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Abstract. The present contribution describes a semi-analytical technique devoted to the nonsmooth modal analysis (natural frequencies and mode shapes) of a non-internally resonant elastic bar of length L subject to a Robin condition at $x = 0$ and a frictionless unilateral contact condition at $x = L$.

Introduction

Modal analysis of nonsmooth mechanical systems, also called *nonsmooth modal analysis*, has been recently performed on a finite elastic bar of length L subject to a Dirichlet boundary condition at $x = 0$ and a unilateral contact constraint at $x = L$ [5]. This system satisfies a *complete internal resonance* condition, *i.e.* all linear natural frequencies are commensurate with the first one, which has drastic consequences on the nonlinear modal response. To further explore the nonlinear dynamics of this one-dimensional contact problem, a non-internally resonant configuration is instead investigated in the present work. Analytical results were only found for the internally resonant bar and restricted to the first nonlinear mode [1]. Moreover, traditional numerical approaches, in the framework of finite-elements, present numerical issues that hinder the calculation of the modes of vibration of contacting systems [2]. In this work, we propose a semi-analytical technique that employs the exact travelling-wave solution to the one-dimensional wave equation [3, p. 77].

Non-internally resonant finite bar

The system of interest is an unforced, homogeneous elastic bar of length $L > 0$ and constant cross-sectional area S subject to a conservative unilateral constraint at its right end. Its left end is connected to a rigid support through a spring of stiffness $\kappa > 0$, as depicted in Fig. 1. The displacement, velocity, strain and stress fields are denoted

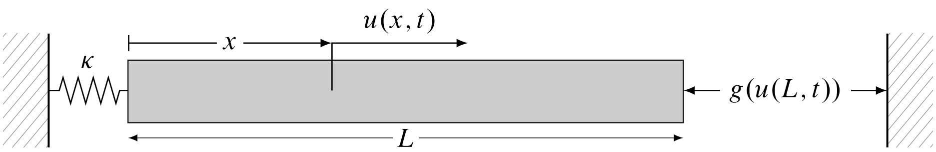

Fig. 1: Elastic bar attached to a spring and subject to unilateral contact constraint.

by $u(x,t)$, $v(x,t)$, $\epsilon(x,t)$ and $\sigma(x,t)$ respectively. Young's modulus is denoted by $E > 0$ and $\rho > 0$ stands for the mass per unit volume. In linear elasticity, the stresses read $\sigma = E\epsilon$ where $\epsilon = \partial_x u$ should be infinitesimally small. The unilateral contact force $r(t)$ is related to the stresses by $\sigma(L,t) = E\partial_x u(L,t) = r(t)/S$. The *gap function* is defined as $g(u(L,t)) = g_0 - u(L,t)$ where g_0 is the signed distance between the unrestricted resting configuration and the obstacle. The full formulation reads:

Wave equation	$\partial^2_{tt}u(x,t) - c^2\partial^2_{xx}u(x,t) = 0,$	$\forall x \in \,]0\,;L[, \quad \forall t > 0,$	(1)
Robin BC	$\partial_x u(0,t) - \alpha u(0,t) = 0,$	$\forall t > 0,$	(2)
Signorini BC	$g(u(L,t)) \geq 0, \quad r(t) \leq 0, \quad r(t)g(u(L,t)) = 0,$	$\forall t > 0,$	(3)
Initial conditions	$u(x,0) = u_0(x), \quad v(x,0) = v_0(x),$	$\forall x \in \,]0\,;L[.$	(4)

where $\alpha = \kappa/(ES)$ and $c = \sqrt{E/\rho}$. The operators $\partial_\xi(\bullet)$ and $\partial^2_{\xi\xi}(\bullet)$ stand for the first and second derivatives of $(\bullet)$ with respect to the argument ξ. This formulation possesses a unique solution which conserves the total energy [4]. Non-trivial solutions of the problem satisfying Eqs. (1) to (4) are successions of free phases (open gap) and contact phases (closed gap) [1]. They can be seen as combinations of travelling wave motions switching at $x = L$ between $\partial_x u(L,\cdot) = 0$ when the gap is open and prescribed displacement $u(L,\cdot) = g_0$ when the gap is closed. The nonlinearity in the formulation arises in the dependence of the solution to the unknown switching time. Consider the general solution to the local equation incorporating the reflection mechanism induced by Eq. (2) at $x = 0$:

$$u(x,t) = f(ct+x) + f(ct-x) - 2\alpha e^{\alpha(x-ct)}\int_0^{ct-x} e^{\alpha s} f(s)\,\mathrm{d}s, \quad \forall x \in [0\,;L], \quad \forall t \geq 0. \tag{5}$$

The displacement $u(x,t)$ is then obtained through Eq. (5) provided that function f is defined everywhere on $\mathbb{R}$. The successive switches in boundary conditions at $x = L$, reflecting Signorini's condition (3), are incorporated through appropriate functional extensions for the free and contact phases.

Periodic solutions and Nonsmooth Modal Analysis

Nonsmooth modes of vibration (NSMs) are defined as continuous families of periodic solutions satisfying the formulation (1)-(4) together with periodicity conditions in displacement and velocity: $\exists T > 0$ such that $u(x, t+T) = u(x, t)$ and $v(x, t+T) = v(x, t)$, $\forall x \in [0\,;L]$ and $\forall t > 0$. Finding such solutions translates into finding corresponding initial conditions u_0 and v_0 and period T which generate periodic motions. *Admissible* T-periodic motions involving one contact phase per period, by assumption, are described by a function f that satisfies the following functional equation, arising from $u_0(x) = u(x, T)$ and $v_0(x) = v(x, T)$, $\forall x \in [0\,;L]$

$$u_0(x) = f(ct_c + x) + f(ct_c - x) - 2\alpha e^{\alpha(x-ct_c)} \int_0^{ct_c - x} e^{\alpha s} f(s)\,ds, \tag{6a}$$

$$\frac{1}{c} v_0(x) = \partial_t f(ct_c + x) + \partial_t f(ct_c - x) - 2\alpha f(ct_c - x) + 2\alpha^2 e^{\alpha(x-ct_c)} \int_0^{ct_c - x} e^{\alpha s} f(s)\,ds. \tag{6b}$$

together with functional inequalities mirroring Signorini's conditions at $x = L$ for all $t \in [0\,;T]$, where $t_c < T$ is the contact phase duration. Solving this constrained formulation is a noticeably challenging task. We propose a simultaneous discretization of the space and time domains in order to accurately mimic the propagation of possibly discontinuous velocity waves along the characteristics lines.

Spectrum of nonsmooth vibration

The response of the autonomous elastic bar is depicted in *frequency-energy plots* (FEPs) where a *backbone curve* represents a NSM. The backbone curves emerging in the range $[\omega_1\,;\Omega_1]$ are shown in Fig. 2 as sets of

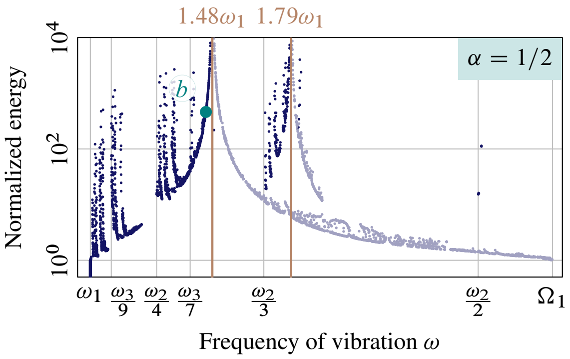

Fig. 2: Backbone curves in $[\omega_1\,;\Omega_1]$ for $\alpha = 1/2$ with $g_0 > 0$ [——], $g_0 < 0$ [——] and $g_0 = 0$ [——].

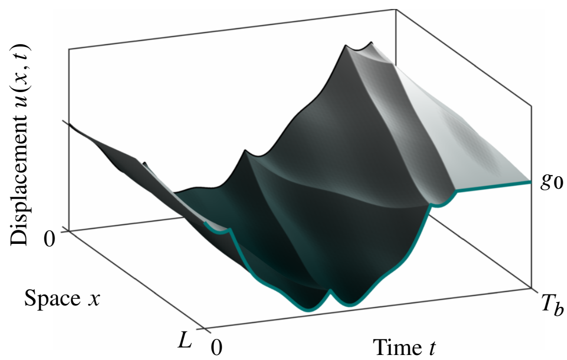

Fig. 3: Periodic displacement field of a NSM motion located at point (b) in Fig. 2.

scattered points supposedly belonging to NSMs, where ω_k and Ω_k for $k \in \mathbb{N}^*$ are the natural frequencies of the linear system involved in the conditional BC switching: Robin–Neumann and Robin–Dirichlet, respectively. In contrast to NSMs of the internally resonant bar [5] where the energy continuously depends on the frequency, the depicted scattered points indicate more complicated backbone curves. A nonsmooth periodic displacement field is depicted in Fig. 3. This point represents a periodic motion belonging to a backbone curve that emerges in the vicinity of ω_1. The complicated pattern of the solution is caused by an intricate interplay between various travelling waves embedding the Robin and Signorini boundary conditions. In contrast to linear modes that are purely harmonic functions, the nonsmooth modes of the non-internally resonant system are nonsmooth piecewise-sinusoidal functions.

Conclusions

The annihilation of the full internal resonance condition with the Robin BC generates complicated modal motions that were computed through a semi-analytical approach based on the exact solution of the wave equation.

References

[1] Henri Cabannes. Mouvements périodiques d'une corde vibrante en présence d'un obstacle ponctuel. *Journal de Mécanique*, 20(1):41–58, 1981. OAI: hal-01620614.

[2] David Doyen, Alexandre Ern, and Serge Piperno. Time-integration schemes for the finite element dynamic Signorini problem. *SIAM Journal of Scientific Computing*, 33(1):223–249, 2011. OAI: hal-00440128.

[3] Karl Graff. *Wave Motion in Elastic Solids*. Dover, 1975.

[4] Michelle Schatzman and Michel Bercovier. Numerical approximation of a wave equation with unilateral constraints. *Mathematics of Computation*, 53(187):55–79, 1989. OAI: hal-01295436.

[5] Carlos Yoong, Anders Thorin, and Mathias Legrand. Nonsmooth modal analysis of an elastic bar subject to a unilateral contact constraint. *Nonlinear Dynamics*, 91(4):2453–2476, 2018. OAI: hal-01471341.