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# 3-anti-power uniform morphisms

Francis Wlazinski

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## Abstract

Words whose three successive factors of the same length are all different i.e. 3-anti-power words are more natural extensions of square-free words (two successive factors of the same length are different) than cube-free words. We give a way to verify whether a uniform morphism preserves 3-anti-power words (the image of a 3-anti-power word is a 3-anti-power word).

A consequence of the existence of such morphisms is the possibility of generating an infinite 3-anti-power word.

## 1 Preliminaries

Let us recall some basic notions of Combinatorics of words.

### 1.1 Words

An *alphabet*  $A$  is a finite set of symbols called *letters*. A *word* over  $A$  is a finite sequence of letters from  $A$ . The *empty word*  $\varepsilon$  is the empty sequence of letters. Equipped with the concatenation operation, the set  $A^*$  of words over  $A$  is a free monoid with  $\varepsilon$  as neutral element and  $A$  as set of generators. Since an alphabet with one element is limited interest to us, we always assume the cardinality of considered alphabets is at least two. Given a non-empty word  $u = a_1 \dots a_n$  with  $a_i \in A$  for any integer  $i$  from 1 to  $n$ , the *length* of  $u$  denoted by  $|u|$  is the integer  $n$  that is the number of letters of  $u$ . By convention, we have  $|\varepsilon| = 0$ . We denote by  $A^+$  the set of words of positive length over  $A$ , i.e.,  $A^+ = A^* \setminus \{\varepsilon\}$ .

An infinite word over  $A$  is a map from  $\mathbb{N}$  to  $A$  that is an infinite sequence of letters  $a_1 \dots a_n \dots$  with  $a_i \in A$ . And  $A^{\mathbb{N}}$  is the set of all infinite words over  $A$ .

A word  $u$  is a *factor* of a word  $v$  if there exist two (possibly empty) words  $p$  and  $s$  such that  $v = pus$ . We denote  $\text{Fcts}(v)$  the set of all factors of  $v$ . If  $u \in \text{Fcts}(v)$ , we also say that  $v$

contains the word  $u$  (as a factor). If  $p = \varepsilon$ ,  $u$  is a *prefix* of  $v$ . If  $s = \varepsilon$ ,  $u$  is a *suffix* of  $v$ . If  $u \neq v$ ,  $u$  is a *proper* factor of  $v$ . If  $u$ ,  $p$  and  $s$  are non-empty,  $u$  is an *internal* factor of  $v$ .

Let  $w$  be a non-empty word and let  $i, j$  be two integers such that  $0 \leq i - 1 \leq j \leq |w|$ . We denote by  $w[i..j]$  the factor of  $w$  such that  $|w[i..j]| = j - i + 1$  and  $w = pw[i..j]s$  for two words  $s$  and  $p$  verifying  $|p| = i - 1$ . When  $j > i$ ,  $w[i..j]$  is simply the factor of  $w$  that starts at the  $i^{\text{th}}$  letter and ends at the  $j^{\text{th}}$ . Note that, when  $j = i - 1$ , we have  $w[i..j] = \varepsilon$ . When  $i = j$ , we also denote by  $w[i]$  the factor  $w[i..i]$  which is the  $i^{\text{th}}$  letter of  $w$ . In particular,  $w[1]$  and  $w[|w|]$  are respectively the first and the last letter of  $w$ .

Powers of a word are defined inductively by  $u^0 = \varepsilon$ , and for any integer  $n \geq 1$ ,  $u^n = uu^{n-1}$ . Given an integer  $k \geq 2$ , since the case  $\varepsilon^k$  is of little interest, we call a  $k$ -power any word  $u^k$  with  $u \neq \varepsilon$ .

Given an integer  $k \geq 2$ , a word is  $k$ -power-free if it does not contain any  $k$ -power as factor. A *primitive* word is a word which is not a  $k$ -power of another word whatever the integer  $k \geq 2$ .

Given two integers  $p > q > 1$  and two words  $x$  and  $y$  with  $xy \neq \varepsilon$ , a word of the form  $(xy)^\alpha x$  with  $\alpha + \frac{|x|}{|xy|} = \frac{p}{q}$  is called a  $\frac{p}{q}$ -power. For instance, the word *anchorman* is a  $(1 + \frac{2}{7} = \frac{9}{7})$ -power and the word *abaabaa* is a  $(2 + \frac{1}{3} = \frac{7}{3})$ -power. In particular, a  $\frac{3}{2}$ -power is a word of the form  $xyx$  with  $|x| = |y| \neq \varepsilon$ . For instance, the word *antman* is a  $\frac{3}{2}$ -power. A word is  $\frac{p}{q}$ -power-free if it does not contain any  $\ell$ -power as factor with  $\ell \geq \frac{p}{q}$ . The word *abcaba* is not  $\frac{3}{2}$ -power-free. Indeed, it contains the word *abcab* which is a  $\frac{5}{3}$ -power.

Given an integer  $k \geq 2$  and an integer  $n \geq 1$ , a  $(k, n)$ -anti-power sequence or simply a  $k$ -anti-power [5] is a concatenation of  $k$  consecutive pairwise different words of the same length  $n$ .

For instance, if  $A = \{a, b\}$ , the words  $u = aa\,ba\,bb\,ab$  and  $v = aba\,bab\,abb\,aaa$  are respectively  $(4, 2)$ -anti-power and  $(4, 3)$ -anti-power sequences. But the prefix *abababab* of  $v$  is not a 4-anti-power sequence: it is even a 4-power.

Given an integer  $k$ , if  $\text{Card}(A) = \alpha \geq 2$  then there exist  $\alpha^n$  different words in  $A^*$  of length  $n \geq 1$ . Among the words of length  $k \times n$ , there are  $\alpha^n$  different  $k$ -powers (of length  $k \times n$ ) and  $A_{\alpha^n}^k = \frac{(\alpha^n)!}{(\alpha^n - k)!}$  different  $(k, n)$ -anti-power sequences if  $\alpha^n \geq k$  and 0 otherwise. It particularly means that there exists an integer  $k_0$  such that there are no  $(k', n)$ -anti-power sequence over  $A$  for any  $k' \geq k_0$ .

For any alphabet  $A$  with  $\text{Card}(A) = \alpha \geq 2$  and for any integer  $k$ , there exists a smallest integer  $p_0$  such that  $\alpha^{p_0} \geq k$ . And, if  $p \geq p_0$ , the set of  $(k, p)$ -anti-power sequences is greater than the set of  $k$ -powers of length  $p \times k$ .

A  $2$ -anti-power word is simply a square-free word. Given an integer  $k \geq 3$ , a word  $w$  is a  $k$ -anti-power word if it is a  $(k - 1)$ -anti-power word and if any factor of  $w$  of length  $k \times \ell$  for every  $1 \leq \ell \leq \left\lfloor \frac{|w|}{k} \right\rfloor$  is a  $(k, \ell)$ -anti-power. By this definition, a word of length  $n$  with

$2 \leq n < k$  is a  $k$ -anti-power word if and only if it is a  $n$ -anti-power word.

An *infinite  $k$ -anti-power word* is an infinite word whose finite factors are all  $k$ -anti-power words. Obviously, the first question is whether such a word exists.

If  $A = \{a, b\}$ , the only 2-anti-power words are  $aba$ ,  $bab$  and their factors. But, for any  $k \geq 3$ , the only  $k$ -anti-power words are  $a$ ,  $b$ ,  $ab$  and  $ba$ .

If  $\text{Card}(A) \geq 3$ , there exist infinite 2-anti-power (square-free) words [1, 9, 10].

If  $k = 3$  and  $A = \{a, b, c\}$ , the only 3-anti-power words are  $abcab$ , the exchange of letters of this word and their factors. Let us note that the word  $abcab$  is not  $\frac{3}{2}$ -power-free.

A  $\frac{3}{2}$ -power-free word contain neither a factor of the form  $xyx$  with  $|x| = |y|$ , nor a factor of the form  $xx$ . Thus a  $\frac{3}{2}$ -power-free word is a 3-anti-power word (but the converse does not hold). Thus a Dejean's word [4, 3, 8] over a four-letter alphabet, which does not contain any  $\ell$ -power with  $\ell > \frac{7}{5}$ -power-free, is a 3-anti-power word.

More generally, a non- $k$ -anti-power word (among  $k$  consecutive factors of the same length of this word, at least two of them are equal) contains at least one fractionnal  $\ell$ -power with  $\ell \geq \frac{k}{k-1}$ . Therefore, a Dejean's word over a  $(k+1)$ -letter alphabet is a  $k$ -anti-power word.

**Remark 1.1** *If we had chosen not to add that a  $k$ -anti-power word must be a  $(k-1)$ -anti-power word, we would have, for instance, that, for  $A = \{a; b; c\}$ , the word  $abcabcab$  would have been a 3-anti-power word but not a 2-anti-power word.*

*More precisely, without the condition that a  $k$ -anti-power word  $w$  must be a  $(k-1)$ -anti-power word, we only could say that all prefixes and all suffixes of  $w$  of length between  $k-1$  and  $\left\lfloor \frac{(k-1)|w|}{k} \right\rfloor$  are  $(k-1)$ -anti-power words.*

*For an infinite word, it does not change anything to add the condition that a  $k$ -anti-power word  $w$  must be a  $(k-1)$ -anti-power word. Indeed, every factor of  $w$  whose length is a multiple of  $k-1$  can be extended to a factor whose length is a multiple of  $k$ . Obviously, if these  $k$  factors are different, the same olds for  $k-1$  ones.*

**Lemma 1.2** [6, 7] *If a non-empty word  $v$  is an internal factor of  $vv$ , i.e., if there exist two non-empty words  $x$  and  $y$  such that  $vv = xvy$ , then there exist a non-empty word  $t$  and two integers  $i, j \geq 1$  such that  $x = t^i$ ,  $y = t^j$ , and  $v = t^{i+j}$ .*

## 1.2 Morphisms

Let  $A$  and  $B$  be two alphabets. A *morphism*  $f$  from  $A^*$  to  $B^*$  is a mapping from  $A^*$  to  $B^*$  such that  $f(uv) = f(u)f(v)$  for all words  $u, v$  over  $A$ . When  $B$  has no importance, we say that  $f$  is a morphism on  $A$  or that  $f$  is defined on  $A$ .

Given an integer  $L \geq 1$ ,  $f$  is  $L$ -uniform if  $|f(a)| = L$  for every letter  $a$  in  $A$ . A morphism  $f$  is *uniform* if it is  $L$ -uniform for some integer  $L \geq 1$ .

A morphism  $f$  on  $A$  is  $k$ -anti-power ( $k \geq 2$ ) if and only if  $f(w)$  is a  $k$ -anti-power word for every  $k$ -anti-power word  $w$  over  $A$ . For instance, the *identity endomorphism*  $Id$  ( $\forall a \in A, Id(a) = a$ ) is a  $k$ -anti-power morphism.

By definition of  $k$ -anti-power words, a  $k$ -anti-power morphism is a  $(k-1)$ -anti-power morphism when  $k \geq 3$ . In particular, all  $k$ -anti-power morphisms are 2-anti-power morphisms, i.e., square-free morphisms. These last morphisms have been characterized in [2].

We say that a morphism is *non-erasing* if, for all letters  $a \in A$ ,  $f(a) \neq \varepsilon$ . A  $k$ -anti-power morphism, as every square-free morphism, is necessarily non-erasing.

A morphism on  $A$  is called *prefix* (resp. *suffix*) if, for all different letters  $a$  and  $b$  in  $A$ , the word  $f(a)$  is not a prefix (resp. not a suffix) of  $f(b)$ . A prefix (resp. suffix) morphism is non-erasing. A morphism is *bifix* if it is prefix and suffix.

Proofs of the following lemmas are left to the reader.

**Lemma 1.3** *Let  $f$  be a bifix morphism on an alphabet  $A$  and let  $u, v, w$ , and  $t$  be words over  $A$ .*

*The equality  $f(u) = f(v)p$  with  $p$  be a prefix of  $f(w)$  implies  $u = vw'$  for a prefix  $w'$  of  $w$  such that  $f(w') = p$ .*

*And the equality  $f(u) = sf(v)$  with  $s$  a suffix of  $f(t)$  implies  $u = t'v$  for a suffix  $t'$  of  $t$  such that  $f(t') = s$ .*

**Lemma 1.4** *Let  $f$  be a prefix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $p_1$  (resp.  $p_2$ ) be a prefix of  $f(a)$  (resp. of  $f(b)$ ). If  $(p_1; p_2) \neq (\varepsilon; f(b))$  and if  $(p_1; p_2) \neq (f(a); \varepsilon)$  then the equality  $f(u)p_1 = f(v)p_2$  implies  $u = v$  and  $p_1 = p_2$ .*

**Lemma 1.5** *Let  $f$  be a suffix morphism on an alphabet  $A$ , let  $u$  and  $v$  be words over  $A$ , and let  $a$  and  $b$  be letters in  $A$ . Furthermore, let  $s_1$  (resp.  $s_2$ ) be a suffix of  $f(a)$  (resp. of  $f(b)$ ). If  $(s_1; s_2) \neq (\varepsilon; f(b))$  and if  $(s_1; s_2) \neq (f(a); \varepsilon)$  then the equality  $s_1f(u) = s_2f(v)$  implies  $u = v$  and  $s_1 = s_2$ .*

Taking  $p_1 = p_2 = \varepsilon$  (resp.  $s_1 = s_2 = \varepsilon$ ) in Lemma 1.4 (resp Lemma 1.5), we get that a prefix (resp. suffix) morphism is injective.

**Definition 1.6** *A morphism  $f$  from  $A^*$  to  $B^*$  is a ps-morphism (Keränen [6] called  $f$  a ps-code) if and only if the equalities*

$$f(a) = ps, f(b) = ps' \text{ and } f(c) = p's$$

*with  $a, b, c \in A$  (possibly  $c = b$ ) and  $p, s, p', s' \in B^*$  imply  $b = a$  or  $c = a$ .*

Obviously, taking  $c = b$ , and  $s = \varepsilon$  in a first time and  $p = \varepsilon$  in a second time, we obtain that a ps-morphism is a bifix morphism.

**Lemma 1.7** [6, 7] *If  $f$  is not a ps-morphism then  $f$  is not a  $k$ -power-free morphism for every integer  $k \geq 2$ .*

Since a  $k$ -anti-power morphism is a square-free morphism, we get the following Corollary.

**Corollary 1.8** *For every integer  $k \geq 2$ , a  $k$ -anti-power morphism is a ps-morphism.*

**Proposition 1.9** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 2$  and let  $f$  be a morphism from  $A^*$  to  $B^*$ . If there exist five letters  $a, b, c, d$  and  $x$  (possibly equal) and four words  $p, s, \pi$  and  $\sigma$  such that  $s$  is a suffix of  $f(a)$ ,  $p$  is a prefix of  $f(b)$ ,  $\sigma$  is a non-empty suffix of  $f(c)$ ,  $\pi$  is a non-empty prefix of  $f(d)$ , and  $sp = \sigma f(x)\pi$  then  $f$  is not a square-free morphism.*

**Proof.**

Since  $|sp| > |\sigma f(x)|$  and  $|sp| > |f(x)\pi|$ , we get  $|s| > |\sigma|$  and  $|p| > |\pi|$ .

Let  $s'$  be the non empty prefix of  $f(x)$  such that  $s = \sigma s'$  and let  $p'$  be the non empty suffix of  $f(x)$  such that  $p = p'\pi$ .

If  $x = a$  and  $x = b$ , then  $f(x)$  is an internal factor of  $f(xx)$ . By Lemma 1.2,  $f(x)$  is not primitive i.e.  $f$  is not a square-free morphism.

Therefore, at least one of the word  $ax$  or  $bx$  is not a square. But  $f(ax)$  contains the square  $s's'$  and  $f(bx)$  contains the square  $p'p'$  i.e.  $f$  is not a square-free morphism.  $\square$

**Proposition 1.10** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 3$  and let  $f$  be a  $L$ -uniform morphism from  $A^*$  to  $B^*$ . If  $L$  is an even number then  $f$  is not a 3-anti-power morphism.*

**Proof.**

Let  $a, b$  and  $c$  be three different letters in  $A$ . Let  $A_1$  and  $A_2$  be the words such that  $f(a) = A_1A_2$  with  $|A_1| = |A_2|$ .

Then  $f(abcab)$  contains the non-3-anti-power sequence  $A_2f(b)f(c)A_1A_2f(b)$  with  $abcab$  a 3-anti-power word.

$\square$

A morphism  $f$  on  $A$  is  $k$ -anti-power up to  $\ell$  ( $k, \ell \geq 2$ ) if and only if  $f(w)$  is a  $k$ -anti-power word for every  $k$ -anti-power word  $w$  over  $A$  of length at most  $\ell$ .

**Proposition 1.11** *Let  $A$  and  $B$  be two alphabets with  $\text{Card}(A) \geq 3$  and let  $f$  be a square-free  $L$ -uniform morphism from  $A^*$  to  $B^*$  (with  $L$  odd).*

*Then  $f$  is a 3-anti-power morphism if and only if it is a 3-anti-power morphism up to 5.*

**Proof.**

By definition of 3-anti-power morphisms, we only have to prove the "if" part of Proposition 1.11.

By Lemma 1.7,  $f$  (square-free) is a ps-morphism and so injective.

By contradiction, we assume that a shortest 3-anti-power word  $w$  (not necessarily unique) such that  $f(w)$  contains a non-3-anti-power satisfy  $|w| \geq 6$ . We will show that this assumption leads to contradictions.

Since the length of  $w$  is minimal, we may assume that there exist five words  $p, s, U_1, U_2$  and  $U_3$  such that  $f(w) = pU_1U_2U_3s$  where  $p$  is a prefix of  $f(w[1])$  different from  $f(w[1])$  and  $s$  is a suffix of  $f(w[|w|])$  different from  $f(w[|w|])$ . Moreover, the words  $U_1, U_2, U_3$  have the same length ( $\geq 1$ ) and two of them are equal.

If  $U_1 = U_2$  or if  $U_2 = U_3$ , it means that  $f(w)$  contains a square with  $w$  a 3-anti-power word so a square-free word. That is  $f$  is not a square-free morphisms: a contradiction with the definition of  $f$ . The only remaining case is  $U_1 = U_3$ . To simplify notations, we denote by  $U$  the words  $U_1$  and  $U_3$  and by  $V$  the word  $U_2$ .

Let  $i_2$  be the shortest integer such that  $f(w) = pU$  is the prefix of  $f(w[1..i_2])$  and let  $i_3$  be the shortest integer such that  $f(w) = pUV$  is the prefix of  $f(w[1..i_3])$ .

If  $i_2 = 1$  then  $|V| = |U| \leq |pU| \leq |f(w[1])| = L$  and  $|pUVUs| < 4L$ . This is impossible since  $|f(w)| \geq 6L$ .

On the same way, by a length criterion, the cases  $i_2 = i_3$  and  $i_3 = |w|$  are impossible.

If we denote  $x = w[2..i_2 - 1]$ ,  $y = w[i_2 + 1..i_3 - 1]$ ,  $z = w[i_3 + 1..|w| - 1]$ ,  $a_1 = w[1]$ ,  $a_2 = w[i_2]$ ,  $a_3 = w[i_3]$  and  $a_4 = w[|w|]$  then  $w = a_1xa_2ya_3za_4$  with  $|w| = 4 + |x| + |y| + |z|$ . It implies that  $|x| + |y| + |z| \geq 2$ .

Moreover, there exists some words  $p_i$  and  $s_i$  ( $1 \leq i \leq 4$ ) such that  $f(a_i) = p_i s_i$  with  $p_1 = p$ ,  $s_4 = s$ . By definition, the words  $s_1, p_2, p_3$  and  $p_4$  are non empty.

In other words, we have  $U = s_1 f(x) p_2 = s_3 f(z) p_4$  and  $V = s_2 f(y) p_3$ .

Since  $\max\{1 + L|x|; L|y|\} < |U| = |V| \leq \min\{L(|x| + 2); L(|y| + 2) - 1\}$ , we get that  $||x| - |y|| < 2$ . It means that, if  $|x| + |y|$  is even then  $|x| = |y|$  and, if  $|x| + |y|$  is odd then  $|x| = |y| \pm 1$ . In the same way, if  $|z| + |y|$  is even then  $|z| = |y|$  and, if  $|z| + |y|$  is odd then  $|z| = |y| \pm 1$ .

• *Case 1* :  $|s_1| = |s_3|$

Since  $2 \times |U| = |U| + |V| = |s_1 f(x) p_2| + |s_2 f(y) p_3| = (|x| + |y| + 2) \times L$ , we get that  $|x| + |y|$  is even i.e.  $|x| = |y|$ .

From the equality  $s_1 f(x) p_2 = s_3 f(z) p_4 (= U)$ , we get  $s_3 = s_1 (\neq \varepsilon)$ . By Lemma 1.4, it also implies  $z = x$  and  $p_2 = p_4 (\neq \varepsilon)$ .

In particular, since  $|x| = |y|$  and since  $w = a_1 x a_2 y a_3 x a_4$  is a 3-anti-power word, we have  $a_1 \neq a_3$  and  $a_2 \neq a_4$ .

Since  $|U| = |V|$ , we get  $|s_1| + |p_2| = |s_2| + |p_3|$ . Since  $|s_1| = |s_3|$ , we get  $|p_1| = |p_3|$  and  $2|p_1| = L + |p_1| - |s_1| = L + |p_3| - |s_1| = L + |p_2| - |s_2| = 2|p_2|$  i.e.  $|p_1| = |p_2|$ .

In a same way, since  $|p_2| = |p_4|$  we get  $|s_2| = |s_4| = |s_3|$ .

If  $a_1 = a_2$  then  $p_1 = p_2 (= p_4)$ ,  $f(a_1) = p_1 s_1$ ,  $f(a_3) = p_3 s_3 = p_3 s_1$  and  $f(a_4) = p_4 s_4 = p_1 s_4$ . It means that  $f(a_3 a_1 a_4)$  contains  $(s_1 p_1)^2$  with  $a_3 a_1 a_4$  square-free since  $a_1 \neq a_3$  and  $a_1 = a_2 \neq a_4$ : a contradiction with the hypothesis that  $f$  is a square-free morphism.

In the same way, if  $a_3 = a_4$ , we get that  $f(a_1 a_4 a_2)$  contains  $(s_4 p_4)^2$  with  $a_1 a_4 a_2$  square-free. And, if  $a_2 = a_3$ , we get that  $f(a_1 a_2 a_4)$  contains  $(s_2 p_2)^2$  with  $a_1 a_2 a_4$  square-free. Both cases lead to the same contradiction.

Thus  $a_1$ ,  $a_2$  and  $a_3$  are three different letters and  $a_2$ ,  $a_3$  and  $a_4$  are also three different letters. It means that  $a_1 a_2 a_3 a_4$  is a 3-anti-power word. But  $f(a_1 a_2 a_3 a_4)$  contains the non-3-anti-power sequence  $s_1 p_2 s_2 p_3 s_3 p_4 = s_1 p_2 s_2 p_3 s_1 p_2$ : a contradiction with the minimality of  $|w|$ .

- *Case 2* :  $s_1 = f(a_1)$  and  $s_3 = \varepsilon$

We have  $|f(a_1) f(x) p_2| = |s_1 f(x) p_2| = |U| = |V| = |s_2 f(y) p_3| = |s_2 f(y) f(a_3)|$  i.e.  $|p_2| = |s_2|$ : this contradicts the fact that  $L$  is odd.

- *Case 3* :  $s_1 \neq s_3$

If  $x = \varepsilon$  and  $z = \varepsilon$  then  $i_2 = 2$ ,  $i_3 = |w| - 1$  and  $|y| = |w[i_2 + 1..i_3 - 1]| = |w| - 4 \geq 2$ : this is impossible.

Thus, in the equality  $s_1 f(x) p_2 = s_3 f(z) p_4 (= U)$ , we have either  $x \neq \varepsilon$  or  $z \neq \varepsilon$ .

Since the equality is symmetric, without loss of generality, we may assume that  $|s_1| < |s_3|$ . In this case, we necessarily have  $x \neq \varepsilon$ . Let  $\chi$  be the first letter of  $x$  and let  $x'$  be the word such that  $x = \chi x'$ . If  $z = \varepsilon$ , let  $P = p_4$  and if  $z \neq \varepsilon$  let  $P = f(\gamma)$  where  $\gamma$  is the first letter of  $z$ . In particular, we have  $P$  non-empty. Let  $\pi$  be the (non empty) prefix of  $f(x') p_2$  such that  $s_1 f(\chi) \pi = s_3 P$ . By proposition 1.9, this last equation implies that  $f$  is not a square-free morphism: a final contradiction.

□

## 2 An example

As stated in the first section, Dejean's words are anti-power words. But we can build 3-power-free words without using fractionnal powers.

According to my computer, the following morphism  $h$  is a 3-anti-power morphism (but I do not really trust my programming skills).



$$\begin{aligned}
h : \{a; b; c; d; e\}^* &\rightarrow \{a; b; c; d; e\}^* \\
a &\mapsto abceacd \\
b &\mapsto abecaed \\
c &\mapsto acbaecd \\
d &\mapsto acbeabd \\
e &\mapsto acebced
\end{aligned}$$

The word  $h^\omega(a) = \lim_{n \rightarrow +\infty} h^n(a) = abceacd abecaed acbaecd acebced abceacd acbaecd acbeabd abceacd abecaed acebced acbaecd abceacd \dots$  generated by  $h$  is thus an infinite 3-anti-power word.

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