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# 1 Packing Arc-Disjoint Cycles in Tournaments \*

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## 24 — Abstract —

25 A tournament is a directed graph in which there is a single arc between every pair of distinct  
26 vertices. Given a tournament  $T$  on  $n$  vertices, we explore the classical and parameterized com-  
27 plexity of the problems of determining if  $T$  has a cycle packing (a set of pairwise arc-disjoint  
28 cycles) of size  $k$  and a triangle packing (a set of pairwise arc-disjoint triangles) of size  $k$ . We  
29 refer to these problems as ARC-DISJOINT CYCLES IN TOURNAMENTS (ACT) and ARC-DISJOINT  
30 TRIANGLES IN TOURNAMENTS (ATT), respectively. Although the maximization version of ACT  
31 can be seen as the linear programming dual of the well-studied problem of finding a minimum  
32 feedback arc set (a set of arcs whose deletion results in an acyclic graph) in tournaments, sur-  
33 prisingly no algorithmic results seem to exist for ACT. We first show that ACT and ATT are  
34 both NP-complete. Then, we show that the problem of determining if a tournament has a cycle  
35 packing and a feedback arc set of the same size is NP-complete. Next, we prove that ACT and  
36 ATT are fixed-parameter tractable, they can be solved in  $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$  time and  $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$   
37 time respectively. Moreover, they both admit a kernel with  $\mathcal{O}(k)$  vertices. We also prove that  
38 ACT and ATT cannot be solved in  $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$  time under the Exponential-Time Hypothesis.

39 **2012 ACM Subject Classification** F.2 Analysis of Algorithms and Problem Complexity

40 **Keywords and phrases** arc-disjoint cycle packing, tournaments, parameterized algorithms, ker-  
41 nelization

\* This paper is based on the two independent manuscripts [9] and [34]. The full version of this extended abstract containing the detailed proofs is appended for the convenience of the reader.



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43 **1 Introduction**

44 Given a (directed or undirected) graph  $G$  and a positive integer  $k$ , the DISJOINT CYCLE  
 45 PACKING problem is to determine whether  $G$  has  $k$  (vertex or arc/edge) disjoint (directed  
 46 or undirected) cycles. Packing disjoint cycles is a fundamental problem in Graph Theory  
 47 and Algorithm Design with applications in several areas. Since the publication of the classic  
 48 Erdős-Pósa theorem in 1965 [22], this problem has received significant scientific attention in  
 49 various algorithmic realms. In particular, VERTEX-DISJOINT CYCLE PACKING in undirected  
 50 graphs is one of the first problems studied in the framework of parameterized complexity.  
 51 In this framework, each problem instance is associated with a non-negative integer  $k$  called  
 52 *parameter*, and a problem is said to be *fixed-parameter tractable* (FPT) if it can be solved in  
 53  $f(k)n^{\mathcal{O}(1)}$  time for some computable function  $f$ , where  $n$  is the input size. For convenience,  
 54 the running time  $f(k)n^{\mathcal{O}(1)}$  is denoted as  $\mathcal{O}^*(f(k))$ . A *kernelization algorithm* is a polynomial-  
 55 time algorithm that transforms an arbitrary instance of the problem to an equivalent instance  
 56 of the same problem whose size is bounded by some computable function  $g$  of the parameter  
 57 of the original instance. The resulting instance is called a *kernel* and if  $g$  is a polynomial  
 58 function, then it is called a *polynomial kernel*. A decidable parameterized problem is FPT  
 59 if and only if it has a kernel (not necessarily of polynomial size). Kernelization typically  
 60 involves applying a set *reduction rules* to the given instance to produce another instance.  
 61 A reduction rule is said to be *safe* if it is sound and complete, i.e., applying it to the given  
 62 instance produces an equivalent instance. In order to classify parameterized problems as  
 63 being FPT or not, the  $W$ -hierarchy is defined:  $\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \dots \subseteq \text{XP}$ . It is believed  
 64 that the subset relations in this sequence are all strict, and a parameterized problem that is  
 65 hard for some complexity class above FPT in this hierarchy is said to be fixed-parameter  
 66 intractable. Further details on parameterized algorithms can be found in [17, 20, 25, 27].

67 VERTEX-DISJOINT CYCLE PACKING in undirected graphs is FPT with respect to the  
 68 solution size  $k$  [11, 38] but has no polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$  [12]. In contrast,  
 69 EDGE-DISJOINT CYCLE PACKING in undirected graphs admits a kernel with  $\mathcal{O}(k \log k)$   
 70 vertices (and is therefore FPT) [12]. On directed graphs, these problems have many practical  
 71 applications (for example in biology [13, 19]) and they have been extensively studied [7, 36].  
 72 It turns out that VERTEX-DISJOINT CYCLE PACKING and ARC-DISJOINT CYCLE PACKING  
 73 are equivalent and are  $W[1]$ -hard [35, 43]. Therefore, studying these problems on a subclass  
 74 of directed graphs is a natural direction of research. Tournaments form a mathematically  
 75 rich subclass of directed graphs with interesting structural and algorithmic properties [6, 40].  
 76 Tournaments have several applications in modeling round-robin tournaments and in the  
 77 study of voting systems and social choice theory [30, 32].

78 FEEDBACK VERTEX SET and FEEDBACK ARC SET are two well-explored algorithmic  
 79 problems on tournaments. A *feedback vertex (arc) set* is a set of vertices (arcs) whose deletion  
 80 results in an acyclic graph. Given a tournament, MINFAST and MINFVST are the problems  
 81 of obtaining a feedback arc set and feedback vertex set of minimum size, respectively. We refer  
 82 to the corresponding decision version of the problems as FAST and FVST. The optimization  
 83 problems MINFAST and MINFVST have numerous practical applications in the areas of  
 84 voting theory [18], machine learning [16], search engine ranking [21] and have been intensively  
 85 studied in various algorithmic areas. MINFAST and MINFVST are NP-hard [3, 14] while  
 86 FAST and FVST are FPT when parameterized by the solution size  $k$  [4, 24, 26, 32]. Further,  
 87 FAST has a kernel with  $\mathcal{O}(k)$  vertices [10] and FVST has a kernel with  $\mathcal{O}(k^{1.5})$  vertices

[37]. Surprisingly, the duals (in the linear programming sense) of MINFAST and MINFVST have not been considered in the literature until recently. Any tournament that has a cycle also has a triangle [7]. Therefore, if a tournament has  $k$  vertex-disjoint cycles, then it also has  $k$  vertex-disjoint triangles. Thus, VERTEX-DISJOINT CYCLE PACKING in tournaments is just packing vertex-disjoint triangles. This problem is NP-hard [8]. A straightforward application of the *colour coding* technique [5] shows that this problem is FPT and a kernel with  $\mathcal{O}(k^2)$  vertices is an immediate consequence of the quadratic element kernel known for 3-SET PACKING [1]. Recently, a kernel with  $\mathcal{O}(k^{1.5})$  vertices was shown for this problem using interesting variants and generalizations of the popular *expansion lemma* [37].

A tournament that has  $k$  arc-disjoint cycles need not necessarily have  $k$  arc-disjoint triangles. This observation hints that packing arc-disjoint cycles could be significantly harder than packing vertex-disjoint cycles. It also hints that packing arc-disjoint cycles and arc-disjoint triangles in tournaments could be problems of different complexities. This is the starting point of our study. Subsequently, we refer to a set of pairwise arc-disjoint cycles as a *cycle packing* and a set of pairwise arc-disjoint triangles as a *triangle packing*. Given a tournament, MAXACT and MAXATT are the problems of obtaining a maximum set of arc-disjoint cycles and triangles, respectively. We refer to the corresponding decision version of the problems as ACT and ATT. Formally, given a tournament  $T$  and a positive integer  $k$ , ACT (resp. ATT) is the task of determining if  $T$  has  $k$  arc-disjoint cycles (resp. triangles). From a structural point of view, the problem of partitioning the arc set of a directed graph into a collection of triangles has been studied for regular tournaments [45], almost regular tournaments [2] and complete digraphs [29]. In this work, we study the classical complexity of MAXACT and MAXATT and the parameterized complexity of ACT and ATT with respect to the solution size (i.e. the number  $k$  of cycles/triangles) as parameter.

### Our main contributions:

- We prove that MAXATT and MAXACT are NP-hard (Theorems 4 and 6). As a consequence, we also show that ACT and ATT do not admit algorithms with  $\mathcal{O}^*(2^{o(\sqrt{k})})$  running time under the Exponential-Time Hypothesis (Theorem 9). Moreover, deciding if a tournament has a cycle packing and a feedback arc set of the same size is NP-complete (Theorem 8).
- A tournament  $T$  has  $k$  arc-disjoint cycles if and only if  $T$  has  $k$  arc-disjoint cycles each of length at most  $2k + 1$  (Theorem 10).
- ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time (Theorem 16) and admits a kernel with  $\mathcal{O}(k)$  vertices (Theorem 15).
- ATT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time and admits a kernel with  $\mathcal{O}(k)$  vertices (Theorem 17).

## 2 Preliminaries

We denote the set  $\{1, 2, \dots, n\}$  of consecutive integers from 1 to  $n$  by  $[n]$ .

**Directed Graphs.** A *directed graph*  $D$  (or *digraph*) is a pair consisting of a finite set  $V(D)$  of *vertices* of  $D$  and a set  $A(D)$  of *arcs* of  $D$ , which are ordered pairs of elements of  $V(D)$ . For a vertex  $v \in V(D)$ , its *out-neighbourhood*, denoted by  $N^+(v)$ , is the set  $\{u \in V(D) : vu \in A(D)\}$  and its *out-degree*, denoted by  $d^+(v)$ , is  $|N^+(v)|$ . For a set  $F$  of arcs,  $V(F)$  denotes the union of the sets of endpoints of arcs in  $F$ . Given a digraph  $D$  and a subset  $X$  of vertices, we denote by  $D[X]$  the digraph induced by the vertices in  $X$ . Moreover, we denote by  $D \setminus X$  the digraph  $D[V(D) \setminus X]$  and say that this digraph is obtained by *deleting*  $X$  from  $D$ .

134 **Paths and Cycles.** A *path*  $P$  in a digraph  $D$  is a sequence  $(v_1, \dots, v_k)$  of distinct  
 135 vertices such that for each  $i \in [k-1]$ ,  $v_i v_{i+1} \in A(D)$ . The set  $\{v_1, \dots, v_k\}$  is denoted by  
 136  $V(P)$  and the set  $\{v_i v_{i+1} : i \in [k-1]\}$  is denoted by  $A(P)$ . A *cycle*  $C$  in  $D$  is a sequence  
 137  $(v_1, \dots, v_k)$  of distinct vertices such that  $(v_1, \dots, v_k)$  is a path and  $v_k v_1 \in A(D)$ . The length  
 138 of a path or cycle  $X$  is the number of vertices in it. A cycle on three vertices is called a  
 139 *triangle*. A digraph is called a *directed acyclic graph* if it has no cycles. A *feedback arc*  
 140 *set* (FAS) is a set of arcs whose deletion results in an acyclic graph. For a digraph  $D$ , let  
 141  $\text{minfas}(D)$  denote the size of a minimum FAS of  $D$ . Any directed acyclic graph  $D$  has an  
 142 ordering  $\sigma(D) = (v_1, \dots, v_n)$  called *topological ordering* of its vertices such that for each  
 143  $v_i v_j \in A(D)$ ,  $i < j$  holds. Given an ordering  $\sigma$  and two vertices  $u$  and  $v$ , we write  $u <_\sigma v$  if  
 144  $u$  is before  $v$  in  $\sigma$ .

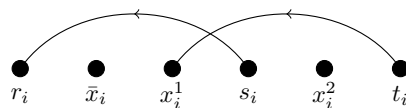
145 **Tournaments.** A *tournament*  $T$  is a digraph in which for every pair  $u, v$  of distinct  
 146 vertices either  $uv \in A(T)$  or  $vu \in A(T)$  but not both. In other words, a tournament  $T$  on  $n$   
 147 vertices is an orientation of the complete graph  $K_n$ . A tournament  $T$  can alternatively be  
 148 defined by an ordering  $\sigma(T) = (v_1, \dots, v_n)$  of its vertices and a set of *backward arcs*  $\overleftarrow{A}_\sigma(T)$   
 149 (which will be denoted  $\overleftarrow{A}(T)$  as the considered ordering is not ambiguous), where each arc  
 150  $a \in \overleftarrow{A}(T)$  is of the form  $v_{i_1} v_{i_2}$  with  $i_2 < i_1$ . Indeed, given  $\sigma(T)$  and  $\overleftarrow{A}(T)$ , we define  $V(T) =$   
 151  $\{v_i : i \in [n]\}$  and  $A(T) = \overleftarrow{A}(T) \cup \overrightarrow{A}(T)$  where  $\overrightarrow{A}(T) = \{v_{i_1} v_{i_2} : (i_1 < i_2) \text{ and } v_{i_2} v_{i_1} \notin \overleftarrow{A}(T)\}$  is  
 152 the set of *forward arcs* of  $T$  in the given ordering  $\sigma(T)$ . The pair  $(\sigma(T), \overleftarrow{A}(T))$  is called a *linear*  
 153 *representation* of the tournament  $T$ . A tournament is called *transitive* if it is a directed acyclic  
 154 graph and a transitive tournament has a unique topological ordering. Given two tournaments  
 155  $T_1, T_2$  defined by  $\sigma(T_l)$  and  $\overleftarrow{A}(T_l)$  with  $l \in \{1, 2\}$ , we denote by  $T = T_1 T_2$  the tournament  
 156 called the *concatenation of  $T_1$  and  $T_2$* , where  $V(T) = V(T_1) \cup V(T_2)$ ,  $\sigma(T) = \sigma(T_1)\sigma(T_2)$  is  
 157 the concatenation of the two sequences, and  $\overleftarrow{A}(T) = \overleftarrow{A}(T_1) \cup \overleftarrow{A}(T_2)$ .

### 158 **3 NP-hardness of MAXACT and MAXATT**

159 This section contains our main results. We prove the NP-hardness of MAXATT using a  
 160 reduction from 3-SAT(3). Recall that 3-SAT(3) corresponds to the specific case of 3-SAT  
 161 where each clause has at most three literals, and each literal appears at most two times  
 162 positively and exactly one time negatively. In the following, denote by  $F$  the input formula  
 163 of an instance of 3-SAT(3). Let  $n$  be the number of its variables and  $m$  be the number of  
 164 its clauses. We may suppose that  $n \equiv 3 \pmod{6}$ . If it is not the case, we can add up to 5  
 165 unused variables  $x$  with the trivial clause  $x \vee \bar{x}$ . This operation guarantees us we keep the  
 166 hypotheses of 3-SAT(3). We can also assume that  $m+1 \equiv 3 \pmod{6}$ . Indeed, if it not the  
 167 case, we add 6 new unused variables  $x_1, \dots, x_6$  with the 6 trivial clauses  $x_i \vee \bar{x}_i$ , and the  
 168 clause  $x_1 \vee x_2$ . This padding process keep both the 3-SAT(3) structure and  $n \equiv 3 \pmod{6}$ .  
 169 From  $F$  we construct a tournament  $T$  which is the concatenation of two tournaments  $T_v$  and  
 170  $T_c$  defined below.

171 In the following, let  $f$  be the reduction that maps an instance  $F$  of 3-SAT(3) to a  
 172 tournament  $T$  we describe now.

173 **The variable tournament  $T_v$ .** For each variable  $v_i$  of  $F$ , we define a tournament  $V_i$   
 174 of order 6 as follows:  $\sigma_i(V_i) = (r_i, \bar{x}_i, x_i^1, s_i, x_i^2, t_i)$  and  $\overleftarrow{A}_\sigma(V_i) = \{s_i r_i, t_i x_i^1\}$ . Figure 1 is  
 175 a representation of one variable gadget  $V_i$ . One can notice that the minimum FAS of  $V_i$   
 176 corresponds exactly to the set of its backward arcs. We now define  $V(T_v)$  be the union  
 177 of the vertex sets of the  $V_i$ s and we equip  $T_v$  with the order  $\sigma_1 \sigma_2 \dots \sigma_n$ . Thus,  $T_v$  has  $6n$   
 178 vertices. We also add the following backward arcs to  $T_v$ . Since  $n \equiv 3 \pmod{6}$ , there is an



■ **Figure 1** The variable gadget  $V_i$ . Only backward arcs are depicted, so all the remaining arcs are forward arcs.

179 edge-disjoint (undirected) triangle packing of  $K_n$  covering all its edges with triangles that  
 180 can be computed in polynomial time [33]. Let  $\{u_1, \dots, u_n\}$  be an arbitrary enumeration of  
 181 the vertices of  $K_n$ . Using a perfect triangle packing  $\Delta_{K_n}$  of  $K_n$ , we create a tournament  
 182  $T_{K_n}$  such that  $\sigma'(T_{K_n}) = (u_1, \dots, u_n)$  and  $\overleftarrow{A}_{\sigma'}(T_{K_n}) = \{u_k u_i : (u_i, u_j, u_k) \text{ is a triangle of}$   
 183  $\Delta_{K_n} \text{ with } i < j < k\}$ . Now we set  $\overleftarrow{A}_{\sigma}(T_v) = \{xy : x \in V(V_i), y \in V(V_j) \text{ for } i \neq j \text{ and}$   
 184  $u_j u_i \in \overleftarrow{A}_{\sigma'}(T_{K_n})\} \cup \bigcup_{i=1}^n \overleftarrow{A}_{\sigma}(V_i)$ . In some way, we “blew up” every vertex  $u_i$  of  $T_{K_n}$  into our  
 185 variable gadget  $V_i$ .

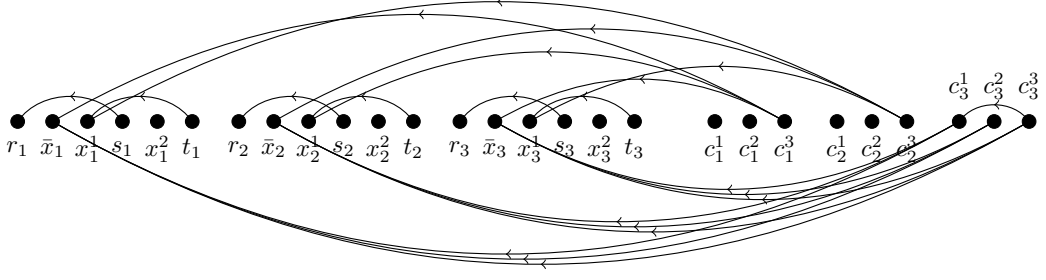
186 **The clause tournament  $T_c$ .** For each of the  $m$  clauses  $c_j$  of  $F$ , we define a tournament  
 187  $C_j$  of order 3 as follows:  $\sigma(C_j) = (c_j^1, c_j^2, c_j^3)$  and  $\overleftarrow{A}_{\sigma}(C_j) = \emptyset$ . In addition, we have a  
 188  $(m+1)^{\text{th}}$  tournament denoted by  $C_{m+1}$  and defined by  $\sigma(C_{m+1}) = (c_{m+1}^1, c_{m+1}^2, c_{m+1}^3)$   
 189 and  $\overleftarrow{A}_{\sigma}(C_{m+1}) = \{c_{m+1}^3 c_{m+1}^1\}$ , that is  $C_{m+1}$  is a triangle. We call this triangle the  
 190 *dummy triangle*, and its vertices the *dummy vertices*. We now define  $T_c$  such that  
 191  $\sigma(T_c)$  is the concatenation of each ordering  $\sigma(C_j)$  in the natural order, that is  $\sigma(T_c) =$   
 192  $(c_1^1, c_1^2, c_1^3, \dots, c_m^1, c_m^2, c_m^3, c_{m+1}^1, c_{m+1}^2, c_{m+1}^3)$ . So  $T_c$  has  $3(m+1)$  vertices. Since  $m+1 \equiv 3$   
 193 (mod 6), we use the same trick as above to add arcs to  $\overleftarrow{A}_{\sigma}(T_c)$  coming from a perfect packing  
 194 of undirected triangles of  $K_{m+1}$ . Once again, we “blew up” every vertex  $u_j$  of  $T_{K_{m+1}}$  into  
 195 our clause gadget  $C_j$ .

196 **The tournament  $T$ .** To define our final tournament  $T$  let us begin with its ordering  
 197  $\sigma$  defined by  $\sigma(T) = \sigma(T_v)\sigma(T_c)$ . Then we construct  $\overleftarrow{A}^{vc}(T)$  the backward arcs between  $T_c$   
 198 and  $T_v$ . For any  $j \in [m]$ , if the clause  $c_j$  in  $F$  has three literals, that is  $c_j = \ell_1 \vee \ell_2 \vee \ell_3$ , then  
 199 we add to  $\overleftarrow{A}^{vc}(T)$  the three backward arcs  $c_j^3 z_u$  where  $u \in [3]$  and such that  $z_u = \bar{x}_{i_u}$  when  
 200  $\ell_u = \bar{v}_{i_u}$ , and  $z_u \in \{x_{i_u}^1, x_{i_u}^2\}$  when  $\ell_u = v_{i_u}$  in such a way that for any  $i \in [n]$ , there exists a  
 201 unique arc  $a \in \overleftarrow{A}^{vc}(T)$  with  $h(a) = x_i^1$ . Informally, in the previous definition, if  $x_{i_u}^1$  is already  
 202 “used” by another clause, we chose  $z_u = x_{i_u}^2$ . Such an orientation will always be possible since  
 203 each variable occurs at most two times positively and once negatively in  $F$ . If the clause  $c_j$   
 204 in  $F$  has only two literals, that is  $c_j = \ell_1 \vee \ell_2$ , then we add in  $\overleftarrow{A}^{vc}(T)$  the two backward arcs  
 205  $c_j^2 z_u$  where  $u \in [2]$  and such that  $z_u = \bar{x}_{i_u}$  when  $\ell_u = \bar{v}_{i_u}$  and  $z_u \in \{x_{i_u}^1, x_{i_u}^2\}$  when  $\ell_u = v_{i_u}$   
 206 in such a way that for any  $i \in [n]$ , there exists a unique arc  $a \in \overleftarrow{A}^{vc}(T)$  with  $h(a) = x_i^1$ .  
 207 Finally, we add in  $\overleftarrow{A}^{vc}(T)$  the backward arcs  $c_{m+1}^u \bar{x}_i$  for any  $u \in [3]$  and  $i \in [n]$ . These arcs  
 208 are called *dummy arcs*. We set  $\overleftarrow{A}_{\sigma}(T) = \overleftarrow{A}_{\sigma}(T_v) \cup \overleftarrow{A}_{\sigma}(T_c) \cup \overleftarrow{A}^{vc}(T)$ . Notice that each  $\bar{x}_i$  has  
 209 exactly four arcs  $a \in \overleftarrow{A}_{\sigma}(T)$  such that  $h(a) = \bar{x}_i$  and  $t(a)$  is a vertex of  $T_c$ . To finish the  
 210 construction, notice also that  $T$  has  $6n + 3(m+1)$  vertices and can be computed in polynomial  
 211 time. Figure 2 is an example of the tournament obtained from a trivial 3-SAT(3) instance.

212 Now, we move on to proving the correctness of the reduction. First of all, observe that in  
 213 each variable gadget  $V_i$ , there are only four triangles: let  $\delta_i^1, \delta_i^2, \delta_i^3$  and  $\delta_i^4$  be the triangles  
 214  $(r_i, \bar{x}_i, s_i)$ ,  $(r_i, x_i^1, s_i)$ ,  $(x_i^1, s_i, t_i)$  and  $(x_i^1, x_i^2, t_i)$ , respectively. Moreover, notice that there are  
 215 only three maximal triangle packings of  $V_i$  which are  $\{\delta_i^1, \delta_i^3\}$ ,  $\{\delta_i^1, \delta_i^4\}$  and  $\{\delta_i^2, \delta_i^4\}$ . We call  
 216 these packings  $\Delta_i^{\top}$ ,  $\Delta_i^{\top'}$  and  $\Delta_i^{\perp}$ , respectively.

217 Given a triangle packing  $\Delta$  of  $T$  and a subset  $X$  of vertices, we define for any  $x \in X$





■ **Figure 2** Example of reduction obtained when  $F = \{c_1, c_2\}$  where  $c_1 = \bar{v}_1 \vee v_2 \vee \bar{v}_3$  and  $c_2 = v_1 \vee \bar{v}_2 \vee v_3$ . Forward arcs are not depicted. In addition to the depicted backward arcs, we have the 36 backward arcs from  $V_3$  to  $V_1$ , and the 9 backward arcs from  $C_3$  to  $C_1$ .

218 the  $\Delta$ -local out-degree of the vertex  $x$ , denoted  $d_{X \setminus \Delta}^+(x)$ , as the remaining out-degree  
 219 of  $x$  in  $T[X]$  when we remove the arcs of the triangles of  $\Delta$ . More formally, we set:  
 220  $d_{X \setminus \Delta}^+(x) = |\{xa : a \in X, xa \in A[X], xa \notin A(\Delta)\}|$ .

221 **► Remark.** Given a variable gadget  $V_i$ , we have:

- 222 (i)  $d_{V_i \setminus \Delta_i^\top}^+(x_i^1) = d_{V_i \setminus \Delta_i^\top}^+(x_i^2) = 1$  and  $d_{V_i \setminus \Delta_i^\top}^+(\bar{x}_i) = 3$ ,
- 223 (ii)  $d_{V_i \setminus \Delta_i^{\top'}}^+(x_i^1) = 1$ ,  $d_{V_i \setminus \Delta_i^{\top'}}^+(x_i^2) = 0$  and  $d_{V_i \setminus \Delta_i^{\top'}}^+(\bar{x}_i) = 3$ ,
- 224 (iii)  $d_{V_i \setminus \Delta_i^\perp}^+(x_i^1) = d_{V_i \setminus \Delta_i^\perp}^+(x_i^2) = 0$  and  $d_{V_i \setminus \Delta_i^\perp}^+(\bar{x}_i) = 4$ ,
- 225 (iv) none of  $\bar{x}_i x_i^1$ ,  $\bar{x}_i x_i^2$ ,  $\bar{x}_i t_i$  belongs to  $\Delta_i^\top$  or  $\Delta_i^\perp$ .

226 Informally, we want to set the variable  $x_i$  to true (resp. false) when one of the locally-  
 227 optimal  $\Delta_i^{\top'}$  or  $\Delta_i^\top$  (resp.  $\Delta_i^\perp$ ) is taken in the variable gadget  $V_i$  in the global solution. Now  
 228 given a triangle packing  $\Delta$  of  $T$ , we partition  $\Delta$  into the following sets:

- 229 ■  $\Delta_{V,V,V} = \{(a, b, c) \in \Delta : a \in V_i, b \in V_j, c \in V_k \text{ with } i < j < k\}$ ,
- 230 ■  $\Delta_{V,V,C} = \{(a, b, c) \in \Delta : a \in V_i, b \in V_j, c \in C_k \text{ with } i < j\}$ ,
- 231 ■  $\Delta_{V,C,C} = \{(a, b, c) \in \Delta : a \in V_i, b \in C_j, c \in C_k \text{ with } j < k\}$ ,
- 232 ■  $\Delta_{C,C,C} = \{(a, b, c) \in \Delta : a \in C_i, b \in C_j, c \in C_k \text{ with } i < j < k\}$ ,
- 233 ■  $\Delta_{2V,C} = \{(a, b, c) \in \Delta : a, b \in V_i, c \in C_j\}$ ,
- 234 ■  $\Delta_{V,2C} = \{(a, b, c) \in \Delta : a \in V_i, b, c \in C_j\}$ ,
- 235 ■  $\Delta_{3V} = \{(a, b, c) \in \Delta : a, b, c \in V_i\}$ ,
- 236 ■  $\Delta_{3C} = \{(a, b, c) \in \Delta : a, b, c \in C_i\}$ .

237 Notice that in  $T$ , there is no triangle with two vertices in a variable gadget  $V_i$  and its  
 238 third vertex in a variable gadget  $V_j$  with  $i \neq j$  since all the arcs between two variable gadgets  
 239 are oriented in the same direction. We have the same observation for clauses.

240 In the two next lemmas, we prove some properties concerning the solution  $\Delta$ , which imply  
 241 the result of Lemma 3.

242 **► Lemma 1.** *There exists a triangle packing  $\Delta^v$  (resp.  $\Delta^c$ ) which uses exactly the arcs between  
 243 distinct variable gadgets (resp. clause gadgets). Therefore, we have  $|\Delta_{V,V,V}| \leq 6n(n-1)$  and  
 244  $|\Delta_{C,C,C}| \leq 3m(m+1)/2$  and these bounds are tight.*

245 **Proof.** First recall that the tournament  $T_v$  is constructed from a tournament  $T_{K_n}$  which  
 246 admits a perfect packing of  $n(n-1)/6$  triangles. Then we replaced each vertex  $u_i$  in  
 247  $T_{K_n}$  by the variable gadget  $V_i$  and kept all the arcs between two variable gadgets  $V_i$

248 and  $V_j$  in the same orientation as between  $u_i$  and  $u_j$ . Let  $u_i u_j u_k$  be a triangle of the  
 249 perfect packing of  $T_{K_n}$ . We temporally relabel the vertices of  $V_i$ ,  $V_j$  and  $V_k$  respectively by  
 250  $\{f_i, i \in [6]\}$ ,  $\{g_i, i \in [6]\}$  and  $\{h_i, i \in [6]\}$  and consider the tripartite tournament  $K_{6,6,6}$  given  
 251 by  $V(K_{6,6,6}) = \{f_i, g_i, h_i, i \in [6]\}$  and  $A(K_{6,6,6}) = \{f_i g_j, g_i h_j, h_i f_j : i, j \in [6]\}$ . Then it is  
 252 easy to check that  $\{(f_i, g_j, h_{i+j \pmod{6}}) : i, j \in [6]\}$  is a perfect triangle packing of  $K_{6,6,6}$ .  
 253 Since every triangle of  $T_{K_n}$  becomes a  $K_{6,6,6}$  in  $T_v$ , we can find a triangle packing  $\Delta^v$  which  
 254 use all the arcs between disjoint variable gadgets. We use the same reasoning to prove that  
 255 there exists a triangle packing  $\Delta^c$  which use all the arcs available in  $T_c$  between two distinct  
 256 clause gadget. ◀

257 ▶ **Lemma 2.** *For any triangle packing  $\Delta$  of the tournament  $T$ , we have:*

- 258 (i)  $|\Delta_{V,V,V}| + |\Delta_{C,C,C}| \leq 6n(n-1) + 3m(m+1)/2$ ,
- 259 (ii)  $|\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{V,C,C}| + |\Delta_{V,V,C}| \leq |\overleftarrow{A}^{vc}(T)|$ ,
- 260 (iii)  $|\Delta_{3V}| \leq 2n$ ,
- 261 (iv)  $|\Delta_{3C}| \leq 1$ .

262 Therefore in total we have  $|\Delta| \leq 6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$ .

263 **Proof.** Let  $\Delta$  be a triangle packing of  $T$ . Recall that we have:  $|\Delta| = |\Delta_{V,V,V}| + |\Delta_{V,V,C}| +$   
 264  $|\Delta_{V,C,C}| + |\Delta_{C,C,C}| + |\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{3V}| + |\Delta_{3C}|$ . First, inequality (i) comes from  
 265 Lemma 1. Then, we have  $|\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{V,C,C}| + |\Delta_{V,V,C}| \leq |\overleftarrow{A}^{vc}(T)|$  since every triangle  
 266 of these sets consumes one backward arc from  $T_c$  to  $T_v$ . We have  $|\Delta_{3V}| \leq 2n$  since we have  
 267 at most 2 disjoint triangles in each variable gadget. Finally we also have  $|\Delta_{3C}| \leq 1$  since the  
 268 dummy triangle is the only triangle lying in a clause gadget. ◀

269 ▶ **Lemma 3.**  *$F$  is satisfiable if and only if there exists a triangle packing  $\Delta$  of size  $6n(n-1) +$   
 270  $3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$  in the tournament  $T$ .*

271 As 3-SAT(3) is NP-hard [41, 44], this implies the following theorem.

272 ▶ **Theorem 4.** *MAXATT is NP-hard.*

273 As mentioned in the introduction, packing arc-disjoint cycles is not necessarily equivalent  
 274 to packing arc-disjoint triangles. Thus, we need to establish the following lemma to transfer  
 275 the previous NP-hardness result to MAXACT.

276 ▶ **Lemma 5.** *Given a 3-SAT(3) instance  $F$ , and  $T$  the tournament constructed from  $F$   
 277 with the reduction  $f$ , we have a triangle packing  $\Delta$  of  $T$  of size  $6n(n-1) + 3m(m+1)/2 +$   
 278  $2n + |\overleftarrow{A}^{vc}(T)| + 1$  if and only if there is a cycle packing  $O$  of the same size.*

279 The previous lemma and Theorem 4 imply the following theorem.

280 ▶ **Theorem 6.** *MAXACT is NP-hard.*

281 Let us now define two special cases TIGHT-ATT (resp. TIGHT-ACT) where, given a  
 282 tournament  $T$  and a linear ordering  $\sigma$  with  $k$  backward arcs, where  $k = \text{minfas}(T)$ , the goal  
 283 is to decide if there is a triangle (resp. cycle) packing of size  $k$ . We call these special cases  
 284 the “tight” versions of the classical packing problems because as the input admits an FAS  
 285 of size  $k$ , any triangle (or cycle) packing has size at most  $k$ . We have the following result,  
 286 directly implying the NP-hardness of TIGHT-ATT and TIGHT-ACT.

287 ▶ **Lemma 7.** *Let  $T$  be a tournament constructed by the reduction  $f$ , and  $k$  be the threshold  
 288 value defined in Lemma 3. Then, we have  $k = \text{minfas}(T)$  and we can construct (in polynomial  
 289 time) an ordering of  $T$  with  $k$  backward arcs.*



290 ► **Theorem 8.** TIGHT-ATT and TIGHT-ACT are NP-hard.

291 Finally, the size  $s$  of the required packing in Lemma 3 satisfies  $s = \mathcal{O}((n+m)^2)$ . Under  
 292 the Exponential-time Hypothesis, the problem 3-SAT cannot be solved in  $2^{o(n+m)}$  [17, 31].  
 293 Then, using the linear reduction from 3-SAT to 3-SAT(3) [44], we also get the following  
 294 result.

295 ► **Theorem 9.** Under the Exponential-time Hypothesis, ATT and ACT cannot be solved in  
 296  $\mathcal{O}^*(2^{o(\sqrt{k})})$  time.

297 In the framework of parameterizing above guaranteed values [39], the above results imply  
 298 that ACT parameterized below the guaranteed value of the size of a minimal feedback arc  
 299 set is fixed-parameter intractable.

## 300 4 Parameterized Complexity of ACT

301 The classical Erdős-Pósa theorem for cycles in undirected graphs states that for each non-  
 302 negative integer  $k$ , every undirected graph either contains  $k$  vertex-disjoint cycles or has a  
 303 feedback vertex set consisting of  $f(k) = \mathcal{O}(k \log k)$  vertices [22]. An interesting consequence  
 304 of this theorem is that it leads to an FPT algorithm for VERTEX-DISJOINT CYCLE PACKING  
 305 (see [38] for more details).

306 Analogous to these results, we prove an Erdős-Pósa type theorem for tournaments and  
 307 show that it leads to an  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time algorithm and a linear vertex kernel for ACT.  
 308 First we obtain the following result.

309 ► **Theorem 10.** Let  $k$  and  $r$  be positive integers such that  $r \leq k$ . A tournament  $T$  contains  
 310 a set of  $r$  arc-disjoint cycles if and only if  $T$  contains a set of  $r$  arc-disjoint cycles each of  
 311 length at most  $2k + 1$ .

312 **Proof.** The reverse direction of the claim holds trivially. Let us now prove the forward  
 313 direction. Let  $\mathcal{C}$  be a set of  $r$  arc-disjoint cycles in  $T$  that minimizes  $\sum_{C \in \mathcal{C}} |C|$ . If every  
 314 cycle in  $\mathcal{C}$  is a triangle, then the claim trivially holds. Otherwise, let  $C$  be a longest cycle in  
 315  $\mathcal{C}$  and let  $\ell$  denote its length. Let  $v_i, v_j$  be a pair of non-consecutive vertices in  $C$ . Then,  
 316 either  $v_i v_j \in A(T)$  or  $v_j v_i \in A(T)$ . In any case, the arc  $e$  between  $v_i$  and  $v_j$  along with  $A(C)$   
 317 forms a cycle  $C'$  of length less than  $\ell$  with  $A(C') \setminus \{e\} \subset A(C)$ . By our choice of  $\mathcal{C}$ , this  
 318 implies that  $e$  is an arc in some other cycle  $\hat{C} \in \mathcal{C}$ . This property is true for the arc between  
 319 any pair of non-consecutive vertices in  $C$ . Therefore, we have  $\binom{\ell}{2} - \ell \leq \ell(k-1)$  leading to  
 320  $\ell \leq 2k + 1$ . ◀

321 This result essentially shows that it suffices to determine the existence of  $k$  arc-disjoint  
 322 cycles in  $T$  each of length at most  $2k + 1$  in order to determine if  $(T, k)$  is a yes-instance  
 323 of ACT. This immediately leads to a quadratic Erdős-Pósa bound. That is, for every  
 324 non-negative integer  $k$ , every tournament  $T$  either contains  $k$  arc-disjoint cycles or has an  
 325 FAS of size  $\mathcal{O}(k^2)$ . Next, we strengthen this result to arrive at a linear bound.

326 We will use the following lemma known from [15] in order to prove Theorem 12<sup>1</sup>. For a  
 327 digraph  $D$ , let  $\Lambda(D)$  denote the number of non-adjacent pairs of vertices in  $D$ . That is,  $\Lambda(D)$   
 328 is the number of pairs  $u, v$  of vertices of  $D$  such that neither  $uv \in A(D)$  nor  $vu \in A(D)$ .

<sup>1</sup> The authors would like to thank F. Havet for pointing out that Lemma 11 was a consequence of a result of [15], as well for an improvement of the constant in Theorem 12.

329 ▶ **Lemma 11.** [15] *Let  $D$  be a triangle-free digraph in which for every pair  $u, v$  of distinct*  
 330 *vertices, at most one of  $uv$  or  $vu$  is in  $A(D)$ . Then, we can compute an FAS of size at most*  
 331  *$\Lambda(D)$  in polynomial time.*

332 ▶ **Theorem 12.** *For every non-negative integer  $k$ , every tournament  $T$  either contains  $k$*   
 333 *arc-disjoint triangles or has an FAS of size at most  $5(k-1)$  that can be obtained in polynomial*  
 334 *time.*

335 **Proof.** Let  $\mathcal{C}$  be a maximal set of arc-disjoint triangles in  $T$  (that can be obtained greedily  
 336 in polynomial time). If  $|\mathcal{C}| \geq k$ , then we have the required set of triangles. Otherwise, let  
 337  $D$  denote the digraph obtained from  $T$  by deleting the arcs that are in some triangle in  
 338  $\mathcal{C}$ . Clearly,  $D$  has no triangle and  $\Lambda(D) \leq 3(k-1)$ . Let  $F$  be an FAS of  $D$  obtained in  
 339 polynomial time using Lemma 11. Then, we have  $|F| \leq 3(k-1)$ . Next, consider a topological  
 340 ordering  $\sigma$  of  $D - F$ . Each triangle of  $\mathcal{C}$  contains at most 2 arcs which are backward in this  
 341 ordering. If we denote by  $F'$  the set of all the arcs of the triangles of  $\mathcal{C}$  which are backward  
 342 in  $\sigma$ , then we have  $|F'| \leq 2(k-1)$  and  $(D - F) - F'$  is acyclic. Thus  $F^* = F \cup F'$  is an FAS  
 343 of  $T$  satisfying  $|F^*| \leq 5(k-1)$ . ◀

344 Next, we show how to obtain a linear kernel for ACT. This kernel is inspired by the  
 345 linear kernelization described in [10] for FAST and uses Theorem 12. Let  $T$  be a tournament  
 346 on  $n$  vertices. First, we apply the following reduction rule.

347 ▶ **Reduction Rule 4.1.** *If a vertex  $v$  is in no cycle, then delete  $v$  from  $T$ .*

348 This rule is clearly safe as our goal is to find  $k$  cycles and  $v$  cannot be in any of them.  
 349 To describe our next rule, we need to state a lemma known from [10]. An *interval* is a  
 350 consecutive set of vertices in a linear representation  $(\sigma(T), \overleftarrow{A}(T))$  of a tournament  $T$ .

351 ▶ **Lemma 13** ([10]). *Let  $T = (\sigma(T), \overleftarrow{A}(T))$  be a tournament on which Reduction Rule 4.1 is*  
 352 *not applicable. If  $|V(T)| \geq 2|\overleftarrow{A}(T)| + 1$ , then there exists a partition  $\mathcal{J}$  of  $V(T)$  into intervals*  
 353 *(that can be computed in polynomial time) such that there are  $|\overleftarrow{A}(T) \cap E| > 0$  arc-disjoint*  
 354 *cycles using only arcs in  $E$  where  $E$  denotes the set of arcs in  $T$  with endpoints in different*  
 355 *intervals.*

356 Our reduction rule that is based on this lemma is as follows.

357 ▶ **Reduction Rule 4.2.** *Let  $T = (\sigma(T), \overleftarrow{A}(T))$  be a tournament on which Reduction Rule*  
 358 *4.1 is not applicable. Let  $\mathcal{J}$  be a partition of  $V(T)$  into intervals satisfying the properties*  
 359 *specified in Lemma 13. Reverse all arcs in  $\overleftarrow{A}(T) \cap E$  and decrease  $k$  by  $|\overleftarrow{A}(T) \cap E|$  where  $E$*   
 360 *denotes the set of arcs in  $T$  with endpoints in different intervals.*

361 ▶ **Lemma 14.** *Reduction Rule 4.2 is safe.*

362 **Proof.** Let  $T'$  be the tournament obtained from  $T$  by reversing all arcs in  $\overleftarrow{A}(T) \cap E$ . Suppose  
 363  $T'$  has  $k - |\overleftarrow{A}(T) \cap E|$  arc-disjoint cycles. Then, it is guaranteed that each such cycle is  
 364 completely contained in an interval. This is due to the fact that  $T'$  has no backward arc  
 365 with endpoints in different intervals. Indeed, if a cycle in  $T'$  uses a forward (backward) arc  
 366 with endpoints in different intervals, then it also uses a back (forward) arc with endpoints in  
 367 different intervals. It follows that for each arc  $uv \in E$ , neither  $uv$  nor  $vu$  is used in these  
 368  $k - |\overleftarrow{A}(T) \cap E|$  cycles. Hence, these  $k - |\overleftarrow{A}(T) \cap E|$  cycles in  $T'$  are also cycles in  $T$ . Then,  
 369 we can add a set of  $|\overleftarrow{A}(T) \cap E|$  cycles obtained from the second property of Lemma 13 to  
 370 these  $k - |\overleftarrow{A}(T) \cap E|$  cycles to get  $k$  cycles in  $T$ . Conversely, consider a set of  $k$  cycles in

371  $T$ . As argued earlier, we know that the number of cycles that have an arc that is in  $E$  is at  
 372 most  $|\overleftarrow{A}(T) \cap E|$ . The remaining cycles (at least  $k - |\overleftarrow{A}(T) \cap E|$  of them) do not contain any  
 373 arc that is in  $E$ , in particular, they do not contain any arc from  $\overleftarrow{A}(T) \cap E$ . Therefore, these  
 374 cycles are also cycles in  $T'$ .  $\blacktriangleleft$

375 Thus, we have the following result.

376 **► Theorem 15.** *ACT admits a kernel with  $\mathcal{O}(k)$  vertices.*

377 **Proof.** Let  $(T, k)$  denote the instance obtained from the input instance by applying Reduction  
 378 Rule 4.1 exhaustively. From Lemma 12, we know that either  $T$  has  $k$  arc-disjoint triangles or  
 379 has an FAS of size at most  $5(k - 1)$  that can be obtained in polynomial time. In the first  
 380 case, we return a trivial yes-instance of constant size as the kernel. In the second case, let  $F$   
 381 be the FAS of size at most  $5(k - 1)$  of  $T$ . Let  $(\sigma(T), \overleftarrow{A}(T))$  be the linear representation of  $T$   
 382 where  $\sigma(T)$  is a topological ordering of the vertices of the directed acyclic graph  $T - F$ . As  
 383  $V(T - F) = V(T)$ ,  $|\overleftarrow{A}(T)| \leq 5(k - 1)$ . If  $|V(T)| \geq 10k - 9$ , then from Lemma 13, there is a  
 384 partition of  $V(T)$  into intervals with the specified properties. Therefore, Reduction Rule 4.2  
 385 is applicable (and the parameter drops by at least 1). When we obtain an instance where  
 386 neither of the Reduction Rules 4.1 and 4.2 is applicable, it follows that the tournament in  
 387 that instance has at most  $10k$  vertices.  $\blacktriangleleft$

388 Finally, we show that ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time. The idea is to reduce  
 389 the problem to the following ARC-DISJOINT PATHS problem in directed acyclic graphs:  
 390 given a digraph  $D$  on  $n$  vertices and  $k$  ordered pairs  $(s_1, t_1), \dots, (s_k, t_k)$  of vertices of  $D$ , do  
 391 there exist arc-disjoint paths  $P_1, \dots, P_k$  in  $D$  such that  $P_i$  is a path from  $s_i$  to  $t_i$  for each  
 392  $i \in [k]$ ? On directed acyclic graphs, ARC-DISJOINT PATHS is known to be NP-complete  
 393 [23], W[1]-hard [43] with respect to  $k$  as parameter and solvable in  $n^{\mathcal{O}(k)}$  time [28]. Despite  
 394 its fixed-parameter intractability, we will show that we can use the  $n^{\mathcal{O}(k)}$  algorithm and  
 395 Theorems 12 and 15 to describe an FPT algorithm for ACT.

396 **► Theorem 16.** *ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time.*

397 **Proof.** Consider an instance  $(T, k)$  of ACT. Using Theorem 15, we obtain a kernel  $\mathcal{I} = (\widehat{T}, \widehat{k})$   
 398 such that  $\widehat{T}$  has  $\mathcal{O}(k)$  vertices. Further,  $\widehat{k} \leq k$ . By definition,  $(T, k)$  is an yes-instance if  
 399 and only if  $(\widehat{T}, \widehat{k})$  is an yes-instance. Using Theorem 12, we know that  $\widehat{T}$  either contains  
 400  $\widehat{k}$  arc-disjoint triangles or has an FAS of size at most  $5(\widehat{k} - 1)$  that can be obtained in  
 401 polynomial time. If Theorem 12 returns a set of  $\widehat{k}$  arc-disjoint triangles in  $\widehat{T}$ , then we declare  
 402 that  $(T, k)$  is an yes-instance.

403 Otherwise, let  $\widehat{F}$  be the FAS of size at most  $5(\widehat{k} - 1)$  returned by Theorem 12. Let  
 404  $D$  denote the (acyclic) digraph obtained from  $\widehat{T}$  by deleting  $\widehat{F}$ . Observe that  $D$  has  $\mathcal{O}(k)$   
 405 vertices. Suppose  $\widehat{T}$  has a set  $\mathcal{C} = \{C_1, \dots, C_{\widehat{k}}\}$  of  $\widehat{k}$  arc-disjoint cycles. For each  $C \in \mathcal{C}$ , we  
 406 know that  $A(C) \cap \widehat{F} \neq \emptyset$  as  $\widehat{F}$  is an FAS of  $\widehat{T}$ . We can guess that subset  $F$  of  $\widehat{F}$  such that  
 407  $F = \widehat{F} \cap A(\mathcal{C})$ . Then, for each cycle  $C_i \in \mathcal{C}$ , we can guess the arcs  $F_i$  from  $F$  that it contains  
 408 and also the order  $\pi_i$  in which they appear. This information is captured as a partition  $\mathcal{F}$  of  
 409  $F$  into  $\widehat{k}$  sets,  $F_1$  to  $F_{\widehat{k}}$  and the set  $\{\pi_1, \dots, \pi_{\widehat{k}}\}$  of permutations where  $\pi_i$  is a permutation  
 410 of  $F_i$  for each  $i \in [\widehat{k}]$ . Any cycle  $C_i$  that has  $F_i \subseteq F$  contains a  $(v, x)$ -path between every  
 411 pair  $(u, v)$ ,  $(x, y)$  of consecutive arcs of  $F_i$  with arcs from  $A(D)$ . That is, there is a path  
 412 from  $h(\pi_i^{-1}(j))$  and  $t(\pi_i^{-1}((j + 1) \bmod |F_i|))$  with arcs from  $D$  for each  $j \in [|F_i|]$ . The total  
 413 number of such paths in these  $\widehat{k}$  cycles is  $\mathcal{O}(|F|)$  and the arcs of these paths are contained in  
 414  $D$  which is a (simple) directed acyclic graph.

415 The number of choices for  $F$  is  $2^{|\widehat{F}|}$  and the number of choices for a partition  $\mathcal{F} =$   
 416  $\{F_1, \dots, F_{\widehat{k}}\}$  of  $F$  and a set  $X = \{\pi_1, \dots, \pi_{\widehat{k}}\}$  of permutations is  $2^{\mathcal{O}(|\widehat{F}| \log |\widehat{F}|)}$ . Once such a  
 417 choice is made, the problem of finding  $\widehat{k}$  arc-disjoint cycles in  $\widehat{T}$  reduces to the problem of  
 418 finding  $\widehat{k}$  arc-disjoint cycles  $\mathcal{C} = \{C_1, \dots, C_{\widehat{k}}\}$  in  $\widehat{T}$  such that for each  $1 \leq i \leq \widehat{k}$  and for each  
 419  $1 \leq j \leq |F_i|$ ,  $C_i$  has a path  $P_{ij}$  between  $h(\pi_i^{-1}(j))$  and  $t(\pi_i^{-1}((j+1) \bmod |F_i|))$  with arcs  
 420 from  $D = \widehat{T} - \widehat{F}$ . This problem is essentially finding  $r = \mathcal{O}(|\widehat{F}|)$  arc-disjoint paths in  $D$  and  
 421 can be solved in  $|V(D)|^{\mathcal{O}(r)}$  time using the algorithm in [28]. Therefore, the overall running  
 422 time of the algorithm is  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  as  $|V(D)| = \mathcal{O}(k)$  and  $r = \mathcal{O}(k)$ . ◀

## 423 5 Parameterized Complexity of ATT

424 It is easy to obtain an  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time algorithm using the classical colour coding technique [5]  
 425 for packing subgraphs of bounded size, and in particular for ATT. Moreover, using matching  
 426 techniques, we also provide a kernel with a linear number of vertices.

427 In this section, we provide an FPT algorithm and a linear vertex kernel for ATT. First,  
 428 it is easy to obtain an  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time algorithm using the classical colour coding technique  
 429 [5] for packing subgraphs of bounded size.

430 ▶ **Theorem 17.** *ATT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time.*

431 **Proof.** Consider an instance  $\mathcal{I} = (T, k)$  of ATT. Let  $n$  denote  $|V(T)|$  and  $m$  denote  $|A(T)|$ .  
 432 Let  $\mathcal{F}$  denote the family of colouring functions  $c : A(T) \rightarrow [3k]$  of size  $2^{\mathcal{O}(k)} \log^2 m$  that  
 433 can be computed in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time using  $3k$ -perfect family of hash functions [?]. For each  
 434 colouring function  $c$  in  $\mathcal{F}$ , we colour  $A(T)$  according to  $c$  and find a triangle packing of size  
 435  $k$  whose arcs use different colours. We use a standard dynamic programming routine to  
 436 finding such a triangle packing. Clearly, if  $\mathcal{I}$  is an yes-instance and  $\mathcal{C}$  is a set of  $k$  arc-disjoint  
 437 triangles in  $T$ , there is a colouring function in  $\mathcal{F}$  that colours the  $3k$  arcs in these triangles  
 438 with distinct colours and our algorithm will find the required triangle packing. Given a  
 439 colouring  $c \in \mathcal{F}$ , we first compute for every set of 3 colours  $\{a, b, c\}$  whether the arcs coloured  
 440 with  $a, b$  or  $c$  induce a triangle using 3 different colours or not. Then, for every set  $S$  of  
 441  $3(p+1)$  colours with  $p \in [k-1]$ , we recursively test if the arcs coloured with the colours in  
 442  $S$  induce  $p+1$  arc-disjoint triangles whose arcs use all the colours of  $S$ . This is achieved by  
 443 iterating over every subset  $\{a, b, c\}$  of  $S$  and checking if there is a triangle using colours  $a, b$   
 444 and  $c$  and a collection of  $p$  arc-disjoint triangles whose arcs use all the colours of  $S \setminus \{a, b, c\}$ .  
 445 For a given  $S$ , we can find this collection of triangles in  $\mathcal{O}(p^3) = \mathcal{O}(k^3)$  time. Therefore, the  
 446 overall running time of the algorithm is  $\mathcal{O}^*(2^{\mathcal{O}(k)})$ . ◀

447 Next, we show that ATT has a linear vertex kernel.

448 ▶ **Theorem 18.** *ATT admits a kernel with  $\mathcal{O}(k)$  vertices.*

449 **Proof.** Let  $\mathcal{X}$  be a maximal collection of arc-disjoint triangles of a tournament  $T$  obtained  
 450 greedily. Let  $V_{\mathcal{X}}$  denote the vertices of the triangles in  $\mathcal{X}$  and  $A_{\mathcal{X}}$  denote the arcs of  $V_{\mathcal{X}}$ .  
 451 Let  $U$  be the remaining vertices of  $V(T)$ , i.e.,  $U = V(T) \setminus V_{\mathcal{X}}$ . If  $|\mathcal{X}| \geq k$ , then  $(T, k)$  is an  
 452 yes-instance of ATT. Otherwise,  $|\mathcal{X}| < k$  and  $|V_{\mathcal{X}}| < 3k$ . Moreover, notice that  $T[U]$  is acyclic  
 453 and  $T$  does not contain a triangle with one vertex in  $V_{\mathcal{X}}$  and two in vertices in  $U$  (otherwise  
 454  $\mathcal{X}$  would not be maximal).

455 Let  $B$  be the (undirected) bipartite graph defined by  $V(B) = A_{\mathcal{X}} \cup U$  and  $E(B) =$   
 456  $\{au : a \in A_{\mathcal{X}}, u \in U \text{ such that } (t(a), h(a), u) \text{ forms a triangle in } T\}$ . Let  $M$  be a maximum  
 457 matching of  $B$  and  $A'$  (resp.  $U'$ ) denote the vertices of  $A_{\mathcal{X}}$  (resp.  $U$ ) covered by  $M$ . Define  
 458  $\overline{A'} = A_{\mathcal{X}} \setminus A'$  and  $\overline{U'} = U \setminus U'$ .

459 We now prove that  $(V_{\mathcal{X}} \cup U', k)$  is a linear kernel of  $(T, k)$ . Let  $\mathcal{C}$  be a maximum sized  
 460 triangle packing that minimizes the number of vertices of  $\overline{U'}$  belonging to a triangle of  $\mathcal{C}$ . By  
 461 previous remarks, we can partition  $\mathcal{C}$  into  $C_{\mathcal{X}} \cup F$  where  $C_{\mathcal{X}}$  are the triangles of  $\mathcal{C}$  included  
 462 in  $T[V_{\mathcal{X}}]$  and  $F$  are the triangles of  $\mathcal{C}$  containing one vertex of  $U$  and two vertices of  $V_{\mathcal{X}}$ . It  
 463 is clear that  $F$  corresponds to a union of vertex-disjoint stars of  $B$  with centres in  $U$ . Denote  
 464 by  $U[F]$  the vertices of  $U$  clause gadget  $g$  to a triangle of  $F$ . If  $U[F] \subseteq U'$  then  $(V_{\mathcal{X}} \cup U', k)$   
 465 is immediately a kernel. Suppose there exists a vertex  $x_0$  such that  $x_0 \in U[F] \cap \overline{U'}$ .

466 We will build a tree rooted in  $x_0$  with edges alternating between  $F$  and  $M$ . For this let  
 467  $H_0 = \{x_0\}$  and construct recursively the sets  $H_{i+1}$  such that

$$468 \quad H_{i+1} = \begin{cases} N_F(H_i) & \text{if } i \text{ is even,} \\ N_M(H_i) & \text{if } i \text{ is odd,} \end{cases}$$

469 where, given a subset  $S \subseteq U$ ,  $N_F(S) = \{a \in A_{\mathcal{X}} : \exists s \in S \text{ s.t. } (t(a), h(a), s) \in F \text{ and } as \notin M\}$   
 470 and given a subset  $S \subseteq A_{\mathcal{X}}$ ,  $N_M(S) = \{u \in U : \exists a \in A_{\mathcal{X}} \text{ s.t. } au \in M\}$ . Notice that  $H_i \subseteq U$   
 471 when  $i$  is even and that  $H_i \subseteq A_{\mathcal{X}}$  when  $i$  is odd, and that all the  $H_i$  are distinct as  $F$  is a  
 472 union of disjoint stars and  $M$  a matching in  $B$ . Moreover, for  $i \geq 1$  we call  $T_i$  the set of edges  
 473 between  $H_i$  and  $H_{i-1}$ . Now we define the tree  $T$  such that  $V(T) = \bigcup_i H_i$  and  $E(T) = \bigcup_i T_i$ .  
 474 As  $T_i$  is a matching (if  $i$  is even) or a union of vertex-disjoint stars with centres in  $H_{i-1}$  (if  
 475  $i$  is odd), it is clear that  $T$  is a tree.

476 For  $i$  being odd, every vertex of  $H_i$  is incident to an edge of  $M$  otherwise  $B$  would contain  
 477 an augmenting path for  $M$ , a contradiction. So every leaf of  $T$  is in  $U$  and incident to an  
 478 edge of  $M$  in  $T$  and  $T$  contains as many edges of  $M$  than edges of  $F$ . Now for every arc  
 479  $a \in A_{\mathcal{X}} \cap V(T)$  we replace the triangle of  $\mathcal{C}$  containing  $a$  and corresponding to an edge of  $F$   
 480 by the triangle  $(t(a), h(a), u)$  where  $au \in M$  (and  $au$  is an edge of  $T$ ). This operation leads  
 481 to another collection of arc-disjoint triangles with the same size as  $\mathcal{C}$  but containing a strictly  
 482 smaller number of vertices in  $\overline{U'}$ , yielding a contradiction.

483 Finally  $V_{\mathcal{X}} \cup U'$  can be computed in polynomial time and we have  $|V_{\mathcal{X}} \cup U'| \leq |V_{\mathcal{X}}| + |M| \leq$   
 484  $2|V_{\mathcal{X}}| \leq 6k$ , which proves that the kernel has  $\mathcal{O}(k)$  vertices.  $\blacktriangleleft$

## 485 6 Concluding Remarks

486 In this work, we studied the classical and parameterized complexity of packing arc-disjoint  
 487 cycles and triangles in tournaments. We showed NP-hardness, fixed-parameter tractability  
 488 and linear kernelization results. An interesting problem could be to find subclasses of  
 489 tournaments where these problems are polynomial-time solvable. For instance, we show  
 490 in the full version of the paper that it is the case for sparse tournaments, that is for  
 491 tournaments which admit an FAS that is a matching. This class of tournaments is worthy of  
 492 attention for these packing problems as packing vertex-disjoint triangles (and hence cycles)  
 493 in sparse tournaments is NP-complete [8]. To conclude, observe that very few problems on  
 494 tournaments are known to admit an  $\mathcal{O}^*(2^{\sqrt{k}})$ -time algorithm when parameterized by the  
 495 standard parameter  $k$  [42] - FAST is one of them [4, 24]. To the best of our knowledge,  
 496 outside bidimensionality theory, there are no packing problems that are known to admit such  
 497 subexponential algorithms. In light of the  $2^{o(\sqrt{k})}$  lower bound shown for ACT and ATT, it  
 498 would be interesting to explore if these problems admit  $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$  algorithms.

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# 1 Packing Arc-Disjoint Cycles in Tournaments \*

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## 24 — Abstract —

25 A tournament is a directed graph in which there is a single arc between every pair of distinct  
26 vertices. Given a tournament  $T$  on  $n$  vertices, we explore the classical and parameterized com-  
27 plexity of the problems of determining if  $T$  has a cycle packing (a set of pairwise arc-disjoint  
28 cycles) of size  $k$  and a triangle packing (a set of pairwise arc-disjoint triangles) of size  $k$ . We  
29 refer to these problems as ARC-DISJOINT CYCLES IN TOURNAMENTS (ACT) and ARC-DISJOINT  
30 TRIANGLES IN TOURNAMENTS (ATT), respectively. Although the maximization version of ACT  
31 can be seen as the linear programming dual of the well-studied problem of finding a minimum  
32 feedback arc set (a set of arcs whose deletion results in an acyclic graph) in tournaments, sur-  
33 prisingly no algorithmic results seem to exist for ACT. We first show that ACT and ATT are  
34 both NP-complete. Then, we show that the problem of determining if a tournament has a cycle  
35 packing and a feedback arc set of the same size is NP-complete. Next, we prove that ACT is  
36 fixed-parameter tractable and admits a polynomial kernel when parameterized by  $k$ . In particu-  
37 lar, we show that ACT has a kernel with  $\mathcal{O}(k)$  vertices and can be solved in  $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$  time.  
38 Then, we show that ATT too has a kernel with  $\mathcal{O}(k)$  vertices and can be solved in  $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$   
39 time. Afterwards, we describe polynomial-time algorithms for ACT and ATT when the input  
40 tournament has a feedback arc set that is a matching. We also prove that ACT and ATT cannot  
41 be solved in  $2^{\mathcal{O}(\sqrt{k})} n^{\mathcal{O}(1)}$  time under the Exponential-Time Hypothesis.

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\* This paper is based on the two independent manuscripts [10] and [38].



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593

## 46 **1** Introduction

47 Given a (directed or undirected) graph  $G$  and a positive integer  $k$ , the DISJOINT CYCLE  
 48 PACKING problem is to determine whether  $G$  has  $k$  (vertex or arc/edge) disjoint (directed  
 49 or undirected) cycles. Packing disjoint cycles is a fundamental problem in Graph Theory  
 50 and Algorithm Design with applications in several areas. Since the publication of the classic  
 51 Erdős-Pósa theorem in 1965 [26], this problem has received significant scientific attention in  
 52 various algorithmic realms. In particular, VERTEX-DISJOINT CYCLE PACKING in undirected  
 53 graphs is one of the first problems studied in the framework of parameterized complexity.  
 54 In this framework, each problem instance is associated with a non-negative integer  $k$  called  
 55 *parameter*, and a problem is said to be *fixed-parameter tractable* (FPT) if it can be solved in  
 56  $f(k)n^{\mathcal{O}(1)}$  time for some computable function  $f$ , where  $n$  is the input size. For convenience,  
 57 the running time  $f(k)n^{\mathcal{O}(1)}$  where  $f$  grows super-polynomially with  $k$  is denoted as  $\mathcal{O}^*(f(k))$ .  
 58 A *kernelization algorithm* is a polynomial-time algorithm that transforms an arbitrary instance  
 59 of the problem to an equivalent instance of the same problem whose size is bounded by some  
 60 computable function  $g$  of the parameter of the original instance. The resulting instance is  
 61 called a *kernel* and if  $g$  is a polynomial function, then it is called a *polynomial kernel* and  
 62 we say that the problem admits a polynomial kernel. A decidable parameterized problem  
 63 is FPT if and only if it has a kernel (not necessarily of polynomial size). Kernelization  
 64 typically involves applying a set of rules (called *reduction rules*) to the given instance to  
 65 produce another instance. A reduction rule is said to be *safe* if it is sound and complete,  
 66 i.e., applying it to the given instance produces an equivalent instance. In order to classify  
 67 parameterized problems as being FPT or not, the  $W$ -hierarchy is defined:  $\text{FPT} \subseteq W[1] \subseteq$   
 68  $W[2] \subseteq \dots \subseteq \text{XP}$ . It is believed that the subset relations in this sequence are all strict, and a  
 69 parameterized problem that is hard for some complexity class above FPT in this hierarchy  
 70 is said to be fixed-parameter intractable. As mentioned before, the set of parameterized  
 71 problems that admit a polynomial kernel is contained in the class FPT and it is believed  
 72 that this subset relation is also strict. Further details on parameterized algorithms can be  
 73 found in [21, 24, 29, 31].

74 VERTEX-DISJOINT CYCLE PACKING in undirected graphs is FPT with respect to the  
 75 solution size  $k$  [12, 43] but has no polynomial kernel unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$  [13]. In contrast,  
 76 EDGE-DISJOINT CYCLE PACKING in undirected graphs admits a kernel with  $\mathcal{O}(k \log k)$   
 77 vertices (and is therefore FPT) [13]. On directed graphs, these problems have many practical  
 78 applications (for example in biology [14, 23]) and they have been extensively studied [7, 40, 44].  
 79 It turns out that VERTEX-DISJOINT CYCLE PACKING and ARC-DISJOINT CYCLE PACKING  
 80 are equivalent and are  $W[1]$ -hard [39, 52]. Therefore, studying these problems on a subclass  
 81 of directed graphs is a natural direction of research. Tournaments form a mathematically  
 82 rich subclass of directed graphs with interesting structural and algorithmic properties [6, 46].  
 83 A *tournament* is a directed graph in which there is a single arc between every pair of distinct  
 84 vertices. Tournaments have several applications in modeling round-robin tournaments and in  
 85 the study of voting systems and social choice theory [34, 36, 42]. Further, the combinatorics  
 86 of inclusion relations of tournaments is reasonably well-understood [16]. A seminal result in  
 87 the theory of undirected graphs is the Graph Minor Theorem (also known as the Robertson

and Seymour theorem) that states that undirected graphs are well-quasi-ordered under the *minor relation* [50]. Developing a similar theory of inclusion relations of directed graphs has been a long-standing research challenge. However, there is such a result known for tournaments that states that tournaments are well-quasi-ordered under the *strong immersion relation* [16].<sup>59†</sup> This is another reason why tournaments is one of the most well-studied classes of directed graphs. In fact, this result on containment theory also holds for a superclass of tournaments, namely, semicomplete digraphs [8]. A *semicomplete digraph* is a directed graph in which there is at least one arc between every pair of distinct vertices. Many results (including some of the ones described in this work) for tournaments straightaway hold for semicomplete digraphs too.

FEEDBACK VERTEX SET and FEEDBACK ARC SET are two well-explored algorithmic problems on tournaments. A *feedback vertex (arc) set* is a set of vertices (arcs) whose deletion results in an acyclic graph. Given a tournament, MINFAST and MINFVST are the problems of obtaining a feedback arc set and feedback vertex set of minimum size, respectively. We refer to the corresponding decision version of the problems as FAST and FVST. The optimization problems MINFAST and MINFVST have numerous practical applications in the areas of voting theory [22, 42], machine learning [18], search engine ranking [25] and have been intensively studied in various algorithmic areas. MINFAST and MINFVST are NP-hard [3, 15, 19, 53] while FAST and FVST are FPT when parameterized by the solution size  $k$  [4, 28, 30, 36, 49]. Further, FAST has a kernel with  $\mathcal{O}(k)$  vertices [11] and FVST has a kernel with  $\mathcal{O}(k^{1.5})$  vertices [41]. Surprisingly, the duals (in the linear programming sense) of MINFAST and MINFVST have not been considered in the literature until recently. Any tournament that has a cycle also has a triangle [7]. Therefore, if a tournament has  $k$  vertex-disjoint cycles, then it also has  $k$  vertex-disjoint triangles. Thus, VERTEX-DISJOINT CYCLE PACKING in tournaments is just packing vertex-disjoint triangles. This problem is NP-hard [9]. A straightforward application of the *colour coding* technique [5] shows that this problem is FPT and a kernel with  $\mathcal{O}(k^2)$  vertices is an immediate consequence of the quadratic element kernel known for 3-SET PACKING [1]. Recently, a kernel with  $\mathcal{O}(k^{1.5})$  vertices was shown for this problem using interesting variants and generalizations of the popular *expansion lemma* [41].

It is easy to verify that a tournament that has  $k$  arc-disjoint cycles need not necessarily have  $k$  arc-disjoint triangles. This observation hints that packing arc-disjoint cycles could be significantly harder than packing vertex-disjoint cycles. Further, it also hints that the problems of packing arc-disjoint cycles and arc-disjoint triangles in tournaments could have different complexities. This is the starting point of our study. Subsequently, we refer to a set of pairwise arc-disjoint cycles as a *cycle packing* and a set of pairwise arc-disjoint triangles as a *triangle packing*. Given a tournament, MAXACT and MAXATT are the problems of obtaining a maximum set of arc-disjoint cycles and triangles, respectively. We refer to the corresponding decision version of the problems as ACT and ATT. Formally, given a tournament  $T$  and a positive integer  $k$ , ACT is the task of determining if  $T$  has  $k$  arc-disjoint cycles and ATT is the task of determining if  $T$  has  $k$  arc-disjoint triangles. MAXATT is a special case of 3-SET PACKING, by creating the hypergraph on the arc set of the tournament and each triangle becomes a hyperedge. The 3-SET PACKING problem admits a  $\frac{4}{3} + \varepsilon$  approximation [20], implying the same result for MAXATT. From a structural point of view, the problem of partitioning the arc set of a directed graph into a collection of triangles has been studied for regular tournaments [55], almost regular tournaments [2] and complete digraphs [33]. In this work, we study the classical complexity of MAXACT and MAXATT and the parameterized complexity of ACT and ATT with respect to the solution

size (i.e. the number  $k$  of cycles/triangles) as parameter. First, we show that MAXACT and MAXATT are NP-hard. Then, we show that ACT is FPT and admits a linear vertex kernel when parameterized by  $k$ . Next, we show that ATT is FPT and admits a linear vertex kernel when parameterized by  $k$ . Finally, we show that MAXACT and MAXATT are polynomial-time solvable on *sparse tournaments* (tournaments that have a feedback arc set that is a matching). This class of tournaments is interesting for cycle packing problems and packing vertex-disjoint triangles (and hence cycles) in sparse tournaments is NP-complete [9]. In particular, we show the following results.

- MAXATT and MAXACT are NP-hard (Theorems 4 and 6). As a consequence, we also show that ACT and ATT do not admit algorithms with  $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$  running time under the Exponential-Time Hypothesis (Theorem 10). Moreover, deciding if a tournament has a cycle packing and a feedback arc set of the same size is NP-complete (Theorem 9).
- A tournament  $T$  has  $k$  arc-disjoint cycles if and only if  $T$  has  $k$  arc-disjoint cycles each of length at most  $2k + 1$  (Theorem 11).
- ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time (Theorem 17) and admits a kernel with  $\mathcal{O}(k)$  vertices (Theorem 16).
- ATT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time (Theorem 18) and admits a kernel with  $\mathcal{O}(k)$  vertices (Theorem 19).
- MAXATT and MAXACT restricted to sparse tournaments is polynomial-time solvable (Theorem 22).

**Road Map.** The paper is organized as follows. In Section 2, we give some definitions related to directed graphs, paths, cycles and tournaments. In Section 3, we show the result on the NP-hardness of the problems considered. In Section 4, we show the parameterized complexity results of ACT. Then, in Section 5, we show the parameterized complexity results of ATT. Then, we show the polynomial-time solvability of MAXATT and MAXACT restricted to sparse tournaments in Section 6. Finally, we conclude with some remarks in Section 7.

## 2 Preliminaries

We denote the set  $\{1, 2, \dots, n\}$  of consecutive integers from 1 to  $n$  by  $[n]$ .

**Directed Graphs.** A *directed graph* (or *digraph*) is a pair consisting of a set  $V$  of vertices and a set  $A$  of arcs. An arc is specified as an ordered pair of vertices (called its endpoints). We will consider only simple unweighted digraphs. For a digraph  $D$ ,  $V(D)$  and  $A(D)$  denote the set of its vertices and the set of its arcs, respectively. Two vertices  $u, v$  are said to be *adjacent* in  $D$  if  $uv \in A(D)$  or  $vu \in A(D)$ . For an arc  $e = uv$ , we define  $h(e) = v$  as the head of  $e$  and  $t(e) = u$  as the tail of  $e$ . For a vertex  $v \in V(D)$ , its *out-neighbourhood*, denoted by  $N^+(v)$ , is the set  $\{u \in V(D) : vu \in A(D)\}$  and its *in-neighbourhood*, denoted by  $N^-(v)$ , is the set  $\{u \in V(D) : uv \in A(D)\}$ . For a set  $F$  of arcs,  $V(F)$  denotes the union of the sets of endpoints of arcs in  $F$ . Given a digraph  $D$  and a subset  $X$  of vertices, we denote by  $D[X]$  the digraph induced by the vertices in  $X$ . Moreover, we denote by  $D \setminus X$  the digraph  $D[V(D) \setminus X]$  and say that this digraph is obtained by *deleting  $X$  from  $D$* . For a set  $F \subseteq A(D)$ ,  $D - F$  denotes the digraph obtained from  $D$  by deleting  $F$ .

**Paths and Cycles.** A *path*  $P$  in a digraph  $D$  is a sequence  $(v_1, \dots, v_k)$  of distinct vertices such that for each  $i \in [k - 1]$ ,  $v_i v_{i+1} \in A(D)$ . The set  $\{v_1, \dots, v_k\}$  is denoted by  $V(P)$  and the set  $\{v_i v_{i+1} : i \in [k - 1]\}$  is denoted by  $A(P)$ . A path  $P = (v_1, \dots, v_k)$  is called an *induced* (or *chordless*) path if  $A(P)$  are the only arcs of  $D[V(P)]$ . A *cycle*  $C$  in  $D$  is a sequence  $(v_1, \dots, v_k)$  of distinct vertices such that  $(v_1, \dots, v_k)$  is a path and  $v_k v_1 \in A(D)$ . The set

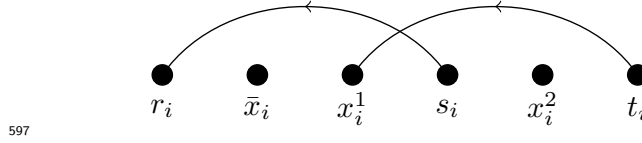
181  $\{v_1, \dots, v_k\}$  is denoted by  $V(C)$  and the set  $\{v_i v_{i+1} : i \in [k-1]\} \cup \{v_k v_1\}$  is denoted by  $A(C)$ .  
 182 A cycle  $C = (v_1, \dots, v_k)$  is called an *induced* (or *chordless*) cycle if  $A(C)$  are the only arcs  
 183 of  $D[V(C)]$ . The length of a path or cycle  $X$  is the number of vertices in it and is denoted  
 184 by  $|X|$ . For a set  $\mathcal{C}$  of paths or cycles,  $V(\mathcal{C})$  denotes the set  $\{v \in V(D) : \exists C \in \mathcal{C}, v \in V(C)\}$   
 185 and  $A(\mathcal{C})$  denotes the set  $\{e \in A(D) : \exists C \in \mathcal{C}, e \in A(C)\}$ . A cycle on three vertices is called  
 186 a *triangle*. A digraph is said to be *triangle-free* if it has no triangles. A set of pairwise  
 187 arc-disjoint cycles is called a *cycle packing* and a set of pairwise arc-disjoint triangles is called  
 188 a *triangle packing*. A digraph is called a *directed acyclic graph* if it has no cycles. A *feedback*  
 189 *arc set* (FAS) is a set of arcs whose deletion results in an acyclic graph. For a digraph  $D$ ,  
 190 let  $\text{minfas}(D)$  denote the size of a minimum FAS of  $D$ . Any directed acyclic graph  $D$  has  
 191 an ordering  $\sigma(D) = (v_1, \dots, v_n)$  called *topological ordering* of its vertices such that for each  
 192  $v_i v_j \in A(D)$ ,  $i < j$  holds. Given an ordering  $\sigma$  and two vertices  $u$  and  $v$ , we write  $u <_\sigma v$  if  
 193  $u$  is before  $v$  in  $\sigma$ .

194 **Tournaments.** A *tournament*  $T$  is a digraph in which for every pair  $u, v$  of distinct vertices  
 195 either  $uv \in A(T)$  or  $vu \in A(T)$  but not both. In other words, a tournament  $T$  on  $n$  vertices  
 196 is an orientation of the complete graph  $K_n$ . A tournament  $T$  can alternatively be defined by  
 197 an ordering  $\sigma(T) = (v_1, \dots, v_n)$  of its vertices and a set of *backward arcs*  $\overleftarrow{A}_\sigma(T)$  (which will  
 198 be denoted  $\overleftarrow{A}(T)$  as the considered ordering is not ambiguous), where each arc  $a \in \overleftarrow{A}(T)$  is of  
 199 the form  $v_{i_1} v_{i_2}$  with  $i_2 < i_1$ . Indeed, given  $\sigma(T)$  and  $\overleftarrow{A}(T)$ , we define  $V(T) = \{v_i : i \in [n]\}$   
 200 and  $A(T) = \overleftarrow{A}(T) \cup \overrightarrow{A}(T)$  where  $\overrightarrow{A}(T) = \{v_{i_1} v_{i_2} : (i_1 < i_2) \text{ and } v_{i_2} v_{i_1} \notin \overleftarrow{A}(T)\}$  is the set  
 201 of *forward arcs* of  $T$  in the given ordering  $\sigma(T)$ . The pair  $(\sigma(T), \overleftarrow{A}(T))$  is called a *linear*  
 202 *representation* of the tournament  $T$ . A tournament is called *transitive* if it is a directed  
 203 acyclic graph and a transitive tournament has a unique topological ordering. It is clear that  
 204 for any linear representation  $(\sigma(T), \overleftarrow{A}(T))$  of  $T$  the set  $\overleftarrow{A}(T)$  is an FAS of  $T$ . A tournament  
 205 is *sparse* if it admits an FAS which is a matching. Given a linear representation  $(\sigma(T), \overleftarrow{A}(T))$   
 206 of a tournament  $T$ , a triangle  $C$  in  $T$  is a triple  $(v_{i_1}, v_{i_2}, v_{i_3})$  with  $i_l < i_{l+1}$  such that either  
 207  $v_{i_3} v_{i_1} \in \overleftarrow{A}(T)$ ,  $v_{i_3} v_{i_2} \notin \overleftarrow{A}(T)$  and  $v_{i_2} v_{i_1} \notin \overleftarrow{A}(T)$  (in this case we call  $C$  a *triangle with*  
 208 *backward arc*  $v_{i_3} v_{i_1}$ ), or  $v_{i_3} v_{i_1} \notin \overleftarrow{A}(T)$ ,  $v_{i_3} v_{i_2} \in \overleftarrow{A}(T)$  and  $v_{i_2} v_{i_1} \in \overleftarrow{A}(T)$  (in this case we  
 209 call  $C$  a *triangle with two backward arcs*  $v_{i_3} v_{i_2}$  and  $v_{i_2} v_{i_1}$ ). Given two tournaments  $T_1, T_2$   
 210 defined by  $\sigma(T_l)$  and  $\overleftarrow{A}(T_l)$  with  $l \in \{1, 2\}$ , we denote by  $T = T_1 T_2$  the tournament called  
 211 the *concatenation of  $T_1$  and  $T_2$* , where  $\overleftarrow{A}(T) = \overleftarrow{A}(T_2) \cup \overleftarrow{A}(T_1)$ ,  $\sigma(T) = \sigma(T_1)\sigma(T_2)$  is the  
 212 concatenation of the two sequences, and  $\overrightarrow{A}(T) = \overrightarrow{A}(T_1) \cup \overrightarrow{A}(T_2)$ .

### 213 3 NP-hardness of MAXACT and MAXATT

214 This section contains our main results. We prove the NP-hardness of MAXATT using a  
 215 reduction from 3-SAT(3). Recall that 3-SAT(3) corresponds to the specific case of 3-SAT  
 216 where each clause has at most three literals, and each literal appears at most two times  
 217 positively and exactly one time negatively. In the following, denote by  $F$  the input formula  
 218 of an instance of 3-SAT(3). Let  $n$  be the number of its variables and  $m$  be the number of  
 219 its clauses. We may suppose that  $n \equiv 3 \pmod{6}$ . If it is not the case, we can add up to 5  
 220 unused variables  $x$  with the trivial clause  $x \vee \bar{x}$ . This operation guarantees us we keep the  
 221 hypotheses of 3-SAT(3). We can also assume that  $m + 1 \equiv 3 \pmod{6}$ . Indeed, if it not the  
 222 case, we add 6 new unused variables  $x_1, \dots, x_6$  with the 6 trivial clauses  $x_i \vee \bar{x}_i$ , and the  
 223 clause  $x_1 \vee x_2$ . This padding process keep both the 3-SAT(3) structure and  $n \equiv 3 \pmod{6}$ .  
 224 From  $F$  we construct a tournament  $T$  which is the concatenation of two tournaments  $T_v$  and  
 225  $T_c$  defined below.





597 **Figure 1** The variable gadget  $V_i$ . Only backward arcs are depicted, so all the remaining arcs are forward arcs.

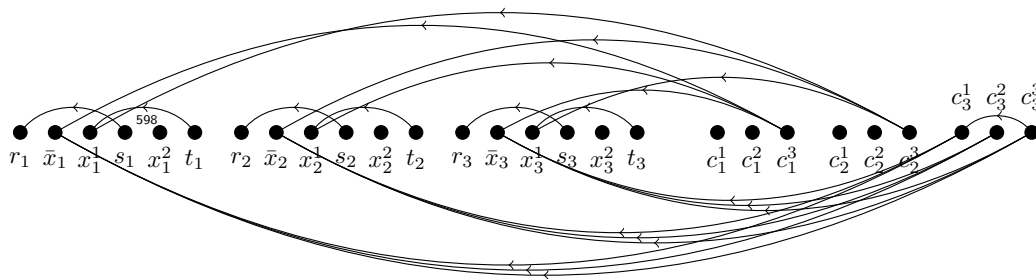
226 In the following, let  $f$  be the reduction that maps an instance  $F$  of 3-SAT(3) to a  
 227 tournament  $T$  we describe now.

228 **The variable tournament  $T_v$ .** For each variable  $v_i$  of  $F$ , we define a tournament  $V_i$  of  
 229 order 6 as follows:  $\sigma_i(V_i) = (r_i, \bar{x}_i, x_i^1, s_i, x_i^2, t_i)$  and  $\overleftarrow{A}_\sigma(V_i) = \{s_i r_i, t_i x_i^1\}$ . Figure 1 is a  
 230 representation of one variable gadget  $V_i$ . One can notice that the minimum FAS of  $V_i$   
 231 corresponds exactly to the set of its backward arcs. We now define  $V(T_v)$  be the union  
 232 of the vertex sets of the  $V_i$ s and we equip  $T_v$  with the order  $\sigma_1 \sigma_2 \dots \sigma_n$ . Thus,  $T_v$  has  $6n$   
 233 vertices. We also add the following backward arcs to  $T_v$ . Since  $n \equiv 3 \pmod{6}$ , there is an  
 234 edge-disjoint (undirected) triangle packing of  $K_n$  covering all its edges with triangles that  
 235 can be computed in polynomial time [37]. Let  $\{u_1, \dots, u_n\}$  be an arbitrary enumeration of  
 236 the vertices of  $K_n$ . Using a perfect triangle packing  $\Delta_{K_n}$  of  $K_n$ , we create a tournament  
 237  $T_{K_n}$  such that  $\sigma'(T_{K_n}) = (u_1, \dots, u_n)$  and  $\overleftarrow{A}_{\sigma'}(T_{K_n}) = \{u_k u_i : (u_i, u_j, u_k) \text{ is a triangle of}$   
 238  $\Delta_{K_n}$  with  $i < j < k\}$ . Now we set  $\overleftarrow{A}_\sigma(T_v) = \{xy : x \in V(V_i), y \in V(V_j) \text{ for } i \neq j \text{ and}$   
 239  $u_j u_i \in \overleftarrow{A}_{\sigma'}(T_{K_n})\} \cup \bigcup_{i=1}^n \overleftarrow{A}_\sigma(V_i)$ . In some way, we “blew up” every vertex  $u_i$  of  $T_{K_n}$  into our  
 240 variable gadget  $V_i$ .

241 **The clause tournament  $T_c$ .** For each of the  $m$  clauses  $c_j$  of  $F$ , we define a tournament  $C_j$  of  
 242 order 3 as follows:  $\sigma(C_j) = (c_j^1, c_j^2, c_j^3)$  and  $\overleftarrow{A}_\sigma(C_j) = \emptyset$ . In addition, we have a  $(m+1)^{th}$  tour-  
 243 nament denoted by  $C_{m+1}$  and defined by  $\sigma(C_{m+1}) = (c_{m+1}^1, c_{m+1}^2, c_{m+1}^3)$  and  $\overleftarrow{A}_\sigma(C_{m+1}) =$   
 244  $\{c_{m+1}^3 c_{m+1}^1\}$ , that is  $C_{m+1}$  is a triangle. We call this triangle the *dummy triangle*, and its ver-  
 245 tices the *dummy vertices*. We now define  $T_c$  such that  $\sigma(T_c)$  is the concatenation of each order-  
 246 ing  $\sigma(C_j)$  in the natural order, that is  $\sigma(T_c) = (c_1^1, c_1^2, c_1^3, \dots, c_m^1, c_m^2, c_m^3, c_{m+1}^1, c_{m+1}^2, c_{m+1}^3)$ .  
 247 So  $T_c$  has  $3(m+1)$  vertices. Since  $m+1 \equiv 3 \pmod{6}$ , we use the same trick as above to  
 248 add arcs to  $\overleftarrow{A}_\sigma(T_c)$  coming from a perfect packing of undirected triangles of  $K_{m+1}$ . Once  
 249 again, we “blew up” every vertex  $u_j$  of  $T_{K_{m+1}}$  into our clause gadget  $C_j$ .

250 **The tournament  $T$ .** To define our final tournament  $T$  let us begin with its ordering  $\sigma$   
 251 defined by  $\sigma(T) = \sigma(T_v)\sigma(T_c)$ . Then we construct  $\overleftarrow{A}^{vc}(T)$  the backward arcs between  $T_c$   
 252 and  $T_v$ . For any  $j \in [m]$ , if the clause  $c_j$  in  $F$  has three literals, that is  $c_j = \ell_1 \vee \ell_2 \vee \ell_3$ ,  
 253 then we add to  $\overleftarrow{A}^{vc}(T)$  the three backward arcs  $c_j^3 z_u$  where  $u \in [3]$  and such that  $z_u = \bar{x}_{i_u}$   
 254 when  $\ell_u = \bar{v}_{i_u}$ , and  $z_u \in \{x_{i_u}^1, x_{i_u}^2\}$  when  $\ell_u = v_{i_u}$  in such a way that for any  $i \in [n]$ , there  
 255 exists a unique arc  $a \in \overleftarrow{A}^{vc}(T)$  with  $h(a) = x_{i_u}^1$ . Informally, in the previous definition, if  $x_{i_u}^1$   
 256 is already “used” by another clause, we chose  $z_u = x_{i_u}^2$ . Such an orientation will always be  
 257 possible since each variable occurs at most two times positively and once negatively in  $F$ . If  
 258 the clause  $c_j$  in  $F$  has only two literals, that is  $c_j = \ell_1 \vee \ell_2$ , then we add in  $\overleftarrow{A}^{vc}(T)$  the two  
 259 backward arcs  $c_j^2 z_u$  where  $u \in [2]$  and such that  $z_u = \bar{x}_{i_u}$  when  $\ell_u = \bar{v}_{i_u}$  and  $z_u \in \{x_{i_u}^1, x_{i_u}^2\}$   
 260 when  $\ell_u = v_{i_u}$  in such a way that for any  $i \in [n]$ , there exists a unique arc  $a \in \overleftarrow{A}^{vc}(T)$  with  
 261  $h(a) = x_{i_u}^1$ .

262 Finally, we add in  $\overleftarrow{A}^{vc}(T)$  the backward arcs  $c_{m+1}^u \bar{x}_i$  for any  $u \in [3]$  and  $i \in [n]$ . These arcs  
 263 are called *dummy arcs*. We set  $\overleftarrow{A}_\sigma(T) = \overleftarrow{A}_\sigma(T_v) \cup \overleftarrow{A}_\sigma(T_c) \cup \overleftarrow{A}^{vc}(T)$ . Notice that each  $\bar{x}_i$  has



■ **Figure 2** Example of reduction obtained when  $F = \{c_1, c_2\}$  where  $c_1 = \bar{v}_1 \vee v_2 \vee \bar{v}_3$  and  $c_2 = v_1 \vee \bar{v}_2 \vee v_3$ . Forward arcs are not depicted. In addition to the depicted backward arcs, we have the 36 backward arcs from  $V_3$  to  $V_1$ , and the 9 backward arcs from  $C_3$  to  $C_1$ .

264 exactly four arcs  $a \in \overleftarrow{A}_\sigma(T)$  such that  $h(a) = \bar{x}_i$  and  $t(a)$  is a vertex of  $T_c$ . To finish the  
 265 construction, notice also that  $T$  has  $6n + 3(m + 1)$  vertices and can be computed in polynomial  
 266 time. Figure 2 is an example of the tournament obtained from a trivial 3-SAT(3) instance.

267 Now, we move on to proving the correctness of the reduction. First of all, observe that in  
 268 each variable gadget  $V_i$ , there are only four triangles: let  $\delta_i^1, \delta_i^2, \delta_i^3$  and  $\delta_i^4$  be the triangles  
 269  $(r_i, \bar{x}_i, s_i)$ ,  $(r_i, x_i^1, s_i)$ ,  $(x_i^1, s_i, t_i)$  and  $(x_i^1, x_i^2, t_i)$ , respectively. Moreover, notice that there are  
 270 only three maximal triangle packings of  $V_i$  which are  $\{\delta_i^1, \delta_i^3\}$ ,  $\{\delta_i^1, \delta_i^4\}$  and  $\{\delta_i^2, \delta_i^4\}$ . We call  
 271 these packings  $\Delta_i^\top$ ,  $\Delta_i^{\top'}$  and  $\Delta_i^\perp$ , respectively.

272 Given a triangle packing  $\Delta$  of  $T$  and a subset  $X$  of vertices, we define for any  $x \in X$   
 273 the  $\Delta$ -local out-degree of the vertex  $x$ , denoted  $d_{X \setminus \Delta}^+(x)$ , as the remaining out-degree  
 274 of  $x$  in  $T[X]$  when we remove the arcs of the triangles of  $\Delta$ . More formally, we set:  
 275  $d_{X \setminus \Delta}^+(x) = |\{xa : a \in X, xa \in A[X], xa \notin A(\Delta)\}|$ .

276 ► **Remark.** Given a variable gadget  $V_i$ , we have:

- 277 (i)  $d_{V_i \setminus \Delta_i^\top}^+(x_i^1) = d_{V_i \setminus \Delta_i^\top}^+(x_i^2) = 1$  and  $d_{V_i \setminus \Delta_i^\top}^+(\bar{x}_i) = 3$ ,  
 278 (ii)  $d_{V_i \setminus \Delta_i^{\top'}}^+(x_i^1) = 1$ ,  $d_{V_i \setminus \Delta_i^{\top'}}^+(x_i^2) = 0$  and  $d_{V_i \setminus \Delta_i^{\top'}}^+(\bar{x}_i) = 3$ ,  
 279 (iii)  $d_{V_i \setminus \Delta_i^\perp}^+(x_i^1) = d_{V_i \setminus \Delta_i^\perp}^+(x_i^2) = 0$  and  $d_{V_i \setminus \Delta_i^\perp}^+(\bar{x}_i) = 4$ ,  
 280 (iv) none of  $\bar{x}_i x_i^1, \bar{x}_i x_i^2, \bar{x}_i t_i$  belongs to  $\Delta_i^\top$  or  $\Delta_i^\perp$ .

281 Informally, we want to set the variable  $x_i$  to true (resp. false) when one of the locally-  
 282 optimal  $\Delta_i^{\top'}$  or  $\Delta_i^\top$  (resp.  $\Delta_i^\perp$ ) is taken in the variable gadget  $V_i$  in the global solution. Now  
 283 given a triangle packing  $\Delta$  of  $T$ , we partition  $\Delta$  into the following sets:

- 284 ■  $\Delta_{V,V,V} = \{(a, b, c) \in \Delta : a \in V_i, b \in V_j, c \in V_k \text{ with } i < j < k\}$ ,  
 285 ■  $\Delta_{V,V,C} = \{(a, b, c) \in \Delta : a \in V_i, b \in V_j, c \in C_k \text{ with } i < j\}$ ,  
 286 ■  $\Delta_{V,C,C} = \{(a, b, c) \in \Delta : a \in V_i, b \in C_j, c \in C_k \text{ with } j < k\}$ ,  
 287 ■  $\Delta_{C,C,C} = \{(a, b, c) \in \Delta : a \in C_i, b \in C_j, c \in C_k \text{ with } i < j < k\}$ ,  
 288 ■  $\Delta_{2V,C} = \{(a, b, c) \in \Delta : a, b \in V_i, c \in C_j\}$ ,  
 289 ■  $\Delta_{V,2C} = \{(a, b, c) \in \Delta : a \in V_i, b, c \in C_j\}$ ,  
 290 ■  $\Delta_{3V} = \{(a, b, c) \in \Delta : a, b, c \in V_i\}$ ,  
 291 ■  $\Delta_{3C} = \{(a, b, c) \in \Delta : a, b, c \in C_i\}$ .

292 Notice that in  $T$ , there is no triangle with two vertices in a variable gadget  $V_i$  and its  
 293 third vertex in a variable gadget  $V_j$  with  $i \neq j$  since all the arcs between two variable gadgets  
 294 are oriented in the same direction. We have the same observation for clauses.

295 In the two next lemmas, we prove some properties concerning the solution  $\Delta$ .

296 ► **Lemma 1.** *There exists a triangle packing  $\Delta^v$  (resp.  $\Delta^c$ ) which uses exactly the arcs between*  
 297 *distinct variable gadgets (resp. clause gadgets). Therefore, we have  $|\Delta_{V,V,V}| \leq 6n(n-1)$  and*  
 298  *$|\Delta_{C,C,C}| \leq 3m(m+1)/2$  and these bounds are tight.*

299 **Proof.** First recall that the tournament  $T_v$  is constructed from a tournament  $T_{K_n}$  which  
 300 admits a perfect packing of  $n(n-1)/6$  triangles. Then we replaced each vertex  $u_i$  in  $T_{K_n}$   
 301 by the variable gadget  $V_i$  and kept all the arcs between two variable gadgets  $V_i$  and  $V_j$   
 302 in the same orientation as between  $u_i$  and  $u_j$ . Let  $u_i u_j u_k$  be a triangle of the perfect packing  
 303 of  $T_{K_n}$ . We temporarily relabel the vertices of  $V_i$ ,  $V_j$  and  $V_k$  respectively by  $\{f_i: i \in [6]\}$ ,  
 304  $\{g_i: i \in [6]\}$  and  $\{h_i: i \in [6]\}$  and consider the tripartite tournament  $K_{6,6,6}$  given by  
 305  $V(K_{6,6,6}) = \{f_i, g_i, h_i: i \in [6]\}$  and  $A(K_{6,6,6}) = \{f_i g_j, g_i h_j, h_i f_j: i, j \in [6]\}$ . Then it is easy  
 306 to check that  $\{(f_i, g_j, h_{i+j \pmod{6}}): i, j \in [6]\}$  is a perfect triangle packing of  $K_{6,6,6}$ . Since  
 307 every triangle of  $T_{K_n}$  becomes a  $K_{6,6,6}$  in  $T_v$ , we can find a triangle packing  $\Delta^v$  which use  
 308 all the arcs between disjoint variable gadgets. We use the same reasoning to prove that there  
 309 exists a triangle packing  $\Delta^c$  which use all the arcs available in  $T_c$  between two distinct clause  
 310 gadget. ◀

311 ► **Lemma 2.** *For any triangle packing  $\Delta$  of the tournament  $T$ , we have the following*  
 312 *inequalities:*

- 313 (i)  $|\Delta_{V,V,V}| + |\Delta_{C,C,C}| \leq 6n(n-1) + 3m(m+1)/2$ ,
- 314 (ii)  $|\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{V,C,C}| + |\Delta_{V,V,C}| \leq |\overleftarrow{A}^{vc}(T)|$ , where  $|\overleftarrow{A}^{vc}(T)| = |\overleftarrow{A}^{vc}(T)|$ ,
- 315 (iii)  $|\Delta_{3V}| \leq 2n$ ,
- 316 (iv)  $|\Delta_{3C}| \leq 1$ .

317 Therefore in total we have  $|\Delta| \leq 6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$ .

318 **Proof.** Let  $\Delta$  be a triangle packing of  $T$ . Recall that we have:  $|\Delta| = |\Delta_{V,V,V}| + |\Delta_{V,V,C}| +$   
 319  $|\Delta_{V,C,C}| + |\Delta_{C,C,C}| + |\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{3V}| + |\Delta_{3C}|$ . First, inequality (i) comes from  
 320 Lemma 1. Then, we have  $|\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{V,C,C}| + |\Delta_{V,V,C}| \leq |\overleftarrow{A}^{vc}(T)|$  since every triangle  
 321 of these sets consumes one backward arc from  $T_c$  to  $T_v$ . We have  $|\Delta_{3V}| \leq 2n$  since we have  
 322 at most 2 disjoint triangles in each variable gadget. Finally we also have  $|\Delta_{3C}| \leq 1$  since the  
 323 dummy triangle is the only triangle lying in a clause gadget. ◀

324 These two lemmas allow us to prove the following.

325 ► **Lemma 3.**  *$F$  is satisfiable if and only if there exists a triangle packing  $\Delta$  of size  $6n(n-$*   
 326  *$1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$  in the tournament  $T$ .*

327 **Proof.** First, let suppose that there exists an assignment  $a$  of the variables which satisfies  $F$ ,  
 328 and let  $a^\top$  (resp.  $a^\perp$ ) be the set of variables set to true (resp. false).

329 We construct a triangle packing  $\Delta$  of  $T$  with the desired number of triangles. First, we  
 330 pick all the disjoint triangles of  $\Delta^v$  and  $\Delta^c$ . By Lemma 2, if we also add the dummy triangle  
 331  $(c_{m+1}^1, c_{m+1}^2, c_{m+1}^3)$  we have  $6n(n-1) + 3m(m+1)/2 + 1$  triangles in  $\Delta$  until now.

332 Then, for any variable  $v_i$  of the formula  $F$ , if  $v_i \in a^\top$ , then we add in  $\Delta$  the triangles  
 333  $\Delta_i^\top$ . Otherwise, we add  $\Delta_i^\perp$ . One can check that in both cases, these triangles are disjoint to  
 334 the triangles we just added. Thus, in each  $V_i$ , we made an locally-optimal solution, so we  
 335 added  $2n$  triangles in  $\Delta$ .

336 Now we add in  $\Delta$  the triangles  $(\bar{x}_i, t_i, c_{m+1}^1)$ ,  $(\bar{x}_i, x_i^1, c_{m+1}^2)$  and  $(\bar{x}_i, x_i^2, c_{m+1}^3)$  which will  
 337 consume all the dummy arcs of the tournament. Recall that in Remark 3 we mentioned  
 338 that the vertices  $x_i^1$  and  $x_i^2$  (resp.  $\bar{x}_i$ ) have an  $\Delta_i^\top$ -local out-degree both equal to 1 (resp.  
 339  $\Delta_i^\perp$ -local out-degree equals to 4). Then given a clause  $c_j$ , let  $\ell$  be one literal which satisfies  
 340  $c_j$ . Assume that the clause is of size 3, since the reasoning is the same for clauses of size 2.

341 If  $\ell$  is a positive literal, say  $v_i$ , then let  $u$  be the number such that  $c_j^3 x_i^u$  is a backward arc  
 342 of  $T$ . By Remark 3, we know that there exists  $v \in V_i$  such that the arc  $x_i^u v$  is available to  
 343 make the triangle  $(x_i^u, v, c_j^3)$ . Otherwise, that is if  $\ell$  is a negative literal, say  $\bar{v}_i$ , then we have  
 344  $d_{V_i \setminus \Delta_i^+}^+(\bar{x}_i) = 4$ . Three of these four available arcs are used in the triangles which consume  
 345 the dummy arcs, then we can still make the triangle  $(\bar{x}_i, s_i, c_j^3)$ . Let also  $\ell_1$  and  $\ell_2$  be the two  
 346 other literals of  $c_j$  (which do not necessarily satisfy  $c_j$ ). Denote by  $a_1$  and  $a_2$  the vertices of  
 347  $T_v$  connected to  $c_j^3$  corresponding to the literals  $\ell_1$  and  $\ell_2$ , respectively. Then we add the  
 348 two following triangles:  $(a_1, c_j^1, c_j^3)$  and  $(a_2, c_j^2, c_j^3)$ . So we used all the backward arc from  $T_c$   
 349 to  $T_v$ , and there are no triangles which use two arcs of  $\overleftarrow{A}^{vc}(T)$ . Then in the packing  $\Delta$  there  
 350 are in total  $6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$  triangles.

351 Conversely let  $\Delta$  be a triangle packing of  $T$  with  $|\Delta| = 6n(n-1) + 3m(m+1)/2 + 2n +$   
 352  $|\overleftarrow{A}^{vc}(T)| + 1$ . In the same way as we already did before, we partition  $\Delta$  into the different subsets  
 353 we defined before. We have  $|\Delta| = |\Delta_{V,V,V}| + |\Delta_{V,V,C}| + |\Delta_{V,C,C}| + |\Delta_{C,C,C}| + |\Delta_{2V,C}| + |\Delta_{V,2C}|$   
 354  $+ |\Delta_{3V}| + |\Delta_{3C}|$ . By Lemma 2 all the upper bounds described above are tight, that is:

- 355  $\blacksquare$   $|\Delta_{V,V,V}| + |\Delta_{C,C,C}| = 6n(n-1) + 3m(m+1)/2,$
- 356  $\blacksquare$   $|\Delta_{2V,C}| + |\Delta_{V,2C}| + |\Delta_{V,C,C}| + |\Delta_{V,V,C}| = |\overleftarrow{A}^{vc}(T)|,$
- 357  $\blacksquare$   $|\Delta_{3V}| = 2n,$
- 358  $\blacksquare$   $|\Delta_{3C}| = 1.$

359 Let us first prove that  $|\Delta_{V,V,C}| + |\Delta_{V,C,C}| = 0$ . Let  $x = |\Delta_{V,V,C}| + |\Delta_{V,C,C}|$ . Since each  
 360 triangle of the sets  $\Delta_{V,V,C}, \Delta_{V,C,C}, \Delta_{2V,C}$  and  $\Delta_{V,2C}$  uses exactly one backward arc of  
 361  $\overleftarrow{A}^{vc}(T)$ , it implies that  $|\Delta_{2V,C}| + |\Delta_{V,2C}| \leq |\overleftarrow{A}^{vc}(T)| - x$ . Moreover, if  $x \neq 0$ , then we have  
 362  $|\Delta_{V,V,V}| < |\Delta^v|$  or  $|\Delta_{C,C,C}| < |\Delta^c|$  because each triangle in  $\Delta_{V,V,C}$  (resp.  $\Delta_{V,C,C}$ ) will use one  
 363 arc between two distinct variable gadgets (resp. clause gadgets) and according to Lemma 1,  $\Delta^v$   
 364 (resp.  $\Delta^c$ ) uses all the arcs between distinct variable gadgets (resp. clause gadgets). Finally,  
 365 we always have  $|\Delta_{3V}| \leq 2n$  and  $|\Delta_{3C}| \leq 1$  by construction. Therefore, if  $x \neq 0$ , we have  $|\Delta| <$   
 366  $|\Delta^v| + |\Delta^c| + x + (|\overleftarrow{A}^{vc}(T)| - x) + 2n + 1$  that is  $|\Delta| < 6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1,$   
 367 which is impossible. So we must have  $x = 0$ , which implies  $\Delta_{V,V,C} = \Delta_{V,C,C} = \emptyset$ .

368 Since  $|\Delta_{3V}| = 2n$  and we have at most two arc-disjoint triangles in each variable gadget  $V_i$ ,  
 369 it implies that  $\Delta[V_i] \in \{\Delta_i^\perp, \Delta_i^\top, \Delta_i^{\top'}\}$ . In the following, we will simply write  $\Delta_i$  instead  
 370 of  $\Delta[V_i]$ . Let us consider the following assignment  $a$ : for any variable  $v_i$ , if  $\Delta_i = \Delta_i^\perp$ , then  
 371  $a(v_i) = false$  and  $a(v_i) = true$  otherwise. Let us see that the assignment  $a$  satisfies the  
 372 formula  $F$ . We have just proved that the backward arcs from  $T_c$  to  $T_v$  are all used in  $\Delta_{2V,C}$   
 373 and  $\Delta_{V,2C}$ . As  $|\Delta_{3C}| = 1$  the dummy triangle  $C_{m+1}$  belongs to  $\Delta$ . So every dummy arc  
 374  $c_{m+1}^u \bar{x}_i$  is contained in a triangle of  $\Delta$  which uses an arc of  $V_i$ . Therefore in each  $V_i$  we have  
 375  $d_{V_i \setminus \Delta_i}^+(\bar{x}_i) \geq 3$ . Moreover, for each clause of size  $q$  with  $q \in \{2, 3\}$ , there are  $q$  triangles which  
 376 use the backward arcs coming from the clause to variable gadgets. Let  $C_j$  be a clause gadget  
 377 of size 3 (we can do the same reasoning if  $C_j$  has size 2). By construction the 3 triangles  
 378 cannot all lie in  $\Delta_{V,2C}$ . Thus, there is at least one of these triangles which is in  $\Delta_{2V,C}$ . Let  $t$   
 379 be one of them,  $V_i$  be the variable gadget where  $t$  has two out of its three vertices and  $\tilde{x}$  be  
 380 the vertex of  $V_i$  which is also the head of the backward arc from  $C_j$  to  $V_i$ . By construction,  
 381  $\tilde{x}$  corresponds to a literal  $\ell$  in the clause  $c_j$ . If  $\ell$  is positive, then  $\tilde{x} = x_i^1$  or  $\tilde{x} = x_i^2$ . In both  
 382 cases, since  $t$  has a second vertex in  $V_i$ , we have  $d_{V_i \setminus \Delta_i}^+(\tilde{x}) > 0$ . Thus, using Figure 3 we  
 383 cannot have  $\Delta_i = \Delta_i^\perp$  so the assignment sets the positive literal  $\ell$  to *true*, which satisfies  $c_j$ .  
 384 Otherwise,  $\ell$  is negative so  $\tilde{x} = \bar{x}_i$ . Since  $\bar{x}_i$  has to use three out-going arcs to consume the  
 385 dummy arcs and one out-going arc to consume  $t$ , we have  $d_{V_i \setminus \Delta_i}^+(\bar{x}_i) \geq 4$  and so  $\Delta_i = \Delta_i^\perp$   
 386 by Figure 3. Therefore,  $c_j$  is satisfied in that case too. Thus, the assignment  $a$  satisfies the  
 387 whole formula  $F$ .  $\blacktriangleleft$

388 As 3-SAT(3) is NP-hard [47, 54], this directly implies the following theorem.

389 ► **Theorem 4.** MAXATT is NP-hard.

390 As mentioned in the introduction, packing arc-disjoint cycles is not necessarily equivalent  
 391 to packing arc-disjoint triangles. Thus, we need to establish the following lemma to transfer  
 392 the previous NP-hardness result to MAXACT.

393 ► **Lemma 5.** Given a 3-SAT(3) instance  $F$ , and  $T$  the tournament constructed from  $F$   
 394 with the reduction  $f$ , we have a triangle packing  $\Delta$  of  $T$  of size  $6n(n-1) + 3m(m+1)/2 +$   
 395  $2n + |\overleftarrow{A}^{vc}(T)| + 1$  if and only if there is a cycle packing  $O$  of the same size.

396 **Proof.** Given a cycle packing  $O$  of  $T$  of size  $6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$ ,  
 397 we partition it into the following sets:

- 398 ■  $O_V = \{(v_1, \dots, v_p) \in O : \exists i \in [n], \forall k \in [p], v_k \in V_i\}$ ,
- 399 ■  $O_C = \{(v_1, \dots, v_p) \in O : \exists j \in [m+1], \forall k \in [p], v_k \in C_j\}$ ,
- 400 ■  $O_{V^*} = \{(v_1, \dots, v_p) \in O : \forall k \in [p], \exists i \in [n], v_k \in V_i \text{ and } (v_1, \dots, v_p) \notin O_V\}$ ,
- 401 ■  $O_{C^*} = \{(v_1, \dots, v_p) \in O : \forall k \in [p], \exists j \in [m+1], v_k \in C_j \text{ and } (v_1, \dots, v_p) \notin O_C\}$ ,
- 402 ■  $O_{V^*, C^*} = \{(v_1, \dots, v_p) \in O : \exists i \in [n], \exists j \in [m+1], \exists k_1, k_2 \in [p], v_{k_1} \in V_i, v_{k_2} \in C_j\}$ .

403 As we did in the previous proof, we begin by finding upper bounds on each of these sets. First,  
 404 recall that the FAS of each  $V_i$  is 2. Thus, we have  $|O_V| \leq 2n$ . By construction, we also have  
 405  $|O_C| \leq 1$ . Secondly, notice that a cycle of  $O_{V^*}$  cannot belong to exactly two distinct variable  
 406 gadgets since the arcs between them are all in the same direction. Thus, the cycles of  $O_{V^*}$   
 407 have at least three vertices which implies  $|O_{V^*}| \leq 6n(n-1)$ . We obtain  $|O_{C^*}| \leq 3m(m+1)/2$   
 408 using the same reasoning on  $O_{C^*}$ . Finally, we have  $|O_{V^*, C^*}| \leq |\overleftarrow{A}^{vc}(T)|$  since each cycle must  
 409 have at least one backward arc.

410 Putting these upper bounds together, we obtain that  $|O| \leq 6n(n-1) + 3m(m+1)/2 +$   
 411  $2n + |\overleftarrow{A}^{vc}(T)| + 1$  which implies that the bounds are tight. In particular, cycles of  $O_{V^*}$  (resp.  
 412  $O_{C^*}$ ) use exactly three arcs that are between distinct variable gadgets (resp. clause gadgets)  
 413 and all these arcs are used. So we can construct a new cycle packing  $O'$  where we replace  
 414 the cycles of  $O_{V^*}$  and  $O_{C^*}$  by the triangle packings  $\Delta^v$  and  $\Delta^c$  defined in Lemma 1. The  
 415 new solution uses a subset of arcs of  $O$  and has the same size.

416 The cycles of  $O_{V^*, C^*}$  use exactly one backward arc of  $\overleftarrow{A}^{vc}(T)$  due to the tight upper  
 417 bound  $|\overleftarrow{A}^{vc}(T)|$ . Moreover, by the previous reasoning, two vertices of a cycle of  $O_{V^*, C^*}$   
 418 cannot belong to two different variable gadgets (resp. clause gadgets). Let  $C_j$  be a clause  
 419 gadget which has three literals (if it has only two literals, the reasoning is analogous). Let  
 420  $\tilde{x}_{i_k} \in V_{i_k}$  be the head of a backward arc from  $c_j^3$  where  $k \in [3]$ . By the previous arguments  
 421 each arc  $c_j^3 \tilde{x}_{i_k}$  is contained in a cycle  $o_k$  of  $O$  for  $k \in [3]$ . There is at least one  $\tilde{x}_{i_k}$  whose  
 422 next vertex in  $o_k$ , say  $y$ , belongs to  $V_{i_k}$  since  $C_j$  has only two other vertices in addition to  
 423  $c_j^3$ . Without loss of generality, we may assume that  $\tilde{x}_{i_3}$  is that vertex. Then, we can replace  
 424  $o_1$  and  $o_2$  by the triangles  $(\tilde{x}_{i_1}, c_j^1, c_j^3)$  and  $(\tilde{x}_{i_2}, c_j^2, c_j^3)$ . The arcs  $c_j^1 c_j^3$  and  $c_j^2 c_j^3$  cannot have  
 425 already been used because  $C_j$  is acyclic and we previously consumed all the arcs between  
 426 clause gadgets. In the same way, we replace the cycle  $o_3$  by the triangle  $(\tilde{x}_{i_3}, y, c_j^3)$ . The arc  
 427  $yc_j^3$  is available since it could have been used only in the cycle  $o_3$ .

428 We now prove that given a  $V_i$ , we can restructure every cycle of  $O_V[V_i]$  into triangles.  
 429 Recall that  $O_V[V_i]$  have exactly 2 cycles, and notice that by construction one cannot have  
 430 two cycles each having a size greater than 3. First, if the two cycles are triangles, we are  
 431 done. Then  $O_V[V_i]$  contains a triangle, say  $\delta$ , and a cycle, say  $o$ , of size greater than 3. If  
 432  $o$  contains the backward arc  $s_i r_i$ , then by construction  $o = (r_i, \bar{x}_i, x_i^1, s_i)$ . In that case, we  
 433 necessary have  $\delta = (x_i^1, x_i^2, t_i)$  and we can restructure  $o$  in the triangle  $(r_i, x_i^1, s_i)$ . The arc

434  $r_i x_i^1$  is not contained in  $O$  since the only arcs inside  $V_i$  we may have imposed until now are  
 435 out-going arcs of  $x_i^1, x_i^2$  and  $\bar{x}_i$ . If  $o$  contains the backward arc  $t_i x_i^1$ , then by construction  
 436  $o = (x_i^1, s_i, x_i^2, t_i)$  and  $t = (r_i, \bar{x}_i, s_i)$ . In the same way, we can restructure  $o$  into  $(x_i^1, s_i, t_i)$   
 437 whose all the arcs are available.

438 As  $O_C$  is <sup>602</sup>already a triangle,  $T$  finally has a triangle packing of size  $6n(n-1) + 3m(m+1)/2 + 2n + |\overleftarrow{A}^{vc}(T)| + 1$ . The other direction of the equivalence is straightforward. ◀

440 The previous lemma and Theorem 4 directly imply the following theorem.

441 ▶ **Theorem 6.** MAXACT is NP-hard.

442 Let us now define two special cases TIGHT-ATT (resp. TIGHT-ACT) where, given a  
 443 tournament  $T$  and a linear ordering  $\sigma$  with  $k$  backward arcs (where  $k = \text{minfas}(T)$ ), the  
 444 goal is to decide if there is a triangle (resp. cycle) packing of size  $k$ . We call these special  
 445 cases the “tight” versions of the classical packing problems because as the input admits an  
 446 FAS of size  $k$ , any triangle (or cycle) packing has size at most  $k$ . We now prove that we  
 447 can construct in polynomial time an ordering of  $T$ , the tournament of the reduction, with  $k$   
 448 backward arcs (where  $k$  is the threshold value defined in Lemma 3).

449 ▶ **Lemma 7.** Let  $T$  be a tournament constructed by the reduction  $f$ , and  $k$  be the threshold  
 450 value defined in Lemma 3. Then, we can construct (in polynomial time) an ordering of  $T$   
 451 with  $k$  backward arcs implying that  $T$  has an FAS of size  $k$ .

452 **Proof.** Let us define a linear representation  $(\sigma(T), \overleftarrow{A}(T))$  such that  $|\overleftarrow{A}(T)| = k$ . Remember  
 453 that since  $n \equiv 3 \pmod{6}$ , the edges of the  $n$ -clique  $K_n$  can be packed into a packing  $O$  of  
 454  $n(n-1)/6$  (undirected) triangles. Let us first prove that there exists an orientation  $T_{K_n}$  of  $K_n$   
 455 and a linear ordering  $\sigma$  of  $T_{K_n}$  with  $|O|$  backward arcs. Let  $\sigma = 1 \dots n$ . For each undirected  
 456 triangle  $ijk$  in  $O$  where  $i < j < k$ , we set  $ki \in \overleftarrow{A}(T_{K_n})$  (implying that  $ij$  and  $jk$  are forward  
 457 arcs). As all edges are used in  $O$  this defines an orientation for all edges. Thus, there is  
 458 only  $|O|$  backward arcs in  $\sigma$ . Thus, when using the previous orientations  $T_{K_n}$  to construct  
 459 the variable tournament  $T_v$  of the reduction (remember that we blow up each vertex  $u_i$  into  
 460 6 vertices  $V_i$ ), we get an ordering with  $36n(n-1)/6 = 6n(n-1)$  backward arcs between  
 461 two different  $V_i$  (more formally,  $|\{a \in \overleftarrow{A}(T_v) : \exists i_1 \neq i_2, h(a) \in V_{i_1}, t(a) \in V_{i_2}\}| = 6n(n-1)$ ).  
 462 Following the same construction for the clause tournament  $T_c$  we get an ordering with  
 463  $3m(m+1)/2$  backward arcs between two distinct  $C_j$ . Now, as there are two backward arcs  
 464 in each  $V_i$ , one backward arc in  $C_{m+1}$ , and  $|\overleftarrow{A}^{vc}(T)|$  backward arcs from  $T_c$  to  $T_v$ , the total  
 465 number of backward arcs is  $k$ . ◀

466 We also prove that  $k = \text{minfas}(T)$ .

467 ▶ **Lemma 8.** Let  $T = (V, A)$  be a tournament constructed by the reduction  $f$  and  $k$  be the  
 468 threshold value defined in Lemma 3. Then,  $\text{minfas}(T) \geq k$ .

469 **Proof.** We suppose that  $T$  is equipped with the ordering defined in Lemma 7. Let  $F$  be an  
 470 optimal FAS of  $T$ . Given an arc  $a$ , let  $v(a) = \{t(a), h(a)\}$ . Let us partition the arcs of  $T$   
 471 into the following sets. For any  $i \in [n], j \in [m+1]$ , let us define

- 472 ■  $A_{V_i} = \{a \in A : v(a) \subseteq V_i\}$
- 473 ■  $A_{C_j} = \{a \in A : v(a) \subseteq C_j\}$
- 474 ■  $A_{V_i C_j} = \{a \in A : |v(a) \cap V_i| = |v(a) \cap C_j| = 1\}$
- 475 ■  $A_{V_i V_{i'}} = \{a \in A : |v(a) \cap V_i| = |v(a) \cap V_{i'}| = 1\}$  where  $i \neq i'$
- 476 ■  $A_{C_j C_{j'}} = \{a \in A : |v(a) \cap C_j| = |v(a) \cap C_{j'}| = 1\}$  where  $j \neq j'$



477 For any  $i, i' \in [n]$ ,  $j, j' \in [m+1]$  and  $X \in \{V_i, C_j, V_i C_j, V_i V_{i'}, C_j C_{j'}\}$ , we also define the  
 478 corresponding sets  $F_X$  in  $F$ , where for example  $F_{V_i} = F \cap A_{V_i}$ . In addition, for any  $j \in [m+1]$   
 479 we define  $F_{*C_j} = \bigcup_{i \in [n]} F_{V_i C_j}$ . Let  $T'_v$  be the directed graph ( $T'_v$  is not a tournament) obtained  
 480 by starting from  $T_v$  and only keeping arcs in  $A_{V_i V_{i'}}$  for any  $i, i' \in [n]$  with  $i \neq i'$ . As  $F$  is FAS  
 481 of  $T$ ,  $F_{VV} = \bigcup_{i, i' \in [n], i \neq i'}^{693} F_{V_i V_{i'}}$  must be an FAS of  $T'_v$ . As according to Lemma 1 there is a  
 482 cycle packing of size  $6n(n-1)$  in  $T'_v$ , we get  $|F_{VV}| \geq 6n(n-1)$ . The same arguments hold for  
 483 the clause part, and thus with  $F_{CC} = \bigcup_{j, j' \in [m+1], j \neq j'} F_{C_j C_{j'}}$ , we get  $|F_{CC}| \geq 3m(m+1)/2$ .  
 484 As  $C_{m+1}$  is a triangle, we also get  $|F_{C_{m+1}}| \geq 1$ .

485 For any  $j \in [m]$ , let  $u_j \in \{2, 3\}$  be equal to the size of the clause  $j$  (we also have  
 486  $u_j = |\{a \in \overleftarrow{A}(T): \exists i \in [n], h(a) \in V_i \text{ and } t(a) \in C_j\}|$ ). Let  $L = \{j \in [m]: |F_{*C_j} \cup F_{C_j}| \geq u_j\}$   
 487 be informally the set of clauses where  $F$  spends a large (in fact larger than the  $u_j$  required)  
 488 amount of arcs, and  $S = [m] \setminus L$ . Let us prove that for any  $j \in S$ ,  $|F_{C_j}| \geq u_j - 1$ . Let us first  
 489 consider the case where  $u_j = 3$ . Suppose by contradiction that  $F_{C_j} = \{a\}$  (arguments will  
 490 also hold for  $F_{C_j} = \emptyset$ ). Remember that  $\sigma(C_j) = (c_j^1, c_j^2, c_j^3)$  (there are only forward arcs). As  
 491  $|F_{*C_j}| \leq 1$ , there exists  $i \in [n]$  and two arcs  $a_1, a_2$  not in  $F$  such that  $t(a_1) = c_j^3$ ,  $h(a_1) \in V_i$ ,  
 492  $t(a_2) = h(a_1)$ , and  $h(a_2) \neq t(a)$ . Thus,  $(t(a_1), t(a_2), h(a_2))$  is a triangle using no arc of  $F$ , a  
 493 contradiction. As the same kind of arguments holds for the case where  $u_j = 2$ , we get that  
 494 for any  $j \in S$ ,  $|F_{C_j}| \geq u_j - 1$  (implying also  $|F_{*C_j}| = 0$ ).

495 Let us now prove that  $|S| \leq 1$ . Suppose by contradiction that  $|S| \geq 2$ . Let  $j_1$  and  $j_2$   
 496 be in  $S$ . For any  $l \in [2]$ , let define  $a_l$  such that there exists  $i_l \in [n]$  with  $t(a_l) \in C_{j_l}$  and  
 497  $h(a_l) \in V_{i_l}$ . Notice that we may have  $i_1 = i_2$ , but we always have  $h(a_1) \neq h(a_2)$ . Moreover,  
 498 as  $a_i$  is the unique backward arc of  $T$  with  $t(a) \in \bigcup_{j \in [m]} C_j$ , we get that  $a_3 = h(a_1)t(a_2)$   
 499 and  $a_4 = h(a_2)t(a_1)$  are forward arcs of  $T$ . As  $|F_{*C_{j_1}}| = |F_{*C_{j_2}}| = 0$  we know that  $a_l \notin F$  for  
 500  $l \in [4]$ . Thus,  $(t(a_1), h(a_1), t(a_2), h(a_2), t(a_1))$  is a cycle using no arc of  $F$ , a contradiction.

501 Let  $L' = \{i \in [n]: \exists a \in T \text{ s.t. } h(a) \in V_i \text{ and } t(a) \in C_j, j \in S\}$ . Notice that if  $S = \emptyset$   
 502 then  $L' = \emptyset$ , and otherwise  $|L'| = u_{j_0}$ , where  $S = \{j_0\}$ . Let  $S' = [n] \setminus L'$ . For any  $i \in [n]$ ,  
 503 let  $\overleftarrow{A}_{V_i C_{m+1}} = \overleftarrow{A}(T) \cap A_{V_i C_{m+1}}$ . Recall that  $\overleftarrow{A}_{V_i C_{m+1}} = c_{m+1}^u \bar{x}_i$  for  $u \in [3]$  where  $\bar{x}_i \in V_i$ .  
 504 Moreover, for any  $x \in \{\bar{x}_i, x_i^1, x_i^2\}$ , let  $A_{xV_i} = \{a \in T: t(a) = x \text{ and } h(a) \in V_i\}$ . Notice that  
 505  $|A_{\bar{x}_i V_i}| = 4$ ,  $|A_{x_i^1 V_i}| = 2$  and  $|A_{x_i^2 V_i}| = 1$ .

506 Let us prove that for any  $i \in S'$ ,  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 5$ . If  $A_{\bar{x}_i V_i} \subseteq F$ , then as  $F_{V_i}$  must be  
 507 an FAS of  $V_i$  and  $A_{\bar{x}_i V_i}$  is not an FAS of  $V_i$ , there exists at least another arc in  $F_{V_i}$  and we  
 508 get  $|F_{V_i}| \geq 5$ . Otherwise,  $\overleftarrow{A}_{V_i C_{m+1}} \subseteq F$  (if it is not the case, there is a cycle  $c_{m+1}^u \bar{x}_i v$  where  
 509  $v \in V_i$  is a out-neighbour of  $\bar{x}_i$ ). Then, as  $\text{minfas}(V_i) \geq 2$ ,  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 5$ .

510 Let us finally prove that for any  $i \in L'$ ,  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 6$ . As  $i \in L'$ , there is an  
 511 arc  $a \in T$  with  $h(a) \in V_i$  and  $t(a) \in C_{j_0}$  where  $S = \{j_0\}$ . Let  $x = h(a)$ . Notice that  
 512  $x \in \{\bar{x}_i, x_i^1, x_i^2\}$ . As  $|F_{*C_{j_0}}| = 0$  we get that  $A_{xV_i} \subseteq F_{V_i}$  (otherwise there would be a cycle  
 513 with one vertex in  $C_{j_0}$ ,  $x$ , and an out-neighbour of  $x$  in  $V_i$ ).

514 **Case 1:**  $x = \bar{x}_i$ . As  $F_{V_i}$  must be an FAS of  $V_i$ ,  $F$  needs two other arcs in  $A_{V_i}$  and we get  
 515  $|F_{V_i}| \geq 6$ .

516 **Case 2:**  $x = x_i^1$ . If  $A_{\bar{x}_i V_i} \subseteq F$  then  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 6$ . Otherwise, as before we get  
 517  $\overleftarrow{A}_{V_i C_{m+1}} \subseteq F$ , and as  $A_{x_i^1 V_i}$  is not an FAS of  $V_i$ ,  $F$  need another arc in  $V_i$ , implying  
 518  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 6$ .

519 **Case 3:**  $x = x_i^2$ . If  $A_{\bar{x}_i V_i} \subseteq F$  then as  $A_{x_i^2 V_i} \cup A_{\bar{x}_i V_i}$  is not an FAS of  $V_i$ ,  $F$  need another arc  
 520 in  $V_i$ , implying  $|F_{V_i}| \geq 6$ . Otherwise, as before we get  $\overleftarrow{A}_{V_i C_{m+1}} \subseteq F$ , and as  $A_{x_i^2 V_i}$  is not an  
 521 FAS of  $V_i$ ,  $F$  need two other arcs in  $V_i$ , implying  $|F_{V_i} \cup F_{V_i C_{m+1}}| \geq 6$ .

522 Putting all the pieces together, we get the following.

$$\begin{aligned}
|F| &= |F_{VV}| + |F_{CC}| + |F_{C_{m+1}}| + \sum_{j \in L} (|F_{*C_j} \cup F_{C_j}|) + \sum_{j \in S} (|F_{*C_j} \cup F_{C_j}|) \\
&+ \sum_{i \in S'} (|F_{V_i} \cup F_{V_i C_{m+1}}|) + \sum_{i \in L'} (|F_{V_i} \cup F_{V_i C_{m+1}}|) \\
&\geq 6n(n-1) + \frac{3m(m+1)}{2} + 1 + \sum_{j \in L} u_j + \sum_{j \in S} (u_j - 1) + 5|S'| + 6|L'| \\
&\geq 6n(n-1) + \frac{3m(m+1)}{2} + 1 + \sum_{j \in [m]} u_j + 5n = k
\end{aligned}$$

Then, using Lemma 7 and Lemma 8, we get the NP-hardness of TIGHT-ATT and TIGHT-ACT.

► **Theorem 9.** TIGHT-ATT and TIGHT-ACT are NP-hard.

Finally, the size  $s$  of the required packing in Lemma 3 satisfies  $s = \mathcal{O}((n+m)^2)$ . Under the Exponential-time Hypothesis, the problem 3-SAT cannot be solved in  $2^{o(n+m)}$  [21, 35]. Then, using the linear reduction from 3-SAT to 3-SAT(3) [54], we also get the following result.

► **Theorem 10.** Under the Exponential-time Hypothesis, ATT and ACT cannot be solved in  $\mathcal{O}^*(2^{o(\sqrt{k})})$  time.

In the framework of parameterizing above guaranteed values [45], the above results imply that ACT parameterized below the guaranteed value of the size of a minimal feedback arc set is fixed-parameter intractable.

## 4 Parameterized Complexity of ACT

The classical Erdős-Pósa theorem for cycles in undirected graphs states that there exists a function  $f(k) = \mathcal{O}(k \log k)$  such that for each non-negative integer  $k$ , every undirected graph either contains  $k$  vertex-disjoint cycles or has a feedback vertex set consisting of  $f(k)$  vertices [26]. An interesting consequence of this theorem is that it leads to an FPT algorithm for VERTEX-DISJOINT CYCLE PACKING. It is well known that the treewidth ( $tw$ ) of a graph is not larger than the size of its feedback vertex set, and that a naive dynamic programming scheme solves VERTEX-DISJOINT CYCLE PACKING in  $\mathcal{O}^*(2^{\mathcal{O}(tw \log tw)})$  time (see, e.g., [21]). Thus, the existence of an  $\mathcal{O}^*(2^{\mathcal{O}(k \log^2 k)})$  time algorithm can be viewed as a direct consequence of the Erdős-Pósa theorem (see [43] for more details). Analogous to these results, we prove an Erdős-Pósa type theorem for tournaments and show that it leads to an  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time algorithm and a linear vertex kernel for ACT.

### 4.1 An Erdős-Pósa Type Theorem

In this section, we show certain interesting combinatorial results on arc-disjoint cycles in tournaments.

► **Theorem 11.** Let  $k$  and  $r$  be positive integers such that  $r \leq k$ . A tournament  $T$  contains a set of  $r$  arc-disjoint cycles if and only if  $T$  contains a set of  $r$  arc-disjoint cycles each of length at most  $2k + 1$ .

559 **Proof.** The reverse direction of the claim holds trivially. Let us now prove the forward  
 560 direction. Let  $\mathcal{C}$  be a set of  $r$  arc-disjoint cycles in  $T$  that minimizes  $\sum_{C \in \mathcal{C}} |C|$ . If every  
 561 cycle in  $\mathcal{C}$  is a triangle, then the claim trivially holds. Otherwise, let  $C$  be a longest cycle in  
 562  $\mathcal{C}$  and let  $\ell$  denote its length. Let  $v_i, v_j$  be a pair of non-consecutive vertices in  $C$ . Then,  
 563 either  $v_i v_j \in A(T)$  or  $v_j v_i \in A(T)$ . In any case, the arc  $e$  between  $v_i$  and  $v_j$  along with  $A(C)$   
 564 forms a cycle  $C'$  of length less than  $\ell$  with  $A(C') \setminus \{e\} \subset A(C)$ . By our choice of  $\mathcal{C}$ , this  
 565 implies that  $e$  is an arc in some other cycle  $\hat{C} \in \mathcal{C}$ . This property is true for the arc between  
 566 any pair of non-consecutive vertices in  $C$ . Therefore, we have  $\binom{\ell}{2} - \ell \leq \ell(k-1)$  leading to  
 567  $\ell \leq 2k+1$ . ◀

568 This result essentially shows that it suffices to determine the existence of  $k$  arc-disjoint  
 569 cycles in  $T$  each of length at most  $2k+1$  in order to determine if  $(T, k)$  is a yes-instance  
 570 of ACT. This immediately leads to a quadratic Erdős-Pósa bound. That is, for every  
 571 non-negative integer  $k$ , every tournament  $T$  either contains  $k$  arc-disjoint cycles or has an  
 572 FAS of size  $\mathcal{O}(k^2)$ . Next, we strengthen this result to arrive at a linear bound.

573 We will use the following lemma known from [17] in the process<sup>1</sup>. For a digraph  $D$ , let  
 574  $\Lambda(D)$  denote the number of non-adjacent pairs of vertices in  $D$ . That is,  $\Lambda(D)$  is the number  
 575 of pairs  $u, v$  of vertices of  $D$  such that neither  $uv \in A(D)$  nor  $vu \in A(D)$ . Recall that for a  
 576 digraph  $D$ ,  $\text{minfas}(D)$  denotes the size of a minimum FAS of  $D$ .

577 ▶ **Lemma 12.** [17] *Let  $D$  be a triangle-free digraph in which for every pair  $u, v$  of distinct*  
 578 *vertices, at most one of  $uv$  or  $vu$  is in  $A(D)$ . Then, we can compute an FAS of size at most*  
 579  *$\Lambda(D)$  in polynomial time.*

580 This leads to the following main result of this section.

581 ▶ **Theorem 13.** *For every non-negative integer  $k$ , every tournament  $T$  either contains  $k$*   
 582 *arc-disjoint triangles or has an FAS of size at most  $5(k-1)$  that can be obtained in polynomial*  
 583 *time.*

584 **Proof.** Let  $\mathcal{C}$  be a maximal set of arc-disjoint triangles in  $T$  (that can be obtained greedily  
 585 in polynomial time). If  $|\mathcal{C}| \geq k$ , then we have the required set of triangles. Otherwise, let  
 586  $D$  denote the digraph obtained from  $T$  by deleting the arcs that are in some triangle in  
 587  $\mathcal{C}$ . Clearly,  $D$  has no triangle and  $\Lambda(D) \leq 3(k-1)$ . Let  $F$  be an FAS of  $D$  obtained in  
 588 polynomial time using Lemma 12. Then, we have  $|F| \leq 3(k-1)$ . Next, consider a topological  
 589 ordering  $\sigma$  of  $D - F$ . Each triangle of  $\mathcal{C}$  contains at most 2 arcs which are backward in this  
 590 ordering. If we denote by  $F'$  the set of all the arcs of the triangles of  $\mathcal{C}$  which are backward  
 591 in  $\sigma$ , then we have  $|F'| \leq 2(k-1)$  and  $(D - F) - F'$  is acyclic. Thus  $F^* = F \cup F'$  is an FAS  
 592 of  $T$  satisfying  $|F^*| \leq 5(k-1)$ . ◀

## 593 4.2 A Linear Vertex Kernel

594 Next, we show that ACT has a linear vertex kernel. This kernel is inspired by the linear  
 595 kernelization described in [11] for FAST and uses Theorem 13. Let  $T$  be a tournament on  $n$   
 596 vertices. First, we apply the following reduction rule.

597 ▶ **Reduction Rule 4.1.** *If a vertex  $v$  is not in any cycle, then delete  $v$  from  $T$ .*

<sup>1</sup> The authors would like to thank F. Havet for pointing out that Lemma 12 was a consequence of a result of [17], as well for an improvement of the constant in Theorem 13.

598 This rule is clearly safe as our goal is to find  $k$  cycles and  $v$  cannot be in any of them.  
 599 To describe our next rule, we need to state a lemma known from [11]. An *interval* is a  
 600 consecutive set of vertices in a linear representation  $(\sigma(T), \overleftarrow{A}(T))$  of a tournament  $T$ .

601 ► **Lemma 14**<sup>2</sup> ([11]). *Let  $T = (\sigma(T), \overleftarrow{A}(T))$  be a tournament on which Reduction Rule 4.1*  
 602 *is not applicable. If  $|V(T)| \geq 2|\overleftarrow{A}(T)| + 1$ , then there exists a partition  $\mathcal{J}$  of  $V(T)$  into intervals*  
 603 *(that can be computed in polynomial time) such that there are  $|\overleftarrow{A}(T) \cap E| > 0$  arc-disjoint*  
 604 *cycles using only arcs in  $E$  where  $E$  denotes the set of arcs in  $T$  with endpoints in different*  
 605 *intervals.*

606 Our reduction rule that is based on this lemma is as follows.

607 ► **Reduction Rule 4.2.** *Let  $T = (\sigma(T), \overleftarrow{A}(T))$  be a tournament on which Reduction Rule*  
 608 *4.1 is not applicable. Let  $\mathcal{J}$  be a partition of  $V(T)$  into intervals satisfying the properties*  
 609 *specified in Lemma 14. Reverse all arcs in  $\overleftarrow{A}(T) \cap E$  and decrease  $k$  by  $|\overleftarrow{A}(T) \cap E|$  where  $E$*   
 610 *denotes the set of arcs in  $T$  with endpoints in different intervals.*

611 ► **Lemma 15.** *Reduction Rule 4.2 is safe.*

612 **Proof.** Let  $T'$  be the tournament obtained from  $T$  by reversing all arcs in  $\overleftarrow{A}(T) \cap E$ . Suppose  
 613  $T'$  has  $k - |\overleftarrow{A}(T) \cap E|$  arc-disjoint cycles. Then, it is guaranteed that each such cycle is  
 614 completely contained in an interval. This is due to the fact that  $T'$  has no backward arc  
 615 with endpoints in different intervals. Indeed, if a cycle in  $T'$  uses a forward (backward) arc  
 616 with endpoints in different intervals, then it also uses a back (forward) arc with endpoints in  
 617 different intervals. It follows that for each arc  $uv \in E$ , neither  $uv$  nor  $vu$  is used in these  
 618  $k - |\overleftarrow{A}(T) \cap E|$  cycles. Hence, these  $k - |\overleftarrow{A}(T) \cap E|$  cycles in  $T'$  are also cycles in  $T$ . Then,  
 619 we can add a set of  $|\overleftarrow{A}(T) \cap E|$  cycles obtained from the second property of Lemma 14 to  
 620 these  $k - |\overleftarrow{A}(T) \cap E|$  cycles to get  $k$  cycles in  $T$ . Conversely, consider a set of  $k$  cycles in  
 621  $T$ . As argued earlier, we know that the number of cycles that have an arc that is in  $E$  is at  
 622 most  $|\overleftarrow{A}(T) \cap E|$ . The remaining cycles (at least  $k - |\overleftarrow{A}(T) \cap E|$  of them) do not contain any  
 623 arc that is in  $E$ , in particular, they do not contain any arc from  $\overleftarrow{A}(T) \cap E$ . Therefore, these  
 624 cycles are also cycles in  $T'$ . ◀

625 Thus, we have the following result.

626 ► **Theorem 16.** *ACT admits a kernel with  $\mathcal{O}(k)$  vertices.*

627 **Proof.** Let  $(T, k)$  denote the instance obtained from the input instance by applying Reduction  
 628 Rule 4.1 exhaustively. From Lemma 13, we know that either  $T$  has  $k$  arc-disjoint triangles or  
 629 has an FAS of size at most  $5(k - 1)$  that can be obtained in polynomial time. In the first  
 630 case, we return a trivial yes-instance of constant size as the kernel. In the second case, let  $F$   
 631 be the FAS of size at most  $5(k - 1)$  of  $T$ . Let  $(\sigma(T), \overleftarrow{A}(T))$  be the linear representation of  $T$   
 632 where  $\sigma(T)$  is a topological ordering of the vertices of the directed acyclic graph  $T - F$ . As  
 633  $V(T - F) = V(T)$ ,  $|\overleftarrow{A}(T)| \leq 5(k - 1)$ . If  $|V(T)| \geq 10k - 9$ , then from Lemma 14, there is a  
 634 partition of  $V(T)$  into intervals with the specified properties. Therefore, Reduction Rule 4.2  
 635 is applicable (and the parameter drops by at least 1). When we obtain an instance where  
 636 neither of the Reduction Rules 4.1 and 4.2 is applicable, it follows that the tournament in  
 637 that instance has at most  $10k$  vertices. ◀

<sup>2</sup> Lemma 14 is Lemma 3.9 of [11] that has been rephrased to avoid the use of several definitions and terminology introduced in [11].

638 **4.3 An FPT Algorithm**

639 Finally, we show that ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time. The idea is to reduce  
 640 the problem to the following ARC-DISJOINT PATHS problem in directed acyclic graphs:  
 641 given a digraph  $D$  on  $n$  vertices and  $k$  ordered pairs  $(s_1, t_1), \dots, (s_k, t_k)$  of vertices of  $D$ , do  
 642 there exist arc-disjoint paths  $P_1, \dots, P_k$  in  $D$  such that  $P_i$  is a path from  $s_i$  to  $t_i$  for each  
 643  $i \in [k]$ ? On directed acyclic graphs, ARC-DISJOINT PATHS is known to be NP-complete  
 644 [27], W[1]-hard [52] with respect to  $k$  as parameter and solvable in  $n^{\mathcal{O}(k)}$  time [32]. Despite  
 645 its fixed-parameter intractability, we will show that we can use the  $n^{\mathcal{O}(k)}$  algorithm and  
 646 Theorems 13 and 16 to describe an FPT algorithm for ACT.

647 ► **Theorem 17.** *ACT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  time.*

648 **Proof.** Consider an instance  $(T, k)$  of ACT. Using Theorem 16, we obtain a kernel  $\mathcal{I} = (\widehat{T}, \widehat{k})$   
 649 such that  $\widehat{T}$  has  $\mathcal{O}(k)$  vertices. Further,  $\widehat{k} \leq k$ . By definition,  $(T, k)$  is a yes-instance if  
 650 and only if  $(\widehat{T}, \widehat{k})$  is a yes-instance. Using Theorem 13, we know that  $\widehat{T}$  either contains  
 651  $\widehat{k}$  arc-disjoint triangles or has an FAS of size at most  $5(\widehat{k} - 1)$  that can be obtained in  
 652 polynomial time. If Theorem 13 returns a set of  $\widehat{k}$  arc-disjoint triangles in  $\widehat{T}$ , then we declare  
 653 that  $(T, k)$  is a yes-instance.

654 Otherwise, let  $\widehat{F}$  be the FAS of size at most  $5(\widehat{k} - 1)$  returned by Theorem 13. Let  
 655  $D$  denote the (acyclic) digraph obtained from  $\widehat{T}$  by deleting  $\widehat{F}$ . Observe that  $D$  has  $\mathcal{O}(k)$   
 656 vertices. Suppose  $\widehat{T}$  has a set  $\mathcal{C} = \{C_1, \dots, C_{\widehat{k}}\}$  of  $\widehat{k}$  arc-disjoint cycles. For each  $C \in \mathcal{C}$ , we  
 657 know that  $A(C) \cap \widehat{F} \neq \emptyset$  as  $\widehat{F}$  is an FAS of  $\widehat{T}$ . We can guess that subset  $F$  of  $\widehat{F}$  such that  
 658  $F = \widehat{F} \cap A(\mathcal{C})$ . Then, for each cycle  $C_i \in \mathcal{C}$ , we can guess the arcs  $F_i$  from  $F$  that it contains  
 659 and also the order  $\pi_i$  in which they appear. This information is captured as a partition  $\mathcal{F}$  of  
 660  $F$  into  $\widehat{k}$  sets,  $F_1$  to  $F_{\widehat{k}}$  and the set  $\{\pi_1, \dots, \pi_{\widehat{k}}\}$  of permutations where  $\pi_i$  is a permutation  
 661 of  $F_i$  for each  $i \in [\widehat{k}]$ . Any cycle  $C_i$  that has  $F_i \subseteq F$  contains a  $(v, x)$ -path between every  
 662 pair  $(u, v), (x, y)$  of consecutive arcs of  $F_i$  with arcs from  $A(D)$ . That is, there is a path  
 663 from  $h(\pi_i^{-1}(j))$  and  $t(\pi_i^{-1}((j+1) \bmod |F_i|))$  with arcs from  $D$  for each  $j \in [|F_i|]$ . The total  
 664 number of such paths in these  $\widehat{k}$  cycles is  $\mathcal{O}(|F|)$  and the arcs of these paths are contained in  
 665  $D$  which is a (simple) directed acyclic graph.

666 The number of choices for  $F$  is  $2^{|\widehat{F}|}$  and the number of choices for a partition  $\mathcal{F} =$   
 667  $\{F_1, \dots, F_{\widehat{k}}\}$  of  $F$  and a set  $X = \{\pi_1, \dots, \pi_{\widehat{k}}\}$  of permutations is  $2^{\mathcal{O}(|\widehat{F}| \log |\widehat{F}|)}$ . Once such a  
 668 choice is made, the problem of finding  $\widehat{k}$  arc-disjoint cycles in  $\widehat{T}$  reduces to the problem of  
 669 finding  $\widehat{k}$  arc-disjoint cycles  $\mathcal{C} = \{C_1, \dots, C_{\widehat{k}}\}$  in  $\widehat{T}$  such that for each  $1 \leq i \leq \widehat{k}$  and for each  
 670  $1 \leq j \leq |F_i|$ ,  $C_i$  has a path  $P_{ij}$  between  $h(\pi_i^{-1}(j))$  and  $t(\pi_i^{-1}((j+1) \bmod |F_i|))$  with arcs  
 671 from  $D = \widehat{T} - \widehat{F}$ . This problem is essentially finding  $r = \mathcal{O}(|\widehat{F}|)$  arc-disjoint paths in  $D$  and  
 672 can be solved in  $|V(D)|^{\mathcal{O}(r)}$  time using the algorithm in [32]. Therefore, the overall running  
 673 time of the algorithm is  $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$  as  $|V(D)| = \mathcal{O}(k)$  and  $r = \mathcal{O}(k)$ . ◀

674 **5 Parameterized Complexity of ATT**

675 In this section, we provide an FPT algorithm and a linear vertex kernel for ATT. First, it is  
 676 easy to obtain an  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time algorithm using the classical colour coding technique [5]  
 677 for packing subgraphs of bounded size.

678 ► **Theorem 18.** *ATT can be solved in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time.*

679 **Proof.** Consider an instance  $\mathcal{I} = (T, k)$  of ATT. Let  $n$  denote  $|V(T)|$  and  $m$  denote  $|A(T)|$ .  
 680 Let  $\mathcal{F}$  denote the family of colouring functions  $c : A(T) \rightarrow [3k]$  of size  $2^{\mathcal{O}(k)} \log^2 m$  that can

681 be computed in  $\mathcal{O}^*(2^{\mathcal{O}(k)})$  time using  $3k$ -perfect family of hash functions [51]. For each  
 682 colouring function  $c$  in  $\mathcal{F}$ , we colour  $A(T)$  according to  $c$  and find a triangle packing of size  
 683  $k$  whose arcs use different colours. We use a standard dynamic programming routine to  
 684 finding such a triangle packing. Clearly, if  $\mathcal{I}$  is an yes-instance and  $\mathcal{C}$  is a set of  $k$  arc-disjoint  
 685 triangles in  $T$ , there is a colouring function in  $\mathcal{F}$  that colours the  $3k$  arcs in these triangles  
 686 with distinct colours and our algorithm will find the required triangle packing. Given a  
 687 colouring  $c \in \mathcal{F}$ , we first compute for every set of 3 colours  $\{a, b, c\}$  whether the arcs coloured  
 688 with  $a, b$  or  $c$  induce a triangle using 3 different colours or not. Then, for every set  $S$  of  
 689  $3(p+1)$  colours with  $p \in [k-1]$ , we recursively test if the arcs coloured with the colours in  
 690  $S$  induce  $p+1$  arc-disjoint triangles whose arcs use all the colours of  $S$ . This is achieved by  
 691 iterating over every subset  $\{a, b, c\}$  of  $S$  and checking if there is a triangle using colours  $a, b$   
 692 and  $c$  and a collection of  $p$  arc-disjoint triangles whose arcs use all the colours of  $S \setminus \{a, b, c\}$ .  
 693 For a given  $S$ , we can find this collection of triangles in  $\mathcal{O}(p^3) = \mathcal{O}(k^3)$  time. Therefore, the  
 694 overall running time of the algorithm is  $\mathcal{O}^*(2^{\mathcal{O}(k)})$ . ◀

695 Next, we show that ATTT has a linear vertex kernel.

696 ▶ **Theorem 19.** *ATTT admits a kernel with  $\mathcal{O}(k)$  vertices.*

697 **Proof.** Let  $\mathcal{X}$  be a maximal collection of arc-disjoint triangles of a tournament  $T$  obtained  
 698 greedily. Let  $V_{\mathcal{X}}$  denote the vertices of the triangles in  $\mathcal{X}$  and  $A_{\mathcal{X}}$  denote the arcs of  $V_{\mathcal{X}}$ .  
 699 Let  $U$  be the remaining vertices of  $V(T)$ , i.e.,  $U = V(T) \setminus V_{\mathcal{X}}$ . If  $|\mathcal{X}| \geq k$ , then  $(T, k)$  is an  
 700 yes-instance of ATTT. Otherwise,  $|\mathcal{X}| < k$  and  $|V_{\mathcal{X}}| < 3k$ . Moreover, notice that  $T[U]$  is acyclic  
 701 and  $T$  does not contain a triangle with one vertex in  $V_{\mathcal{X}}$  and two in vertices in  $U$  (otherwise  
 702  $\mathcal{X}$  would not be maximal).

703 Let  $B$  be the (undirected) bipartite graph defined by  $V(B) = A_{\mathcal{X}} \cup U$  and  $E(B) =$   
 704  $\{au : a \in A_{\mathcal{X}}, u \in U \text{ such that } (t(a), h(a), u) \text{ forms a triangle in } T\}$ . Let  $M$  be a maximum  
 705 matching of  $B$  and  $A'$  (resp.  $U'$ ) denote the vertices of  $A_{\mathcal{X}}$  (resp.  $U$ ) covered by  $M$ . Define  
 706  $\bar{A}' = A_{\mathcal{X}} \setminus A'$  and  $\bar{U}' = U \setminus U'$ .

707 We now prove that  $(V_{\mathcal{X}} \cup U', k)$  is a linear kernel of  $(T, k)$ . Let  $\mathcal{C}$  be a maximum sized  
 708 triangle packing that minimizes the number of vertices of  $\bar{U}'$  belonging to a triangle of  $\mathcal{C}$ . By  
 709 previous remarks, we can partition  $\mathcal{C}$  into  $C_{\mathcal{X}} \cup F$  where  $C_{\mathcal{X}}$  are the triangles of  $\mathcal{C}$  included  
 710 in  $T[V_{\mathcal{X}}]$  and  $F$  are the triangles of  $\mathcal{C}$  containing one vertex of  $U$  and two vertices of  $V_{\mathcal{X}}$ . It  
 711 is clear that  $F$  corresponds to a union of vertex-disjoint stars of  $B$  with centres in  $U$ . Denote  
 712 by  $U[F]$  the vertices of  $U$  which belong to a triangle of  $F$ . If  $U[F] \subseteq U'$  then  $(V_{\mathcal{X}} \cup U', k)$  is  
 713 immediately a kernel. Suppose there exists a vertex  $x_0$  such that  $x_0 \in U[F] \cap \bar{U}'$ .

714 We will build a tree rooted in  $x_0$  with edges alternating between  $F$  and  $M$ . For this let  
 715  $H_0 = \{x_0\}$  and construct recursively the sets  $H_{i+1}$  such that

$$716 \quad H_{i+1} = \begin{cases} N_F(H_i) & \text{if } i \text{ is even,} \\ N_M(H_i) & \text{if } i \text{ is odd,} \end{cases}$$

717 where, given a subset  $S \subseteq U$ ,  $N_F(S) = \{a \in A_{\mathcal{X}} : \exists s \in S \text{ s.t. } (t(a), h(a), s) \in F \text{ and } as \notin M\}$   
 718 and given a subset  $S \subseteq A_{\mathcal{X}}$ ,  $N_M(S) = \{u \in U : \exists a \in A_{\mathcal{X}} \text{ s.t. } au \in M\}$ . Notice that  $H_i \subseteq U$   
 719 when  $i$  is even and that  $H_i \subseteq A_{\mathcal{X}}$  when  $i$  is odd, and that all the  $H_i$  are distinct as  $F$  is a  
 720 union of disjoint stars and  $M$  a matching in  $B$ . Moreover, for  $i \geq 1$  we call  $T_i$  the set of edges  
 721 between  $H_i$  and  $H_{i-1}$ . Now we define the tree  $T$  such that  $V(T) = \bigcup_i H_i$  and  $E(T) = \bigcup_i T_i$ .  
 722 As  $T_i$  is a matching (if  $i$  is even) or a union of vertex-disjoint stars with centres in  $H_{i-1}$  (if  $i$   
 723 is odd), it is clear that  $T$  is a tree.

724 For  $i$  being odd, every vertex of  $H_i$  is incident to an edge of  $M$  otherwise  $B$  would contain  
 725 an augmenting path for  $M$ , a contradiction. So every leaf of  $T$  is in  $U$  and incident to an



726 edge of  $M$  in  $T$  and  $T$  contains as many edges of  $M$  than edges of  $F$ . Now for every arc  
 727  $a \in A_{\mathcal{X}} \cap V(T)$  we replace the triangle of  $\mathcal{C}$  containing  $a$  and corresponding to an edge of  $F$   
 728 by the triangle  $(t(a), h(a), u)$  where  $au \in M$  (and  $au$  is an edge of  $T$ ). This operation leads  
 729 to another collection of arc-disjoint triangles with the same size as  $\mathcal{C}$  but containing a strictly  
 730 smaller number of vertices in  $\overline{U'}$ , yielding a contradiction.

731 Finally  $V_{\mathcal{X}} \cup U'$  can be computed in polynomial time and we have  $|V_{\mathcal{X}} \cup U'| \leq |V_{\mathcal{X}}| + |M| \leq$   
 732  $2|V_{\mathcal{X}}| \leq 6k$ , which proves that the kernel has  $\mathcal{O}(k)$  vertices.  $\blacktriangleleft$

733 **6 MAXACT and MAXATT in Sparse Tournaments**

734 Recall that a tournament is *sparse* if it admits an FAS which is a matching. In this section,  
 735 we show that MAXACT and MAXATT are polynomial-time solvable on sparse tournaments.  
 736 Note that packing vertex-disjoint triangles (and hence cycles) in sparse tournaments is  
 737 NP-complete [9].

738 Let  $T$  be a sparse tournament according to the ordering of its vertices  $\sigma(T)$ , that is the  
 739 set of its backward arcs  $\overleftarrow{A}(T)$  is a matching. If a backward arc  $xy$  of  $T$  lies between two  
 740 consecutive vertices, then we can exchange the position of  $x$  and  $y$  in  $\sigma(T)$  to obtain a sparse  
 741 tournament with fewer backward arc. So we can assume that the backward arcs of  $T$  do not  
 742 contain consecutive vertices. Moreover, if a vertex  $x$  of  $T$  is contained in no backward arc  
 743 of  $T$  then call  $A$  (resp.  $B$ ) the vertices of  $T$  which are before (resp. after)  $x$  in  $\sigma(T)$ . Let  
 744  $X_0$  be the set of triangles made from a backward arc from  $B$  to  $A$  and the vertex  $x$ . As  
 745  $T$  is sparse it is clear that  $X_0$  is a set of disjoint triangles. Moreover, it can easily be seen  
 746 that there exists an optimal packing of triangles (resp. cycles) of  $T$  which is the union of  
 747 an optimal packing of triangles (resp. cycles) of  $T[A]$ , one of  $T[B]$  and  $X_0$ . Thus to solve  
 748 MAXATT or MAXACT on  $T$  we can solve the problem on  $T[A]$  and on  $T[B]$  and build the  
 749 optimal solution for  $T$ . Therefore we can focus on the case where every vertex of  $T$  is the  
 750 beginning or the end of a backward arc  $\overleftarrow{A}(T)$ . We will call such a tournament a *fully sparse*  
 751 *tournament*. So we focus on solving MAXATT in fully sparse tournaments. In the following,  
 752 let  $\Pi$  be the problem of finding a collection of arc-disjoint triangles of maximum size on fully  
 753 sparse tournament.

754 Now order the arcs  $e_1, \dots, e_b$  of  $\overleftarrow{A}(T)$  such that for any  $i \in [b-1]$ ,  $h(e_i) <_{\sigma} h(e_{i+1})$ .  
 755 Moreover, let  $G'$  be the digraph with vertex set  $V' = \{e_i : i \in [b]\}$  and arc set  $A'$  defined  
 756 by:  $(e_i e_j) \in A'$  if  $(h(e_i), h(e_j), t(e_i))$  or  $(h(e_i), t(e_j), t(e_i))$  is a triangle of  $T$ . Let  $\Pi'$  be the  
 757 problem such that, given a digraph  $G' = (V', A')$ , the objective is to find a maximum sized  
 758 subset of  $A'$  such that the digraph induced by the arcs of the subset is a functional and  
 759 digon-free digraph. Remind that a functional digraph is a digraph such that any of its  
 760 vertices has out-degree at most 1.

761 Let  $X$  be a solution (not necessary optimal) of  $\Pi'(G')$ , and  $e_i e_j$  an arc of  $X$ . We denote  
 762 by  $\Pi(e_i e_j)$  the triangle  $(h(e_i), h(e_j), t(e_i))$  if  $i < j$  and otherwise. Given a triangle  $\Pi(e_i e_j)$ ,  
 763 let  $s(e_j)$  be the second vertex of  $\Pi(e_i e_j)$ ; in other words, if  $\Pi(e_i e_j) = (h(e_i), t(e_j), t(e_i))$ , then  
 764  $s(e_j) = t(e_j)$  and  $s(e_j) = h(e_j)$  otherwise. Informally,  $\Pi(e_i e_j)$  corresponds to the triangle  
 765 formed by the backward arc  $e_i$  and one vertex of  $e_j$ , that vertex being  $s(e_j)$ . In the same  
 766 way, we define  $\Pi(X) = \bigcup_{x \in X} \Pi(x)$ .

767 **► Claim 19.1.** *Let  $X$  be a solution of  $\Pi'(G')$ . The set  $X$  is an optimal solution if and only*  
 768 *if  $\Pi(X)$  is an optimal solution of  $\Pi(T)$ .*

769 **Proof.** Let  $e_i e_j$  and  $e_k e_l$  be two distinct arcs of  $X$ . We cannot have  $e_i = e_k$  as  $X$  induces  
 770 a functional digraph in  $G'$ . Without loss of generality, we may assume that  $i < k$ , that is

771  $h(e_i) <_{\sigma} h(e_k)$ . Moreover, we cannot have  $t(e_i) = t(e_k)$  without contradicting that  $T$  is a  
 772 sparse tournament. As  $h(e_i) <_{\sigma} h(e_k)$  the arc  $h(e_i)s(e_j)$  is not an arc of  $\Pi(e_k e_l)$ . Thus if  
 773  $\Pi(e_i e_j)$  and  $\Pi(e_k e_l)$  share a common arc, it means that  $s(e_j)t(e_i) = h(e_k)s(e_l)$ . But in this  
 774 case  $e_i = e_l$  and  $e_j = e_k$ , implying  $\{e_i e_j, e_k e_l\}$  is a digon of  $G'$ , which contradict the fact  
 775 that  $X$  is a solution  $\Pi'(G')$ . So, if  $X$  is a solution of  $\Pi'(G')$ , then  $\Pi(X)$  is an solution of  
 776  $\Pi(T)$ . Notice that the size of the solution does not change.

777 On the other hand, if  $X$  is a subset of the arcs of  $G'$  such that  $\Pi(X)$  is a solution of  
 778  $\Pi(T)$ . We cannot have a vertex  $e_i$  of  $G'$  such that  $d_X^+(e_i) > 1$ , since it would imply that the  
 779 backward arc  $e_i$  of  $T$  is covered by at least two triangles of  $\Pi(X)$ . So  $X$  induces a functional  
 780 subdigraph of  $G'$ . As previously the digraph induced by  $X$  is also digon-free otherwise we  
 781 would have two arc-disjoint triangles on only four vertices in  $\Pi(X)$ , which is impossible.  
 782 Thus,  $X$  is a solution of  $\Pi'(G')$ , and the solution of the same size.

783 The two problems  $\Pi$  and  $\Pi'$  being both maximization problems, they have the same  
 784 optimal solution. ◀

785 Now we show how to solve  $\Pi'$  in polynomial time.

786 ▶ **Claim 19.2.** *If  $G'$  is strongly connected and has a cycle  $C$  of size at least 3 then the*  
 787 *solution of  $\Pi'(G')$  is the number of vertices of  $G'$ .*

788 **Proof.** We construct the arc set  $X$  as follows: we start by taking the arcs of  $C$ . Then, while  
 789 there is a vertex  $x$  which is not covered by any arcs of  $X$ , we add to  $X$  the arcs of the  
 790 shortest path from  $x$  to any vertex of  $X$ . By construction, every vertex  $x$  of every arc of  $X$   
 791 verify  $d_X^+(x) = 1$ , and  $X$  is digon free. Since  $X$  covers every vertex of  $G'$ ,  $|X|$  is a maximum  
 792 solution of  $\Pi'(G')$ , that is the number of vertices of  $G'$ . ◀

793 A digraph  $D$  is a *digoned tree* if  $D$  arises from a non-trivial tree whose each edge is  
 794 replaced by a digon.

795 ▶ **Claim 19.3.** *If  $G'$  is strongly connected and has only cycles of size 2 then  $G'$  is a digoned*  
 796 *tree.*

797 **Proof.** Since  $G'$  is strongly connected, then for any arc  $xy$  of  $G'$  there exists a path from  
 798  $y$  to  $x$ . As  $G'$  only contains cycles of size 2, the only path from  $y$  to  $x$  is the directed arc  
 799  $yx$ . So every arc of  $G'$  is contained in a digon. If  $H$  is the underlying graph of  $G'$  (without  
 800 multiple edges) then it is clear that  $H$  is a tree otherwise  $G'$  would contain a cycle of size  
 801 more than 2. ◀

802 ▶ **Claim 19.4.** *If  $G'$  is a digoned tree or if  $|V(G')|=1$ , then the optimal solution of  $\Pi'(G')$*   
 803 *is  $|V(G')|-1$ .*

804 **Proof.** The case  $|V(G')|=1$  is clear. So assume that  $G'$  is a digoned tree and let  $X$  be a set  
 805 of arcs of  $G'$  corresponding to an optimal solution of  $\Pi'(G')$ . Then  $X$  is acyclic and then  
 806 has size at most  $|V(G')|-1$ . Moreover, any in-branching of  $G'$  provides a solution of size  
 807  $|V(G')|-1$ . ◀

808 ▶ **Lemma 20.** *Let  $G'$  be a digraph with  $n$  vertices. Denote by  $S_1, \dots, S_p$  terminal strong*  
 809 *components of  $G'$  such that for any  $i$  with  $1 \leq i \leq k$ ,  $S_i$  is a digoned tree or an isolated*  
 810 *vertex and for any  $i > k$ ,  $S_i$  contains a cycle of length at least 3. Then an optimal solution*  
 811 *of  $\Pi'(G')$  has size  $n - k$  and we can construct one in polynomial time.*

812 **Proof.** We can assume that  $G'$  is connected otherwise we apply the result on every connected  
 813 component of  $G'$  and the disjoint union of the solutions produces an optimal solution on the  
 814 whole digraph  $G'$ .

815 So assume that  $G'$  is connected and let  $S$  be a terminal strong component of  $G'$ . If  $X$  is  
 816 an optimal solution of  $\Pi'(G')$  then the restriction of  $X$  to the arcs of  $G'[S]$  is an optimal  
 817 solution of  $\Pi'(G'[S])$ . Indeed otherwise we could replace this set of arcs in  $X$  by an optimal  
 818 solution of  $\Pi'(G'[S])$  and obtain a better solution for  $\Pi'(G')$ , a contradiction.

819 So by Claim 19.2 and Claim 19.4 the set  $X$  contains at most  $\sum_{i=1,\dots,p} |S_i| - k$  arcs lying  
 820 in a terminal component of  $G'$ . Now as every vertex of  $G' \setminus \bigcup_{i=1,\dots,p} S_i$  is the beginning of at  
 821 most one arc of  $X$ , the set  $X$  has size at most  $n - k$ . Conversely by growing in-branchings  
 822 in  $G'$  from the union of the optimal solutions of  $\Pi'(G'[S_i])$  for  $i = 1, \dots, p$ , by Claim 19.2  
 823 and 19.4 we obtain a solution of  $\Pi'(G')$  of size  $n - k$  which is then optimal. Moreover, this  
 824 solution can clearly be built in polynomial time. ◀

825 Using Claim 19.1 and Lemma 20 we can solve MAXATT in polynomial time.

826 ► **Lemma 21.** *In a fully sparse tournament  $T$  the size of a maximum cycle packing is equal  
 827 to the size of a maximum triangle packing.*

828 **Proof.** First if  $T$  has an optimal triangle packing of size  $|\overleftarrow{A}(T)|$  then as  $\overleftarrow{A}(T)$  is an FAS of  $T$ ,  
 829 every optimal cycle packing of  $T$  has size  $|\overleftarrow{A}(T)|$ . Otherwise, we build from  $T$  the digraph  $G'$   
 830 as previously. By Lemma 20,  $G'$  has some terminal components  $S_1, \dots, S_k$  which are either  
 831 a single vertex or induces a digoned tree and every optimal triangle packing of  $T$  has size  
 832  $|\overleftarrow{A}(T)| - k$ . Let see that no  $S_i$  can be a single vertex. Indeed if  $S_i = \{e\}$  where  $e$  is a backward  
 833 arc of  $T$ , it means that no backward of  $T$  begins or ends between  $h(e)$  and  $t(e)$  in  $\sigma(T)$ . As  $T$   
 834 is fully sparse, it means that  $h(e)$  and  $t(e)$  are consecutive in  $\sigma(T)$  what we forbid previously.  
 835 Now consider a component  $S_i$  which induces a digoned tree in  $G'$ . Let  $\pi_i$  be the order  $\sigma(T)$   
 836 restricted to the heads and tails of the arcs of  $T$  corresponding to the vertices of  $S_i$ . First  
 837 notice that  $\pi_i$  is an interval of the order  $\sigma(T)$ . Indeed otherwise there exists two backward  
 838 arcs  $a$  and  $b$  of  $T$  such that  $a \in S_i$ ,  $b \notin S_i$  and  $h(a)$  is before the head or the of  $b$  which is  
 839 before  $t(a)$  in  $\sigma(T)$ . But in this case there is an arc in  $G'$  from  $a$  to  $b$  contradicting the fact  
 840 that  $S_i$  is a terminal component of  $G'$ . So we denote  $\pi_i$  by  $(x_1, x_2, \dots, x_l)$  and notice that  
 841  $x_1$  and  $x_2$  are then forced to be the heads of backward arcs belonging to  $S_i$ . If  $x_3$  is also  
 842 the head of backward arc of  $S_i$ , then we obtain that the three corresponding backward arcs  
 843 form a 3-cycle in  $G'$  contradicting the fact that  $S_i$  induces a digoned tree in  $G'$ . Repeating  
 844 the same argument we show that  $l$  is even and that the backward arcs corresponding to the  
 845 elements of  $S_i$  are exactly  $x_3x_1$ ,  $x_lx_{l-2}$  and  $x_jx_{j-3}$  for all odd  $j \in [l] \setminus \{1, 3\}$ . In other words  
 846  $S_i$  induces a 'digoned path' in  $G'$ . Now consider  $\Delta$  an optimal cycle packing of  $T$ . Let  $X_1$   
 847 be the set of backward arcs of  $\overleftarrow{A}(T)$  with head strictly before  $x_1$  and tail strictly after  $x_l$  in  
 848  $\sigma(T)$ . And let  $\Delta_1$  be the cycles of  $\Delta$  using at least one arc of  $X_1$ . It is easy to check that  
 849  $\Delta' = (\Delta \setminus \Delta_1) \cup \{(h(e), x_1, t(e)) : e \in X_1\}$  is also an optimal cycle packing of  $T$ . Now every  
 850 cycle of  $\Delta'$  which uses a backward arc of  $S_i$  only uses backward arcs of  $S_i$  (otherwise it must  
 851 one arc of  $X_1$ , which is not possible). Let  $\Delta_i$  be the set of cycles of  $\Delta$  using backward arcs  
 852 of  $S_i$ . It is easy to see that  $\{x_i x_{i+1} : i \text{ even and } i \in [l-2]\}$  is an FAS of  $T[\{x_1, \dots, x_l\}]$  and  
 853 has size  $l/2 - 1 = |S_i| - 1$ . So we have  $|\Delta_i| \leq |S_i| - 1$ .

854 Repeating this argument for  $i = 1, \dots, k$  we obtain that  $|\Delta| \leq |\overleftarrow{A}(T)| - k$ . Thus by Lemma 20  
 855  $\Delta$  has the same size than an optimal triangle packing of  $T$ . ◀

856 This leads to the following main result of this section.

857 ► **Theorem 22.** MAXATT and MAXACT restricted to sparse tournaments can be solved in  
858 polynomial time.

## 859 **7** Concluding Remarks

860 In this work, we studied the classical and parameterized complexity of packing arc-disjoint  
861 cycles and triangles in tournaments. We showed NP-hardness, fixed-parameter tractability and  
862 linear kernelization results. We also showed that these problems are polynomial-time solvable  
863 in sparse tournaments. To conclude, observe that very few problems on tournaments are  
864 known to admit an  $\mathcal{O}^*(2^{\sqrt{k}})$ -time algorithm when parameterized by the standard parameter  
865  $k$  [48] - FAST is one of them [4, 28]. To the best of our knowledge, outside bidimensionality  
866 theory, there are no packing problems that are known to admit such subexponential algorithms.  
867 In light of the  $2^{o(\sqrt{k})}$  lower bound shown for ACT and ATT, it would be interesting to  
868 explore if these problems admit  $\mathcal{O}^*(2^{\mathcal{O}(\sqrt{k})})$  algorithms.

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