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A weak solution theory for stochastic Volterra equations of convolution type*

Eduardo Abi Jaber[†] Christa Cuchiero[‡] Martin Larsson[§] Sergio Pulido[¶]

October 29, 2019

Abstract

We obtain general weak existence and stability results for stochastic convolution equations with jumps under mild regularity assumptions, allowing for non-Lipschitz coefficients and singular kernels. Our approach relies on weak convergence in L^p spaces. The main tools are new a priori estimates on Sobolev–Slobodeckij norms of the solution, as well as a novel martingale problem that is equivalent to the original equation. This leads to generic approximation and stability theorems in the spirit of classical martingale problem theory. We also prove uniqueness and path regularity of solutions under additional hypotheses. To illustrate the applicability of our results, we consider scaling limits of nonlinear Hawkes processes and approximations of stochastic Volterra processes by Markovian semimartingales.

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[†]Ecole Polytechnique, eduardo.abi-jaber@polytechnique.edu

[‡]Vienna University of Economics and Business, christa.cuchiero@wu.ac.at

[§]Carnegie Mellon University, martinl@andrew.cmu.edu

[¶]ENSIIE & Université Paris-Saclay, sergio.pulidonino@ensiie.fr

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1 Introduction and main results

A stochastic Volterra equation of convolution type is a stochastic equation of the form

$$X_t = g_0(t) + \int_{[0,t)} K(t-s) dZ_s, \quad (1.1)$$

where X is the d -dimensional process to be solved for, g_0 is a given function, K is a given $d \times k$ matrix-valued convolution kernel, and Z is a k -dimensional Itô semimartingale whose differential characteristics are given functions of X . The solution concept is described in detail below. In particular, conditions are needed to ensure that the stochastic integral on the right-hand side of (1.1) is well-defined.

This type of equation appears in multiple applications, for example turbulence ([Barndorff-Nielsen and Schmiegel, 2008](#)), energy markets ([Barndorff-Nielsen et al., 2013](#)), and rough volatility modeling in finance ([El Euch and Rosenbaum, 2019](#); [Gatheral et al., 2018](#)). In the latter context the kernel is singular, $K(t) = t^{\gamma-1}$ with $\gamma \in (\frac{1}{2}, 1)$, and the driving semimartingale is continuous with coefficients that are just continuous functions without any Lipschitz-type regularity. Such examples fall outside the scope of classical theory, such as the results of [Berger and Mizel \(1980\)](#); [Protter \(1985\)](#); [Coutin and Decreusefond \(2001\)](#); [Wang \(2008\)](#); [Zhang \(2010\)](#). This motivated the work of [Abi Jaber et al. \(2017\)](#), although their results only apply in the path-continuous case. Equations like (1.1) also occur in the study of fractional Brownian motion.

There are however many important examples with jumps. The most basic ones are Lévy driven moving averages where the characteristics of the driving semimartingale are constant and thus do not depend on X ([Basse and Pedersen, 2009](#); [Marquardt, 2006](#)). A more complex example is the intensity λ of a Hawkes process N . Here the driving semimartingale is the Hawkes process itself, which is a counting process, and the intensity satisfies

$$\lambda_t = g_0(t) + \int_{[0,t)} K(t-s) dN_s.$$

Various multivariate and nonlinear generalizations have also been studied and applied; see Brémaud and Massoulié (1996); Daley and Vere-Jones (2003); Delattre et al. (2016) and the references there.

Solutions of (1.1) are neither semimartingales nor Markov processes in general. Classically, they are constructed using Picard iteration, but only under Lipschitz or near-Lipschitz assumptions. Alternatively, one can use scaling limits of Hawkes-type processes to generate continuous solutions for well-chosen kernels and affine characteristics (Jaisson and Rosenbaum, 2016; Gatheral and Keller-Ressel, 2019). Yet another approach is to use projections of Markovian solutions to certain degenerate stochastic partial differential equations (Abi Jaber and El Euch, 2019b; Benth et al., 2019; Cuchiero and Teichmann, 2018, 2019; Mytnik and Salisbury, 2015). In the case of affine characteristics a unified theory is presented by Cuchiero and Teichmann (2018), by lifting Volterra processes to so-called generalized Feller processes in infinite dimension. Their construction builds on approximating Brownian or complicated jump drivers by finite activity jump processes.

In this paper we also use approximation by jumps, but not via scaling limits of Hawkes processes, nor infinite dimensional lifts. Instead we work with a priori L^p estimates for solutions of (1.1), combined with a novel “Volterra” martingale problem in $\mathbb{R}^d$ that allows us to pass to weak limits in (1.1). In view of the irregular path behavior that occurs, in particular, in the presence of jumps, this identifies L^p spaces as a natural environment for the weak convergence analysis. With this approach we obtain

- existence of weak solutions for singular kernels, non-Lipschitz coefficients and general jump behavior;
- strong existence and pathwise uniqueness under Lipschitz conditions (but still singular kernels and jumps);
- convergence and stability theorems in the spirit of classical martingale problem theory, allowing for instance to study scaling limits of nonlinear Hawkes processes and to approximate stochastic Volterra processes by Markovian semimartingales;
- path regularity under certain additional conditions on the kernel and the characteristics.

Let us now describe the solution concept for (1.1). For $p \in [2, \infty)$ we denote by $L_{\text{loc}}^p = L_{\text{loc}}^p(\mathbb{R}_+, \mathbb{R}^n)$ the space of locally p -integrable functions from $\mathbb{R}_+$ to $\mathbb{R}^n$, where the dimension n of the image space will depend on the context. Let $d, k \in \mathbb{N}$ and consider the following data:

(D1) an *initial condition* $g_0: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ in L_{loc}^p ,

(D2) a *convolution kernel* $K: \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times k}$ in L_{loc}^p ,

(D3) a *characteristic triplet* (b, a, ν) of measurable maps $b: \mathbb{R}^d \rightarrow \mathbb{R}^k$ and $a: \mathbb{R}^d \rightarrow \mathbb{S}_+^k$ as well as a kernel $\nu(x, d\zeta)$ from $\mathbb{R}^d$ into $\mathbb{R}^k$ such that $\nu(x, \{0\}) = 0$ for all $x \in \mathbb{R}^d$ and, for some $c \in \mathbb{R}_+$,

$$|b(x)| + |a(x)| + \int_{\mathbb{R}^k} (1 \wedge |\zeta|^2) \nu(x, d\zeta) \leq c(1 + |x|^p), \quad x \in \mathbb{R}^d. \quad (1.2)$$

Given this data, we can now state the following key definition.

Definition 1.1. A weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) is an $\mathbb{R}^d$ -valued predictable process X , defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, that has trajectories in L_{loc}^p and satisfies

$$X_t = g_0(t) + \int_{[0,t)} K(t-s) dZ_s \quad \mathbb{P} \otimes dt\text{-a.e.} \quad (1.3)$$

for some $\mathbb{R}^k$ -valued Itô semimartingale Z with $Z_0 = 0$ whose differential characteristics (with respect to some given truncation function) are $b(X)$, $a(X)$, $\nu(X, d\zeta)$. For convenience we often refer to the pair (X, Z) as a weak L^p solution.

Due to condition (1.2), the stochastic integral in (1.3) is well-defined for almost every $t \in \mathbb{R}_+$, confirming that the definition of L^p solution makes sense. This is shown in Lemma A.3.

Throughout this section we assume $\int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) < \infty$ for all $x \in \mathbb{R}^d$ so we can use the “truncation function” $\chi(\zeta) = \zeta$. The characteristics of Z are therefore understood with respect to this function. We can now state our main result on existence of weak L^p solutions.

Theorem 1.2. Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider data (g_0, K, b, a, ν) as in (D1)–(D3). Assume b and a are continuous, and $x \mapsto |\zeta|^2 \nu(x, d\zeta)$ is continuous from $\mathbb{R}^d$ into the finite positive measures on $\mathbb{R}^k$ with the topology of weak convergence. In addition, assume there exist a constant $\eta \in (0, 1)$, a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a constant c_{LG} such that

$$\int_0^T \frac{|K(t)|^p}{t^{\eta p}} dt + \int_0^T \int_0^T \frac{|K(t) - K(s)|^p}{|t - s|^{1+\eta p}} ds dt \leq c_K(T), \quad T \geq 0, \quad (1.4)$$

and

$$\begin{aligned} |b(x)|^2 + |a(x)| + \int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) \\ + \left(\int_{\mathbb{R}^k} |\zeta|^p \nu(x, d\zeta) \right)^{2/p} \leq c_{\text{LG}}(1 + |x|^2), \quad x \in \mathbb{R}^d. \end{aligned} \quad (1.5)$$

Then there is a weak L^p solution (X, Z) of (1.1) for the data (g_0, K, b, a, ν) .

An overview of the proof of Theorem 1.2 is given below, and the formal argument is in Section 4. However, let us first mention several kernels of interest that satisfy (1.4).

Example 1.3. (i) Consider the kernel $K(t) = t^{\gamma-1}$ with $\gamma > \frac{1}{2}$, which is singular when $\gamma < 1$. Then with $\eta \in (0, (\gamma - \frac{1}{2}) \wedge 1)$ one has $2\gamma - 2\eta - 1 > 0$, and therefore

$$\int_0^T |K(t)|^2 t^{-2\eta} dt = \frac{T^{2\gamma-2\eta-1}}{2\gamma - 2\eta - 1}$$

and

$$\int_0^T \int_0^T \frac{|K(t) - K(s)|^2}{|t - s|^{1+2\eta}} ds dt = \frac{2T^{2\gamma-2\eta-1}}{2\gamma - 2\eta - 1} \int_0^1 \frac{(u^{\gamma-1} - 1)^2}{(1 - u)^{1+2\eta}} du.$$

These expressions are locally bounded in T , so (1.4) holds with $p = 2$.

(ii) Consider a locally Lipschitz kernel K with optimal Lipschitz constant L_T over $[0, T]$. Let $p \in [2, \infty)$ and choose $\eta < \frac{1}{p}$. Then

$$\int_0^T |K(t)|^p t^{-\eta p} dt \leq \max_{t \in [0, T]} |K(t)|^p \frac{T^{1-\eta p}}{1 - \eta p}$$

and

$$\int_0^T \int_0^T \frac{|K(t) - K(s)|^p}{|t - s|^{1+2\eta}} ds dt \leq L_T^p \int_0^T \int_0^T |t - s|^{p-1-2\eta} ds dt.$$

Since $1 - \eta p > 0$ and hence $p - 2\eta > 0$, these expressions are locally bounded in T . Thus (1.4) holds.

(iii) Consider two kernels K_1 and K_2 . Suppose $K_1 \in L_{\text{loc}}^p$ satisfies (1.4) for some $p \in [2, \infty)$ and $\eta \in (0, 1)$, and K_2 is locally Lipschitz. Then it is not hard to check that the product $K = K_1 K_2$ satisfies (1.4) with the same p and η as K_1 . An example of this kind is the exponentially dampened singular kernel $K(t) = t^{\gamma-1} e^{-\beta t}$ with $\gamma \in (\frac{1}{2}, 1)$ and $\beta \geq 0$. For this kernel one can take $p = 2$ and any $\eta \in (0, \gamma - \frac{1}{2})$.

The proof of Theorem 1.2 is based on approximation and weak convergence of laws on suitable function spaces. The semimartingale Z has trajectories in the Skorokhod space $D = D(\mathbb{R}_+, \mathbb{R}^k)$ of càdlàg functions. Weak convergence in D is a classical tool used, for example, to obtain weak solutions of stochastic differential equations with jumps (see, e.g., Ethier and Kurtz (2005)). However, as explained in Section 6, the trajectories of X need not be càdlàg, only locally p -integrable. Thus it is natural to regard X as a random element of the Polish space $L_{\text{loc}}^p = L_{\text{loc}}^p(\mathbb{R}_+, \mathbb{R}^d)$. It is in this space—or rather, the product space $L_{\text{loc}}^p \times D$ —that our weak convergence analysis takes place.

Relative compactness in L^p is characterized by the Kolmogorov–Riesz–Fréchet theorem; see e.g. Brezis (2010, Theorem 4.26). A more convenient criterion in our context uses the

Sobolev–Slobodeckij norms, defined for any measurable function $f: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ by

$$\|f\|_{W^{\eta,p}(0,T)} = \left(\int_0^T |f(t)|^p dt + \int_0^T \int_0^T \frac{|f(t) - f(s)|^p}{|t - s|^{1+\eta p}} ds dt \right)^{1/p},$$

where $p \geq 1$, $\eta \in (0, 1)$, $T \geq 0$ are parameters. The relation between these norms and L^p spaces is somewhat analogous to the relation between Hölder norms and spaces of continuous functions. In particular, balls with respect to $\|\cdot\|_{W^{\eta,p}(0,T)}$ are relatively compact in $L^p(0, T)$; see e.g. [Flandoli and Gatarek \(1995, Theorem 2.1\)](#). The following a priori estimate clarifies the role of the conditions (1.4) and (1.5) in Theorem 1.2, and is the key tool that allows us to obtain convergent sequences of approximate L^p solutions. The proof is given in Section 2.

Theorem 1.4. *Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider data (g_0, K, b, a, ν) as in (D1)–(D3). Assume there exists a constant c_{LG} such that (1.5) holds. Then any weak L^p solution X of (1.1) for the data (g_0, K, b, a, ν) satisfies*

$$\mathbb{E}[\|X\|_{L^p(0,T)}^p] \leq c, \quad (1.6)$$

where $c < \infty$ only depends on $d, k, p, c_{\text{LG}}, T, \|g_0\|_{L^p(0,T)}$, and, L^p -continuously, on $K|_{[0,T]}$. If in addition there exist a constant $\eta \in (0, 1)$ and a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (1.4) holds, then

$$\mathbb{E}[\|X - g_0\|_{W^{\eta,p}(0,T)}^p] \leq c, \quad (1.7)$$

where $c < \infty$ only depends on $d, k, p, \eta, c_K, c_{\text{LG}}, T$.

An immediate corollary is the following tightness result.

Corollary 1.5. *Fix $d, k, p, \eta, c_K, c_{\text{LG}}$ as in Theorem 1.4, and let $G_0 \subset L_{\text{loc}}^p$ be relatively compact. Let $\mathcal{X}$ be the set of all weak L^p solutions X of (1.1) as g_0 ranges through G_0 , K ranges through all kernels that satisfy (1.4) with the given η and c_K , and (b, a, ν) ranges through all characteristic triplets that satisfy (1.5) with the given c_{LG} . Then $\mathcal{X}$ is tight, in the sense that the family $\{\text{Law}(X): X \in \mathcal{X}\}$ is tight in $\mathcal{P}(L_{\text{loc}}^p)$.*

Proof. Fix $T \in \mathbb{R}_+$ and let c be the constant in (1.7). For any $m > 0$, Markov’s inequality gives

$$\sup_{X \in \mathcal{X}} \mathbb{P}(\|X - g_0\|_{W^{\eta,p}(0,T)} > m) \leq \frac{c}{m^p}.$$

The balls $\{f: \|f\|_{W^{\eta,p}(0,T)} \leq m\}$ are relatively compact in $L^p(0, T)$, so the above estimate implies that the family $\{(X - g_0)|_{[0,T]}: X \in \mathcal{X}\}$ is tight in $L^p(0, T)$. Since T was arbitrary, it follows that $\mathcal{X}_0 = \{X - g_0: X \in \mathcal{X}\}$ is tight in L_{loc}^p . Since G_0 is relatively compact, $G_0 + \mathcal{X}_0$ is tight as well, and it contains $\mathcal{X}$. Thus $\mathcal{X}$ is tight. $\square$

The second main ingredient in the proof of Theorem 1.2 relies on a reformulation of (1.1) as a certain martingale problem. This martingale problem is introduced in Section 3, and it is shown in Lemma 3.3 that weak L^p solutions of (1.1) can equivalently be understood as solutions of the martingale problem. This point of view is useful because it leads to the following stability result, which under appropriate conditions asserts that the weak limit of a sequence of solutions is again a solution. The proof is given at the end of Section 3. Recall that D denotes the Skorokhod space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^k$.

Theorem 1.6. *Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$. For each $n \in \mathbb{N}$, let (X^n, Z^n) be a weak L^p solution of (1.1) given data $(g_0^n, K^n, b^n, a^n, \nu^n)$ as in (D1)–(D3). Assume the triplets (b^n, a^n, ν^n) all satisfy (1.5) with a common constant c_{LG} . Assume also, for some (g_0, K, b, a, ν) and limiting process (X, Z) , that*

- $g_0^n \rightarrow g_0$ in L^p_{loc} ,
- $K^n \rightarrow K$ in L^p_{loc} ,
- $(b^n, a^n, \nu^n) \rightarrow (b, a, \nu)$ in the sense that $A^n f \rightarrow Af$ locally uniformly on $\mathbb{R}^d \times \mathbb{R}^k$ for every $f \in C_c^2(\mathbb{R}^k)$, where Af is defined in terms of the characteristic triplet by

$$\begin{aligned} Af(x, z) &= b(x)^\top \nabla f(z) + \frac{1}{2} \text{tr}(a(x) \nabla^2 f(z)) \\ &\quad + \int_{\mathbb{R}^k} (f(z + \zeta) - f(z) - \zeta^\top \nabla f(z)) \nu(x, d\zeta), \end{aligned}$$

and $A^n f$ is defined analogously,

- $(X^n, Z^n) \Rightarrow (X, Z)$ in $L^p_{\text{loc}} \times D$.

Then (X, Z) is a weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) .

It is important to appreciate that no pointwise convergence of characteristic triplets is required in Theorem 1.6. For example, it may happen that $a^n = 0$ for all n , but the limiting triplet has $a \neq 0$. This is because diffusion can be approximated by small jumps, and we indeed make use of this in a crucial manner.

By combining the tightness and stability results with an approximation scheme for the characteristic triplet, we reduce the existence question to the pure jump case where Z is piecewise constant with finite activity jumps. A solution X can then be constructed directly. The details are given in Section 4.

At this point it is natural to ask about uniqueness of solutions to (1.1). Standard counterexamples for SDEs reveal that no reasonable uniqueness statement will hold at the level of generality of Theorem 1.2. Additional assumptions are needed. In Section 5 we prove a pathwise uniqueness theorem under suitable Lipschitz conditions; see Theorem 5.3. This in turn yields uniqueness in law via the abstract machinery of Kurtz (2014) and, as

a by-product, strong existence. As for SDEs, uniqueness in the non-Lipschitz case is more delicate and not treated here. In certain situations, uniqueness in law can still be established; see for instance [Abi Jaber et al. \(2017\)](#) for the case of affine characteristics and continuous trajectories.

In Section 6 we turn to path regularity of solutions X of (1.1). Basic examples show that X can be as irregular as the kernel K itself. However, often additional information is available that allows one to assert better path regularity. Criteria of this kind are collected in Theorem 6.1.

At this stage let us mention various path regularity results for stochastic convolutions that already exist in the literature. For one-dimensional continuous kernels K , stochastic convolutions $\int_0^t K(t-s)dW_s$ with W a standard Brownian motion may fail to be locally bounded in t ([Brzeźniak et al., 2001](#), Theorem 1). However, under appropriate conditions on K , allowing in particular for certain singular kernels, a version with Hölder sample paths exists ([Abi Jaber et al., 2017](#), Lemma 2.4). If W is replaced by a pure jump process, [Rosinski \(1989, Theorem 4\)](#) showed that the stochastic convolution fails to be locally bounded whenever the kernel is singular. Similar results appear in infinite dimensions, see [Brzeźniak and Zabczyk \(2010, Theorem 7.1\)](#). Under additional regularity of the kernel, existence of Hölder continuous versions for fractional Lévy processes has been established by [Marquardt \(2006\)](#) and [Mytnik and Neuman \(2011\)](#).

Finally, in Section 7 we sketch how our results can be applied to scaling limits of Hawkes processes (Subsection 7.1) and approximations of solutions of (1.1) by means of finite-dimensional systems of Markovian SDEs (Subsection 7.2).

Some basic auxiliary results are gathered in the appendix.

2 Sobolev–Slobodeckij a priori estimate

This section is devoted to the proof of Theorem 1.4. We will need the following inequality, taken from [Marinelli and Röckner \(2014, Theorem 1\)](#). It first appeared in [Novikov \(1975, Theorem 1\)](#), but is also known as the *Bichteler–Jacod inequality* or *Kunita estimate*. We refer to [Marinelli and Röckner \(2014\)](#) for a historical survey of these maximal inequalities.

Lemma 2.1. *Let μ be a random measure with compensator ν , and define $\bar{\mu} = \mu - \nu$. For any $T \in \mathbb{R}_+$ and g such that the integral*

$$M_t = \int_{[0,t] \times \mathbb{R}^k} g(s, \zeta) \bar{\mu}(ds, d\zeta)$$

is well-defined for all $t \in [0, T]$, one has the inequality

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} |M_t|^p \right] &\leq C(p, T) \mathbb{E} \left[\int_{[0,T] \times \mathbb{R}^k} |g(s, \zeta)|^p \nu(ds, d\zeta) \right. \\ &\quad \left. + \left(\int_{[0,T] \times \mathbb{R}^k} |g(s, \zeta)|^2 \nu(ds, d\zeta) \right)^{p/2} \right], \end{aligned}$$

for any $p \geq 2$, where $C(p, T)$ only depends on p and T .

We now proceed to the proof of Theorem 1.4. Let therefore $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider (g_0, K, b, a, ν) as in (D1)–(D3). We assume there exists a constant c_{LG} such that (1.5) holds, and let (X, Z) be a weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) .

Proof of (1.6). Observe that Z admits the representation

$$Z_t = \int_0^t b(X_s) ds + M_t^c + M_t^d, \quad t \geq 0,$$

where M^c is a continuous local martingale with quadratic variation $\langle M^c \rangle = \int_0^\cdot a(X_s) ds$ and M^d is a purely discontinuous local martingale with compensator $\int_{\mathbb{R}^k} \zeta \nu(X, d\zeta)$. Define $\tau_n = \inf\{t: \int_0^t |X_s|^p ds \geq n\} \wedge T$. Since X is predictable with sample paths in L_{loc}^p , the process $\int_0^\cdot |X_s|^p ds$ is continuous, adapted, and increasing. Thus τ_n is a stopping time for every n , and $\tau_n \rightarrow \infty$. Define the process X^n by $X_t^n = X_t \mathbf{1}_{t < \tau_n}$. We then have

$$\begin{aligned} & \|X^n\|_{L^p(0, T)}^p \\ & \leq 4^{p-1} \left(\|g_0\|_{L^p(0, T)}^p + \int_0^T \left| \int_{[0, t)} K(t-s) b(X_s^n) ds \right|^p dt \right) \\ & \quad + 4^{p-1} \left(\int_0^T \left| \int_{[0, t)} K(t-s) dM_s^{c, n} \right|^p dt + \int_0^T \left| \int_{[0, t)} K(t-s) dM_s^{d, n} \right|^p dt \right) \\ & = 4^{p-1} \left(\|g_0\|_{L^p(0, T)}^p + \int_0^T (\mathbf{I}_t + \mathbf{II}_t + \mathbf{III}_t) dt \right), \end{aligned}$$

where $M^{c, n}$ has quadratic variation equal to $\int_0^\cdot a(X_s^n) ds$, and the jump measure of $M^{d, n}$ has compensator $\nu(X^n, d\zeta)$. An application of the Jensen and BDG inequalities combined with Fubini's theorem and (1.5) leads to

$$\mathbb{E}[\mathbf{I}_t] + \mathbb{E}[\mathbf{II}_t] \leq C(c_{\text{LG}}, p, T) \int_0^t |K(t-s)|^p (1 + \mathbb{E}[|X_s^n|^p]) ds,$$

for every $t \leq T$. Thanks to Novikov's inequality, see Lemma 2.1, we have

$$\begin{aligned} \mathbb{E}[\mathbf{III}_t] & \leq \mathbb{E} \left[\sup_{r \leq t} \left| \int_{[0, r)} K(t-s) dM_s^{d, n} \right|^p \right] \\ & \leq C(p, t) \int_0^t |K(t-s)|^p \mathbb{E} \left[\int_{\mathbb{R}^k} |\zeta|^p \nu(X_s^n, d\zeta) + \left(\int_{\mathbb{R}^k} |\zeta|^2 \nu(X_s^n, d\zeta) \right)^{p/2} \right] ds \\ & \leq C(c_{\text{LG}}, p, t) \int_0^t |K(t-s)|^p (1 + \mathbb{E}[|X_s^n|^p]) ds, \end{aligned}$$

for every $t \leq T$, where the last inequality follows from (1.5). Combining the above yields

$$\begin{aligned} \mathbb{E}[\|X^n\|_{L^p(0,T)}^p] &\leq C(c_{LG}, p, T) \\ &\times \left(\|g_0\|_{L^p(0,T)}^p + \int_0^T \int_0^t |K(t-s)|^p (1 + \mathbb{E}[|X_s^n|^p]) ds dt \right). \end{aligned}$$

Multiple changes of variables and applications of Tonelli's theorem yield

$$\begin{aligned} &\int_0^T \int_0^t |K(t-s)|^p (1 + \mathbb{E}[|X_s^n|^p]) ds dt \\ &= \int_0^T |K(s)|^p \int_0^{T-s} (1 + \mathbb{E}[|X_t^n|^p]) dt ds \\ &\leq T \|K\|_{L^p(0,T)}^p + \int_0^T |K(T-s)|^p \mathbb{E}[\|X^n\|_{L^p(0,s)}^p] ds. \end{aligned}$$

We deduce that the function $f_n(t) = \mathbb{E}[\|X^n\|_{L^p(0,t)}^p]$ satisfies the convolution inequality

$$f_n(t) \leq C(c_{LG}, p, T) \left(\|g_0\|_{L^p(0,T)}^p + T \|K\|_{L^p(0,T)}^p \right) - (\widehat{K} * f_n)(t),$$

where $\widehat{K} = -C(c_{LG}, p, T)|K|^p$ lies in $L^1(0, T)$. The resolvent $\widehat{R}$ of $\widehat{K}$ is nonpositive and lies in $L^1(0, T)$; see Gripenberg et al. (1990, Theorem 2.3.1 and its proof). Moreover, $f_n \leq n$ by construction. Thus the Gronwall lemma for convolution inequalities applies; see Lemma A.2. In particular, we have

$$f_n(T) \leq C(c_{LG}, p, T) \left(\|g_0\|_{L^p(0,T)}^p + T \|K\|_{L^p(0,T)}^p \right) \left(1 + \|\widehat{R}\|_{L^1(0,T)} \right).$$

As $n \rightarrow \infty$ we have $\tau_n \rightarrow \infty$, and hence $f_n(T) \rightarrow \mathbb{E}[\|X\|_{L^p(0,T)}^p]$ by monotone convergence. We deduce (1.6), as desired. Finally, the continuous dependence on $K|_{[0,T]}$ follows from Gripenberg et al. (1990, Theorem 2.3.1), which implies that the map from $L^p(0, T)$ to $\mathbb{R}$ that takes $K|_{[0,T]}$ to $\|\widehat{R}\|_{L^1(0,T)}$ is continuous. $\square$

For the proof of the second part of Theorem 1.4, namely (1.7), will need the following estimate.

Lemma 2.2. *Let $K: \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times k}$ be measurable. For any $\eta > 0$, $T \in \mathbb{R}_+$, $p \geq 2$ and nonnegative measurable function f , one has*

$$\begin{aligned} &\int_0^T \int_0^T \int_{s \wedge t}^{s \vee t} \frac{|K(s \vee t - u)|^p}{|t - s|^{1+p\eta}} f(u) du ds dt \\ &\leq \|f\|_{L^1(0,T)} \frac{1}{\eta} \int_0^T |K(t)|^p t^{-\eta p} dt \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} & \int_0^T \int_0^T \int_0^{s \wedge t} \frac{|K(t-u) - K(s-u)|^p}{|t-s|^{1+\eta p}} f(u) du ds dt \\ & \leq \|f\|_{L^1(0,T)} \int_0^T \int_0^T \frac{|K(t) - K(s)|^p}{|t-s|^{1+\eta p}} ds dt \end{aligned} \quad (2.2)$$

Proof. We first prove (2.1). Since $\int_{s \wedge t}^{s \vee t} (\dots) du = \int_0^T (\mathbf{1}_{s < u < t} + \mathbf{1}_{t < u < s})(\dots) du$, we may re-write the left-hand side of (2.1) as

$$2 \int_0^T \int_0^T \int_0^{s \wedge t} \mathbf{1}_{s < u < t} \frac{|K(t-u)|^p}{(t-s)^{1+\eta p}} f(u) du ds dt.$$

By Tonelli's theorem this equals

$$\begin{aligned} & 2 \int_0^T f(u) \int_u^T |K(t-u)|^p \int_0^u \frac{1}{(t-s)^{1+\eta p}} ds dt du \\ & = \frac{2}{\eta p} \int_0^T f(u) \int_u^T |K(t-u)|^p ((t-u)^{-\eta p} - t^{-\eta p}) dt du \\ & \leq \frac{1}{\eta} \int_0^T f(u) \int_u^T |K(t-u)|^p (t-u)^{-\eta p} dt du. \end{aligned}$$

Since $\int_u^T |K(t-u)|^p (t-u)^{-\eta p} dt = \int_0^{T-u} |K(v)|^p v^{-\eta p} dv \leq \int_0^T |K(v)|^p v^{-\eta p} dv$, it follows that (2.1) holds, as claimed.

We now prove (2.2). Since $\int_0^{s \wedge t} (\dots) du = \int_0^T \mathbf{1}_{u < s} \mathbf{1}_{u < t} (\dots) du$, and by using Tonelli's theorem, we find that the left-hand side of (2.2) is equal to

$$\int_0^T f(u) \int_u^T \int_u^T \frac{|K(t-u) - K(s-u)|^p}{|t-s|^{1+\eta p}} ds dt du.$$

By a change of variables one sees that this is bounded by the right-hand side of (2.2), as claimed. $\square$

The proof of the second part of Theorem 1.4 is now straightforward. In addition to the above, we assume there exist a constant $\eta \in (0, 1)$ and a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (1.4) holds.

Proof of (1.7). Set $\bar{X} = X - g_0$ and observe that

$$\begin{aligned} |\bar{X}_t - \bar{X}_s| & \leq \left| \int_{[0, s \wedge t)} (K(t-u) - K(s-u)) dZ_u \right| \\ & \quad + \left| \int_{[s \wedge t, s \vee t)} K(s \vee t - u) dZ_u \right|, \quad \mathbb{P} \otimes dt \otimes ds\text{-a.e.} \end{aligned}$$

A similar argument as in the proof of (1.6) shows that $\mathbb{E}[\int_0^T \int_0^T \frac{|\bar{X}_t - \bar{X}_s|^p}{|t-s|^{1+\eta p}} ds dt]$ is bounded above by

$$C(c_{LG}, p, T) \left(\int_0^T \int_0^T \int_0^{s \wedge t} \frac{|K(t-u) - K(s-u)|^p (1 + \mathbb{E}[|X_u|^p])}{|t-s|^{1+\eta p}} du ds dt \right. \\ \left. + \int_0^T \int_0^T \int_{s \wedge t}^{s \vee t} \frac{|K(s \vee t - u)|^p (1 + \mathbb{E}[|X_u|^p])}{|t-s|^{1+\eta p}} du ds dt \right).$$

Applying (1.6), as well as Lemma 2.2 with $f(u) = 1 + \mathbb{E}[|X_u|^p]$, we obtain the bound (1.7) with a constant $c < \infty$ that depends on $d, k, p, \eta, c_K, c_{LG}, T$ as well as, L^p -continuously, on $K|_{[0,T]}$. Note that the set of restrictions $K|_{[0,T]}$ of kernels that satisfy (1.4) with the given c_K is relatively compact in $L^p(0, T)$. By maximizing the bound over all such K , we obtain a bound that only depends on $d, k, p, \eta, c_K, c_{LG}, T$. $\square$

3 Martingale problem and stability

We consider initial conditions g_0 and convolution kernels K as in (D1)–(D2) of Section 1, as well as linear operators A that map functions $f \in C_c^2(\mathbb{R}^k)$ to measurable functions $Af: \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$, and satisfy the following growth bound for some $p \in [1, \infty)$:

$$\text{For every } f \in C_c^2(\mathbb{R}^k) \text{ there is a finite constant } c_f \text{ such that} \quad (3.1) \\ |Af(x, z)| \leq c_f(1 + |x|^p) \text{ for all } (x, z) \in \mathbb{R}^d \times \mathbb{R}^k.$$

Note that (3.1) ensures that $Af(\mathbf{x}, \mathbf{z}) \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R})$ for any pair of functions $(\mathbf{x}, \mathbf{z}) \in L_{\text{loc}}^p \times D$.

Definition 3.1. *Let $p \in [1, \infty)$. A solution of the local martingale problem for (g_0, K, A) is a pair (X, Z) of processes with trajectories in $L_{\text{loc}}^p \times D$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, such that X is predictable, Z is adapted with $Z_0 = 0$, the process*

$$M_t^f = f(Z_t) - \int_0^t Af(X_s, Z_s) ds, \quad t \geq 0, \quad (3.2)$$

is a local martingale for every $f \in C_c^2(\mathbb{R}^k)$, and one has the equality

$$\int_0^t X_s ds = \int_0^t g_0(s) ds + \int_0^t K(t-s) Z_s ds, \quad t \geq 0. \quad (3.3)$$

Note that both the left- and right-hand sides of (3.3) are continuous in t and equal to zero for $t = 0$. For the convolution $\int_0^t K(t-s) Z_s ds$, this follows because K is in L_{loc}^1 and the trajectories of Z are in L_{loc}^∞ ; see Gripenberg et al. (1990, Corollary 2.2.3).

Our first goal is to establish the equivalence between weak L^p solutions of (1.1) and solutions of the local martingale problem. The relevant operator A is given by

$$\begin{aligned} Af(x, z) &= b(x)^\top \nabla f(z) + \frac{1}{2} \operatorname{tr}(a(x) \nabla^2 f(z)) \\ &\quad + \int_{\mathbb{R}^k} \left(f(z + \zeta) - f(z) - \chi(\zeta)^\top \nabla f(z) \right) \nu(x, d\zeta), \end{aligned} \quad (3.4)$$

where (b, a, ν) is the given characteristic triplet and χ is the truncation function. This equivalence will allow us to establish Theorem 1.6 by proving a stability theorem for solutions of local martingale problems; see Theorem 3.4 below. The latter is easier, because the conditions (3.2) and (3.3) are more easily shown to be closed with respect to suitable perturbations of X , Z , g_0 , K and A .

Lemma 3.2. *Let $p \in [2, \infty)$. Consider a kernel $K \in L^2_{\text{loc}}$ and a characteristic triplet (b, a, ν) satisfying (1.2). Let X be a predictable process with trajectories in L^p_{loc} and let Z be an Itô semimartingale whose differential characteristics with respect to some given truncation function χ are $b(X)$, $a(X)$, $\nu(X, d\zeta)$. Then $\int_{[0, t)} K(t-s) dZ_s$ is well-defined for almost every $t \in \mathbb{R}_+$, and*

$$\int_0^t \left(\int_{[0, s)} K(s-u) dZ_u \right) ds = \int_0^t K(t-s) Z_s ds, \quad t \geq 0.$$

Proof. The stochastic integral $\int_{[0, t)} K(t-s) dZ_s$ is well-defined for a.e. $t \in \mathbb{R}_+$ by Lemma A.3. Define $\kappa(x) = |b(x)| + |a(x)| + \int_{\mathbb{R}^k} (1 \wedge |\zeta|^2) \nu(x, d\zeta)$. The bound (1.2) and a change of variables yield

$$\begin{aligned} &\int_0^t \left(\int_0^t |K(s-u)|^2 1_{\{u < s\}} ds \right) \kappa(X_u) du \\ &\leq c \int_0^t \left(\int_0^t |K(v)|^2 1_{\{v < t-u\}} dv \right) (1 + |X_u|^p) du \\ &\leq c \|K\|_{L^2(0, t)}^2 (t + \|X\|_{L^p(0, t)}^p) < \infty. \end{aligned}$$

This implies that the stochastic integral

$$\int_0^t \left(\int_0^t |K(s-u)|^2 1_{\{u < s\}} ds \right)^{\frac{1}{2}} dZ_u = \int_0^t \left(\int_0^{t-u} |K(v)|^2 dv \right)^{\frac{1}{2}} dZ_u$$

is well defined. Two applications of the stochastic Fubini theorem in Protter (2005, Theo-

rem 65) along with two changes of variables yield

$$\begin{aligned}
\int_0^t \left(\int_{[0,s)} K(s-u) dZ_u \right) ds &= \int_0^t \left(\int_0^t K(s-u) 1_{\{u < s\}} ds \right) dZ_u \\
&= \int_0^t \left(\int_0^{t-u} K(v) dv \right) dZ_u \\
&= \int_0^t K(v) \left(\int_0^{t-v} dZ_u \right) dv \\
&= \int_0^t K(t-s) Z_s ds.
\end{aligned}$$

This completes the proof. $\square$

We can now prove the equivalence of weak L^p solutions and solutions of the local martingale problem.

Lemma 3.3. *Let $p \in [2, \infty)$ and consider data (g_0, K, b, a, ν) as in (D1)–(D3) and a truncation function χ . A pair (X, Z) is a weak L^p solution of (1.1) if and only if it is a solution of the local martingale problem for (g_0, K, A) , where A is given by (3.4).*

Proof. Suppose first (X, Z) is a weak L^p solution of (1.1). Itô's formula applied to Z shows that the process M^f in (3.2) is a local martingale for every $f \in C_c^2(\mathbb{R}^k)$; see Jacod and Shiryaev (2003, Theorem II.2.42 (a) $\Rightarrow$ (c)). Furthermore, integrating both sides of (1.3) and invoking Lemma 3.2 yields (3.3). Thus (X, Z) is a solution of the local martingale problem.

Conversely, suppose (X, Z) is a solution of the local martingale problem for (g_0, K, A) . Lemma 3.2 and (3.3) yield

$$\int_0^T X_t dt = \int_0^T \left(g_0(t) + \int_{[0,t)} K(t-s) dZ_s \right) dt$$

for any $T > 0$. This implies (1.3). It remains to check that Z is a semimartingale with differential characteristics $b(X), a(X), \nu(X, d\zeta)$ with respect to χ . This will follow from Jacod and Shiryaev (2003, Theorem II.2.42 (c) $\Rightarrow$ (a)), once we prove that M^f given in (3.2) is a local martingale not only for all $f \in C_c^2(\mathbb{R}^k)$, but for all $f \in C_b^2(\mathbb{R}^k)$, i.e. bounded functions which are continuously twice differentiable. Observe that M^f remains well-defined thanks to (1.2). We adapt the proof of Cheridito et al. (2005, Proposition 3.2). Consider the stopping times

$$\begin{aligned}
T_m &= \inf\{t \geq 0: \int_0^t (1 + |X_s|^p) ds \geq m\}, \\
S_m &= \inf\{t \geq 0: |Z_{t-}| \geq m \text{ or } |Z_t| \geq m\}, \\
\tau_m &= T_m \wedge S_m,
\end{aligned}$$

for $m \geq 1$. It is clear that $\tau_m \rightarrow \infty$ as $m \rightarrow \infty$. Fix any function $f \in C_b^2(\mathbb{R}^k)$. Fix also functions $\varphi_n \in C_c^2(\mathbb{R}^k)$ taking values in $[0, 1]$ and equal to one on the centered ball $B(0, n)$ of radius n . Then $f\varphi_n \in C_c^2(\mathbb{R}^k)$, so that $M^{f\varphi_n}$ defined as in (3.2) is a local martingale for each n . Write $M_t^{n,m} = M_{t \wedge \tau_m}^{f\varphi_n}$. We then have for $n, m \in \mathbb{N}$

$$|M_t^{n,m}| \leq \|f\|_\infty + mc_n, \quad t \geq 0,$$

where the constant c_n comes from (1.2) and depends on n . Hence, $M^{n,m}$ is a true martingale for each $m, n \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and set $M_t^m = M_{t \wedge \tau_m}^f$. For all $n > m$, by definition of T_m and the fact that $\varphi_n = 1$ on $B(0, n)$ we have

$$M_t^m - M_t^{n,m} = \int_{(0, t \wedge \tau_m] \times \mathbb{R}^k} (f(Z_s + \zeta) - (f\varphi_n)(Z_s + \zeta)) \nu(X_s, d\zeta) ds.$$

Thus

$$|M_t^m - M_t^{n,m}| \leq \|f\|_\infty \int_{(0, t \wedge \tau_m] \times \mathbb{R}^k} \mathbf{1}_{|\zeta| \geq n-m} \nu(X_s, d\zeta) ds.$$

As $n \rightarrow \infty$, the right-hand side tends to zero in $L^1(\mathbb{P})$, by virtue of the dominated convergence theorem. Indeed, $1 \wedge |\zeta|^2 \geq \mathbf{1}_{|\zeta| \geq n-m} \rightarrow 0$ as $n \rightarrow \infty$, and it follows from (1.2) that

$$\int_{(0, t \wedge \tau_m] \times \mathbb{R}^d} (1 \wedge |\zeta|^2) \nu(X_s, d\zeta) ds \leq c \int_0^{t \wedge \tau_m} (1 + |X_s|^p) ds \leq cm.$$

We conclude that $\mathbb{E}[|M_t^{n,m} - M_t^m|] \rightarrow 0$ as $n \rightarrow \infty$. Thus $M_t^m = M_{t \wedge \tau_m}^f$ is a martingale being an $L^1(\mathbb{P})$ -limit of martingales. Thus M^f is a local martingale, as required. $\square$

The following is our main result on stability for solutions of local martingale problems. Together with Lemma 3.3, it will imply Theorem 1.6. We let $\mathbf{x} = (\mathbf{x}(t))_{t \geq 0}$ and $\mathbf{z} = (\mathbf{z}(t))_{t \geq 0}$ denote generic elements of L_{loc}^p and D , respectively.

Theorem 3.4. *Let $d, k \in \mathbb{N}$, $p \in (1, \infty)$. Consider data (g_0^n, K^n, A^n) for $n \in \mathbb{N}$ and (g_0, K, A) , and assume that the A^n satisfy (3.1) with constants c_f that do not depend on n . For each n , let (X^n, Z^n) be a solution of the local martingale problem for (g_0^n, K^n, A^n) . Assume that*

- $g_0^n \rightarrow g_0$ in L_{loc}^p ,
- $K^n \rightarrow K$ in L_{loc}^p ,
- $A^n f \rightarrow A f$ locally uniformly on $\mathbb{R}^d \times \mathbb{R}^k$ for every $f \in C_c^2(\mathbb{R}^k)$,
- $(X^n, Z^n) \Rightarrow (X, Z)$ in $L_{\text{loc}}^p \times D$ for some limiting process (X, Z) .

Then (X, Z) is a solution of the local martingale problem for (g_0, K, A) .

Proof. Let $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t \geq 0}, \mathbb{P}^n)$ be the filtered probability space where (X^n, Z^n) is defined. We may assume without loss of generality that this space supports an $\mathcal{F}_0^n$ -measurable standard uniform random variable U^n that is independent of (X^n, Z^n) . We then have $(U^n, X^n, Z^n) \Rightarrow (U, X, Z)$ in $[0, 1] \times L_{\text{loc}}^p \times D$, where U is standard uniform and independent of (X, Z) . The standard uniform random variable U will be used below as a randomization device to avoid the jumps of Z .

Fix $f \in C_c^2(\mathbb{R}^k)$ and $m \in \mathbb{N}$. For any $(u, \mathbf{x}) \in [0, 1] \times L_{\text{loc}}^p$, define

$$\tau(u, \mathbf{x}) = \inf\{t \geq 0: \int_0^t (u + 1 + |\mathbf{x}(s)|^p) ds \geq m\}.$$

Then $\tau(U^n, X^n)$ is a stopping time in $(\mathcal{F}_t^n)_{t \geq 0}$, and the growth bound (3.1) yields

$$\int_0^{t \wedge \tau(U^n, X^n)} |A^n f(X_s^n, Z_s^n)| ds \leq c_f \int_0^{t \wedge \tau(U^n, X^n)} (1 + |X_s^n|^p) ds \leq mc_f.$$

Thus the local martingale

$$M_t^n = f(Z_{t \wedge \tau(U^n, X^n)}^n) - \int_0^{t \wedge \tau(U^n, X^n)} A^n f(X_s^n, Z_s^n) ds, \quad t \geq 0,$$

satisfies

$$|M_t^n| \leq \|f\|_\infty + mc_f, \quad t \geq 0. \quad (3.5)$$

In particular it is a true martingale, so for any time points $0 \leq t_1 < \dots < t_k \leq s < t$, and functions $h \in C([0, 1])$ and $g_i \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, $i = 1, \dots, k$, we have

$$\mathbb{E} \left[(M_t^n - M_s^n) h(U^n) \prod_{i=1}^k g_i \left(\int_0^{t_i} X_r^n dr, Z_{t_i}^n \right) \right] = 0, \quad (3.6)$$

where $\mathbb{E}$ is understood as expectation under $\mathbb{P}^n$.

Next, by Skorokhod's representation theorem (see Billingsley (1999, Theorem 6.7)), we may assume that all the triplets (U^n, X^n, Z^n) and (U, X, Z) are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that $(U^n, X^n, Z^n) \rightarrow (U, X, Z)$ in $[0, 1] \times L_{\text{loc}}^p \times D$ almost surely, and that each triplet has the same law under $\mathbb{P}$ as it did under $\mathbb{P}^n$.¹ In particular, (3.6) still holds, now with $\mathbb{E}$ understood as expectation under $\mathbb{P}$.

We now prepare to pass to the limit in (3.6). One easily checks that the map $(u, \mathbf{x}) \mapsto \tau(u, \mathbf{x})$ is continuous. Combined with Lemma 3.6 below, it follows that

$$\int_0^{t \wedge \tau(U^n, X^n)} A^n f(X_r^n, Z_r^n) dr \rightarrow \int_0^{t \wedge \tau(U, X)} A f(X_r, Z_r) dr$$

¹We can however *not* assume that the filtrations $(\mathcal{F}_t^n)_{t \geq 0}$ are the same.

almost surely for any $t \geq 0$. Moreover, Z is continuous at $\tau(U, X)$, almost surely. To see this, let $\{T_i(\mathbf{z}): i \in \mathbb{N}\}$ denote an enumeration of the countably many jump times of the function $\mathbf{z} \in D$. We choose $T_i(\mathbf{z})$ measurable in $\mathbf{z}$. Since U and (X, Z) are independent, and since for any $\mathbf{x} \in L^p_{\text{loc}}$ the law of $\tau(U, \mathbf{x})$ has no atoms, we get

$$\mathbb{P}(\tau(U, X) = T_i(Z)) = \mathbb{E} [\mathbb{P}(\tau(U, \mathbf{x}) = T_i(\mathbf{z}))|_{(\mathbf{x}, \mathbf{z})=(X, Z)}] = 0.$$

Thus

$$\mathbb{P}(\tau(U, X) \in \{T_i(\mathbf{z}): i \in \mathbb{N}\}) \leq \sum_{i \in \mathbb{N}} \mathbb{P}(\tau(U, X) = T_i(Z)) = 0,$$

showing that Z is indeed continuous at $\tau(U, X)$, almost surely. We conclude that

$$M_t^n \rightarrow M_t$$

almost surely for any $t \in \mathcal{C}(Z) = \{r \in \mathbb{R}_+ : \mathbb{P}(Z_r = Z_{r-}) = 1\}$, where we define

$$M_t = f(Z_{t \wedge \tau(U, X)}) - \int_0^{t \wedge \tau(U, X)} Af(X_s, Z_s) ds, \quad t \geq 0.$$

Selecting $0 \leq t_1 < \dots < t_k \leq s < t$ from $\mathcal{C}(Z)$, we may thus use the bounded convergence theorem, justified by (3.5), to pass to the limit in (3.6) to obtain

$$\mathbb{E} \left[(M_t - M_s) h(U) \prod_{i=1}^k g_i \left(\int_0^{t_i} X_r dr, Z_{t_i} \right) \right] = 0. \quad (3.7)$$

By Ethier and Kurtz (2005, Theorem 3.7.7), $\mathcal{C}(Z)$ is dense in $\mathbb{R}_+$. Along with right-continuity of M and Z , this implies that (3.7) actually holds for any choice of times points $0 \leq t_1 < \dots < t_k \leq s < t$. Thus M is a martingale with respect to the filtration given by

$$\mathcal{F}_t = \sigma(U) \vee \sigma \left(\int_0^s X_r dr, Z_s : s \leq t \right), \quad t \geq 0.$$

Since $\tau(U, X)$ is a stopping time for this filtration, and since the constant m in the definition of $\tau(U, X)$ was arbitrary, the process M^f in (3.2) is a local martingale.

We must also verify (3.3). This is immediate from L^p convergence of g_0^n and X^n as well as Lemma 3.5 below. This lets us pass to the limit in the identity $\int_0^t X_s^n ds = \int_0^t g_0^n(s) ds + \int_0^t K^n(t-s) Z_s^n ds$, which is valid by assumption.

It only remains to ensure that Z is adapted and X is predictable. Adaptedness of Z holds by definition of the filtration. It is however not clear that X is predictable. Therefore, we replace X by the process $\tilde{X} = \liminf_{h \downarrow 0} \tilde{X}^h$, where for each $h > 0$ we define

$$\tilde{X}_t^h = \frac{1}{h} \int_{(t-h) \vee 0}^t X_s ds, \quad t \geq 0.$$

Note that $\tilde{X}$ is predictable, being the pointwise liminf of the continuous and adapted processes $\tilde{X}^h$. Moreover, for every fixed ω , the trajectory $\tilde{X}(\omega)$ coincides with $X(\omega)$ almost everywhere by Lebesgue's differentiation theorem. Replacing X by $\tilde{X}$ therefore does not affect either (3.3) or the local martingale property in (3.2). $\square$

The following two lemmas were used in the proof of Theorem 3.4. The first one uses the convolution notation $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Lemma 3.5. *Fix $p \in (1, \infty)$. If $K^n \rightarrow K$ in $L_{\text{loc}}^p(\mathbb{R}_+, \mathbb{R}^{d \times k})$ and $\mathbf{z}_n \rightarrow \mathbf{z}$ in D , then $K^n * \mathbf{z}_n \rightarrow K * \mathbf{z}$ locally uniformly.*

Proof. Fix any $T \in \mathbb{R}_+$ and let $q \in (1, \infty)$ satisfy $p^{-1} + q^{-1} = 1$. The triangle inequality and Young's inequality, see Lemma A.1 with $r = \infty$, give

$$\begin{aligned} \|K * \mathbf{z} - K^n * \mathbf{z}_n\|_{L^\infty(0, T)} &\leq \|K * (\mathbf{z} - \mathbf{z}_n)\|_{L^\infty(0, T)} + \|(K - K^n) * \mathbf{z}_n\|_{L^\infty(0, T)} \\ &\leq \|K\|_{L^p(0, T)} \|\mathbf{z} - \mathbf{z}_n\|_{L^q(0, T)} + \|K - K^n\|_{L^p(0, T)} \|\mathbf{z}_n\|_{L^q(0, T)}. \end{aligned}$$

Since $\mathbf{z}_n \rightarrow \mathbf{z}$ in D , we have $\sup_n \|\mathbf{z} - \mathbf{z}_n\|_{L^\infty(0, T)} < \infty$ and $\mathbf{z}_n(t) \rightarrow \mathbf{z}(t)$ for almost every $t \in [0, T]$. Hence $\mathbf{z}_n \rightarrow \mathbf{z}$ in $L^q(0, T)$ by the dominated convergence theorem. Since $K * \mathbf{z}$ and $K^n * \mathbf{z}_n$ are continuous functions due to Gripenberg et al. (1990, Corollary 2.2.3), the $L^\infty(0, T)$ norm coincides with the supremum norm on $[0, T]$. The result follows. $\square$

Lemma 3.6. *Fix $d, k \in \mathbb{N}$, $p \in [1, \infty)$. Let $g_n: \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ be continuous functions satisfying the following polynomial growth condition: For every compact subset $Q \subset \mathbb{R}^k$, there exists a constant $c_Q \in \mathbb{R}_+$ such that*

$$|g_n(x, z)| \leq c_Q(1 + |x|^p), \quad (n, x, z) \in \mathbb{N} \times \mathbb{R}^d \times Q. \quad (3.8)$$

Assume that $g_n \rightarrow g$ locally uniformly for some function $g: \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$. Then, whenever $(\mathbf{x}_n, \mathbf{z}_n) \rightarrow (\mathbf{x}, \mathbf{z})$ in $L_{\text{loc}}^p \times D$, we have

$$\int_0^t g_n(\mathbf{x}_n(s), \mathbf{z}_n(s))ds \rightarrow \int_0^t g(\mathbf{x}(s), \mathbf{z}(s))ds$$

locally uniformly in $t \in \mathbb{R}_+$.

Proof. Suppose $(\mathbf{x}_n, \mathbf{z}_n) \rightarrow (\mathbf{x}, \mathbf{z})$ in $L_{\text{loc}}^p \times D$. Fix $T \in \mathbb{R}_+$, let $Q \subset \mathbb{R}^k$ be a compact set that contains the values attained by $\mathbf{z}$ and $\mathbf{z}_n$, $n \in \mathbb{N}$, over $[0, T]$, and let c_Q be the

associated constant in (3.8). Let $R \in [1, \infty)$ be an arbitrary constant, and write

$$\begin{aligned}
& \int_0^T |g_n(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}(s), \mathbf{z}(s))| ds \\
& \leq \int_0^T |g_n(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}_n(s), \mathbf{z}_n(s))| \mathbf{1}_{|\mathbf{x}_n(s)| \leq R} ds \\
& \quad + \int_0^T |g_n(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}_n(s), \mathbf{z}_n(s))| \mathbf{1}_{|\mathbf{x}_n(s)| > R} ds \\
& \quad + \int_0^T |g(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}(s), \mathbf{z}_n(s))| \mathbf{1}_{|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)| \leq R} ds \\
& \quad + \int_0^T |g(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}(s), \mathbf{z}_n(s))| \mathbf{1}_{|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)| > R} ds \\
& \quad + \int_0^T |g(\mathbf{x}(s), \mathbf{z}_n(s)) - g(\mathbf{x}(s), \mathbf{z}(s))| ds \\
& = \mathbf{I}_n + \mathbf{II}_n + \mathbf{III}_n + \mathbf{IV}_n + \mathbf{V}_n.
\end{aligned}$$

We bound these terms individually. First, defining the compact set $Q_R = \overline{B(0, R)} \times Q$, where $\overline{B(0, R)} = \{x \in \mathbb{R}^d: |x| \leq R\}$ is the centered closed ball of radius R , we have

$$\mathbf{I}_n \leq T \sup_{(x, z) \in Q_R} |g_n(x, z) - g(x, z)| \rightarrow 0 \quad (n \rightarrow \infty).$$

Next, consider the restrictions $\mathbf{x}_n|_{[0, T]}$, again denoted by $\mathbf{x}_n$ for simplicity; they are convergent in $L^p(0, T)$. The Vitali convergence theorem implies that $\{|\mathbf{x}_n|^p: n \in \mathbb{N}\}$ is uniformly integrable. Since g satisfies the same polynomial growth condition (3.8) as the g_n and since $R \geq 1$, we then get

$$\mathbf{II}_n \leq 4c_Q \int_0^T |\mathbf{x}_n(s)|^p \mathbf{1}_{|\mathbf{x}_n(s)| > R} ds \leq \varphi_{\mathbf{II}}(R^p),$$

where $\varphi_{\mathbf{II}}(R^p) = 4c_Q \sup_n \int_0^T |\mathbf{x}_n(s)|^p \mathbf{1}_{|\mathbf{x}_n(s)|^p > R^p} ds$ converges to zero as $R \rightarrow \infty$ by the definition of uniform integrability. In a similar manner, we get

$$\mathbf{IV}_n \leq 4c_Q \int_0^T (|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)|)^p \mathbf{1}_{|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)| > R} ds \leq \varphi_{\mathbf{IV}}(R^p),$$

where $\varphi_{\mathbf{IV}}(R^p) = 4c_Q \sup_n \int_0^T (|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)|)^p \mathbf{1}_{|\mathbf{x}_n(s)| \vee |\mathbf{x}(s)| > R} ds$ also converges to zero as $R \rightarrow \infty$.

We now turn to $\mathbf{III}_n$. Let $\omega_R: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous strictly increasing concave function with $\omega_R(0) = 0$ such that

$$\sup_{z \in Q} |g(x, z) - g(y, z)| \leq \omega_R(|x - y|), \quad x, y \in \overline{B(0, R)}.$$

Such a function exists because g is uniformly continuous on the compact set Q_R . Its inverse ω_R^{-1} exists and is convex, so by using Jensen's inequality we get

$$\begin{aligned} \mathbf{III}_n &\leq \int_0^T \omega_R(|\mathbf{x}_n(s) - \mathbf{x}(s)|) ds \\ &= T \omega_R \circ \omega_R^{-1} \left(\int_0^T \omega_R(|\mathbf{x}_n(s) - \mathbf{x}(s)|) \frac{ds}{T} \right) \\ &\leq T \omega_R \left(\int_0^T |\mathbf{x}_n(s) - \mathbf{x}(s)| \frac{ds}{T} \right) \\ &\rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Finally, consider $\mathbf{V}_n$. Since $\mathbf{z}_n \rightarrow \mathbf{z}$ in D , we have $\mathbf{z}_n(s) \rightarrow \mathbf{z}(s)$ for almost every $s \in \mathbb{R}_+$. Thus the integrand in $\mathbf{V}_n$ converges to zero for almost every $s \in \mathbb{R}_+$. Moreover, the polynomial growth condition (3.8) implies that the integrand is bounded by $2c_Q(1+|\mathbf{x}(s)|^p)$, which has finite $L^1([0, T], \mathbb{R}^d)$ -norm. The dominated convergence theorem now shows that $\mathbf{V}_n \rightarrow 0$ as $n \rightarrow \infty$.

Combining the above bounds, we obtain

$$\limsup_{n \rightarrow \infty} \int_0^T |g_n(\mathbf{x}_n(s), \mathbf{z}_n(s)) - g(\mathbf{x}(s), \mathbf{z}(s))| ds \leq \varphi_{\mathbf{II}}(R^p) + \varphi_{\mathbf{IV}}(R^p).$$

Sending R to infinity shows that the left-hand side is actually equal to zero. This completes the proof. $\square$

The proof of Theorem 1.6 is now straightforward.

Proof of Theorem 1.6. This is a consequence of Lemma 3.3 and Theorem 3.4. We only need to observe that the “truncation function” $\chi(\zeta) = \zeta$ can be used under the stronger integrability condition (1.5), and that the A^n satisfy (3.1) with constants c_f that do not depend on n . To see this, observe that any $f \in C_c^2(\mathbb{R}^k)$ satisfies

$$|f(z + \zeta) - f(z) - \zeta^\top \nabla f(z)| \leq \frac{1}{2} \|\nabla^2 f\|_\infty |\zeta|^2. \quad (3.9)$$

Therefore,

$$\begin{aligned} |A^n f(x, z)| &\leq \left(\|\nabla f\|_\infty + \frac{1}{2} \|\nabla^2 f\|_\infty \right) \\ &\quad \times \left(|b^n(x)| + |a^n(x)| + \int_{\mathbb{R}^k} |\zeta|^2 \nu^n(x, d\zeta) \right). \end{aligned}$$

Since (b^n, a^n, ν^n) satisfy (1.5) with a common constant c_{LG} , and due to the bounds $|b^n(x)| \leq 1 + |b^n(x)|^2$ and $|x|^2 \leq 1 + |x|^p$, we deduce that $|A^n f(x, z)| \leq c_f(1 + |x|^p)$ holds with

$$c_f = 2(1 + c_{\text{LG}}) \left(\|\nabla f\|_\infty + \frac{1}{2} \|\nabla^2 f\|_\infty \right). \quad (3.10)$$

This does not depend on n , as required. The proof is complete. $\square$

4 Existence of weak L^p solutions

This section is devoted to the proof of Theorem 1.2. We first give an elementary existence result for the simple pure jump case where the diffusion part of the characteristic triplet vanishes, and the jump kernel is uniformly bounded.

Lemma 4.1. *Let $K: \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times k}$ and $g_0: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ be measurable functions. Let $\nu(x, d\zeta)$ be a bounded kernel from $\mathbb{R}^d$ into $\mathbb{R}^k$, meaning that $\sup_{x \in \mathbb{R}^d} \nu(x, \mathbb{R}^k) < \infty$. Then there exists a filtered probability space with a predictable process X and a càdlàg piecewise constant semimartingale Z such that*

$$X_t = g_0(t) + \int_{[0,t)} K(t-s) dZ_s, \quad t \geq 0,$$

and the differential characteristics of Z are $b(X) = \int_{\mathbb{R}^k} \zeta \nu(X, d\zeta)$, $a(X) = 0$, $\nu(X, d\zeta)$.

Proof. Let $\{(U_n, E_n): n \in \mathbb{N}\}$ be a collection of independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with U_n standard uniform and E_n standard exponential. Define

$$T_0 = 0, \quad X_t^0 = g_0(t), \quad Z_t^0 = 0, \quad t \geq 0.$$

We now construct processes X^n , Z^n and random times T_n recursively as follows. For each $n \in \mathbb{N}$, if X^{n-1} and Z^{n-1} have already been constructed, define a jump time T_n and jump size J_n as follows. First set

$$T_n = \inf\{t > T_{n-1}: \int_{T_{n-1}}^t \nu(X_s^{n-1}, \mathbb{R}^k) ds \geq E_n\},$$

and note that $T_n > T_{n-1}$ since the kernel $\nu(x, d\zeta)$ is bounded. Then let $F: \mathbb{R}^d \times [0, 1] \rightarrow \mathbb{R}^k$ be a measurable function with the following property: If U is standard uniform, then $F(x, U)$ has distribution $\nu(x, \cdot) / \nu(x, \mathbb{R}^k)$ if $\nu(x, \mathbb{R}^k) > 0$, and $F(x, U) = 0$ otherwise. Set $J_n = F(X_{T_n}^{n-1}, U_n)$. We can now define

$$\begin{aligned} X_t^n &= X_t^{n-1} + K(t - T_n) J_n \mathbf{1}_{t > T_n} \\ Z_t^n &= Z_t^{n-1} + J_n \mathbf{1}_{t \geq T_n} \end{aligned}$$

for $t \geq 0$. Note that (X^n, Z^n) coincides with (X^{n-1}, Z^{n-1}) on $[0, T_n)$.

Since the kernel $\nu(x, d\zeta)$ is bounded, we have $\sup_{x \in \mathbb{R}^d} \nu(x, \mathbb{R}^k) \leq c$ for some constant c , and thus $T_n - T_{n-1} \geq \inf\{t > 0: ct \geq E_n\} = E_n/c$. It follows from the Borel–Cantelli lemma that $\lim_{n \rightarrow \infty} T_n = \sum_{n \in \mathbb{N}} (T_n - T_{n-1}) = \infty$. We can thus define (X_t, Z_t) for all $t \geq 0$

by setting $(X_t, Z_t) = (X_t^n, Z_t^n)$ for $t < T_n$. It follows from the construction that Z is càdlàg and piecewise constant, and that

$$X_t = g_0(t) + \sum_{n: t > T_n} K(t - T_n) \Delta Z_{T_n}, \quad t \geq 0.$$

This is the desired convolution equation.

Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by Z , so that in particular Z is a semimartingale. It follows from the construction of Z that its jump characteristic is $\nu(X_t, d\zeta)dt$, provided X is predictable. We now show that this is the case. Indeed, any process of the form $f(t)g(T_n, J_n)\mathbf{1}_{t > T_n}$ is predictable, so by a monotone class argument the same is true for $K(t - T_n)J_n\mathbf{1}_{t > T_n}$. Since $X^0 = g_0$ is predictable, it follows by induction that X^n is predictable for each n . Thus X is predictable, and the proof is complete. $\square$

We now proceed with the proof of Theorem 1.2. Throughout the rest of this section, we therefore consider $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and (g_0, K, b, a, ν) as in (D1)–(D3). We assume that b and a are continuous, and that $x \mapsto |\zeta|^2 \nu(x, d\zeta)$ is continuous from $\mathbb{R}^d$ to $M_+(\mathbb{R}^k)$, the finite positive measures on $\mathbb{R}^k$ with the topology of weak convergence. We also assume there exist a constant $\eta \in (0, 1)$, a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a constant c_{LG} such that (1.4) and (1.5) hold.

Lemma 3.3 connects (1.1) to the local martingale problem for (g_0, K, A) , where the operator A is given by

$$\begin{aligned} Af(x, z) = & b(x)^\top \nabla f(z) + \frac{1}{2} \text{tr}(a(x) \nabla^2 f(z)) \\ & + \int_{\mathbb{R}^k} (f(z + \zeta) - f(z) - \zeta^\top \nabla f(z)) \nu(x, d\zeta). \end{aligned} \quad (4.1)$$

By the same arguments as in the proof of Theorem 1.6, the inequality (3.9) and the growth bound (1.5), A satisfies (3.1) with the constants c_f given by (3.10). In the following lemma, we construct approximations of A .

Lemma 4.2. *Let A be as in (4.1). Then there exist kernels $\nu^n(x, d\zeta)$ from $\mathbb{R}^d$ into $\mathbb{R}^k$ with the following properties.*

- (i) *boundedness and compact support:* $\sup_{x \in \mathbb{R}^d} \nu^n(x, \mathbb{R}^k) < \infty$, and $\nu^n(x, \cdot)$ is compactly supported for every $x \in \mathbb{R}^d$,
- (ii) *linear growth uniformly in n :* with $b^n(x) = \int_{\mathbb{R}^k} \zeta \nu^n(x, d\zeta)$, one has

$$|b^n(x)|^2 + \int_{\mathbb{R}^k} |\zeta|^2 \nu^n(x, d\zeta) + \left(\int_{\mathbb{R}^k} |\zeta|^p \nu^n(x, d\zeta) \right)^{2/p} \leq c'_{\text{LG}}(1 + |x|^2), \quad (4.2)$$

where $c'_{\text{LG}} = (5 + 2\sqrt{d})c_{\text{LG}}$,

(iii) *locally uniform approximation: for every $f \in C_c^2(\mathbb{R}^k)$, defining*

$$A^n f(x, z) = \int_{\mathbb{R}^k} (f(z + \zeta) - f(z)) \nu^n(x, d\zeta),$$

we have $A^n f \in C(\mathbb{R}^d \times \mathbb{R}^k)$ and $A^n f \rightarrow Af$ locally uniformly.

Proof. Multiplying by a continuous cutoff function if necessary, we may assume that $b(x)$, $a(x)$, and $\nu(x, d\zeta)$ are zero for all x outside some compact set Q . Moreover, we can approximate the b , a , and ν parts separately and then add up the approximations (observing that the left-hand side of (4.2) is subadditive in (b^n, ν^n) , so that we may simply add up the corresponding constants c'_{LG}).

Suppose first that a and ν are zero, and let

$$\nu^n(x, d\zeta) = \frac{1}{\varepsilon} \delta_{\varepsilon b(x)}(d\zeta) \mathbf{1}_{\zeta \neq 0},$$

where $\varepsilon = n^{-1}$. Clearly (i) holds. Moreover, $A^n f(x, z) = \varepsilon^{-1}(f(z + \varepsilon b(x)) - f(z))$ lies in $C(\mathbb{R}^d \times \mathbb{R}^k)$, and converges to $Af(x, z) = b(x)^\top \nabla f(z)$. The convergence is locally uniform, since the difference quotients converge locally uniformly for $f \in C^2(\mathbb{R}^k)$. Thus (iii) holds. Finally, note that $b^n(x) = b(x)$, and that $\int_{\mathbb{R}^k} |\zeta|^q \nu^n(x, d\zeta) = \varepsilon^{q-1} |b(x)|^q$ for any $q \geq 2$. Thus it follows from (1.5) that (4.2) holds with $c'_{\text{LG}} = 3c_{\text{LG}}$.

Suppose instead that b and ν are zero. Write $\sigma(x) = a(x)^{1/2}$ using the positive semidefinite square root. Then $x \mapsto \sigma(x)$ is again continuous and compactly supported. So are its columns, denoted by $\sigma_1(x), \dots, \sigma_d(x)$. Let

$$\nu^n(x, d\zeta) = \frac{1}{2\varepsilon^2} \sum_{i=1}^d (\delta_{\varepsilon \sigma_i(x)}(d\zeta) + \delta_{-\varepsilon \sigma_i(x)}(d\zeta)) \mathbf{1}_{\zeta \neq 0},$$

where again $\varepsilon = n^{-1}$. As before, (i) holds. Moreover,

$$\begin{aligned} A^n f(x, z) &= \frac{1}{2} \sum_{i=1}^d \frac{f(z + \varepsilon \sigma_i(x)) - 2f(z) + f(z - \varepsilon \sigma_i(x))}{\varepsilon^2} \\ &\rightarrow \frac{1}{2} \sum_{i=1}^d \sigma_i(x)^\top \nabla^2 f(z) \sigma_i(x) = \frac{1}{2} \text{tr}(a(x) \nabla^2 f(z)). \end{aligned}$$

Again, $A^n f$ lies in $C(\mathbb{R}^d \times \mathbb{R}^k)$ and the convergence is locally uniform since f is C^2 and the σ_i are continuous. This gives (iii). Next, we have $b^n(x) = 0$. Also, writing $\sigma_i^j(x)$ for the j th component of $\sigma_i(x)$, we have

$$\int_{\mathbb{R}^k} |\zeta|^q \nu^n(x, d\zeta) = \varepsilon^{q-2} \sum_{i=1}^d |\sigma_i(x)|^q \leq \left(\sum_{i,j=1}^d |\sigma_i^j(x)|^2 \right)^{q/2} = \text{tr}(a(x))^{q/2}$$

for any $q \geq 2$. Since also $\text{tr}(a(x)) \leq \sqrt{d}|a(x)|$, it follows from (1.5) that (4.2) holds with $c'_{\text{LG}} = 2\sqrt{d}c_{\text{LG}}$.

Finally, suppose that b and a are zero. Let φ_n be a continuous cutoff function supported on $[n^{-1}, n]$ and equal to one on $[2n^{-1}, n/2]$. We arrange so that $\varphi_{n+1} \geq \varphi_n$ for all n . Let

$$\nu^n(x, B) = \int_{\mathbb{R}^k} \left(\delta_\zeta(B) + \frac{1}{\varepsilon} \delta_{-\varepsilon\zeta}(B) \right) \varphi_n(|\zeta|) \nu(x, d\zeta),$$

where again $\varepsilon = n^{-1}$. Clearly $\nu^n(x, \cdot)$ has compact support. Moreover,

$$\begin{aligned} \nu^n(x, \mathbb{R}^k) &\leq \left(1 + \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^k} n^2 |\zeta|^2 \nu(x, d\zeta) \\ &\leq c_{\text{LG}}(1+n)n^2 \sup_{x \in Q} (1 + |x|^2) < \infty, \end{aligned}$$

due to the growth bound (1.5) and recalling that we assumed $\nu(x, d\zeta) = 0$ for all x outside some compact set Q . We deduce that (i) holds. Next, we have

$$b^n(x) = \int_{\mathbb{R}^k} \left(\zeta + \frac{1}{\varepsilon} (-\varepsilon\zeta) \right) \varphi_n(|\zeta|) \nu(x, d\zeta) = 0$$

and

$$\int_{\mathbb{R}^k} |\zeta|^q \nu^n(x, d\zeta) = 2 \int_{\mathbb{R}^k} |\zeta|^q \varphi_n(|\zeta|) \nu(x, d\zeta) \leq 2 \int_{\mathbb{R}^k} |\zeta|^q \nu(x, d\zeta).$$

Thus it follows from (1.5) that (4.2) holds with $c'_{\text{LG}} = 2c_{\text{LG}}$. It remains to show that $A^n f \rightarrow Af$ locally uniformly. Write

$$\begin{aligned} Af(x, z) - A^n f(x, z) &= \int_{\mathbb{R}^k} \left(f(z + \zeta) - f(z) - \zeta^\top \nabla f(z) \right) (1 - \varphi_n(|\zeta|)) \nu(x, d\zeta) \\ &\quad + \int_{\mathbb{R}^k} \frac{1}{\varepsilon} \left(f(z) - f(z - \varepsilon\zeta) - \varepsilon\zeta^\top \nabla f(z) \right) \varphi_n(|\zeta|) \nu(x, d\zeta). \end{aligned}$$

Due to (3.9) and the bound

$$|f(z) - f(z - \varepsilon\zeta) - \varepsilon\zeta^\top \nabla f(z)| \leq \frac{\varepsilon^2}{2} \|\nabla^2 f\|_\infty |\zeta|^2,$$

we obtain

$$|Af(x, z) - A^n f(x, z)| \leq c \int_{\mathbb{R}^k} (1 - \varphi_n(|\zeta|)) \tilde{\nu}(x, d\zeta) + \frac{c}{n} \tilde{\nu}(x, \mathbb{R}^k) \quad (4.3)$$

for the constant $c = \frac{1}{2} \|\nabla^2 f\|_\infty$ and the finite kernel

$$\tilde{\nu}(x, d\zeta) = |\zeta|^2 \nu(x, d\zeta).$$

Thanks to the growth bound (1.5) and the assumption that $\nu(x, d\zeta) = 0$ for all x outside a compact set Q , we have $\int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) \leq c_{\text{LG}} \sup_{x \in Q} (1 + |x|^2) < \infty$. Thus the second term on the right-hand side of (4.3) tends to zero uniformly as $n \rightarrow \infty$. To bound the first term, write

$$\int_{\mathbb{R}^d} (1 - \varphi_n(|\zeta|)) \tilde{\nu}(x, d\zeta) \leq \int_{\mathbb{R}^d} \psi_n(|\zeta|) \tilde{\nu}(x, d\zeta) + \int_{\mathbb{R}^d} \mathbf{1}_{|\zeta| \geq n/2} \tilde{\nu}(x, d\zeta), \quad (4.4)$$

where $\psi_n = (1 - \varphi_n) \mathbf{1}_{[0, 2n^{-1}]}$ is continuous and supported on $[0, 2n^{-1}]$. We bound the two terms on the right-hand side of (4.4) separately.

First, by assumption, $x \mapsto \tilde{\nu}(x, d\zeta)$ is continuous from $\mathbb{R}^d$ to $M_+(\mathbb{R}^k)$. Moreover, $\tilde{\nu}(x, d\zeta)$ is zero for x outside a compact set Q . Thus the set $P = \{\tilde{\nu}(x, d\zeta) : x \in \mathbb{R}^d\} = \{\tilde{\nu}(x, d\zeta) : x \in Q\}$ is a compact subset of $M_+(\mathbb{R}^d)$, being a continuous image of a compact set. Therefore P is tight, so that

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}_{|\zeta| \geq n/2} \tilde{\nu}(x, d\zeta) = \sup_{\mu \in P} \mu(B(0, n/2)^c) \rightarrow 0, \quad n \rightarrow \infty. \quad (4.5)$$

Next, we claim that

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \psi_n(|\zeta|) \tilde{\nu}(x, d\zeta) = 0. \quad (4.6)$$

Let v denote the limsup in (4.6). For each n , $x \mapsto \int_{\mathbb{R}^d} \psi_n(|\zeta|) \tilde{\nu}(x, d\zeta)$ is continuous and supported on Q , hence maximized at some $x_n \in Q$. After passing to a subsequence, we have $x_n \rightarrow \bar{x}$ for some $\bar{x} \in Q$, and $\int_{\mathbb{R}^d} \psi_n(|\zeta|) \tilde{\nu}(x_n, d\zeta) \rightarrow v$. By the choice of φ_n , we have $\psi_{n+1} \leq \psi_n$ for all n . As a result, for each fixed m ,

$$v \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \psi_m(|\zeta|) \tilde{\nu}(x_n, d\zeta) = \int_{\mathbb{R}^d} \psi_m(|\zeta|) \tilde{\nu}(\bar{x}, d\zeta).$$

This tends to zero as $m \rightarrow \infty$ by dominated convergence, since $\tilde{\nu}(\bar{x}, \{0\}) = 0$. Thus $v = 0$, that is, (4.6) holds. Combining (4.4), (4.5), and (4.6), it follows that also the first term on the right-hand side of (4.3) tends to zero uniformly as $n \rightarrow \infty$. This gives (iii) and completes the proof of the lemma. $\square$

We can now complete the proof of existence of weak L^p solutions.

Proof of Theorem 1.2. Consider the kernels $\nu^n(x, d\zeta)$ and corresponding triplets $(b^n, 0, \nu^n)$ given by Lemma 4.2. Apply the basic existence result Lemma 4.1 with each kernel $\nu^n(x, d\zeta)$ and the given g_0 and K to obtain processes (X^n, Z^n) . Note that the differential characteristics of Z^n with respect to the “truncation function” $\chi(\zeta) = \zeta$ are $b^n(X^n)$, $a^n(X^n) = 0$, $\nu^n(X^n, d\zeta)$. Thus (X^n, Z^n) is a weak L^p solution of (1.1) for the data $(g_0, K, b^n, 0, \nu^n)$.

The triplets $(b^n, 0, \nu^n)$ satisfy the growth bound in Lemma 4.2(ii) with a common constant c'_{LG} . Corollary 1.5 thus implies that the sequence $\{X^n\}_{n \in \mathbb{N}}$ is tight in L^p_{loc} . By passing to a subsequence, we assume that $X^n \Rightarrow X$ in L^p_{loc} for some limiting process X .

We claim that the sequence $\{Z^n\}_{n \in \mathbb{N}}$ is tight in D . To prove this, first note that for any $T \in \mathbb{R}_+$, $m > 0$, $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{P} \left(\int_0^T \int_{\mathbb{R}^k} \mathbf{1}_{|\zeta| > m} \nu^n(X_t^n, d\zeta) > \varepsilon \right) &\leq \frac{1}{m^2 \varepsilon} \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^k} |\zeta|^2 \nu^n(X_t^n, d\zeta) \right] \\ &\leq \frac{1}{m^2 \varepsilon} c'_{\text{LG}} \left(T + \mathbb{E}[\|X^n\|_{L^2(0,T)}^2] \right). \end{aligned}$$

Theorem 1.4 shows that the expectation on the right-hand side is bounded by a constant that does not depend on n . Therefore,

$$\lim_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P} \left(\int_0^T \int_{\mathbb{R}^k} \mathbf{1}_{|\zeta| > m} \nu^n(X_t^n, d\zeta) > \varepsilon \right) = 0.$$

Furthermore, the increasing process

$$\int_0^t \left(|b^n(X_s^n)| + \int_{\mathbb{R}^k} |\zeta|^2 \nu^n(X_s^n, d\zeta) \right) ds, \quad t \geq 0, \quad (4.7)$$

is *strongly majorized* by $c'_{\text{LG}} \int_0^\cdot (1 + |X_s^n|^2) ds$ in the sense that the difference of the two is increasing; see Jacod and Shiryaev (2003, Definition VI.3.34). The latter process converges weakly to the continuous increasing process $c'_{\text{LG}} \int_0^\cdot (1 + |X_s|^2) ds$. Thus (4.7) is tight with only continuous limit points; see Jacod and Shiryaev (2003, Proposition VI.3.35). With these observations we may now apply Jacod and Shiryaev (2003, Theorem VI.4.18 and Remark VI.4.20(2)) to conclude that $\{Z^n\}_{n \in \mathbb{N}}$ is tight in D .

Finally, by passing to a further subsequence, we now have $(X^n, Z^n) \Rightarrow (X, Z)$ in $L^p_{\text{loc}} \times D$ for some limiting process (X, Z) . An application of Theorem 1.6 then shows that (X, Z) is a weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) , as desired. The proof of Theorem 1.2 is complete. $\square$

5 Uniqueness of weak L^p solutions

We now turn to pathwise uniqueness and uniqueness in law under suitable Lipschitz conditions.

Let (X, Z) be a weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) , where $\int_{\mathbb{R}^k} |\zeta|^2 \nu(x, d\zeta) < \infty$. The characteristics are understood with respect to the “truncation function” $\chi(\zeta) = \zeta$. Standard representation theorems for semimartingales allow us to express Z as a stochastic integral with respect to time, Brownian motion, and a compensated Poisson random measure; see Jacod and Protter (2011, Theorem 2.1.2) and El Karoui and Lepeltier (1977);

Lepeltier and Marchal (1976). It follows that X satisfies a d -dimensional stochastic Volterra equation of the form

$$\begin{aligned} X_t = & g_0(t) + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s \\ & + \int_{[0,t) \times \mathbb{R}^m} K(t-s)\gamma(X_s, \xi)(\mu(ds, d\xi) - F(d\xi)ds), \quad \mathbb{P} \otimes dt\text{-a.e.} \end{aligned} \quad (5.1)$$

for some d' -dimensional Brownian motion W , Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{R}^m$ with compensator $dt \otimes F(d\xi)$, and some measurable functions $\sigma: \mathbb{R}^d \rightarrow \mathbb{R}^{k \times d'}$ and $\gamma: \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^k$ such that

$$a(x) = \sigma(x)\sigma(x)^\top \quad \text{and} \quad \nu(x, B) = \int_{\mathbb{R}^m} \mathbf{1}_B(\gamma(x, \xi))F(d\xi).$$

Both W and μ are defined on some extension $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ of the filtered probability space where X and Z are defined.

Conversely, given $(g_0, K, b, \sigma, \gamma, F)$ along with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ equipped with a d' -dimensional Brownian motion W and Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{R}^m$ with compensator $dt \otimes F(d\xi)$, a *solution of (5.1)* is any predictable process X on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with trajectories in L_{loc}^p such that (5.1) holds. We are now in position to define pathwise uniqueness for such solutions.

Definition 5.1. Fix $(g_0, K, b, \sigma, \gamma, F)$ as above. We say that pathwise uniqueness holds for (5.1) if for any $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, W , μ as above and any two solutions X and Y of (5.1), we have $X = Y$, $\mathbb{P} \otimes dt$ -a.e.

The powerful abstract machinery of Kurtz (2014) can be used in this setting to relate pathwise uniqueness and weak existence to strong existence and uniqueness in law. A *strong solution* of (5.1) in the sense of Kurtz (2014, Definition 1.2) is a weak L^p solution X which is $\mathbb{P} \otimes dt$ -a.e. equal to a Borel measurable function of W and $N = \int_0^\cdot \xi(\mu(d\xi, ds) - F(d\xi)ds)$ from (5.1).

Theorem 5.2. The following are equivalent:

- (i) There exists a weak L^p solution of (1.1), and pathwise uniqueness holds for (5.1).
- (ii) There exists a strong solution of (5.1), and joint uniqueness in law of (X, W, N) holds.

Proof. Let $S_1 = L_{\text{loc}}^p$ and $S_2 = D \times D$. Then the statement follows from Kurtz (2014, Theorem 1.5 and Lemma 2.10). Indeed, Kurtz (2014, Lemma 2.10) clarifies that our notion of pathwise uniqueness is equivalent to the one used in Kurtz (2014, Theorem 1.5). Note that the definitions in Kurtz (2014, Definition 1.4, Definition 2.9) have to be adapted to replace $\mathbb{P}$ -a.s. assertions by $\mathbb{P} \otimes dt$ -a.e. assertions. $\square$

As for standard SDEs, pathwise uniqueness holds under Lipschitz conditions on the coefficients.

Theorem 5.3. *Let $K \in L^2_{\text{loc}}$ and suppose there exists a constant c_{Lip} such that b, σ, γ, F in (5.1) satisfy*

$$|b(x) - b(y)|^2 + |\sigma(x) - \sigma(y)|^2 + \int_{\mathbb{R}^m} |\gamma(x, \xi) - \gamma(y, \xi)|^2 F(d\xi) \leq c_{\text{Lip}} |x - y|^2$$

for all $x, y \in \mathbb{R}^d$. Then pathwise uniqueness holds for (5.1), and hence also uniqueness in law of weak L^p solutions of (1.1).

Proof. The argument is similar to the proof of (1.6), so we only give a sketch. Let X and Y be two solutions of (5.1) with trajectories in L^2_{loc} . Define $\tau_n = \inf\{t: \int_0^t (|X_s|^2 + |Y_s|^2) ds \geq n\} \wedge T$ as well as $X_t^n = X_t \mathbf{1}_{t < \tau_n}$ and $Y_t^n = Y_t \mathbf{1}_{t < \tau_n}$. As in the proof of (1.6), but relying on the Lipschitz assumption rather than linear growth, one shows that

$$\mathbb{E}[\|X^n - Y^n\|_{L^2(0,T)}^2] \leq c \int_0^T \int_0^t |K(t-s)|^2 \mathbb{E}[|X_s^n - Y_s^n|^2] ds dt$$

for all $T \geq 0$ and some $c = c(T, c_{\text{Lip}}) < \infty$ that depends continuously on T and c_{Lip} . Multiple changes of variables and applications of Tonelli's theorem then show that $f_n(t) = \mathbb{E}[\|X^n - Y^n\|_{L^2(0,t)}^2]$ satisfies the convolution inequality $f_n(t) \leq -(\hat{K} * f_n)(t)$ on $[0, T]$ with $\hat{K} = -c(T, c_{\text{Lip}})|K|^2$. The Gronwall lemma for convolution inequalities (see Lemma A.2) yields $f_n(T) \leq 0$, and monotone convergence gives $f_n(T) \rightarrow \mathbb{E}[\|X - Y\|_{L^2(0,T)}^2]$. Thus $\mathbb{E}[\|X - Y\|_{L^2(0,T)}^2] = 0$, which implies pathwise uniqueness in the sense of Definition 5.1. Uniqueness in law now follows from Theorem 5.2. $\square$

6 Path regularity

Solutions X of (1.1) can be very irregular. Consider for example the simple case

$$X_t = \int_{[0,t)} K(t-s) dN_s = \sum_{t > T_n} K(t - T_n),$$

where N is a standard Poisson process with jump times T_n , $n \in \mathbb{N}$. Without further information about K , nothing can be said about the path regularity of X beyond measurability. Even with singular but otherwise “nice” kernels such as those in Example 1.3(i), X fails to have càdlàg or even làdlàg trajectories. This is why L^p spaces are useful for the solution theory. Nonetheless, one frequently does have additional information that implies better path regularity.

The following result yields Hölder continuity in many cases, also when the driving semimartingale has jumps. The result relies on a combination of the estimates (1.6)-(1.7)

with Sobolev embedding theorems. For any $T > 0$ and $\eta > 0$, we denote by $C^\eta(0, T)$ the space of Hölder continuous functions of order η on $[0, T]$. Thus $f \in C^\eta(0, T)$ if

$$\|f\|_{C^\eta(0, T)} = \|f\|_{L^\infty(0, T)} + \sup_{\substack{t, s \in [0, T] \\ t \neq s}} \frac{|f(t) - f(s)|}{|t - s|^\eta} < \infty.$$

Theorem 6.1. *Let $d, k \in \mathbb{N}$, $p \in [2, \infty)$, and consider data (g_0, K, b, a, ν) as in (D1)–(D3). Assume there exist a constant $\eta \in (0, 1)$, a locally bounded function $c_K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and a constants c_{LG} such that (1.4) and (1.5) hold. Then for any weak L^p solution X of (1.1) the following statements hold:*

- (i) *if $\eta p > 1$, then $X - g_0$ admits a version whose sample paths lie in $C^{(\eta p - 1)/p}(0, T)$ almost surely.*
- (ii) *if $p = 2$ and $\nu \equiv 0$, then $X - g_0$ admits a version whose sample paths lie in $C^\beta(0, T)$ for all $\beta < \eta$ almost surely.*
- (iii) *if $K(0) < \infty$ and if $K - K(0)$ (instead of K) satisfies (1.4) with $\eta p > 1$, then $X - g_0$ admits a version with càglàd sample paths.*
- (iv) *without assuming (1.4) and (1.5), but rather that K is differentiable with derivative $K' \in L^2_{\text{loc}}$, we have that $X - g_0$ is a semimartingale and thus admits a version with càglàd sample paths.²*

Proof. Assertion (i) follows from (1.7) and the Sobolev embedding theorem, see Di Nezza et al. (2012, Theorem 8.2). To prove (ii), one can adapt the proof of Theorem 1.4 to get that (1.6)–(1.7) hold for all $p \geq 2$. Applying Di Nezza et al. (2012, Theorem 8.2) for sufficiently large values of p yields the claimed statement. For (iii), we write

$$X_t - g_0(t) = K(0)Z_{t-} + \int_{[0, t)} (K(t - s) - K(0))dZ_s.$$

The claimed regularity follows on observing that the first term on the right-hand side is càglàd and that, similarly to (i), the second term admits a version with continuous sample paths. For (iv) one applies a Fubini theorem, see Lemma 3.2, to get that

$$X_t - g_0(t) = K(0)Z_{t-} + \int_0^t \left(\int_{[0, s)} K'(s - u)dZ_u \right) ds.$$

This completes the proof. □

²Note that (1.4) is implied by the given assumption on K , for any $\eta < 1/p$.

7 Applications

In this section, we illustrate our results with two applications: scaling limits of Hawkes processes and approximation of stochastic Volterra equations by Markovian semimartingales.

7.1 Generalized nonlinear Hawkes processes and their scaling limits

Fix $d, k \in \mathbb{N}$ along with functions $g_0: \mathbb{R}_+ \rightarrow \mathbb{R}^d$, $b: \mathbb{R}^d \rightarrow \mathbb{R}^k$, $\Lambda: \mathbb{R}^d \rightarrow \mathbb{R}_+^k$, and a kernel $K: \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times k}$. We fix $p \geq 2$ and assume that g_0 and K lie in L_{loc}^p , that K satisfies (1.4) for some $\eta \in (0, 1)$ and locally bounded function c_K , and that b and Λ are continuous and satisfy the linear growth condition

$$|b(y)| + |\Lambda(y)| \leq c(1 + |y|), \quad y \in \mathbb{R}^d, \quad (7.1)$$

for some constant $c \in \mathbb{R}_+$. Consider a k -dimensional counting process N with no simultaneous jumps, whose intensity vector is given by $\Lambda(Y)$ with Y a d -dimensional predictable process with trajectories in L_{loc}^p that satisfies

$$Y_t = g_0(t) + \int_0^t K(t-s)b(Y_s)ds + \int_{[0,t)} K(t-s)dN_s \quad \mathbb{P} \otimes dt\text{-a.e.} \quad (7.2)$$

We call such a process N a *generalized nonlinear Hawkes process*. The existence of Y and N follows immediately from Theorem 1.2. Indeed, (7.2) is a stochastic Volterra equation of the form (1.1) whose driving semimartingale Z has differential characteristics $b(Y)$, $a(Y) = 0$, and $\nu(Y, d\zeta) = \sum_{i=1}^d \Lambda_i(Y) \delta_{e_i}(d\zeta)$, where $e_1, \dots, e_d$ are the canonical basis vectors in $\mathbb{R}^d$.

Example 7.1. For $k = d$, $b = 0$, and $\Lambda(y) = y$ we obtain a multivariate Hawkes process, and nonlinear Hawkes processes for more general Λ ; see Brémaud and Massoulié (1996); Daley and Vere-Jones (2003); Delattre et al. (2016) and the references there.

We now establish convergence of rescaled generalized nonlinear Hawkes processes toward stochastic Volterra equations with no jump part, as those studied by Abi Jaber et al. (2017). In the following theorem we consider given inputs g_0, K as well as g_0^n, K^n indexed by $n \in \mathbb{N}$, that satisfy the assumptions described in the beginning of this subsection. We consider a fixed function $\Lambda = (\Lambda_1, \dots, \Lambda_d)$ as above and take $b = -\Lambda$. We continue to assume (7.1) (with $b = -\Lambda$). For each n , denote the corresponding generalized nonlinear Hawkes process by N^n . Its intensity vector is $\Lambda(Y^n)$, where Y^n satisfies

$$\begin{aligned} Y_t^n &= g_0^n(t) + \int_{[0,t)} K^n(t-s)dM_s^n, \\ M_t^n &= N_t^n - \int_0^t \Lambda(Y_s^n)ds. \end{aligned}$$

Theorem 7.2. *For each $n \in \mathbb{N}$, consider a diagonal matrix of rescaling parameters, $\varepsilon^n = \text{diag}(\varepsilon_1^n, \dots, \varepsilon_d^n) \in \mathbb{R}^{d \times d}$. Assume for all i that*

$$n(\varepsilon_i^n)^2 \Lambda_i((\varepsilon^n)^{-1}x) \leq c_i(1 + |x|^2), \quad x \in \mathbb{R}^d, \quad (7.3)$$

for some constant $c_i > 0$ independent of n , and that

$$n(\varepsilon_i^n)^2 \Lambda_i((\varepsilon^n)^{-1}x) \rightarrow \bar{\Lambda}_i(x) \quad (7.4)$$

locally uniformly in x for some function $\bar{\Lambda}: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Assume also that

- (i) $\varepsilon^n g_0^n(n \cdot) \rightarrow g_0$ in L_{loc}^p ,
- (ii) $\varepsilon^n K^n(n \cdot)(\varepsilon^n)^{-1} \rightarrow K$ in L_{loc}^p ,
- (iii) $\varepsilon^n K^n(n \cdot)(\varepsilon^n)^{-1}$ satisfy (1.4) with the same η and c_K as K .

Then the rescaled sequence (X^n, Z^n) given by $X_t^n = \varepsilon^n Y_{nt}^n$, $Z_t^n = \varepsilon^n M_{nt}^n$ is tight in $L_{\text{loc}}^p \times D$, and every limit point (X, Z) is a weak L^p solution of

$$X_t = g_0(t) + \int_0^t K(t-s) dZ_s, \quad (7.5)$$

where Z admits the representation $Z_t = \int_0^t \sqrt{\text{diag}(\bar{\Lambda}(X_s))} dW_s$ for some d -dimensional Brownian motion W .

Proof. One verifies that the rescaled intensity X^n satisfies the equation

$$X_t^n = \varepsilon^n g_0^n(nt) + \int_0^t \varepsilon^n K^n(n(t-s))(\varepsilon^n)^{-1} dZ_s^n,$$

where Z^n has differential characteristics $b^n(X^n) = 0$, $a^n(X^n) = 0$, $\nu^n(X^n, d\zeta)$ with jump kernel given by $\nu^n(x, d\zeta) = \sum_{i=1}^d n \Lambda_i((\varepsilon^n)^{-1}x) \delta_{\varepsilon_i^n e_i}(d\zeta)$. Here $e_1, \dots, e_d$ are the canonical basis vectors in $\mathbb{R}^d$. The associated operator is given by

$$A^n f(x, z) = \sum_{i=1}^d n \Lambda_i((\varepsilon^n)^{-1}x) \left(f(z + \varepsilon_i^n e_i) - f(z) - \varepsilon_i^n \nabla f(z)^\top e_i \right),$$

which converges locally uniformly to $\frac{1}{2} \text{tr}(\text{diag}(\bar{\Lambda}(x)) \nabla^2 f(z))$ due to (7.4). Consequently, provided (X^n, Z^n) is tight, Theorem 3.4 shows that every limit point (X, Z) is a weak L^p solution of (7.5), where Z has differential characteristics $b(X) = 0$, $a(X) = \text{diag}(\bar{\Lambda}(X))$, $\nu(X, d\zeta) = 0$. The representation of Z in terms of a Brownian motion is standard. It remains to prove tightness. First, by virtue of (7.3), we have $\int_{\mathbb{R}^d} |\zeta|^2 \nu^n(x, d\zeta) \leq c(1 + |x|^2)$ for all $x \in \mathbb{R}^d$ and some constant c . Thus, (1.5) is satisfied uniformly in n . Recalling (i) and (iii), Corollary 1.5 yields tightness of $(X^n)_{n \geq 1}$. Tightness of $(Z^n)_{n \geq 1}$ in D is then obtained by reiterating the arguments in the proof of Theorem 1.2 at the end of Section 4. Since marginal tightness implies joint tightness the proof is complete. $\square$

Example 7.3. Let K, g_0 be as described in the beginning of this subsection and let $\varepsilon^n = \text{diag}(\varepsilon_1^n, \dots, \varepsilon_d^n) \in \mathbb{R}^{d \times d}$ as above. Then the functions g_0^n and K^n given by

$$g_0^n(t) = (\varepsilon^n)^{-1} g_0\left(\frac{t}{n}\right), \quad K^n(t) = (\varepsilon^n)^{-1} K\left(\frac{t}{n}\right) \varepsilon^n$$

satisfy (i)–(iii). There are other ways of constructing such kernels, as illustrated in Jaisson and Rosenbaum (2015, 2016) for linear Hawkes processes.

Theorem 7.2 is in the same spirit as the results of Erny et al. (2019), who obtain square-root type processes as limits of mean field interactions of multi-dimensional nonlinear Hawkes processes. The following example provides a concrete specification for the special case of fractional powers, extending results in Jaisson and Rosenbaum (2015, 2016) to nonlinear Hawkes processes.

Example 7.4. Let $\beta_i \in (0, 2)$, $i = 1, \dots, d$, and take $\Lambda(y) = (y_1^{\beta_1}, \dots, y_d^{\beta_d})$. Let $\varepsilon^n = \text{diag}(\varepsilon_1^n, \dots, \varepsilon_d^n)$ satisfy $n(\varepsilon_i^n)^{2-\beta_i} \rightarrow \nu_i$ for some constants $\nu_i \geq 0$. Then (7.3)–(7.4) are satisfied with $\bar{\Lambda} = \Lambda$. The limiting process (X, Z) produced by Theorem 7.2 takes the form

$$X_t = g_0(t) + \int_0^t K(t-s) \sqrt{\text{diag}(\nu_1 |X_s^1|^{\beta_1}, \dots, \nu_d |X_s^d|^{\beta_d})} dW_s,$$

where W is a d -dimensional Brownian motion.

7.2 Approximation by Markovian semimartingales

It is sometimes useful, for example for numerical purposes, to replace a singular kernel with a smooth approximation. Theorem 3.4 can be used to analyze this procedure; see also the stability result of Abi Jaber and El Euch (2019a, Theorem 3.6) for the case without jumps. An approximation scheme that is useful in practice is to consider weighted sums of exponentials.

Theorem 7.5. Fix $d, k \in \mathbb{N}$, $p \geq 2$ and (g_0, K, b, a, ν) as in (D1)–(D3), and assume (1.5) holds. For each $n \in \mathbb{N}$, let $g_0^n \in L_{\text{loc}}^p$ and consider the kernel

$$K^n(t) = \sum_{i=1}^n c_i^n e^{-\lambda_i^n t}$$

for some $c_i^n \in \mathbb{R}^{d \times k}$ and $\lambda_i^n \geq 0$, $i = 1, \dots, n$. By Example 1.3(ii) and Theorem 1.2 there exists a weak L^p -solution (X^n, Z^n) for the data (g_0^n, K^n, b, a, ν) . Moreover, X^n admits the representation

$$X_t^n = g_0^n(t) + \sum_{i=1}^n c_i^n Y_t^{n,i}$$

$$dY_t^{n,i} = -\lambda_i^n Y_t^{n,i} dt + dZ_t^n, \quad Y_0^{n,i} = 0, \quad i = 1, \dots, n.$$

Assume in addition that

- (i) $g_0^n \rightarrow g_0$ in L_{loc}^p ,
- (ii) $K^n \rightarrow K$ in L_{loc}^p ,
- (iii) K^n satisfy (1.4) with the same η and c_K as K .

Then $(X^n, Z^n)_{n \geq 1}$ is tight in $L_{\text{loc}}^p \times D$, and every limit point (X, Z) is a weak L^p solution of (1.1) for the data (g_0, K, b, a, ν) .

Proof. Defining $Y_t^{n,i} = \int_0^t e^{-\lambda_i^n(t-s)} dZ_s^n$, the representation of X^n follows from Itô's formula. Corollary 1.5 yields tightness of $(X^n)_{n \geq 1}$. Tightness of $(Z^n)_{n \geq 1}$ in D is then obtained by reiterating the arguments in the proof of Theorem 1.2 at the end of Section 4. The claimed convergence follows from Theorem 3.4. $\square$

Remark 7.6. If K is the Laplace transform of a $\mathbb{R}^{d \times d}$ -valued measure μ ,

$$K(t) = \int_{\mathbb{R}_+} e^{-\lambda t} \mu(d\lambda), \quad t > 0,$$

then K can indeed be approximated by weighted sums of exponentials. Constructions of such weighted sums are given by [Abi Jaber and El Euch \(2019a\)](#).

A Auxiliary results

We occasionally use the following version of Young's inequality on subintervals. It uses the convolution notation $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Lemma A.1. Fix $T \in \mathbb{R}_+$ and $p, q, r \in [1, \infty]$ with $p^{-1} + q^{-1} = r^{-1} + 1$. For any matrix-valued measurable functions f, g on $[0, T]$ of compatible size, one has the Young type inequality $\|f * g\|_{L^r(0, T)} \leq \|f\|_{L^p(0, T)} \|g\|_{L^q(0, T)}$.

Proof. This follows from the Young inequality for convolutions on the whole real line applied to the functions $|f|\mathbf{1}_{[0, T]}$ and $|g|\mathbf{1}_{[0, T]}$ that equal $|f(t)|$ and $|g(t)|$ for $t \in [0, T]$ and zero elsewhere. $\square$

For ease of reference, we give the following well-known Gronwall type lemma for convolution inequalities; see [Gripenberg et al. \(1990, Lemma 9.8.2\)](#) for the case of non-convolution kernels.

Lemma A.2. Let $T \in \mathbb{R}_+$ and suppose $f, g, k \in L^1(0, T)$. Assume k has a nonpositive resolvent $r \leq 0$. If $f \leq g - k * f$, then $f \leq g - r * g$.

Proof. Write $f + k * f = g - h$ for $h \geq 0$. By the definition of resolvent, one then has $f = (g - h) - r * (g - h) \leq g - r * g$. $\square$

Lemma A.3. *Let $p \in [2, \infty)$. Consider a convolution kernel $K \in L^p_{\text{loc}}$ and a characteristic triplet (b, a, ν) satisfying (1.2). Let X be a predictable process with trajectories in L^p_{loc} , and let Z be an Itô semimartingale whose differential characteristics (with respect to some given truncation function χ) are $b(X), a(X), \nu(X, d\zeta)$. Then for almost every $t \in \mathbb{R}_+$, the stochastic integral $\int_{[0,t)} K(t-s)dZ_s$ is well-defined.*

Proof. Define $\kappa(x) = |b(x)| + |a(x)| + \int_{\mathbb{R}^k} (1 \wedge |\zeta|^2) \nu(x, d\zeta)$ and set $\tau_n = \inf\{t: \int_0^t |X_s|^p ds > n\}$. Due to the bound (1.2) and the definition of τ_n , we have $\int_0^{T \wedge \tau_n} \kappa(X_s) ds \leq c(T + n)$. Thus, for any $T \in \mathbb{R}_+$, Young's inequality, see Lemma A.1, gives

$$\begin{aligned} & \int_0^T \left(\int_0^{t \wedge \tau_n} |K(t-s)|^2 \kappa(X_s) ds \right)^{p/2} dt \\ & \leq \left(\int_0^T |K(t)|^p dt \right) \left(\int_0^{T \wedge \tau_n} \kappa(X_s) dt \right)^{p/2} \\ & \leq \left(\int_0^T |K(t)|^p dt \right) (c(T + n))^{p/2}. \end{aligned}$$

The right-hand side is deterministic; call it c_n . Taking expectations and using Tonelli's theorem yields

$$\int_0^T \mathbb{E} \left[\left(\int_0^{t \wedge \tau_n} |K(t-s)|^2 \kappa(X_s) ds \right)^{p/2} \right] dt \leq c_n.$$

Therefore, for each n , there is a nullset $N_n \subset [0, T]$ such that the expectation is finite for all $t \in [0, T] \setminus N_n$. The union $N = \bigcup_n N_n$ is still a nullset, and for each $t \in [0, T] \setminus N$,

$$\int_0^{t \wedge \tau_n} |K(t-s)|^2 \kappa(X_s) ds < \infty \text{ for all } n, \mathbb{P}\text{-a.s.}$$

Since X has trajectories in L^p_{loc} , we have $\tau_n \rightarrow \infty$. We infer that, for each $t \in [0, T] \setminus N$, $\int_0^t |K(t-s)|^2 \kappa(X_s) ds < \infty$, $\mathbb{P}$ -a.s. This implies that the random variable $\int_{[0,t)} K(t-s)dZ_s$ is well-defined. $\square$

References

- Eduardo Abi Jaber and Omar El Euch. Multifactor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2):309–349, 2019a.
- Eduardo Abi Jaber and Omar El Euch. Markovian structure of the Volterra Heston model. *Statistics & Probability Letters*, 149:63–72, 2019b.
- Eduardo Abi Jaber, Martin Larsson, and Sergio Pulido. Affine Volterra processes. *Annals of Applied Probability*, to appear, 2017.

- Ole E Barndorff-Nielsen and Jürgen Schmiegel. Time change, volatility, and turbulence. In *Mathematical Control Theory and Finance*, pages 29–53. Springer, 2008.
- Ole E Barndorff-Nielsen, Fred Espen Benth, Almut ED Veraart, et al. Modelling energy spot prices by volatility modulated lévy-driven volterra processes. *Bernoulli*, 19(3):803–845, 2013.
- Andreas Basse and Jan Pedersen. Lévy driven moving averages and semimartingales. *Stochastic Processes and their Applications*, 119(9):2970–2991, 2009.
- Fred Espen Benth, Nils Detering, and Paul Kruehner. Stochastic Volterra integral equations and a class of first order stochastic partial differential equations. *arXiv preprint arXiv:1903.05045*, 2019.
- Marc A. Berger and Victor J. Mizel. Volterra equations with Itô integrals. I. *J. Integral Equations*, 2(3):187–245, 1980. ISSN 0163-5549. URL <http://www.jstor.org/stable/26164035>.
- Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. ISBN 0-471-19745-9. doi: 10.1002/9780470316962. URL <http://dx.doi.org/10.1002/9780470316962>. A Wiley-Interscience Publication.
- Pierre Brémaud and Laurent Massoulié. Stability of nonlinear Hawkes processes. *The Annals of Probability*, pages 1563–1588, 1996.
- Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.
- Zdzisław Brzeźniak and Jerzy Zabczyk. Regularity of Ornstein–Uhlenbeck processes driven by a Lévy white noise. *Potential Analysis*, 32(2):153–188, 2010.
- Zdzisław Brzeźniak, Szymon Peszat, and Jerzy Zabczyk. Continuity of stochastic convolutions. *Czechoslovak Mathematical Journal*, 51(4):679–684, 2001.
- Patrick Cheridito, Damir Filipović, and Marc Yor. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Annals of Applied Probability*, pages 1713–1732, 2005.
- Laure Coutin and Laurent Decreusefond. Stochastic Volterra equations with singular kernels. In *Stochastic Analysis and Mathematical Physics*, volume 50 of *Progr. Probab.*, pages 39–50. Birkhäuser Boston, Boston, MA, 2001.
- Christa Cuchiero and Josef Teichmann. Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case. *arXiv preprint arXiv:1804.10450*, 2018.

- Christa Cuchiero and Josef Teichmann. Markovian lifts of positive semidefinite affine volterra type processes. *arXiv preprint arXiv:1907.01917*, 2019.
- Daryl J Daley and David Vere-Jones. An introduction to the theory of point processes. vol. i. probability and its applications, 2003.
- Sylvain Delattre, Nicolas Fournier, Marc Hoffmann, et al. Hawkes processes on large networks. *The Annals of Applied Probability*, 26(1):216–261, 2016.
- Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques*, 136(5):521–573, 2012.
- Omar El Euch and Mathieu Rosenbaum. The characteristic function of rough Heston models. *Mathematical Finance*, 29(1):3–38, 2019. doi: 10.1111/mafi.12173. URL <https://doi.org/10.1111/mafi.12173>.
- Nicole El Karoui and Jean-Pierre Lepeltier. Représentation des processus ponctuels multivariés à l’aide d’un processus de poisson. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 39(2):111–133, 1977.
- Xavier Erny, Eva Löcherbach, and Dasha Loukianova. Mean field limits for interacting Hawkes processes in a diffusive regime. *arXiv preprint arXiv:1904.06985*, 2019.
- Stewart N Ethier and Thomas G Kurtz. *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Statistics. Wiley, 2005.
- Franco Flandoli and Dariusz Gatarek. Martingale and stationary solutions for stochastic Navier-stokes equations. *Probability Theory and Related Fields*, 102(3):367–391, 1995.
- Jim Gatheral and Martin Keller-Ressel. Affine forward variance models. *Finance and Stochastics*, 23(3):501–533, 2019.
- Jim Gatheral, Thibault Jaisson, and Mathieu Rosenbaum. Volatility is rough. *Quantitative Finance*, 18(6):933–949, 2018. doi: 10.1080/14697688.2017.1393551. URL <https://doi.org/10.1080/14697688.2017.1393551>.
- Gustaf Gripenberg, Stig-Olof Londen, and Olof Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-37289-5. doi: 10.1017/CBO9780511662805. URL <http://dx.doi.org/10.1017/CBO9780511662805>.
- Jean Jacod and Philip Protter. *Discretization of processes*, volume 67. Springer Science & Business Media, 2011.

- Jean Jacod and Albert N. Shiryaev. *Limit Theorems for Stochastic Processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2003.
- Thibault Jaisson and Mathieu Rosenbaum. Limit theorems for nearly unstable hawkes processes. *The Annals of Applied Probability*, 25(2):600–631, 2015.
- Thibault Jaisson and Mathieu Rosenbaum. Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes. *The Annals of Applied Probability*, 26(5):2860–2882, 2016.
- Thomas Kurtz. Weak and strong solutions of general stochastic models. *Electronic Communications in Probability*, 19(58):1–16, 2014.
- Jean-Pierre Lepeltier and Bernard Marchal. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégral-différentiel. In *Annales de l’IHP Probabilités et statistiques*, volume 12, pages 43–103, 1976.
- Carlo Marinelli and Michael Röckner. On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*, pages 293–315. Springer, 2014.
- Tina Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.
- Leonid Mytnik and Eyal Neuman. Sample path properties of Volterra processes. *arXiv preprint arXiv:1101.4969*, 2011.
- Leonid Mytnik and Thomas S. Salisbury. Uniqueness for Volterra-type stochastic integral equations. *arXiv preprint arXiv:1502.05513*, 2015.
- Aleksandr Aleksandrovich Novikov. On discontinuous martingales. *Theory of Probability & Its Applications*, 20(1):11–26, 1975.
- Philip Protter. Volterra equations driven by semimartingales. *Ann. Probab.*, 13(2):519–530, 1985. ISSN 0091-1798. URL [http://links.jstor.org/sici?sici=0091-1798\(198505\)13:2<519:VEDBS>2.0.CO;2-3&origin=MSN](http://links.jstor.org/sici?sici=0091-1798(198505)13:2<519:VEDBS>2.0.CO;2-3&origin=MSN).
- Philip E. Protter. *Stochastic Integration and Differential Equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.
- Jan Rosinski. On path properties of certain infinitely divisible processes. *Stochastic Processes and their Applications*, 33(1):73–87, 1989.

Zhidong Wang. Existence and uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients. *Statistics & Probability Letters*, 78 (9):1062–1071, 2008. doi: <https://doi.org/10.1016/j.spl.2007.10.007>. URL <http://www.sciencedirect.com/science/article/pii/S0167715207003665>.

Xicheng Zhang. Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J. Funct. Anal.*, 258(4):1361–1425, 2010. ISSN 0022-1236. doi: 10.1016/j.jfa.2009.11.006. URL <http://dx.doi.org/10.1016/j.jfa.2009.11.006>.