Homogenization of perforated elastic structures

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Abstract

The paper is dedicated to the asymptotic behavior of periodically perforated elastic domains (3D, plate-like or beam-like). We homogenize these structures, passing to the limit w.r.t. the period. In case of plate-like or beam-like structures we simultaneously proceed to a dimension reduction. These periodic structures can be made e.g. of balls or cylinders glued, so that the surface in contact has a non-zero measure. Since the boundaries of these structures might be non-Lipschitz, the classical extension approach does not serve. We will proceed using interpolations. The Korn inequalities in the case of thin structures are based on the decomposition of beam or plate displacements. For the asymptotic behavior the unfolding and rescaling operators are used.

Keyword: Homogenization, periodic unfolding method, dimension reduction, linear elasticity, variational inequality, perforated non-Lipschitz domains, plates, beams, extension operators and Korn inequalities.

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1 Introduction

This paper concerns the linearized elasticity problem posed in periodic domains. These domains are obtained by reproducing periodically a cell of size ε in order to get a beam-like, plate-like or $3D$ structures fixed on a part Γ_ε of their exterior boundary. The ε-cells are made by elastic materials. The reference cell is denoted $\mathbf{C}$. We assume that $\mathbf{C}$ has a Lipschitz boundary and that two neighboring cells $\overline{\mathbf{C}} \cup (\overline{\mathbf{C}} + \mathbf{e}_i)$, $i = \overline{1, N}$ are connected. Under these assumptions, the whole periodic structure might not have a Lipschitz boundary.

Our aim is to give the asymptotic behavior of these elastic periodic structures as ε tends to 0. Since these structures might not be Lipschitz, one of the main difficulties is to obtain a priori estimates. The classical extension approach (see [16]) and the Korn inequalities for Lipschitz domains (see [4, 5]) cannot be used. Thus, in order to derive a priori estimates we used interpolations suggested in [9, Section 5.5]. This makes it possible to prove Korn inequalities with constants independent of ε. Note that in the case of a beam-like and a plate-like domains the derivation of Korn inequalities are also based on the decomposition of beam or plate displacements. These decompositions have been introduced in [3, 11].

To derive the limit problems we use the periodic unfolding method introduced in [7]. Since then this method has been applied to a vast number of different problems such as problems in perforated domains [6], contact problems [13, 15], problems including a thin layer [14], problems for structures made of curved rods [10], problems in domain with "rough boundary" [1, 2], to name but a few. In this context we would like to mention the first book [9] devoted to the periodic unfolding method. It contains not only the detailed theory underlying this method but a lot of examples of its application to different partial differential problems. Application procedure of the periodic unfolding method that we used here is standard and includes of the following steps: depending on the problem introduce and apply an appropriate unfolding operator, using a priori estimates for the displacement obtain uniform estimate for the unfolded displacement, which, in turn, are used to pass to a weak limit in fixed space, establish an unfolded limit problem, which can be used for extracting a homogenized problem.

As a general reference for the homogenization of elasticity problems in 3D periodically perforated domains with Lipschitz boundary we refer to [16]. In the case of heterogeneous plate-like domain we mention [9, Chapter V] where the interaction of homogenization and domain reduction, involving two small parameters such as plate thickness δ and periodicity ε in its large dimensions was investigated. For similar results in the case of a beam-like domain we refer to [12]. The novelty of this paper is the extension of the results to non-Lipschitz perforated domains.

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The paper is organized as follows. Sections 2, 3 and 4 deal with a N-dimensional ($3D$-like domain particular), plate-like and beam-like domains respectively. For each of them we begin by introducing the notations and by describing the geometry. Then, taking into account of the particularity of the geometry we introduce the unfolding operator for every case, derive weak limits of the fields, specify the limit problem and finally characterize the limit fields. Towards the end of the paper in Appendix we derive Korn's type inequalities for N–dimensional, plate-like and beam-like domains which are used to obtain the a priori estimates for the displacements.

Throughout this paper we use Einstein's summation convention. Moreover, in all the estimates the constants do not depend on ε.

2 N-dimensional periodic domain

2.1 Notations and geometric setting

Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N} \setminus \{0,1\}$, be a bounded domain with a Lipschitz boundary and Γ be a subset of $\partial\Omega$ with non null measure. We assume that there exists an open set Ω' with a Lipschitz boundary such that $\Omega \subset \Omega'$ and $\Omega' \cap \partial\Omega = \Gamma$.
Denote

- $Y \doteq (-1/2, 1/2)^N$,
- $\mathbf{C} \subset Y$ a domain with Lipschitz boundary such that interior$\big(\overline{\mathbf{C}} \cup (\overline{\mathbf{C}} + \mathbf{e}_i)\big)$, $i = \overline{1,N}$, is connected,
- $\Xi_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^N \ \mid \ \varepsilon(\xi + Y) \cap \Omega \neq \emptyset\big\}$, $\Xi'_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^N \ \mid \ \varepsilon(\xi + Y) \cap \Omega' \neq \emptyset\big\}$, $\widehat{\Xi}_{\varepsilon,i} \doteq \big\{\xi \in \Xi_\varepsilon \ \mid \ \xi + \mathbf{e}_i \in \Xi_\varepsilon\big\}$, $i = \overline{1,N}$, observe that $\Xi_\varepsilon = \bigcup_{i=1}^{N} \Xi_{\varepsilon,i}$,
- $\Omega^*_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{\mathbf{C}})\Big)$, $\Omega^{ext}_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\Big)$, $\Omega'^*_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi'_\varepsilon} \varepsilon(\xi + \overline{\mathbf{C}})\Big)$,
- $\Omega_1 \doteq \big\{x \in \mathbb{R}^N \mid \text{dist}(x, \Omega) < 1\big\}$,
- for a.e. $x \in \mathbb{R}^N$, one has
$$x = [x] + \{x\}, \qquad \text{where } \ [x] \in \mathbb{Z}^N, \ \ \{x\} \in Y.$$

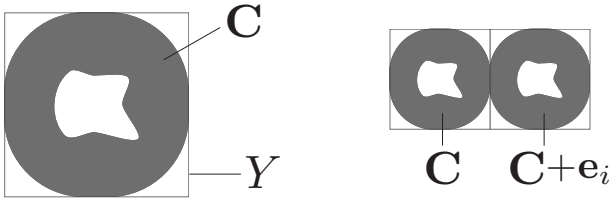
Note that the domains Ω^*_ε, Ω'^*_ε are connected.

We are interested in the elastic behavior of a structure occupying the domain Ω^*_ε which is fixed on the part of its boundary. The space of all admissible displacements is denoted $\mathbf{V}_\varepsilon$

$$\mathbf{V}_\varepsilon = \big\{u \in H^1(\Omega^*_\varepsilon)^N \ \mid \ \exists u' \in H^1(\Omega'^*_\varepsilon)^N \text{ such that } u'_{|\Omega^*_\varepsilon} = u \text{ and } u' = 0 \text{ in } \Omega'^*_\varepsilon \setminus \overline{\Omega^*_\varepsilon}\big\}.$$

It means that the displacements belonging to $\mathbf{V}_\varepsilon$ "vanish" on a part Γ_ε of $\partial\Omega^*_\varepsilon$.

Remark 2.1. *Note that the domain Ω^*_ε might not be Lipschitz (see e.g. Figure 2.1). In this case one can not extend the displacement in the holes of this domain.*



(a) Sets $\mathbf{C}$ and Y. (b) Sets $\mathbf{C}$ and $\mathbf{C} + \mathbf{e}_i$.

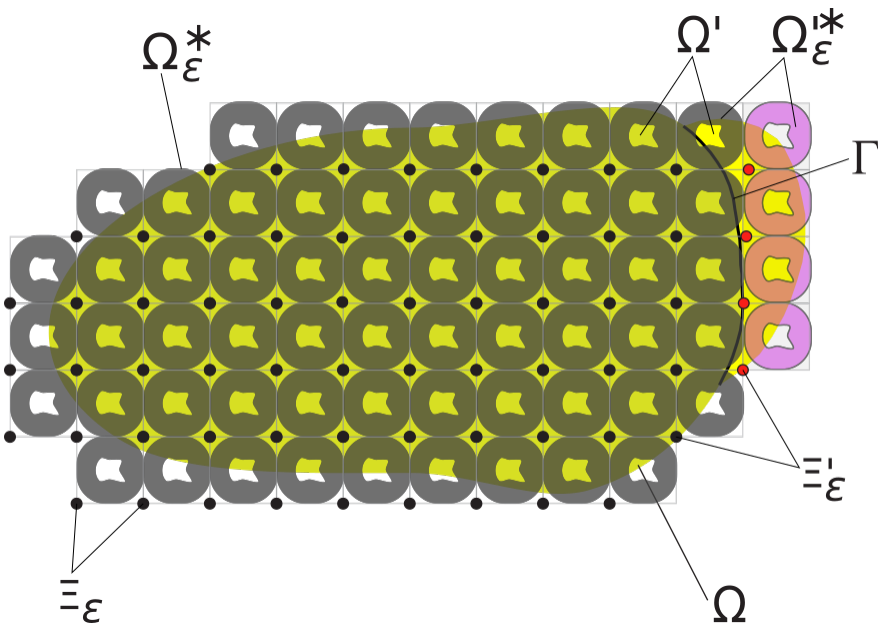


Figure 1: Sets Ω, Ω_ε^*, Ω', $\Omega_\varepsilon'^*$ and Ξ_ε, Ξ'_ε.

For $u \in H^1(\Omega_\varepsilon^*)^N$ we denote by e the stress tensor

$$e(u) \doteq \frac{1}{2}\Big(\nabla u + (\nabla u)^T\Big), \quad e_{ij}(u) \doteq \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{2.1}$$

Let $a_{ijkl} \in L^\infty(\mathbf{C}), i,j,k,l = \overline{1,N}$ be functions that satisfy the following conditions:

- $a_{ijkl}(X) = a_{jikl}(X) = a_{klij}(X) \quad$ for a.e. $X \in \mathbf{C}$
- for any $\tau \in M_s^{N\times N}$, where $M_s^{N\times N}$ is the space of $N \times N$ symmetric matrices, there exists $c_0 > 0$ such that

$$a_{ijkl}(X)\tau_{ij}\tau_{kl} \geq c_0\tau_{ij}\tau_{ij} \quad \text{for a.e. } X \in \mathbf{C}. \tag{2.2}$$

The constitutive law for the material occupying the domain Ω_ε^* is given by the relation between the strain tensor and the stress tensor

$$\sigma_{ij}^\varepsilon(u) \doteq a_{ijkl}^\varepsilon e_{kl}(u), \qquad \forall u \in \mathbf{V}_\varepsilon, \tag{2.3}$$

where the coefficients a_{ijkl}^ε are given by

$$a_{ijkl}^\varepsilon(x) = a_{ijkl}\Big(\Big\{\frac{x}{\varepsilon}\Big\}\Big) \quad \text{for a.e. } x \in \Omega_\varepsilon^*.$$

Let f be in $L^2(\Omega_1)^N$, one defines the applied forces f_ε by

$$f_\varepsilon = f|_{\Omega_\varepsilon^*}. \tag{2.4}$$

The unknown displacement $u_\varepsilon : \Omega_\varepsilon^* \to \mathbb{R}^N$ is the solution to the linearized elasticity system:

$$\begin{cases} \nabla \cdot \sigma^\varepsilon(u_\varepsilon) = -f_\varepsilon & \text{in} \quad \Omega_\varepsilon^*, \\ u_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon, \\ \sigma^\varepsilon(u_\varepsilon)\cdot\nu_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon^* \setminus \Gamma_\varepsilon, \end{cases} \tag{2.5}$$

where ν_ε is the outward normal vector to $\partial\Omega_\varepsilon^*$. The variational formulation of (2.5) is:

$$\begin{cases} \text{Find } u_\varepsilon \in \mathbf{V}_\varepsilon \text{ such that,} \\ \displaystyle\int_{\Omega_\varepsilon^*} \sigma^\varepsilon(u_\varepsilon) : e(v)\,dx = \int_{\Omega_\varepsilon^*} f_\varepsilon \cdot v\,dx, \qquad \forall v \in \mathbf{V}_\varepsilon. \end{cases} \tag{2.6}$$

2.2 Preliminary results

The following lemma is a simple consequence of Proposition 5.2 in Appendix.

Lemma 2.1. *For every $u \in \mathbf{V}_\varepsilon$ one has*

$$\|u\|_{H^1(\Omega_\varepsilon^*)} \le C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \tag{2.7}$$

Lemma 2.2. *The solution u_ε of problem (2.5) satisfies*

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon^*)} \le C\|f\|_{L^2(\Omega_1)}. \tag{2.8}$$

Proof. Taking into account (2.7), we have

$$\Big|\int_{\Omega_\varepsilon^*} f_\varepsilon \cdot u_\varepsilon \, dx\Big| \le \|f\|_{L^2(\Omega_\varepsilon^*)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \le C\|f\|_{L^2(\Omega_1)}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}$$

and thus (2.8) follows from (2.2) and (2.7). □

2.3 The unfolding operator

The unfolding operator is defined in a similar way as for domains with holes (see [9]).

Definition 2.1. *For every measurable function $\phi : \Omega_\varepsilon^* \to \mathbb{R}$ the unfolding operator $\mathcal{T}_\varepsilon^* : \Omega_\varepsilon^{ext} \times \mathbf{C} \to \mathbb{R}$ is defined as follows:*

$$\mathcal{T}_\varepsilon^*(\phi)(x,X) = \phi\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big] + \varepsilon X\Big) \quad \textit{for a.e. } (x,X) \in \Omega_\varepsilon^{ext} \times \mathbf{C}.$$

Below are some properties of $\mathcal{T}_\varepsilon^*$, they are similar to those of the unfolding operators introduced [9]. That is due to the fact that

$$\Lambda_\varepsilon^{ext} \doteq \Omega_\varepsilon^{ext} \setminus \overline{\Omega} \quad \text{satisfies} \quad \lim_{\varepsilon\to 0} |\Lambda_\varepsilon^{ext}| = 0.$$

Proposition 2.1. *For every $\phi \in L^1(\Omega_\varepsilon^*)$*

$$\begin{aligned} &\int_{\Omega_\varepsilon^{ext}\times\mathbf{C}} \mathcal{T}_\varepsilon^*(\phi)(x,X)\,dx\,dX = \int_{\Omega_\varepsilon^*} \phi(x)\,dx, \\ &\|\mathcal{T}_\varepsilon^*(\phi)\|_{L^1(\Omega_\varepsilon^{ext}\times\mathbf{C})} = \|\phi\|_{L^1(\Omega_\varepsilon^*)}. \end{aligned} \tag{2.9}$$

For every $\phi \in H^1(\Omega_\varepsilon^*)$

$$\mathcal{T}_\varepsilon^*(\nabla\phi)(x,X) = \frac{1}{\varepsilon}\nabla_X \mathcal{T}_\varepsilon^*(\phi)(x,X) \quad \textit{for a.e. } (x,X) \in \Omega_\varepsilon^{ext} \times \mathbf{C}. \tag{2.10}$$

As a consequence of the estimate (2.8) and Proposition 2.1, one obtains that the solution to (2.5) satisfies

$$\|\mathcal{T}_\varepsilon^*(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^{\rm ext}\times\mathbf{C})} + \|\mathcal{T}_\varepsilon^*(\nabla u_\varepsilon)\|_{L^2(\Omega_\varepsilon^{\rm ext}\times\mathbf{C})} \le C\|f\|_{L^2(\Omega_1)}.$$

For more properties see [9].

2.4 Weak limits of the fields and the limit problem

Set

$$(\mathbb{R}^N)_\varepsilon^* \doteq \text{interior}\Big(\bigcup_{\xi\in\mathbb{Z}^N} \varepsilon(\xi + \overline{\mathbf{C}})\Big).$$

Denote by $H^1_{N,per}(\mathbf{C})$ the subspace of the periodic functions belonging to $H^1_{loc}((\mathbb{R}^N)_\varepsilon^*)$

$$H^1_{N,per}(\mathbf{C}) \doteq \Big\{\psi \in H^1_{loc}((\mathbb{R}^N)_\varepsilon^*) \;\mid\; \psi(\cdot+\xi) = \psi(\cdot) \text{ a.e. in } (\mathbb{R}^N)_\varepsilon^*,\ \forall \xi \in \mathbb{Z}^N\Big\},$$

by $H^1_{N,per,0}(\mathbf{C})$ the subspace of the functions in $H^1_{N,per}(\mathbf{C})$ with zero mean

$$H^1_{N,per,0}(\mathbf{C}) \doteq \Big\{\psi \in H^1_{N,per}(\mathbf{C}) \;\mid\; \int_{\mathbf{C}} \psi(X)\,dX = 0\Big\}$$

and by $H^1_\Gamma(\Omega)$ the space of the functions in $H^1(\Omega)$ that vanish on Γ[1]

$$H^1_\Gamma(\Omega) \doteq \big\{\phi \in H^1(\Omega) \mid \phi = 0 \quad \text{on } \Gamma\big\}.$$

Lemma 2.3. *Let u_ε be the solution of problem (2.5). There exists $u \in H^1_\Gamma(\Omega)^N$ and $\widehat{u} \in L^2(\Omega; H^1_{N,per,0}(\mathbf{C}))^N$ such that*

$$\begin{aligned} \mathcal{T}^*_\varepsilon(u_\varepsilon) &\to u \quad \textit{strongly in} \quad L^2(\Omega; H^1(\mathbf{C}))^N, \\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) &\rightharpoonup \nabla u + \nabla_X \widehat{u} \quad \textit{weakly in} \quad L^2(\Omega \times \mathbf{C})^{N\times N}, \\ \mathcal{T}^*_\varepsilon\big(e(u_\varepsilon)\big) &\to e(u) + e_X(\widehat{u}) \quad \textit{strongly in} \quad L^2(\Omega \times \mathbf{C})^{N\times N}, \end{aligned} \tag{2.11}$$

and the pair $(u, \widehat{u})$ is the unique solution to the following unfolded problem:

$$\begin{cases} \displaystyle\int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(u) + e_{X,kl}(\widehat{u})\big)\big(e_{ij}(\Psi) + e_{X,ij}(\widehat{\Phi})\big)\,dx\,dX = |\mathbf{C}| \displaystyle\int_\Omega f\cdot\Psi\,dx, \\ \forall \Psi \in H^1_\Gamma(\Omega)^N, \ \ \forall \widehat{\Phi} \in L^2(\Omega; H^1_{N,per,0}(\mathbf{C}))^N \end{cases} \tag{2.12}$$

where for all $\widehat{\Phi} \in H^1(\mathbf{C})^N$

$$e_{X,kl}(\widehat{\Phi}) = \frac{1}{2}\Big(\frac{\partial \widehat{\Phi}_k}{\partial X_l} + \frac{\partial \widehat{\Phi}_l}{\partial X_k}\Big), \quad k,l = \overline{1,N}.$$

Proof. Taking into account (2.8) by [9, Theorem 4.43] there exists a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, $u \in H^1_\Gamma(\Omega)^N$ and $\widehat{u} \in L^2(\Omega; H^1_{N,per,0}(\mathbf{C}))^N$ such that

$$\begin{aligned} \mathcal{T}^*_\varepsilon(u_\varepsilon) &\to u \quad \text{strongly in} \quad L^2(\Omega; H^1(\mathbf{C}))^N, \\ \mathcal{T}^*_\varepsilon(\nabla u_\varepsilon) &\rightharpoonup \nabla u + \nabla_X \widehat{u} \quad \text{weakly in} \quad L^2(\Omega \times \mathbf{C})^{N\times N}. \end{aligned} \tag{2.13}$$

In order to obtain the limit problem (2.12) we use the same approach as in [6, Theorem 4.3]. Let us introduce the following fields:

$$\Psi \in H^1(\Omega_1)^N \ \text{ s.t. } \Psi = 0 \ \text{ in } \ \Omega_1 \cap \big(\Omega' \setminus \overline{\Omega}\big), \quad \varphi \in \mathcal{D}(\Omega), \ \ \psi \in H^1_{N,per,0}(\mathbf{C})^N$$

and take $v_\varepsilon(x) = \Psi(x) + \varepsilon\psi_\varepsilon(x)\varphi(x)$ as a test function in (2.6), where $\psi_\varepsilon(x) = \psi\Big(\dfrac{x}{\varepsilon}\Big)$. Note that

$$\begin{aligned} e_{ij}(v_\varepsilon)(x) &= e_{ij}(\Psi)(x) + \varepsilon e_{ij}(\psi_\varepsilon\varphi)(x) \\ &= e_{ij}(\Psi)(x) + e_{X,ij}(\psi)\Big(\frac{x}{\varepsilon}\Big)\varphi(x) + \frac{\varepsilon}{2}\Big(\psi_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial\varphi}{\partial x_i}(x) + \psi_i\Big(\frac{x}{\varepsilon}\Big)\frac{\partial\varphi}{\partial x_j}(x)\Big) \\ &= e_{ij}(\Psi)(x) + e_{X,ij}(\psi)(X)\varphi(x) + \frac{\varepsilon}{2}\Big(\psi_j(X)\frac{\partial\varphi}{\partial x_i}(x) + \psi_i(X)\frac{\partial\varphi}{\partial x_j}(x)\Big), \quad x \in \Omega^*_\varepsilon, \ i,j = \overline{1,N}. \end{aligned}$$

Then, transform by $\mathcal{T}^*_\varepsilon$, that gives

$$\begin{aligned} &\mathcal{T}^*_\varepsilon(v_\varepsilon) \to \Psi \quad \text{strongly in} \quad L^2(\Omega\times\mathbf{C})^N, \\ &\mathcal{T}^*_\varepsilon(e(v_\varepsilon)) \to e(\Psi) + e_X(\psi)\varphi \quad \text{strongly in} \quad L^2(\Omega\times\mathbf{C})^{N\times N}. \end{aligned}$$

Unfolding the left hand side of (2.6) and using $\|e(v_\varepsilon)\|_{L^2(\Lambda^{ext}_\varepsilon)} = \|e(\Psi)\|_{L^2(\Lambda^{ext}_\varepsilon)} \longrightarrow 0$, then passing to the limit we obtain

$$\begin{aligned} \int_{\Omega^*_\varepsilon} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\,dx &= \int_{\Omega^{ext}_\varepsilon\times\mathbf{C}} \mathcal{T}^*_\varepsilon(\sigma^\varepsilon(u_\varepsilon)) : \mathcal{T}^*_\varepsilon(e(v_\varepsilon))\,dx\,dX \\ &= \int_{\Omega\times\mathbf{C}} \mathcal{T}^*_\varepsilon(\sigma^\varepsilon(u_\varepsilon)) : \mathcal{T}^*_\varepsilon(e(v_\varepsilon))\,dx\,dX + \int_{\Lambda^{\text{ext}}} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\,dx \\ &\longrightarrow \int_{\Omega\times\mathbf{C}} a_{ijkl}(e_{kl}(u) + e_{X,kl}(\widehat{u}))(e_{ij}(\Psi) + e_{X,ij}(\psi)\varphi)\,dx\,dX. \end{aligned}$$

[1]Every function in $H^1_\Gamma(\Omega)$ is extended in a function in $H^1_\Gamma(\Omega')$ which vanishes in $\Omega' \setminus \overline{\Omega}$.

Taking into account (2.4) and using $\|v_\varepsilon\|_{L^2(\Lambda_\varepsilon^{ext})} = \|\Psi\|_{L^2(\Lambda_\varepsilon^{ext})} \longrightarrow 0$ we have

$$\int_{\Omega_\varepsilon^*} f\cdot v_\varepsilon\,dx = \int_{\Omega_\varepsilon^{ext}\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f)\cdot\mathcal{T}_\varepsilon^*(v_\varepsilon)\,dxdX = \int_{\Omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f)\cdot\mathcal{T}_\varepsilon^*(v_\varepsilon)\,dxdX + \int_{\Lambda^{ext}} f\cdot v_\varepsilon\,dx$$
$$\longrightarrow \int_{\Omega\times\mathbf{C}} f(x)\cdot\Psi(x)\,dx\,dX = |\mathbf{C}|\int_\Omega f(x)\cdot\Psi(x)\,dx.$$

Hence, the above convergences lead to

$$\int_{\Omega\times\mathbf{C}} a_{ijkl}(e_{kl}(u)+e_{X,kl}(\widehat{u}))(e_{ij}(\Psi)+e_{X,ij}(\psi)\varphi)\,dx\,dX = |\mathbf{C}|\int_\Omega f\cdot\Psi\,dx.$$

Finally, since the functions $\Psi\in H^1(\Omega_1)^N$ such that $\Psi=0$ in $\Omega_1\cap\big(\Omega'\setminus\overline{\Omega}\big)$ are dense in $H^1_\Gamma(\Omega)$ and the tensor product $\mathcal{D}(\Omega)\otimes H^1_{N,per,0}(\mathbf{C})$ is dense in $L^2(\Omega;H^1_{N,per,0}(\mathbf{C}))$ we obtain (2.12).
The solution to the variational problem (2.12) is unique. Indeed, if there are two solutions $(u_1,\widehat{u}_1)$ and $(u_2,\widehat{u}_2)$ to this problem, denote $v=u_1-u_2$ and $\widehat{v}=\widehat{u}_1-\widehat{u}_2$. Taking into account the respective equalities from (2.12) and choosing the test functions v, $\widehat{v}$, we obtain

$$\int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(v)+e_{X,kl}(\widehat{v})\big)\big(e_{ij}(v)+e_{X,ij}(\widehat{v})\,dx\,dX=0.$$

Using property (2.2) of tensor $\{a_{ijkl}\}$ yields

$$c_0\|e(v)+e(\widehat{v})\|^2_{L^2(\Omega\times\mathbf{C})} \le \int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(v)+e_{X,kl}(\widehat{v})\big)\big(e_{ij}(v)+e_{X,ij}(\widehat{v})\big)\,dx\,dX=0.$$

So $e(\widehat{v})=-e(v)$ and thus the field $\widehat{v}$ is an affine function with respect to X. Since it is periodic with respect to X and belongs to $L^2(\Omega;H^1_{N,per,0}(\mathbf{C}))^N$ it is equal to 0. Hence, $e(v)=0$ and due to the boundary conditions $v=0$. Finally, the whole sequences in (2.13) converge to their limits.
Now, we prove the strong convergence $(2.11)_3$. By Proposition 2.1, (2.6), (2.12) we have

$$\begin{aligned}
&\int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(u)+e_{X,kl}(\widehat{u})\big)\big(e_{kl}(u)+e_{X,kl}(\widehat{u})\big)\,dx\,dX\\
&\le \liminf_{\varepsilon\to0}\int_{\Omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(a^\varepsilon_{ijkl})\mathcal{T}_\varepsilon^*(e_{kl}(u_\varepsilon))\mathcal{T}_\varepsilon^*(e_{ij}(u_\varepsilon))\,dx\,dX\\
&\le \liminf_{\varepsilon\to0}\int_{\Omega\times\mathbf{C}} a_{ijkl}\,\mathcal{T}_\varepsilon^*(e_{kl}(u_\varepsilon))\mathcal{T}_\varepsilon^*(e_{ij}(u_\varepsilon))\,dx\,dX + \liminf_{\varepsilon\to0}\int_{\Lambda^{\rm ext}}\sigma^\varepsilon(u_\varepsilon):e(v_\varepsilon)\,dx\\
&\le \liminf_{\varepsilon\to0}\int_{\Omega_\varepsilon^*}\sigma(u_\varepsilon):e(u_\varepsilon)\,dx \le \limsup_{\varepsilon\to0}\int_{\Omega_\varepsilon^*}\sigma(u_\varepsilon):e(u_\varepsilon)\,dx = \limsup_{\varepsilon\to0}\int_{\Omega_\varepsilon^*} f\cdot u_\varepsilon\,dx\\
&= |\mathbf{C}|\int_{\Omega\times\mathbf{C}} f\cdot u\,dx = \int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(u)+e_{X,kl}(\widehat{u})\big)\big(e_{kl}(u)+e_{X,kl}(\widehat{u})\big)\,dx\,dX.
\end{aligned}$$

The strong convergence $(2.11)_3$ holds. □

2.5 Homogenization

In this section we give the expressions of the microscopic field $\widehat{u}$ in terms of the macroscopic displacement u. First, taking $\Psi=0$ as a test function in (2.12), we obtain

$$\int_{\mathbf{C}} a_{ijkl}\big(e_{kl}(u)+e_{X,kl}(\widehat{u})\big)e_{X,ij}(\widehat{\Phi})\,dx\,dX=0,\quad \forall\,\widehat{\Phi}\in H^1_{per,0}(\mathbf{C})^N,\quad \text{a.e. in }\Omega.$$

This shows that the displacement $\widehat{u}$ can be written in terms of the elements of the tensor $e(u)$.
Denote by $\mathbf{M}^{np}$ the $N\times N$ symmetric matrix with following coefficients

$$\mathbf{M}^{np}_{kl}=\frac{1}{2}\big(\delta_{kn}\delta_{lp}+\delta_{kp}\delta_{ln}\big),\qquad n,p,k,l\in\overline{1,N},$$

where δ_{ij} is the Kronecker's symbol.

Since the tensor $e(u)$ has N^2 components, we introduce the N^2 correctors

$$\widehat{\chi}_{np} \in H^1_{per,0}(\mathbf{C})^N \qquad n,p=\overline{1,N},$$

which are the solutions to the following cell problems:

$$\int_{\mathbf{C}} a_{ijkl}\big(e_{X,kl}(\widehat{\chi}_{np})+\mathbf{M}^{np}_{kl}\big)e_{X,ij}(\widehat{\Phi})\,dX=0, \qquad \forall\widehat{\Phi}\in H^1_{per,0}(\mathbf{C})^N. \tag{2.14}$$

Observe that $\widehat{\chi}_{np}=\widehat{\chi}_{pn}$. As a consequence, the function $\widehat{u}$ is written in the form

$$\widehat{u}(x,X)=\sum_{n,p=1}^{N} e_{np}(u)(x)\widehat{\chi}_{np}(X) \qquad \text{for a.e. } (x,X)\in\Omega\times\mathbf{C}. \tag{2.15}$$

Theorem 2.1. *The limit displacement $u\in H^1_\Gamma(\Omega)^N$ is the solution to the following homogenized problem:*

$$\int_\Omega a^{hom}_{ijkl}\,e_{kl}(u)e_{ij}(\Psi)\,dx=\int_\Omega f\cdot\Psi\,dx \quad \forall\Psi\in H^1_\Gamma(\Omega)^N, \tag{2.16}$$

where

$$a^{hom}_{ijkl}=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} a_{ijnp}\left(\mathbf{M}^{kl}_{np}+e_{X,np}(\widehat{\chi}_{kl})\right)dX. \tag{2.17}$$

Proof. Taking $\widehat{\Phi}=0$ as a test function in (2.12) and using (2.15) give

$$\int_{\Omega\times\mathbf{C}} a_{ijkl}\big(e_{kl}(u)+e_{np}(u)e_{X,kl}(\widehat{\chi}_{np})\big)e_{ij}(\Psi)\,dx\,dX=|\mathbf{C}|\int_\Omega f\cdot\Psi\,dx \quad \forall\Psi\in H^1_\Gamma(\Omega)^N.$$

After straightforward calculations we have

$$\begin{aligned}&\int_{\Omega\times\mathbf{C}} a_{ijkl}\left(M^{np}_{kl}e_{np}(u)+e_{X,kl}(\widehat{\chi}_{np})e_{np}(u)\right)e_{ij}(\Psi)\,dx\,dX=|\mathbf{C}|\int_\Omega f\cdot\Psi\,dx,\\ &\int_\Omega\Big(\int_{\mathbf{C}} a_{ijkl}\big(\mathbf{M}^{np}_{kl}+e_{X,kl}(\widehat{\chi}_{np})\big)\,dX\Big)e_{np}(u)\,e_{ij}(\Psi)\,dx=|\mathbf{C}|\int_\Omega f\cdot\Psi\,dx, \quad \forall\Psi\in H^1_\Gamma(\Omega)^N\end{aligned}$$

and the assertion of the theorem follows. □

Lemma 2.4. *The left-hand side operator in problem* (2.16) *is uniformly elliptic.*

Proof. Using formulas (2.17) of the homogenized coefficients and (2.14), we obtain

$$a^{hom}_{npn'p'}\tau_{np}\tau_{n'p'}=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} a_{ijkl}\Big(e_{X,kl}\big(\Psi\big)+M_{kl}\Big)\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\,dX, \quad \tau\in M^{N\times N}_s$$

where

$$M=\tau_{np}\mathbf{M}^{np}, \qquad \Psi=\tau_{np}\widehat{\chi}_{np}.$$

Then, in view of (2.2) and following the proof of [9, Lemma 11.19], we have

$$a^{hom}_{npn'p'}\tau_{np}\tau_{n'p'}\geq\frac{c_0}{|\mathbf{C}|}\int_{\mathbf{C}}\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\,dX\geq c_0\tau_{np}\tau_{np}$$

which ends the proof. □

3 Periodic plate

3.1 Notations and geometric setting

In this section, we consider a bounded domain ω in $\mathbb{R}^2$ with Lipschitz boundary.
Denote:

- γ a subset of $\partial\omega$ with a non null measure. We assume that there exists a bounded domain ω' with Lipschitz boundary such that
$$\omega \subset \omega' \quad \text{and} \quad \omega' \cap \partial\omega = \gamma.$$
- $\Gamma_\varepsilon \doteq \gamma \times (-\varepsilon/2, \varepsilon/2)$,
- $Y' \doteq (-1/2, 1/2)^2$, $Y \doteq Y' \times (-1/2, 1/2) = (-1/2, 1/2)^3$,
- $\mathbf{C} \subset Y$ a domain with Lipschitz boundary such that $\text{interior}\big(\overline{\mathbf{C}} \cup (\overline{\mathbf{C}} + \mathbf{e}_i)\big)$, $i = \overline{1,2}$, is connected,
- $\Xi_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^2 \ \big| \ (\varepsilon\xi + \varepsilon Y') \cap \omega \neq \emptyset\big\}$, $\Xi'_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^2 \ \big| \ (\varepsilon\xi + \varepsilon Y') \cap \omega' \neq \emptyset\big\}$,
- $\Omega^*_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big)$, $\Omega'^*_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi'_\varepsilon}(\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big)$,
- $\omega^{\text{ext}}_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon}(\varepsilon\xi + \varepsilon\overline{Y'})\Big)$, $\quad \omega_1 = \big\{x \in \mathbb{R}^2 \ \big| \ \text{dist}(x, \omega) < 1\big\}$, $\omega \subset \omega_1$,
- $\omega'^{int}_{3\varepsilon} = \big\{x \in \omega \ \big| \ \text{dist}(x, \partial\omega') > 3\varepsilon\big\}$, $\Xi'^{int}_\varepsilon \doteq \big\{\xi \in \Xi_\varepsilon \ \big| \ (\varepsilon\xi + \varepsilon Y') \cap \omega'^{int}_{3\varepsilon} \neq \emptyset\big\}$,
- $\Omega'^{int}_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi'^{int}_\varepsilon}(\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big)$.

Note that the domain Ω^*_ε is a connected open set and if ε is small enough, we have $\Omega^*_\varepsilon \subset \omega_1 \times (-\varepsilon/2, \varepsilon/2)$.
The space of all admissible displacements is denoted by $\mathbf{V}_\varepsilon$:

$$\mathbf{V}_\varepsilon \doteq \Big\{v \in H^1(\Omega^*_\varepsilon)^3 \ \Big| \ \exists v' \in H^1(\Omega'^*_\varepsilon)^3, \ \ v = v'_{|\Omega^*_\varepsilon}, \ \ v' = 0 \text{ in } \Omega'^*_\varepsilon \setminus \overline{\Omega^*_\varepsilon}\Big\}.$$

We are interested in the elastic behavior of a structure occupying the domain Ω^*_ε and fixed on a part of its boundary (see above).

The constitutive law for the material occupying the domain Ω^*_ε is given by the relation between the strain tensor and the stress tensor (see also (2.3))

$$\sigma^\varepsilon_{ij}(u) \doteq a^\varepsilon_{ijkl} e_{kl}(u) \quad \forall u \in \mathbf{V}_\varepsilon,$$

where the coefficients a^ε_{ijkl} the same as in Subsection 2.1.
Let f be in $L^2(\omega_1)^3$. We define the applied forces f_ε as follows

$$f_{\varepsilon,\alpha} = \varepsilon^2 f_\alpha|_{\Omega^*_\varepsilon}, \quad f_{\varepsilon,3} = \varepsilon^3 f_3|_{\Omega^*_\varepsilon}, \qquad \alpha = 1, 2. \tag{3.1}$$

The unknown displacement $u_\varepsilon : \Omega^*_\varepsilon \to \mathbb{R}^3$ is the solution to the linearized elasticity system

$$\begin{cases} \nabla \cdot \sigma^\varepsilon(u_\varepsilon) = -f_\varepsilon & \text{in} \quad \Omega^*_\varepsilon, \\ u_\varepsilon = 0 & \text{on} \quad \Gamma_\varepsilon \cap \overline{\Omega^*_\varepsilon}, \\ \sigma^\varepsilon(u_\varepsilon) \cdot \nu_\varepsilon = 0 & \text{on} \quad \partial\Omega^*_\varepsilon \setminus \Gamma_\varepsilon, \end{cases} \tag{3.2}$$

where ν_ε is the outward normal vector to $\partial\Omega^*_\varepsilon$. The variational formulation of problem (3.2) is

$$\begin{cases} \text{Find } u_\varepsilon \in \mathbf{V}_\varepsilon \text{ such that,} \\ \displaystyle\int_{\Omega^*_\varepsilon} \sigma^\varepsilon(u_\varepsilon) : e(v)\, dx = \int_{\Omega^*_\varepsilon} f_\varepsilon \cdot v\, dx, \qquad \forall v \in \mathbf{V}_\varepsilon. \end{cases} \tag{3.3}$$

3.2 The unfolding-rescaling operator

Definition 3.1. *For every measurable function $u : \Omega^*_\varepsilon \to \mathbb{R}^3$ the unfolding operator $\mathcal{T}^*_\varepsilon$ is defined as follows:*

$$\mathcal{T}^*_\varepsilon(u)(x', X) = u\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big] + \varepsilon X', \varepsilon X_3\Big) \quad \textit{for a.e. } (x', X) \in \omega^{ext}_\varepsilon \times \mathbf{C},$$

where $x' = (x_1, x_2)$, $X = (X', X_3) = (X_1, X_2, X_3)$.

Below we recall some properties of $\mathcal{T}^*_\varepsilon$ (for further results see [9]).

Lemma 3.1. *For every $u \in L^2(\Omega_\varepsilon^*)$*

$$\begin{aligned} &\int_{\omega_\varepsilon^{ext}\times\mathbf{C}} \mathcal{T}_\varepsilon^*(u)(x',X)\,dx'dX = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon^*} u(x)\,dx, \\ &\|\mathcal{T}_\varepsilon^P(u)\|_{L^2(\omega_\varepsilon^{ext}\times\mathbf{C})} = \frac{1}{\sqrt{\varepsilon}}\|u\|_{L^2(\Omega_\varepsilon^*)}. \end{aligned} \tag{3.4}$$

For every $u \in H^1(\Omega_\varepsilon^*)$

$$\mathcal{T}_\varepsilon^*(\nabla u)(x',X) = \frac{1}{\varepsilon}\nabla_X \mathcal{T}_\varepsilon^*(u)(x',X) \quad \textit{for a.e. } (x',X) \in \omega_\varepsilon^{ext}\times\mathbf{C}. \tag{3.5}$$

3.3 Weak limits of the fields and the limit problem

Denote by $H^1_\gamma(\omega)$ the space of functions in $H^1(\omega)$ that vanish on γ

$$H^1_\gamma(\omega) \doteq \Big\{u \in H^1(\omega) \mid u = 0 \text{ on } \gamma\Big\},$$

and by $H^2_\gamma(\omega)$ the space of functions in $H^2(\omega)$ that vanish on γ and their derivatives vanish on γ as well

$$H^2_\gamma(\omega) \doteq \Big\{u \in H^2(\omega) \mid u = 0 \text{ and } \nabla u = 0 \text{ on } \gamma\Big\}.$$

Lemma 3.2. *The solution u_ε of the problem (3.2) satisfies*

$$\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^{5/2}\Big(\|f\|_{L^2(\omega_1)} + \|g\|_{L^2(\omega_1)}\Big). \tag{3.6}$$

Proof. Taking into account the decomposition of the displacements introduced in Subsection 5.3 of the appendix, the Cauchy–Schwarz inequality, the estimates (5.14) and (5.21) of Corollary 5.1, we have

$$\Big|\int_{\Omega_\varepsilon^*} f_\varepsilon\cdot u_\varepsilon\,dx\Big| = \Big|\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot u_\varepsilon\,dx\Big| \le \sum_{\xi\in\Xi_\varepsilon}\Big|\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot(u_\varepsilon - \mathbf{R}_{\varepsilon\xi})\,dx\Big| + \sum_{\xi\in\Xi_\varepsilon}\Big|\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot\mathbf{R}_{\varepsilon\xi}\,dx\Big|. \tag{3.7}$$

Each term in the right-hand side of (3.7) can be estimated as follows:

$$\begin{aligned} \sum_{\xi\in\Xi_\varepsilon}\Big|\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot(u_\varepsilon - \mathbf{R}_{\varepsilon\xi})\,dx\Big| &\le \sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+\mathbf{C})}\Big|f_\varepsilon\cdot(u_\varepsilon - \mathbf{R}_{\varepsilon\xi})\Big|\,dx \\ &\le \sum_{\xi\in\Xi_\varepsilon}\|f_\varepsilon\|_{L^2(\varepsilon(\xi+\mathbf{C}))}\|u_\varepsilon - \mathbf{R}_{\varepsilon\xi}\|_{L^2(\varepsilon(\xi+\mathbf{C}))} \\ &\le \sqrt{\sum_{\xi\in\Xi_\varepsilon}\|f_\varepsilon\|^2_{L^2(\varepsilon(\xi+\mathbf{C}))}}\sqrt{\sum_{\xi\in\Xi_\varepsilon}\|u_\varepsilon - \mathbf{R}_{\varepsilon\xi}\|^2_{L^2(\varepsilon(\xi+\mathbf{C}))}} \le C\varepsilon^{7/2}\|f\|_{L^2(\omega_1)}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}, \end{aligned}$$

$$
\begin{aligned}
\sum_{\xi\in\Xi_\varepsilon}\Big|\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot\mathbf{R}_{\varepsilon\xi}\,dx\Big| &= \sum_{\xi\in\Xi_\varepsilon}\Big|\int_{\varepsilon(\xi+\mathbf{C})} f_\varepsilon\cdot(\boldsymbol{\mathcal{U}}(\varepsilon\xi)+\boldsymbol{\mathcal{R}}(\varepsilon\xi)\wedge(x-\varepsilon\xi))\,dx\Big| \\
&\le \varepsilon^3 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}|\boldsymbol{\mathcal{U}}_1(\varepsilon\xi)f_1(x')|\,dx' + \varepsilon^3 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}|\boldsymbol{\mathcal{U}}_2(\varepsilon\xi)f_2(x')|\,dx' \\
&+ \varepsilon^4 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}|\boldsymbol{\mathcal{U}}_3(\varepsilon\xi)f_3(x')|\,dx' \\
&+ \varepsilon^4 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}\big|\boldsymbol{\mathcal{R}}_1(\varepsilon\xi)\big|\Big(\varepsilon\big|f_3(x')\big|+\big|f_2(x')\big|\Big)\,dx' \\
&+ \varepsilon^4 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}\big|\boldsymbol{\mathcal{R}}_2(\varepsilon\xi)\big|\Big(\big|f_1(x')\big|+\varepsilon\big|f_3(x')\big|\Big)\,dx' \\
&+ \varepsilon^4 C\sum_{\xi\in\Xi_\varepsilon}\int_{\varepsilon(\xi+Y')}\big|\boldsymbol{\mathcal{R}}_3(\varepsilon\xi)\big|\Big(\big|f_2(x')\big|+\big|f_1(x')\big|\Big)\,dx' \\
&\le \varepsilon^3 C\|f_1\|_{L^2(\omega_1)}\sqrt{\sum_{\xi\in\Xi_\varepsilon}|\boldsymbol{\mathcal{U}}_1(\varepsilon\xi)|^2\varepsilon^2} + \varepsilon^3 C\|f_2\|_{L^2(\omega_1)}\sqrt{\sum_{\xi\in\Xi_\varepsilon}|\boldsymbol{\mathcal{U}}_2(\varepsilon\xi)|^2\varepsilon^2} \\
&+ \varepsilon^4 C\|f_3\|_{L^2(\omega_1)}\sqrt{\sum_{\xi\in\Xi_\varepsilon}|\boldsymbol{\mathcal{U}}_3(\varepsilon\xi)|^2\varepsilon^2} + C\varepsilon^4\sqrt{\sum_{\xi\in\Xi_\varepsilon}|\boldsymbol{\mathcal{R}}(\varepsilon\xi)|^2\varepsilon^2}\|f\|_{L^2(\omega_1)} \\
&\le C\varepsilon^{5/2}\|f\|_{L^2(\omega_1)}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}.
\end{aligned}
$$

And finally,

$$
\Big|\int_{\Omega_\varepsilon^*} f_\varepsilon\cdot u_\varepsilon\,dx\Big| \le C\varepsilon^{5/2}\|f\|_{L^2(\omega_1)}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}.
$$

Using this estimate, we obtain (3.6). □

Taking into account the result in Subsection 5.3 of the Appendix the following decomposition holds:

$$
u_\varepsilon(x)=\mathbf{U}_\varepsilon^e(x)+\overline{u}_\varepsilon(x)=\begin{pmatrix}\mathcal{U}_{\varepsilon,1}(x')+x_3\mathcal{R}_{\varepsilon,2}(x')\\ \mathcal{U}_{\varepsilon,2}(x')-x_3\mathcal{R}_{\varepsilon,1}(x')\\ \mathcal{U}_{\varepsilon,3}(x')\end{pmatrix}+\overline{u}_\varepsilon(x),\quad x=(x',x)=(x_1,x_2,x_3)\in\Omega_\varepsilon^{\prime\,int}, \tag{3.8}
$$

here $\mathcal{U}_\varepsilon\in H^1_\gamma(\omega_{3\varepsilon}^{\prime\,int})^3$, $\mathcal{R}_\varepsilon\in H^1_\gamma(\omega_{3\varepsilon}^{\prime\,int})^2$ and $\overline{u}_\varepsilon\in H^1\big(\omega_{3\varepsilon}^{\prime\,int}\times(-\varepsilon/2,\varepsilon/2)\big)^3$.
Moreover, the strain tensor of the displacement u_ε is the symmetric matrix

$$
\begin{aligned}
e(u_\varepsilon)=e(\mathbf{U}_\varepsilon^e)+e(\overline{u}_\varepsilon)=&\begin{pmatrix}\frac{\partial\mathcal{U}_{\varepsilon,1}}{\partial x_1}+x_3\frac{\partial\mathcal{R}_{\varepsilon,1}}{\partial x_1} & \frac12\Big(\frac{\partial\mathcal{U}_{\varepsilon,1}}{\partial x_2}+\frac{\partial\mathcal{U}_{\varepsilon,2}}{\partial x_1}\Big)+\frac{x_3}{2}\Big(\frac{\partial\mathcal{R}_{\varepsilon,2}}{\partial x_2}-\frac{\partial\mathcal{R}_{\varepsilon,1}}{\partial x_1}\Big) & \frac12\Big(\mathcal{R}_{\varepsilon,2}+\frac{\partial\mathcal{U}_{\varepsilon,3}}{\partial x_1}\Big)\\ * & \frac{\partial\mathcal{U}_{\varepsilon,2}}{\partial x_2}-x_3\frac{\partial\mathcal{R}_{\varepsilon,1}}{\partial x_2} & \frac12\Big(-\mathcal{R}_{\varepsilon,1}+\frac{\partial\mathcal{U}_{\varepsilon,3}}{\partial x_2}\Big)\\ * & * & 0\end{pmatrix}+\\
&+\begin{pmatrix}\frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_1} & \frac12\Big(\frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_2}+\frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_1}\Big) & \frac12\Big(\frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_3}+\frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_1}\Big)\\ * & \frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_2} & \frac12\Big(\frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_3}+\frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_2}\Big)\\ * & * & \frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_3}\end{pmatrix}.
\end{aligned}
\tag{3.9}
$$

From Lemmas 3.2, 5.9 and Proposition 5.2 we obtain the following estimates of the terms $(\mathcal{U}_\varepsilon,\,\mathcal{R}_\varepsilon,\overline{u}_\varepsilon)$:

Lemma 3.3. *For every displacement $u_\varepsilon\in\mathbf{V}_\varepsilon$ one has*

$$
\begin{aligned}
&\sum_\alpha\|\mathcal{U}_{\varepsilon,\alpha}\|_{H^1(\omega_{3\varepsilon}^{\prime\,int})}+\varepsilon\|\mathcal{U}_{\varepsilon,3}\|_{H^1(\omega_{3\varepsilon}^{\prime\,int})}+\varepsilon\sum_\alpha\|\mathcal{R}_{\varepsilon,\alpha}\|_{H^1(\omega_{3\varepsilon}^{\prime\,int})}+\|\mathcal{R}_{\varepsilon,3}\|_{H^1(\omega_{3\varepsilon}^{\prime\,int})}\le\frac{C}{\varepsilon^{1/2}}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)},\\
&\|u_{\varepsilon,1}\|_{L^2(\Omega_\varepsilon^*)}+\|u_{\varepsilon,2}\|_{L^2(\Omega_\varepsilon^*)}+\varepsilon\|u_{\varepsilon,3}\|_{L^2(\Omega_\varepsilon^*)}+\varepsilon\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^*)}\le C\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)},\\
&\Big\|\frac{\partial\mathcal{U}_\varepsilon}{\partial x_\alpha}-\mathcal{R}_\varepsilon\wedge\mathbf{e}_\alpha\Big\|_{L^2(\omega_{3\varepsilon}^{\prime\,int})}\le\frac{C}{\varepsilon^{1/2}}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)},\ \alpha=1,2,\\
&\|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^{\prime\,int})}+\varepsilon\|\nabla\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^{\prime\,int})}\le C\varepsilon\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)}.
\end{aligned}
\tag{3.10}
$$

We extend $\mathcal{U}_\varepsilon$, $\mathcal{R}_\varepsilon$ and their gradients by 0 in $\omega' \setminus \overline{\omega'^{int}_{3\varepsilon}}$ and the field $\overline{u}_\varepsilon$ by 0 in $\Omega'_\varepsilon \setminus \overline{\Omega'^{int}_\varepsilon}$.

Lemma 3.4. *For a subsequence, still denoted $\{\varepsilon\}$,*
(i) there exist $\mathcal{U}_\alpha \in H^1(\omega')$, $\alpha = 1,2$, $\mathcal{U}_3 \in H^2(\omega')$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{U}_{\varepsilon,\alpha} \longrightarrow \mathcal{U}_\alpha \quad \textit{strongly in} \quad L^2(\omega'),\\
&\frac{1}{\varepsilon}\mathcal{U}_{\varepsilon,3} \longrightarrow \mathcal{U}_3 \quad \textit{strongly in} \quad L^2(\omega'),\\
&\frac{1}{\varepsilon^2}\nabla\mathcal{U}_{\varepsilon,\alpha}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{U}_\alpha \quad \textit{weakly in} \quad L^2(\omega')^2,\\
&\frac{1}{\varepsilon}\nabla\mathcal{U}_{\varepsilon,3}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{U}_3 \quad \textit{weakly in} \quad L^2(\omega')^2,
\end{aligned} \tag{3.11}$$

(ii) there exists $\mathcal{R} \in H^1(\omega')^2$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha} \longrightarrow \mathcal{R}_\alpha \quad \textit{strongly in} \quad L^2(\omega'),\\
&\frac{1}{\varepsilon}\nabla\mathcal{R}_{\varepsilon,\alpha}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{R}_\alpha \quad \textit{weakly in} \quad L^2(\omega')^2,
\end{aligned} \tag{3.12}$$

and

$$\mathcal{R}_1 = -\frac{\partial\mathcal{U}_3}{\partial x_2}, \quad \mathcal{R}_2 = \frac{\partial\mathcal{U}_3}{\partial x_1} \quad \textit{a.e. in } \omega',\ \alpha = 1,2; \tag{3.13}$$

furthermore, the fields $\mathcal{U}_\alpha$, $\mathcal{R}$, $\mathcal{U}_3$ and $\nabla\mathcal{U}_3$ vanish in $\omega' \setminus \overline{\omega}$,
(iii) there exists $\overline{u} \in L^2(\omega'; H^1_{2,per}(\mathbf{C}))^3$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}^*_\varepsilon(\overline{u}_\varepsilon) \longrightarrow 0 \quad \textit{strongly in } L^2(\omega' \times \mathbf{C})^3,\\
&\frac{1}{\varepsilon^2}\mathcal{T}^*_\varepsilon(\nabla\overline{u}_\varepsilon\mathbf{1}_{\Omega'^*_\varepsilon}) \rightharpoonup \nabla_X\overline{u} \quad \textit{weakly in } L^2(\omega' \times \mathbf{C})^9.
\end{aligned} \tag{3.14}$$

Proof. In order to prove (i)-(ii) we note that from estimates (3.6) and $(3.10)_1$ in Lemma 3.3 and Lemma 5.1, it follows that there exist functions $\mathcal{U} \in H^1(\omega')^3$ and $\mathcal{R} \in H^1(\omega')^2$ such that following convergences hold

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{U}_{\varepsilon,\alpha} \rightharpoonup \mathcal{U}_\alpha \quad \text{weakly in} \quad L^2(\omega'), \quad \frac{1}{\varepsilon^2}\nabla\mathcal{U}_{\varepsilon,\alpha}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{U}_\alpha \quad \text{weakly in} \quad L^2(\omega')^2,\\
&\frac{1}{\varepsilon}\mathcal{U}_{\varepsilon,3} \rightharpoonup \mathcal{U}_3 \quad \text{weakly in} \quad L^2(\omega'), \quad \frac{1}{\varepsilon}\nabla\mathcal{U}_{\varepsilon,3}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{U}_3 \quad \text{weakly in} \quad L^2(\omega')^2,\\
&\frac{1}{\varepsilon}\mathcal{R}_{\varepsilon,\alpha} \rightharpoonup \mathcal{R}_\alpha \quad \text{weakly in} \quad L^2(\omega'), \quad \frac{1}{\varepsilon}\nabla\mathcal{R}_{\varepsilon,\alpha}\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \nabla\mathcal{R}_\alpha \quad \text{weakly in} \quad L^2(\omega')^2.
\end{aligned}$$

Now we prove that the fields $\mathcal{U}_\alpha$, $\mathcal{R}$, $\mathcal{U}_3$ and $\nabla\mathcal{U}_3$ vanish in $\omega' \setminus \overline{\omega}$.
Let $\mathcal{O}$ be an open subset such that $\mathcal{O} \Subset \omega' \setminus \overline{\omega}$. Since u_ε vanishes in $\Omega'^*_\varepsilon \setminus \overline{\Omega^*_\varepsilon}$, then the fields $\mathcal{U}_\varepsilon$, $\mathcal{R}_\varepsilon$ vanish in $\omega'_\varepsilon \setminus \overline{\omega'^{int}_{3\varepsilon}}$. If ε is small enough then $\mathcal{O} \subset \omega'_\varepsilon \setminus \overline{\omega'^{int}_{3\varepsilon}}$. Thus by construction the fields $\mathcal{U}_{\varepsilon,\alpha}$, $\mathcal{R}_\varepsilon$, $\mathcal{U}_{\varepsilon,3}$ and $\nabla\mathcal{U}_{\varepsilon,3}$ vanish in $\mathcal{O}$. As a consequence, their weak limits also vanish in $\mathcal{O}$. Since that is true for every open set $\mathcal{O}$ strictly included in $\omega' \setminus \overline{\omega}$, thus that is also satisfied in the full set $\omega' \setminus \overline{\omega}$.

Estimate $(3.10)_3$ leads to

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(\frac{\partial\mathcal{U}_{3,\varepsilon}}{\partial x_2} + \mathcal{R}_{1,\varepsilon}\right)\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \longrightarrow 0 \quad \text{strongly in } L^2(\omega'),\\
&\frac{1}{\varepsilon}\left(\frac{\partial\mathcal{U}_{3,\varepsilon}}{\partial x_1} - \mathcal{R}_{2,\varepsilon}\right)\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \longrightarrow 0 \quad \text{strongly in } L^2(\omega').
\end{aligned}$$

From convergences $(3.11)_4$ and $(3.12)_1$ we also have

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(\frac{\partial\mathcal{U}_{3,\varepsilon}}{\partial x_2} + \mathcal{R}_{1,\varepsilon}\right)\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \frac{\partial\mathcal{U}_3}{\partial x_2} + \mathcal{R}_1 \quad \text{weakly in } L^2(\omega'),\\
&\frac{1}{\varepsilon}\left(\frac{\partial\mathcal{U}_{3,\varepsilon}}{\partial x_1} - \mathcal{R}_{2,\varepsilon}\right)\mathbf{1}_{\omega'^{int}_{3\varepsilon}} \rightharpoonup \frac{\partial\mathcal{U}_3}{\partial x_1} - \mathcal{R}_2 \quad \text{weakly in } L^2(\omega')
\end{aligned}$$

and then we get the equalities (3.13). Thus, one has $\mathcal{U}_3 \in H^2(\omega')$

(iii) From estimate $(3.10)_4$, we obtain

$$\|\mathcal{T}_\varepsilon^*(\overline{u}_\varepsilon)\|_{L^2(\omega'\times\mathbf{C})} \le \frac{1}{\sqrt{\varepsilon}}\|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^{int})} \le C\varepsilon^{1/2}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^3,$$
$$\|\nabla_X \mathcal{T}_\varepsilon^*(\overline{u}_\varepsilon \mathbf{1}_{\Omega_\varepsilon'})\|_{L^2(\omega'\times\mathbf{C})} = \varepsilon\|\mathcal{T}_\varepsilon^*(\nabla\overline{u}_\varepsilon \mathbf{1}_{\Omega_\varepsilon'^*})\|_{L^2(\omega'\times\mathbf{C})} \le \varepsilon^{1/2}\|\nabla\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon'^{int})} \le C\varepsilon^{1/2}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^3.$$

Thus, for a subsequence, still denoted by $\{\varepsilon\}$, there exists $\overline{u} \in L^2(\omega'; H^1_{2,per}(\mathbf{C}))$ such that convergences $(3.14)_{1,2}$ hold. □

Since the fields $\mathcal{U}_\alpha$, $\mathcal{R}$, $\mathcal{U}_3$ and gradient $\nabla\mathcal{U}_3$ vanish in $\omega' \setminus \overline{\omega}$ then

$$\mathcal{U}_\alpha \in H^1_\gamma(\omega), \quad \mathcal{U}_3 \in H^2_\gamma(\omega), \quad \mathcal{R} \in H^1_\gamma(\omega)^2.$$

Lemma 3.5. *For a subsequence, still denoted by $\{\varepsilon\}$, we have*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(\mathcal{U}_{\varepsilon,\alpha}) \to \mathcal{U}_\alpha \quad \textit{strongly in} \quad L^2(\omega'\times\mathbf{C}), \quad \alpha = 1,2,\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\mathcal{U}_{\varepsilon,3}) \to \mathcal{U}_3 \quad \textit{strongly in} \quad L^2(\omega'\times\mathbf{C}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{U}_{\varepsilon,\alpha}}{\partial x_\beta}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup \frac{\partial\mathcal{U}_\alpha}{\partial x_\beta} \quad \textit{weakly in} \quad L^2(\omega'\times\mathbf{C}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{U}_{\varepsilon,3}}{\partial x_\beta}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \longrightarrow \frac{\partial\mathcal{U}_3}{\partial x_\beta}\ (\alpha,\beta = 1,2) \quad \textit{strongly in} \quad L^2(\omega'\times\mathbf{C});
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\mathcal{R}_{\varepsilon,1}) \longrightarrow -\frac{\partial\mathcal{U}_3}{\partial x_2}, \quad \frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\mathcal{R}_{\varepsilon,2}) \longrightarrow \frac{\partial\mathcal{U}_3}{\partial x_1} \quad \textit{strongly in} \quad L^2(\omega'\times\mathbf{C}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{R}_{\varepsilon,1}}{\partial x_\alpha}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup -\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha\partial x_2}, \quad \frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{R}_{\varepsilon,2}}{\partial x_\alpha}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup \frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_\alpha}, \quad \textit{weakly in} \quad L^2(\omega'\times\mathbf{C}).
\end{aligned} \tag{3.16}$$

Proof. Applying [8, Proposition 2.9] and equality (3.13) we have convergences $(3.15)_{1,2}$, $(3.16)_1$ and there exist functions $\widehat{\mathcal{R}}_\alpha$, $\widehat{\mathcal{U}}_\alpha$, $\widehat{\mathcal{U}}_3 \in L^2(\omega'; H^1_{2,per,0}(\mathbf{C}))$, $(\alpha = 1,2)$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{R}_{\varepsilon,\alpha}}{\partial x_\beta}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup -\frac{\partial^2\mathcal{U}_3}{\partial x_\alpha\partial x_\beta} + \frac{\partial\widehat{\mathcal{R}}_\alpha}{\partial X_\beta} \quad \text{weakly in} \quad L^2(\omega'\times\mathbf{C}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{U}_{\varepsilon,\alpha}}{\partial x_\beta}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup \frac{\partial\mathcal{U}_\alpha}{\partial x_\beta} + \frac{\partial\widehat{\mathcal{U}}_\alpha}{\partial X_\beta} \quad \text{weakly in} \quad L^2(\omega'\times\mathbf{C}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{\partial\mathcal{U}_{\varepsilon,3}}{\partial x_\beta}\mathbf{1}_{\omega_{3\varepsilon}'^{int}}\Big) \rightharpoonup \frac{\partial\mathcal{U}_3}{\partial x_\beta} + \frac{\partial\widehat{\mathcal{U}}_3}{\partial X_\beta} \quad \alpha,\beta = \overline{1,2} \quad \text{weakly in} \quad L^2(\omega'\times\mathbf{C}).
\end{aligned} \tag{3.17}$$

From Remark 5.1 the functions $\mathcal{R}_{\varepsilon,\alpha}$, $\mathcal{U}_{\varepsilon,\alpha}$, $\mathcal{U}_{\varepsilon,3}$ are piecewise linear with respect to the variables X_β $(\beta = \overline{1,2})$. Thus, the functions $\widehat{\mathcal{R}}_\alpha$, $\widehat{\mathcal{U}}_\alpha$, $\widehat{\mathcal{U}}_3$ are also piecewise linear. As they are periodic, these fields are independent on X_β, $\beta \in \{1,2\}$. Hence

$$\frac{\partial\widehat{\mathcal{R}}_\alpha}{\partial X_\beta} = \frac{\partial\widehat{\mathcal{U}}_\alpha}{\partial X_\beta} = \frac{\partial\widehat{\mathcal{U}}_3}{\partial X_\beta} = 0$$

and convergences $(3.15)_3$, $(3.16)_2$ hold. □

For any $u \in H^1(\omega)^2$, $v \in H^2(\omega)$ we denote

$$E^M(u) \doteq \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) & 0\\ \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) & \frac{\partial u_2}{\partial x_2} & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad E^B(v) \doteq \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2} & \frac{\partial^2 v}{\partial x_1\partial x_2} & 0\\ \frac{\partial^2 v}{\partial x_1\partial x_2} & \frac{\partial^2 v}{\partial x_2^2} & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

Lemma 3.6. *Let u_ε be the solution to (3.2). Then the following convergences hold:*

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(u_{\varepsilon,1}) \to \mathcal{U}_1 - X_3\frac{\partial \mathcal{U}_3}{\partial x_1} \quad \textit{strongly in} \quad L^2(\omega\times\mathbf{C}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(u_{\varepsilon,2}) \to \mathcal{U}_2 - X_3\frac{\partial \mathcal{U}_3}{\partial x_2} \quad \textit{strongly in} \quad L^2(\omega\times\mathbf{C}),\\
&\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(u_{\varepsilon,3}) \to \mathcal{U}_3 \quad \textit{strongly in} \quad L^2(\omega\times\mathbf{C}).
\end{aligned} \tag{3.18}$$

Moreover

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(e(u_\varepsilon)\mathbf{1}_{\omega'^{int}_{3\varepsilon}}) \rightharpoonup E^M(\mathcal{U}_M) - X_3E^B(\mathcal{U}_3) + e_X(\widehat{u}) \quad \textit{weakly in } L^2(\omega\times\mathbf{C})^9,\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(\sigma^\varepsilon_{ij}(u_\varepsilon\mathbf{1}_{\omega'^{int}_{3\varepsilon}})) \rightharpoonup a_{ijkl}\Big(E^M_{kl}(\mathcal{U}_m) - X_3E^B_{kl}(\mathcal{U}_3) + e_{X,kl}(\widehat{u})\Big) \quad \textit{weakly in } \ L^2(\omega\times\mathbf{C}),
\end{aligned} \tag{3.19}$$

where the functions $\mathcal{U}_m \doteq (\mathcal{U}_1,\mathcal{U}_2)$, $\mathcal{U}_3$, $\widehat{u}$ are the solution to the following unfolded problem:

$$\begin{cases}
\displaystyle\int_{\omega\times\mathbf{C}} a_{ijkl}\Big(E^M_{kl}(\mathcal{U}_m) - X_3E^B_{kl}(\mathcal{U}_3) + e_{X,kl}(\widehat{u})\Big)\Big(E^M_{ij}(V_m) - X_3E^B_{ij}(V_3) + e_{X,ij}(\Phi)\Big)\,dx'dX\\
\displaystyle= |\mathbf{C}|\int_\omega f_\alpha(x')V_\alpha(x')\,dx' + \int_{\mathbf{C}} X_3\,dX\int_\omega \Big[f_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x') + g_\alpha(x')V_\alpha(x')\Big]\,dx'\\
\displaystyle+\int_{\mathbf{C}} X_3^2\,dX\int_\omega g_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x')\,dx', \qquad \forall\Phi\in L^2(\omega;H^1_{2,per}(\mathbf{C}))^3,\ V_\alpha\in H^1_\gamma(\omega),\ \alpha=1,2,\ V_3\in H^2_\gamma(\omega).
\end{cases} \tag{3.20}$$

Proof. From $(3.15)_{1,2}$, $(3.16)_1$, $(3.14)_2$, we obtain the convergences (3.18).
From estimate $(3.10)_3$ and (3.6) one has

$$\Big\|\Big(\frac{\partial \mathcal{U}_\varepsilon}{\partial x_\alpha} - \mathcal{R}_\varepsilon\wedge\mathbf{e}_\alpha\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}}\Big\|_{L^2(\omega)} \le C\varepsilon^2.$$

Then there exists $\mathcal{X}\in L^2(\omega)^2$ such that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_1} - \mathcal{R}_{\varepsilon,2}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}} \rightharpoonup \mathcal{X}_1 \quad \text{weakly in} \quad L^2(\omega),\\
&\frac{1}{\varepsilon^2}\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_2} + \mathcal{R}_{\varepsilon,1}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}} \rightharpoonup \mathcal{X}_2 \quad \text{weakly in} \quad L^2(\omega).
\end{aligned} \tag{3.21}$$

Due to $(3.16)_2$, (3.21) and [9, Lemma 11.11] there exists a function $\widehat{\mathcal{Z}}\in L^2(\omega;H^1_{2,per}(\mathbf{C}))$ such that, up to subsequence,

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_1} - \mathcal{R}_{\varepsilon,2}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}}\Big) \rightharpoonup \mathcal{X}_1 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_1} - \widehat{\mathcal{R}}_2 \quad \text{weakly in} \quad L^2(\omega\times\mathbf{C}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_2} + \mathcal{R}_{\varepsilon,1}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}}\Big) \rightharpoonup \mathcal{X}_2 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_2} + \widehat{\mathcal{R}}_1 \quad \text{weakly in} \quad L^2(\omega\times\mathbf{C}),
\end{aligned}$$

where the field $\widehat{\mathcal{R}}_\alpha$ is introduced in Lemma 3.5 (see (3.17)). Since $\widehat{\mathcal{R}}$ is independent of X_1 and X_2 with mean value on a cell equal to zero. Hence, one has

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_1} - \mathcal{R}_{\varepsilon,2}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}}\Big) \rightharpoonup \mathcal{X}_1 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_1} \quad \text{weakly in} \quad L^2(\omega\times\mathbf{C}),\\
&\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\Big(\Big(\frac{\partial \mathcal{U}_{\varepsilon,3}}{\partial x_2} + \mathcal{R}_{\varepsilon,1}\Big)\mathbf{1}_{\omega^{int}_{3\varepsilon}}\Big) \rightharpoonup \mathcal{X}_2 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_2} \quad \text{weakly in} \quad L^2(\omega\times\mathbf{C}),
\end{aligned} \tag{3.22}$$

In order to prove $(3.19)_1$ note that from (3.9) and convergences (3.15), (3.16) we have

$$\begin{aligned}
\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*\big(e(u_\varepsilon\mathbf{1}_{\omega'^{int}_{3\varepsilon}})\big) &\rightharpoonup \begin{pmatrix} \frac{\partial\mathcal{U}_1}{\partial x_1} - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1^2} & \frac12\Big(\frac{\partial\mathcal{U}_1}{\partial x_2} + \frac{\partial\mathcal{U}_2}{\partial x_1}\Big) - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & \frac12\Big(\mathcal{X}_1 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_1}\Big)\\ * & \frac{\partial\mathcal{U}_2}{\partial x_2} - X_3\frac{\partial^2\mathcal{U}_3}{\partial x_2^2} & \frac12\Big(\mathcal{X}_2 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_2}\Big)\\ * & * & 0\end{pmatrix} + e_X(\overline{u})\\
&= \begin{pmatrix} \frac{\partial\mathcal{U}_1}{\partial x_1} & \frac12\Big(\frac{\partial\mathcal{U}_1}{\partial x_2} + \frac{\partial\mathcal{U}_2}{\partial x_1}\Big) & 0\\ * & \frac{\partial\mathcal{U}_2}{\partial x_2} & 0\\ * & * & 0\end{pmatrix} - X_3\begin{pmatrix} \frac{\partial^2\mathcal{U}_3}{\partial x_1^2} & \frac{\partial^2\mathcal{U}_3}{\partial x_1\partial x_2} & 0\\ * & \frac{\partial^2\mathcal{U}_3}{\partial x_2^2} & 0\\ * & * & 0\end{pmatrix} + \frac12\begin{pmatrix} 0 & 0 & \mathcal{X}_1 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_1}\\ * & 0 & \mathcal{X}_2 + \frac{\partial\widehat{\mathcal{Z}}}{\partial X_2}\\ * & * & 0\end{pmatrix} + e_X(\overline{u}).
\end{aligned}$$

Set

$$\widehat{u}(x,X) \doteq \begin{pmatrix} \overline{u}_1(x,X) + X_3\mathcal{X}_1(x) \\ \overline{u}_2(x,X) + X_3\mathcal{X}_2(x) \\ \overline{u}_3(x,X) + \widehat{\mathcal{Z}}(x,X) \end{pmatrix}$$

and thus $(3.19)_1$ follows. Then, taking into account definition (2.3), we have $(3.19)_2$.
To obtain the limit problem (3.20) let us define the following fields

$$V_\alpha \in H^1_\gamma(\omega),\ \alpha = 1,2, \quad V_3 \in H^2_\gamma(\omega), \quad \varphi \in \mathcal{D}(\omega), \quad \psi \in H^1_{2,per}(\mathbf{C})^3$$

and take the test function in (3.3) as

$$v_\varepsilon(x) = \varepsilon^2 \begin{pmatrix} V_1(x') - \frac{x_3}{\varepsilon}\frac{\partial V_3}{\partial x_1}(x') \\ V_2(x') - \frac{x_3}{\varepsilon}\frac{\partial V_3}{\partial x_2}(x') \\ \frac{1}{\varepsilon}V_3(x') \end{pmatrix} + \varepsilon^3 \begin{pmatrix} \varphi(x')\psi_{\varepsilon,1}(x) \\ \varphi(x')\psi_{\varepsilon,2}(x) \\ \varphi(x')\psi_{\varepsilon,3}(x) \end{pmatrix},$$

where $\psi_\varepsilon(x) = \psi\Big(\dfrac{x}{\varepsilon}\Big)$. Then

$$\begin{aligned} e(v_\varepsilon) = {} & \varepsilon^2 \begin{pmatrix} \frac{\partial V_1}{\partial x_1} & \frac{1}{2}\Big(\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\Big) & 0 \\ \frac{1}{2}\Big(\frac{\partial V_1}{\partial x_2} + \frac{\partial V_2}{\partial x_1}\Big) & \frac{\partial V_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \varepsilon x_3 \begin{pmatrix} \frac{\partial^2 V_3}{\partial x_1^2} & \frac{\partial^2 V_3}{\partial x_1 \partial x_2} & 0 \\ \frac{\partial^2 V_3}{\partial x_1 \partial x_2} & \frac{\partial^2 V_3}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + \varepsilon^3 \begin{pmatrix} \frac{\partial \varphi}{\partial x_1}\psi_1 & \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_2}\psi_1 + \frac{\partial \varphi}{\partial x_1}\psi_2\Big) & \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_3}\psi_1 + \frac{\partial \varphi}{\partial x_1}\psi_3\Big) \\ \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_2}\psi_1 + \frac{\partial \varphi}{\partial x_1}\psi_2\Big) & \frac{\partial \varphi}{\partial x_2}\psi_2 & \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_2}\psi_3 + \frac{\partial \varphi}{\partial x_3}\psi_2\Big) \\ \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_3}\psi_1 + \frac{\partial \varphi}{\partial x_1}\psi_3\Big) & \frac{1}{2}\Big(\frac{\partial \varphi}{\partial x_2}\psi_3 + \frac{\partial \varphi}{\partial x_3}\psi_2\Big) & \frac{\partial \varphi}{\partial x_3}\psi_3 \end{pmatrix} \\ & + \varepsilon^2\varphi \begin{pmatrix} \frac{\partial \psi_1}{\partial X_1} & \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_2} + \frac{\partial \psi_2}{\partial X_1}\Big) & \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_3} + \frac{\partial \psi_3}{\partial X_1}\Big) \\ \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_2} + \frac{\partial \psi_2}{\partial X_1}\Big) & \frac{\partial \psi_2}{\partial X_2} & \Big(\frac{\partial \psi_3}{\partial X_2} + \frac{\partial \psi_{\varepsilon,2}}{\partial X_3}\Big) \\ \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_3} + \frac{\partial \psi_3}{\partial X_1}\Big) & \Big(\frac{\partial \psi_3}{\partial X_2} + \frac{\partial \psi_{\varepsilon,2}}{\partial X_3}\Big) & \frac{\partial \psi_3}{\partial X_3} \end{pmatrix} \end{aligned} \tag{3.23}$$

Applying the unfolding operator $\mathcal{T}_\varepsilon^*$ to the stress tensor $e(v_\varepsilon)$ (3.23) and passing to the limit as $\varepsilon \to 0$, we obtain

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(e(v_\varepsilon)) \longrightarrow E^M(V_m) - X_3E^B(V_3) + e_X(\psi)\varphi \quad \text{strongly in } L^2(\omega \times \mathbf{C})^9, \tag{3.24}$$

where $V_m = (V_1, V_2)$.
Unfold the left hand side of (3.3) and taking into account that by virtue of (3.6), (3.23) and Cauchy-Schwarz inequality

$$\int_{\Omega_\varepsilon^* \setminus \Omega_\varepsilon'^{int}} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\,dx \le \|\sigma^\varepsilon(u_\varepsilon)\|_{L^2(\Omega_\varepsilon'^*)} \|e(v_\varepsilon)\|_{L^2(\Omega_\varepsilon^* \setminus \Omega_\varepsilon'^{int})} = O\Big(\varepsilon^{5/2}\Big)O\Big(\varepsilon^{7/2}\Big) = o\Big(\varepsilon^5\Big)$$

we have

$$\begin{aligned} & \int_{\Omega_\varepsilon^*} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\,dx = \varepsilon \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(\sigma^\varepsilon(u_\varepsilon \mathbf{1}_{\omega_{3\varepsilon}'^{,int}}) : \mathcal{T}_\varepsilon^*(e(v_\varepsilon))\,dx'dX + \int_{\Omega_\varepsilon^* \setminus \Omega_\varepsilon'^{int}} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\,dx \\ & = \varepsilon^5 \int_{\omega\times\mathbf{C}} \frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(\sigma^\varepsilon(u_\varepsilon \mathbf{1}_{\omega_{3\varepsilon}'^{,int}}) : \frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(e(v_\varepsilon))\,dx'dX + o(\varepsilon^5). \end{aligned}$$

Unfold the right hand side of (3.3)

$$\begin{aligned} \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\,dx = & \varepsilon \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_\varepsilon)\mathcal{T}_\varepsilon^*(v_\varepsilon)\,dx'dX \\ = & \varepsilon \sum_{\alpha=1}^{2} \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_{\varepsilon,\alpha})\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dx'dX + \varepsilon \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_{\varepsilon,3})\mathcal{T}_\varepsilon^*(v_{\varepsilon,3})\,dx'dX. \end{aligned} \tag{3.25}$$

Taking into account the form of the applied forces (3.1) the first term in the right-hand side of (3.25) can be rewritten as follows

$$\varepsilon \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_{\varepsilon,\alpha})\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dxdX = \varepsilon^5 \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_\alpha)\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dx'dX + \varepsilon^5 \int_{\omega\times\mathbf{C}} X_3\mathcal{T}_\varepsilon^*(g_\alpha)\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dx'dX, \quad \alpha = 1,2$$

and, thus, as $\varepsilon \to 0$ we have

$$\begin{aligned}
&\int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_\alpha)\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dx'dX + \int_{\omega\times\mathbf{C}} X_3\mathcal{T}_\varepsilon^*(g_\alpha)\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(v_{\varepsilon,\alpha})\,dx'dX\\
&\to \int_{\omega\times\mathbf{C}} f_\alpha(x')\Big(V_\alpha(x') + X_3\frac{\partial V_3}{\partial x_\alpha}(x')\Big)\,dx'dX\\
&+ \int_{\omega\times\mathbf{C}} X_3 g_\alpha(x')\Big(V_\alpha(x') + X_3\frac{\partial V_3}{\partial x_\alpha}(x')\Big)\,dx'dX\\
&= |\mathbf{C}|\int_\omega f_\alpha(x')V_\alpha(x')\,dx' + \int_{\mathbf{C}} X_3\,dX\int_\omega f_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x')\,dx'\\
&+ \int_{\mathbf{C}} X_3\,dX\int_\omega g_\alpha(x')V_\alpha(x')\,dx' + \int_{\mathbf{C}} X_3^2\,dX\int_\omega g_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x')\,dx', \quad \alpha = 1,2.
\end{aligned}$$

Using (3.1) the second term in the right-hand side of (3.25) can be rewritten as follows

$$\varepsilon \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_{\varepsilon,3})\mathcal{T}_\varepsilon^*(v_{\varepsilon,3})\,dx'dX = \varepsilon^5 \int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_3)\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(v_{\varepsilon,3})\,dx'dX$$

and, thus, as $\varepsilon \to 0$

$$\int_{\omega\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_3)\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(v_{\varepsilon,3})\,dx'dX \to \int_{\omega\times\mathbf{C}} f_3(x')V_3(x')\,dx'dX = |\mathbf{C}|\int_\omega f_3(x')V_3(x')\,dx'.$$

Hence, taking into account (3.19), (3.24) and the convergences obtained above, we can pass to the limit as $\varepsilon \to 0$

$$\begin{aligned}
&\int_{\omega\times\mathbf{C}} a_{ijkl}(E_{kl}^M(\mathcal{U}_M) - X_3E_{kl}^B(\mathcal{U}_3) + e_{X,kl}(\widehat{u}))(E_{ij}^M(V_M) - X_3E_{ij}^B(V_3) + \varphi e_{X,ij}(\psi))\,dx'dX\\
&= |\mathbf{C}|\int_\omega f_i(x')V_i(x')\,dx' + \int_{\mathbf{C}} X_3\,dX\int_\omega [f_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x')\,dx' + g_\alpha(x')V_\alpha(x')]\,dx' + \int_{\mathbf{C}} X_3^2\,dX\int_\omega g_\alpha(x')\frac{\partial V_3}{\partial x_\alpha}(x')\,dx'.
\end{aligned}$$

Finally, since the tensor product $\mathcal{D}(\omega)\otimes H^1_{2,per}(\mathbf{C})$ is dense in $L^2(\omega; H^1_{2,per}(\mathbf{C}))$, we obtain the limit problem (3.20). □

3.4 Homogenization

In this section we give the expressions of the microscopic displacement $\widehat{u}$ in terms of the membrane displacements $\mathcal{U}_m$ and the bending $\mathcal{U}_3$.

Taking $V = 0$ as a test function in (3.20), we obtain

$$\int_{\mathbf{C}} a_{ijkl}(E_{kl}^M(\mathcal{U}_m) - X_3E_{kl}^B(\mathcal{U}_3) + e_{X,kl}(\widehat{u}))e_{X,ij}(\Phi)\,dX = 0 \quad \forall\Phi \in H^1_{2,per}(\mathbf{C})^3.$$

This shows that the microscopic displacement $\widehat{u}$ can be written in terms of the tensors E^M, E^B.

Set

$$\mathbf{M}^{11} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \mathbf{M}^{12} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \mathbf{M}^{22} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Since the tensors E^M, E^B have 6 components we introduce 6 correctors

$$\chi_{\alpha\beta}^M,\ \chi_{\alpha\beta}^B \in H^1_{2,per}(\mathbf{C})^3 \quad (\alpha,\beta) \in (1,1),\ (1,2),\ (2,2),$$

which are the solutions to the following cell problems

$$\begin{aligned}
&\int_{\mathbf{C}} a_{ijkl}\left(e_{X,kl}(\chi^M_{\alpha\beta}) + \mathbf{M}^{\alpha\beta}_{kl}\right) e_{X,ij}(\Phi(X))\,dX = 0,\\
&\int_{\mathbf{C}} a_{ijkl}\left(e_{X,kl}(\chi^B_{\alpha\beta}) - X_3\mathbf{M}^{\alpha\beta}_{kl}\right) e_{X,ij}(\Phi(X))\,dX = 0, \quad \alpha,\beta = 1,2.
\end{aligned} \tag{3.26}$$

for all $\Phi(X) \in L^2(\omega; H^1_{2,per}(\mathbf{C}))^3$.

As a consequence, the function $\widehat{u}$ from (3.19) is given in terms of $\mathcal{U}$ as follows

$$\widehat{u}(x', X) = \sum_{\alpha,\beta=1}^{2} \Big[e_{\alpha\beta}(\mathcal{U}_m(x'))\chi^M_{\alpha\beta}(X) + \frac{\partial^2 \mathcal{U}_3(x')}{\partial x_\alpha \partial x_\beta}\chi^B_{\alpha\beta}(X)\Big] \qquad \text{for a.e. } (x', X) \in \omega \times \mathbf{C}. \tag{3.27}$$

Theorem 3.1. *The limit displacement*

$$\mathcal{U} = \big(\mathcal{U}_m, \mathcal{U}_3\big) \in H^1_\gamma(\omega)^2 \times H^2_\gamma(\omega)$$

is the solution to homogenized problem

$$\begin{cases}
\displaystyle\int_\omega \Big(a^{hom}_{\alpha\beta\alpha'\beta'} e_{\alpha\beta}(\mathcal{U}_m) e_{\alpha'\beta'}(V_m) + b^{hom}_{\alpha\beta\alpha'\beta'} \frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta} e_{\alpha'\beta'}(V_m)\\
\displaystyle + b^{hom}_{\alpha\beta\alpha'\beta'} e_{\alpha\beta}(\mathcal{U}_m) \frac{\partial^2 V_3}{\partial x_{\alpha'} \partial x_{\beta'}} + c^{hom}_{\alpha\beta\alpha'\beta'} \frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 V_3}{\partial x_{\alpha'} \partial x_{\beta'}}\Big)\,dx'\\
\displaystyle = \int_\omega f_\alpha V_\alpha\,dx' + \frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} X_3\,dX \int_\omega \Big[f_\alpha \frac{\partial V_3}{\partial x_\alpha} + g_\alpha V_\alpha\Big]\,dx'\\
\displaystyle + \frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} X_3^2\,dX \int_\omega g_\alpha \frac{\partial V_3}{\partial x_\alpha}\,dx', \qquad \forall\, V_m \in H^1_\gamma(\omega),\, m = 1,2,\, V_3 \in H^2_\gamma(\omega).
\end{cases} \tag{3.28}$$

where

$$\begin{aligned}
a^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} a_{ijkl}\Big(e_{X,kl}\big(\chi^M_{\alpha\beta}\big) + M^{\alpha\beta}_{kl}\Big)\Big(e_{X,ij}\big(\chi^M_{\alpha'\beta'}\big) + M^{\alpha'\beta'}_{ij}\Big)\,dX,\\
b^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} a_{ijkl}\Big(e_{X,kl}\big(\chi^B_{\alpha\beta}\big) - X_3 M^{\alpha\beta}_{kl}\Big)\Big(e_{X,ij}\big(\chi^M_{\alpha'\beta'}\big) + M^{\alpha'\beta'}_{ij}\Big)\,dX,\\
c^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{|\mathbf{C}|}\int_{\mathbf{C}} a_{ijkl}\Big(e_{X,kl}\big(\chi^B_{\alpha\beta}\big) - X_3 M^{\alpha\beta}_{kl}\Big)\Big(e_{X,ij}\big(\chi^B_{\alpha'\beta'}\big) - X_3 M^{\alpha'\beta'}_{ij}\Big)\,dX.
\end{aligned} \tag{3.29}$$

Proof. Take $\Phi = 0$ as a test function in (3.20). Replacing $\widehat{u}$ by its expression (3.27), yields

$$\begin{aligned}
&\int_{\omega\times\mathbf{C}} a_{ijkl}\Big(e_{\alpha\beta}(\mathcal{U}_m)\Big(e_{X,kl}\big(\chi^M_{\alpha\beta}\big) + M^{\alpha\beta}_{kl}\Big) + \frac{\partial^2 \mathcal{U}_3}{\partial x_\alpha \partial x_\beta}\Big(e_{X,kl}\big(\chi^B_{\alpha\beta}\big) - X_3 M^{\alpha\beta}_{kl}\Big)\Big)\\
&\times M^{\alpha'\beta'}_{ij}\Big(e_{\alpha'\beta'}(V_m) - X_3\frac{\partial^2 V_3}{\partial x_{\alpha'}\partial x_{\beta'}}\Big)\,dx'dX\\
&= |\mathbf{C}|\int_\omega f_\alpha V_\alpha\,dx' + \int_{\mathbf{C}} X_3\,dX\int_\omega \Big[f_\alpha\frac{\partial V_3}{\partial x_\alpha} + g_\alpha V_\alpha\Big]\,dx' + \int_{\mathbf{C}} X_3^2\,dX\int_\omega g_\alpha\frac{\partial V_3}{\partial x_\alpha}\,dx'.
\end{aligned}$$

Taking into account the variational problems (3.26) satisfied by the correctors, the problem (3.28) with the homogenized coefficients given by (3.29) is obtained by a simple computation. □

Lemma 3.7. *The left-hand side operator in Problem* (3.28) *is uniformly elliptic.*

Proof. Using formulas (3.29) of the homogenized coefficients, we obtain

$$\begin{aligned}
&a^{hom}_{\alpha\beta\alpha'\beta'}\tau^m_{\alpha\beta}\tau^m_{\alpha'\beta'} + b^{hom}_{\alpha\beta\alpha'\beta'}\tau^b_{\alpha\beta}\tau^m_{\alpha'\beta'} + b^{hom}_{\alpha\beta\alpha'\beta'}\tau^m_{\alpha\beta}\tau^b_{\alpha'\beta'} + c^{hom}_{\alpha\beta\alpha'\beta'}\tau^b_{\alpha\beta}\tau^b_{\alpha'\beta'}\\
&= \int_{\mathbf{C}} a_{ijkl}\Big(e_{X,kl}\big(\Psi\big) + M_{kl}\Big)\Big(e_{X,ij}\big(\Psi\big) + M_{ij}\Big)\,dX, \qquad \tau^m_{\alpha\beta},\ \tau^b_{\alpha\beta} \in M^{2\times 2}_s
\end{aligned}$$

where

$$M = \big(\tau^m_{\alpha\beta} - X_3\tau^b_{\alpha\beta}\big)M^{\alpha\beta}, \qquad \Psi = \tau^m_{\alpha\beta}\chi^M_{\alpha\beta} + \tau^b_{\alpha\beta}\chi^B_{\alpha\beta}.$$

Then, in view of (2.2) and following the proof of [9][Lemma 11.19], we obtain

$$\begin{aligned}&a^{hom}_{\alpha\beta\alpha'\beta'}\tau^m_{\alpha\beta}\tau^m_{\alpha'\beta'} + b^{hom}_{\alpha\beta\alpha'\beta'}\tau^b_{\alpha\beta}\tau^m_{\alpha'\beta'} + b^{hom}_{\alpha\beta\alpha'\beta'}\tau^m_{\alpha\beta}\tau^b_{\alpha'\beta'} + c^{hom}_{\alpha\beta\alpha'\beta'}\tau^b_{\alpha\beta}\tau^b_{\alpha'\beta'}\\ \geq& c_0\int_{\mathbf{C}}\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\,dX \geqslant C\big(\tau^m_{\alpha\beta}\tau^m_{\alpha\beta}+\tau^b_{\alpha\beta}\tau^b_{\alpha\beta}\big).\end{aligned}$$ □

4 Periodic beam

4.1 Notations and geometric setting

Let $\mathbf{C}\in\mathbb{R}^3$ be a bounded domain with Lipschitz boundary and L be a fixed positive constant. In this section, we also assume that the interior of $\overline{\mathbf{C}}\cup(\overline{\mathbf{C}}+\mathbf{e}_3)$ is connected and $\mathbf{C}\cap(\mathbf{C}+\mathbf{e}_3)=\emptyset$. The beam-like structure is

$$\Omega^*_\varepsilon = \text{interior}\Big(\bigcup_{i=0}^{N-1}\varepsilon\big(i\mathbf{e}_3+\overline{\mathbf{C}}\big)\Big),\qquad \varepsilon=\frac{L}{N}.$$

We choose as centerline of the structure the segment whose direction is $\mathbf{e}_3$ and origin the center of mass of the first cell (thus the other centers of mass are also on this segment). The orthonormal basis $(\mathbf{e}_1,\,\mathbf{e}_2,\mathbf{e}_3)$ is chosen in such way to get $\displaystyle\int_{\mathbf{C}} x_1x_2\,dx=0$ and we set

$$I_\alpha=\int_{\mathbf{C}}x_\alpha^2\,dx.$$

Concerning the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ it is important to note that they do not necessary correspond to the principal axes of inertia.

The space of all admissible displacements is denoted by $\mathbf{V}_\varepsilon$

$$\mathbf{V}_\varepsilon=\big\{u\in H^1(\Omega^*_\varepsilon)^3\mid u=0\text{ on }\Gamma_\varepsilon\big\},\quad\text{where}\quad \Gamma_\varepsilon\doteq(\varepsilon\overline{\mathbf{C}}-\varepsilon\mathbf{e}_3)\cap\varepsilon\overline{\mathbf{C}}.$$

Here also, we are interested in the elastic behavior of a structure occupying the domain Ω^*_ε and fixed on the part Γ_ε of its boundary. The constitutive law for the material occupying the domain Ω^*_ε is given by the relation between the strain tensor and the stress tensor

$$\sigma^\varepsilon_{ij}(u)\doteq a^\varepsilon_{ijkl}e_{kl}(u)\qquad\forall u\in\mathbf{V}_\varepsilon,$$

where the coefficients a^ε_{ijkl} are given in Subsection 2.1.
Let f and g be in $L^2(0,L)^3$, we define the applied forces $f_\varepsilon\in L^2(\Omega^*_\varepsilon)^3$ by

$$\begin{aligned}f_{\varepsilon,1}(x)&=\big(\varepsilon^2f_1(x_1)+x_2g_3(x_1)\big)|_{\Omega^*_\varepsilon},\\ f_{\varepsilon,2}(x)&=\big(\varepsilon^2f_2(x_1)-x_1g_3(x_1)\big)|_{\Omega^*_\varepsilon},\qquad\qquad\text{for a.e. } x\in\Omega^*_\varepsilon.\\ f_{\varepsilon,3}(x)&=\big(\varepsilon f_3(x_1)-x_1g_1(x_1)-x_2g_2(x_1)\big)|_{\Omega^*_\varepsilon},\end{aligned}\tag{4.1}$$

The unknown displacement $u_\varepsilon:\Omega^*_\varepsilon\to\mathbb{R}^3$ is the solution to the linearized elasticity system

$$\begin{cases}\nabla\cdot\sigma^\varepsilon(u_\varepsilon)=-f_\varepsilon & \text{in}\quad\Omega^*_\varepsilon,\\ u_\varepsilon=0 & \text{on}\quad\Gamma_\varepsilon\cap\overline{\Omega^*_\varepsilon},\\ \sigma^\varepsilon(u_\varepsilon)\cdot\nu_\varepsilon=0 & \text{on}\quad\partial\Omega^*_\varepsilon\setminus\Gamma_\varepsilon,\end{cases}\tag{4.2}$$

where ν_ε is the outward normal vector to $\partial\Omega^*_\varepsilon$.

The variational formulation of problem (4.2) is

$$\begin{cases}\text{Find } u_\varepsilon\in\mathbf{V}_\varepsilon\text{ such that,}\\ \displaystyle\int_{\Omega^*_\varepsilon}\sigma^\varepsilon(u_\varepsilon):e(v)\,dx=\int_{\Omega^*_\varepsilon}f_\varepsilon\cdot v\,dx,\qquad\forall v\in\mathbf{V}_\varepsilon.\end{cases}\tag{4.3}$$

4.2 The unfolding-rescaling operator

Definition 4.1. *For every measurable function $\phi : \Omega_\varepsilon^* \to \mathbb{R}^3$ the unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as follows:*

$$\mathcal{T}_\varepsilon^*(\phi)(x_3, X) = \phi\Big(\varepsilon X_1, \varepsilon X_2, \varepsilon\Big[\frac{x_3}{\varepsilon}\Big] + \varepsilon X_3\Big) \quad \text{for a.e. } (x_3, X) \in (0, L) \times \mathbf{C}.$$

Lemma 4.1 (Properties of the operator $\mathcal{T}_\varepsilon^*$)**.**
(a) For every $\phi \in L^2(\Omega_\varepsilon^*)$

$$\begin{aligned} &\int_{(0,L)\times\mathbf{C}} \mathcal{T}_\varepsilon^*(\phi)(x_3, X)\, dx_3 dX = \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^*} \phi(x)\, dx, \\ &\|\mathcal{T}_\varepsilon^*(\phi)\|_{L^2((0,L)\times\mathbf{C})} = \frac{1}{\varepsilon}\|\phi\|_{L^2(\Omega_\varepsilon^*)}. \end{aligned} \tag{4.4}$$

(b) For every $\phi \in H^1(\Omega_\varepsilon^*)$

$$\mathcal{T}_\varepsilon^*(\nabla\phi)(x_3, X) = \frac{1}{\varepsilon}\nabla_X \mathcal{T}_\varepsilon^*(\phi)(x_3, X) \quad \text{for a.e. } (x_3, X) \in (0, L) \times \mathbf{C}. \tag{4.5}$$

4.3 Weak limits of the fields and the limit problem

Denote by $H^1_\Gamma(0, L)$, $H^2_\Gamma(0, L)$ the space of functions from $H^1(0, L)$, $H^2(0, L)$ respectively that vanish at 0 :

$$H^1_\Gamma(0, L) \doteq \Big\{u \in H^1(0, L) \mid u(0) = 0\Big\}, \qquad H^2_\Gamma(0, L) \doteq \Big\{u \in H^2(0, L) \mid u(0) = u'(0) = 0\Big\}.$$

Taking into account the results from Appendix (see Subsection 5.4) the following decomposition holds:

$$u_\varepsilon(x) = \mathbf{U}^e_\varepsilon(x) + \overline{u}_\varepsilon(x) = \mathcal{U}_\varepsilon(x_3) + \mathcal{R}_\varepsilon(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \overline{u}_\varepsilon(x), \quad \text{for a.e. } x = (x_1, x_2, x_3) \in \Omega_\varepsilon^*, \tag{4.6}$$

where $\mathcal{U}_\varepsilon, \mathcal{R}_\varepsilon \in W^{1,\infty}(0, L)^3$ and satisfy $\mathcal{U}_\varepsilon(0) = \mathcal{R}_\varepsilon(0) = 0$. The displacement $\overline{u}_\varepsilon$ belongs to $\mathbf{V}_\varepsilon$.

Lemma 4.2. *The solution u_ε to problem (4.2) satisfies the following estimate:*

$$\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^2\big(\|f\|_{L^2(0,L)} + \|g\|_{L^2(0,L)}\big). \tag{4.7}$$

Proof. Taking into account the estimates in Lemma 5.12, we have

$$\begin{aligned} \Big|\int_{\Omega_\varepsilon^*} f_\varepsilon \cdot u_\varepsilon\, dx\Big| &= \Big|\int_{\Omega_\varepsilon^*} \Big(\big(\varepsilon^2 f_1(x_3) + x_2 g_3(x_3)\big)\big(\mathcal{U}_{\varepsilon,1}(x_3) - x_2\mathcal{R}_{\varepsilon,3}(x_3) + \overline{u}_{\varepsilon,1}(x)\big) \\ &+ \big(\varepsilon^2 f_2(x_3) - x_1 g_3(x_3)\big)\big(\mathcal{U}_{\varepsilon,2}(x_3) + x_1\mathcal{R}_{\varepsilon,3}(x_3) + \overline{u}_{\varepsilon,2}(x)\big) \\ &+ \big(\varepsilon f_3(x_3) - x_1 g_1(x_3) - x_2 g_2(x_3)\big)\big(\mathcal{U}_{\varepsilon,3}(x_3) + x_2\mathcal{R}_{\varepsilon,1}(x_3) - x_1\mathcal{R}_{\varepsilon,2}(x_3) + \overline{u}_{\varepsilon,3}(x)\big)\Big)\, dx\Big| \\ &\le \varepsilon^2\Big|\int_{\Omega_\varepsilon^*} f_1(x_3)\mathcal{U}_{\varepsilon,1}(x_3)\, dx\Big| + \varepsilon^2\Big|\int_{\Omega_\varepsilon^*} f_1(x_3)\overline{u}_{\varepsilon,1}(x)\, dx\Big| + \Big|\int_{\Omega_\varepsilon^*} x_2^2 g_3(x_3)\mathcal{R}_{\varepsilon,3}(x_3)\, dx\Big| \\ &+ \varepsilon^2\Big|\int_{\Omega_\varepsilon^*} f_2(x_3)\mathcal{U}_{\varepsilon,2}(x_3)\, dx\Big| + \varepsilon^2\Big|\int_{\Omega_\varepsilon^*} f_2(x_3)\overline{u}_{\varepsilon,2}(x)\, dx\Big| + \Big|\int_{\Omega_\varepsilon^*} x_1^2 g_3(x_3)\mathcal{R}_{\varepsilon,3}(x_3)\, dx\Big| \\ &+ \varepsilon\Big|\int_{\Omega_\varepsilon^*} f_3(x_3)\mathcal{U}_{\varepsilon,3}(x_3)\, dx\Big| + \varepsilon\Big|\int_{\Omega_\varepsilon^*} f_3(x_3)\overline{u}_{\varepsilon,3}(x)\, dx\Big| + \Big|\int_{\Omega_\varepsilon^*} x_2^2 g_2(x_3)\mathcal{R}_{\varepsilon,1}(x_3)\, dx\Big| \\ &+ \Big|\int_{\Omega_\varepsilon^*} x_1^2 g_1(x_3)\mathcal{R}_{\varepsilon,2}(x_3)\, dx\Big| \\ &\le C\varepsilon^4\|f_1\|_{L^2(0,L)}\|\mathcal{U}_{\varepsilon,1}\|_{L^2(0,L)} + C\varepsilon^3\|f_1\|_{L^2(0,L)}\|\overline{u}_{\varepsilon,1}\|_{L^2(\Omega_\varepsilon^*)} + C\varepsilon^4\|g_3\|_{L^2(0,L)}\|\mathcal{R}_{\varepsilon,3}\|_{L^2(0,L)} \\ &+ C\varepsilon^4\|f_2\|_{L^2(0,L)}\|\mathcal{U}_{\varepsilon,2}\|_{L^2(0,L)} + C\varepsilon^3\|f_2\|_{L^2(0,L)}\|\overline{u}_{\varepsilon,2}\|_{L^2(\Omega_\varepsilon^*)} + C\varepsilon^4\|g_3\|_{L^2(0,L)}\|\mathcal{R}_{\varepsilon,3}\|_{L^2(0,L)} \\ &+ C\varepsilon^3\|f_3\|_{L^2(0,L)}\|\mathcal{U}_{\varepsilon,3}\|_{L^2(0,L)} + C\varepsilon^2\|f_3\|_{L^2(0,L)}\|\overline{u}_{\varepsilon,3}\|_{L^2(\Omega_\varepsilon^*)} + C\varepsilon^4\|g_2\|_{L^2(0,L)}\|\mathcal{R}_{\varepsilon,1}\|_{L^2(0,L)} \\ &+ C\varepsilon^4\|g_1\|_{L^2(0,L)}\|\mathcal{R}_{\varepsilon,2}\|_{L^2(0,L)} \le C\varepsilon^2\Big(\|f\|_{L^2(0,L)} + \|g\|_{L^2(0,L)}\Big)\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \end{aligned}$$

and thus (4.7) follows. □

The strain tensor for the displacement u_ε is

$$\begin{aligned} e(u_\varepsilon) &= e(\mathbf{U}^e_\varepsilon) + e(\overline{u}_\varepsilon) \\ &= \begin{pmatrix} 0 & 0 & \frac{1}{2}\Big(\frac{d\mathcal{U}_{\varepsilon,1}}{dx_3} - \mathcal{R}_{\varepsilon,2} - x_2\frac{d\mathcal{R}_{\varepsilon,3}}{dx_3}\Big) \\ * & 0 & \frac{1}{2}\Big(\frac{d\mathcal{U}_{\varepsilon,2}}{dx_3} + \mathcal{R}_{\varepsilon,1} + x_1\frac{d\mathcal{R}_{\varepsilon,3}}{dx_3}\Big) \\ * & * & \frac{d\mathcal{U}_{\varepsilon,3}}{dx_3} + x_2\frac{d\mathcal{R}_{\varepsilon,1}}{dx_3} - x_1\frac{d\mathcal{R}_{\varepsilon,2}}{dx_3} \end{pmatrix} + \begin{pmatrix} \frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_1} & \frac{1}{2}\Big(\frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_2} + \frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_1}\Big) & \frac{1}{2}\Big(\frac{\partial\overline{u}_{\varepsilon,1}}{\partial x_3} + \frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_1}\Big) \\ * & \frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_2} & \frac{1}{2}\Big(\frac{\partial\overline{u}_{\varepsilon,2}}{\partial x_3} + \frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_2}\Big) \\ * & * & \frac{\partial\overline{u}_{\varepsilon,3}}{\partial x_3} \end{pmatrix} \end{aligned} \tag{4.8}$$

In order to simplify the expression of the strain tensor $e(\mathbf{U}^e_\varepsilon)$, we define a new triplet $(\underline{u}_\varepsilon, \mathbb{U}_\varepsilon, \Theta_\varepsilon)$. Set

$$\mathbb{U}_\varepsilon(x_3) = \int_0^{x_3} \mathcal{R}_\varepsilon(t) \wedge \mathbf{e}_3\, dt, \qquad \underline{u}_\varepsilon(x_3) = \mathcal{U}_\varepsilon(x_3) - \mathbb{U}_\varepsilon(x_3), \qquad \Theta_\varepsilon = \mathcal{R}_{\varepsilon,3} \quad \text{for a.e. } x_3 \in (0, L).$$

Then, one has

$$\begin{cases} \dfrac{d\mathcal{R}_{\varepsilon,1}}{dx_3} = -\dfrac{d^2\mathbb{U}_{\varepsilon,2}}{dx_3^2}, \quad \dfrac{d\mathcal{R}_{\varepsilon,2}}{dx_3} = \dfrac{d^2\mathbb{U}_{\varepsilon,1}}{dx_3^2}, \\ \dfrac{d\mathcal{U}_{\varepsilon,1}}{dx_3} - \mathcal{R}_{\varepsilon,2} = \dfrac{d(\mathcal{U}_{\varepsilon,1} - \mathbb{U}_{\varepsilon,1})}{dx_3} = \dfrac{d\underline{u}_{\varepsilon,1}}{dx_3}, \quad \dfrac{d\mathcal{U}_{\varepsilon,2}}{dx_3} + \mathcal{R}_{\varepsilon,1} = \dfrac{d(\mathcal{U}_{\varepsilon,2} - \mathbb{U}_{\varepsilon,2})}{dx_3} = \dfrac{d\underline{u}_{\varepsilon,2}}{dx_3}, \\ \mathbb{U}_{\varepsilon,3} \equiv 0. \end{cases}$$

From now on, we have a new decomposition of the fields $\mathbf{U}^e_\varepsilon(x)$

$$\begin{aligned} \mathbf{U}^e_\varepsilon(x) &= \underline{u}_\varepsilon(x_3) + \mathbb{U}_\varepsilon(x_3) + \begin{pmatrix} -\frac{d\mathbb{U}_{\varepsilon,2}}{dx_3}(x_3) \\ \frac{d\mathbb{U}_{\varepsilon,1}}{dx_3}(x_3) \\ \Theta_\varepsilon(x_3) \end{pmatrix} \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= \begin{pmatrix} \mathbb{U}_{\varepsilon,1}(x_3) + \underline{u}_{\varepsilon,1}(x_3) - x_2\Theta_\varepsilon(x_3) \\ \mathbb{U}_{\varepsilon,2}(x_3) + \underline{u}_{\varepsilon,2}(x_3) + x_1\Theta_\varepsilon(x_3) \\ \underline{u}_{\varepsilon,3}(x_3) - x_1\frac{d\mathbb{U}_{\varepsilon,1}}{dx_3}(x_3) - x_2\frac{d\mathbb{U}_{\varepsilon,2}}{dx_3}(x_3) \end{pmatrix} \quad \text{for a.e.} \quad x \in \Omega^*_\varepsilon \end{aligned} \tag{4.9}$$

and the strain tensor of the displacement $\mathbf{U}^e_\varepsilon$ is

$$e(\mathbf{U}^e_\varepsilon) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\Big(\frac{d\underline{u}_{\varepsilon,1}}{dx_3} - x_2\frac{d\Theta_\varepsilon}{dx_3}\Big) \\ * & 0 & \frac{1}{2}\Big(\frac{d\underline{u}_{\varepsilon,2}}{dx_3} + x_1\frac{d\Theta_\varepsilon}{dx_3}\Big) \\ * & * & \frac{d\underline{u}_{\varepsilon,3}}{dx_3} - x_1\frac{d^2\mathbb{U}_{\varepsilon,1}}{dx_3^2} - x_2\frac{d^2\mathbb{U}_{\varepsilon,2}}{dx_3^2} \end{pmatrix} \tag{4.10}$$

We note that boundary conditions on the decomposition are

$$\underline{u}_\varepsilon(0) = \mathbb{U}_\varepsilon(0) = \frac{d\mathbb{U}_\varepsilon}{dx_3}(0) = \Theta_\varepsilon(0) = 0, \quad \overline{u}_\varepsilon = 0 \text{ for a.e. } x \in \Gamma_\varepsilon \cap \overline{\Omega^*_\varepsilon}$$

and also note that, since $\mathcal{R}_{\varepsilon,\alpha} \in H^1_\Gamma(0, L)$, one has $\mathbb{U}_{\varepsilon,\alpha} \in H^2_\Gamma(0, L)$.
Lemmas 5.11, 5.12, 4.2 lead to the following estimates for the terms appearing in the decomposition of u_ε:

Lemma 4.3. *For every displacement $u_\varepsilon \in \mathbf{V}_\varepsilon$ one has*

$$\begin{aligned} &\|\underline{u}_\varepsilon\|_{H^1(0,L)} \le \frac{C}{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\Omega^*_\varepsilon)}, \\ &\|\mathbb{U}_{\varepsilon,\alpha}\|_{H^2(0,L)} + \|\Theta_\varepsilon\|_{H^1(0,L)} \le \frac{C}{\varepsilon^2}\|e(u_\varepsilon)\|_{L^2(\Omega^*_\varepsilon)} \quad \alpha = 1,\, 2 \end{aligned} \tag{4.11}$$

and

$$\|u_{\varepsilon,3}\|_{L^2(\Omega^*_\varepsilon)} + \|\nabla\overline{u}_\varepsilon\|_{L^2(\Omega^*_\varepsilon)} + \frac{1}{\varepsilon}\|\overline{u}_\varepsilon\|_{L^2(\Omega^*_\varepsilon)} + \varepsilon\big(\|u_{\varepsilon,1}\|_{L^2(\Omega^*_\varepsilon)} + \|u_{\varepsilon,2}\|_{L^2(\Omega^*_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^*_\varepsilon)}\big) \le C\|e(u_\varepsilon)\|_{L^2(\Omega^*_\varepsilon)}. \tag{4.12}$$

Lemma 4.4. *For a subsequence, still denoted by* $\{\varepsilon\}$,
(i) there exists $\mathbb{U} \in H^2_\Gamma(0,L)^2$ *such that the following convergences hold:*

$$\mathbb{U}_\varepsilon \rightharpoonup \mathbb{U} \quad \text{weakly in} \quad H^2(0,L)^2, \tag{4.13}$$

$$\begin{aligned} &\mathcal{T}_\varepsilon^*(\mathbb{U}_\varepsilon) \to \mathbb{U} \quad \text{strongly in} \quad L^2((0,L);H^2(0,1))^2,\\ &\mathcal{T}_\varepsilon^*\Big(\frac{d\mathbb{U}_\varepsilon}{dx_3}\Big) \to \frac{d\mathbb{U}}{dx_3} \quad \text{strongly in} \quad L^2((0,L),H^1(0,1))^2,\\ &\mathcal{T}_\varepsilon^*\Big(\frac{d^2\mathbb{U}_\varepsilon}{dx_3^2}\Big) \rightharpoonup \frac{d^2\mathbb{U}}{dx_3^2} \quad \text{weakly in} \quad L^2((0,L)\times(0,1))^2; \end{aligned} \tag{4.14}$$

(ii) there exists $\Theta \in H^1_\Gamma(0,L)$ *such that the following convergences hold:*

$$\Theta_\varepsilon \rightharpoonup \Theta \quad \text{weakly in} \quad H^1(0,L), \tag{4.15}$$

$$\begin{aligned} &\mathcal{T}_\varepsilon^*(\Theta_\varepsilon) \to \Theta \quad \text{strongly in} \quad L^2((0,L),H^1(0,1)),\\ &\mathcal{T}_\varepsilon^*\Big(\frac{d\Theta_\varepsilon}{dx_3}\Big) \rightharpoonup \frac{d\Theta}{dx_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)); \end{aligned} \tag{4.16}$$

(iii) there exist $\underline{u} \in H^1_\Gamma(0,L)^3$, $\widehat{\underline{u}}_\alpha \in L^2((0,L),H^1_{1,per}(0,1))$ $(\alpha=1,2)$ *such that*

$$\frac{1}{\varepsilon}\underline{u}_\varepsilon \rightharpoonup \underline{u} \quad \text{weakly in} \quad H^1(0,L)^3, \tag{4.17}$$

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\underline{u}_\varepsilon) \to \underline{u} \quad \text{strongly in} \quad L^2((0,L),H^1(0,1))^3,\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{d\underline{u}_{\varepsilon,\alpha}}{dx_3}\Big) \rightharpoonup \frac{d\underline{u}_\alpha}{dx_3} + \frac{\partial\widehat{\underline{u}}_\alpha}{\partial X_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)), \quad \alpha=1,2,\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{d\underline{u}_{\varepsilon,3}}{dx_3}\Big) \rightharpoonup \frac{d\underline{u}_3}{dx_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)), \quad i=1,...,3; \end{aligned} \tag{4.18}$$

(iv) there exists $\overline{u} \in L^2((0,L);H^1_{1,per}(\mathbf{C}))^3$ *such that*

$$\begin{aligned} &\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^*(\overline{u}_\varepsilon) \rightharpoonup \overline{u} \quad \text{weakly in } L^2((0,L);H^1(\mathbf{C}))^3\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\nabla\overline{u}_\varepsilon) \rightharpoonup \nabla_X\overline{u} \quad \text{weakly in } L^2((0,L)\times\mathbf{C})^9\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(e(\overline{u}_\varepsilon)) \rightharpoonup e_X(\overline{u}) \quad \text{weakly in } L^2((0,L)\times\mathbf{C})^9. \end{aligned} \tag{4.19}$$

Proof. (i)-(iii) From Lemma 4.3 (formula $(4.11)_{1,3}$) and [8, Theorem 3.6], [9, Corollary 1.37] it follows that there exist functions $\mathbb{U} \in H^2_\Gamma(0,L)$, $\Theta \in H^1_\Gamma(0,L)$, $\underline{u} \in H^1_\Gamma(0,L)^3$ such that convergences (4.13), $(4.14)_{1,2}$, (4.15), $(4.16)_1$, (4.17) and $(4.18)_1$ hold.
The functions $\mathcal{R}_\varepsilon$, $\underline{u}_{\varepsilon,3}$ are piecewise linear with respect to the variable x_3, hence

$$\mathcal{T}_\varepsilon^*\Big(\frac{d\mathcal{R}_\varepsilon}{dx_3}\Big) \rightharpoonup \frac{d\mathcal{R}}{dx_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)).$$

As a consequence

$$\begin{aligned} &\mathcal{T}_\varepsilon^*\Big(\frac{d^2\mathbb{U}_\varepsilon}{dx_3^2}\Big) = \mathcal{T}_\varepsilon^*\Big(\frac{d\mathcal{R}_\varepsilon}{dx_3}\wedge\mathbf{e}_3\Big) \rightharpoonup \frac{d\mathcal{R}}{dx_3}\wedge\mathbf{e}_3 = \frac{d^2\mathbb{U}}{dx_3^2} \quad \text{weakly in} \quad L^2((0,L)\times(0,1))^3,\\ &\mathcal{T}_\varepsilon^*\Big(\frac{d\Theta_\varepsilon}{dx_3}\Big) = \mathcal{T}_\varepsilon^*\Big(\frac{d\mathcal{R}_\varepsilon}{dx_3}\cdot\mathbf{e}_3\Big) \rightharpoonup \frac{d\Theta}{dx_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)),\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*\Big(\frac{d\underline{u}_{\varepsilon,3}}{dx_3}\Big) \rightharpoonup \frac{d\underline{u}_3}{dx_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)). \end{aligned}$$

From estimates (4.7) and (4.11), there exists $\widehat{\underline{u}}_\alpha \in L^2((0,L),H^1_{1,per}(0,1))$ $(\alpha\in\{1,2\})$ such that

$$\mathcal{T}_\varepsilon^*\Big(\frac{d\underline{u}_{\varepsilon,\alpha}}{dx_3}\Big) \rightharpoonup \frac{d\underline{u}_\alpha}{dx_3} + \frac{\partial\widehat{\underline{u}}_\alpha}{\partial X_3} \quad \text{weakly in} \quad L^2((0,L)\times(0,1)).$$

(iv) From (4.12), $(4.4)_2$ and (4.5) it follows that

$$\|\mathcal{T}_\varepsilon^*(\overline{u}_\varepsilon)\|_{L^2((0,L)\times\mathbf{C})} = \frac{1}{\varepsilon}\|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^2$$
$$\|\nabla_X\mathcal{T}_\varepsilon^*(\overline{u}_\varepsilon)\|_{L^2((0,L)\times\mathbf{C})} = \varepsilon\|\mathcal{T}_\varepsilon^*(\nabla\overline{u}_\varepsilon)\|_{L^2((0,L)\times\mathbf{C})} = \|\nabla\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon^2.$$

and, thus, for a subsequence, still denoted by $\{\varepsilon\}$, there exists $\overline{u} \in L^2((0,L); H^1(\mathbf{C}))$ such that convergence $(4.19)_1$ holds. The periodicity of $\overline{u}$, that is $\overline{u} \in L^2((0,L), H^1_{1,per}(\mathbf{C}))$, can be proved in a similar way as in [6, Theorem 2.1].

From $(4.19)_1$ and $(4.4)_2$ we have $(4.19)_2$ and $(4.19)_3$. □

Let us introduce the following vector space:

$$\mathbf{V}_M \doteq \left\{ \underline{u} \in H^1(0,L)^3,\, \mathbb{U} \in H^2(0,L)^2,\, \Theta \in H^1(0,L) \big|\, \underline{u}(0) = \mathbb{U}(0) = \frac{d\mathbb{U}}{dx_3}(0) = \Theta(0) = 0 \right\}.$$

For every $(\underline{u},\, \mathbb{U},\, \Theta) \in \mathbf{V}_M$, we define the symmetric tensor E by

$$E(\underline{u},\, \mathbb{U},\, \Theta) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\left(\frac{d\underline{u}_1}{dx_3} - X_2\frac{d\Theta}{dx_3}\right) \\ * & 0 & \frac{1}{2}\left(\frac{d\underline{u}_2}{dx_3} + X_1\frac{d\Theta}{dx_3}\right) \\ * & * & \frac{d\underline{u}_3}{dx_3} - X_1\frac{d^2\mathbb{U}_1}{dx_3^2} - X_2\frac{d^2\mathbb{U}_2}{dx_3^2} \end{pmatrix}$$

Lemma 4.5. *Let u_ε be the solution to (4.2). Then there are functions $(\underline{u},\, \mathbb{U},\, \Theta) \in \mathbf{V}_M$, $\widehat{u} \in L^2((0,L), H^1_{1,per}C)^3$ that the following convergences hold*

$$\begin{aligned} &\mathcal{T}_\varepsilon^*(u_{\varepsilon,\alpha}) \rightharpoonup \mathbb{U}_\alpha \quad \textit{weakly in} \quad L^2((0,L), H^1(\mathbf{C})), \\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(u_{\varepsilon,\alpha} - \mathbb{U}_\alpha) \rightharpoonup \underline{u}_\alpha - X_2\Theta \quad \textit{weakly in} \quad L^2((0,L), H^1(\mathbf{C})), \\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(u_{\varepsilon,3}) \rightharpoonup \underline{u}_3 - X_1\frac{d\mathbb{U}_1}{dx_3} - X_2\frac{d\mathbb{U}_2}{dx_3} \quad \textit{weakly in} \quad L^2((0,L), H^1(\mathbf{C})), \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(e(u_\varepsilon)) \rightharpoonup E(\underline{u},\, \mathbb{U},\, \Theta) + e_X(\widehat{u}) \quad \textit{weakly in } L^2((0,L)\times\mathbf{C})^9, \\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\sigma_{ij}^\varepsilon(u_\varepsilon)) \rightharpoonup a_{ijkl}\Big(E_{kl}(\underline{u},\, \mathbb{U},\, \Theta) + e_{X,kl}(\widehat{u})\Big) \quad \textit{weakly in } L^2((0,L)\times\mathbf{C}), \end{aligned} \tag{4.21}$$

where the functions $\underline{u}$, $\mathbb{U}$, Θ, $\widehat{u}$ are the solution to the following unfolded problem:

$$\begin{cases} \displaystyle\int_{(0,L)\times\mathbf{C}} a_{ijkl}(E_{kl}(\underline{u},\, \mathbb{U},\, \Theta) + e_{X,kl}(\widehat{u}))(E_{ij}(V,W,Z) + e_{X,ij}(\Phi))\, dx_3 dX \\ \displaystyle = \sum_{\alpha=1}^{2}\left\{ |\mathbf{C}| \int_{(0,L)} f_\alpha W_\alpha\, dx_3 + I_\alpha \int_{(0,L)} \left[g_\alpha \frac{dW_\alpha}{dx_3} - g_3 Z \right] dx_3 \right\} + |\mathbf{C}| \int_{(0,L)} f_3 V_3\, dx_3, \\ \qquad\qquad \forall\, \Phi \in L^2((0,L); H^1_{1,per}(\mathbf{C}))^3,\ (V,\, W,\, Z) \in \mathbf{V}_M. \end{cases} \tag{4.22}$$

Proof. From $(4.14)_{1,2}$, $(4.16)_1$, $(4.18)_1$, $(4.19)_1$, we obtain convergences (4.20).

By virtue of (4.10), $(4.14)_3$, $(4.16)_2$, $(4.18)_{2,3}$ we have

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(e(\mathbf{U}_\varepsilon^e)) \rightharpoonup \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{d\underline{u}_1}{dx_3} - X_2\frac{d\Theta}{dx_3} \\ * & 0 & \frac{d\underline{u}_2}{dx_3} + X_1\frac{d\Theta}{dx_3} \\ * & * & 2\frac{d\underline{u}_3}{dx_3} - 2X_1\frac{d^2\mathbb{U}_1}{dx_3^2} - 2X_2\frac{d^2\mathbb{U}_2}{dx_3^2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & \frac{\partial\widehat{u}_1}{\partial X_3} \\ * & 0 & \frac{\partial\widehat{u}_2}{\partial X_3} \\ * & * & 0 \end{pmatrix}$$

Set

$$\widehat{u}(x,X) \doteq \begin{pmatrix} \overline{u}_1(x,X) + \widehat{\underline{u}}_1(x_3,X_3) \\ \overline{u}_2(x,X) + \widehat{\underline{u}}_2(x_3,X_3) \\ \overline{u}_3(x,X) \end{pmatrix}$$

and thus $(4.21)_1$ follows. Then, taking into account definition (2.3), we have $(4.21)_2$.

To obtain the limit problem (4.22). Let us introduce the following fields

$$(V,\, W,\, Z) \in \mathbf{V}_M$$

and take the test function in (4.3) as

$$v_\varepsilon(x) = \varepsilon \begin{pmatrix} \frac{1}{e} W_1(x_3) + V_1(x_3) - \frac{x_2}{\varepsilon} Z(x_3) \\ \frac{1}{e} W_2(x_3) + V_2(x_3) + \frac{x_1}{\varepsilon} Z(x_3) \\ V_3(x_3) - \frac{x_1}{\varepsilon} \frac{dW_1}{dx_3}(x_3) - \frac{x_2}{\varepsilon} \frac{dW_2}{dx_3}(x_3) \end{pmatrix} + \varepsilon^2 \varphi(x_3) \begin{pmatrix} \psi_{\varepsilon,1}(x) \\ \psi_{\varepsilon,2}(x) \\ \psi_{\varepsilon,3}(x) \end{pmatrix},$$

where $\psi_\varepsilon(x) = \psi\Big(\dfrac{x}{\varepsilon}\Big)$, $\psi \in H^1_{1,per}(\mathbf{C})^3$. Then

$$\begin{aligned} e(v_\varepsilon) = \frac{\varepsilon}{2} \begin{pmatrix} 0 & 0 & \frac{dV_1}{dx_3}(x_3) - \frac{x_2}{\varepsilon} \frac{dZ}{dx_3} \\ * & 0 & \frac{dV_2}{dx_3}(x_3) + \frac{x_1}{\varepsilon} \frac{dZ}{dx_3} \\ * & * & 2\frac{dV_3}{dx_3} - 2\frac{x_1}{\varepsilon} \frac{d^2W_1}{dx_3^2} - 2\frac{x_2}{\varepsilon} \frac{d^2W_2}{dx_3^2} \end{pmatrix} + \frac{\varepsilon^2}{2} \begin{pmatrix} 0 & 0 & \frac{\partial \varphi}{\partial x_3}\psi_1 \\ * & 0 & \frac{\partial \varphi}{\partial x_3}\psi_2 \\ * & * & 2\frac{\partial \varphi}{\partial x_3}\psi_3 \end{pmatrix} \\ + \varepsilon\varphi \begin{pmatrix} \frac{\partial \psi_1}{\partial X_1} & \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_2} + \frac{\partial \psi_2}{\partial X_1}\Big) & \frac{1}{2}\Big(\frac{\partial \psi_1}{\partial X_3} + \frac{\partial \psi_3}{\partial X_1}\Big) \\ * & \frac{\partial \psi_2}{\partial X_2} & \frac{1}{2}\Big(\frac{\partial \psi_3}{\partial X_2} + \frac{\partial \psi_2}{\partial X_3}\Big) \\ * & * & \frac{\partial \psi_3}{\partial X_3} \end{pmatrix} \end{aligned} \tag{4.23}$$

Applying the unfolding operator $\mathcal{T}_\varepsilon^*$ to the stress tensor $e(v_\varepsilon)$ (4.23) and passing to the limit as $\varepsilon \to 0$, we obtain

$$\frac{1}{\varepsilon} \mathcal{T}_\varepsilon^*(e(v_\varepsilon)) \to E(V,W,Z) + e_X(\psi)\varphi \quad \text{strongly in } L^2((0,L)\times \mathbf{C})^9. \tag{4.24}$$

Unfold the left hand side of (4.3)

$$\int_{\Omega_\varepsilon^*} \sigma^\varepsilon(u_\varepsilon) : e(v_\varepsilon)\, dx = \varepsilon \int_{(0,L)\times\mathbf{C}} \mathcal{T}_\varepsilon^*(\sigma^\varepsilon(u_\varepsilon)) : \mathcal{T}_\varepsilon^*(e(v_\varepsilon))\, dx_3\, dX = \varepsilon^3 \int_{(0,L)\times\mathbf{C}} \frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(\sigma^\varepsilon(u_\varepsilon)) : \frac{1}{\varepsilon}\mathcal{T}_\varepsilon^*(e(v_\varepsilon))\, dx_3\, dX.$$

Unfolding the right hand side of (4.3) and applying (4.1), we have

$$\begin{aligned} \int_{\Omega_\varepsilon^*} f_\varepsilon v_\varepsilon\, dx &= \varepsilon \int_{(0,L)\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_\varepsilon)\mathcal{T}_\varepsilon^*(v_\varepsilon)\, dx_3 dX = \varepsilon \sum_{i=1}^3 \int_{(0,L)\times\mathbf{C}} \mathcal{T}_\varepsilon^*(f_i)\mathcal{T}_\varepsilon^*(v_{\varepsilon,i})\, dx_3 dX \\ &= \varepsilon^3 \int_{(0,L)\times\mathbf{C}} \Big[f_1 W_1 + f_2 W_2 + f_3 V_3 - X_2^2 g_3 Z - X_1^2 g_3 Z + X_1^2 g_1 \frac{dW_1}{dx_3} + X_2^2 g_2 \frac{dW_2}{dx_3} \Big]\, dx_3 dX \end{aligned}$$

Hence, taking into account (4.21), (4.24) and the convergences obtained above, we can pass to the limit as $\varepsilon \to 0$

$$\begin{aligned} &\int_{(0,L)\times\mathbf{C}} a_{ijkl}(E_{kl}(\underline{u},\mathbb{U},\Theta) + e_{X,kl}(\widehat{u}))(E_{ij}(V,W,Z) + \varphi e_{X,ij}(\psi))\, dx_3 dX \\ &= \sum_{\alpha=1}^2 \Big\{ |\mathbf{C}| \int_{(0,L)} f_\alpha W_\alpha\, dx_3 + I_\alpha \int_{(0,L)} \Big[g_\alpha \frac{dW_\alpha}{dx_3} - g_3 Z \Big]\, dx_3 \Big\} + |\mathbf{C}| \int_{(0,L)} f_3 V_3\, dx_3. \end{aligned}$$

Finally, since the tensor product $\mathcal{D}(0,L) \otimes H^1_{1,per}(\mathbf{C})$ is dense in $L^2((0,L); H^1_{1,per}(\mathbf{C}))$, we obtain the limit problem (4.22). □

4.4 Homogenization

In this section we give the expressions of the microscopic displacement $\widehat{u}$ in terms of the macroscopic fields $\underline{u}$, $\mathbb{U}$ and Θ.

Taking $(V,\, W,\, Z) = 0$ as a test function in (4.22), we obtain

$$\int_{(0,L)\times\mathbf{C}} a_{ijkl}\big(E_{kl}(\underline{u},\mathbb{U},\Theta) + e_{X,kl}(\widehat{u})\big) e_{X,ij}(\Phi))\, dx_3 dX = 0.$$

This shows that the microscopic displacement $\widehat{u}$ can be written in terms of the tensor E.

Set

$$\mathbf{M}^{13}=\frac{1}{2}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \mathbf{M}^{23}=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \mathbf{M}^{33}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

The tensors $E(\underline{u},\,\mathbb{U},\,\Theta)$ have 6 components

$$E(\underline{u},\,\mathbb{U},\,\Theta)=\sum_{\beta=1}^{3}\frac{d\underline{u}_\beta}{dx_3}\mathbf{M}^{\beta 3}-\sum_{\alpha=1}^{2}X_\alpha\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\mathbf{M}^{33}+\Big(X_1\mathbf{M}^{23}-X_2\mathbf{M}^{13}\Big)\frac{d\Theta}{dx_3}$$

and introduce 6 correctors

$$\chi^{\underline{u}}_i,\,\chi^{\mathbb{U}}_\alpha,\,\chi^{\Theta}\in H^1_{1,per,0}(\mathbf{C})^3\quad \alpha=1,2,\,i=1,2,3,$$

which are the solutions to the following cell problems

$$\begin{aligned}
&\int_{\mathbf{C}}\chi^{\underline{u}}_\beta dX=0,\qquad &&\int_{\mathbf{C}}a_{ijkl}\big(e_{X,kl}\big(\chi^{\underline{u}}_\beta\big)+\mathbf{M}^{\beta 3}_{kl}\big)e_{X,ij}(\Phi)\,dX=0,\quad \beta=1,2,3,\\
&\int_{\mathbf{C}}\chi^{\mathbb{U}}_\alpha dX=0,\qquad &&\int_{\mathbf{C}}a_{ijkl}\left(e_{X,kl}\big(\chi^{\mathbb{U}}_\alpha\big)-X_\alpha\mathbf{M}^{33}_{kl}\right)e_{X,ij}(\Phi)\,dX=0,\quad \alpha=1,2,\\
&\int_{\mathbf{C}}\chi^{\Theta}dX=0,\qquad &&\int_{\mathbf{C}}a_{ijkl}\left(e_{X,kl}\big(\chi^{\Theta}\big)+X_1\mathbf{M}^{23}_{kl}-X_2\mathbf{M}^{13}_{kl}\right)e_{X,ij}(\Phi)\,dX=0
\end{aligned}\tag{4.25}$$

for all $\Phi\in H^1_{1,per}(\mathbf{C})^3$.

As a consequence, the function $\widehat{u}$ is given in terms of $\underline{u}$, $\mathbb{U}$, Θ as follows

$$\widehat{u}(x_3,X)=\sum_{\beta=1}^{3}\frac{d\underline{u}_\beta}{dx_3}\chi^{\underline{u}}_\beta-\sum_{\alpha=1}^{2}X_\alpha\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\chi^{\mathbb{U}}_\alpha+\frac{d\Theta}{dx_3}\chi^{\Theta}\quad\text{for a.e. }(x_3,X)\in(0,L)\times\mathbf{C}.\tag{4.26}$$

Theorem 4.1. *The limit displacements $(\underline{u},\,\mathbb{U},\,\Theta)\in\mathbf{V}_M$ is the solution to the homogenized problem*

$$\begin{cases}
\displaystyle\int_0^L\Big\{a^{hom}_{\alpha\alpha'}\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\frac{d^2W_{\alpha'}}{dx_3^2}+b^{hom}_{\beta\beta'}\frac{d\underline{u}_\beta}{dx_3}\frac{dV_{\beta'}}{dx_3}+c^{hom}\frac{d\Theta}{dx_3}\frac{dZ}{dx_3}-ab^{hom}_{\alpha\beta}\left(\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\frac{dV_\beta}{dx_3}+\frac{d\underline{u}_\beta}{dx_3}\frac{d^2W_\alpha}{dx_3^2}\right)+\\
\displaystyle-ac^{hom}_\alpha\left(\frac{d\Theta}{dx_3}\frac{d^2W_\alpha}{dx_3^2}+\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\frac{dZ}{dx_3}\right)+bc^{hom}_\beta\left(\frac{d\underline{u}_\beta}{dx_3}\frac{dZ}{dx_3}+\frac{d\Theta}{dx_3}\frac{dV_\beta}{dx_3}\right)\Big\}\,dx_3\\
\displaystyle=\Big\{\int_0^L f_\alpha W_\alpha\,dx_3+\frac{I_\alpha}{|\mathbf{C}|}\int_0^L\Big[g_\alpha\frac{dW_\alpha}{dx_3}-g_3Z\Big]\,dx_3\Big\}+\int_0^L f_3V_3\,dx_3,\quad\forall\,(V,\,W,\,Z)\in\mathbf{V}_M.
\end{cases}\tag{4.27}$$

where

$$\begin{aligned}
a^{hom}_{\alpha\alpha'}&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(\mathbf{M}^{33}_{kl}+e_{X,kl}\big(\chi^{\mathbb{U}}_\alpha\big)\right)\left(\mathbf{M}^{33}_{ij}+e_{X,ij}\big(\chi^{\mathbb{U}}_{\alpha'}\big)\right)X_\alpha X_{\alpha'}\,dX,\\
b^{hom}_{\beta\beta'}&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(\mathbf{M}^{\beta 3}_{kl}+e_{X,kl}\big(\chi^{\underline{u}}_\beta\big)\right)\left(\mathbf{M}^{\beta' 3}_{ij}+e_{X,ij}\big(\chi^{\underline{u}}_{\beta'}\big)\right)\,dX,\\
c^{hom}&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(X_1\mathbf{M}^{23}_{kl}-X_2\mathbf{M}^{13}_{kl}+e_{X,kl}\big(\chi^{\Theta}\big)\right)\left(X_1\mathbf{M}^{23}_{ij}-X_2\mathbf{M}^{13}_{ij}+e_{X,kl}\big(\chi^{\Theta}\big)\right)\,dX,\\
ab^{hom}_{\alpha\beta}&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(\mathbf{M}^{33}_{kl}+e_{X,kl}\big(\chi^{\mathbb{U}}_\alpha\big)\right)\left(\mathbf{M}^{\beta 3}_{ij}+e_{X,ij}\big(\chi^{\underline{u}}_\beta\big)\right)X_\alpha\,dX,\\
ac^{hom}_\alpha&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(\mathbf{M}^{33}_{kl}+e_{X,kl}\big(\chi^{\mathbb{U}}_\alpha\big)\right)\left(X_1\mathbf{M}^{23}_{ij}-X_2\mathbf{M}^{13}_{ij}+e_{X,ij}\big(\chi^{\Theta}\big)\right)X_\alpha\,dX,\\
bc^{hom}_\beta&=\frac{1}{|\mathbf{C}|}\int_{\mathbf{C}}a_{ijkl}\left(\mathbf{M}^{\beta 3}_{kl}+e_{X,kl}\big(\chi^{\underline{u}}_\beta\big)\right)\left(X_1\mathbf{M}^{23}_{ij}-X_2\mathbf{M}^{13}_{ij}+e_{X,ij}\big(\chi^{\Theta}\big)\right)\,dX.
\end{aligned}\tag{4.28}$$

Proof. We take $\Phi=0$ in (4.22). Replacing $\widehat{u}$ by its expression (4.26), for every $(V,\,W,\,Z)\in\mathbf{V}_M$ yields

$$\begin{aligned}
&\int_{(0,L)\times\mathbf{C}}a_{ijkl}\Big(\frac{d\underline{u}_\beta}{dx_3}\Big(\mathbf{M}^{\beta 3}_{kl}+e_{X,kl}\big(\chi^{\underline{u}}_\beta\big)\Big)-X_\alpha\frac{d^2\mathbb{U}_\alpha}{dx_3^2}\Big(\mathbf{M}^{33}_{kl}+e_{X,kl}\big(\chi^{\mathbb{U}}_\alpha\big)\Big)+\frac{d\Theta}{dx_3}\Big(X_1\mathbf{M}^{23}_{kl}-X_2\mathbf{M}^{13}_{kl}+e_{X,kl}\big(\chi^{\Theta}\big)\Big)\Big)\\
&\times\Big(\frac{dV_\beta}{dx_3}\mathbf{M}^{\beta 3}_{ij}-X_\alpha\frac{d^2W_\alpha}{dx_3^2}\mathbf{M}^{33}_{ij}+\Big(X_1\mathbf{M}^{23}_{ij}-X_2\mathbf{M}^{13}_{ij}\Big)\frac{dZ}{dx_3}\Big)\,dx_3dX\\
&=\sum_{\alpha=1}^{2}\Big\{|\mathbf{C}|\int_{(0,L)}f_\alpha W_\alpha\,dx_3+I_\alpha\int_{(0,L)}\Big[g_\alpha\frac{dW_\alpha}{dx_3}-g_3Z\Big]\,dx_3\Big\}+|\mathbf{C}|\int_{(0,L)}f_3V_3\,dx_3,\quad .
\end{aligned}$$

Taking into account the variational problems (4.25) satisfied by the correctors, the problem (4.27) with the homogenized coefficients given by (4.28) is obtained by a simple computation. □

Lemma 4.6. *The left-hand side operator in Problem* (4.27) *is uniformly elliptic.*

Proof. Using formulas (4.28) of the homogenized coefficients, one obtains

$$\begin{aligned}&a^{hom}_{\alpha\alpha'}\tau^{\mathbb{U}}_{\alpha}\tau^{\mathbb{U}}_{\alpha'}+b^{hom}_{\beta\beta'}\tau^{u}_{\beta}\tau^{u}_{\beta'}+c^{hom}\tau^{\Theta}\tau^{\Theta}+2ab^{hom}_{\alpha\beta}\tau^{\mathbb{U}}_{\alpha}\tau^{u}_{\beta}+2ac^{hom}_{\alpha}\tau^{\mathbb{U}}_{\alpha}\tau^{\Theta}+2bc^{hom}_{\beta}\tau^{u}_{\beta}\tau^{\Theta}\\&=\int_{\mathbf{C}}a_{ijkl}\Big(e_{X,kl}\big(\Psi\big)+M_{kl}\Big)\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\,dX,\end{aligned}$$

where

$$M=\Big(\tau^{u}_{1}-X_2\tau^{\Theta}\Big)\mathbf{M}^{13}+\Big(\tau^{u}_{2}+X_1\tau^{\Theta}\Big)\mathbf{M}^{23}+\Big(\tau^{\mathbb{U}}_{1}+\tau^{\mathbb{U}}_{2}+\tau^{u}_{3}\Big)\mathbf{M}^{33}$$

$$\Psi=\chi^{\mathbb{U}}_{\alpha}\tau^{\mathbb{U}}_{\alpha}+\chi^{u}_{\beta}\tau^{u}_{\beta}+\chi^{\Theta}\tau^{\Theta},\qquad \tau^{\mathbb{U}}_{\alpha},\ \tau^{u}_{\beta},\ \tau^{\Theta}\in\mathbb{R},\quad \alpha,\,\beta=1,2.$$

Then, in view of (2.2) and following the proof of [9, Lemma 11.19], we obtain

$$\begin{aligned}&a^{hom}_{\alpha\alpha'}\tau^{\mathbb{U}}_{\alpha}\tau^{\mathbb{U}}_{\alpha'}+b^{hom}_{\beta\beta'}\tau^{u}_{\beta}\tau^{u}_{\beta'}+c^{hom}\tau^{\Theta}\tau^{\Theta}+2ab^{hom}_{\alpha\beta}\tau^{\mathbb{U}}_{\alpha}\tau^{u}_{\beta}+2ac^{hom}_{\alpha}\tau^{\mathbb{U}}_{\alpha}\tau^{\Theta}+2bc^{hom}_{\beta}\tau^{u}_{\beta}\tau^{\Theta}\geqslant\\&\geqslant\int_{\mathbf{C}}c_0\Big(e_{X,kl}\big(\Psi\big)+M_{kl}\Big)\Big(e_{X,ij}\big(\Psi\big)+M_{ij}\Big)\,dX\geqslant C\big(\tau^{\mathbb{U}}_{\alpha}+\tau^{u}_{\beta}+\tau^{\Theta}\big)\big(\tau^{\mathbb{U}}_{\alpha'}+\tau^{u}_{\beta'}+\tau^{\Theta}\big)\end{aligned}$$

which ends the proof. □

5 Appendix

5.1 A lemma

For every open set $\mathcal{O}$ in $\mathbb{R}^N$ and $\delta>0$, denote $\mathcal{O}^{int}_{\delta}=\big\{x\in\mathcal{O}\mid \mathrm{dist}(x,\partial\mathcal{O})>\delta\big\}$.

Lemma 5.1. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^N$ and $\{\phi_\varepsilon\}_\varepsilon$ be a sequence of functions belonging to $H^1(\mathcal{O}^{int}_{\kappa\varepsilon})$ (κ is a fixed strictly positive constant) satisfying*

$$\|\phi_\varepsilon\|_{H^1(\mathcal{O}^{int}_{\kappa\varepsilon})}\leq C \tag{5.1}$$

where C does not depend on ε. We extend ϕ_ε and $\nabla\phi_\varepsilon$ by 0 in $\mathbb{R}^N\setminus\overline{\mathcal{O}^{int}_{\kappa\varepsilon}}$ (extensions with the same names). Then, there exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $\phi\in H^1(\mathcal{O})$ such that

$$\begin{aligned}&\phi_\varepsilon\longrightarrow\phi\quad \textit{weakly in } L^2(\mathcal{O}),\\&\nabla\phi_\varepsilon\mathbf{1}_{\mathcal{O}^{int}_{\kappa\varepsilon}}\rightharpoonup\nabla\phi\quad \textit{weakly in } L^2(\mathcal{O})^N.\end{aligned}$$

Proof. It follows from (5.1) that there exist $\phi\in L^2(\mathcal{O})$ and $\Phi\in L^2(\mathcal{O})^N$ such that (up to a subsequence still denoted $\{\varepsilon\}$)

$$\begin{aligned}&\phi_\varepsilon\rightharpoonup\phi\quad \text{weakly in } L^2(\mathcal{O}),\\&\nabla\phi_\varepsilon\mathbf{1}_{\mathcal{O}^{int}_{\kappa\varepsilon}}\rightharpoonup\Phi\quad \text{weakly in } L^2(\mathcal{O})^N.\end{aligned}$$

Now, we show that $\nabla\phi=\Phi$, so ϕ belongs to $H^1(\mathcal{O})$. Let $\mathcal{O}'$ be an open subset of $\mathcal{O}$ such that $\mathcal{O}'\Subset\mathcal{O}$. If ε is small enough, one has $\mathcal{O}'\subset\mathcal{O}^{int}_{\kappa\varepsilon}$. For all $\psi\in\mathcal{D}(\mathcal{O}')^N$, using the above convergences we obtain

$$\int_{\mathcal{O}'}\nabla\phi_\varepsilon\cdot\psi\,dx=-\int_{\mathcal{O}'}\phi_\varepsilon\,\mathrm{div}\,(\psi)\,dx\longrightarrow-\int_{\mathcal{O}'}\phi\,\mathrm{div}\,(\psi)\,dx=\int_{\mathcal{O}'}\Phi\cdot\psi\,dx.$$

Hence $\Phi=\nabla\phi$ for every open set $\mathcal{O}'$ strictly included in $\mathcal{O}$. Thus $\Phi=\nabla\phi$ a.e. in $\mathcal{O}$. So, we have $\phi\in H^1(\mathcal{O})$. □

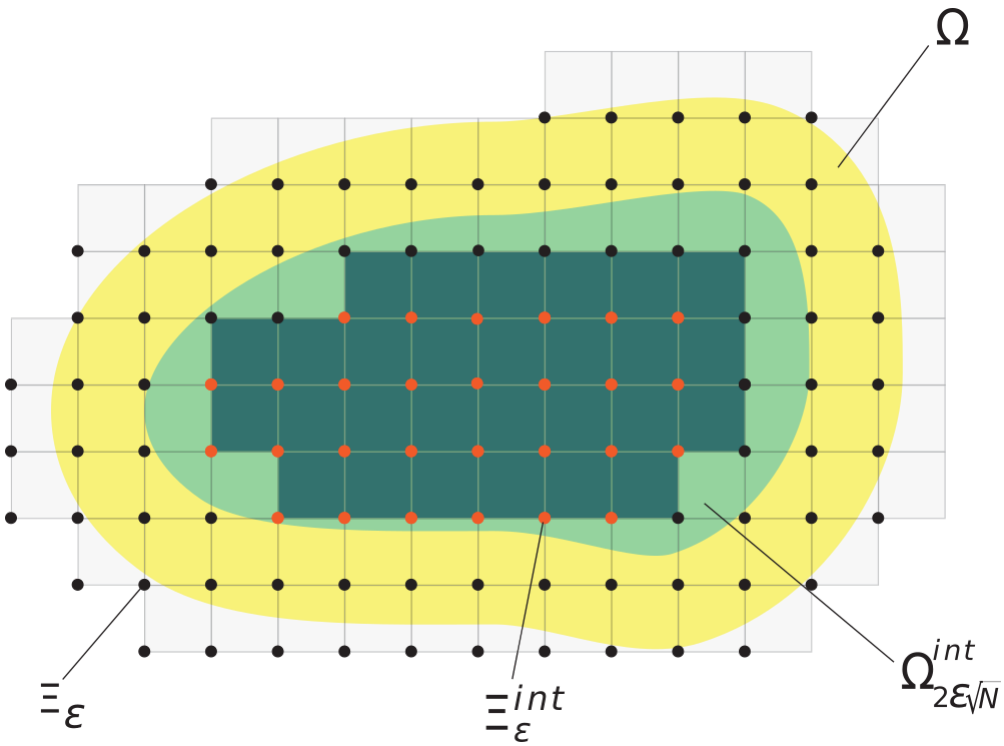


Figure 2: Sets Ω, $\Omega^{int}_{2\varepsilon\sqrt{N}}$, Ξ_ε and Ξ^{int}_ε.

5.2 Korn's type inequality

See Section 2.1 for the main notations. We also denote (see figure 5.2)

$$\Xi^{int}_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega^{int}_{2\varepsilon\sqrt{N}}\big\}.$$

First, we recall the following lemmas proved in [9, Lemmas 5.22 and 5.35]:

Lemma 5.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ the sets Ω^{int}_δ are uniformly Lipschitz.*

Lemma 5.3. *Suppose $p \in [1, +\infty)$. Let ℓ be a function defined on Ξ_ε. There exists a constant C which only depends on p and $\partial\Omega$ such that*

$$\sum_{\xi\in\Xi_\varepsilon} |\ell(\xi)|^p \le C\Big(\sum_{\xi\in\Xi^{int}_\varepsilon} |\ell(\xi)|^p + \sum_{i=1}^N \sum_{\xi\in\Xi_{\varepsilon,i}} |\ell(\xi+\mathbf{e}_i) - \ell(\xi)|^p\Big).$$

Proposition 5.1 (Poincaré-Wirtinger inequality)**.** *Assume Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary. Then, the domains Ω^{int}_δ (resp. Ω^*_ε) with $\delta \in (0, \delta_0]$ satisfy a uniform Poincaré-Wirtinger inequality for every $p \in [1, +\infty)$, i.e., there exists a constant C independent of δ (resp. ε) (it depends only on p and $\partial\Omega$) such that*

$$\begin{aligned} &\|\varphi - \mathcal{M}_{\Omega^{int}_\delta}(\varphi)\|_{L^p(\Omega^{int}_\delta)} \le C\|\nabla\varphi\|_{L^p(\Omega^{int}_\delta)}, \quad \forall\varphi \in W^{1,p}(\Omega^{int}_\delta),\\ &\textit{(resp.}\quad \|\varphi - \mathcal{M}_{\Omega^*_\varepsilon}(\varphi)\|_{L^p(\Omega^*_\varepsilon)} \le C\|\nabla\varphi\|_{L^p(\Omega^*_\varepsilon)}, \quad \forall\varphi \in W^{1,p}(\Omega^*_\varepsilon)\textit{)}, \end{aligned} \tag{5.2}$$

where

$$\mathcal{M}_{\Omega^{int}_\delta}(\varphi) \doteq \frac{1}{|\Omega^{int}_\delta|}\int_{\Omega^{int}_\delta} \varphi(x)\,dx, \qquad \textit{(resp.}\quad \mathcal{M}_{\Omega^*_\varepsilon}(\varphi) \doteq \frac{1}{|\Omega^*_\varepsilon|}\int_{\Omega^*_\varepsilon} \varphi(x)\,dx\textit{)}.$$

Let Φ be a displacement in $W^{1,p}(\mathbf{C}_j)^N$, $p \in (1, +\infty)$ and $j \in \{1, \dots, N\}$. Applying the Korn inequality in $\mathbf{C}$ and $\mathbf{C} + \mathbf{e}_j$ gives two rigid displacements $\mathbf{R}_{j,0}$, $\mathbf{R}_{j,1}$

$$\begin{aligned} &\mathbf{R}_{j,0}(x) = \mathbf{a}_{j,0} + \mathbf{B}_{j,0}\,x,\\ &\mathbf{R}_{j,1}(x) = \mathbf{a}_{j,1} + \mathbf{B}_{j,1}\,(x - \mathbf{e}_j), \end{aligned} \qquad \mathbf{a}_{j,0},\ \mathbf{a}_{j,1} \in \mathbb{R}^N, \quad x \in \mathbb{R}^N,$$

where $\mathbf{B}_{j,0}$, $\mathbf{B}_{j,1}$ are antisymmetric $N \times N$ matrices. One has

$$\begin{aligned} &\|\Phi - \mathbf{R}_{j,0}\|_{W^{1,p}(\mathbf{C})} \le C\|e(\Phi)\|_{L^p(\mathbf{C})}, \\ &\|\Phi - \mathbf{R}_{j,1}\|_{W^{1,p}(\mathbf{C}+\mathbf{e}_j)} \le C\|e(\Phi)\|_{L^p(\mathbf{C}+\mathbf{e}_j)}, \end{aligned} \tag{5.3}$$

where the constant depends only on $\mathbf{C}$.

Lemma 5.4. *The following estimates hold:*

$$\begin{aligned} &|\mathbf{B}_{j,1} - \mathbf{B}_{j,0}| \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}, \\ &|\mathbf{a}_{j,1} - \mathbf{a}_{j,0} - \mathbf{B}_{j,1}\,\mathbf{e}_j| \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}, \end{aligned} \qquad j = 1, \dots, N, \tag{5.4}$$

where the constant C depends only on $\mathbf{C}$.

Proof. Since the domain $\mathbf{C}_j$ is connected with a Lipschitz boundary, it satisfies the Korn inequality. Hence, there exists a rigid displacement $\mathbf{R}_j$

$$\mathbf{R}_j(x) = \mathbf{a}_j + \mathbf{B}_j\,(x - \mathbf{e}_j/2) \qquad \mathbf{a}_j \in \mathbb{R}^N, \qquad x \in \mathbb{R}^N$$

where $\mathbf{B}_j$ is an antisymmetric $N \times N$ matrix. It satisfies

$$\|\Phi - \mathbf{R}_j\|_{W^{1,p}(\mathbf{C}_j)} \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}. \tag{5.5}$$

where the constant C depends on $\mathbf{C}_j$. Hence, by (5.3) and (5.5)

$$\|\nabla(\mathbf{R}_j - \mathbf{R}_{j,0})\|_{L^p(\mathbf{C})} + \|\nabla(\mathbf{R}_j - \mathbf{R}_{j,1})\|_{L^p(\mathbf{C}+\mathbf{e}_j)} \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}. \tag{5.6}$$

Taking into account the inequality (5.6), we obtain

$$\begin{aligned} &|\mathbf{B}_j - \mathbf{B}_{j,0}| \le C\|\nabla(\mathbf{R}_j - \mathbf{R}_{j,0})\|_{L^p(\mathbf{C})} \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}, \\ &|\mathbf{B}_j - \mathbf{B}_{j,1}| \le C\|\nabla(\mathbf{R}_j - \mathbf{R}_{j,1})\|_{L^p(\mathbf{C}+\mathbf{e}_j)} \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}. \end{aligned} \tag{5.7}$$

Subtracting yields $(5.4)_1$.

Now we prove $(5.4)_2$. First observe that

$$\Big\|\mathbf{a}_j - \mathbf{a}_{j,0} - \frac{1}{2}\mathbf{B}_j\,\mathbf{e}_j\Big\|_{L^p(\mathbf{C})} \le \Big\|\mathbf{a}_j + \mathbf{B}_j\,(\cdot - \frac{1}{2}\mathbf{e}_j) - \big(\mathbf{a}_{j,0} + \mathbf{B}_{j,0}\big)\Big\|_{L^p(\mathbf{C})} + \big\|\mathbf{B}_j - \mathbf{B}_{j,0}\big\|_{L^p(\mathbf{C})}. \tag{5.8}$$

Besides, one has

$$\begin{aligned} \Big\|\mathbf{a}_j + \mathbf{B}_j\,(\cdot - \frac{1}{2}\mathbf{e}_j) - \big(\mathbf{a}_{j,0} + \mathbf{B}_{j,0}\big)\Big\|_{L^p(\mathbf{C})} &= \big\|\mathbf{R}_j - \mathbf{R}_{j,0}\big\|_{L^p(\mathbf{C})} \le \big\|\Phi - \mathbf{R}_j\big\|_{L^p(\mathbf{C})} + \big\|\Phi - \mathbf{R}_{j,0}\big\|_{L^p(\mathbf{C})} \\ &\le \big\|\Phi - \mathbf{R}_j\big\|_{L^p(\mathbf{C}_j)} + \big\|\Phi - \mathbf{R}_{j,0}\big\|_{L^p(\mathbf{C})} \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}. \end{aligned}$$

The previous estimate together with (5.8) and (5.7) give

$$\Big|\mathbf{a}_j - \mathbf{a}_{j,0} - \frac{1}{2}\mathbf{B}_j\,\mathbf{e}_j\Big| \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}.$$

Similarly we obtain

$$\Big|\mathbf{a}_j - \mathbf{a}_{j,1} + \frac{1}{2}\mathbf{B}_j\,\mathbf{e}_j\Big| \le C\|e(\Phi)\|_{L^p(\mathbf{C}_j)}.$$

Hence $(5.4)_2$ holds. □

Now, let u be a displacement in $W^{1,p}(\Omega_\varepsilon^*)^N$. By the Korn inequality in $\varepsilon(\xi + \mathbf{C})$ there exist rigid displacements $\mathbf{R}_{\varepsilon\xi}$ $(\xi \in \Xi_\varepsilon)$

$$\mathbf{R}_{\varepsilon\xi}(x) = \mathbf{a}(\varepsilon\xi) + \mathbf{B}(\varepsilon\xi)\,(x - \varepsilon\xi), \qquad x \in \mathbb{R}^N$$

such that (using (5.3) and after ε-scaling)

$$\begin{aligned} &\|\nabla(u - \mathbf{R}_{\varepsilon\xi})\|_{L^p(\varepsilon(\xi+\mathbf{C}))} \le C\|e(u)\|_{L^p(\varepsilon(\xi+\mathbf{C}))}, \\ &\|u - \mathbf{R}_{\varepsilon\xi}\|_{L^p(\varepsilon(\xi+\mathbf{C}))} \le C\varepsilon\|e(u)\|_{L^p(\varepsilon(\xi+\mathbf{C}))}. \end{aligned} \tag{5.9}$$

As above we obtain the following estimates for every $\xi \in \Xi_{\varepsilon,i}$:

$$|\mathbf{B}(\varepsilon\xi + \varepsilon\mathbf{e}_i) - \mathbf{B}(\varepsilon\xi)| \leq C\varepsilon^{-N/p}\|e(u)\|_{L^p(\varepsilon\mathbf{C}_i^\xi)},$$
$$|\mathbf{a}(\varepsilon\xi + \varepsilon\mathbf{e}_i) - \mathbf{a}(\varepsilon\xi) - \varepsilon\mathbf{B}(\varepsilon\xi + \varepsilon\mathbf{e}_i)\mathbf{e}_i| \leq C\varepsilon^{1-N/p}\|e(u)\|_{L^p(\varepsilon\mathbf{C}_i^\xi)},$$

where $\mathbf{C}_i^\xi = \text{interior}\big((\overline{\mathbf{C}} + \xi) \cup (\mathbf{e}_i + \xi + \overline{\mathbf{C}})\big)$.

As immediate consequence of Lemma 5.4, we have

Lemma 5.5. *The following estimates hold:*

$$\sum_{i=1}^N \sum_{\xi\in\Xi_{\varepsilon,i}} |\mathbf{B}(\varepsilon\xi + \varepsilon\mathbf{e}_i) - \mathbf{B}(\varepsilon\xi)|^p\varepsilon^N \leq C\|e(u)\|^p_{L^p(\Omega_\varepsilon^*)},$$
$$\sum_{i=1}^N \sum_{\xi\in\Xi_{\varepsilon,i}} |\mathbf{a}(\varepsilon\xi + \varepsilon\mathbf{e}_i) - \mathbf{a}(\varepsilon\xi) - \varepsilon\mathbf{B}(\varepsilon\xi + \varepsilon\mathbf{e}_i)\,\mathbf{e}_i|^p\varepsilon^N \leq C\varepsilon^p\|e(u)\|^p_{L^p(\Omega_\varepsilon^*)},$$

where the constant C depends only on $\mathbf{C}$.

Let ξ be in Ξ_ε. If all the vertices of the parallelotope $\varepsilon(\xi + \overline{Y})$ belong to Ξ_ε, we extend the field $\mathbf{a}$ (resp. $\mathbf{B}$) in this parallelotope as the Q_1 interpolate of its values on the vertices of the parallelotope.

We obtain a field, still denoted $\mathbf{a}$ (resp. $\mathbf{B}$), defined at least in $\Omega^{int}_{2\varepsilon\sqrt{N}}$. It belongs to $W^{1,\infty}(\Omega^{int}_{2\varepsilon\sqrt{N}})^N$ (resp. $W^{1,\infty}(\Omega^{int}_{2\varepsilon\sqrt{N}})^{N\times N}$).

Lemma 5.6. *For every displacement $u \in W^{1,p}(\Omega_\varepsilon^*)^N$ one has*

$$\begin{aligned}
&\|\nabla\mathbf{B}\|_{L^p(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^p(\Omega_\varepsilon^*)},\\
&\big\|\nabla\mathbf{a} - \mathbf{B}\big\|_{L^p(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq C\|e(u)\|_{L^p(\Omega_\varepsilon^*)},\\
&\big\|e(\mathbf{a})\big\|_{L^p(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq C\|e(u)\|_{L^p(\Omega_\varepsilon^*)},
\end{aligned} \tag{5.10}$$

where the constants do not depend on ε.

Proof. A straightforward calculation and the estimates in Lemma 5.5 yield $(5.10)_{1,2}$. Then $(5.10)_2$ gives $(5.10)_3$ (recall that $\mathbf{B}$ is an antisymmetric $N \times N$ matrix). □

As a consequence of Lemmas 5.2 and 5.6, one has

Lemma 5.7. *There exists an antisymmetric $N \times N$ matrix $\mathcal{B}$ such that*

$$\sum_{\xi\in\Xi_\varepsilon} |\mathbf{B}(\varepsilon\xi) - \mathcal{B}|^p\varepsilon^N \leq C\|e(u)\|^p_{L^p(\Omega_\varepsilon^*)}, \tag{5.11}$$

where the constant does not depend on ε.

Proof. Since the boundary of $\Omega^{int}_{2\varepsilon\sqrt{N}}$ is uniformly Lipschitz, the Korn inequality and $(5.10)_3$ give a rigid displacement $\mathbf{R}$ such that

$$\big\|\mathbf{a} - \mathbf{R}\big\|_{W^{1,p}(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq C\big\|e(\mathbf{a})\big\|_{L^p(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq C\|e(u)\|_{L^p(\Omega_\varepsilon^*)}.$$

Then $(5.10)_2$ and the previous estimate lead to

$$\big\|\mathbf{B} - \nabla\mathbf{R}\big\|_{L^p(\Omega^{int}_{2\varepsilon\sqrt{N}})} \leq C\|e(u)\|_{L^p(\Omega_\varepsilon^*)}.$$

Set $\mathcal{B} = \nabla\mathbf{R}$. Hence

$$\sum_{\xi\in\Xi_\varepsilon^{int}} |\mathbf{B}(\varepsilon\xi) - \mathcal{B}|^p\varepsilon^N \leq C\|e(u)\|^p_{L^p(\Omega_\varepsilon^*)}.$$

The first estimate in Lemma 5.5 together with the above and Lemma 5.3 yield (5.11). □

We assume that there exists a domain Ω' with a Lipschitz boundary such that $\Omega \subset \Omega'$ and $\Omega' \cap \partial\Omega = \Gamma$.
Set

$$W^{1,p}_{\Gamma}(\Omega^*_\varepsilon) \doteq \Big\{\psi \in W^{1,p}(\Omega^*_\varepsilon) \;\big|\; \exists \psi' \in W^{1,p}(\Omega'^*_\varepsilon),\ \psi = \psi'_{|\Omega^*_\varepsilon},\ \psi' = 0 \text{ in } \Omega'^*_\varepsilon \setminus \overline{\Omega^*_\varepsilon}\Big\},$$

where

$$\Omega'^*_\varepsilon \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi'_\varepsilon} (\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big), \quad \Xi'_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^3 \;\big|\; (\varepsilon\xi + \varepsilon Y) \cap \Omega' \neq \emptyset\big\}$$

Proposition 5.2 (Korn inequality)**.** *For every displacement $u \in W^{1,p}(\Omega^*_\varepsilon)^N$, $p \in (1, +\infty)$, there exists a rigid displacement $\mathbf{R}$ such that*

$$\|u - \mathbf{R}\|_{W^{1,p}(\Omega^*_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega^*_\varepsilon)}. \tag{5.12}$$

Furthermore, if $u \in W^{1,p}_{\Gamma}(\Omega^*_\varepsilon)^N$ then

$$\|u\|_{W^{1,p}(\Omega^*_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega^*_\varepsilon)}, \tag{5.13}$$

where the constants do not depend on ε.

Proof. Estimates (5.9) and (5.11) lead to

$$\|\nabla u - \mathcal{B}\|_{L^p(\Omega^*_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega^*_\varepsilon)}.$$

Then, using this estimate and the Poincaré-Wirtinger inequality (5.2) we obtain (5.12).

If u belongs to $W^{1,p}_{\Gamma}(\Omega^*_\varepsilon)^N$, applying the previous result (5.12) with u' (resp. Ω') in place of u (resp. Ω) gives a rigid displacement $\mathbf{R}'$ such that

$$\|u' - \mathbf{R}'\|_{W^{1,p}(\Omega'^*_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega^*_\varepsilon)}.$$

Let $\mathcal{O}$ be an open set such that $\mathcal{O} \Subset \big(\Omega' \setminus \overline{\Omega}\big)$. For ε small enough, the function u' vanishes in $\mathcal{O} \cap \Omega'^*_\varepsilon$. Hence

$$\|\mathbf{R}'\|_{W^{1,p}(\mathcal{O}\cap\Omega'^*_\varepsilon)} \leq C\|e(u)\|_{L^p(\Omega^*_\varepsilon)}$$

which allows to obtain an estimate, independent of ε, for the components of $\mathbf{R}'$. And thus the estimate (5.13) follows. □

5.3 Korn's type inequality in a plate-like domain

In this subsection the proofs of the lemmas follow the same lines as the proofs of those in the previous subsection.

The notations are those of Subsection 3.1. We recall that $\mathbf{C}$ is a domain with Lipschitz boundary included in $Y = (-1/2, 1/2)^3$ and satisfying $\mathbf{C}_j = \text{interior}\big(\overline{\mathbf{C}} \cup (\mathbf{e}_j + \overline{\mathbf{C}})\big)$, $j = 1, 2$, connected.

Here the notations are

- $\Xi_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^2 \;\big|\; (\varepsilon\xi + \varepsilon Y') \cap \omega \neq \emptyset\big\}$, $\Xi_{\varepsilon,\alpha} \doteq \big\{\xi \in \Xi_\varepsilon \;\big|\; \xi + \mathbf{e}_\alpha \in \Xi_\varepsilon\big\}$, $\alpha = \overline{1,2}$,
- $\Xi^{int}_\varepsilon \doteq \big\{\xi \in \mathbb{Z}^2 \;\big|\; (\varepsilon\xi + \varepsilon Y') \subset \omega^{int}_{3\varepsilon}\big\}$,
- $\Omega^*_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi_\varepsilon} (\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big)$, $\Omega^{int}_\varepsilon = \text{interior}\Big(\bigcup_{\xi\in\Xi^{int}_\varepsilon} (\varepsilon\xi + \varepsilon\overline{\mathbf{C}})\Big)$.

Let u be in $H^1(\Omega^*_\varepsilon)^3$. For every $\xi \in \Xi_\varepsilon$ there exists a rigid displacement $\mathbf{R}_{\varepsilon\xi}$

$$\mathbf{R}_{\varepsilon\xi}(x) = \boldsymbol{\mathcal{U}}(\varepsilon\xi) + \boldsymbol{\mathcal{R}}(\varepsilon\xi) \wedge (x - \varepsilon\xi) \qquad x \in \mathbb{R}^3,$$

such that

$$\|\nabla(u - \mathbf{R}_{\varepsilon\xi})\|_{L^2(\varepsilon(\xi+\mathbf{C}))} \leq C\|e(u)\|_{L^2(\varepsilon(\xi+\mathbf{C}))}, \quad \|u - \mathbf{R}_{\varepsilon\xi}\|_{L^2(\varepsilon(\xi+\mathbf{C}))} \leq C\varepsilon\|e(u)\|_{L^2(\varepsilon(\xi+\mathbf{C}))}. \tag{5.14}$$

Remark 5.1. *By construction, the fields $\boldsymbol{\mathcal{U}}$, $\boldsymbol{\mathcal{R}}$ are piecewise linear in each cell.*

In a similar way as in Lemma 5.5 we obtain

Lemma 5.8. *The following estimates hold:*

$$\begin{aligned}
&\sum_{\alpha=1}^{2}\sum_{\xi\in\Xi_{\varepsilon,\alpha}}|\boldsymbol{\mathcal{R}}(\varepsilon\xi+\varepsilon\mathbf{e}_\alpha)-\boldsymbol{\mathcal{R}}(\varepsilon\xi)|^2\varepsilon^3\le C\|e(u)\|^2_{L^2(\Omega^*_\varepsilon)},\quad \alpha=1,2,\\
&\sum_{\alpha=1}^{2}\sum_{\xi\in\Xi_{\varepsilon,\alpha}}|\boldsymbol{\mathcal{U}}(\varepsilon\xi+\varepsilon\mathbf{e}_\alpha)-\boldsymbol{\mathcal{U}}(\varepsilon\xi)-\varepsilon\boldsymbol{\mathcal{R}}(\varepsilon\xi+\varepsilon\mathbf{e}_\alpha)\wedge\mathbf{e}_\alpha|^2\varepsilon^3\le C\varepsilon^2\|e(u)\|^2_{L^2(\Omega^*_\varepsilon)}.
\end{aligned}\tag{5.15}$$

The constant C depends only on $\mathbf{C}$.

As in the previous subsection, using Q_1 interpolation we extend the fields $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{R}}$ to the whole domain $\omega^{int}_{3\varepsilon}$ and obtain two fields $\mathcal{U}\in W^{1,\infty}(\omega^{int}_{3\varepsilon})^3$ and $\mathcal{R}\in W^{1,\infty}(\omega^{int}_{3\varepsilon})^2$ satisfying

$$\mathcal{U}(\varepsilon\xi)=\boldsymbol{\mathcal{U}}(\varepsilon\xi),\quad \mathcal{R}(\varepsilon\xi)=\boldsymbol{\mathcal{R}}(\varepsilon\xi)\qquad \forall\xi\in\Xi_\varepsilon\cap\omega^{int}_{3\varepsilon}.$$

Define the displacement $\mathbf{U}^e$ by

$$\mathbf{U}^e(x)=\mathcal{U}(x')+\mathcal{R}(x')\wedge x_3\mathbf{e}_3,\qquad \forall x\in\Omega^{int}_\varepsilon,\quad x'=(x_1,x_2).$$

Lemma 5.9. *For every displacement $u\in H^1(\Omega^*_\varepsilon)^3$ we have*

$$\begin{aligned}
&\|\nabla\mathcal{R}\|_{L^2(\omega^{int}_{3\varepsilon})}\le\frac{C}{\varepsilon^{3/2}}\|e(u)\|_{L^2(\Omega^*_\varepsilon)},\\
&\Big\|\frac{\partial\mathcal{U}}{\partial x_\alpha}-\mathcal{R}\wedge\mathbf{e}_\alpha\Big\|_{L^2(\omega^{int}_{3\varepsilon})}\le\frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega^*_\varepsilon)},\quad \alpha=1,2.
\end{aligned}\tag{5.16}$$

The constant C depends only on $\mathbf{C}$.

Proof. Estimates (5.16) are the consequences of (5.15) and the fact that the fields $\mathcal{U}$ and $\mathcal{R}$ are piecewise linear on every cell. □

Theorem 5.1. *For every displacement $u\in H^1(\Omega^*_\varepsilon)^3$ there exists a rigid displacement $\mathbf{R}$ such that*

$$\|u_\alpha-\mathbf{R}_\alpha\|_{L^2(\Omega^*_\varepsilon)}\le C\|e(u)\|_{L^2(\Omega^*_\varepsilon)},\qquad \|u_3-\mathbf{R}_3\|_{L^2(\Omega^*_\varepsilon)}+\|\nabla(u-\mathbf{R})\|_{L^2(\Omega^*_\varepsilon)}\le\frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega^*_\varepsilon)}.$$

The constant C does not depend on ε.

Proof. From Proposition 5.1, there exits $(b_1,b_2)\in\mathbb{R}^2$ such that

$$\|\mathcal{R}_\alpha-b_\alpha\|_{L^2(\omega^{int}_{3\varepsilon})}\le\frac{C}{\varepsilon^{3/2}}\|e(u)\|_{L^2(\Omega^*_\varepsilon)}.\tag{5.17}$$

Then, the above estimate, $(5.15)_1$ and Lemma 5.3 yield

$$\sum_{\xi\in\Xi_\varepsilon}|\mathcal{R}_\alpha(\varepsilon\xi)-b_\alpha|^2\varepsilon^3\le\frac{C}{\varepsilon^3}\|e(u)\|_{L^2(\Omega^*_\varepsilon)}.$$

Besides, $(5.16)_2$ and (5.17) lead to

$$\Big\|\frac{\partial\mathcal{U}_3}{\partial x_1}+b_2\Big\|_{L^2(\omega^{int}_{3\varepsilon})}+\Big\|\frac{\partial\mathcal{U}_3}{\partial x_2}-b_1\Big\|_{L^2(\omega^{int}_{3\varepsilon})}\le\frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega^*_\varepsilon)},\quad \alpha=1,2.$$

Proceeding as above, there exists $a_3\in\mathbb{R}$ such that

$$\sum_{\xi\in\Xi_\varepsilon}|\mathcal{U}_3(\varepsilon\xi)-a_3+b_2\varepsilon\xi_1-b_1\varepsilon\xi_2|^2\varepsilon^3\le\frac{C}{\varepsilon^3}\varepsilon\|e(u)\|_{L^2(\Omega^*_\varepsilon)}.$$

From $(5.16)_2$ we also obtain

$$\Big\|\frac{\partial\mathcal{U}_1}{\partial x_1}\Big\|_{L^2(\omega^{int}_{3\varepsilon})}+\Big\|\frac{\partial\mathcal{U}_2}{\partial x_2}\Big\|_{L^2(\omega^{int}_{3\varepsilon})}+\Big\|\frac{\partial\mathcal{U}_1}{\partial x_2}+\frac{\partial\mathcal{U}_2}{\partial x_1}\Big\|_{L^2(\omega^{int}_{3\varepsilon})}\le\frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega^*_\varepsilon)}.$$

Since the boundary of $\omega_{3\varepsilon}^{int}$ is uniformly Lipschitz, the $2D$-Korn inequality gives a rigid displacement $r(x_1,x_2) = (a_1 - b_3x_2)\mathbf{e}_1 + (a_2 + b_3x_1)\mathbf{e}_2$ such that

$$\|\mathcal{U}_1 - r_1\|_{H^1(\omega_{3\varepsilon}^{int})} + \|\mathcal{U}_2 - r_2\|_{H^1(\omega_{3\varepsilon}^{int})} \leq \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon^*)}.$$

These estimates and $(5.16)_2$ imply that

$$\|\mathcal{R}_3 - b_3\|_{L^2(\omega_{3\varepsilon}^{int})} \leq \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon^*)}.$$

Then, as above we obtain

$$\sum_{\xi\in\Xi_\varepsilon} |\mathcal{R}_3(\varepsilon\xi) - b_3|^2\varepsilon^3 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_1(\varepsilon\xi) - a_1 + b_3\varepsilon\xi_2|^2\varepsilon^3 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_2(\varepsilon\xi) - a_2 - b_3\varepsilon\xi_1|^2\varepsilon^3 \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon^*)}.$$

By choosing $\mathbf{R}(x) = a + b \wedge x$ and using (5.14) that ends the proof of the theorem. □

Let γ be a subset of $\partial\omega$ with a non null measure. Assume that there exists a domain ω' with Lipschitz boundary such that

$$\omega \subset \omega' \quad \text{and} \quad \omega' \cap \partial\omega = \gamma.$$

Denote

$$\mathbf{V}_\varepsilon \doteq \Big\{ v \in H^1(\Omega_\varepsilon^*)^3 \;\Big|\; \exists v' \in H^1(\Omega_\varepsilon'^*)^3,\ \ v = v'_{|\Omega_\varepsilon^*},\ \ v' = 0 \text{ in } \Omega_\varepsilon'^* \setminus \overline{\Omega_\varepsilon^*} \Big\},$$

where

$$\Omega_\varepsilon'^* \doteq \text{interior}\Big(\bigcup_{\xi\in\Xi'_\varepsilon} (\varepsilon\xi + \varepsilon\overline{\mathbf{C}}) \Big), \quad \Xi'_\varepsilon \doteq \big\{ \xi \in \mathbb{Z}^2 \;\big|\; (\varepsilon\xi + \varepsilon Y') \cap \omega' \neq \emptyset \big\}.$$

Theorem 5.2. *For every displacement u in $\mathbf{V}_\varepsilon$ one has*

$$\|u_1\|_{L^2(\Omega_\varepsilon^*)} + \|u_2\|_{L^2(\Omega_\varepsilon^*)} \leq C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}, \qquad \|u_3\|_{L^2(\Omega_\varepsilon^*)} + \|\nabla u\|_{L^2(\Omega_\varepsilon^*)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \tag{5.18}$$

The constant C does not depend on ε.

Proof. Since u belongs to $\mathbf{V}_\varepsilon$, there exists $u' \in H^1(\Omega_\varepsilon'^*)^3$ such that $u = u'_{|\Omega_\varepsilon^*}$, $u' = 0$ in $\Omega_\varepsilon'^* \setminus \overline{\Omega_\varepsilon^*}$. Then, applying Theorem 5.1 with u' (resp. Ω') in place of u (resp. Ω) gives a rigid displacement $\mathbf{R}'$ such that

$$\|u'_\alpha - \mathbf{R}'_\alpha\|_{L^2(\Omega_\varepsilon'^*)} \leq C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}, \qquad \|u'_3 - \mathbf{R}'_3\|_{L^2(\Omega_\varepsilon'^*)} + \|\nabla(u' - \mathbf{R}')\|_{L^2(\Omega_\varepsilon'^*)} \leq \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \tag{5.19}$$

Let $\mathcal{O}$ be an open set such that $\mathcal{O} \Subset \big(\omega' \setminus \overline{\omega}\big)$. For ε small enough, the function u' vanishes in $\mathcal{O} \times (-\varepsilon/2, \varepsilon/2) \cap \Omega_\varepsilon'^*$. Then the terms of its decomposition $\mathcal{U}'$ and $\mathcal{R}'$ vanish in $\mathcal{O}$ too. Hence, one can choose $\mathbf{R}' = 0$ without changing the estimates (5.19). So, (5.18) follows. □

As a consequence of the two previous theorems, one has

Corollary 5.1. *For every displacement u in $\mathbf{V}_\varepsilon$ the following estimates hold:*

$$\begin{aligned}
&\|\mathcal{U}_1\|_{H^1(\omega_{3\varepsilon}^{int})} + \|\mathcal{U}_2\|_{H^1(\omega_{3\varepsilon}^{int})} + \|\mathcal{R}_3\|_{H^1(\omega_{3\varepsilon}^{int})} \leq \frac{C}{\varepsilon^{1/2}}\|e(u)\|_{L^2(\Omega_\varepsilon^*)},\\
&\|\mathcal{U}_3\|_{H^1(\omega_{3\varepsilon}^{int})} + \|\mathcal{R}_1\|_{H^1(\omega_{3\varepsilon}^{int})} + \|\mathcal{R}_2\|_{H^1(\omega_{3\varepsilon}^{int})} \leq \frac{C}{\varepsilon^{3/2}}\|e(u)\|_{L^2(\Omega_\varepsilon^*)},\\
&\|\overline{u}\|_{L^2(\Omega_\varepsilon^{int})} \leq C\varepsilon\|e(u)\|_{L^2(\Omega_\varepsilon^*)}, \quad \|\nabla\overline{u}\|_{L^2(\Omega_\varepsilon^{int})} \leq C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}.
\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}
&\sum_{\xi\in\Xi_\varepsilon} |\mathcal{R}_1(\varepsilon\xi)|^2\varepsilon^2 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{R}_2(\varepsilon\xi)|^2\varepsilon^2 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_3(\varepsilon\xi)|^2\varepsilon^2 \leq \frac{C}{\varepsilon^3}\|e(u)\|^2_{L^2(\Omega_\varepsilon^*)},\\
&\sum_{\xi\in\Xi_\varepsilon} |\mathcal{R}_3(\varepsilon\xi)|^2\varepsilon^2 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_1(\varepsilon\xi)|^2\varepsilon^2 + \sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_2(\varepsilon\xi)|^2\varepsilon^2 \leq \frac{C}{\varepsilon}\|e(u)\|^2_{L^2(\Omega_\varepsilon^*)}.
\end{aligned} \tag{5.21}$$

The constants do not depend on ε.

5.4 Korn's type inequality in a beam-like domain

In this subsection the notations are those of Subsection 4.1.

For every displacement $u \in H^1(\Omega_\varepsilon^*)^3$, the Korn inequality applied in the domain $\varepsilon(\xi + \mathbf{C})$, $\xi \in \Xi_\varepsilon$, gives a rigid displacement $\mathbf{R}_{\varepsilon\xi}$

$$\mathbf{R}_{\varepsilon\xi}(x) = \boldsymbol{\mathcal{U}}(\varepsilon\xi) + \boldsymbol{\mathcal{R}}(\varepsilon\xi) \wedge (x - \varepsilon\xi) \qquad x \in \mathbb{R}^3$$

such that

$$\|\nabla(u - \mathbf{R}_{\varepsilon\xi})\|_{L^2(\varepsilon(\xi+\mathbf{C}))} \le C\|e(u)\|_{L^2(\varepsilon(\xi+\mathbf{C}))}, \quad \|u - \mathbf{R}_{\varepsilon\xi}\|_{L^2(\varepsilon(\xi+\mathbf{C}))} \le C\varepsilon\|e(u)\|_{L^2(\varepsilon(\xi+\mathbf{C}))}. \tag{5.22}$$

Remark 5.2. *By construction, the fields $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{R}}$ are piecewise constant.*

In a similar way as in Lemma 5.5 we obtain

Lemma 5.10. *The following estimates hold:*

$$\begin{aligned}
&\sum_{\xi\in\Xi_\varepsilon} |\boldsymbol{\mathcal{R}}(\varepsilon\xi + \varepsilon\mathbf{e}_3) - \boldsymbol{\mathcal{R}}(\varepsilon\xi)|^2\varepsilon^3 \le C\|e(u)\|^2_{L^2(\Omega_\varepsilon^*)},\\
&\sum_{\xi\in\Xi_\varepsilon} |\boldsymbol{\mathcal{U}}(\varepsilon\xi + \varepsilon\mathbf{e}_3) - \boldsymbol{\mathcal{U}}(\varepsilon\xi) - \varepsilon\boldsymbol{\mathcal{R}}(\varepsilon\xi + \varepsilon\mathbf{e}_3) \wedge \mathbf{e}_3|^2\varepsilon^3 \le C\varepsilon^2\|e(u)\|^2_{L^2(\Omega_\varepsilon^*)}.
\end{aligned} \tag{5.23}$$

The constant C depends only on $\mathbf{C}$.

Set

$$\boldsymbol{\mathcal{R}}(N\varepsilon) = \boldsymbol{\mathcal{R}}((N-1)\varepsilon), \quad \boldsymbol{\mathcal{U}}(N\varepsilon) = \boldsymbol{\mathcal{U}}((N-1)\varepsilon) + \varepsilon\boldsymbol{\mathcal{R}}(N\varepsilon) \wedge \mathbf{e}_3.$$

Now, using Q_1 interpolation, we extend the fields $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{R}}$ and we obtain two fields $\mathcal{U}$, $\mathcal{R}$ belonging to $W^{1,\infty}(0,L)^3$ and such that

$$\mathcal{U}(\varepsilon\xi) = \boldsymbol{\mathcal{U}}(\varepsilon\xi), \qquad \mathcal{R}(\varepsilon\xi) = \boldsymbol{\mathcal{R}}(\varepsilon\xi), \qquad \forall \xi \in \{0,\dots,N\}.$$

Define the displacement $\mathbf{U}^e$ by

$$\mathbf{U}^e(x) = \mathcal{U}(x_3) + \mathcal{R}(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2), \qquad \forall x \in \Omega_\varepsilon^*.$$

Lemma 5.11. *For every displacement $u \in H^1(\Omega_\varepsilon^B)^3$ one has*

$$\begin{aligned}
&\Big\|\frac{d\mathcal{R}}{dx_3}\Big\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e(u)\|_{L^2(\Omega_\varepsilon^*)},\\
&\Big\|\frac{d\mathcal{U}}{dx_3} - \mathcal{R}\wedge\mathbf{e}_3\Big\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon^*)},\\
&\big\|e(\mathbf{U}^e)\big\|_{L^2(\Omega_\varepsilon^*)} \le C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}.
\end{aligned} \tag{5.24}$$

Moreover,

$$\|\nabla\big(u - \mathbf{U}^e\big)\|_{L^2(\Omega_\varepsilon^*)} \le C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}, \qquad \|u - \mathbf{U}^e\|_{L^2(\Omega_\varepsilon^*)} \le C\varepsilon\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \tag{5.25}$$

The constant C depends only on $\mathbf{C}$.

Proof. Estimates (5.23) yield $(5.24)_{1,2}$. A straightforward calculation and $(5.24)_{1,2}$ lead to $(5.24)_3$. Then taking into account (5.22) we obtain (5.25). □

Denote

$$\mathbf{H}(0,L) \doteq \big\{\phi \in H^1(0,L) \mid \phi(0) = 0\big\}.$$

Lemma 5.12. *For every displacement $u \in \mathbf{V}_\varepsilon$ one has*

$$\|\mathcal{U}_3\|_{H^1(0,L)} + \varepsilon\big(\|\mathcal{U}_1\|_{H^1(0,L)} + \|\mathcal{U}_2\|_{H^1(0,L)} + \|\mathcal{R}\|_{L^2(0,L)}\big) \le \frac{C}{\varepsilon}\|e(u)\|_{L^2(\Omega_\varepsilon^*)} \tag{5.26}$$

and

$$\|u_3\|_{L^2(\Omega_\varepsilon^*)} + \varepsilon\big(\|u_1\|_{L^2(\Omega_\varepsilon^*)} + \|u_2\|_{L^2(\Omega_\varepsilon^*)} + \|\nabla u\|_{L^2(\Omega_\varepsilon^*)}\big) \le C\|e(u)\|_{L^2(\Omega_\varepsilon^*)}. \tag{5.27}$$

The constant C does not depend on ε.

Proof. We extend u by 0 in the cell $\varepsilon\big(-\mathbf{e}_3+\mathbf{C}\big)$. Then, proceeding as in Lemma 5.4 we obtain

$$|\mathcal{R}(0)|^2\varepsilon^3 \le C\|e(u)\|^2_{L^2(\Omega^*_\varepsilon)}, \qquad |\mathcal{U}(0)|\varepsilon^3 \le C\varepsilon^2\|e(u)\|^2_{L^2(\Omega^*_\varepsilon)}.$$

Without losing the estimates (5.24), we set $\mathcal{U}(0)=\mathcal{R}(0)=0$. Estimates (5.26) are the immediate consequences of $(5.24)_{1,2}$ and the Poincaré inequality. Finally (5.22) and (5.26) lead to (5.27). □

As a consequence of the previous lemma and (5.22), we have the following decomposition of a displacement $u\in\mathbf{V}_\varepsilon$:

$$u=\mathbf{U}^e+\overline{u},$$

where

$$\mathbf{U}^e(x)=\mathcal{U}(x_3)+\mathcal{R}(x_3)\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2), \qquad \forall x\in\Omega^*_\varepsilon, \qquad \mathcal{U},\ \mathcal{R}\in\mathbf{H}(0,L)^3$$

and the displacement $\overline{u}\in\mathbf{V}_\varepsilon$ satisfies the estimates

$$\|\overline{u}\|_{L^2(\Omega^*_\varepsilon)}\le C\varepsilon\|e(u)\|_{L^2(\Omega^*_\varepsilon)}, \quad \|\nabla\overline{u}\|_{L^2(\Omega^*_\varepsilon)}\le C\|e(u)\|_{L^2(\Omega^*_\varepsilon)}. \tag{5.28}$$

The constant C does not depend on ε.

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