text{sol}^{\mathbb{R}_+}(\phi)$, can be computed in at most exponential time in the size of the system ϕ .*

365 The algorithm from the proof Corollary 40 can be improved so that it becomes sufficiently
 366 efficient for practical use. For this the two steps with exponential worst case complexity must
 367 be made polynomial for the particular instances. First note that the computation of the
 368 elementary modes (Theorem 15) is known to be computationally feasible. Various algorithms
 369 for this purpose were implemented [7, 12, 2, 3] and applied successfully to problems in
 370 systems biology [9]. The second exponential step concerns the enumeration of all boolean
 371 variable assignments. This enumeration may be avoided by using constraint programming
 372 techniques for computing the solution set $\text{sol}^{\mathbb{B}}(\phi'')$. For those $h_{\mathbb{B}}$ -mixed systems for which
 373 both steps can be done in polynomial time, we can compute the boolean abstraction of
 374 the \mathbb{R}_+ -solution set in polynomial time too. The practical feasibility of this approach was
 375 demonstrated recently at an application to knockout prediction in systems biology [1], where
 376 previously only over-approximations could be computed.

377 9 Computing Sign Abstractions

378 We next show how to compute the sign abstraction $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi)$ of the \mathbb{R} -solutions set of a
 379 linear equation system ϕ . For this, we convert ϕ into a first-order formula ϕ' based on our
 380 main Theorem 39 such that $h_{\mathbb{S}} \circ \text{sol}^{\mathbb{R}}(\phi)$ can be computed from $\text{sol}^{\mathbb{B}}(\phi')$ in polynomial time.
 381 In order to do so, we relate in a first step the sign abstraction to the boolean abstraction,
 382 then show in a second step that this relationship can be defined in first-order logic, so that
 383 our Main Theorem for the boolean abstraction can be applied.

In the first step, we relate the sign abstraction to the boolean abstraction. For doing so,
 we decompose any real number r into two positive numbers, its negative part $\ominus(r)$ and its
 positive part $\oplus(r)$, such that if $r \geq 0$ then $\ominus(r) = 0$ and $\oplus(r) = r$ and otherwise $\ominus(r) = -r$
 and $\oplus(r) = 0$. The decomposition function $\text{dec} : \mathbb{R} \rightarrow (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \{0\})$ is defined as
 follows for $r \in \mathbb{R}$:

$$\text{dec}(r) = (\ominus(r), \oplus(r))$$

384 This function is a bijection, so it has an inverse function $\text{dec}^{-1} : (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \{0\}) \rightarrow \mathbb{R}$,
 385 which satisfies $\text{dec}^{-1}((r_1, r_2)) = r_2 - r_1$ for all pairs (r_1, r_2) in its domain.

386 ► **Lemma 41 Decomposition.** $h_{\mathbb{S}} = \text{dec}^{-1} \circ h_{\mathbb{B}}^2 \circ \text{dec}$

387 **Proof** If r is negative then $\text{dec}^{-1}(h_{\mathbb{B}}^2(\text{dec}(r))) = \text{dec}^{-1}(h_{\mathbb{B}}^2((-r, 0))) = \text{dec}^{-1}((h_{\mathbb{B}}(-r), 0)) =$
 388 $-h_{\mathbb{B}}(-r) = h_{\mathbb{S}}(r)$. Otherwise if r is positive then $\text{dec}^{-1}(h_{\mathbb{B}}^2(\text{dec}(r))) = \text{dec}^{-1}(h_{\mathbb{B}}^2((0, r))) =$
 389 $\text{dec}^{-1}((0, h_{\mathbb{B}}(r))) = h_{\mathbb{B}}(r) = h_{\mathbb{S}}(r)$.

We will show in a first step that first-order formulas over the reals can be rewritten, such
 that interpretation over the positive reals is enough. We start with a definition of positivity
 of reals in first-order logic. For any variable $x \in \mathcal{V}$ we define the formulas $\text{pos}(x) \in \mathcal{F}_{\Sigma}$ by:

$$\text{pos}(x) =_{\text{def}} \exists z. x \stackrel{\circ}{=} z * z$$

390 Clearly, if $\alpha \in \text{sol}^{\mathbb{R}}(\text{pos}(x))$ then $\alpha(x) \in \mathbb{R}_+$. We can use this formula to relate \mathbb{R}_+ -solutions
 391 to \mathbb{R} -solutions of particular formulas.

392 ► **Definition 42.** *A formula $\phi \in \mathcal{F}_{\Sigma}$ is called domain-positive if ϕ has the form $\phi' \wedge$
 393 $\bigwedge_{y \in \mathcal{V}(\phi')} \text{pos}(y)$ as well as all formulas ϕ'' for which $\exists x. \phi''$ is a subformula of ϕ .*

394 ► **Lemma 43.** *All domain-positive formulas $\phi \in \mathcal{F}_{\Sigma}$ satisfy $\text{sol}^{\mathbb{R}_+}(\phi) = \text{sol}^{\mathbb{R}}(\phi)$.*

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395 **Proof** By induction on the structure of formulas. The induction step is straightforward
 396 from due to the fact that $pos(y)$ imposes the positivity of the value y .

397 We next show how to make first-order formulas domain-positive based on the decompo-
 398 sition of real numbers into two positive numbers presented earlier. We fix two generators
 399 of fresh variable $\nu_{\ominus}, \nu_{\oplus} : \mathcal{V} \rightarrow \mathcal{V}$. For any $x \in \mathcal{V}$, the intention is that $\nu_{\oplus}(x)$ stands
 400 for the positive part of x and $\nu_{\ominus}(x)$ for its negative part. We will preserve the invariant
 401 that $x = \nu_{\oplus}(x) - \nu_{\ominus}(x)$ and $\nu_{\oplus}(x) * \nu_{\ominus}(x) = 0$. Furthermore, let $\nu : \mathcal{V} \rightarrow \mathcal{V}^2$ such that
 402 $\nu(x) = (\nu_{\oplus}(x), \nu_{\ominus}(x))$ for all $x \in \mathcal{V}$.

► **Proposition 44 Positivity.** *For any formula $\phi \in \mathcal{F}_{\Sigma}$ we can compute in linear time a formula $dec_{\nu}(\phi) \in \mathcal{F}_{\Sigma}$ such that:*

$$dec \circ sol^{\mathbb{R}}(\phi) = \{\alpha^2 \circ \nu|_{\mathcal{V}(\phi)} \mid \alpha \in sol^{\mathbb{R}+}(dec_{\nu}(\phi) \wedge \bigwedge_{x \in \mathcal{V}(\phi)} \nu_{\oplus}(x) * \nu_{\ominus}(x) \stackrel{\circ}{=} 0)\}$$

403 *Furthermore, if ϕ was a linear equation system, then $dec_{\nu}(\phi)$ is a $h_{\mathbb{B}}$ -mixed system.*

Proof We can assume without loss of generality that all equations of ϕ are flat, i.e., of the form $x \stackrel{\circ}{=} x_1 + x_2$, $x \stackrel{\circ}{=} x_1 * x_2$, $x \stackrel{\circ}{=} 0$, or $x \stackrel{\circ}{=} 1$. We define the formulas $dec_{\nu}(\phi), dec'_{\nu}(\phi) \in \mathcal{F}_{\Sigma}$ for all formulas ϕ in \mathcal{F}_{Σ} with flat equations recursively as follows:

$$\begin{aligned} dec_{\nu}(\exists x.\phi) &= \exists \nu_{\ominus}(x). \exists \nu_{\oplus}(x). dec_{\nu}(\phi) & dec_{\nu}(\phi \wedge \phi') &= dec_{\nu}(\phi) \wedge dec_{\nu}(\phi') \\ &\wedge \nu_{\oplus}(x) * \nu_{\ominus}(x) \stackrel{\circ}{=} 0 & dec_{\nu}(\neg\phi) &= \neg dec_{\nu}(\phi) \\ &\wedge pos(\nu_{\oplus}(x)) \wedge pos(\nu_{\ominus}(x)) \\ dec_{\nu}(x \stackrel{\circ}{=} x_1 + x_2) &= & dec_{\nu}(x \stackrel{\circ}{=} x_1 * x_2) &= \\ \nu_{\oplus}(x) + \nu_{\ominus}(x_1) + \nu_{\ominus}(x_2) \stackrel{\circ}{=} & & \nu_{\oplus}(x) + \nu_{\oplus}(x_1) * \nu_{\ominus}(x_2) + \nu_{\ominus}(x_1) * \nu_{\oplus}(x_2) \stackrel{\circ}{=} & \\ \nu_{\ominus}(x) + \nu_{\oplus}(x_1) + \nu_{\oplus}(x_2) & & \nu_{\ominus}(x) + \nu_{\oplus}(x_1) * \nu_{\oplus}(x_2) + \nu_{\ominus}(x_1) * \nu_{\ominus}(x_2) & \\ dec_{\nu}(x \stackrel{\circ}{=} 0) &= \nu_{\oplus}(x) \stackrel{\circ}{=} \nu_{\ominus}(x) & dec_{\nu}(x \stackrel{\circ}{=} 1) &= \nu_{\oplus}(x) \stackrel{\circ}{=} \nu_{\ominus}(x) + 1 \end{aligned}$$

The rewriting for equations with multiplication relies on the distributivity law. We also use inverses for addition in the structure of the reals. Let

$$dec'_{\nu}(\phi) = dec_{\nu}(\phi) \wedge \bigwedge_{x \in \mathcal{V}(\phi)} \nu_{\oplus}(x) * \nu_{\ominus}(x) \stackrel{\circ}{=} 0 \wedge pos(\nu_{\ominus}(x)) \wedge pos(\nu_{\oplus}(x)).$$

404 We can show by induction on the structure of formulas that all formulas $dec'_{\nu}(\phi)$ are domain-
 405 positive and satisfy $dec \circ sol^{\mathbb{R}}(\phi) = \{\alpha^2 \circ \nu|_{\mathcal{V}(\phi)} \mid \alpha \in sol^{\mathbb{R}}(dec'_{\nu}(\phi))\}$. The proposition follows
 406 with \mathbb{R}_+ instead of \mathbb{R} from Lemma 43 and the domain-positivity of $dec'_{\nu}(\phi)$.

► **Theorem 45 Computing Sign Abstractions.** *For any linear equation system $\phi \in \mathcal{F}_{\Sigma}$ we can compute in at most exponential time a formula $\phi' \in \mathcal{F}_{\Sigma}$ such that:*

$$h_{\mathbb{S}} \circ sol^{\mathbb{R}}(\phi) = \{[y/\beta(\nu_{\oplus}(y)) - \beta(\nu_{\ominus}(y))] \mid y \in \mathcal{V}(\phi') \mid \beta \in sol^{\mathbb{B}}(\phi')\}$$

Proof Let $\phi \in \mathcal{F}_{\Sigma}$ be a system of linear equations. The formula $dec_{\nu}(\phi) \wedge \bigwedge_{x \in \mathcal{V}(\phi)} \nu_{\oplus}(x) * \nu_{\ominus}(x) \stackrel{\circ}{=} 0$ is a $h_{\mathbb{B}}$ -mixed system by Positivity Proposition 44, so that we can apply the Main Theorem 39. Therefore, we can compute in at most exponential time an \mathbb{R}_+ -equivalent formula ϕ' that is $h_{\mathbb{B}}$ -exact.

$$\begin{aligned} &h_{\mathbb{S}} \circ sol^{\mathbb{R}}(\phi) \\ &= dec^{-1} \circ h_{\mathbb{B}}^2 \circ dec \circ sol^{\mathbb{R}}(\phi) && \text{by Decomposition Lemma 41} \\ &= dec^{-1} \circ h_{\mathbb{B}}^2 \circ \{\alpha^2 \circ \nu|_{\mathcal{V}(\phi)} \mid \alpha \in sol^{\mathbb{R}+}(dec_{\nu}(\phi) \wedge \bigwedge_{x \in \mathcal{V}(\phi)} \nu_{\oplus}(x) * \nu_{\ominus}(x) \stackrel{\circ}{=} 0)\} && \text{by Positivity Proposition 44} \\ &= dec^{-1} \circ h_{\mathbb{B}}^2 \circ \{\alpha^2 \circ \nu|_{\mathcal{V}(\phi')} \mid \alpha \in sol^{\mathbb{R}+}(\phi')\} && \text{where } \phi' \text{ from Main Theorem 39} \\ &= \{dec^{-1} \circ \beta^2 \circ \nu|_{\mathcal{V}(\phi')} \mid \beta \in sol^{\mathbb{B}}(\phi')\} && \text{by } h_{\mathbb{B}}\text{-exactness of } \phi' \\ &= \{[y/\beta(\nu_{\oplus}(y)) - \beta(\nu_{\ominus}(y))] \mid y \in \mathcal{V}(\phi') \mid \beta \in sol^{\mathbb{B}}(\phi')\} && \text{by definition of } dec^{-1} \end{aligned}$$

407

408 ——— **References** ———

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Interpretation of expressions as sets of elements $\llbracket e \rrbracket^{\alpha, S} \subseteq \text{dom}(S)$, where S is a Σ -structure and $\alpha : V \rightarrow \text{dom}(S)$ where V contains all free variables.

$$\llbracket c \rrbracket^{\alpha, S} = c^S \quad \llbracket x \rrbracket^{\alpha, S} = \{\alpha(x)\} \quad \llbracket e \odot e' \rrbracket^{\alpha, S} = \cup \{s \odot^S s' \mid s \in \llbracket e \rrbracket^{\alpha, S}, s' \in \llbracket e' \rrbracket^{\alpha, S}\}$$

Interpretation of formulas as truth values $\llbracket \phi \rrbracket^{\alpha, S} \in \mathbb{B}$:

$$\begin{aligned} \llbracket e \overset{\circ}{=} e' \rrbracket^{\alpha, S} &= \begin{cases} 1 & \text{if } \llbracket e \rrbracket^{\alpha, S} \cap \llbracket e' \rrbracket^{\alpha, S} \neq \emptyset \\ 0 & \text{else} \end{cases} & \llbracket \phi \wedge \phi' \rrbracket^{\alpha, S} &= \llbracket \phi \rrbracket^{\alpha, S} \wedge^{\mathbb{B}} \llbracket \phi' \rrbracket^{\alpha, S} \\ \llbracket \neg \phi \rrbracket^{\alpha, S} &= \neg^{\mathbb{B}}(\llbracket \phi \rrbracket^{\alpha, S}) & \llbracket \exists x. \phi \rrbracket^{\alpha, S} &= \begin{cases} 1 & \text{if exists } s \in \text{dom}(S). \\ & \llbracket \phi \rrbracket^{\alpha[x/s], S} = 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

■ **Figure 4** Semantics of Σ -expressions and formulas over a Σ -structure S with respect to a variable assignment $\alpha : V \rightarrow \text{dom}(S)$.

A Proofs for Section 3 (The Boolean and the Sign Abstraction)

► **Lemma 7.** *The function $h_{\mathbb{B}} : \mathbb{R}_+ \rightarrow \mathbb{B}$ is a Σ -abstraction between Σ -algebras.*

Proof. For all $r, r' \in \mathbb{R}_+$ we have:

$$\begin{aligned} h_{\mathbb{B}}(r +^{\mathbb{R}_+} r') = 1 &\Leftrightarrow r +^{\mathbb{R}_+} r' \neq 0 \Leftrightarrow r \neq 0 \vee r' \neq 0 \Leftrightarrow h_{\mathbb{B}}(r) = 1 \vee h_{\mathbb{B}}(r') = 1 \\ h_{\mathbb{B}}(r *^{\mathbb{R}_+} r') = 1 &\Leftrightarrow r *^{\mathbb{R}_+} r' \neq 0 \Leftrightarrow r \neq 0 \wedge r' \neq 0 \Leftrightarrow h_{\mathbb{B}}(r) = 1 \wedge h_{\mathbb{B}}(r') = 1 \end{aligned}$$

Hence $h_{\mathbb{B}}(r +^{\mathbb{R}_+} r') = h_{\mathbb{B}}(r) +^{\mathbb{B}} h_{\mathbb{B}}(r')$ and $h_{\mathbb{B}}(r *^{\mathbb{R}_+} r') = h_{\mathbb{B}}(r) *^{\mathbb{B}} h_{\mathbb{B}}(r')$. Finally, for both constants $c \in C$ we have that $h_{\mathbb{B}}(c^{\mathbb{R}_+}) = h_{\mathbb{B}}(c) = c = c^{\mathbb{B}}$. ◀

► **Lemma 9.** *$h_{\mathbb{S}} : \mathbb{R} \rightarrow \mathbb{S}$ is a Σ -abstraction into a Σ -structure (that is not a Σ -algebra).*

Proof. For any $r, r' \in \mathbb{R}$ the second condition for homomorphism follows for all $\odot \in F^{(2)}$: $h_{\mathbb{S}}(r \odot r') \in h_{\mathbb{S}}(r) \odot^{\mathbb{S}} h_{\mathbb{S}}(r')$. And for all constants $c \in C$ we have by definition that $h_{\mathbb{S}}(c^{\mathbb{R}}) = c^{\mathbb{S}}$. ◀

B Proofs for Section 4 (Abstractions and First-Order Logic)

The following two lemmas are classical. Let R be a subset of assignments of type $V' \rightarrow \text{dom}(S)$ and $V \cap V' = \emptyset$ two subsets of \mathcal{V} . We define: $\text{ext}_V^S(R) = \{\alpha \cup \alpha' \mid \alpha' : V' \cup V \rightarrow \text{dom}(S), \alpha \in R\}$.

► **Lemma 46 Cylindrification.** *If $V \cap \mathcal{V}(\phi) = \emptyset$ then: $\text{sol}_{V \cup \mathcal{V}(\phi)}^S(\phi) = \text{ext}_V^S(\text{sol}^S(\phi))$.*

Proof. We can show for all expressions $e \in \mathcal{E}_{\Sigma}$ with $\mathcal{V}(e)$ disjoint to V and any variables assignment $\alpha : \mathcal{V}(e) \cup V \rightarrow \text{dom}(S)$ that $\llbracket e \rrbracket^{\alpha, S} = \llbracket e \rrbracket^{\alpha|_{\mathcal{V}(e)}, S}$. This is by induction on the structure of expressions. It follows for all formulas $\phi \in \mathcal{F}_{\Sigma}$ such that $\mathcal{V}(\phi)$ disjoint from V and $\alpha : \mathcal{V}(\phi) \cup V \rightarrow \text{dom}(S)$ that $\llbracket \phi \rrbracket^{\alpha, S} = \llbracket \phi \rrbracket^{\alpha|_{\mathcal{V}(\phi)}, S}$. This is by induction on the structure of formulas. It implies the lemma. ◀

► **Lemma 10 Quantification is projection.** *$\text{sol}^S(\exists x. \phi) = \pi_x(\text{sol}^S(\phi))$.*

Proof. This is straightforward from the semantics of existential quantifiers: $\text{sol}^S(\exists x. \phi) = \{\alpha|_{\mathcal{V}(\phi) \setminus \{x\}} \mid \alpha \in \text{sol}^S(\phi)\} = \pi_x(\text{sol}^S(\phi))$. ◀

505 **C Proofs for Section 6 (Abstraction Exactness)**

506 In order to do so we first show that h -exactness is preserved when adding variables. For
 507 this we have to assume that the abstraction h is surjective, which will be the case of all
 508 abstraction of interest.

509 ► **Lemma 47 Variable extension preserves exactness.** *Let $h : S \rightarrow \Delta$ be a Σ -abstraction
 510 that is surjective, $\phi \in \mathcal{F}_\Sigma$ a formula, and $V \supseteq \mathcal{V}(\phi)$. Then the h -exactness of ϕ implies the
 511 h -exactness of ϕ with respect to V .*

512 **Proof.** Essentially this follows from that solutions of formulas can be extended arbitrarily to
 513 variables that do not appear freely in the formula, as stated by the following claim.

514 ► **Claim 48.** *Any variable assignment $\sigma : V \rightarrow \Delta$ satisfies $\sigma \in h \circ \text{sol}_V^S(\phi)$ iff $\sigma|_{\mathcal{V}(\phi)} \in$
 515 $h \circ \text{sol}^S(\phi)$.*

516 For the one direction let $\sigma \in h \circ \text{sol}_V^S(\phi)$. Then there exists $\alpha \in \text{sol}_V^S(\phi)$ such that
 517 $\sigma = h \circ \alpha$. Since $V \supseteq \mathcal{V}(\phi)$ it follows that $\alpha|_{\mathcal{V}(\phi)} \in \text{sol}^S(\phi)$. Furthermore $\sigma|_{\mathcal{V}(\phi)} = h \circ \alpha|_{\mathcal{V}(\phi)}$
 518 and thus $\sigma|_{\mathcal{V}(\phi)} \in h \circ \text{sol}^S(\phi)$.

519 For the other direction let $\sigma|_{\mathcal{V}(\phi)} \in h \circ \text{sol}^S(\phi)$. Then there exists $\alpha \in \text{sol}^S(\phi)$ such that
 520 $\sigma|_{\mathcal{V}(\phi)} = h \circ \alpha$. For any $y \in V \setminus \mathcal{V}(\phi)$ let $s_y \in \text{dom}(S)$ be such that $h(s_y) = \sigma(y)$. Such
 521 values exists since h is surjective. Now define $\alpha' = \alpha[y/s_y \mid y \in V \setminus \mathcal{V}(\phi)]$. Since $V \supseteq \mathcal{V}(\phi)$ it
 522 follows that $\alpha' \in \text{sol}_V^S(\phi)$. Furthermore, $\sigma = h \circ \alpha'$, so $\sigma \in h \circ \text{sol}_V^S(\phi)$. ◀

523 ► **Lemma 17.** *Let V be a set of variables, R_1 and R_2 be subsets of assignments of type
 524 $V \rightarrow \text{dom}(S)$ and $h : S \rightarrow \Delta$ be a Σ -abstraction. $h \circ (R_1 \cup R_2) = h \circ R_1 \cup h \circ R_2$.*

Proof. This lemma follows from the following equivalences:

$$\begin{aligned} \beta \in h \circ (R_1 \cup R_2) &\Leftrightarrow \exists \alpha. \alpha \in R_1 \cup R_2 \wedge \beta = h \circ \alpha \\ &\Leftrightarrow \exists \alpha. (\alpha \in R_1 \vee \alpha \in R_2) \wedge \beta = h \circ \alpha \\ &\Leftrightarrow \exists \alpha. (\alpha \in R_1 \wedge \beta = h \circ \alpha) \vee (\alpha \in R_2 \wedge \beta = h \circ \alpha) \\ &\Leftrightarrow \beta \in h \circ R_1 \vee \beta \in h \circ R_2 \\ &\Leftrightarrow \beta \in h \circ R_1 \cup h \circ R_2 \end{aligned}$$

525 ◀

526 ► **Proposition 18.** *The disjunction of h -exact formulas is h -exact.*

Proof. Let ϕ_1 and ϕ_2 be negation free formulas that are h -exact. Let $V = \mathcal{V}(\phi_1) \cup \mathcal{V}(\phi_2)$.
 Lemma 47 shows that ϕ_1 and ϕ_2 are also h -exact with respect to the extended variable set
 V , i.e., for both $i \in \{1, 2\}$:

$$h \circ \text{sol}_V^S(\phi_i) = \text{sol}_V^\Delta(\phi_i)$$

The h -exactness of the disjunction $\phi_1 \vee \phi_2$ can now be shown as follows:

$$\begin{aligned} h \circ \text{sol}^S(\phi_1 \vee \phi_2) &= h \circ (\text{sol}_V^S(\phi_1) \cup \text{sol}_V^S(\phi_2)) \\ &= h \circ \text{sol}_V^S(\phi_1) \cup h \circ \text{sol}_V^S(\phi_2) && \text{by Lemma 17} \\ &= \text{sol}_V^\Delta(\phi_1) \cup \text{sol}_V^\Delta(\phi_2) && \text{by } h\text{-exactness of } \phi_1 \text{ and } \phi_2 \text{ wrt. } V \\ &= \text{sol}^\Delta(\phi_1 \vee \phi_2) \end{aligned}$$

527 ◀

528 ► **Lemma 19 Projection commutes with abstraction.** For any Σ -abstraction $h : S \rightarrow \Delta$,
 529 subset R of assignments of type $V \rightarrow S$, and variable $x \in \mathcal{V}$: $h \circ \pi_x(R) = \pi_x(h \circ R)$.

530 **Proof.** For all $\alpha : V \rightarrow \text{dom}(S)$ we have $h \circ \pi_x(\alpha) = h \circ \alpha|_{V \setminus \{x\}} = (h \circ \alpha)|_{V \setminus \{x\}} = \pi_x(h \circ \alpha)$. ◀

531 ► **Proposition 20 Quantification preserves exactness.** For any surjective Σ -abstraction
 532 $h : S \rightarrow \Delta$ and formula $\exists x.\phi \in \mathcal{F}_\Sigma$, if ϕ is h -exact then $\exists x.\phi$ is h -exact.

Proof. Let ϕ be h -exact. By definition ϕ is h -exact with respect to V . Since h is assumed to be surjective, Lemma 47 implies that ϕ is h -exact with respect to $V \cup \{x\}$ (independently of whether x occurs freely in ϕ or not). Hence:

$$\begin{aligned} h(\text{sol}^S(\exists x.\phi)) &= h(\pi_x(\text{sol}^S(\phi))) && \text{by Lemma 10} \\ &= \pi_x(h(\text{sol}^S(\phi))) && \text{by Lemma 19} \\ &= \pi_x(\text{sol}^\Delta(\phi)) && \text{since } \phi \text{ is } h\text{-exact} \\ &= \text{sol}^\Delta(\exists x.\phi) && \text{by Lemma 10} \end{aligned}$$

533

534 ► **Lemma 21 Singleton property.** If S is a Σ -algebra, $e \in \mathcal{E}_\Sigma$, and $\alpha : \mathcal{V}(e) \rightarrow S$ a variable
 535 assignment, then the set $\llbracket e \rrbracket^{\alpha, S}$ is a singleton.

536 **Proof.** By induction on the structure of expressions $e \in \mathcal{E}$:

537 **Case** of constants $c \in \{0, 1\}$. The set $\llbracket c \rrbracket^{\alpha, S} = \{c^S\}$ is a singleton.

538 **Case** of variables $x \in \mathcal{V}$. The set $\llbracket x \rrbracket^{\alpha, S} = \{\alpha(x)\}$ is a singleton.

Case $e_1 \odot e_2$ where $e_1, e_2 \in \mathcal{E}_\Sigma$ and $\odot \in \{+, *\}$.

$$\llbracket e_1 \odot e_2 \rrbracket^{\alpha, S} = \{s \odot^S s' \mid s \in \llbracket e_1 \rrbracket^{\alpha, S}, s' \in \llbracket e_2 \rrbracket^{\alpha, S}\}$$

539 This set is a singleton since $\llbracket e_1 \rrbracket^{\alpha, S}$ and $\llbracket e_2 \rrbracket^{\alpha, S}$ are singletons by induction hypothesis,
 540 meaning that $s \odot^S s'$ is also a singleton since S is a Σ -algebra. ◀

► **Lemma 23.** Let $\phi \in \mathcal{F}_\Sigma$ be a Σ -formula and S a Σ -algebra with unique division. For
 nonzero natural number n , variable $y \notin \mathcal{V}(\phi)$, and expression $e \in \mathcal{E}_\Sigma$ with $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$:

$$\text{sol}^S(\phi \wedge ny \overset{\circ}{=} e) = \{\alpha[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}] \mid \alpha \in \text{sol}^S(\phi)\}$$

541 **Proof.** We fix some $\alpha : \mathcal{V}(\phi) \rightarrow \text{dom}(S)$ arbitrarily. Since S is a Σ -algebra, $\llbracket e \rrbracket^{\alpha, S}$ is a
 542 singleton and $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$, $\text{ele}(\llbracket e \rrbracket^{\alpha, S})$ is defined uniquely. Furthermore S has unique
 543 division, so that $\frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}$ is well defined element of $\text{dom}(S)$. Therefore and since $y \notin \mathcal{V}(\phi)$,
 544 $\alpha[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}]$ is the unique solution of the equation $ny \overset{\circ}{=} e$ that extends on α .

545 First we prove the inclusion “ \supseteq ”. Let $\alpha \in \text{sol}^S(\phi)$, $y \notin \mathcal{V}(\phi)$, and $\alpha[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}]$ is a
 546 solution of $ny \overset{\circ}{=} e$, it follows that $\alpha[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}]$ is a solution of $\phi \wedge ny \overset{\circ}{=} e$.

547 Second, we prove the inverse inclusion “ \subseteq ”. Let $\alpha \in \text{sol}^S(\phi \wedge ny \overset{\circ}{=} e)$. Since $\alpha[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}]$
 548 is the unique solution of the equation $ny \overset{\circ}{=} e$ that extends on $\alpha' = \alpha|_{\mathcal{V}(\phi)}$ it follows that
 549 $\alpha(y) = \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}$ so that $\alpha = \alpha'[y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}]$ while $\alpha' \in \text{sol}^S(\phi)$. ◀

550 ► **Proposition 24.** Let $\phi \in \mathcal{F}_\Sigma$ a Σ -formula, $n \neq 0$ a natural number, $e \in \mathcal{E}_\Sigma$ a Σ -expression
 551 with $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$ and $y \notin \mathcal{V}(\phi)$ and the Σ -abstraction $h : S \rightarrow \Delta$ with S and Δ two Σ -algebras
 552 with unique division. Then if ϕ is h -exact implies that $\phi \wedge e \overset{\circ}{=} ny$ is h -exact.

553 **Proof.** Let $e \in \mathcal{E}_\Sigma$ a Σ -expression.

554 ▶ **Claim 49.** For any $\alpha : V \rightarrow \mathbb{R}_+$ with $V \supseteq \mathcal{V}(e) : h(\text{ele}(\llbracket e \rrbracket^{\alpha, S})) = \text{ele}(\llbracket e \rrbracket^{h \circ \alpha, \Delta})$.

555 This can be seen as follows. For any $\alpha : \mathcal{V}(\phi) \rightarrow S$ Theorem 11 on homomorphism yields
 556 $h(\llbracket e \rrbracket^{\alpha, S}) \subseteq \llbracket e \rrbracket^{h \circ \alpha, \Delta}$. Since S and Δ are both Σ -algebras, the sets $\llbracket e \rrbracket^{\alpha, S}$ and $\llbracket e \rrbracket^{h \circ \alpha, \Delta}$ are
 557 both singletons by Lemma 21, so that $h(\text{ele}(\llbracket e \rrbracket^{\alpha, S})) = \text{ele}(\llbracket e \rrbracket^{h \circ \alpha, \Delta})$.

558 ▶ **Claim 50.** For any $s \in \text{dom}(S)$ and $n \neq 0$ a natural number: $h(\frac{s}{n}) = \frac{h(s)}{n}$.

559 Since S is assumed to have unique division $s' = \frac{s}{n}$ is well-defined as the unique element
 560 of $\text{dom}(S)$ such that $\underbrace{s' +^S \dots +^S s'}_n = s$. Hence, $h(\underbrace{s' +^S \dots +^S s'}_n) = h(s)$ and since h is a
 561 homomorphism, it follows that $\underbrace{h(s') +^\Delta \dots +^\Delta h(s')}_n = h(s)$. Since Δ is assumed to have
 562 unique division, this implies that $h(s') = \frac{h(s)}{n}$.

The Proposition can now be shown based on these two claims. Let ϕ be h -exact, $y \notin \mathcal{V}(\phi)$,
 and $\mathcal{V}(e) \subseteq \mathcal{V}(\phi)$. We have to show that $\phi \wedge ny \stackrel{\circ}{=} e$ is h -exact too:

$$\begin{aligned}
 h \circ \text{sol}^S(\phi \wedge e \stackrel{\circ}{=} ny) &= h \circ \{ \alpha [y / \frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n}] \mid \alpha \in \text{sol}^S(\phi) \} && \text{by Lemma 23} \\
 &= \{ (h \circ \alpha) [y / h(\frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n})] \mid \alpha \in \text{sol}^S(\phi) \} && \text{elementary} \\
 &= \{ \sigma [y / h(\frac{\text{ele}(\llbracket e \rrbracket^{\alpha, S})}{n})] \mid \sigma \in \text{sol}^\Delta(\phi) \} && h\text{-exactness of } \phi \\
 &= \{ \sigma [y / \frac{h(\text{ele}(\llbracket e \rrbracket^{\alpha, S}))}{n}] \mid \sigma \in \text{sol}^\Delta(\phi) \} && \text{by Claim 50} \\
 &= \{ \sigma [y / \frac{\text{ele}(\llbracket e \rrbracket^{h \circ \alpha, \Delta})}{n}] \mid \sigma \in \text{sol}^\Delta(\phi) \} && \text{by Claim 49} \\
 &= \text{sol}^\Delta(\phi \wedge e \stackrel{\circ}{=} ny) && \text{by Lemma 23}
 \end{aligned}$$

563 ◀

564 ▶ **Proposition 25.** Let $h : S \rightarrow \Delta$ be a Σ -abstraction between Σ -algebras with unique division.
 565 Then any strongly-triangular system of Σ -equations with natural coefficients is h -exact.

566 **Proof.** Any strongly-triangular system of equations has the form $\bigwedge_{i=1}^n e_i \stackrel{\circ}{=} n_i y_i$ where n and
 567 $n_i \neq 0$ are naturals and y_i is i -fresh for all $1 \leq i \leq n$. The proof is by induction on n . In
 568 the case $n = 0$, the conjunction is equal to true which is h -exact since $h(\text{sol}^S(\text{true})) = h(\{\})$
 569 $= \text{sol}^\Delta(\text{true})$. In the case $n > 0$, we have by induction hypothesis that $\bigwedge_{j=1}^{i-1} e_j \stackrel{\circ}{=} n_j y_j$ is
 570 h -exact. Since $n_i \neq 0$ it follows from Proposition 24 that that $e_i \stackrel{\circ}{=} n_i y_i \wedge \bigwedge_{j=1}^{i-1} e_j \stackrel{\circ}{=} n_j y_j$ is
 571 h -exact. ◀

572 **D Proofs for Section 7 (Abstraction Invariance)**

573 ▶ **Lemma 29.** If either R_1 or R_2 are h -invariant then: $h \circ (R_1 \cap R_2) = h \circ R_1 \cap h \circ R_2$.

Proof. The one inclusion is straightforward without invariance:

$$\begin{aligned}
 h \circ (R_1 \cap R_2) &= \{ h \circ \alpha \mid \alpha \in R_1, \alpha \in R_2 \} \\
 &\subseteq \{ h \circ \alpha \mid \alpha \in R_1 \} \cap \{ h \circ \alpha \mid \alpha \in R_2 \} \\
 &= h \circ R_1 \cap h \circ R_2
 \end{aligned}$$

574 For the other inclusion, we can assume with loss of generality that R_1 is h -invariant. So let
 575 $\beta \in h \circ R_1 \cap h \circ R_2$. Then there exist $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$ such that $\beta = h \circ \alpha_1 = h \circ \alpha_2$. By
 576 h -invariance of R_1 it follows that $\alpha_1 \in R_2$. So $\alpha_1 \in R_1 \cap R_2$, and hence, $\beta \in h \circ (R_1 \cap R_2)$. ◀

577 ► **Lemma 30 Algebraic characterization.** *A subset of R variables assignment of type $V \rightarrow$*
 578 *$dom(S)$ is h -invariant for an Σ -abstraction $h : S \rightarrow \Delta$ iff $h \circlearrowleft (h \circ R) \subseteq R$.*

579 **Proof.** “ \Rightarrow ”. Let R be h -invariant and $\alpha \in h \circlearrowleft (h \circ R)$. Then there exists $\alpha' \in R$ such that
 580 $h \circ \alpha = h \circ \alpha'$. The h -invariance of R thus implies that $\alpha \in R$.

581 “ \Leftarrow ”. Suppose that $h \circlearrowleft (h \circ R) \subseteq R$. Let $\alpha, \alpha' : V \rightarrow dom(S)$ such that $h \circ \alpha = h \circ \alpha'$ and
 582 $\alpha \in R$. We have to show that $\alpha' \in R$. From $h \circ \alpha = h \circ \alpha'$ and $\alpha \in R$ it follows that
 583 $\alpha' \in h \circlearrowleft (h \circ R)$ and thus $\alpha' \in R$ as required. ◀

584 ► **Lemma 51 Variable extension preserves invariance.** *Let h be a surjective abstraction*
 585 *and R a subset of functions of type $V' \rightarrow dom(S)$ and V a subset of variables disjoint from*
 586 *V' . If R is h -invariant then $ext_V^S(R)$ is h -invariant too.*

587 **Proof.** This will follow straightforwardly from the characterization of h -invariance in Lemma
 588 30 and the following two claims:

589 ► **Claim 52.** *If h is surjective then $h \circ ext_V^S(R) = ext_V^\Delta(h \circ R)$.*

590 This follows from $h \circ ext_V^S(R) = \{h \circ \alpha \mid \alpha \in ext_V^S(R)\} = ext_V^\Delta(\{h \circ \alpha' \mid \alpha' \in R\})$ where we
 591 use the surjectivity of h in the last step.

592 ► **Claim 53.** *$h \circ ext_V^\Delta(R') = ext_V^S(h \circlearrowleft R')$ for any subset R' of functions of type $V' \rightarrow dom(\Delta)$.*

$$\begin{aligned} h \circ ext_V^\Delta(R') &= \{\alpha : V \cup V' \rightarrow dom(S) \mid h \circ \alpha \in ext_V^\Delta(R')\} \\ &= \{\alpha : V \cup V' \rightarrow dom(S) \mid h \circ \alpha|_{V'} \in R'\} \\ &= ext_V^S(\{\alpha' : V' \rightarrow dom(S) \mid h \circ \alpha' \in R'\}) \\ &= ext_V^S(h \circlearrowleft R') \end{aligned}$$

593 ◀

594 ► **Lemma 54.** *Let $h : S \rightarrow \Delta$ be a surjective Σ -abstraction, ϕ be a Σ -formula, and $V \supseteq \mathcal{V}(\phi)$.*
 595 *Then the h -invariance of ϕ implies the h -invariance of $sol_V^S(\phi)$.*

596 **Proof.** This follows from the cylindrification Lemma 46 and that extension preserves h -
 597 invariance as shown in Lemma 51. ◀

598 ► **Lemma 32.** *For any Σ -abstraction $h : S \rightarrow \Delta$, any subsets of assignments of type*
 599 *$V \rightarrow dom(S)$ R_1 and R_2 and V a subset of variables:*

600 ■ $h \circlearrowleft (R_1 \cap R_2) = h \circlearrowleft R_1 \cap h \circlearrowleft R_2$.

601 ■ $h \circlearrowleft (R_1 \cup R_2) = h \circlearrowleft R_1 \cup h \circlearrowleft R_2$.

Proof. The case for unions follows straightforwardly from the definitions:

$$\begin{aligned} h \circlearrowleft (R_1 \cup R_2) &= \{\alpha \mid h \circ \alpha \in R_1 \cup R_2\} \\ &= \{\alpha \mid h \circ \alpha \in R_1 \vee h \circ \alpha \in R_2\} \\ &= \{\alpha \mid h \circ \alpha \in R_1\} \cup \{\alpha \mid h \circ \alpha \in R_2\} \\ &= h \circlearrowleft R_1 \cup h \circlearrowleft R_2 \end{aligned}$$

The case of intersection is symmetric:

$$\begin{aligned} h \circlearrowleft (R_1 \cap R_2) &= \{\alpha \mid h \circ \alpha \in R_1 \cap R_2\} \\ &= \{\alpha \mid h \circ \alpha \in R_1 \wedge h \circ \alpha \in R_2\} \\ &= \{\alpha \mid h \circ \alpha \in R_1\} \cap \{\alpha \mid h \circ \alpha \in R_2\} \\ &= h \circlearrowleft R_1 \cap h \circlearrowleft R_2 \end{aligned}$$

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602

603 ► **Lemma 33 Intersection and union preserve invariance.** *Let $h : S \rightarrow \Delta$ be a Σ -abstraction.*
 604 *Then the intersection and union of any two h -invariant subsets R_1 and R_2 of variables*
 605 *assignments of type $V \rightarrow \text{dom}(S)$ is h -invariant.*

606 **Proof.** This follows from the algebraic characterization Lemma 30 for invariance, in combi-
 607 nation with the algebraic properties of composition and decomposition given in Lemmas 17,
 608 29, and 32. ◀

609 ► **Lemma 34 Projection commutes with left-decomposition.** $h \circ \pi_x(R) = \pi_x(h \circ R)$.

610 **Proof.** For all $\alpha : V \rightarrow \text{dom}(\Delta)$ we have $h \circ \pi_x(\alpha) = h \circ \alpha|_{V \setminus \{x\}} = (h \circ \alpha)|_{V \setminus \{x\}} = \pi_x(h \circ \alpha)$. ◀

611 ► **Proposition 35 Invariance is preserved by conjunction, disjunction, and quantification.**
 612 *If h is a surjective abstraction then the class of h -invariant FO-formulas is closed under*
 613 *conjunction, disjunction, and existential quantification.*

614 **Proof.** Let $h : S \rightarrow \Delta$ be a Σ -abstraction.

Case of conjunction: Let ϕ_1 and ϕ_2 be h -invariant and $V = \mathcal{V}(\phi_1 \wedge \phi_2)$. By Lemma 54 the sets $\text{sol}_V^S(\phi_1)$ and $\text{sol}_V^S(\phi_2)$ are both h -invariant, and so by Lemma 33 is their intersection. Hence:

$$\begin{aligned} h \circ (h \circ \text{sol}^S(\phi_1 \wedge \phi_2)) &= h \circ (h \circ (\text{sol}_V^S(\phi_1) \cap \text{sol}_V^S(\phi_2))) \\ &\subseteq \text{sol}_V^S(\phi_1) \cap \text{sol}_V^S(\phi_2) && \text{by } h\text{-invariance and Lemma 30} \\ &= \text{sol}^S(\phi_1 \wedge \phi_2) \end{aligned}$$

615 By Lemma 30 in the other direction, this implies that $\phi_1 \wedge \phi_2$ is h -invariant.

616 **Case of disjunction:** Analogous to the case of conjunction.

Case of existential quantification:

$$\begin{aligned} h \circ (h \circ \text{sol}^S(\exists x.\phi_1)) &= h \circ (h \circ \pi_x(\text{sol}^S(\phi_1))) && \text{by Lemma 10} \\ &= h \circ (\pi_x(h \circ \text{sol}^S(\phi_1))) && \text{by Lemma 19} \\ &= \pi_x(h \circ (h \circ \text{sol}^S(\phi_1))) && \text{by Lemma 34} \\ &\subseteq \pi_x(\text{sol}^S(\phi_1)) && \text{by } h\text{-invariance of } \phi_1 \text{ and Lemma 30} \\ &= \text{sol}^S(\exists x.\phi_1) && \text{by Lemma 10} \end{aligned}$$

617 By Lemma 30, this implies that $\exists x.\phi_1$ is h -invariant. ◀

618 ► **Proposition 36.** *Let h be a surjective Σ -abstractions. Then the class of h -exact and h -*
 619 *invariant Σ -formulas is closed under conjunction, disjunction and existential quantification.*

620 **Proof.** Closure under conjunction follows from Propositions 31 and 35, closure under disjunc-
 621 tion from Propositions 18 and 35, and closure under existential quantification by Propositions
 622 20 and 35. ◀

623 ► **Theorem 37 Invariance.** *Any positive polynomial equation $p \stackrel{\circ}{=} 0$ such that p has no constant*
 624 *term is $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant.*

625 **Proof.** Consider a positive polynomial equation $p \stackrel{\circ}{=} 0$ such that p has no constant term and
 626 only positive coefficients. Thus p has the form $\sum_{j=1}^l n_j \prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}} \stackrel{\circ}{=} 0$ where $l \geq 0$, and
 627 $n_j, i_j, m_{j,k} > 0$.

628 ▶ **Claim 55.** For both algebras $S \in \{\mathbb{B}, \mathbb{R}_+\}$: $\text{sol}^S(p \stackrel{\circ}{=} 0) = \text{sol}^S(\bigwedge_{j=1}^l \bigvee_{k=1}^{i_j} x_{j,k} \stackrel{\circ}{=} 0)$.

629 The polynomial has values zero if and only if all its monomials do, that is: $\prod_{k=1}^{i_j} x_{j,k}^{m_{j,k}} = 0$
 630 for all $1 \leq j \leq l$. Since constant terms are ruled out, we have $i_j \neq 0$. Furthermore, we
 631 assumed for all polynomials that $m_{j,k} \neq 0$. So for all $1 \leq j \leq l$ there must exist $1 \leq k \leq i_j$
 632 such that $x_{j,k} = 0$.

633 ▶ **Claim 56.** The equation $x \stackrel{\circ}{=} 0$ is $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant.

This is straightforward from the definitions. With these two claims we are now in the position to prove the lemma. Since the class of $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant formulas is closed under conjunction and disjunction by Proposition 36, it follows from by Claim 56 that $\bigwedge_{j=1}^l \bigvee_{k=1}^{i_j} x_{j,k} \stackrel{\circ}{=} 0$ is both $h_{\mathbb{B}}$ -exact and $h_{\mathbb{B}}$ -invariant. Since this formula is equivalent over \mathbb{R}_+ to the polynomial equation by Claim 55, the $h_{\mathbb{B}}$ -invariance carries over to $p \stackrel{\circ}{=} 0$. The $h_{\mathbb{B}}$ -exactness also carries over based on the equivalence for both structures \mathbb{R}_+ and \mathbb{B} :

$$\begin{aligned} h_{\mathbb{B}} \circ \text{sol}^{\mathbb{R}_+}(p \stackrel{\circ}{=} 0) &= h_{\mathbb{B}} \circ \text{sol}_V^{\mathbb{R}_+}(\bigwedge_{j=1}^l \bigvee_{k=1}^{i_j} x_{j,k} \stackrel{\circ}{=} 0) && \text{by Claim 55 for } \mathbb{R}_+ \\ &= \text{sol}^{\mathbb{B}}(\bigwedge_{j=1}^l \bigvee_{k=1}^{i_j} x_{j,k} \stackrel{\circ}{=} 0) && \text{by } h_{\mathbb{B}} \text{ exactness} \\ &= \text{sol}^{\mathbb{B}}(p \stackrel{\circ}{=} 0) && \text{by Claim 55 for } \mathbb{B}. \end{aligned}$$

634

635 **E** Proofs for Section 8 (Boolean Abstractions of $h_{\mathbb{B}}$ -Mixed Systems)

636 ▶ **Corollary 40.** The $h_{\mathbb{B}}$ -abstraction of the \mathbb{R}_+ -solution set of an $h_{\mathbb{B}}$ -mixed system ϕ , that is
 637 $h_{\mathbb{B}} \circ \text{sol}^{\mathbb{R}_+}(\phi)$, can be computed in at most exponential time in the size of the system ϕ .

638 **Proof.** Given an $h_{\mathbb{B}}$ -mixed system ϕ , we can apply Theorem 39 to compute in at most
 639 exponential time an \mathbb{R}_+ -equivalent formula ϕ'' that is $h_{\mathbb{B}}$ -exact. It is then sufficient to
 640 compute $\text{sol}^{\mathbb{B}}(\phi'')$ in exponential time in the size of ϕ . This can be done in the naive
 641 manner, that is by evaluating the formula ϕ'' – which may be of exponential size – over all
 642 possible boolean variable assignments – of which there may be exponentially many. For each
 643 assignment the evaluation can be done in PSPACE and thus in exponential time. The overall
 644 time required is thus a product of two exponentials, which remains exponential. ◀