



**HAL**  
open science

## 3-anti-power uniform morphisms

Francis Wlazinski

► **To cite this version:**

| Francis Wlazinski. 3-anti-power uniform morphisms. 2019. hal-02279489v3

**HAL Id: hal-02279489**

**<https://hal.science/hal-02279489v3>**

Preprint submitted on 9 Dec 2019 (v3), last revised 21 Dec 2023 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# 3-anti-power uniform morphisms

Francis Wlazinski

November 24, 2019

## Abstract

Words whose three successive factors of the same length are all different i.e. 3-anti-power words are more natural extensions of square-free words (two successive factors of the same length are different) than cube-free words. We give a way to verify whether a uniform morphism preserves 3-anti-power words (the image of a 3-anti-power word is a 3-anti-power word).

A consequence of the existence of such morphisms is the possibility of generating infinite 3-anti-power words.

## 1 Preliminaries

Let us recall some basic notions of Combinatorics of words.

### 1.1 Words

An *alphabet*  $A$  is a finite set of symbols called *letters*. A *word* over  $A$  is a finite sequence of letters from  $A$ . The *empty word*  $\varepsilon$  is the empty sequence of letters. Equipped with the concatenation operation, the set  $A^*$  of words over  $A$  is a free monoid with  $\varepsilon$  as neutral element and  $A$  as set of generators. Since an alphabet with one element is limited interest to us, we always assume the cardinality of considered alphabets is at least two. Given a non-empty word  $u = a_1 \dots a_n$  with  $a_i \in A$  for any integer  $i$  from 1 to  $n$ , the *length* of  $u$  denoted by  $|u|$  is the integer  $n$  that is the number of letters of  $u$ . By convention, we have  $|\varepsilon| = 0$ . We denote by  $A^+$  the set of words of positive length over  $A$ , i.e.,  $A^+ = A^* \setminus \{\varepsilon\}$ .

An infinite word over  $A$  is a map from  $\mathbb{N}$  to  $A$  that is an infinite sequence of letters  $a_1 \dots a_n \dots$  with  $a_i \in A$ . And  $A^{\mathbb{N}}$  is the set of all infinite words over  $A$ .

A word  $u$  is a *factor* of a word  $v$  if there exist two (possibly empty) words  $p$  and  $s$  such that  $v = pus$ . We denote  $\text{Fcts}(v)$  the set of all factors of  $v$ . If  $u \in \text{Fcts}(v)$ , we also say that  $v$

contains the word  $u$  (as a factor). If  $p = \varepsilon$ ,  $u$  is a *prefix* of  $v$ . If  $s = \varepsilon$ ,  $u$  is a *suffix* of  $v$ . If  $u \neq v$ ,  $u$  is a *proper* factor of  $v$ . If  $u$ ,  $p$  and  $s$  are non-empty,  $u$  is an *internal* factor of  $v$ .

Let  $w$  be a non-empty word and let  $i, j$  be two integers such that  $0 \leq i - 1 \leq j \leq |w|$ . We denote by  $w[i..j]$  the factor of  $w$  such that  $|w[i..j]| = j - i + 1$  and  $w = pw[i..j]s$  for two words  $s$  and  $p$  verifying  $|p| = i - 1$ . When  $j > i$ ,  $w[i..j]$  is simply the factor of  $w$  that starts at the  $i^{\text{th}}$  letter of  $w$  and ends at the  $j^{\text{th}}$ . Note that, when  $j = i - 1$ , we have  $w[i..j] = \varepsilon$ . When  $i = j$ , we also denote by  $w[i]$  the factor  $w[i..i]$  which is the  $i^{\text{th}}$  letter of  $w$ . In particular,  $w[1]$  and  $w[|w|]$  are respectively the first and the last letter of  $w$ .

Powers of a word are defined inductively by  $u^0 = \varepsilon$ , and for any integer  $n \geq 1$ ,  $u^n = uu^{n-1}$ . Given an integer  $k \geq 2$ , since the case  $\varepsilon^k$  is of little interest, we call a *k-power* any word  $u^k$  with  $u \neq \varepsilon$ .

Given an integer  $k \geq 2$ , a word is *k-power-free* if it does not contain any  $k$ -power as factor. A *primitive* word is a word which is not a  $k$ -power of another word whatever the integer  $k \geq 2$ .

Given two integers  $p > q > 1$  and two words  $x$  and  $y$  with  $xy \neq \varepsilon$ , a word of the form  $(xy)^\alpha x$  with  $\alpha + \frac{|x|}{|xy|} = \frac{p}{q}$  is called a  $\frac{p}{q}$ -power. For instance, the word *anchorman* is a  $(1 + \frac{2}{7} = \frac{9}{7})$ -power and the word *abaabaa* is a  $(2 + \frac{1}{3} = \frac{7}{3})$ -power. In particular, a  $\frac{3}{2}$ -power is a word of the form  $xyx$  with  $|x| = |y| \neq \varepsilon$ . For instance, the word *antman* is a  $\frac{3}{2}$ -power. A word is  $\frac{p}{q}$ -power-free if it does not contain any  $\ell$ -power as factor with  $\ell \geq \frac{p}{q}$ . The word *abcaba* is not  $\frac{3}{2}$ -power-free. Indeed, it contains the word *abcab* which is a  $\frac{5}{3}$ -power.

Given an integer  $k \geq 2$  and an integer  $n \geq 1$ , a  $(k, n)$ -anti-power sequence or simply a *k-anti-power* [5] is a concatenation of  $k$  consecutive pairwise different words of the same length  $n$ .

For instance, if  $A = \{a, b\}$ , the words  $u = aa\,ba\,bb\,ab$  and  $v = aba\,bab\,abb\,aaa$  are respectively  $(4, 2)$ -anti-power and  $(4, 3)$ -anti-power sequences. But the prefix *abababab* of  $v$  is not a 4-anti-power sequence: it is even a 4-power.

Given an integer  $k$ , if  $\text{Card}(A) = \alpha \geq 2$  then there exist  $\alpha^n$  different words in  $A^*$  of length  $n \geq 1$ . Among the words of length  $k \times n$ , there are  $\alpha^n$  different  $k$ -powers (of length  $k \times n$ ) and  $A_{\alpha^n}^k = \frac{(\alpha^n)!}{(\alpha^n - k)!}$  different  $(k, n)$ -anti-power sequences if  $\alpha^n \geq k$  and 0 otherwise. It particularly means that there exists an integer  $k_0$  such that there are no  $(k', n)$ -anti-power sequence over  $A$  for any  $k' \geq k_0$ .

For any alphabet  $A$  with  $\text{Card}(A) = \alpha \geq 2$  and for any integer  $k$ , there exists a smallest integer  $p_0$  such that  $\alpha^{p_0} \geq k$ . And, if  $p \geq p_0$ , the set of  $(k, p)$ -anti-power sequences is greater than the set of  $k$ -powers of length  $p \times k$ .

A *2-anti-power word* is simply a square-free word. Given an integer  $k \geq 3$ , a word  $w$  is a *k-anti-power word* if it is a  $(k - 1)$ -anti-power word and if any factor of  $w$  of length  $k \times \ell$  for every  $1 \leq \ell \leq \left\lfloor \frac{|w|}{k} \right\rfloor$  is a  $(k, \ell)$ -anti-power. By this definition, a word of length  $n$  with

$2 \leq n < k$  is a  $k$ -anti-power word if and only if it is a  $n$ -anti-power word.

An *infinite  $k$ -anti-power word* is an infinite word whose finite factors are all  $k$ -anti-power words. Obviously, the first question is whether such a word exists.

If  $A = \{a, b\}$ , the only 2-anti-power words are  $aba$ ,  $bab$  and their factors. But, for any  $k \geq 3$ , the only  $k$ -anti-power words are  $a$ ,  $b$ ,  $ab$  and  $ba$ .

If  $\text{Card}(A) \geq 3$ , there exist infinite 2-anti-power (square-free) words [1, 9, 10].

If  $k = 3$  and  $A = \{a, b, c\}$ , the only 3-anti-power words are  $abcab$ , the exchange of letters of this word and their factors. Let us note that the word  $abcab$  is not  $\frac{3}{2}$ -power-free.

A  $\frac{3}{2}$ -power-free word contain neither a factor of the form  $xyx$  with  $|x| = |y|$ , nor a factor of the form  $xx$ . Thus a  $\frac{3}{2}$ -power-free word is a 3-anti-power word (but the converse does not hold). Thus a Dejean's word [4, 3, 8] over a four-letter alphabet, which does not contain any  $\ell$ -power with  $\ell > \frac{7}{5}$ -power-free, is a 3-anti-power word.

More generally, a non- $k$ -anti-power word (among  $k$  consecutive factors of the same length of this word, at least two of them are equal) contains at least one fractionnal  $\ell$ -power with  $\ell \geq \frac{k}{k-1}$ . Therefore, when  $k \geq 3$ , a Dejean's word over a  $(k + 1)$ -letter alphabet is a  $k$ -anti-power word.

**Remark 1.1** *If we had chosen not to add that a  $k$ -anti-power word must be a  $(k - 1)$ -anti-power word, we would have, for instance, that, for  $A = \{a; b; c\}$ , the word  $abcabcab$  would have been a 3-anti-power word but not a 2-anti-power word.*

*More precisely, without the condition that a  $k$ -anti-power word  $w$  must be a  $(k - 1)$ -anti-power word, we only could say that all prefixes and all suffixes of  $w$  of length between  $k - 1$  and  $\left\lfloor \frac{(k - 1)|w|}{k} \right\rfloor$  are  $(k - 1)$ -anti-power words.*

*For an infinite word, it does not change anything to add the condition that a  $k$ -anti-power word  $w$  must be a  $(k - 1)$ -anti-power word. Indeed, every factor of  $w$  whose length is a multiple of  $k - 1$  can be extended to a factor whose length is a multiple of  $k$ . Obviously, if these  $k$  factors are different, the same holds for  $k - 1$  ones.*

**Lemma 1.2** [6, 7] *If a non-empty word  $v$  is an internal factor of  $vv$ , i.e., if there exist two non-empty words  $x$  and  $y$  such that  $vv = xvy$ , then there exist a non-empty word  $t$  and two integers  $i, j \geq 1$  such that  $x = t^i$ ,  $y = t^j$ , and  $v = t^{i+j}$ .*

## 1.2 Morphisms

Let  $A$  and  $B$  be two alphabets. A *morphism*  $f$  from  $A^*$  to  $B^*$  is a mapping from  $A^*$  to  $B^*$  such that  $f(uv) = f(u)f(v)$  for all words  $u, v$  over  $A$ . When  $B$  has no importance, we say that  $f$  is a morphism on  $A$  or that  $f$  is defined on  $A$ .

Given an integer  $L \geq 1$ ,  $f$  is  $L$ -uniform if  $|f(a)| = L$  for every letter  $a$  in  $A$ . A morphism  $f$  is *uniform* if it is  $L$ -uniform for some integer  $L \geq 1$ .

A 2-anti-power morphism is a square-free morphism.

Let  $A$  be an alphabet of cardinality at least  $k + 1 \geq 3$ . A morphism  $f$  on  $A$  is  $k$ -anti-power if and only if  $f$  is a  $(k - 1)$ -anti-power morphism and  $f(w)$  is a  $k$ -anti-power word for every  $k$ -anti-power word  $w$  over  $A$ . For instance, the *identity endomorphism*  $Id$  ( $\forall a \in A, Id(a) = a$ ) is a  $k$ -anti-power morphism.

In particular, all  $k$ -anti-power morphisms are square-free morphisms. These last morphisms have been characterized in [2].

We say that a morphism is *non-erasing* if, for all letters  $a \in A$ ,  $f(a) \neq \varepsilon$ . A  $k$ -anti-power morphism, as every square-free morphism, is necessarily non-erasing.

A morphism on  $A$  is called *prefix* (resp. *suffix*) if, for all different letters  $a$  and  $b$  in  $A$ , the word  $f(a)$  is not a prefix (resp. not a suffix) of  $f(b)$ . A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Proofs of the two following lemmas are left to the reader.

**Lemma 1.3** *Let  $f$  be a prefix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $p_1$  (resp.  $p_2$ ) be a prefix of  $f(a)$  (resp. of  $f(b)$ ). If  $(p_1; p_2) \neq (\varepsilon; f(b))$  and if  $(p_1; p_2) \neq (f(a); \varepsilon)$  then the equality  $f(u)p_1 = f(v)p_2$  implies  $u = v$  and  $p_1 = p_2$ .*

**Lemma 1.4** *Let  $f$  be a suffix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $s_1$  (resp.  $s_2$ ) be a suffix of  $f(a)$  (resp. of  $f(b)$ ). If  $(s_1; s_2) \neq (\varepsilon; f(b))$  and if  $(s_1; s_2) \neq (f(a); \varepsilon)$  then the equality  $s_1f(u) = s_2f(v)$  implies  $u = v$  and  $s_1 = s_2$ .*

Taking  $p_1 = p_2 = \varepsilon$  (resp.  $s_1 = s_2 = \varepsilon$ ) in Lemma 1.3 (resp Lemma 1.4), we get that a prefix (resp. suffix) morphism is injective.

**Definition 1.5** *A morphism  $f$  from  $A^*$  to  $B^*$  is a ps-morphism (Keränen [6] called  $f$  a ps-code) if and only if the equalities*

$$f(a) = ps, f(b) = ps' \text{ and } f(c) = p's$$

*with  $a, b, c \in A$  (possibly  $c = b$ ) and  $p, s, p', s' \in B^*$  imply  $b = a$  or  $c = a$ .*

Obviously, taking  $c = b$ , and  $s = \varepsilon$  in a first time and  $p = \varepsilon$  in a second time, we obtain that a ps-morphism is a bifix morphism.

**Lemma 1.6** [6, 7] *If  $f$  is not a ps-morphism then  $f$  is not a  $k$ -power-free morphism for every integer  $k \geq 2$ .*

Since a  $k$ -anti-power morphism is a square-free morphism, we get the following Corollary.

**Corollary 1.7** *For every integer  $k \geq 2$ , a  $k$ -anti-power morphism is a ps-morphism.*

**Proposition 1.8** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 2$  and let  $f$  be a morphism from  $A^*$  to  $B^*$ . If there exist five letters  $a, b, c, d$  and  $x$  (possibly equal) and four words  $p, s, \pi$  and  $\sigma$  such that  $s$  is a suffix of  $f(a)$ ,  $p$  is a prefix of  $f(b)$ ,  $\sigma$  is a non-empty suffix of  $f(c)$ ,  $\pi$  is a non-empty prefix of  $f(d)$ , and  $sp = \sigma f(x)\pi$  then  $f$  is not a square-free morphism.*

**Proof.**

Since  $|sp| > |\sigma f(x)|$  and  $|sp| > |f(x)\pi|$ , we get  $|s| > |\sigma|$  and  $|p| > |\pi|$ .

Let  $s'$  be the non empty prefix of  $f(x)$  such that  $s = \sigma s'$  and let  $p'$  be the non empty suffix of  $f(x)$  such that  $p = p'\pi$ .

If  $x = a$  and  $x = b$ , then  $f(x)$  is an internal factor of  $f(ax)$ . By Lemma 1.2,  $f(x)$  is not primitive i.e.  $f$  is not a square-free morphism.

Therefore, at least one of the word  $ax$  or  $bx$  is not a square. But  $f(ax)$  contains the square  $s's'$  and  $f(bx)$  contains the square  $p'p'$  i.e.  $f$  is not a square-free morphism.  $\square$

**Proposition 1.9** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 3$  and let  $f$  be a  $L$ -uniform morphism from  $A^*$  to  $B^*$ . If  $L$  is an even number then  $f$  is not a 3-anti-power morphism.*

**Proof.**

Let  $a, b$  and  $c$  be three different letters in  $A$ . Let  $A_1$  and  $A_2$  be the words such that  $f(a) = A_1A_2$  with  $|A_1| = |A_2|$ .

Then  $f(abcab)$  contains the non-3-anti-power sequence  $A_2f(b)f(c)A_1A_2f(b)$  with  $abcab$  a 3-anti-power word.

$\square$

A morphism  $f$  on  $A$  is  $k$ -anti-power up to  $\ell$  ( $k, \ell \geq 2$ ) if and only if  $f(w)$  is a  $k$ -anti-power word for every  $k$ -anti-power word  $w$  over  $A$  of length at most  $\ell$ .

**Proposition 1.10** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 3$  and let  $f$  be a square-free  $L$ -uniform morphism from  $A^*$  to  $B^*$  (with  $L$  odd).*

*Then  $f$  is a 3-anti-power morphism if and only if it is a 3-anti-power morphism up to 5.*

**Proof.**

By definition of 3-anti-power morphisms, we only have to prove the "if" part of Proposition 1.10.

By Lemma 1.6,  $f$  (square-free) is a ps-morphism and so injective.

By contradiction, we assume that a shortest 3-anti-power word  $w$  (not necessarily unique) such that  $f(w)$  contains a non-3-anti-power satisfy  $|w| \geq 6$ . We will show that this assumption leads to contradictions.

Since the length of  $w$  is minimal, we may assume that there exist five words  $p, s, U_1, U_2$  and  $U_3$  such that  $f(w) = pU_1U_2U_3s$  where  $p$  is a prefix of  $f(w[1])$  different from  $f(w[1])$  and  $s$  is a suffix of  $f(w[|w|])$  different from  $f(w[|w|])$ . Moreover, the words  $U_1, U_2, U_3$  have the same length ( $\geq 1$ ) and two of them are equal.

If  $U_1 = U_2$  or if  $U_2 = U_3$ , it means that  $f(w)$  contains a square with  $w$  a 3-anti-power word so a square-free word. That is  $f$  is not a square-free morphisms: a contradiction with the definition of  $f$ . The only remaining case is  $U_1 = U_3$ . To simplify notations, we denote by  $U$  the words  $U_1$  and  $U_3$  and by  $V$  the word  $U_2$ .

Let  $i_2$  be the shortest integer such that  $pU$  is the prefix of  $f(w[1..i_2])$  and let  $i_3$  be the shortest integer such that  $pUV$  is the prefix of  $f(w[1..i_3])$ .

If  $i_2 = 1$  then  $|V| = |U| \leq |pU| \leq |f(w[1])| = L$  and  $|pUVUs| < 4L$ . This is impossible since  $|f(w)| \geq 6L$ .

On a the same way, by a length criterion, the cases  $i_2 = i_3$  and  $i_3 = |w|$  are impossible.

If we denote  $x = w[2..i_2 - 1]$ ,  $y = w[i_2 + 1..i_3 - 1]$ ,  $z = w[i_3 + 1..|w| - 1]$ ,  $a_1 = w[1]$ ,  $a_2 = w[i_2]$ ,  $a_3 = w[i_3]$  and  $a_4 = w[|w|]$  then  $w = a_1xa_2ya_3za_4$  with  $|w| = 4 + |x| + |y| + |z|$ . It implies that  $|x| + |y| + |z| \geq 2$ .

Moreover, there exists some words  $p_i$  and  $s_i$  ( $1 \leq i \leq 4$ ) such that  $f(a_i) = p_i s_i$  with  $p_1 = p$ ,  $s_4 = s$ . By definition, the words  $s_1, p_2, p_3$  and  $p_4$  are non empty.

In other words, we have  $U = s_1 f(x) p_2 = s_3 f(z) p_4$  and  $V = s_2 f(y) p_3$ .

Since  $\max\{1+L|x|; L|y|\} < |U| = |V| \leq \min\{L(|x|+2); L(|y|+2)-1\}$ , we get that  $||x|-|y|| < 2$ . It means that, if  $|x| + |y|$  is even then  $|x| = |y|$  and, if  $|x| + |y|$  if odd then  $|x| = |y| \pm 1$ . In the same way, if  $|z| + |y|$  is even then  $|z| = |y|$  and, if  $|z| + |y|$  if odd then  $|z| = |y| \pm 1$ .

• *Case 1* :  $|s_1| = |s_3|$

Since  $2 \times |U| = |U| + |V| = |s_1 f(x) p_2| + |s_2 f(y) p_3| = (|x| + |y| + 2) \times L$ , we get that  $|x| + |y|$  is even i.e.  $|x| = |y|$ .

From the equality  $s_1 f(x) p_2 = s_3 f(z) p_4 (= U)$ , we get  $s_3 = s_1 (\neq \varepsilon)$ . By Lemma 1.3, it also implies  $z = x$  and  $p_2 = p_4 (\neq \varepsilon)$ .

In particular, since  $|x| = |y|$  and since  $w = a_1 x a_2 y a_3 x a_4$  is a 3-anti-power word, we have  $a_1 \neq a_3$  and  $a_2 \neq a_4$ .

Since  $|U| = |V|$ , we get  $|s_1| + |p_2| = |s_2| + |p_3|$ . Since  $|s_1| = |s_3|$ , we get  $|p_1| = |p_3|$  and  $2|p_1| = L + |p_1| - |s_1| = L + |p_3| - |s_1| = L + |p_2| - |s_2| = 2|p_2|$  i.e.  $|p_1| = |p_2|$ .

In a same way, since  $|p_2| = |p_4|$  we get  $|s_2| = |s_4| = |s_3|$ .

If  $a_1 = a_2$  then  $p_1 = p_2 (= p_4)$ ,  $f(a_1) = p_1 s_1$ ,  $f(a_3) = p_3 s_3 = p_3 s_1$  and  $f(a_4) = p_4 s_4 = p_1 s_4$ . It means that  $f(a_3 a_1 a_4)$  contains  $(s_1 p_1)^2$  with  $a_3 a_1 a_4$  square-free since  $a_1 \neq a_3$  and  $a_1 = a_2 \neq a_4$ : a contradiction with the hypothesis that  $f$  is a square-free morphism.

In the same way, if  $a_3 = a_4$ , we get that  $f(a_1 a_4 a_2)$  contains  $(s_4 p_4)^2$  with  $a_1 a_4 a_2$  square-free. And, if  $a_2 = a_3$ , we get that  $f(a_1 a_2 a_4)$  contains  $(s_2 p_2)^2$  with  $a_1 a_2 a_4$  square-free. Both cases lead to the same contradiction.

Thus  $a_1, a_2$  and  $a_3$  are three different letters and  $a_2, a_3$  and  $a_4$  are also three different letters. It means that  $a_1 a_2 a_3 a_4$  is a 3-anti-power word. But  $f(a_1 a_2 a_3 a_4)$  contains the non-3-anti-power sequence  $s_1 p_2 s_2 p_3 s_3 p_4 = s_1 p_2 s_2 p_3 s_1 p_2$ : a contradiction with the minimality of  $|w|$ .

- *Case 2* :  $s_1 = f(a_1)$  and  $s_3 = \varepsilon$

We have  $|f(a_1) f(x) p_2| = |s_1 f(x) p_2| = |U| = |V| = |s_2 f(y) p_3| = |s_2 f(y) f(a_3)|$  i.e.  $|p_2| = |s_2|$ : this contradicts the fact that  $L$  is odd.

- *Case 3* :  $s_1 \neq s_3$

If  $x = \varepsilon$  and  $z = \varepsilon$  then  $i_2 = 2$ ,  $i_3 = |w| - 1$  and  $|y| = |w[i_2 + 1..i_3 - 1]| = |w| - 4 \geq 2$ : this is impossible.

Thus, in the equality  $s_1 f(x) p_2 = s_3 f(z) p_4 (= U)$ , we have either  $x \neq \varepsilon$  or  $z \neq \varepsilon$ .

Since the equality is symmetric, without loss of generality, we may assume that  $|s_1| < |s_3|$ . In this case, we necessarily have  $x \neq \varepsilon$ . Let  $\chi$  be the first letter of  $x$  and let  $x'$  be the word such that  $x = \chi x'$ . If  $z = \varepsilon$ , let  $P = p_4$  and if  $z \neq \varepsilon$  let  $P = f(\gamma)$  where  $\gamma$  is the first letter of  $z$ . In particular, we have  $P$  non-empty. Let  $\pi$  be the (non empty) prefix of  $f(x') p_2$  such that  $s_1 f(\chi) \pi = s_3 P$ . By proposition 1.8, this last equation implies that  $f$  is not a square-free morphism: a final contradiction.

□

## 2 An example

As stated in the first section, Dejean's words are anti-power words. But we can build 3-power-free words without using fractionnal powers.

According to my computer, the following morphism  $h$  is a 3-anti-power morphism (but I do not really trust my programming skills).

$$\begin{aligned}
 h : \{a; b; c; d; e\}^* &\rightarrow \{a; b; c; d; e\}^* \\
 a &\mapsto abceacd \\
 b &\mapsto abecaed \\
 c &\mapsto acbaecd \\
 d &\mapsto acbeabd \\
 e &\mapsto acebcde
 \end{aligned}$$



The word  $h^\omega(a) = \lim_{n \rightarrow +\infty} h^n(a) = abceacd abecaed acbaecd acebced abceacd acbaecd acbeabd abceacd abecaed acebced acbaecd abceacd \dots$  generated by  $h$  is thus an infinite 3-anti-power word.

Let us remark that  $h^\omega(abcac)$  is also a 3-anti-power word. But it contains an infinite number of factors that are  $\frac{5}{3}$ -powers.

### Acknowledgment.

I would like to thank James D. Currie for his comments on the first version of this paper.

## References

- [1] J. Berstel. Axel Thue's papers on repetition in words: a translation. Technical Report 20, Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec, Montréal, 1995.
- [2] M. Crochemore. Sharp characterizations of squarefree morphisms. *Theoretical Computer Science*, 18:221–226, 1982.
- [3] James D. Currie and Narad Rampersad. A proof of dejean's conjecture. *Mathematics of computation*, 80:1063–1070, 2011.
- [4] F. Dejean. Sur un théorème de Thue. *J. Comb. Theory*, 13:90–99, 1972. series A.
- [5] Gabriele Fici, Antonio Restivo, Manuel Silva, and Luca Q. Zamboni. Anti-powers in infinite words. *Journal of Combinatorial Theory, Series A*, 157:109 – 119, 2018.
- [6] V. Keränen. On the  $k$ -freeness of morphisms on free monoids. *Annales Academiae Scientiarum Fennicae* 61, Series A, 1986.
- [7] M. Leconte. *Codes sans répétition*. PhD thesis, LITP Université Paris 6, october 1985.
- [8] Michaël Rao. Last cases of dejean's conjecture. *Theoretical Computer Science*, 412(27):3010 – 3018, 2011. Combinatorics on Words (WORDS 2009).
- [9] A. Thue. Über unendliche zeichenreihen. *Kristiania Videnskapselskapets Skrifter Klasse I. Mat.-naturv*, 7:1–22, 1906.
- [10] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Kristiania Videnskapselskapets Skrifter Klasse I. Mat.-naturv*, 1:1–67, 1912.