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Abstract. We study the moments of even order of the generating function $\prod_{0\le r<n}(1-e(2^r x))$ of the Thue–Morse sequence and we present several conjectures related to these moments.

1. Introduction

For any nonnegative integer m we denote by $s(m)$ the number of distinct powers of 2 in the binary representation of m. Then the Thue–Morse sequence (or Prouhet–Thue–Morse sequence) is the sequence $((-1)^{s(m)})_{m\in\mathbb{N}}$. This sequence occurs in many questions related to combinatorics, algebra, number theory, harmonic analysis, geometry, dynamical systems, ergodic theory, etc. (see for example [1] or [14]). Mahler introduced it for the first time in the context of harmonic analysis in [13] in order to illustrate the results obtained by Wiener [19]. He showed in particular that for any $k\in\mathbb{N}$ the sequence $\left(\frac{1}{N}\sum_{m<N}(-1)^{s(m)}(-1)^{s(m+k)}\right)_{N\in\mathbb{N}}$ converges, and that the limit is nonzero for infinitely many integers k. This can be interpreted as the fact that the correlation measure of the symbolic dynamical system associated to the Thue–Morse sequence is the Riesz product $\prod_{r\ge 0}(1-\cos 2^r t)$ (see [12]).

It follows from the definition of s that for $|z|<1$ we have

$$\prod_{r=0}^{\infty}\left(1-z^{2^r}\right) \;=\; \sum_{m=0}^{\infty}(-1)^{s(m)}z^m\,.$$

For any $n\in\mathbb{N}$ we consider the function T_n defined for $x\in\mathbb{R}$ by

$$T_n(x)=\prod_{0\le r<n}(1-e(2^r x))=\sum_{0\le m<2^n}(-1)^{s(m)}e(mx), \tag{1.1}$$

where $e(\theta)=e^{2\pi i\theta}$.

The study of the values of $\|T_n\|_p$, $1\le p\le+\infty$ plays an important role in many problems. In particular, sharp estimates for $\|T_n\|_1$ and $\|T_n\|_\infty$ allowed Mauduit and Rivat to prove a Prime Number Theorem for the sum-of-digits function in [15] (see [16] and [5] for the study of the sum-of-digits function along polynomial sequences) and the study of the ratios $\|T_{n+1}\|_p/\|T_n\|_p$ (for p an even integer) allowed Kurths, Pikowsky, and Zaks to compute the generalized dimension for the Fourier spectrum of the Thue–Morse sequence in [21] (see also [20]).

Here we consider the moments

$$M_k(n)=\int_0^1 |T_n(x)|^{2k}\,dx \tag{1.2}$$

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for positive integers k, which turn out to have some interesting and unexpected properties. By Parseval's identity it is clear that $M_k(n)$ is an integer, and that $M_1(n) = 2^n$. We show that for any fixed k, the sequence of moments $M_k(n)$ satisfies a linear recurrence of order k.

Theorem 1. *For positive integers k let $A_k = [a_{ij}]$ be the $k \times k$ matrix with integral entries*

$$a_{ij} = (-1)^i 2^{2k-2i-1} \binom{j+k-i}{i} \frac{j+k}{j+k-i} \qquad (0 \le i, j < k), \tag{1.3}$$

and let

$$p_k(z) = \det(zI - A_k) = z^k - c_{k-1,k} z^{k-1} - \cdots - c_{0,k} \ \in \ \mathbb{Z}[z] \tag{1.4}$$

be the characteristic polynomial of A_k. Then

$$M_k(n) = c_{k-1,k} M_k(n-1) + \cdots + c_{0,k} M_k(n-k) \tag{1.5}$$

for all integers $n \ge k$.

The fact that the a_{ij} are integers is not immediately clear from (1.3), but their integrality will be established in the course of the proof of Theorem 1.

Theorem 2. *For all positive integers k, there exists a $C_k > 0$ such that for $n \to +\infty$,*

$$M_k(n) = C_k \ \rho(A_k)^n (1 + o(1)) \tag{1.6}$$

where $\rho(A_k)$ denotes the spectral radius of A_k.

It is well known that $\|T_n\|_\infty \le 2 \times 3^{(n-1)/2}$ (see for example [8, Lemme II], [17, Lemma p. 72-73] or [6, Formule (2.10)]) so that

$$M_k(n) \le \left(\frac{4}{3}\right)^k \times 3^{kn},$$

and it follows from (1.6) that

$$\rho(A_k) \le 3^k.$$

The next theorem gives a better upper bound for $\rho(A_k)$.

Theorem 3. *We have*

$$\rho(A_k) \le \frac{1}{2}(3^k + 4^{2k/3}) = \frac{3^k}{2}(1 + o(1)).$$

The method used in the proof of Theorem 2 yields better upper bounds for $\rho(A_k)$ for small values of k. For example it leads to $\rho(A_2) \le 4\sqrt{2}$ and $\rho(A_3) \le 16$, while in fact numerical computation gives $\rho(A_2) = 5.1231$ and $\rho(A_4) = 14.2191$ and we conjecture that $\rho(A_k) \sim \frac{1}{2} 3^k$ (see section 7).

It would be nice to have more explicit formulas for the $c_{j,k}$ defined by (1.4). In this direction, we determine the trace and determinant of A_k, which yields $c_{k-1,k}$ and $c_{0,k}$.

Theorem 4. *With the matrix A_k defined as in (1.3),*

$$\operatorname{tr} A_k = 3^{k-1} + (-1)^{k-1}, \tag{1.7}$$

and

$$\det A_k = \varepsilon_k 2^{k^2} \tag{1.8}$$

where $\varepsilon_k = 1$ if $k \equiv 0$ or $1 \pmod 4$, and $\varepsilon_k = -1$ if $k \equiv 2$ or $3 \pmod 4$.

From (1.7) it follows that if $t_k = \operatorname{tr} A_k$, then

$$t_k = 2t_{k-1} + 3t_{k-2}\,. \tag{1.9}$$

Since $c_{0,k} \neq 0$, the linear recurrence (1.5) is genuinely of order k (not less). However, it could still be the case that M_k satisfies a linear recurrence of lower order—in which case the polynomial p_k would be reducible. Numerical experimentation suggests that the p_k are all irreducible, but this is far from proven. Indeed, numerical experimentation suggests that the eigenvalues of the A_k, and the coefficients of p_k have many striking properties. Our conjectures on this issue are collected in §7.

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2. A combinatorial identity

In the course of proving Theorem 1, we encounter the following combinatorial sum, which can be written in closed form.

Lemma 1. *For integers i and n with $0 \le i < n$,*

$$s_0(n,i) := \sum_{i \le m \le n/2} \binom{n}{2m}\binom{m}{i} \;=\; 2^{n-2i-1}\binom{n-i}{i}\frac{n}{n-i}\,. \tag{2.1}$$

This formula is asserted as item (3.120) in Gould [9, p. 36], but with no indication as to where a proof might be found. A complicated proof can be pieced together by combining several exercises from various chapters of Riordan [18]. A machine proof might be constructed using an implementation of Zeilberger's algorithm, but we have not achieved that. In fact, the formula is ancient, as it is a special case of a formula known to Chu in the thirteenth century (see Askey [2, Chapter 7]).

Proof. Let $r = m - i$. Then

$$s_0(n,i) = \binom{n}{2i}\sum_{r=0}^{n/2-i}\frac{\binom{n}{2r+2i}\binom{r+i}{i}}{\binom{n}{2i}} = \binom{n}{2i}\sum_{r=0}^{n/2-i} a_r,$$

say. By simple algebra we see that

$$\frac{a_{r+1}}{a_r} = \frac{(r+i-n/2)(r+i-(n-1)/2)}{(r+1)(r+i+1/2)}\,.$$

Since $a_0 = 1$, it follows that

$$\begin{aligned} s_0(n,i) &= \binom{n}{2i}\sum_{r=0}^{n/2-i}\frac{(i-n/2)_r(i-(n-1)/2)_r}{(i+1/2)_r r!} \\ &= \binom{n}{2i}\,{}_2F_1\!\left(\begin{matrix} i-n/2\,,\, i-(n-1)/2 \\ i+1/2 \end{matrix}\;\Big|\; 1\right). \end{aligned} \tag{2.2}$$

Here $(x)_r = x(x+1)(x+2)\cdots(x+r-1)$ is the Pochhammer symbol.

The Chu–Vandermonde identity asserts that if s is a nonnegative integer, then

$$\sum_{r=0}^{s}\frac{(-s)_r(a)_r}{(c)_r r!} = \frac{(c-a)_s}{(c)_s} \tag{2.3}$$

for arbitrary a and c. In Andrews–Askey–Roy [3, Corollary 2.2.3] this arises as a special case of a hypergeometric identity due to Gauss [7].

Suppose that n is even, say $n = 2t$. In (2.3) we take $s = t - i$, $a = i - t + 1/2$, and $c = i + 1/2$ to see that

$$s_0(n,i) = \binom{n}{2i} \frac{(t)_{t-i}}{(i+1/2)_{t-i}}\,.$$

Now $(a+1/2)_m = (2a+1)_{2m}/\big(2^{2m}(a+1)_m\big)$, so that

$$s_0(n,i) = 2^{2t-2i} \binom{n}{2i} \frac{(t)_{t-i}(i+1)_{t-i}}{(2i+1)_{2t-2i}}\,.$$

As $(t)_{t-i} = (n-i-1)!/(t-1)!$, $(i+1)_{t-i} = t!/i!$, and $(2i+1)_{2t-2i} = n!/(2i)!$, we get

$$s_0(n,i) = 2^{n-2i} \frac{(n-i-1)!\,t}{i!(n-2i)!} = 2^{n-2i-1} \binom{n-i}{i} \frac{n}{n-i},$$

as was to be shown.

Suppose that n is odd, say $n = 2t+1$. In (2.3) we take $s = t-i$, $a = i-t-1/2$, and $c = i+1/2$ to see that

$$\begin{aligned} s_0(n,i) &= \binom{n}{2i} \frac{(t+1)_{t-i}}{(i+1/2)_{t-i}} = 2^{2t-2i} \binom{n}{2i} \frac{(t+1)_{t-i}(i+1)_{t-i}}{(2i+1)_{2t-2i}} \\ &= 2^{n-2i-1} \frac{n(2t-i)!}{i!(n-2i)!} = 2^{n-2i-1} \binom{n-i}{i} \frac{n}{n-i}, \end{aligned}$$

which completes the proof. □

The sum $s_0(n,i)$ has a companion, namely

$$s_1(n,i) := \sum_{i \le m \le (n-1)/2} \binom{n}{2m+1} \binom{m}{i} \;=\; 2^{n-2i-1} \binom{n-i-1}{i}. \tag{2.4}$$

This evaluation in closed form is also an easy consequence of the Chu–Vandermonde identity (2.3). By using familiar properties of binomial coefficients it is easy to show that

$$\begin{aligned} s_0(n+1,i) &= s_0(n,i) + s_1(n,i) + s_1(n,i-1), \\ s_1(n+1,i) &= s_1(n,i) + s_0(n,i)\,. \end{aligned} \tag{2.5}$$

These identities make it possible to prove (2.1) and (2.4) simultaneously by a double induction. This is a little tedious, since various bases of induction need to be checked. In addition, this ignores the fact that both (2.1) and (2.4) are simple consequences of an ancient formula.

3. Proof of Theorem 1

Following Fouvry & Mauduit [6], for $f \in L^2(\mathbb{T})$ we define the operators

$$P_k\, f(x) = \frac{1}{2}(2\sin \pi x/2)^{2k} f(x/2) + \frac{1}{2}(2\cos \pi x/2)^{2k} f((x+1)/2), \tag{3.1}$$

$$Q_k\, f(x) = (2 \sin \pi x)^{2k} f(2x). \tag{3.2}$$

Thus

$$M_k(n) = \int_0^1 Q_k^n 1\, dx\,. \tag{3.3}$$

For $f, g \in L^2(\mathbb{T})$ we note that

$$\begin{aligned}\langle Q_k\, f, g\rangle &= \int_0^1 (2\sin \pi x)^{2k} f(2x)\overline{g(x)}\, dx \\ &= \frac{1}{2}\int_0^2 (2\sin \pi u/2)^{2k} f(u)\overline{g(u/2)}\, du \\ &= \frac{1}{2}\int_0^1 f(u)\big((2\sin \pi u/2)^{2k}\overline{g(u/2)} + (2\cos \pi u/2)^{2k}\overline{g((u+1)/2)}\big)\, du \\ &= \langle f, P_k\, g\rangle\,.\end{aligned}$$

Thus P_k is the adjoint of Q_k, $P_k = Q_k^*$. In particular,

$$M_k(n) = \langle Q_k^n\, 1, 1\rangle = \langle 1, P_k^n\, 1\rangle = \int_0^1 P_k^n\, 1\, dx\,. \tag{3.4}$$

Let E_k denote the vector space of even trigonometric polynomials with period 1 and degree $< k$. Of course $\cos 2\pi jx$ for $0 \le j < k$ is a basis for E_k, but we note that

$$\sin^{2j} \pi x = (-1)^j 2^{-2j} \sum_{n=-j}^{j} (-1)^n \binom{2j}{j-n} e(nx) \tag{3.5}$$

is an even trigonometric polynomial with period 1 and degree j, so 1, $\sin^2 \pi x$, $\sin^4 \pi x$, …, $\sin^{2(k-1)} \pi x$ is also a basis for E_k. Suppose that $0 \le j < k$. Then

$$P_k\, \sin^{2j} \pi x = 2^{2k-1} \sin^{2(j+k)} \frac{\pi x}{2} + 2^{2k-1} \cos^{2(j+k)} \frac{\pi x}{2}\,.$$

By the half angle formulæ this is

$$= 2^{2k-1}\Big(\frac{1-\cos \pi x}{2}\Big)^{j+k} + 2^{2k-1}\Big(\frac{1+\cos \pi x}{2}\Big)^{j+k}\,.$$

By the binomial theorem this is

$$= 2^{k-j} \sum_{0\le m\le (j+k)/2} \binom{j+k}{2m} \cos^{2m} \pi x, \tag{3.6}$$

which is an even trigonometric polynomial with period 1 and degree $[(j+k)/2] < k$. Thus P_k maps E_k to itself. Let $\widetilde{P_k}$ denote the restriction of P_k to E_k. Continuing from (3.6), we find that

$$\begin{aligned}\widetilde{P_k}\, \sin^{2j} \pi x &= 2^{k-j} \sum_{0\le m\le (j+k)/2} \binom{j+k}{2m} (1-\sin^2 \pi x)^m \\ &= 2^{k-j} \sum_{0\le m\le (j+k)/2} \binom{j+k}{2m} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \sin^{2i} \pi x \\ &= 2^{k-j} \sum_{0\le i\le (j+k)/2} (-1)^i \sin^{2i} \pi x \sum_{i\le m\le (j+k)/2} \binom{j+k}{2m}\binom{m}{i}\,.\end{aligned}$$

Here it is clear that the coefficient of $\sin^{2i} \pi x$ is an integer. From Lemma 1 with $n = j+k$, we see that

$$\widetilde{P_k}\, \sin^{2j} \pi x = \sum_{i=0}^{k-1} a_{ij} \sin^{2i} \pi x$$

with the a_{ij} defined in (1.3). Let p_k be the characteristic polynomial of the matrix $A_k = [a_{ij}]$, as defined in (1.4). By the Cayley–Hamilton theorem we know that $p_k(A_k) = 0$. Thus

$$A_k^n = c_{k-1,k}A_k^{n-1} + c_{k-2,k}A_k^{n-2} + \cdots + c_{0,k}A_k^{n-k}$$

for $n \geq k$, and hence

$$\widetilde{P_k}^n = c_{k-1,k}\widetilde{P_k}^{n-1} + c_{k-2,k}A\widetilde{P_k}^{n-2} + \cdots + c_{0,k}\widetilde{P_k}^{n-k}.$$

Thus

$$\begin{aligned} M_k(n) &= \int_0^1 \widetilde{P_k}^n 1\,dx \\ &= \int_0^1 c_{k-1,k}\widetilde{P_k}^{n-1} 1 + \cdots + c_{0,k}\widetilde{P_k}^{n-k} 1\,dx \\ &= c_{k-1,k}M_k(n-1) + c_{k-2,k}M_k(n-2) + \cdots + c_{0,k}M_k(n-k), \end{aligned}$$

which completes the proof.

4. Proof of Theorem 2

The operator P_k that we introduced in (3.1) is a special case of positive quasi-compact transfer operators that have been studied by many authors in ergodic theory (see in particular [4, 10, 11]).

When the transfer function is a trigonometric polynomial (in our case $(2\sin\pi x)^{2k}$) the quasi-compactness of P_k is trivial. Indeed, as we saw in §3 the operator P_k acts on the k dimensional vector space E_k and A_k is the matrix of P_k in the basis $(1, \sin^2\pi x, \sin^4\pi x, \ldots, \sin^{2(k-1)}\pi x)$.

Proposition 1. *The spectral radius of P_k is equal to*

$$\rho(A_k) = \lim_{n\to+\infty} \|P_k^n 1\|_\infty^{1/n} \tag{4.1}$$

and is the only eigenvalue of P_k with modulus $\rho(A_k)$. The eigenfunction ψ_k associated to $\rho(A_k)$ is strictly positive on $[0,1]$. We have the following spectral decomposition of P_k:

$$E_k = \ker(P_k - \rho(A_k)\mathrm{Id}) \oplus F_k, \tag{4.2}$$

where F_k is a subspace of E_k stabilized by P_k and such that the spectral radius of the restriction of P_k to F_k is strictly less than $\rho(A_k)$. Moreover for any $x \in [0,1]$ we have

$$\rho(A_k) = \lim_{n\to+\infty} \frac{P_k^{n+1}1(x)}{P_k^n 1(x)}. \tag{4.3}$$

Proof. Proposition 1 follows from the study made by Hervé in [11] in the context of wavelet theory (see in particular Théorème 3.1 and Théorème 4.2 from [11]). In order to apply the results from [11]) it is enough to check that the function $(2\sin\pi x)^{2k}$ does not admit any invariant periodic cycle (i.e. there exists no positive integer Q such that

$$\forall q \in \{1,\ldots,Q\},\ \forall \ell \in \{0,\ldots,2^q-1\},\quad \sin\pi\left(\frac{\ell}{2^q-1}+\frac{1}{2}\right) = 0),$$

The fact that ψ_k is stricly positive is a consequence of the study by Conze and Raugi in [4] on the invariant compact sets associated to the transformations $x \mapsto \frac{x}{2}$ and $x \mapsto \frac{x+1}{2}$ of the interval $[0,1]$ and the description by Hervé in [10] of the zeros of the eigenvalues of this class of operators. Moreover the fact that $\rho(A_k)$ is the only eigenvalue of P_k with modulus $\rho(A_k)$ follows from the proof of Théorème 4.2 in [11]. ∎

Writing the function 1 according to (4.2), it follows that

$$1 = \alpha_k \psi_k + f_k,$$

with $\alpha_k \in \mathbb{R}$ ($\alpha_k \neq 0$ because of (4.1)) and $f_k \in F_k$.

This implies that

$$\begin{aligned} M_k(n) &= \int_0^1 P_k^n \, 1 \, dx \\ &= \alpha_k \rho(A_k)^n \int_0^1 \psi_k(x) dx (1 + O(\theta_k^{-n})), \end{aligned}$$

where $\theta_k > 1$ is the ratio of $\rho(A_k)$ to the spectral radius of the restriction of P_k to F_k.

5. Proof of Theorem 3

Applying the method of [17, Lemma pages 72-73] or [6, page 583] we have for all integer $n \geq 2$

$$T_n(x) = \prod_{0 \leq r < n} |2 \sin(\pi 2^r x)| \leq 2 \prod_{0 \leq r < n-1} u(2^r x),$$

where

$$u(x) = 2 \left|\sin(\pi x)\right|^{2/3} \left|\sin(2\pi x)\right|^{1/3}.$$

It follows that

$$M_k(n) \leq 4^k I_k(n-1)$$

where for $n \geq 0$,

$$I_k(n) = \int_0^1 \prod_{0 \leq r < n} u(2^r x)^{2k} \, dx$$

(with the convention that an empty product is equal to 1, so that $I_k(0) = 1$). We have for $n \geq 1$

$$\begin{aligned} I_k(n) &= \int_0^{1/2} u(x)^{2k} \prod_{1 \leq r < n} u(2^r x)^{2k} \, dx + \int_{1/2}^1 u(x)^{2k} \prod_{1 \leq r < n} u(2^r x)^{2k} \, dx \\ &= \int_0^1 \left(\frac{1}{2} u(x/2)^{2k} + \frac{1}{2} u((x+1)/2)^{2k} \right) \prod_{0 \leq r < n-1} u(2^r x)^{2k} \, dx \\ &\leq w_k I_k(n-1), \end{aligned}$$

where

$$w_k = \max_{x \in \mathbb{R}} \tfrac{1}{2} \left(u(x/2)^{2k} + u((x+1)/2)^{2k} \right).$$

By induction we get

$$I_k(n) \leq w_k^n I_k(0) = w_k^n.$$

For all $x \in \mathbb{R}$ we have $u(x/2)^3 + u((x+1)/2)^3 = 8 \left|\sin \pi x\right| \leq 8$, so that $\min(u(x/2), u((x+1)/2)) \leq 4^{1/3}$. Furthermore for all $x \in \mathbb{R}$ we have $u(x) \leq u(1/3) = \sqrt{3}$, hence

$$w_k \leq \frac{1}{2}(3^k + 4^{2k/3}).$$

It follows that $M_k(n) \leq 4^k I_k(n-1) \leq 4^k w_k^{n-1}$ and from (1.6) we deduce that

$$\rho(A_k) \leq w_k,$$

which completes the proof of Theorem 3.

6. Proof of Theorem 4

From the definition (1.3) of the a_{ij} it is clear that

$$\begin{aligned}\operatorname{tr} A_k &= \sum_{i=0}^{k-1} a_{ii} = \frac{1}{k}\sum_{i=0}^{k-1}(-1)^i 2^{2k-2i-1}\binom{k}{i}(k+i)\\ &= \sum_{i=0}^{k-1}(-1)^i 2^{2k-2i-1}\binom{k}{i} + \sum_{i=0}^{k-1}(-1)^i 2^{2k-2i-1}\binom{k-1}{i-1}\\ &= \frac{1}{2}\big((4-1)^k-(-1)^k\big) - \frac{1}{2}\big((4-1)^{k-1}-(-1)^{k-1}\big)\\ &= 3^{k-1}+(-1)^{k-1}.\end{aligned}$$

As for the second assertion of Theorem 4, let $B_k = [b_{ijk}]$ be the $k\times k$ matrix with entries

$$b_{ijk} = \binom{j+k-i}{i} + \binom{j+k-i-1}{i-1} = \binom{j+k-i}{i}\frac{j+k}{j+k-i} \qquad (0 \le j,k < k).$$

Thus

$$\det A_k = \det B_k \times \prod_{i=0}^{k-1}\big((-1)^i 2^{2k-2i-1}\big) = \varepsilon_k 2^{k^2}\det B_k,$$

so it suffices to show that $\det B_k = 1$. We induct on k. We know that $B_1 = [1]$, so $\det B_1 = 1$. Let $\mathbf{b}_{0\,k}, \dots, \mathbf{b}_{k-1\,k}$ denote the columns of B_k. Our first task is to show that if $0 < j < k$, then

$$\mathbf{b}_{j\,k} - \mathbf{b}_{j-1\,k} = \left[\begin{array}{c}\mathbf{0}\\ \hline \mathbf{b}_{j-1\,k-1}\end{array}\right]. \tag{6.1}$$

To this end we note first that $b_{0\,j\,k} = 1$ for all j, so that $b_{0\,j\,k} - b_{0\,j-1\,k} = 0$ for $0 < j < k$. If $2i \le j+k-1$, then

$$\begin{aligned}b_{i\,j\,k} - b_{i\,j-1\,k} &= \binom{j+k-i}{i} + \binom{j+k-i-1}{i-1}\\ &\quad - \binom{j+k-i-1}{i-1} - \binom{j+k-i-2}{i-2}\\ &= b_{i-1\,j-1\,k-1}.\end{aligned}$$

If $2i = j+k$, then $b_{i\,j-1,k} = 0$, and so $b_{i\,j\,k} - b_{i\,j-1\,k} = b_{i\,j\,k} = 2 = b_{i-1\,j-1\,k-1}$. If $2i > j+k$, then $b_{i\,j\,k} = b_{i\,j-1\,k} = b_{i-1\,j-1\,k-1} = 0$. Thus we have (6.1).

We now operate on B_k as follows: We subtract column $\mathbf{b}_{k-2\,k}$ from $\mathbf{b}_{k-1\,k}$, then subtract $\mathbf{b}_{k-3\,k}$ from $b_{k-2\,k}$, and so on, until finally we subtract $\mathbf{b}_{0\,k}$ from $\mathbf{b}_{1\,k}$. The result is a matrix of the form

$$\begin{bmatrix}1 & \mathbf{0}\\ * & B_{k-1}\end{bmatrix}.$$

Thus $\det B_k = \det B_{k-1}$, so the induction is complete.

7. Conjectures and Questions

Based on some experimentation with the matrices A_k and B_k, their characteristic polynomials and their eigenvalues, we propose the following:

Conjecture 1. All eigenvalues of A_k are real.

Conjecture 2. If $k = 2r$, then A_k has r positive eigenvalues, and r negative eigenvalues. If $k = 2r+1$, then A_k has $r+1$ positive eigenvalues, and r negative eigenvalues.

Conjecture 3. If the negative eigenvalues of A_k are replaced by their negatives, then the resulting numbers interlace with the positive eigenvalues, e.g., when $k = 5$, the eigenvalues are 122.32, 37.02, 6.14, −18.59, −64.91.

Conjecture 4. For each k, the characteristic polynomial p_k is irreducible over $\mathbb{Q}$.

Conjecture 5. With $c_{k-r,k}$ defined in Theorem 1 and ε_r defined in Theorem 4,

$$\operatorname{sgn} c_{k-r,k} = \varepsilon_{r-1}\,.$$

Conjecture 6. The zeros of p_k interlace with those of p_{k+1}.

Conjecture 7. Let B_k be defined as in the Proof of Theorem 4. The eigenvalues of B_k are all positive real.

Conjecture 8. The spectral radius of A_k satisfies

$$\rho(A_k) = \frac{3^k}{2}(1 + o(1)).$$

Numerical computations suggest the stronger conjecture that $\rho(A_k) = \frac{1}{2}\,3^k + O(k^2)$.

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