Circular flows for the Euler equations in two-dimensional annular domains

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Abstract

In this paper, we consider steady Euler flows in two-dimensional bounded annuli, as well as in exterior circular domains, in punctured disks and in the punctured plane. We always assume rigid wall boundary conditions. We prove that, if the flow does not have any stagnation point, and if it satisfies further conditions at infinity in the case of an exterior domain or at the center in the case of a punctured disk or the punctured plane, then the flow is circular, namely the streamlines are concentric circles. In other words, the flow then inherits the radial symmetry of the domain. The proofs are based on the study of the trajectories of the flow and the orthogonal trajectories of the gradient of the stream function, which is shown to satisfy a semilinear elliptic equation in the whole domain. In exterior or punctured domains, the method of moving planes is applied to some almost circular domains located between some streamlines of the flow, and the radial symmetry of the stream function is shown by a limiting argument. The paper also contains two Serrin-type results in simply or doubly connected bounded domains with free boundaries. Here, the flows are further assumed to have constant norm on each connected component of the boundary and the domains are then proved to be disks or annuli.

Contents

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1 Introduction and main results

Throughout this paper, $|\ |$ denotes the Euclidean norm in $\mathbb{R}^2$ and, for $0 \le a < b \le \infty$, $\Omega_{a,b}$ denotes the two-dimensional domain defined by

$$\Omega_{a,b} = \big\{x \in \mathbb{R}^2 :\ a < |x| < b\big\}.$$

When $a < b$ are two positive real numbers, $\Omega_{a,b}$ is a bounded smooth annulus. When $0 < a < b = \infty$, $\Omega_{a,\infty}$ is an exterior domain which is the complement of a closed disk. When $0 = a < b < \infty$, $\Omega_{0,b}$ is a punctured disk. When $0 = a < b = \infty$, $\Omega_{0,\infty}$ is the punctured plane $\mathbb{R}^2\setminus\{0\}$, where we denote $0 = (0,0)$ with a slight abuse of notation.

We also denote

$$e_r(x) = \frac{x}{|x|} \quad \text{and} \quad e_\theta(x) = e_r(x)^\perp = \Big(-\frac{x_2}{|x|}, \frac{x_1}{|x|}\Big)$$

for $x = (x_1, x_2) \in \mathbb{R}^2\setminus\{0\}$. Moreover, for $x \in \mathbb{R}^2$ and $r > 0$,

$$B(x,r) = \{y \in \mathbb{R}^2 : |y - x| < r\}$$

denotes the open Euclidean disk with center x and radius r. We also write $B_r = B(0,r)$ and

$$C_r = \partial B_r = \{x \in \mathbb{R}^2 : |x| = r\}$$

for $r > 0$.

In $\Omega_{a,b}$, we consider steady flows

$$v = (v_1, v_2)$$

of an inviscid fluid, solving the system of the Euler equations:

$$\begin{cases} v \cdot \nabla\, v + \nabla\, p = 0 & \text{in } \Omega_{a,b}, \\ \text{div } v = 0 & \text{in } \Omega_{a,b}, \end{cases} \tag{1.1}$$

where the solutions v and p are always understood in the classical sense, that is, they are (at least) of class C^1 in $\Omega_{a,b}$ and therefore satisfy (1.1) everywhere in $\Omega_{a,b}$. We always assume rigid wall boundary conditions, that is, v is (at least) continuous up to the regular parts of $\partial\Omega_{a,b}$ and tangential there:

$$\begin{cases} v \cdot e_r = 0 \text{ on } C_a \text{ if } a > 0, \\ v \cdot e_r = 0 \text{ on } C_b \text{ if } b < \infty. \end{cases} \tag{1.2}$$

A flow v in $\Omega_{a,b}$ is called a circular flow if $v(x)$ is parallel to the vector $e_\theta(x)$ at every point $x \in \Omega_{a,b}$, that is, $v \cdot e_r = 0$ in $\Omega_{a,b}$. The main goal in this paper is to show that, under some conditions, the flow is a circular flow and thus inherits the radial symmetry of the domain $\Omega_{a,b}$. We obtain such results in the four cases $0 < a < b < \infty$, $0 < a < b = \infty$, $0 = a < b < \infty$, and $0 = a < b = \infty$.

We also consider Euler flows in simply or doubly connected bounded domains whose boundaries are free: under the additional condition that the flow has constant norm on the boundary, we show that both the domain and the flow are then circular.

The case of bounded smooth annuli $\Omega_{a,b}$ with $0 < a < b < \infty$

The first result is concerned with flows having no stagnation point in the closed annulus $\overline{\Omega_{a,b}}$. Throughout the paper, the stagnation points of a flow v are the points x for which $|v(x)| = 0$.

Theorem 1.1 *Assume $0 < a < b < \infty$. Let v be a $C^2(\overline{\Omega_{a,b}})$ flow solving (1.1)-(1.2) and such that $|v| > 0$ in $\overline{\Omega_{a,b}}$. Then v is a circular flow, and there is a $C^2([a,b])$ function V with constant strict sign such that*

$$v(x) = V(|x|)\, e_\theta(x) \quad \textit{for all } x \in \overline{\Omega_{a,b}}.$$

It actually turns out that the assumption $|v| > 0$ in $\overline{\Omega_{a,b}}$ can be slightly relaxed. Namely, if $|v| > 0$ in the open annulus $\Omega_{a,b}$ and if the set of stagnation points is assumed to be properly included in one of the connected components of $\partial\Omega_{a,b}$, then the same conclusion holds, and then in fact $|v| > 0$ in $\overline{\Omega_{a,b}}$. This is the purpose of the following result.

Theorem 1.2 *Assume $0 < a < b < \infty$. Let v be a $C^2(\overline{\Omega_{a,b}})$ flow solving (1.1)-(1.2) and such that*

$$\left\{x \in \overline{\Omega_{a,b}} : |v(x)| = 0\right\} \subsetneq C_a \quad \textit{or} \quad \left\{x \in \overline{\Omega_{a,b}} : |v(x)| = 0\right\} \subsetneq C_b.^1 \tag{1.3}$$

Then $|v| > 0$ in $\overline{\Omega_{a,b}}$ and the conclusion of Theorem 1.1 holds.

Theorem 1.2 is clearly stronger than Theorem 1.1, but we preferred to state Theorem 1.1 separately since the assumption is simpler.

Several further comments are in order. First of all, despite the fact that $\Omega_{a,b}$ is not simply connected, the flow v has a stream function $u : \overline{\Omega_{a,b}} \to \mathbb{R}$ of class $C^3(\overline{\Omega_{a,b}})$ defined by

$$\nabla^\perp u = v, \quad \text{that is,} \quad \frac{\partial u}{\partial x_1} = v_2 \quad \text{and} \quad \frac{\partial u}{\partial x_2} = -v_1 \tag{1.4}$$

[1] Throughout the paper, by $E \subsetneq F$, we mean that $E \subset F$ and $E \neq F$.

in $\overline{\Omega_{a,b}}$, since v is divergence free and tangential on C_a. Notice that the stream function u is uniquely defined in $\overline{\Omega_{a,b}}$ up to an additive constant. Theorems 1.1 and 1.2 can then be viewed as Liouville-type symmetry results since their conclusion means that the stream function u is radially symmetric (and strictly monotone with respect to $|x|$ in $\overline{\Omega_{a,b}}$). Furthermore, if for x in $\overline{\Omega_{a,b}}$ one calls ξ_x the solution of

$$\begin{cases} \dot{\xi}_x(t) = v(\xi_x(t)), \\ \xi_x(0) = x, \end{cases} \tag{1.5}$$

the conclusion of Theorems 1.1 and 1.2 then implies that each function ξ_x is defined in $\mathbb{R}$ and periodic, and that the streamlines $\Xi_x = \xi_x(\mathbb{R})$ of the flow are concentric circles.

Theorems 1.1 and 1.2 also mean equivalently that any $C^2(\overline{\Omega_{a,b}})$ non-circular flow for (1.1)-(1.2) must either have a stagnation point in the open annulus $\Omega_{a,b}$, or must have stagnation points in both circles C_a and C_b, or in the whole circle C_a, or in the whole circle C_b.

Without the assumption $|v| > 0$ in $\overline{\Omega_{a,b}}$ or the weaker one (1.3), the conclusion of Theorems 1.1 and 1.2 obviously does not hold in general, in the sense that there are non-circular flows which do not fulfill (1.3). To construct such flows explicitly, we first point out that, for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and any non-radial $C^2(\overline{\Omega_{a,b}})$ solution u of

$$\Delta u + f(u) = 0 \tag{1.6}$$

in $\Omega_{a,b}$ which is constant on C_a and on C_b and which has a critical point in $\overline{\Omega_{a,b}}$, the $C^1(\overline{\Omega_{a,b}})$ field

$$v = \nabla^\perp u$$

is a non-circular solution of (1.1)-(1.2) with a stagnation point in $\overline{\Omega_{a,b}}$: notice indeed that $v = \nabla^\perp u$ satisfies the boundary condition $v \cdot e_r = -\nabla u \cdot e_\theta = 0$ on $\partial\Omega_{a,b}$ since u is constant on C_a and on C_b, and v solves (1.1) with pressure

$$p = -\frac{|v|^2}{2} - F(u) = -\frac{|\nabla u|^2}{2} - F(u),$$

where $F' = f$. As an example, let $\lambda \in \mathbb{R}$ and $\varphi \in C^\infty([a,b])$ be the principal eigenvalue and the principal eigenfunction of the eigenvalue problem

$$-\varphi''(r) - r^{-1}\varphi'(r) + r^{-2}\varphi(r) = \lambda\,\varphi(r) \ \text{ in } [a,b]$$

with $\varphi > 0$ in (a,b) and Dirichlet boundary condition $\varphi(a) = \varphi(b) = 0$ (the principal eigenvalue λ is unique and the principal eigenfunction φ is unique up to multiplication by positive constants). The $C^\infty(\overline{\Omega_{a,b}})$ function u defined by $u(x) = \varphi(|x|)\,x_1/|x|$ (that is, $u(x) = \varphi(r)\cos(\theta)$ in the usual polar coordinates) satisfies

$$\Delta u + \lambda u = 0 \ \text{ in } \overline{\Omega_{a,b}}$$

and it has some critical points in $\overline{\Omega_{a,b}}$ (since $\min_{\overline{\Omega_{a,b}}} u < 0 < \max_{\overline{\Omega_{a,b}}} u$ and $u = 0$ on $\partial\Omega_{a,b}$). Actually, it can easily be seen that φ has only one critical point in $[a,b]$ and that u has exactly

6 critical points in $\overline{\Omega_{a,b}}$ (2 in $\Omega_{a,b}$, 2 on C_a, and 2 on C_b). Then the $C^\infty(\overline{\Omega_{a,b}})$ flow $v = \nabla^\perp u$ is a non-circular flow solving (1.1)-(1.2) and having 2 stagnation points in $\Omega_{a,b}$ and 4 on $\partial\Omega_{a,b}$.

However, we do not know whether the hypothesis (1.3) could be more relaxed for the conclusion of Theorems 1.1 and 1.2 to still hold. For instance, would it be sufficient to assume that v has no stagnation point in $\Omega_{a,b}$? We refer to Remark 2.5 in Section 2 below for further comments on this question.

On the other hand, we point out that the sufficient conditions $|v| > 0$ in $\overline{\Omega_{a,b}}$ or the more general one (1.3) are obviously not equivalent to being a circular flow, in the sense that there are circular flows for (1.1)-(1.2) which do not fulfill (1.3) (besides the trivial flow $v = (0,0)$!). Actually, any $C^1(\overline{\Omega_{a,b}})$ circular flow $v(x) = V(|x|)\, e_\theta(x)$ solving (1.1)-(1.2) and for which $V \in C^1([a,b])$ does not have a constant strict sign, has a set of stagnation points containing at least a circle. For instance, let $\mu \in \mathbb{R}$ and $\phi \in C^\infty([a,b])$ be the principal eigenvalue and the principal eigenfunction of the eigenvalue problem

$$-\phi''(r) - r^{-1}\phi'(r) = \mu\,\phi(r) \quad \text{in } [a,b],$$

with $\phi > 0$ in (a,b) and Dirichlet boundary condition $\phi(a) = \phi(b) = 0$, and let $u = \phi(|\cdot|)$. Then $v = \nabla^\perp u = \phi'(|\cdot|)\, e_\theta$ is a $C^\infty(\overline{\Omega_{a,b}})$ non-trivial circular flow solving (1.1)-(1.2) with pressure $p(x) = -\phi'(|x|)^2/2 - \mu\phi(|x|)^2/2$ and with a circle of stagnation points in $\overline{\Omega_{a,b}}$: more precisely, if $r^* \in (a,b)$ denotes a real number such that $\phi(r^*) = \max_{[a,b]} \phi$ (it is easy to see that r^* is the only critical point of ϕ in $[a,b]$), then the set of stagnation points of the flow v is equal to the whole circle C_{r^*}.

Lastly, the assumption on the $C^2(\overline{\Omega_{a,b}})$ smoothness of v is a technical assumption which is used in the proof. It warrants the C^1 smoothness of the vorticity $\omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$, satisfying $v \cdot \nabla\omega = 0$ in $\Omega_{a,b}$, and the C^1 smoothness of the vorticity function f arising in the semilinear elliptic equation of the type (1.6) satisfied by the stream function u. We refer to the proofs in Section 2 and especially Lemma 2.4 below for further details.

The case of exterior domains $\Omega_{a,\infty}$ with $0 < a < \infty$

Theorem 1.3 *Assume $0 < a < \infty$ and $b = \infty$. Let v be a $C^2(\overline{\Omega_{a,\infty}})$ flow solving (1.1)-(1.2) and such that*

$$\big\{x \in \overline{\Omega_{a,\infty}} : |v(x)| = 0\big\} \subsetneq C_a \quad \text{and} \quad \liminf_{|x|\to+\infty} |v(x)| > 0. \tag{1.7}$$

Assume moreover that

$$v(x) \cdot e_r(x) = o\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to +\infty. \tag{1.8}$$

Then $|v| > 0$ in $\overline{\Omega_{a,\infty}}$ and v is a circular flow, namely there is a $C^2([a,+\infty))$ function V with constant strict sign such that $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \overline{\Omega_{a,\infty}}$.

As for Theorems 1.1 and 1.2, the conclusion of Theorem 1.3 says that the stream function u is radially symmetric and strictly monotone with respect to $|x|$ in $\overline{\Omega_{a,\infty}}$, and that the streamlines of the flow v are concentric circles.

As far as the behavior of v at infinity is concerned, we do not know what could be the critical behavior of $v(x) \cdot e_r(x)$ as $|x| \to +\infty$, or another type of asymptotic condition at

infinity, for the conclusion of Theorem 1.3 to hold. However, we can say that without the condition (1.8) the conclusion of Theorem 1.3 does not hold in general. For instance, consider the $C^\infty(\overline{\Omega_{a,\infty}})$ function u defined by $u(x) = 2(|x|^2/a^2 - 1) + (|x|/a - a/|x|)x_1/|x|$, that is,

$$u = 2\left(\frac{r^2}{a^2} - 1\right) + \left(\frac{r}{a} - \frac{a}{r}\right)\cos\theta$$

in the usual polar coordinates. The function u satisfies $\Delta u - 8/a^2 = 0$ in $\overline{\Omega_{a,\infty}}$ with Dirichlet boundary condition $u = 0$ on C_a, and the $C^\infty(\overline{\Omega_{a,\infty}})$ field $v = \nabla^\perp u$ satisfies (1.1)-(1.2) with pressure $p = -|v|^2/2 + 8u/a^2$. In the usual polar coordinates, the field v is given by

$$v = \left[\frac{4r}{a^2} + \left(\frac{1}{a} + \frac{a}{r^2}\right)\cos\theta\right] e_\theta + \left[\left(\frac{1}{a} - \frac{a}{r^2}\right)\sin\theta\right] e_r. \tag{1.9}$$

It satisfies condition (1.7) (and even $\inf_{\Omega_{a,\infty}} |v| \geq 2/a > 0$). But

$$v(x)\cdot e_r(x) = \left(\frac{1}{a} - \frac{a}{|x|^2}\right)\frac{x_2}{|x|} \neq o\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to +\infty,$$

and v is not a circular flow. However, since $u(x) \to +\infty$ as $|x| \to +\infty$ and $u = 0$ on C_a and since u has no critical point, it is easily seen that all solutions ξ_x of (1.5) are defined in $\mathbb{R}$ and periodic and that all streamlines $\Xi_x = \xi_x(\mathbb{R})$ (which are level sets of u) surround the origin.[2] Nevertheless, the streamlines do not converge to any family of circles at infinity since a calculation yields $\max_{y\in\Xi_x} |y| - \min_{y\in\Xi_x} |y| = \max_{\mathbb{R}} |\xi_x(\cdot)| - \min_{\mathbb{R}} |\xi_x(\cdot)| \to a/2 > 0$ as $|x| \to +\infty$.

We point out that, in Theorem 1.3, the flow v is not assumed to be bounded. Actually, there are unbounded circular flows satisfying all assumptions of Theorem 1.3: consider for instance the $C^\infty(\overline{\Omega_{a,\infty}})$ unbounded circular flow v defined by

$$v(x) = |x|\, e_\theta(x),$$

solving (1.1)-(1.2) with stream function $u(x) = |x|^2/2$ and pressure $p(x) = |x|^2/2$, and satisfying $\inf_{\Omega_{a,\infty}} |v| = a > 0$.

Notice lastly that the condition (1.7) is fulfilled in particular when $\inf_{\Omega_{a,\infty}} |v| > 0$. Furthermore, as soon as $|v| > 0$ on C_a (that holds if $\inf_{\Omega_{a,\infty}} |v| > 0$), the boundary condition (1.2) and the continuity of v imply in particular that $v \cdot e_\theta$ has a constant strict sign on C_a. Under the condition $\inf_{\Omega_{a,\infty}} |v| > 0$, the following result then provides some estimates on the infimum or the supremum of the vorticity $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ in $\Omega_{a,\infty}$, in terms of the sign of $v \cdot e_\theta$ on C_a.

Theorem 1.4 *Assume $0 < a < \infty$ and $b = \infty$. Let v be a $C^2(\overline{\Omega_{a,\infty}})$ flow solving (1.1)-(1.2) and such that $\inf_{\Omega_{a,\infty}} |v| > 0$. If $v \cdot e_\theta > 0$ on C_a (respectively if $v \cdot e_\theta < 0$ on C_a), then*

$$\sup_{\Omega_{a,\infty}}\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) > 0 \quad (\textit{respectively } \inf_{\Omega_{a,\infty}}\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) < 0).$$

[2]Throughout the paper, we say that a Jordan curve $\mathcal{C}$ surrrounds the origin if the bounded connected component of $\mathbb{R}^2 \setminus \mathcal{C}$ contains the origin.

The flow v given by (1.9) is an example of a flow satisfying the assumptions of Theorem 1.4, with $v \cdot e_\theta > 0$ on C_a, and for which the vorticity (namely Δu) is actually equal to the positive constant $8/a^2$ everywhere in $\overline{\Omega_{a,\infty}}$.

Theorem 1.4 can also be viewed as a Liouville-type result. Namely, we show in its proof that, if $\inf_{\Omega_{a,\infty}} |v| > 0$, if $v \cdot e_\theta > 0$ on C_a, and if the vorticity is nonpositive everywhere in $\Omega_{a,\infty}$, then v is a circular flow of the type $v = V(|\cdot|)\, e_\theta$ with $V : [a, +\infty) \to [\eta, +\infty)$ for some $\eta > 0$. Therefore, the vorticity $\frac{\partial v_2}{\partial x_1}(x) - \frac{\partial v_1}{\partial x_2}(x) = V'(|x|) + V(|x|)/|x|$ can not be nonpositive everywhere (since otherwise the function $r \mapsto r\, V(r)$ $(\geq \eta r)$ would be nonincreasing in $[a, +\infty)$, leading to a contradiction).[3]

Notice that Theorem 1.4 does not hold good if the assumption $\inf_{\Omega_{a,\infty}} |v| > 0$ is dropped. There are actually some circular flows v satisfying (1.1)-(1.2) such that $|v| > 0$ in $\overline{\Omega_{a,\infty}}$ and $v \cdot e_\theta > 0$ on C_a, but $\inf_{\Omega_{a,\infty}} |v| = 0$ and for which the vorticity is negative everywhere. Consider for instance the $C^\infty(\overline{\Omega_{a,\infty}})$ circular flow

$$v(x) = \frac{1}{|x|^2}\, e_\theta(x),$$

solving (1.1)-(1.2) with stream function $u(x) = -1/|x|$ and pressure $p(x) = -1/(4|x|^2)$: one has $|v| > 0$ in $\overline{\Omega_{a,\infty}}$ and $v \cdot e_\theta > 0$ on C_a, but $\inf_{\Omega_{a,\infty}} |v| = 0$ and $\frac{\partial v_2}{\partial x_1}(x) - \frac{\partial v_1}{\partial x_2}(x) = -1/|x|^3 < 0$ in $\overline{\Omega_{a,\infty}}$.

The case of punctured disks $\Omega_{0,b}$ with $0 < b < \infty$

Theorem 1.5 *Assume $a = 0$ and $0 < b < \infty$. Let v be a $C^2(\overline{\Omega_{0,b}} \setminus \{0\})$ flow solving (1.1)-(1.2) and such that*

$$\big\{x \in \overline{\Omega_{0,b}} \setminus \{0\} : |v(x)| = 0\big\} \subsetneq C_b \quad \text{and} \quad \int_{C_\varepsilon} |v \cdot e_r| \to 0 \ \text{ as } \varepsilon \stackrel{>}{\to} 0. \tag{1.10}$$

Then $|v| > 0$ in $\overline{\Omega_{0,b}} \setminus \{0\}$ and v is a circular flow, namely there is a $C^2((0,b])$ function V with constant strict sign such that $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \overline{\Omega_{0,b}} \setminus \{0\}$.

Notice that the condition $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} |v \cdot e_r| = 0$ is fulfilled in particular if $v(x) \cdot e_r(x) = o(1/|x|)$ as $|x| \stackrel{>}{\to} 0$. We do not know what could be the critical behavior of $v \cdot e_r$ at 0, or another type of asymptotic condition at the origin, for the conclusion of Theorem 1.5 to hold. However we can say that, without the condition $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} |v \cdot e_r| = 0$, the conclusion of Theorem 1.5 does not hold in general. Let us give a counter-example similar to (1.9) above (which was there defined in $\overline{\Omega_{a,\infty}}$). More precisely, consider the $C^\infty(\overline{\Omega_{0,b}} \setminus \{0\})$ function u defined by $u(x) = (|x|/b - b/|x|)x_1/|x|$, that is,

$$u = \Big(\frac{r}{b} - \frac{b}{r}\Big) \cos\theta$$

in the usual polar coordinates. The function u satisfies $\Delta u = 0$ in $\overline{\Omega_{0,b}} \setminus \{0\}$ with Dirichlet boundary condition $u = 0$ on C_b, and the $C^\infty(\overline{\Omega_{0,b}} \setminus \{0\})$ field $v = \nabla^\perp u$ satisfies (1.1)-(1.2)

[3] The same arguments do not lead to any contradiction in the case of bounded annuli $\Omega_{a,b}$ and $\Omega_{0,b}$ with $b < \infty$, see Remark 1.6.

with pressure $p = -|v|^2/2$ (and vorticity equal to 0). In the usual polar coordinates, the field v is given by

$$v = \Big[\Big(\frac{1}{b} + \frac{b}{r^2}\Big)\cos\theta\Big]\, e_\theta + \Big[\Big(\frac{1}{b} - \frac{b}{r^2}\Big)\,\sin\theta\Big]\, e_r. \tag{1.11}$$

It has only two stagnation points in $\overline{\Omega_{0,b}} \setminus \{0\}$ and they both lie on C_b. Hence, the first part of condition (1.10) is fulfilled. But $\int_{C_\varepsilon} |v \cdot e_r| = 4(\varepsilon/b - b/\varepsilon) \not\to 0$ as $\varepsilon \stackrel{>}{\to} 0$, and v is not a circular flow.

Lastly, in Theorem 1.5, the flow v is not assumed to be bounded. Actually, there are unbounded circular flows satisfying all assumptions of Theorem 1.5: consider for instance the $C^\infty(\overline{\Omega_{0,b}} \setminus \{0\})$ unbounded circular flow v defined by

$$v(x) = \frac{1}{|x|}\, e_\theta(x) \tag{1.12}$$

solving (1.1)-(1.2) with stream function $u(x) = \ln|x|$ and pressure $p(x) = -1/(2|x|^2)$, and satisfying $|v| > 0$ in $\overline{\Omega_{0,b}} \setminus \{0\}$ and then (1.10).

Remark 1.6 A result similar to Theorem 1.4 does not hold in the punctured disk $\Omega_{0,b}$. For instance, the $C^\infty(\overline{\Omega_{0,b}} \setminus \{0\})$ flow (1.12) satisfies (1.1)-(1.2), $v \cdot e_\theta > 0$ on C_b, $\inf_{\Omega_{0,b}} |v| > 0$, but $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \equiv 0$ in $\Omega_{0,b}$. The same observation holds good in a smooth annulus $\Omega_{a,b}$ with $0 < a < b < \infty$.

The case of the punctured plane $\Omega_{0,\infty}$

The last geometric configuration considered in the paper is that the punctured plane

$$\Omega_{0,\infty} = \mathbb{R}^2 \setminus \{0\}.$$

Theorem 1.7 *Let v be a $C^2(\Omega_{0,\infty})$ flow solving (1.1) and such that $|v| > 0$ in $\Omega_{0,\infty}$ and $\liminf_{|x|\to+\infty} |v(x)| > 0$. Assume moreover that*

$$v(x)\cdot e_r(x) = o\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to +\infty \quad \text{and} \quad \int_{C_\varepsilon} |v\cdot e_r| \to 0 \text{ as } \varepsilon \stackrel{>}{\to} 0, \tag{1.13}$$

and that the flow has one streamline which is a Jordan curve surrounding the origin. Then v is a circular flow. Furthermore, there is a $C^2((0,+\infty))$ function V with constant strict sign such that $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \Omega_{0,\infty}$.

The conclusion says that, under some conditions on $|v|$ and under the same conditions as in Theorems 1.3 and 1.5 on the behavior of the radial component of v at infinity and at the origin, the existence of a streamline surrounding the origin implies that all streamlines surround the origin and are actually all concentric circles.

Remark 1.8 Let us mention here other rigidity results for the stationary solutions of (1.1) in various geometrical configurations. The analyticity of the streamlines under a condition of the type $v_1 > 0$ in the unit disk was shown in [16]. The local correspondence between

the vorticities of the solutions of (1.1) and the co-adjoint orbits of the vorticities for the non-stationary version of (1.1) in more general annular domains was investigated in [8]. In a previous paper [12] (see also [13]), we considered the case of a two-dimensional strip with bounded section and the case of bounded flows in a half-plane, assuming in both cases that the flows v are tangential on the boundary and that $\inf |v| > 0$: all streamlines are then proved to be lines which are parallel to the boundary of the domain (in other words the flow is a parallel flow). Earlier results by Kalisch [15] were concerned with flows in two-dimensional strips under the additional assumption $v \cdot e \neq 0$, where e is the main direction of the strip. Lastly, in [14], we considered the case of the whole plane $\mathbb{R}^2$ and we showed that any $C^2(\mathbb{R}^2)$ bounded flow v is still a parallel flow under the condition $\inf_{\mathbb{R}^2} |v| > 0$.

Some Serrin-type free boundary problems with overdetermined boundary conditions

The last main results on the solutions of the Euler equations (1.1) are two Serrin-type results in smooth simply or doubly connected bounded domains whose boundaries are free but on which the flow is assumed to satisfiy an additional condition.

Theorem 1.9 *Let Ω be a C^2 non-empty simply connected bounded domain of $\mathbb{R}^2$. Let $v \in C^2(\overline{\Omega})$ satisfy the Euler equations* (1.1) *and assume that $v \cdot n = 0$ and $|v|$ is constant on $\partial\Omega$, where n denotes the outward unit normal on $\partial\Omega$. Assume moreover that v has a unique stagnation point in $\overline{\Omega}$. Then, up to a shift,*

$$\Omega = B_R$$

for some $R > 0$. Furthermore, the unique stagnation point of v is the center of the disk and v is a circular flow, that is, there is a $C^2([0, R])$ function $V : [0, R] \to \mathbb{R}$ such that $V \neq 0$ in $(0, R]$, $V(0) = 0$, and $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \overline{B_R} \setminus \{0\}$.

In the proof, we will show that the $C^3(\overline{\Omega})$ stream function u defined by (1.4) satisfies a semilinear elliptic equation $\Delta u + f(u) = 0$ in $\overline{\Omega}$. Furthermore, up to normalization, the function u vanishes on $\partial\Omega$ and is positive in Ω. Lastly, since $|v|$ is assumed to be constant along $\partial\Omega$, the normal derivative $\frac{\partial u}{\partial n}$ of u along $\partial\Omega$ is constant. This problem is therefore an elliptic equation with overdetermined boundary conditions. Since the celebrated paper by Serrin [22], it has been known that these overdetermined boundary conditions on $\partial\Omega$ determine the geometry of Ω, namely, Ω is then a ball and the function u is radially symmetric (hence, here, v would then be a circular flow). The proof is based on the method of moving planes developed in [3, 6, 10, 22] and on the maximum principle, and it relies on the Lipschitz continuity of the function f. In our case, the function f is given in terms of the function u itself and it may not be Lipschitz continuous on the whole range $[0, \max_{\overline{\Omega}} u]$. More precisely, it may not be Lipschitz continuous in a neighborhood of the maximal value $\max_{\overline{\Omega}} u$. One therefore has to adapt the proof to this case by removing small neighborhoods of size ε around the maximal point of u (which is the unique stagnation point of v): one shows the symmetry of the domain in all directions up to ε and one concludes by passing to the limit as $\varepsilon \stackrel{>}{\to} 0$.

In connection with Theorems 1.5 and 1.9, we state the following conjecture.

Conjecture 1.10 *Let D be an open non-empty disk and let $z \in D$. Let v be a $C^2(\overline{D}\setminus\{z\})$ and bounded flow solving* (1.1) *and $v \cdot n = 0$ on ∂D, where n denotes the outward unit normal on ∂D. Assume that $|v| > 0$ in $\overline{D}\setminus\{z\}$. Then z is the center of the disk and the flow is circular with respect to z.*

Up to shift, one can assume that $D = \Omega_{0,b}$ for some $b \in (0, +\infty)$, hence $n = e_r$ on ∂D. If the point z is a priori assumed to be the center of the disk, namely the origin, then Theorem 1.5 implies that v is a circular flow. Up to rotation, assume now that $z = (\alpha, 0)$ for some $\alpha \in (0, b)$ and, without loss of generality, that the stream function u is positive in $D\setminus\{z\}$ and vanishes on ∂D. The goal would be to reach a contradiction. As far as Theorem 1.9 is concerned, the method of proof described in the paragraph following the statement shows simultaneously the symmetry of the domain and the symmetry of the function u (which obeys an equation of the type $\Delta u + f(u) = 0$), thanks to the overdetermined boundary conditions satisfied by u. Here in Conjecture 1.10, the same technics based on the method of moving method implies for instance on the one hand that the function u is even in x_2 in $\overline{\Omega_{0,b}}\setminus\{z\}$, and on the other hand that $u(x_1, x_2) < u(2\alpha - x_1, x_2)$ for all $(x_1, x_2) \in \overline{\Omega_{0,b}}$ such that $x_1 > \alpha$. But, regarding the second property, the Hopf lemma might not apply to the function $(x_1, x_2) \mapsto u(x_1, x_2) - u(2\alpha - x_1, x_2)$ at the point $z = (\alpha, 0)$ since the vorticity function f might not be Lipschitz continuous around the limiting value of u at z (see also Remark 2.5 below, and notice that u is not differentiable at z, unless one further assumes that $|v(x)| \to 0$ as $x \to z$). Therefore, the same arguments as the ones in the proof of Theorem 1.9 do not lead to an obvious contradiction if z is not the center of the disk. However, Conjecture 1.10 seems natural and will be the purpose of further investigation.

A related weaker conjecture (with stronger assumptions) can also be formulated: if D is an open non-empty disk, if $z \in D$, if $v \in C^2(\overline{D})$ solves (1.1), if $v \cdot n = 0$ on ∂D and if z is the only stagnation point of v in $\overline{D}$, then z is the center of the disk and v is circular with respect to z. For the same reasons as in the previous paragraph (since the vorticity function f might not be Lipschitz continuous around $u(z)$), the proof of that second conjecture not clear either.

The last main result of the paper is concerned with the case of doubly connected bounded domains.

Theorem 1.11 *Let ω_1 and ω_2 be two C^2 non-empty simply connected bounded domains of $\mathbb{R}^2$ such that $\overline{\omega_1} \subset \omega_2$, and denote*

$$\Omega = \omega_2 \setminus \overline{\omega_1}.$$

Let $v \in C^2(\overline{\Omega})$ satisfy the Euler equations (1.1)*. Assume that $v \cdot n = 0$ on $\partial\Omega = \partial\omega_1 \cup \partial\omega_2$, where n denotes the outward unit normal on $\partial\Omega$, and that $|v|$ is constant on $\partial\omega_1$ and on $\partial\omega_2$. Assume moreover that $|v| > 0$ in $\overline{\Omega}$. Then ω_1 and ω_2 are two concentric disks and, up to shift,*

$$\Omega = \Omega_{a,b}$$

for some $0 < a < b < \infty$ and v is a circular flow satisfying the conclusion of Theorem 1.1 *in $\overline{\Omega} = \overline{\Omega_{a,b}}$.*

In this case, by using similar arguments as in the proof of Theorem 1.1 in smooth annuli $\Omega_{a,b}$ with $0 < a < b < \infty$, it follows that the stream function u of the flow v satisfies a semilinear

elliptic equation $\Delta u + f(u) = 0$ in $\overline{\Omega}$, with $u = c_1$ on $\partial\omega_1$ and $u = c_2$ on $\partial\omega_2$, for some real numbers $c_1 \neq c_2$. Furthermore, $\min(c_1, c_2) < u < \max(c_1, c_2)$ in Ω and $\frac{\partial u}{\partial n}$ is constant along $\partial\omega_1$ and along $\partial\omega_2$. Since v has no stagnation point in $\overline{\Omega}$, the function f is then shown to be Lipschitz continuous in the whole interval $[\min(c_1, c_2), \max(c_1, c_2)]$, and known results of Reichel [18] and Sirakov [24] then imply that, up to shift, $\Omega = \Omega_{a,b}$ for some $0 < a < b < \infty$, and u is radially symmetric.

Further symmetry results have been obtained for nonlinear elliptic equations of the type $\Delta u + f(u) = 0$ or more general ones in exterior domains with overdetermined boundary conditions (see e.g. [1, 19, 24]), or in the whole space (see e.g. [11, 17, 23]), in both cases with further assumptions on the solution u at infinity and on the function f. Such conditions are in general not satisfied by the stream function u and the vorticity function f of a flow v that would be defined in the complement of a simply connected bounded domain or in the whole or punctured plane. Lastly, we refer to [5, 9, 20, 21] for further references on overdetermined boundary value elliptic problems in domains with more complex topology or in unbounded epigraphs.

Outline of the paper

In Section 2, we prove Theorems 1.1 and 1.2 dealing with the case of bounded smooth annuli $\Omega_{a,b}$. Section 3 is devoted to the proof of Theorems 1.3 and 1.4 in the exterior domains $\Omega_{a,\infty}$. Sections 4 and 5 are concerned with the proof of Theorems 1.5 and 1.7 in the punctured disks $\Omega_{0,b}$ and in the punctured plane $\Omega_{0,\infty}$. Lastly, the proof of the Serrin-type Theorems 1.9 and 1.11 is carried out in Section 6. The strategies of the proofs of Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and 1.7 share some common features: we show some properties of the streamlines of the flow and we prove some symmetry results for the equation satisfied by the stream function u, after checking that this equation is well defined. However, the cases of the exterior domains $\Omega_{a,\infty}$ and the punctured disks $\Omega_{0,b}$ and plane $\Omega_{0,\infty}$ involve some additional technicalities and require specific additional assumptions. They also require some further Liouville type results for the semilinear elliptic equations $\Delta u + f(u) = 0$ in these domains. For the sake of clarity of the paper, that is why we preferred to first deal with the case of smooth annuli $\Omega_{a,b}$ (with $0 < a < b < \infty$) and to carry out the whole proof of Theorems 1.1 and 1.2 separately in Section 2.

2 The case of bounded annuli $\Omega_{a,b}$: proof of Theorems 1.1 and 1.2

This section is devoted to the proof of Theorem 1.2 (we recall that Theorem 1.1 is a particular case of Theorem 1.2). Throughout this section, we consider two positive real numbers $a < b$ and a $C^2(\overline{\Omega_{a,b}})$ solution v of (1.1)-(1.2) satisfying (1.3), namely

$$\big\{x \in \overline{\Omega_{a,b}} : |v(x)| = 0\big\} \subsetneq C_a \quad \text{or} \quad \big\{x \in \overline{\Omega_{a,b}} : |v(x)| = 0\big\} \subsetneq C_b.$$

Before going into further details, let us first explain the general strategy of the proof of Theorem 1.2. As already mentioned in the introduction, since $\operatorname{div} v = 0$ in the two-dimensional

annulus $\Omega_{a,b}$ and since $v \cdot e_r = 0$ on C_a, there is a $C^3(\overline{\Omega_{a,b}})$ stream function $u : \overline{\Omega_{a,b}} \to \mathbb{R}$ satisfying (1.4), that is,

$$\nabla^\perp u = v \ \text{ in } \overline{\Omega_{a,b}}.$$

By definition, the stream function u is constant along the streamlines of the flow, parametrized by the solutions ξ_x of (1.5). In order to show that the flow v is circular, one will show that the stream function u is radially symmetric, that is, there is a $C^3([a,b])$ function U such that

$$u(x) = U(|x|) \ \text{ for all } x \in \overline{\Omega_{a,b}}.$$

Indeed, this means that

$$v(x) = V(|x|)\, e_\theta(x) \ \text{ for all } x \in \overline{\Omega_{a,b}},$$

with $V = U' \in C^2([a,b])$. Furthermore, since $|v|$ is continuous and does not vanish in $\Omega_{a,b}$ nor in the whole circle C_a nor in the whole circle C_b, the function V then has a constant strict sign in $[a,b]$.

To show that u is radially symmetric, we will prove that, up to changing v into $-v$, u satisfies a semilinear elliptic equation of the type

$$\left\{ \begin{array}{l} \Delta u + f(u) = 0 \ \text{ in } \Omega_{a,b}, \\ c_1 < u < c_2 \ \text{ in } \Omega_{a,b}, \\ u = c_1 \text{ on } C_a, \ \ u = c_2 \text{ on } C_b, \end{array} \right. \tag{2.1}$$

for some real numbers $c_1 < c_2$ and some $C^1([c_1,c_2])$ function f. Lastly, we use a result of Sirakov [24] to complete the proof.

The first step of the proof consists in proving that u strictly ranges between its, different, values on C_a and C_b.

Lemma 2.1 *There are two real numbers $c_1 \neq c_2$ such that $u = c_1$ on C_a, $u = c_2$ on C_b, and*

$$\min(c_1,c_2) < u < \max(c_1,c_2) \ \textit{in } \Omega_{a,b}.$$

Proof. First of all, since u satisfies (1.2), the $C^3(\mathbb{R})$ functions $t \mapsto u(a\cos t, a\sin t)$ and $t \mapsto u(b\cos t, b\sin t)$ are constant. There are then two real numbers c_1 and c_2 such that

$$u = c_1 \text{ on } C_a \ \text{ and } \ u = c_2 \text{ on } C_b. \tag{2.2}$$

For each $x \in \Omega_{a,b}$, let σ_x be the solution of

$$\left\{ \begin{array}{rcl} \dot\sigma_x(t) & = & \nabla u(\sigma_x(t)), \\ \sigma_x(0) & = & x. \end{array} \right. \tag{2.3}$$

Since ∇u is (at least) Lipschitz-continuous in $\overline{\Omega_{a,b}}$, each σ_x is defined in a neighborhood of 0 and the quantities

$$\left\{ \begin{array}{l} t_x^+ = \sup\big\{t > 0 : \sigma_x([0,t]) \subset \Omega_{a,b}\big\}, \\ t_x^- = \inf\big\{t < 0 : \sigma_x([t,0]) \subset \Omega_{a,b}\big\} \end{array} \right.$$

are such that $-\infty \le t_x^- < 0 < t_x^+ \le +\infty$. The curve $\sigma_x((t_x^-, t_x^+))\,(\subset \Omega_{a,b})$ is the trajectory of the gradient flow in $\Omega_{a,b}$ containing x. Notice that the functions σ_x and $u \circ \sigma_x$ are of class $C^1((t_x^-, t_x^+))$ with

$$(u \circ \sigma_x)'(t) = |\nabla u(\sigma_x(t))|^2 = |v(\sigma_x(t))|^2 > 0 \text{ for all } t \in (t_x^-, t_x^+),$$

since $|v| > 0$ in $\Omega_{a,b}$ by (1.3). Observe also that $\sigma_x((t_x^-, t_x^+))$ meets the streamline of the flow v containing x (parametrized by the solution ξ_x of (1.5)), orthogonally at x, since $v = \nabla^\perp u$.

We now claim that, for each $x \in \Omega_{a,b}$,

$$\mathrm{dist}(\sigma_x(t), \partial\Omega_{a,b}) \to 0 \text{ as } t \overset{<}{\to} t_x^+ \tag{2.4}$$

and

$$\mathrm{dist}(\sigma_x(t), \partial\Omega_{a,b}) \to 0 \text{ as } t \overset{>}{\to} t_x^-, \tag{2.5}$$

meaning that $\min\big(|\sigma_x(t)| - a, b - |\sigma_x(t)|\big) \to 0$ as $t \to t_x^+$ with $t < t_x^+$ and as $t \to t_x^-$ with $t > t_x^-$. Assume first by way of contradiction that (2.4) does not hold. There exist then an increasing sequence of positive real numbers $(t_n)_{n\in\mathbb{N}}$ and a point $y \in \Omega_{a,b}$ such that $t_n \to t_x^+$ and $\sigma_x(t_n) \to y$ as $n \to +\infty$. Since $y \in \Omega_{a,b}$ and the continuous field $|\nabla u| = |v|$ does not vanish in $\Omega_{a,b}$ by (1.3), there are three real numbers $r > 0$, $\eta > 0$ and $\tau > 0$ such that

$$\begin{cases} \overline{B(y,r)} \subset \Omega_{a,b}, \quad |\nabla u| \ge \eta \text{ in } \overline{B(y,r)}, \\ \sigma_z(t) \in \overline{B(y,r)} \text{ for all } z \in \overline{B(y,r/2)} \text{ and } t \in [-\tau, \tau]. \end{cases}$$

Since $\sigma_x(t_n) \to y$ as $n \to +\infty$, one has $\sigma_x(t_n) \in \overline{B(y,r/2)}$ for all n large enough, hence σ_x is defined in $[t_n - \tau, t_n + \tau]$ with $\sigma_x(t) \in \overline{B(y,r)} \subset \Omega_{a,b}$ for all $t \in [t_n - \tau, t_n + \tau]$ and n large enough. This implies that $t_x^+ = +\infty$. Furthermore, for all n large enough, one has $(u \circ \sigma_x)'(t) = |\nabla u(\sigma_x(t))|^2 \ge \eta^2$ for all $t \in [t_n - \tau, t_n + \tau]$, hence

$$u(\sigma_x(t_n + \tau)) \ge u(\sigma_x(t_n - \tau)) + 2\eta^2\tau.$$

Since $u \circ \sigma_x$ is increasing in (t_x^-, t_x^+) and since $t_n \to t_x^+ = +\infty$ as $n \to +\infty$, one then gets that $u(\sigma_x(t)) \to +\infty$ as $t \to t_x^+ = +\infty$, contradicting the boundedness of u (u is continuous in the compact set $\overline{\Omega_{a,b}}$). Therefore, (2.4) has been proved. Similarly, (2.5) holds.

Finally, for each $x \in \Omega_{a,b}$, since the function $u \circ \sigma_x$ is increasing in (t_x^-, t_x^+), it then follows from (2.2) and (2.4)-(2.5) that $u(\sigma_x(t)) \to c_1$ or c_2 as $t \to t_x^\pm$ and that $c_1 \ne c_2$. Furthermore, if $c_1 < c_2$ (resp $c_1 > c_2$), then $|\sigma_x(t)| \to a$ as $t \to t_x^-$ and $|\sigma_x(t)| \to b$ as $t \to t_x^+$ (resp. $|\sigma_x(t)| \to b$ as $t \to t_x^-$ and $|\sigma_x(t)| \to a$ as $t \to t_x^+$). In both cases, one has

$$u(\sigma_x(t)) \to \min(c_1, c_2) \text{ as } t \to t_x^- \text{ and } u(\sigma_x(t)) \to \max(c_1, c_2) \text{ as } t \to t_x^+.$$

Using again that $u \circ \sigma_x$ is increasing, one infers that $\min(c_1, c_2) < u(\sigma_x(t)) < \max(c_1, c_2)$ for each $t \in (t_x^-, t_x^+)$. In particular, at $t = 0$, one concludes that $\min(c_1, c_2) < u(x) < \max(c_1, c_2)$. This property holds for every $x \in \Omega_{a,b}$ and the proof of Lemma 2.1 is thereby complete. □

The next lemma shows that all streamlines of the flow which are included in the open annulus $\Omega_{a,b}$ are closed and surround the origin.

Lemma 2.2 *For every $x \in \Omega_{a,b}$, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic. Furthermore, there are a continuous periodic function $\rho_x : \mathbb{R} \to (a,b)$ and a continuous function $\theta_x : \mathbb{R} \to \mathbb{R}$ such that*

$$\xi_x(t) = (\rho_x(t)\cos\theta_x(t), \rho_x(t)\sin\theta_x(t)) \text{ for all } t \in \mathbb{R}, \quad \text{and} \quad \theta_x(\mathbb{R}) = \mathbb{R}. \tag{2.6}$$

In other words, the Jordan curve $\xi_x(\mathbb{R})$ surrounds the origin.

Proof. Throughout the proof, we fix any $x \in \Omega_{a,b}$. Since v is (at least) Lipschitz continuous in $\overline{\Omega_{a,b}}$, there is a maximal open interval $(\tau_x^-, \tau_x^+) \subset \mathbb{R}$ containing 0 such that ξ_x is C^1 in (τ_x^-, τ_x^+) and ranges in $\Omega_{a,b}$. Furthermore, owing to the definitions of ξ_x and u, the function $u \circ \xi_x$ is constant in (τ_x^-, τ_x^+). It then follows from Lemma 2.1 and the uniform continuity of u in $\overline{\Omega_{a,b}}$ that

$$a < \inf_{t\in(\tau_x^-,\tau_x^+)} |\xi_x(t)| \le \sup_{t\in(\tau_x^-,\tau_x^+)} |\xi_x(t)| < b. \tag{2.7}$$

Furthermore, the function ξ_x is then defined in $\mathbb{R}$, that is, $(\tau_x^-, \tau_x^+) = \mathbb{R}$. In the sequel, we call

$$\Xi_x = \xi_x(\mathbb{R})$$

the streamline containing x.

Let us now show that ξ_x is periodic. Consider the sequence $(\xi_x(n))_{n\in\mathbb{N}}$ in $\overline{\Omega_{a,b}}$. Since $\overline{\Omega_{a,b}}$ is compact and since $a < \inf_{t\in\mathbb{R}} |\xi_x(t)| \le \sup_{t\in\mathbb{R}} |\xi_x(t)| < b$, there are $y \in \Omega_{a,b}$ and an increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\xi_x(\varphi(n)) \to y$ as $n \to +\infty$. Since u is constant along Ξ_x, one has $u(x) = u(\xi_x(0)) = u(\xi_x(\varphi(n))) \to u(y)$ as $n \to +\infty$, hence $u(x) = u(y)$ and $u(\xi_x(\varphi(n))) = u(y)$ for all $n \in \mathbb{N}$. Furthermore, since $|\nabla u(y)| = |v(y)| > 0$ by (1.3) (remember that $y \in \Omega_{a,b}$), there are some real numbers $r > 0$ and $\tau_- < 0 < \tau_+$ such that $\overline{B(y,r)} \subset \Omega_{a,b}$ and

$$B(y,r) \cap \Xi_y = B(y,r) \cap u^{-1}(\{u(y)\}) = \xi_y((\tau_-, \tau_+)).$$

Therefore, $\xi_x(\varphi(n)) \in \xi_y((\tau_-, \tau_+))$ for all n large enough, that is, $\xi_x(\varphi(n)) = \xi_y(\tau_n)$ with $\tau_n \in (\tau_-, \tau_+)$ (notice in particular that this implies that the streamlines Ξ_x and Ξ_y coincide). Since

$$\xi_x(\varphi(n) - \tau_n) = \xi_y(0) = \xi_x(\varphi(2n) - \tau_{2n})$$

for all n large enough and $\varphi(2n) - \varphi(n) \ge n$ for all $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ large enough such that $T_N := (\varphi(2N) - \tau_{2N}) - (\varphi(N) - \tau_N)) > 0$, and $\xi_x(\varphi(N) - \tau_N) = \xi_x(\varphi(N) - \tau_N + T_N)$. As a consequence, the function ξ_x is T_N-periodic.

Finally, remembering that $|\dot\xi_x(t)| = |v(\xi_x(t))| > 0$ for all $t \in \mathbb{R}$ by (1.3) and (2.7), we claim that the non-empty open connected subset of $\mathbb{R}^2$ surrounded by the curve Ξ_x is not included in $\Omega_{a,b}$. Indeed, otherwise, the function u, which is constant on Ξ_x, would have a critical point in this domain: that is impossible since $|\nabla u| = |v| > 0$ in $\Omega_{a,b}$. As a conclusion, the streamline Ξ_x surrounds the origin. That implies that $\theta_x(\mathbb{R}) = \mathbb{R}$, where $\theta_x : \mathbb{R} \to \mathbb{R}$ is any continuous function given as in (2.6). In (2.6), the function $\rho_x = |\xi_x|$ is necessarily continuous and periodic, as is ξ_x. The proof of Lemma 2.2 is thereby complete. □

Remark 2.3 Lemma 2.2 was concerned with the description of the streamlines Ξ_x when x belongs to the open annulus $\Omega_{a,b}$. If $|v| > 0$ on C_a (resp. on C_b), then the boundary conditions (1.2) imply that, for any $x \in C_a$ (resp. $x \in C_b$), ξ_x is still defined and periodic in $\mathbb{R}$ with $\Xi_x = C_a$ (resp. $\Xi_x = C_b$). If $x \in \partial\Omega_{a,b}$ and $|v(x)| = 0$, then $\xi_x(t) = x$ for all $t \in \mathbb{R}$ and $\Xi_x = \{x\}$. If $x \in C_a$ (resp. $x \in C_b$) with $|v(x)| > 0$ and if v has some stagnation points on C_a (resp. C_b), then it is easy to see that ξ_x is still defined in $\mathbb{R}$, but it is not periodic anymore and Ξ_x is a proper arc of C_a (resp. C_b) which is open relatively to C_a (resp. C_b).

Up to changing v into $-v$ and u into $-u$, one can assume without loss of generality that the real numbers $c_1 \neq c_2$ given in Lemma 2.1 are such that

$$c_1 < c_2. \tag{2.8}$$

The last preliminary lemma is the derivation of an equation of the type (1.6) in $\overline{\Omega_{a,b}}$.

Lemma 2.4 *There is a $C^1([c_1, c_2])$ function $f : [c_1, c_2] \to \mathbb{R}$ such that $\Delta u + f(u) = 0$ in $\overline{\Omega_{a,b}}$.*

Proof. From assumption (1.3), either there is a point $A \in C_a$ such that $|v(A)| > 0$ and $|v| > 0$ on C_b, or there is a point $B \in C_b$ such that $|v(B)| > 0$ and $|v| > 0$ on C_a. Let us consider the first case only (the second one can be handled similarly). Then, let σ_A be the solution of (2.3) with $x = A$. Since $|\nabla u(A)| = |v(A)| > 0$ and $\nabla u(A) \cdot e_\theta(A) = -v(A) \cdot e_r(A) = 0$ by (1.2), the vector $\nabla u(A)$ is non-zero and parallel to $e_r(A)$. Furthermore, since $u = c_1$ on C_a and $c_1 < u < c_2$ in $\Omega_{a,b}$ by Lemma 2.1, one infers that $\nabla u(A) = |\nabla u(A)|\, e_r(A)$, that σ_A is defined in (at least) some interval $[0, t_*)$ with $t_* > 0$ and $\sigma_A((0, t_*)) \subset \Omega_{a,b}$. Denote

$$t_A^+ = \sup\{t > 0 : \sigma_A((0, t]) \subset \Omega_{a,b}\} \in (0, +\infty].$$

The function σ_A is of class $C^1([0, t_A^+))$ and, for every $t \in (0, t_A^+)$, one has $\sigma_A(t) \in \Omega_{a,b}$ with

$$(u \circ \sigma_A)'(t) = |\nabla u(\sigma_A(t))|^2 = |v(\sigma_A(t))|^2 > 0.$$

Since $u(\sigma_A(t)) \to u(\sigma_A(0)) = u(A) = c_1$ as $t \to 0$, the proof of Lemma 2.1 then implies that $|\sigma_A(t)| \to b$ and $u(\sigma_A(t)) \to c_2$ as $t \to t_A^+$. On the other hand, the function $|\nabla u \circ \sigma_A|$ is continuous in $[0, t_A^+)$, positive at 0 (since $|\nabla u(\sigma_A(0))| = |v(A)| > 0$), positive in $(0, t_A^+)$ (since $\sigma_A((0, t_A^+)) \subset \Omega_{a,b}$), and $\liminf_{t \to t_A^+} |\nabla u(\sigma_A(t))| = \liminf_{t \to t_A^+} |v(\sigma_A(t))| > 0$ (since the uniformly continuous field $|v|$ is positive on C_b and $|\sigma_A(t)| \to b$ as $t \to t_A^+$). As a consequence, there is $\eta > 0$ such that

$$|\dot\sigma_A(t)| = |\nabla u(\sigma_A(t))| \geq \eta \ \text{ for all } t \in [0, t_A^+).$$

Therefore, $(u \circ \sigma_A)'(t) = |\nabla u(\sigma_A(t))|^2 \geq \eta^2$ for all $t \in [0, t_A^+)$ and t_A^+ is positive a real number, since u is bounded in $\overline{\Omega_{a,b}}$. Moreover, for every $t \in [0, t_A^+)$, one has

$$c_2 - c_1 \geq u(\sigma_A(t)) - u(A) = u(\sigma_A(t)) - u(\sigma_A(0)) = \int_0^t |\nabla u(\sigma_A(s))|^2 ds \geq \eta \int_0^t |\dot\sigma_A(s)|\, ds,$$

hence the length of the curve $\sigma_A([0, t_A^+))$ is finite. Finally, there is a point $A^+ \in C_b$ such that $\sigma_A(t) \to A^+$ as $t \to t_A^+$. By setting $\sigma_A(t_A^+) = A^+$ and remembering that the field ∇u

is (at least) continuous in $\overline{\Omega_{a,b}}$, it follows that the function $\sigma_A : [0, t_A^+] \to \overline{\Omega_{a,b}}$ is then of class $C^1([0, t_A^+])$ and that the function $g := u \circ \sigma_A$ is a C^1 diffeomorphism from $[0, t_A^+]$ onto $[c_1, c_2]$ (with $g' \geq \eta^2 > 0$ in $[0, t_A^+]$). Let $g^{-1} : [c_1, c_2] \to [0, t_A^+]$ be the reciprocal of g_A and define

$$f(\tau) = -\Delta u(\sigma_A(g^{-1}(\tau))) \ \text{ for } \tau \in [c_1, c_2]. \tag{2.9}$$

The function $f : [c_1, c_2] \to \mathbb{R}$ is of class $C^1([c_1, c_2])$ (since Δu, σ_A and g^{-1} are C^1, respectively in $\overline{\Omega_{a,b}}$, $[0, t_A^+]$, and $[c_1, c_2]$), and

$$\Delta u(\sigma_A(s)) + f(u(\sigma_A(s)) = 0 \ \text{ for all } s \in [0, t_A^+]. \tag{2.10}$$

Consider now any $x \in \Omega_{a,b}$. Since $|\sigma_A(0)| = |A| = a$ and $|\sigma_A(t_A^+)| = |A^+| = b$, the streamline Ξ_x necessarily meets the curve $\sigma_A([0, t_A^+])$ at a point lying in $\Omega_{a,b}$, by Lemma 2.2. Hence, there are $t \in \mathbb{R}$ and $s \in (0, t_A^+)$ such that $\xi_x(t) = \sigma_A(s)$, while $u(x) \in (c_1, c_2)$. On the other hand, the stream function u is constant along the streamline Ξ_x, and the $C^1(\overline{\Omega_{a,b}})$ vorticity $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$ satisfies $v \cdot \nabla(\Delta u) = 0$ in $\overline{\Omega_{a,b}}$ from the Euler equations (1.1), hence Δu is constant along the streamline Ξ_x too. As a consequence, (2.10) yields

$$\Delta u(x) + f(u(x)) = \Delta u(\xi_x(t)) + f(u(\xi_x(t))) = \Delta u(\sigma_A(s)) + f(u(\sigma_A(s))) = 0.$$

Lastly, since the functions Δu and $f \circ u$ are (at least) continuous in $\overline{\Omega_{a,b}}$, the equation $\Delta u + f(u) = 0$ is satisfied every in $\overline{\Omega_{a,b}}$ and the proof of Lemma 2.4 is thereby complete. □

Proof of Theorem 1.2. It follows from Lemmas 2.1 and 2.4, together with (2.8) that the function u is a classical solution of (2.1) for some real numbers $c_1 < c_2$ and some (at least) Lipschitz-continuous function $f : [c_1, c_2] \to \mathbb{R}$. If then follows from [24, Theorem 5][4] that u is radially symmetric and increasing with respect to $|x|$ in $\overline{\Omega_{a,b}}$. Therefore, there is a $C^3([a, b])$ increasing function $U : [a, b] \to \mathbb{R}$ such that $u(x) = U(|x|)$ for all $x \in \overline{\Omega_{a,b}}$. The flow $v = \nabla^\perp u$ is then given by

$$v(x) = V(|x|)\, e_\theta(x)$$

for all $x \in \overline{\Omega_{a,b}}$, with $V = U' \in C^2([a, b])$. Lastly, since $|v|$ is continuous in $\overline{\Omega_{a,b}}$ and does not vanish in $\Omega_{a,b}$ nor in the whole circle C_a nor in the whole circle C_b, the function V then has a constant strict (positive) sign in $[a, b]$. The proof of Theorem 1.2 is thereby complete. □

Remark 2.5 For a $C^2(\overline{\Omega_{a,b}})$ flow v solving (1.1)-(1.2), could the assumption (1.3) be slightly relaxed for v still to be necessarily a circular flow? As we mentioned in the introduction, the conclusion does not hold in general if v has stagnation points in $\Omega_{a,b}$. So a natural question is the following one: if $|v| > 0$ in $\Omega_{a,b}$, then is v a circular flow? It is easy to see from their proofs that Lemmas 2.1 and 2.2 hold good if (1.3) is replaced by $|v| > 0$ in $\Omega_{a,b}$. Consider then any point $y \in \Omega_{a,b}$. With the same notations as in Lemma 2.1, and assuming without loss of

[4]Notice that this result holds in any dimension $n \geq 2$. It is similar to the classical radial symmetry property proved in [10] in the case where u is a positive solution of the equation $\Delta u + f(u) = 0$ in a ball, with Dirichlet condition $u = 0$ on the boundary.

generality that $c_1 < c_2$, there are some quantities $t_y^\pm$ such that $-\infty \le t_y^- < 0 < t_y^+ \le +\infty$ and the solution σ_y of (2.3) with y instead of x is of class $C^1((t_y^-, t_y^+))$ and ranges in $\Omega_{a,b}$, with

$$\begin{cases} |\sigma_y(t)| \to a \text{ and } u(\sigma_y(t)) \to c_1 \text{ as } t \to t_y^-, \\ |\sigma_y(t)| \to b \text{ and } u(\sigma_y(t)) \to c_2 \text{ as } t \to t_y^+. \end{cases} \tag{2.11}$$

The $C^1((t_y^-, t_y^+))$ function $g := u \circ \sigma_y$ is increasing (since $(u \circ \sigma_y)'(t) = |\nabla u(\sigma_y(t))|^2 = |v(\sigma_y(t))|^2 > 0$ for all $t \in (t_y^-, t_y^+)$), and g is then an increasing homeomorphism from (t_y^-, t_y^+) onto (c_1, c_2). The function $f : (c_1, c_2) \to \mathbb{R}$ defined by

$$f(\tau) = -\Delta u(\sigma_y(g^{-1}(\tau))) \text{ for } \tau \in (c_1, c_2) \tag{2.12}$$

is of class $C^1((c_1, c_2))$ and, since for every $x \in \Omega_{a,b}$ the streamline Ξ_x intersects $\sigma_y((t_y^-, t_y^+))$ by Lemma 2.2, the same arguments as in the proof of Lemma 2.4 imply that

$$\Delta u + f(u) = 0 \text{ in } \Omega_{a,b}.$$

Furthermore, remembering from Lemma 2.2 that for each $x \in \Omega_{a,b}$, the C^1 solution ξ_x of (1.5) is periodic and ranges in $\Omega_{a,b}$, we claim that

$$\max_{t \in \mathbb{R}} |\xi_x(t)| \to a \text{ as } |x| \stackrel{>}{\to} a. \tag{2.13}$$

Indeed, otherwise, there would exist some sequences $(x_n)_{n\in\mathbb{N}}$ in $\Omega_{a,b}$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, and a point z such that $a < |z| \le b$ together with $|x_n| \to a$ and $\xi_{x_n}(t_n) \to z$ as $n \to +\infty$. Hence, $u(x_n) \to c_1$ by Lemma 2.1 and the uniform continuity of u, while $u(x_n) = u(\xi_{x_n}(t_n)) \to u(z) > c_1$ by Lemma 2.1 again, leading to a contradiction. Therefore, (2.13) holds and, similarly, one has $\min_{t\in\mathbb{R}} |\xi_x(t)| \to b$ as $|x| \to b$ with $|x| < b$. Since the function Δu is constant along any streamline of the flow from the Euler equations (1.1) and since Δu is uniformly continuous in $\overline{\Omega_{a,b}}$, it then follows from the previous observations and Lemma 2.2 that Δu is constant on C_a and constant on C_b. Call d_1 and d_2 the values of Δu on C_a and C_b, respectively, and set $f(c_1) = -d_1$ and $f(c_2) = -d_2$. One then infers from (2.11) and (2.12) that $f : [c_1, c_2] \to \mathbb{R}$ is continuous in $[c_1, c_2]$ and that the equation $\Delta u + f(u) = 0$ holds in the closed annulus $\overline{\Omega_{a,b}}$ (u is then a classical $C^2(\overline{\Omega_{a,b}})$ solution of (2.1)). However, since

$$f'(\tau) = -\frac{\nabla(\Delta u)(\sigma_y(g^{-1}(\tau))) \cdot \nabla u(\sigma_y(g^{-1}(\tau)))}{|\nabla u(\sigma_y(g^{-1}(\tau)))|^2} \text{ for all } \tau \in (c_1, c_2)$$

and since $|\nabla u(\sigma_y(g^{-1}(\tau)))|$ may converge to 0 as $\tau \to c_1$ or c_2 (this happens if $|v| = 0$ on C_a or if $|v| = 0$ on C_b), it is not sure whether the function f' is bounded in (c_1, c_2) or not (it is not sure whether or not there exists a maximal curve $\sigma_X((t_X^-, t_X^+))$ lying in $\Omega_{a,b}$, for some $X \in \Omega_{a,b}$, along which $|\nabla u|$ is bounded from below by a positive constant). The argument used in the proof of Theorem 1.2 to conclude that the solution u of (2.1) is radially symmetric relies on [24, Theorem 5], which itself uses the Lipschitz-continuity of f over the range of u. Thus, the same argument can not be applied as such in general in the case where v is just assumed to have no stagnation point in $\Omega_{a,b}$, without the assumption (1.3). Other arguments should then be used to prove that v is circular or to disprove this property in general. We leave this question open for a further work.

3 The case of unbounded annuli $\Omega_{a,\infty}$: proof of Theorems 1.3 and 1.4

This section is devoted to the proof of Theorems 1.3 and 1.4. Throughout this section, we fix a positive real number a and we consider a $C^2(\overline{\Omega_{a,\infty}})$ flow v solving (1.1)-(1.2) and such that

$$\left\{x \in \overline{\Omega_{a,\infty}} : |v(x)| = 0\right\} \subsetneq C_a \quad \text{and} \quad |v| \geq \eta > 0 \text{ in } \overline{\Omega_{a+1,\infty}} \tag{3.1}$$

for some positive real number $\eta > 0$. Notice that these conditions are fulfilled in both Theorems 1.3 and 1.4. The $C^3(\overline{\Omega_{a,\infty}})$ stream function u given by (1.4) is well defined since v is divergence free and tangent on C_a, and u satisfies

$$|\nabla u| \geq \eta > 0 \quad \text{in } \overline{\Omega_{a+1,\infty}}. \tag{3.2}$$

Let $A \in C_a$ be a point such that

$$|v(A)| > 0, \tag{3.3}$$

hence $v(A) \cdot e_\theta(A) \neq 0$ since $v \cdot e_r = 0$ on C_a. Up to changing v into $-v$ and u into $-u$, one can assume without loss of generality that $v(A) \cdot e_\theta(A) > 0$, that is,

$$\nabla u(A) \cdot e_r(A) > 0. \tag{3.4}$$

Since $\nabla u \cdot e_\theta = -v \cdot e_r = 0$ on C_a, the function u is constant on C_a and, since u is unique up to an additive constant, one can also assume without loss of generality that

$$u = 0 \quad \text{on } C_a. \tag{3.5}$$

We first show in Section 3.1 a preliminary lemma, namely Lemma 3.1 below, holding for both Theorems 1.3 and 1.4. It is concerned with the limit of u along the trajectory of the gradient flow starting from the point A. Then Sections 3.2 and 3.3 are devoted to the proof of Theorems 1.3 and 1.4. In Section 3.4, we do the proof of an independent lemma, Lemma 3.8 below, which is itself used not only in the proof of Theorems 1.3 and 1.4, but also for Theorems 1.5 and 1.7 as well as for the Serrin-type Theorem 1.9.

3.1 A preliminary common lemma

Let us consider here the trajectory of ∇u starting from the boundary point A satisfying (3.3)-(3.4). More precisely, let σ be the solution of (2.3) with $x = A$, that is,

$$\begin{cases} \dot{\sigma}(t) &= \nabla u(\sigma(t)), \\ \sigma(0) &= A. \end{cases} \tag{3.6}$$

Lemma 3.1 *There is $T \in (0, +\infty]$ such that σ is defined and of class C^1 in $[0, T)$, and*

$$|\sigma(t)| \to +\infty \quad \textit{and} \quad u(\sigma(t)) \to +\infty \quad \textit{as } t \stackrel{<}{\to} T.$$

Proof. Since ∇u is (at least) of class $C^1(\overline{\Omega_{a,\infty}})$ and $\nabla u(A)\cdot e_r(A)>0$ by (3.4), there is $t_*\in(0,+\infty)$ such that σ is defined and of class C^1 at least in $[0,t_*)$, and $\sigma(s)\in\Omega_{a,\infty}$ for all $0<s<t_*$. Define

$$T=\sup\big\{t>0:\sigma \text{ is defined and of class } C^1 \text{ in } [0,t) \text{ and } \sigma((0,t))\subset\Omega_{a,\infty}\big\}.$$

There holds $0<t_*\le T\le+\infty$ and the function σ is of class $C^1([0,T))$ with $\sigma((0,T))\subset\Omega_{a,\infty}$. Furthermore, $(u\circ\sigma)'(t)=|\nabla u(\sigma(t))|^2=|v(\sigma(t))|^2>0$ for all $t\in[0,T)$ by (3.1) and (3.3). Since $\sigma(0)=A\in C_a$ and u is continuous in $\overline{\Omega_{a,\infty}}$ and constant on C_a, one infers that $\liminf_{t\to T,\,t<T}|\sigma(t)|>a$.

Assume now by way of contradiction that $|\sigma(t)|$ does not converge to $+\infty$ as $t\stackrel{<}{\to}T$. Then there are a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,T)$ and a point $y\in\Omega_{a,\infty}$ such that $t_n\to T$ and $\sigma(t_n)\to y$ as $n\to+\infty$. As in the proof of Lemma 2.1, there are three positive real numbers $r>0$, $\rho>0$ and $\tau>0$ such that $\overline{B(y,r)}\subset\Omega_{a,\infty}$, $|v|\ge\rho$ in $\overline{B(y,r)}$, and σ_z is defined (at least) in $[-\tau,\tau]$ and ranges in $\overline{B(y,r)}$ for all $z\in\overline{B(y,r/2)}$. Owing to the definition of T, one gets that $T=+\infty$ and $u(\sigma(t_n+\tau))\ge u(\sigma(t_n-\tau))+2\rho^2\tau$ for all n large enough, hence $u(\sigma(t))\to+\infty$ as $t\to+\infty$ since $u\circ\sigma$ is increasing on $[0,T)=[0,+\infty)$. This leads to a contradiction since $u(\sigma(t_n))\to u(y)\in\mathbb{R}$ as $n\to+\infty$. Therefore, $|\sigma(t)|\to+\infty$ as $t\stackrel{<}{\to}T$.

Lastly, let $T_0\in(0,T)$ such that $|\sigma(s)|\ge a+1$ for all $s\in[T_0,T)$. It follows from (3.2) that, for all $t\in[T_0,T)$,

$$u(\sigma(t))-u(\sigma(T_0))=\int_{T_0}^t|\nabla u(\sigma(s))|^2ds\ge\eta\int_{T_0}^t|\dot\sigma(s)|\,ds\ge\eta\,(|\sigma(t)|-|\sigma(T_0)|).$$

Consequently, $u(\sigma(t))\to+\infty$ as $t\stackrel{<}{\to}T$, and the proof of Lemma 3.1 is thereby complete. □

Remark 3.2 Notice that in Theorems 1.3 and 1.4, the flow v is not assumed to be bounded in $\Omega_{a,\infty}$. If v is assumed to be bounded in $\Omega_{a,\infty}$, say $|v|\le M$ in $\Omega_{a,\infty}$ for some positive real number M (notice that $M\ge\eta>0$), then

$$u(\sigma(t))-u(\sigma(0))=\int_0^t|\nabla u(\sigma(s))|^2ds\le M^2t$$

for all $t\in[0,T)$, hence $T=+\infty$ by Lemma 3.1. However, T might be finite in general if the flow v is not bounded. As an example, consider the smooth unbounded flow defined by $v(x)=|x|^2e_\theta(x)$ in $\overline{\Omega_{a,\infty}}$. It solves (1.1)-(1.2) with pressure $p(x)=|x|^4/4$ and stream function $u(x)=|x|^3/3$ (up to additive constants). Furthermore, $|v|\ge a^2>0$ in $\overline{\Omega_{a,\infty}}$. It is immediate to check that the solution σ of (3.6) is given by $\sigma(t)=(1-ta)^{-1}A$ for all $t\in[0,1/a)$, and that $T=1/a$.

3.2 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. In addition to the assumptions (3.1)-(3.3) and the normalizations (3.4)-(3.5), we assume in this section that

$$v(x)\cdot e_r(x)=o\Big(\frac{1}{|x|}\Big)\ \text{ as } |x|\to+\infty. \tag{3.7}$$

Let us describe in this paragraph the main scheme of the proof of Theorem 1.3. We first show that the stream function u is positive in $\Omega_{a,\infty}$ and converges to $+\infty$ at infinity (see Lemma 3.3 below). This implies that all streamlines of the flow surround the origin and we further show that the far streamlines converge to families of concentric circles at infinity (Lemma 3.4). Therefore, u satisfies a semilinear elliptic equation of the type $\Delta u + f(u) = 0$ in $\overline{\Omega_{a,\infty}}$ with Dirichlet boundary conditions on C_a, for some function f of class $C^1([0,+\infty))$ (Lemma 3.6). If some streamlines were true circles centered at the origin, then [24, Theorem 5] would imply that the stream function u is radially symmetric in the bounded region between C_a and these streamlines. To circumvent the fact that the streamlines are not known to be true circles a priori, we use Lemmas 3.7 and 3.8 to compare the stream function u with its reflection with respect to some lines approximating any line containing the origin. We then proceed by passing to the limit as the approximation parameter goes to 0. With Lemma 3.8, it then easily follows that u is radially symmetric and that all streamlines are truly circular, thus completing the proof of Theorem 1.3.

The first lemma is concerned with the positivity of u and with its limit at infinity.

Lemma 3.3 *The function u satisfies $u > 0$ in $\Omega_{a,\infty}$ and $u(x) \to +\infty$ as $|x| \to +\infty$.*

Proof. For every $r \geq a$, the $C^3(\mathbb{R})$ function $w_r : \theta \mapsto w_r(\theta) = u(r\cos\theta, r\sin\theta)$ is 2π-periodic and

$$w_r'(\theta) = r\nabla u(r\cos\theta, r\sin\theta)\cdot e_\theta(r\cos\theta, r\sin\theta) = -r\, v(r\cos\theta, r\sin\theta)\cdot e_r(r\cos\theta, r\sin\theta) \quad (3.8)$$

for all $\theta \in \mathbb{R}$. Hence, (3.7) implies that $\max_{\mathbb{R}} |w_r'| = \max_{[0,2\pi]} |w_r'| \to 0$ as $r \to +\infty$ and

$$\max_{C_r} u - \min_{C_r} u \to 0 \quad \text{as } r \to +\infty.^5 \quad (3.9)$$

Furthermore, it follows from Lemma 3.1 that, for every $r \geq a$, there is $s_r \in [0,T)$ such that $|\sigma(s_r)| = r$. Therefore, $s_r \to T$ as $r \to +\infty$ (since σ is at least continuous in $[0,T)$) and $u(\sigma(s_r)) \to +\infty$ by Lemma 3.1. Together with (3.9), there holds $\min_{C_r} u \to +\infty$ as $r \to +\infty$. In other words, $u(x) \to +\infty$ as $|x| \to +\infty$.

Let now $R > a$ be any large real number such that $\min_{C_R} u > 0$. Since $u = 0$ on C_a and u has no critical point in $\Omega_{a,\infty}$, one gets that $u > 0$ in $\Omega_{a,R}$. Since R can be as large as wanted, one concludes that $u > 0$ in $\Omega_{a,\infty}$. □

Before stating the next lemma on the property of all streamlines and the almost radial symmetry of the far streamlines, we recall that the streamlines of the flow can be parametrized by the solutions ξ_x of (1.5).

Lemma 3.4 *For each $x \in \Omega_{a,\infty}$, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic, and the streamline $\Xi_x = \xi_x(\mathbb{R})$ surrounds the origin. Furthermore,*

$$\max_{\mathbb{R}} |\xi_x| - \min_{\mathbb{R}} |\xi_x| \to 0 \quad \text{as } |x| \to +\infty.$$

[5] Notice that the property $\lim_{r\to+\infty}(\max_{C_r} u - \min_{C_r} u) = 0$ still holds if (3.7) is replaced by the weaker condition $\lim_{\rho\to+\infty} \int_{C_\rho} |v\cdot e_r| = 0$, since in this case one still has $\max_{[0,2\pi]} w_r - \min_{[0,2\pi]} w_r \leq \int_{C_r} |v\cdot e_r| \to 0$ as $r \to +\infty$. However, the condition (3.7) will be used in the proof of Lemma 3.4 (there we actually use $v\cdot e_r = o(1/|x|) = o(1)$ as $|x| \to +\infty$) and of Lemma 3.7 below.

Proof. Consider any $x \in \Omega_{a,\infty}$. From Lemma 3.3, there is $r > |x|$ such that $\min_{C_r} u > u(x) > 0$. Since $u = 0$ on C_a and u equal to the constant $u(x)$ along Ξ_x, it follows from the continuity of u that $a < \inf_{y\in\Xi_x} |y| \leq \sup_{y\in\Xi_x} |y| < r$. Therefore, as in Lemma 2.2, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic, and the streamline $\Xi_x = \xi_x(\mathbb{R})$ surrounds the origin.

On the other hand, it follows from (3.1) and (3.7) that there is a real number $R_0 \geq a + 1$ such that

$$|\nabla u \cdot e_r| = |v \cdot e_\theta| \geq \frac{\eta}{2} \ \text{ in } \overline{\Omega_{R_0,\infty}}. \tag{3.10}$$

Together with the continuity of ∇u and Lemma 3.3, one can even say that

$$\nabla u \cdot e_r \geq \frac{\eta}{2} \ \text{ in } \overline{\Omega_{R_0,\infty}}. \tag{3.11}$$

Let us now show that $\max_{\mathbb{R}} |\xi_x| - \min_{\mathbb{R}} |\xi_x| \to 0$ as $|x| \to +\infty$. Consider any $\varepsilon > 0$. From (3.9) and (3.11), there is $R_\varepsilon \geq a + \varepsilon$ such that

$$\max_{C_{|x|-\varepsilon}} u < u(x) - \frac{\varepsilon\,\eta}{4} \ \text{ and } \ \min_{C_{|x|+\varepsilon}} u > u(x) + \frac{\varepsilon\,\eta}{4}, \ \text{ for all } |x| \geq R_\varepsilon.$$

Therefore, for every $x \in \overline{\Omega_{R_\varepsilon,\infty}}$, one has $\Xi_x \subset \Omega_{|x|-\varepsilon,|x|+\varepsilon}$ and $\max_{\mathbb{R}} |\xi_x| - \min_{\mathbb{R}} |\xi_x| < 2\varepsilon$. In other words, $\max_{\mathbb{R}} |\xi_x| - \min_{\mathbb{R}} |\xi_x| \to 0$ as $|x| \to +\infty$, and the proof of Lemma 3.4 is thereby complete. □

Remark 3.5 In addition to Lemma 3.3, with $R_0 \geq a + 1$ as in (3.11), there is then $R_1 \geq R_0$ such that, for every $x \in \overline{\Omega_{R_1,\infty}}$, one has $\Xi_x \subset \overline{\Omega_{R_0,\infty}}$ and $\nabla u \cdot e_r \geq \eta/2 > 0$ on Ξ_x. Pick any such x with $|x| \geq R_1$. Remembering that Ξ_x surrounds the origin, it then follows that, for every $\theta \in \mathbb{R}$, there is a unique $\varrho_x(\theta) \geq R_0$ such that $(\varrho_x(\theta)\cos\theta, \varrho_x(\theta)\sin\theta) \in \Xi_x$, and moreover

$$\Xi_x = \big\{(\varrho_x(\theta)\cos\theta, \varrho_x(\theta)\sin\theta) : \theta \in \mathbb{R}\big\}. \tag{3.12}$$

Notice also that the 2π-periodic function ϱ_x is of class $C^3(\mathbb{R})$ from the implicit function theorem.

Lemma 3.6 *There is a $C^1([0,+\infty))$ function $f : [0,+\infty) \to \mathbb{R}$ such that*

$$\Delta u + f(u) = 0 \quad \textit{in } \overline{\Omega_{a,\infty}}.$$

Proof. Remember first from the proof of Lemma 3.1 that the function $g = u \circ \sigma$ is of class $C^1([0,T))$ and satisfies $g'(t) = |\nabla u(\sigma(t))|^2 = |v(\sigma(t))|^2 > 0$ for all $t \in [0,T)$, due to (3.1) and (3.3). Furthermore, $g(0) = u(\sigma(0)) = u(A) = 0$ and $g(t) \to +\infty$ as $t \stackrel{<}{\to} T$. The function g is then a C^1 diffeomorphism from $[0,T)$ onto $[0,+\infty)$. Denote $g^{-1} : [0,+\infty) \to [0,T)$ its reciprocal. From the chain rule, the function f defined by

$$\begin{array}{rcl} f : [0,+\infty) & \to & \mathbb{R} \\ s & \mapsto & f(s) := -\Delta u(\sigma(g^{-1}(s))) \end{array} \tag{3.13}$$

is of class $C^1([0,+\infty))$, and $\Delta u(\sigma(t)) + f(u(\sigma(t))) = 0$ for all $t \in [0,T)$. In other words, $\Delta u + f(u) = 0$ along the curve $\sigma([0,T))$.

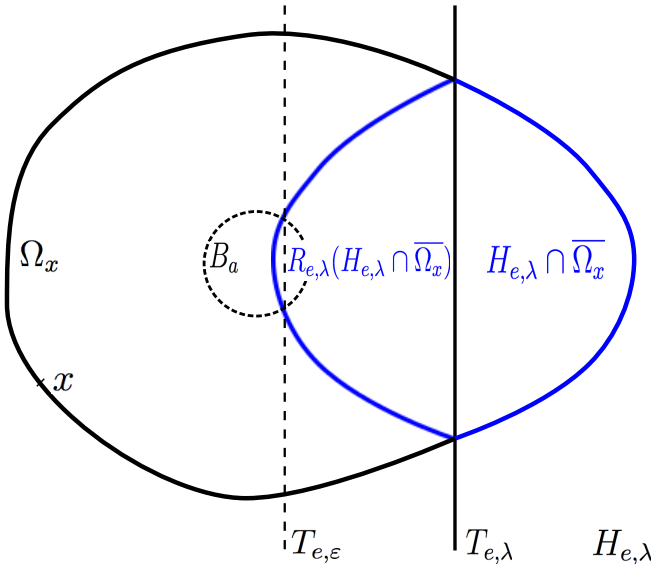

Figure 1: The sets Ω_x, $H_{e,\lambda}\cap\overline{\Omega_x}$ and $R_{e,\lambda}(H_{e,\lambda}\cap\overline{\Omega_x})$

Consider finally any point $x\in\Omega_{a,\infty}$. It follows from Lemmas 3.1 and 3.3 that there is a real number $s'\in[0,T)$ such that $\sigma(s')\in\Xi_x$. Since both u and Δu are constant along Ξ_x, one gets that $\Delta u(x)+f(u(x))=\Delta u(\sigma(s'))+f(u(\sigma(s')))=0$. The equation $\Delta u+f(u)=0$ then holds in $\overline{\Omega_{a,\infty}}$ since u is at least of class $C^2(\overline{\Omega_{a,\infty}})$ and $f\circ u$ is at least continuous in $\overline{\Omega_{a,\infty}}$. The proof of Lemma 3.6 is thereby complete. □

Let us now introduce a few notations which will be used in this section, as well as in the proof of Theorems 1.4, 1.5, 1.7 and 1.9 in the following sections. For $e\in\mathbb{S}^1=C_1$ and $\lambda\in\mathbb{R}$, we denote

$$T_{e,\lambda}=\big\{x\in\mathbb{R}^2:x\cdot e=\lambda\big\},\quad H_{e,\lambda}=\big\{x\in\mathbb{R}^2:x\cdot e>\lambda\big\},\tag{3.14}$$

and, for $x\in\mathbb{R}^2$,

$$R_{e,\lambda}(x)=x_{e,\lambda}=x-2(x\cdot e-\lambda)e.\tag{3.15}$$

In other words, $R_{e,\lambda}$ is the orthogonal reflection with respect to the line $T_{e,\lambda}$. For $x\in\Omega_{a,\infty}$, let Ω_x denote the bounded connected component of $\mathbb{R}^2\setminus\Xi_x$. Notice that Ω_x is well defined and contains the origin, by Lemma 3.4. Notice also that u is equal to the positive constant $u(x)$ along Ξ_x, while u vanishes along C_a and has no critical point in $\Omega_{a,\infty}$. Hence,

$$0<u(y)<u(x)\ \text{ for all } y\in\Omega_x\cap\Omega_{a,\infty},\tag{3.16}$$

where $\Omega_x\cap\Omega_{a,\infty}$ is the bounded domain located between Ξ_x and C_a. As a consequence, $\nabla u(z)$ points outwards Ω_x at each point $z\in\Xi_x$.

The following lemma says that, for any $\varepsilon>0$, the set $\Omega_x\cap H_{e,\lambda}$ will be an admissible set for the method of moving planes for any $e\in\mathbb{S}^1$ and $\lambda>\varepsilon>0$, provided $|x|$ is large enough.

Lemma 3.7 *For each $\varepsilon>0$, there exists $R_\varepsilon>a$ such that*

$$R_{e,\lambda}\big(H_{e,\lambda}\cap\overline{\Omega_x}\big)\subset\Omega_x$$

for all $e \in \mathbb{S}^1$, $\lambda > \varepsilon$ *and* $|x| \geq R_\varepsilon$ (*see Fig.* 1).

Proof. Fix $\varepsilon > 0$, and assume by way of contradiction that the conclusion of the lemma does not hold. Then there are some sequences $(x_n)_{n\in\mathbb{N}}$ in $\Omega_{a,\infty}$, $(e_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^1$, $(\lambda_n)_{n\in\mathbb{N}}$ in $(\varepsilon, +\infty)$ and $(y_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} |x_n| = +\infty, \quad \text{and} \quad y_n \in H_{e_n,\lambda_n} \cap \overline{\Omega_{x_n}} \text{ and } z_n := R_{e_n,\lambda_n}(y_n) \notin \Omega_{x_n} \text{ for all } n \in \mathbb{N}.$$

By Lemma 3.4, there is a sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 such that $B_{|x_n|-r_n} \subset \Omega_{x_n} \subset B_{|x_n|+r_n}$ for all $n \in \mathbb{N}$, hence $|y_n| \leq |x_n| + r_n$. On the other hand, since $y_n \cdot e_n > \lambda_n > \varepsilon > 0$, one has

$$|y_n|^2 - |z_n|^2 = |y_n|^2 - |R_{e_n,\lambda_n}(y_n)|^2 = 4\lambda_n(y_n \cdot e_n - \lambda_n) > 0,$$

hence $|y_n| > |z_n| \geq |x_n| - r_n$ since $z_n \notin \Omega_{x_n}$. As a consequence, $|x_n| - r_n \leq |z_n| < |y_n| \leq |x_n| + r_n$ for all $n \in \mathbb{N}$, and $\lim_{n\to+\infty}(|y_n| - |x_n|) = \lim_{n\to+\infty}(|y_n| - |z_n|) = 0$. The inequality $|y_n|^2 - |z_n|^2 = 4\lambda_n(y_n \cdot e_n - \lambda_n) > 4\varepsilon(y_n \cdot e_n - \lambda_n) > 0$ then yields $\lim_{n\to+\infty}(y_n \cdot e_n - \lambda_n) = 0$. Hence,

$$\mathrm{dist}(y_n, \Xi_{x_n} \cap T_{e_n,\lambda_n}) \to 0 \quad \text{and} \quad |y_n - z_n| \to 0 \quad \text{as } n \to +\infty.$$

For each $n \in \mathbb{N}$, let $\varphi_n \in \mathbb{R}$ be such that $e_n = (\cos\varphi_n, \sin\varphi_n)$. Since $y_n \cdot e_n > \lambda_n > \varepsilon > 0$, there is a unique $\theta_n \in (-\pi/2, \pi/2)$ such that

$$\frac{y_n}{|y_n|} = (\cos(\varphi_n + \theta_n), \sin(\varphi_n + \theta_n)).$$

Similarly, since $(z_n - y_n) \cdot e_n \to 0$ as $n \to +\infty$, one has $z_n \cdot e_n > \varepsilon/2$ for all large n and there is a unique $\theta'_n \in (-\pi/2, \pi/2)$ such that

$$\frac{z_n}{|z_n|} = (\cos(\varphi_n + \theta'_n), \sin(\varphi_n + \theta'_n)).$$

Since $\lim_{n\to+\infty} |y_n - z_n| = 0$ and $\lim_{n\to+\infty} |y_n| = \lim_{n\to+\infty} |z_n| = \lim_{n\to+\infty} |x_n| = +\infty$, one also infers that $\theta_n - \theta'_n \to 0$ as $n \to +\infty$. We also recall from (3.12) that

$$\Xi_{x_n} = \big\{(\varrho_{x_n}(\theta)\cos\theta, \varrho_{x_n}(\theta)\sin\theta) : \theta \in \mathbb{R}\big\}$$

for all n large enough. It then follows from Lemma 3.4 and from the assumptions on y_n and z_n that $|y_n| \leq \varrho_{x_n}(\varphi_n + \theta_n)$ and $|z_n| \geq \varrho_{x_n}(\varphi_n + \theta'_n)$ for all n large enough. Denote, for n large enough,

$$\begin{cases} y'_n = (\varrho_{x_n}(\varphi_n + \theta_n)\cos(\varphi_n + \theta_n), \varrho_{x_n}(\varphi_n + \theta_n)\sin(\varphi_n + \theta_n)) \in \Xi_{x_n}, \\ z'_n = (\varrho_{x_n}(\varphi_n + \theta'_n)\cos(\varphi_n + \theta'_n), \varrho_{x_n}(\varphi_n + \theta'_n)\sin(\varphi_n + \theta'_n)) \in \Xi_{x_n}, \end{cases}$$

and observe that $y_n \in (0, y'_n]$ and $z'_n \in (0, z_n]$.

We now claim that $\theta'_n \neq \theta_n$ for all n large enough. Indeed, otherwise, up to extraction of a subsequence, $y'_n = z'_n$ and the four points 0, y_n, $y'_n = z'_n$ and z_n would be aligned in that order. But since $y_n - z_n = 2(y_n \cdot e_n - \lambda_n)e_n$ with $y_n \cdot e_n - \lambda_n > 0$, the vectors y_n and z_n would

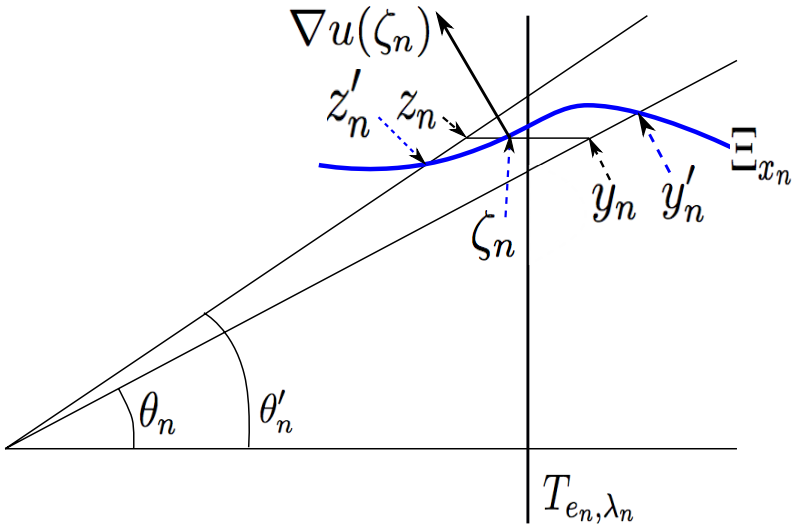

Figure 2: The points y_n, y'_n, z_n, z'_n, and ζ_n (with here $e_n = (1,0)$ and $\varphi_n = 0$)

be parallel to e_n. Hence, $y_n = (y_n \cdot e_n)e_n$ with $y_n \cdot e_n > \lambda_n > \varepsilon > 0$ and $z_n = (z_n \cdot e_n)e_n$ with $z_n \cdot e_n = 2\lambda_n - y_n \cdot e_n < \lambda_n < y_n \cdot e_n$. This contradicts the fact that 0, y_n and z_n lie on the half-line $\mathbb{R}_+ e_n$ in that order. Thus, $\theta'_n \neq \theta_n$ for all n large enough, thus for all n without loss of generality. Notice that the same arguments also imply that $\theta_n \neq 0$ and $\theta'_n \neq 0$ for all n large enough (since otherwise in either case one would have $\theta_n = \theta'_n = 0$ up to extraction of a subsequence), thus for all n without loss of generality. In particular, either $0 < \theta_n < \pi/2$ or $-\pi/2 < \theta_n < 0$.

Assume first that, up to extraction of a subsequence, $0 < \theta_n < \pi/2$ for all n. One then infers from the definition of $z_n = R_{e_n \lambda_n}(y_n)$ and the previous paragraph that

$$0 < \theta_n < \theta'_n < \frac{\pi}{2}.$$

Remember now that $0 < u(y) < u(x_n)$ for every y in the domain $\Omega_{x_n} \cap \Omega_{a,\infty}$ between Ξ_{x_n} and C_a, and $\nabla u(z)$ points outwards Ω_{x_n} at each point $z \in \Xi_{x_n}$. For each $n \in \mathbb{N}$, since $y_n \in (0, y'_n]$, $z'_n \in (0, z_n]$ and since $y_n - z_n = \varsigma_n e_n$ with $\varsigma_n := 2(y_n \cdot e_n - \lambda_n) > 0$, there is then an angle

$$\phi_n \in [\theta_n, \theta'_n] \subset \left(0, \frac{\pi}{2}\right)$$

such that

$$\zeta_n := (\varrho_{x_n}(\varphi_n + \phi_n)\cos(\varphi_n + \phi_n), \varrho_{x_n}(\varphi_n + \phi_n)\sin(\varphi_n + \phi_n)) \in \Xi_{x_n} \cap [y_n, z_n]$$

and $\nabla u(\zeta_n) \cdot e_n \leq 0$, see Fig. 2. The point ζ_n can be defined as the first point on $\Xi_{x_n} \cap [y_n, z_n]$ when going from y'_n to z'_n along Ξ_{x_n} with increasing angle. Notice that $|\zeta_n| \to +\infty$ since $|y_n| \to +\infty$ and $|y_n - z_n| \to 0$ as $n \to +\infty$. Call $v_{1,n} = v(\zeta_n) \cdot e_n$ and $v_{2,n} = v(\zeta_n) \cdot e_n^\perp$. The inequality $\nabla u(\zeta_n) \cdot e_n \leq 0$ means that $v_{2,n} \leq 0$. Therefore,

$$v(\zeta_n) \cdot e_r(\zeta_n) = v_{1,n}\cos\phi_n + v_{2,n}\sin\phi_n \leq v_{1,n}\cos\phi_n, \tag{3.17}$$

while

$$0 < \frac{\eta}{2} \le |v(\zeta_n) \cdot e_\theta(\zeta_n)| = |-v_{1,n} \sin\phi_n + v_{2,n} \cos\phi_n|$$

for all n large enough, from (3.10) and $|\zeta_n| \to +\infty$ as $n \to +\infty$. Furthermore, for each $x \in \Omega_{a,\infty}$, since $u(y) < u(x)$ for every $y \in \Omega_x \cap \Omega_{a,\infty}$, one has $\nabla u(z) \cdot e_r(z) \ge 0$ at a point $z \in \Xi_x$ such that $|z| = \max_{\mathbb{R}} |\xi_x|$. Since the continuous function $v \cdot e_\theta = \nabla u \cdot e_r$ has a constant strict sign at infinity, it follows that $v(\zeta) \cdot e_\theta(\zeta) = \nabla u(\zeta) \cdot e_r(\zeta) > 0$ for all $|\zeta|$ large enough. As a consequence,

$$\frac{\eta}{2} \le v(\zeta_n) \cdot e_\theta(\zeta_n) = -v_{1,n} \sin\phi_n + v_{2,n} \cos\phi_n$$

for all n large enough. Since $v_{2,n} \le 0$ and $0 < \phi_n < \pi/2$, one gets that $-v_{1,n} \sin\phi_n \ge \eta/2$, hence $v_{1,n} \le -\eta/2$. Together with (3.17), it follows that, for all n large enough,

$$v(\zeta_n) \cdot e_r(\zeta_n) \le -\frac{\eta}{2} \cos\phi_n.$$

On the other hand, since $\zeta_n \in [y_n, z_n]$ and $\lim_{n\to+\infty} |z_n - y_n| = \lim_{n\to+\infty} (y_n \cdot e_n - \lambda_n) = 0$, there holds $\zeta_n \cdot e_n - \lambda_n \to 0$, hence $\zeta_n \cdot e_n \ge \varepsilon/2$ for all n large enough (since $\lambda_n > \varepsilon > 0$ for all n). Finally,

$$\cos\phi_n = \frac{\zeta_n \cdot e_n}{|\zeta_n|} \ge \frac{\varepsilon}{2|\zeta_n|} \quad \text{and} \quad v(\zeta_n) \cdot e_r(\zeta_n) \le -\frac{\eta\,\varepsilon}{4|\zeta_n|}$$

for all n large enough. That last inequality contradicts the assumption (3.7) and the limit $\lim_{n\to+\infty} |\zeta_n| = +\infty$.

The second case, for which, up to extraction of a subsequence, $-\pi/2 < \theta_n < 0$ for all n (and then $-\pi/2 < \theta'_n < \theta_n < 0$) can be handled similarly and leads to a contradiction as well. The proof of Lemma 3.7 is thereby complete. □

Let us finally state the following important Lemma 3.8, that will be used in the proof of Theorems 1.3, 1.4, 1.5, 1.7 and 1.9.

Lemma 3.8 *Let Ξ and Ξ' be two C^1 Jordan curves surrounding the origin, and let Ω and Ω' be the bounded connected components of $\mathbb{R}^2 \setminus \Xi$ and $\mathbb{R}^2 \setminus \Xi'$, respectively. Assume that $\overline{\Omega'} \subset \Omega$ and let*

$$\omega = \Omega \setminus \overline{\Omega'}$$

be the non-empty and doubly connected domain located between Ξ and Ξ', with boundary

$$\partial\omega = \Xi \cup \Xi'.$$

Call $R' = \min_{x\in\Xi'} |x| > 0$ and $R = \max_{x\in\Xi} |x| > R'$. Let $e \in \mathbb{S}^1$, let $\overline{\lambda} = \max_{x\in\Xi} x \cdot e > 0$ and let $\varepsilon \in [0, \overline{\lambda})$. Let $c_1 < c_2 \in \mathbb{R}$ and let $\varphi \in C^2(\overline{\omega})$ be a solution of

$$\left\{ \begin{array}{rcl} \Delta\varphi + F(|x|, \varphi) & = & 0 \quad \textit{in } \overline{\omega}, \\ c_1 < \varphi & < & c_2 \quad \textit{in } \omega, \\ \varphi = c_1 \textit{ on } \Xi, \quad \varphi & = & c_2 \quad \textit{on } \Xi', \end{array} \right. \tag{3.18}$$

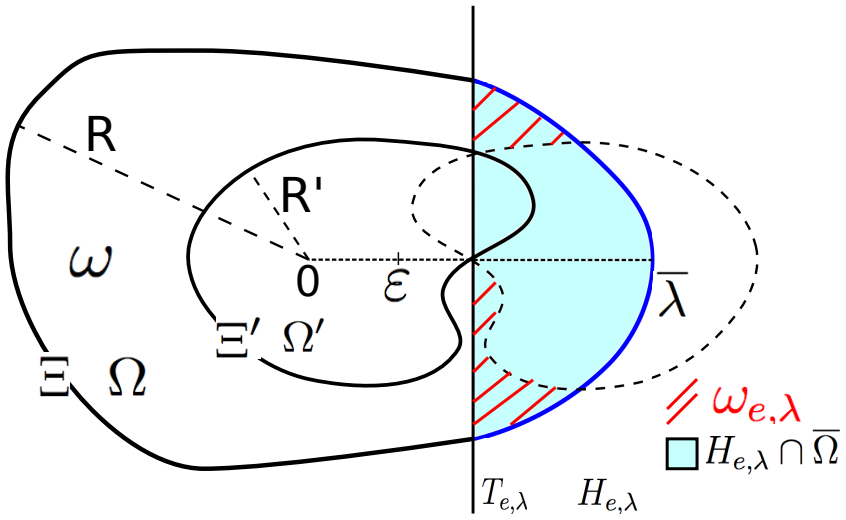

Figure 3: The sets Ω, Ω', $\omega = \Omega \setminus \overline{\Omega'}$, $H_{e,\lambda} \cap \overline{\Omega}$ (light blue background), $R_{e,\lambda}(\Omega')$ (with dashed boundary), and $\omega_{e,\lambda}$ (dashed red)

with a continuous function $F : [R', R] \times [c_1, c_2] \to \mathbb{R}$ *that is nonincreasing with respect to its first variable and uniformly Lipschitz continuous with respect to its second variable. Assume that*

$$R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega}) \subset \Omega \quad \textit{for all } \lambda > \varepsilon \tag{3.19}$$

and that

$$R_{e,\lambda}(H_{e,\lambda} \cap \Xi') \subset \Omega' \quad \textit{for all } \lambda > \varepsilon, \tag{3.20}$$

see Fig. 3. Then, for every $\lambda \in [\varepsilon, \overline{\lambda})$*, there holds*

$$\varphi(x) \leq \varphi_{e,\lambda}(x) = \varphi(x_{e,\lambda}) \quad \textit{for all } x \in \overline{\omega_{e,\lambda}}, \tag{3.21}$$

with

$$\omega_{e,\lambda} = (H_{e,\lambda} \cap \omega) \setminus R_{e,\lambda}(\overline{\Omega'}).^{6}$$

The proof of Lemma 3.8 is postponed in Section 3.4. Let us now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We shall show that the stream function u is radially symmetric in $\overline{\Omega_{a,\infty}}$. Notice that we already know that $u = 0$ on C_a. Let then $x \neq y \in \Omega_{a,\infty}$ be such that

$$|x| = |y| \, (> a).$$

Call

$$e = \frac{y - x}{|y - x|} \in \mathbb{S}^1. \tag{3.22}$$

[6]Notice that $\omega_{e,\lambda}$ is open by definition, and it is non-empty for each $\lambda \in [0, \overline{\lambda})$: indeed, for such λ, the set $T_{e,\lambda} \cap \Xi$ is not empty and, for any $x \in T_{e,\lambda} \cap \Xi$ and $r > 0$, $\omega_{e,\lambda} \cap B(x, r) \neq \emptyset$. But $\omega_{e,\lambda}$ may not be connected, as in Fig. 3.

Consider an arbitrary real number ε such that $0 < \varepsilon < a$. Let $R_\varepsilon > a$ be as in Lemma 3.7. From Lemma 3.4, there is a point $x_\varepsilon \in \Omega_{a,\infty}$ such that $|x_\varepsilon| \geq R_\varepsilon$ and $\min_{\mathbb{R}} |\xi_{x_\varepsilon}| > |x| = |y|$. Lemma 3.7 then yields

$$R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega_{x_\varepsilon}}) \subset \Omega_{x_\varepsilon} \ \text{ for all } \lambda > \varepsilon. \tag{3.23}$$

We are now going to apply Lemma 3.8 with

$$\begin{cases} \Xi = \Xi_{x_\varepsilon},\ \Omega = \Omega_{x_\varepsilon},\ \Xi' = C_a,\ \Omega' = B_a,\ \omega = \Omega_{x_\varepsilon} \setminus \overline{B_a}, \\ R' = a,\ R = \max\limits_{\mathbb{R}} |\xi_{x_\varepsilon}| > a,\ \overline{\lambda} = \max\limits_{z \in \Xi_{x_\varepsilon}} z \cdot e > a > \varepsilon > 0, \\ \varphi = -u \in C^3(\overline{\omega}),\ c_1 = -u(x_\varepsilon) = -u_{|\Xi_{x_\varepsilon}} < 0,\ c_2 = 0 = -u_{|C_a}, \\ F(r,s) = F(s) = -f(-s) \text{ for } (r,s) \in [a,R] \times [-u(x_\varepsilon), 0]. \end{cases}$$

Notice immediately that assumption (3.20) is automatically satisfied. The function F clearly satisfies the assumptions of Lemma 3.8 since f is of class $C^1([0,+\infty))$. The function φ satisfies $\Delta\varphi + F(\varphi) = 0$ in $\overline{\omega}$, with $c_1 < \varphi < c_2$ in ω (since $0 < u < u(x_\varepsilon)$ in $\Omega_{x_\varepsilon} \cap \Omega_{a,\infty}$ by (3.16)). Together with (3.23), all assumptions of Lemma 3.8 are satisfied.

Lemma 3.8 applied with $\lambda = \varepsilon$ then implies that $\varphi \leq \varphi_{e,\varepsilon}$, namely $u \geq u_{e,\varepsilon}$, in $\overline{\omega_{e,\varepsilon}}$ with

$$\omega_{e,\varepsilon} = (H_{e,\varepsilon} \cap \omega) \setminus R_{e,\varepsilon}(\overline{\Omega'}) = \big(H_{e,\varepsilon} \cap (\Omega_{x_\varepsilon} \setminus \overline{B_a})\big) \setminus R_{e,\varepsilon}(\overline{B_a}).$$

Observe now that $y \cdot e = (|y|^2 - x \cdot y)/|y - x| > 0$ since $|x| = |y|$ and $x \neq y$, and remember that $a < |y| < \min_{\mathbb{R}} |\xi_{x_\varepsilon}|$, hence $y \in \omega$. Therefore, $y \in \omega_{e,\varepsilon}$ for all $\varepsilon > 0$ small enough, and

$$u(y) \geq u_{e,\varepsilon}(y) = u(y_{e,\varepsilon}) = u(y - 2(y \cdot e - \varepsilon)e)$$

for all $\varepsilon > 0$ small enough. By passing to the limit $\varepsilon \stackrel{>}{\to} 0$ and using the definition of e and the assumption $|x| = |y|$, one infers that

$$u(y) \geq u(y - 2(y \cdot e)e) = u(x).$$

Since the last inequality holds for all $x \neq y \in \Omega_{a,\infty}$ such that $|x| = |y|$ (and also for all x, $y \in C_a$), the $C^3(\overline{\Omega_{a,\infty}})$ function u is radially symmetric in $\overline{\Omega_{a,\infty}}$. Together with (3.1), (3.5) and Lemma 3.3, there is then a $C^3([a,+\infty))$ function U such that $U(a) = 0$, $U' > 0$ in $[a,+\infty)$ and $u(x) = U(|x|)$ for all $x \in \overline{\Omega_{a,\infty}}$. This means that $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \overline{\Omega_{a,\infty}}$ with $V = U' \in C^2([a,+\infty))$ and $V > 0$ in $[a,+\infty)$. The proof of Theorem 1.3 is thereby complete. □

3.3 Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. Instead of (3.1)-(3.2), we assume the stronger condition

$$|v| = |\nabla u| \geq \eta > 0 \ \text{ in } \overline{\Omega_{a,\infty}} \tag{3.24}$$

for some $\eta > 0$. Properties (3.3)-(3.5) still hold (A can now be any point on C_a), as well as Lemma 3.1. Notice that the normalization condition (3.4) and the conditions (1.2) and (3.24) imply that

$$v \cdot e_\theta = \nabla u \cdot e_r \geq \eta > 0 \ \text{ on } C_a. \tag{3.25}$$

To prove Theorem 1.4, we then have to show that the supremum of the vorticity is positive, namely

$$\sup_{\Omega_{a,\infty}} \Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \Big) > 0. \tag{3.26}$$

We first show that all streamlines of the flow surround the origin and that the stream function u satisfies an equation of the type (1.6) in $\overline{\Omega_{a,\infty}}$, with $u > 0$ in $\Omega_{a,\infty}$. Then, we prove that if (3.26) does not hold, namely if we assume by way of contradiction that the vorticity $\Delta u = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$ of the flow is nonpositive in $\Omega_{a,\infty}$, then a Kelvin transform of the variables applied to the stream function u leads to a semilinear heterogeneous equation $\Delta w + |x|^{-4} f(w) = 0$ in the punctured disk $\Omega_{0,1/a}$ with a nonnegative function f. Lemma 3.8, using the good monotonicity of $|x|^{-4} f(w)$ with respect to $|x|$, implies that w is radially symmetric, hence u is radially symmetric and the assumption $\Delta u \leq 0$ contradicts (3.4) and (3.24), leading to the desired conclusion.

As a first step of this scheme, let us consider the solutions ξ_x of (1.5) with $x \in \overline{\Omega_{a,\infty}}$ and let us recall that Ξ_x denotes the streamline of the flow containing x.

Lemma 3.9 *Let $T \in (0, +\infty]$ be as in Lemma 3.1. Then, for every $s \in [0, T)$, the streamline $\Xi_{\sigma(s)}$ surrounds the origin.*

Proof. Denote

$$E = \big\{ s \in [0, T) : \text{the streamline } \Xi_{\sigma(s)} \text{ surrounds the origin} \big\}.$$

Our goal is to show that $E = [0, T)$. To do so, we prove that E is not empty (it contains 0), open relatively to $[0, T)$ and that the largest interval containing 0 and contained in E is actually equal to $[0, T)$.

Note first that, since $v \cdot e_r = 0$ and $v \cdot e_\theta \neq 0$ on C_a, the streamline $\Xi_{\sigma(0)} = \Xi_A$ is equal to the circle C_a and it surrounds the origin. In other words, $0 \in E$.

Let us now show that E is open relatively to $[0, T)$. Let $s_0 \in E$ and denote $x = \sigma(s_0) \in \overline{\Omega_{a,\infty}}$. By definition, the function ξ_x is periodic, with some period $T_x > 0$. Remember also that u is constant along each streamline of the flow. Therefore, as in Lemma 2.2, since v is (at least) continuous and $|v(x)| = |\nabla u(x)| > 0$, there are some real numbers $r > 0$ and $\tau \in (0, T_x)$ such that, for every $y \in B(x, r) \cap \overline{\Omega_{a,\infty}}$, there are some real numbers $t_y^\pm$ such that

$$-\tau < t_y^- < 0 < t_y^+ < \tau \quad \text{and} \quad B(x, r) \cap \Xi_y = B(x, r) \cap u^{-1}(\{u(y)\}) = \xi_y((t_y^-, t_y^+)).$$

On the other hand, since $\xi_x(T_x) = \xi_x(0) = x$, the Cauchy-Lipschitz theorem provides the existence of a real number $r' \in (0, r]$ such that, for every $z \in B(x, r') \cap \overline{\Omega_{a,\infty}}$, the function ξ_z is defined (and of class C^1) at least on the interval $[0, T_x]$ and $\xi_z(T_x) \in B(x, r) \cap \overline{\Omega_{a,\infty}}$. Furthermore, by continuity of σ, there is $\varepsilon > 0$ such that $s_0 + \varepsilon < T$ and

$$\sigma(s) \in B(\sigma(s_0), r') \cap \overline{\Omega_{a,\infty}} = B(x, r') \cap \overline{\Omega_{a,\infty}} \quad \text{for all } s \in [\max(0, s_0 - \varepsilon), s_0 + \varepsilon].$$

As a consequence, for every $s \in [\max(0, s_0 - \varepsilon), s_0 + \varepsilon]$, the points $z := \sigma(s) \in B(x, r') \cap \overline{\Omega_{a,\infty}}$ and $y := \xi_z(T_x) \in B(x, r) \cap \overline{\Omega_{a,\infty}}$ satisfy $u(z) = u(y)$, hence

$$z \in B(x, r') \cap u^{-1}(\{u(y)\})$$

and $z = \xi_y(t)$ for some $t \in (t_y^-, t_y^+)\,(\subset (-\tau,\tau))$. Thus, $\xi_y(-T_x) = z = \xi_y(t)$ and since $|t| < \tau < T_x$, the function ξ_y is defined in $\mathbb{R}$ and $(T_x + t)$-periodic. So is ξ_z since $z \in \Xi_y$. In other words, for every $s \in [\max(0, s_0 - \varepsilon), s_0 + \varepsilon]$, the function $\xi_{\sigma(s)} = \xi_z$ is defined in $\mathbb{R}$ and periodic. One then concludes as in the last paragraph of the proof of Lemma 2.2 that $\Xi_{\sigma(s)}$ surrounds the origin. Finally, the set E is open relatively to $[0, T)$.

Denote

$$T_* = \sup\big\{t \in [0,T) : [0,t] \subset E\big\}.$$

The previous paragraphs imply that $0 < T_* \leq T$. The proof of Lemma 3.9 will be complete once we show that $T_* = T$. Assume by way of contradiction that $T_* < T$ (in particular, T_* is then a positive real number).

Consider any increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $(0, T_*)$ and converging to T_*. Owing to the definition of T_*, each function $\xi_{\sigma(s_n)}$ is periodic and each streamline $\Xi_{\sigma(s_n)}$ surrounds the origin. Furthermore, since each s_n is positive and $u \circ \sigma$ is increasing in $[0, T)$ and u is constant on $C_a\,(\ni A = \sigma(0))$, each streamline $\Xi_{\sigma(s_n)}$ is included in the open set $\Omega_{a,\infty}$. Consider now any $n \in \mathbb{N}$ and any point $x \in \Xi_{\sigma(s_n)}$. Since ∇u is (at least) Lipschitz-continuous and $x \in \Omega_{a,\infty}$, the solution σ_x of (2.3) is defined and of class C^1 in at least a neighborhood of 0. Let

$$t_x^- = \inf\big\{t < 0 : \sigma_x([t,0]) \subset \Omega_{a,\infty}\big\} \quad \text{and} \quad t_x^+ = \sup\big\{t > 0 : \sigma_x([0,t]) \subset \Omega_{a,\infty}\big\}.$$

One has $-\infty \leq t_x^- < 0 < t_x^+ \leq +\infty$ and the function σ_x is of class $C^1((t_x^-, t_x^+))$. Notice that $u(x) = u(\sigma(s_n))$ since $x \in \Xi_{\sigma(s_n)}$, and that $u \circ \sigma$ is increasing on $[0, T)$, hence

$$0 = u(A) = u(\sigma(0)) < u(\sigma(s_n)) < u(\sigma(T_*)).$$

Therefore, $0 < u(x) < u(\sigma(T_*))$. Furthermore, $\dot\sigma_x(0) = \nabla u(\sigma_x(0)) = \nabla u(x)$ is orthogonal to $\Xi_{\sigma(s_n)}$ at x by definition of u. Since $u\circ\sigma_x$ is increasing on (t_x^-, t_x^+), since $u(\sigma_x(0)) = u(x) > 0$ with $u = 0$ on C_a, and since $\Xi_{\sigma(s_n)}$ surrounds the origin, it then follows as in the proof of Lemma 2.1 that $|\sigma_x(t)| \to a$ as $t \to t_x^-$ and $u(\sigma_x(t)) > 0$ for all $t \in (t_x^-, t_x^+)$. Then, for any $t \in (t_x^-, 0)$, there holds

$$\begin{aligned} u(\sigma(T_*)) > u(x) = u(\sigma_x(0)) > u(\sigma_x(0)) - u(\sigma_x(t)) = \int_t^0 |\nabla u(\sigma_x(s))|^2 ds &\geq \eta \int_t^0 |\dot\sigma_x(s)|\, ds \\ &\geq \eta\,\big(|x| - |\sigma_x(t)|\big). \end{aligned}$$

By passing to the limit as $t \to t_x^-$, one gets that $|x| \leq a + u(\sigma(T_*))/\eta$. This property holds for any $n \in \mathbb{N}$ and any $x \in \Xi_{\sigma(s_n)}$, hence

$$\sup_{n\in\mathbb{N}} \Big(\max_{\mathbb{R}} |\xi_{\sigma(s_n)}| \Big) \leq a + \frac{u(\sigma(T_*))}{\eta} =: M.$$

Lastly, consider the streamline $\Xi_{\sigma(T_*)}$ parametrized by $t \mapsto \xi_{\sigma(T_*)}(t)$. If there is a real number t such that $|\xi_{\sigma(T_*)}(t)| > M$, then $|\xi_{\sigma(s_n)}(t)| > M$ for all n large enough, by the Cauchy-Lipschitz theorem. Therefore, $\Xi_{\sigma(T_*)} \subset \overline{B_M}$ and, as in the proof of Lemma 2.2, it follows that $\xi_{\sigma(T_*)}$ is defined in $\mathbb{R}$ and periodic, and it surrounds the origin. In other words, $T_* \in E$.

Since E is open relatively to $[0, T)$ from the previous paragraph, one is led to a contradiction with the definition of T_* if $T_* < T$. Eventually, $T_* = T$ and the proof of Lemma 3.9 is

thereby complete. □

The next lemma gives the same conclusion as the previous lemma, for any streamline. It also implies that each level set of the stream function u has only one connected component.

Lemma 3.10 *There holds* $\min_{\mathbb{R}} |\xi_{\sigma(s)}| \to +\infty$ *as* $s \overset{<}{\to} T$ *and*

$$\overline{\Omega_{a,\infty}} = \bigcup_{s\in[0,T)} \Xi_{\sigma(s)}.$$

Furthermore, for every $x \in \overline{\Omega_{a,\infty}}$*, the solution* ξ_x *of* (1.5) *is defined in* $\mathbb{R}$ *and periodic, and* Ξ_x *surrounds the origin. Lastly,*

$$\min_{\mathbb{R}} |\xi_x| \to +\infty \quad \text{as } |x| \to +\infty.$$

Proof. Fix any $R > a$, and let $C \in [0, +\infty)$ be such that $|u| \le C$ in $\overline{\Omega_{a,R}}$. Since $u(\sigma(t)) \to +\infty$ as $t \overset{<}{\to} T$ by Lemma 3.1, there is $\tau \in (0,T)$ such that $u(\sigma(s)) > C$ for all $s \in (\tau, T)$, hence $u(\xi_{\sigma(s)}(t)) = u(\sigma(s)) > C$ and $|\xi_{\sigma(s)}(t)| > R$ for all $s \in (\tau, T)$ and $t \in \mathbb{R}$. Thus, $\min_{\mathbb{R}} |\xi_{\sigma(s)}| > R$ for all $s \in (\tau, T)$ (notice that the minimum is well defined by Lemma 3.9). This shows that $\min_{\mathbb{R}} |\xi_{\sigma(s)}| \to +\infty$ as $s \overset{<}{\to} T$.

Consider now any point $x \in \overline{\Omega_{a,\infty}}$, and let $s \in (0,T)$ be such that $\min_{\mathbb{R}} |\xi_{\sigma(s)}| > |x|$. Therefore, the point x belongs to the bounded open set surrounded by the Jordan curve $\Xi_{\sigma(s)}$. It follows as in the proof of Lemma 2.2 that ξ_x is defined in $\mathbb{R}$ and periodic, and that it surrounds the origin. Lemma 3.1 then implies that the streamline Ξ_x crosses $\sigma([0,T))$: there are $t \in \mathbb{R}$ and $s' \in [0,T)$ such that $\xi_x(t) = \sigma(s')$ (notice that such a s' is unique since $u \circ \sigma$ is increasing in $[0,T)$ and u is constant along Ξ_x). One then gets that $\Xi_x = \Xi_{\sigma(s')}$ and $x \in \bigcup_{s''\in[0,T)} \Xi_{\sigma(s'')}$. Finally,

$$\overline{\Omega_{a,\infty}} \subset \bigcup_{s''\in[0,T)} \Xi_{\sigma(s'')},$$

and both sets are equal since the other inclusion is obvious by definition.

It only remains to show that $\min_{\mathbb{R}} |\xi_x| \to +\infty$ as $|x| \to +\infty$. Fix again any $R \ge a$. From the first paragraph of the proof, there is $s \in [0,T)$ such that $\min_{\mathbb{R}} |\xi_{\sigma(s)}| > R$. Define $R' = \max_{\mathbb{R}} |\xi_{\sigma(s)}|$ (one has $R' > R$). For any x with $|x| > R'$, the streamlines Ξ_x and $\Xi_{\sigma(s)}$ do not intersect, and both of them surround the origin. Therefore, $\min_{\mathbb{R}} |\xi_x| > \min_{\mathbb{R}} |\xi_{\sigma(s)}|$, hence $\min_{\mathbb{R}} |\xi_x| > R$ for every $|x| > R'$. The proof of Lemma 3.10 is thereby complete. □

Lemma 3.11 *The function* u *satisfies* $u > 0$ *in* $\Omega_{a,\infty}$ *and* $u(x) \to +\infty$ *as* $|x| \to +\infty$.

Proof. Fix any $x \in \Omega_{a,\infty}$. By Lemmas 3.1 and 3.10, the streamline Ξ_x crosses $\sigma([0,T))$, at some point $\sigma(s)$ with $s \in [0,T)$, and, since u is constant along Ξ_x, one has $u(x) = u(\sigma(s))$. The real number s cannot be 0, since otherwise $\sigma(0) = A$ would belong to Ξ_x, that is, $\Xi_x = \Xi_A = C_a$, which is impossible since x is taken in the open set $\Omega_{a,\infty}$. Thus, $s > 0$ and, since $u \circ \sigma$ is increasing on $[0,T)$, one infers that $u(x) = u(\sigma(s)) > u(\sigma(0)) = u(A) = 0$ from the normalization (3.5).

Furthermore, as in the proof of Lemmas 2.1 and 3.9, the solution σ_x of (2.3) is defined in a maximal interval (t_x^-, t_x^+) with image in $\Omega_{a,\infty}$, with $-\infty \le t_x^- < 0 < t_x^+ \le +\infty$ and $|\sigma_x(t)| \to a$ as $t \stackrel{>}{\to} t_x^-$. For any $t \in (t_x^-, 0)$, there holds $u(\sigma_x(t)) > 0$ (by applying the result of previous paragraph at the point $\sigma_x(t) \in \Omega_{a,\infty}$). Therefore, for any $t \in (t_x^-, 0)$, one has

$$\begin{aligned} u(x) = u(\sigma_x(0)) > u(\sigma_x(0)) - u(\sigma_x(t)) = \int_t^0 |\nabla u(\sigma_x(s))|^2 ds &\ge \eta \int_t^0 |\dot\sigma_x(s)|\, ds \\ &\ge \eta \big(|x| - |\sigma_x(t)|\big), \end{aligned}$$

hence

$$u(x) \ge \eta\,(|x| - a)$$

by passing to the limit as $t \stackrel{>}{\to} t_x^-$. Since this last inequality holds for any $x \in \Omega_{a,\infty}$, one concludes that $u(x) \to +\infty$ as $|x| \to +\infty$, and the proof of Lemma 3.11 is complete. □

The last preliminary lemma provides the existence of a function f such that the elliptic equation (1.6) holds in $\overline{\Omega_{a,\infty}}$.

Lemma 3.12 *There is a C^1 function $f : [0, +\infty) \to \mathbb{R}$ such that*

$$\Delta u + f(u) = 0 \quad \text{in } \overline{\Omega_{a,\infty}}.$$

Proof. The $C^1([0,T))$ function $g := u \circ \sigma$ satisfies $g'(t) = |\nabla u(\sigma(t))|^2 \ge \eta^2$ for all $t \in [0,T)$ and it converges to $+\infty$ as $t \stackrel{<}{\to} T$ by Lemma 3.1. Furthermore, $g(0) = u(\sigma(0)) = u(A) = 0$. The function g is then a C^1 diffeomorphism from $[0,T)$ onto $[0,+\infty)$. As in the proof of Lemma 3.6, the function f defined by (3.13) is of class $C^1([0,+\infty))$, and $\Delta u(\sigma(t)) + f(u(\sigma(t))) = 0$ for all $t \in [0,T)$. Now, for any point $x \in \overline{\Omega_{a,\infty}}$, it follows from Lemma 3.10 that $\sigma(s') \in \Xi_x$ for some $s' \in [0,T)$. Since both u and Δu are constant along Ξ_x, one gets that $\Delta u(x) + f(u(x)) = \Delta u(\sigma(s')) + f(u(\sigma(s'))) = 0$. The proof of Lemma 3.12 is thereby complete. □

Proof of Theorem 1.4. Given the conditions (3.24)-(3.25), our goal is to show that the property (3.26) holds. To do so, let us assume by way of contradiction that

$$\Delta u = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \le 0 \ \text{ in } \overline{\Omega_{a,\infty}}. \tag{3.27}$$

We will show that u is radially symmetric, and this will easily lead to a contradiction. To prove the radially symmetry of u, let us use a Kelvin transform of the variables by setting

$$w(x) = u\Big(\frac{x}{|x|^2}\Big) \ \text{ for } x \in \overline{\Omega_{0,1/a}}\backslash\{0\},$$

and let us show that the $C^3(\overline{\Omega_{0,1/a}}\backslash\{0\})$ function w is radially symmetric in $\overline{\Omega_{0,1/a}}\backslash\{0\}$. Notice from (3.5) and Lemma 3.11 that

$$w = 0 \text{ on } C_{1/a}, \quad w > 0 \text{ in } \Omega_{0,1/a} \ \text{ and } \ w(x) \to +\infty \text{ as } x \stackrel{>}{\to} 0.$$

Furthermore, it follows from Lemma 3.12 and a straightforward calculation that

$$\Delta w(x) + \frac{1}{|x|^4} f(w(x)) = 0 \ \text{ for all } x \in \overline{\Omega_{0,1/a}} \setminus \{0\},$$

that is, $\Delta w(x) + F(|x|, w(x)) = 0$ in $\overline{\Omega_{0,1/a}} \setminus \{0\}$ with

$$\begin{array}{rcl} F : (0, 1/a] \times [0, +\infty) & \to & \mathbb{R} \\ (r, s) & \mapsto & F(r, s) = r^{-4} f(s). \end{array}$$

The function F is of class $C^1((0, 1/a] \times [0, +\infty))$. Furthermore, the range of u is equal to the whole interval $[0, +\infty)$ by (3.5) and Lemma 3.11, and $f \geq 0$ in $[0, +\infty)$ by (3.27) and Lemma 3.12. Therefore, the function F is nonincreasing with respect to its first variable in $(0, 1/a] \times [0, +\infty)$.

Consider now any two points $x \neq y \in \overline{\Omega_{0,1/a}} \setminus \{0\}$ with $|x| = |y|$. As in the proof of Theorem 1.3, denote

$$e = \frac{y - x}{|y - x|} \in \mathbb{S}^1$$

and consider an arbitrary real number ε such that

$$0 < \varepsilon < |x| = |y| \leq \frac{1}{a}.$$

By Lemma 3.10, there is a point $x_\varepsilon \in \Omega_{a,\infty}$ such that $\min_{\mathbb{R}} |\xi_{x_\varepsilon}| > 1/\varepsilon > a$. One knows that the streamline Ξ_{x_ε} surrounds the origin and that $u = u(x_\varepsilon) > 0$ along Ξ_{x_ε}. Furthermore, as in (3.16), one has $0 < u < u(x_\varepsilon)$ in the domain $\Omega_{x_\varepsilon} \cap \Omega_{a,\infty}$ between Ξ_{x_ε} and C_a, since $u = 0$ on C_a and u has no critical point in $\Omega_{a,\infty}$.

Denote $\Xi = C_{1/a}$ and

$$\Xi' = \left\{ x \in \mathbb{R}^2 : \frac{x}{|x|^2} \in \Xi_{x_\varepsilon} \right\}.$$

Notice that the Jordan curve Ξ' surrounds the origin and $\Xi' \subset B_\varepsilon \, (\subset B_{1/a})$ by definition of x_ε. Call $\Omega = B_{1/a}$, let Ω' be the bounded connected component of $\mathbb{R}^2 \setminus \Xi'$, and let

$$\omega = \Omega \setminus \overline{\Omega'} = B_{1/a} \setminus \overline{\Omega'} \ (\supset \Omega_{\varepsilon,1/a}).$$

Denote $R = 1/a$,

$$0 < R' = \min_{x \in \Xi'} |x| = \frac{1}{\max\limits_{\mathbb{R}} |\xi_{x_\varepsilon}|} < \varepsilon < R,$$

and $\overline{\lambda} = \max_{x \in \Xi} x \cdot e = 1/a > 0$. One has $0 < \varepsilon < 1/a$, hence $\varepsilon \in [0, \overline{\lambda})$. The function $\varphi = w$ is of class $C^3(\overline{\omega})$ with

$$\varphi = c_1 = 0 \text{ on } \Xi = C_{1/a}, \ \varphi = c_2 = u(x_\varepsilon) > 0 \text{ on } \Xi', \text{ and } 0 < \varphi < u(x_\varepsilon) \text{ in } \omega$$

(since $0 < u < u(x_\varepsilon)$ in $\Omega_{x_\varepsilon} \cap \Omega_{a,\infty}$). Furthermore, φ satisfies $\Delta\varphi + F(|x|, \varphi) = 0$ in $\overline{\omega}$, with $F(r, s) = r^{-4} f(s)$ and, here, $(r, s) \in [R', 1/a] \times [0, u(x_\varepsilon)]$. The function F then satisfies the

conditions of Lemma 3.8. Lastly, the condition (3.19) is immediately satisfied since $\Omega = B_{1/a}$ and the condition (3.20) also holds since $H_{e,\lambda} \cap \Xi' = \emptyset$ for all $\lambda > \varepsilon$ (because $\Xi' \subset B_\varepsilon$).

To sum up, all assumptions of Lemma 3.8 are fulfilled. Its conclusion with $\lambda = \varepsilon$ yields $w \le w_{e,\varepsilon}$ in $\overline{\omega_{e,\varepsilon}}$, with

$$\omega_{e,\varepsilon} = \big(H_{e,\varepsilon} \cap (B_{1/a} \setminus \overline{\Omega'})\big) \setminus R_{e,\varepsilon}(\overline{\Omega'}).$$

Since $y \cdot e > 0$ and since $\overline{\Omega'} \subset B_\varepsilon$ and $R_{e,\varepsilon}(\overline{\Omega'}) \subset B_{3\varepsilon}$, it follows that $y \in \omega_{e,\varepsilon}$ for all $\varepsilon > 0$ small enough. As a consequence, $w(y) \le w_{e,\varepsilon}(y) = w(y_{e,\varepsilon}) = w(y - 2(y \cdot e - \varepsilon)e)$ for all $\varepsilon > 0$ small enough and the passage to the limit as $\varepsilon \to 0$ yields

$$w(y) \le w(y - 2(y \cdot e)e) = w(x)$$

by definition of e. Since this holds for all $x \neq y \in \overline{\Omega_{0,1/a}} \setminus \{0\}$ with $|x| = |y|$, this means that w is radially symmetric in $\overline{\Omega_{0,1/a}} \setminus \{0\}$, hence u is radially symmetric in $\overline{\Omega_{a,\infty}}$. Together with (3.25), there is then a $C^3([a,+\infty))$ function U such that $u(x) = U(|x|)$ and $U' \ge \eta > 0$ in $[a,+\infty)$. But $\Delta u \le 0$ in $\overline{\Omega_{a,\infty}}$ by (3.27). Hence $U''(r) + r^{-1}U'(r) \le 0$ in $[a,+\infty)$ and the function $r \mapsto rU'(r)$ is nonincreasing in $[a,+\infty)$, a contradiction with $U' \ge \eta > 0$.

As a conclusion, (3.27) can not hold, that is, (3.26) holds and the proof of Theorem 1.4 is thereby complete. □

3.4 Proof of Lemma 3.8

It is based on the method of moving planes developed in [3, 6, 10, 22]. The idea is to compare the function φ to its reflection $\varphi_{e,\lambda}$ in $\overline{\omega_{e,\lambda}}$ by moving the lines $T_{e,\lambda}$ and decreasing λ from the value $\overline{\lambda}$ to the value ε. We recall that

$$\omega_{e,\lambda} = (H_{e,\lambda} \cap \omega) \setminus R_{e,\lambda}(\overline{\Omega'}).$$

Notice in particular that $R' \le |x| \le R$ for all $\lambda \in [\varepsilon, \overline{\lambda})$ and $x \in \overline{\omega_{e,\lambda}}$, since $\omega_{e,\lambda} \subset \omega = \Omega \setminus \overline{\Omega'}$.

Consider first any $\lambda \in (\varepsilon, \overline{\lambda})$. For each $x \in \omega_{e,\lambda}$, there holds

$$x_{e,\lambda} = R_{e,\lambda}(x) \in R_{e,\lambda}(H_{e,\lambda} \cap \omega) \subset R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega}) \subset \Omega$$

by (3.19), and $x_{e,\lambda} \notin \overline{\Omega'}$, hence, $x_{e,\lambda} \in \omega$. Thus

$$R_{e,\lambda}(\overline{\omega_{e,\lambda}}) \subset \overline{\omega}$$

and the function $\varphi_{e,\lambda}$ given in (3.21) is well defined and of class C^2 in $\overline{\omega_{e,\lambda}}$. Furthermore, $\Delta\varphi_{e,\lambda} + F(|x_{e,\lambda}|, \varphi_{e,\lambda}) = 0$ in $\overline{\omega_{e,\lambda}}$. Since $|x| \ge |x_{e,\lambda}|$ for all $x \in \overline{\omega_{e,\lambda}}$ (remember that $\lambda > \varepsilon \ge 0$) and since F is nonincreasing with respect to its first variable, it follows that

$$\Delta\varphi_{e,\lambda} + F(|x|, \varphi_{e,\lambda}) \le 0 \ \text{ in } \overline{\omega_{e,\lambda}}.$$

Let

$$\Phi_{e,\lambda} = \varphi_{e,\lambda} - \varphi,$$

which is well defined and of class C^2 in $\overline{\omega_{e,\lambda}}$. There holds

$$\Delta\Phi_{e,\lambda} + c_{e,\lambda}\Phi_{e,\lambda} \le 0 \ \text{ in } \overline{\omega_{e,\lambda}}, \tag{3.28}$$

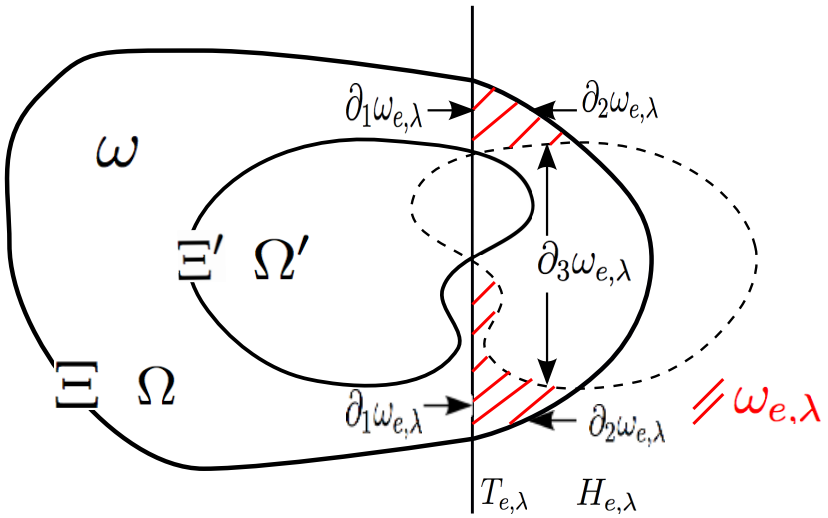

Figure 4: The three parts $\partial_1\omega_{e,\lambda}$, $\partial_2\omega_{e,\lambda}$ and $\partial_3\omega_{e,\lambda}$ of the boundary of the set $\omega_{e,\lambda}$ (dashed red)

where, say,

$$c_{e,\lambda}(x) = \begin{cases} \dfrac{F(|x|,\varphi_{e,\lambda}(x)) - F(|x|,\varphi(x))}{\varphi_{e,\lambda}(x) - \varphi(x)} & \text{if } \varphi_{e,\lambda}(x) \neq \varphi(x), \\ 0 & \text{if } \varphi_{e,\lambda}(x) = \varphi(x). \end{cases}$$

Since the function F is assumed to be Lipschitz continuous with respect to its second variable, uniformly with respect to the first one, the function $c_{e,\lambda}$ is in $L^\infty(\omega_{e,\lambda})$ and, moreover, there is a constant $M \geq 0$ such that

$$|c_{e,\lambda}(x)| \leq M \quad \text{for all } \lambda \in (\varepsilon, \overline{\lambda}) \text{ and for all } x \in \overline{\omega_{e,\lambda}}. \tag{3.29}$$

Consider again any $\lambda \in (\varepsilon, \overline{\lambda})$ and let us decompose the boundary of $\omega_{e,\lambda}$ into three parts. More precisely, since

$$\partial(A \cap B \cap C) \ \subset \ \big(\partial A \cap \overline{B} \cap \overline{C}\big) \cup \big(A \cap \partial B \cap \overline{C}\big) \cup \big(A \cap B \cap \partial C\big)$$

for any three sets A, B and C, since $\partial\omega = \Xi \cup \Xi'$ and since $(H_{e,\lambda} \cap \Xi') \setminus R_{e,\lambda}(\Omega') = \emptyset$ by assumption (3.20), one has (with $A = H_{e,\lambda}$, $B = \omega$ and $C = \mathbb{R}^2 \setminus R_{e,\lambda}(\overline{\Omega'})$)

$$\partial\omega_{e,\lambda} \subset \underbrace{\big((T_{e,\lambda} \cap \overline{\omega}) \setminus R_{e,\lambda}(\Omega')\big)}_{=:\partial_1\omega_{e,\lambda}} \cup \underbrace{\big((H_{e,\lambda} \cap \Xi) \setminus R_{e,\lambda}(\Omega')\big)}_{=:\partial_2\omega_{e,\lambda}} \cup \underbrace{\big(H_{e,\lambda} \cap \omega \cap R_{e,\lambda}(\Xi')\big)}_{=:\partial_3\omega_{e,\lambda}}, \tag{3.30}$$

see Fig. 4. Notice that, since $T_{e,\lambda} \cap \Xi = T_{e,\lambda} \cap \partial\Omega$ is not empty (because $\lambda \in (\varepsilon, \overline{\lambda}) \subset [0, \overline{\lambda})$), both sets $\partial_1\omega_{e,\lambda}$ and $\partial_2\omega_{e,\lambda}$ are not empty (however, $\partial_3\omega_{e,\lambda}$ may be empty). Furthermore, even if $\omega_{e,\lambda}$ may not be connected (as in Fig. 4), the boundary of each connected component of $\omega_{e,\lambda}$ intersects $\partial_2\omega_{e,\lambda} \cup \partial_3\omega_{e,\lambda}$.

Let us now study the sign of $\Phi_{e,\lambda}$ on $\partial\omega_{e,\lambda}$, for any $\lambda \in (\varepsilon, \overline{\lambda})$. Firstly, on $\partial_1\omega_{e,\lambda}$ $(\subset T_{e,\lambda})$, one has $\varphi_{e,\lambda} = \varphi$, hence $\Phi_{e,\lambda} = 0$. Secondly, for each $x \in \partial_2\omega_{e,\lambda}$, one has

$$x_{e,\lambda} \in R_{e,\lambda}(H_{e,\lambda} \cap \Xi) \subset R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega}) \subset \Omega$$

by (3.19), hence $x_{e,\lambda} \in \omega \cup \Xi'$ and $\varphi_{e,\lambda}(x) = \varphi(x_{e,\lambda}) > c_1$ by (3.18), while $x \in \Xi$ and $\varphi(x) = c_1$. Thus, $\Phi_{e,\lambda}(x) = \varphi_{e,\lambda}(x) - \varphi(x) > 0$ for each $x \in \partial_2\omega_{e,\lambda}$. Thirdly, for each $x \in \partial_3\omega_{e,\lambda}$, one has $x_{e,\lambda} \in \Xi'$ and $\varphi_{e,\lambda}(x) = \varphi(x_{e,\lambda}) = c_2$, while $x \in \omega$ and $\varphi(x) < c_2$, by (3.18). Thus, $\Phi_{e,\lambda}(x) = \varphi_{e,\lambda}(x) - \varphi(x) > 0$ for each $x \in \partial_3\omega_{e,\lambda}$. As a consequence, $\Phi_{e,\lambda} \geq 0$ on $\partial\omega_{e,\lambda}$ and even $\Phi_{e,\lambda} > 0$ on $\partial_2\omega_{e,\lambda} \cup \partial_3\omega_{e,\lambda}$ ($\neq \emptyset$), hence

$$\Phi_{e,\lambda} \geq\not\equiv 0 \text{ on the boundary of each connected component of } \omega_{e,\lambda}. \tag{3.31}$$

Let us now consider $\lambda \simeq \overline{\lambda}$ with $\lambda < \overline{\lambda}$. Since the functions $\Phi_{e,\lambda}$ satisfy (3.28)-(3.31), since the sets $\omega_{e,\lambda}$ are all included in the given bounded domain Ω and since the Lebesgue measure $|\omega_{e,\lambda}|$ of $\omega_{e,\lambda}$ goes to 0 as $\lambda \stackrel{<}{\to} \overline{\lambda}$ owing to the definition of $\overline{\lambda}$, it follows for instance from the maximum principle in sets with bounded diameter and small Lebesgue measure and from the strong maximum principle [7], that there is $\lambda_0 \in (\varepsilon, \overline{\lambda})$ such that $\Phi_{e,\lambda} > 0$ in $\omega_{e,\lambda}$ for all $\lambda \in (\lambda_0, \overline{\lambda})$.

Let us finally define

$$\lambda_* = \inf\big\{\lambda \in (\varepsilon, \overline{\lambda}) : \Phi_{e,\lambda'} > 0 \text{ in } \omega_{e,\lambda'} \text{ for all } \lambda' \in (\lambda, \overline{\lambda})\big\},$$

and notice that $\varepsilon \leq \lambda_* \leq \lambda_0 < \overline{\lambda}$. Our goal is to show that $\lambda_* = \varepsilon$. Assume by way of contradiction that $\lambda_* > \varepsilon$. Notice that $\Phi_{e,\lambda_*} \geq 0$ in $\overline{\omega_{e,\lambda_*}}$ by continuity (indeed, for each $x \in \omega_{e,\lambda_*}$, there holds $x \in \omega_{e,\lambda}$ for $\lambda - \lambda_* > 0$ small, hence $\varphi(x) < \varphi_{e,\lambda}(x)$ for $\lambda - \lambda_* > 0$ small, and $\varphi(x) \leq \varphi_{e,\lambda_*}(x)$ by passing to the limit $\lambda \stackrel{>}{\to} \lambda_*$ and by continuity of φ; therefore, $\varphi \leq \varphi_{e,\lambda_*}$ in $\overline{\omega_{e,\lambda_*}}$ again by continuity of φ). On the other hand, $\Phi_{e,\lambda_*} \geq\not\equiv 0$ on the boundary of each connected component of ω_{e,λ_*}, because $\lambda_* \in (\varepsilon, \overline{\lambda})$. Hence, $\Phi_{e,\lambda_*} > 0$ in ω_{e,λ_*} from the strong maximum principle. As in the previous paragraph, from [7], there exists $\delta > 0$ such that the weak maximum principle holds in any open set $\omega' \subset \omega$ for the solutions $\Phi \in C^2(\omega') \cap C(\overline{\omega'})$ of $\Delta\Phi + c\Phi \leq 0$ in ω' with $\Phi \geq 0$ on $\partial\omega'$ and $\|c\|_{L^\infty(\omega')} \leq M$, as soon as $|\omega'| \leq \delta$. Let then K be a compact subset of ω_{e,λ_*} such that

$$|\omega_{e,\lambda_*} \setminus K| < \frac{\delta}{2}.$$

Since $\min_K \Phi_{e,\lambda_*} > 0$, it follows from the continuity of φ in $\overline{\omega}$ that there exists $\underline{\lambda} \in (\varepsilon, \lambda_*)$ such that, for all $\lambda \in [\underline{\lambda}, \lambda_*]$,

$$\min_K \Phi_{e,\lambda} > 0, \quad \partial(\omega_{e,\lambda} \setminus K) = \partial\omega_{e,\lambda} \cup \partial K \quad \text{and} \quad |\omega_{e,\lambda} \setminus K| < \delta.$$

For any such $\lambda \in [\underline{\lambda}, \lambda_*]$, one then has $\Phi_{e,\lambda} \geq\not\equiv 0$ on the boundary of each connected component of $\omega_{e,\lambda} \setminus K$ and one then infers from the choice of δ and from the strong maximum principle that $\Phi_{e,\lambda} > 0$ in $\omega_{e,\lambda} \setminus K$, and finally $\Phi_{e,\lambda} > 0$ in $\omega_{e,\lambda}$. This last property contradicts the definition of λ_*.

As a conclusion, $\lambda_* = \varepsilon$. Therefore, for every $\lambda \in (\varepsilon, \overline{\lambda})$, one has $\Phi_{e,\lambda} > 0$ in $\omega_{e,\lambda}$, namely $\varphi < \varphi_{e,\lambda}$ in $\omega_{e,\lambda}$ and $\varphi \leq \varphi_{e,\lambda}$ in $\overline{\omega_{e,\lambda}}$ by continuity of φ. As in the previous paragraph, it also follows by continuity that $\varphi \leq \varphi_{e,\varepsilon}$ in $\overline{\omega_{e,\varepsilon}}$. The proof of Lemma 3.8 is thereby complete. □

4 The case of punctured disks $\Omega_{0,b}$: proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. Throughout this section, we fix a positive real number b and we consider a $C^2(\overline{\Omega_{0,b}}\setminus\{0\})$ flow v solving (1.1)-(1.2) and such that

$$\big\{x\in\overline{\Omega_{0,b}}\setminus\{0\}:|v(x)|=0\big\}\subsetneq C_b \ \text{ and } \ \int_{C_\varepsilon}|v\cdot e_r|\to 0 \text{ as } \varepsilon\stackrel{>}{\to}0. \tag{4.1}$$

Since v is divergence free, together with the second condition in (4.1), it follows that there is a $C^3(\overline{\Omega_{0,b}}\setminus\{0\})$ stream function u, namely $\nabla^\perp u=v$ in $\overline{\Omega_{0,b}}\setminus\{0\}$. Let $B\in C_b$ be a point such that

$$|v(B)|>0, \tag{4.2}$$

hence $v(B)\cdot e_\theta(B)\neq 0$ since $v\cdot e_r=0$ on C_b. Up to changing v into $-v$ and u into $-u$, one can assume without loss of generality that $v(B)\cdot e_\theta(B)<0$, that is,

$$\nabla u(B)\cdot e_r(B)<0. \tag{4.3}$$

Since $\nabla u\cdot e_\theta=-v\cdot e_r=0$ on C_b, the function u is constant on C_b and, since u is unique up to an additive constant, one can also assume without loss of generality that

$$u=0 \ \text{ on } C_b. \tag{4.4}$$

Following the general scheme of the proof of Theorem 1.3, we will show that the function u is positive in $\Omega_{0,b}$, that it has a limit at 0, that all streamlines of the flow in $\Omega_{0,b}$ surround the origin, and that u satisfies a semilinear elliptic equation of the type (1.6) in $\overline{\Omega_{0,b}}\setminus\{0\}$. Finally, we will apply Lemma 3.8 in suitable domains to prove the radial symmetry of u.

The first lemma, analogue to Lemma 3.1, is concerned with the trajectory of the gradient flow starting from the point B. We denote σ the solution of (2.3) with $x=B$, that is,

$$\left\{\begin{array}{rcl}\dot\sigma(t) &=& \nabla u(\sigma(t)),\\ \sigma(0) &=& B.\end{array}\right. \tag{4.5}$$

Lemma 4.1 *There is $T\in(0,+\infty]$ such that σ is defined and of class C^1 in $[0,T)$, and*

$$|\sigma(t)|\to 0 \ \textit{as}\ t\stackrel{<}{\to}T.$$

Proof. Since ∇u is (at least) of class $C^1(\overline{\Omega_{0,b}}\setminus\{0\})$ and $\nabla u(B)\cdot e_r(B)<0$ by (4.3), there is $t_*\in(0,+\infty)$ such that σ is defined and of class C^1 at least in $[0,t_*)$, and $\sigma(s)\in\Omega_{0,b}$ for all $0<s<t_*$. Define

$$T=\sup\big\{t>0:\sigma \text{ is defined and of class } C^1 \text{ in } [0,t) \text{ and } \sigma((0,t))\subset\Omega_{0,b}\big\}.$$

There holds $0<t_*\le T\le+\infty$ and the function σ is of class $C^1([0,T))$ with $\sigma((0,T))\subset\Omega_{0,b}$. Furthermore, $(u\circ\sigma)'(t)=|\nabla u(\sigma(t))|^2=|v(\sigma(t))|^2>0$ for all $t\in[0,T)$ by (4.1)-(4.2). Since $\sigma(0)=B\in C_b$ and u is continuous in $\overline{\Omega_{0,b}}\setminus\{0\}$ and constant on C_b, one infers that $\limsup_{t\to T,\,t<T}|\sigma(t)|<b$. By arguing by way of contradiction as in the proof of Lemma 3.1, it

follows that $|\sigma(t)| \to 0$ as $t \stackrel{<}{\to} T$. $\square$

Since $u \circ \sigma$ is increasing in $[0,T)$ and $u(\sigma(0)) = u(B) = 0$, there is $L \in (0,+\infty]$ such that

$$u(\sigma(t)) \to L \text{ as } t \stackrel{<}{\to} T. \tag{4.6}$$

Remark 4.2 The quantities T and L in Lemma 4.1 and in (4.6) may be finite or infinite. As a first example, consider the $C^\infty(\overline{\Omega_{0,b}}\setminus\{0\})$ flow v defined by

$$v(x) = -\frac{1}{|x|} e_\theta(x)$$

in $\overline{\Omega_{0,b}}\setminus\{0\}$. It solves (1.1)-(1.2), with $|v| > 0$ and $v \cdot e_r = 0$ in $\overline{\Omega_{0,b}}\setminus\{0\}$, hence (1.10) is fulfilled. The stream function u and the pressure p are given by $u(x) = \ln b - \ln|x|$ and $p(x) = -(2|x|)^{-1}$ (up to additive constants). In this case, for any $B \in C_b$, the solution σ of (4.5) is given by $\sigma(t) = \sqrt{1-2t/b^2}B$, with $T = b^2/2$ and $L = +\infty$. As a second example, consider the $C^\infty(\overline{\Omega_{0,b}}\setminus\{0\})$ flow v defined by

$$v(x) = -|x|\, e_\theta(x)$$

in $\overline{\Omega_{0,b}}\setminus\{0\}$. It solves (1.1)-(1.2), with $|v| > 0$ and $v \cdot e_r = 0$ in $\overline{\Omega_{0,b}}\setminus\{0\}$, hence (1.10) is fulfilled. The stream function u and the pressure p are given by $u(x) = b^2/2 - |x|^2/2$ and $p(x) = |x|^2/2$ (up to additive constants). In this case, for any $B \in C_b$, the solution σ of (4.5) is given by $\sigma(t) = e^{-t}B$, with $T = +\infty$ and $L = b^2/2$.

The next lemma shows that u has the limit L at the origin and that u is positive in $\Omega_{0,b}$.

Lemma 4.3 *Let $L \in (0,+\infty]$ be defined by (4.6). Then the function u satisfies $0 < u < L$ in $\Omega_{0,b}$ and $u(x) \to L$ as $|x| \stackrel{>}{\to} 0$.*

Proof. For every $r \in (0,b]$, the $C^3(\mathbb{R})$ function $w_r : \theta \mapsto w_r(\theta) = u(r\cos\theta, r\sin\theta)$ is 2π-periodic and, as in (3.8), one has $w_r'(\theta) = -r\, v(r\cos\theta, r\sin\theta) \cdot e_r(r\cos\theta, r\sin\theta)$ for all $\theta \in \mathbb{R}$. Hence, (4.1) implies that

$$\max_{C_r} u - \min_{C_r} u = \max_{[0,2\pi]} w_r - \min_{[0,2\pi]} w_r \le \int_{C_r} |v \cdot e_r| \to 0 \quad \text{as } r \stackrel{>}{\to} 0. \tag{4.7}$$

Furthermore, it follows from Lemma 4.1 that, for every $r \in (0,b]$, there is $s_r \in [0,T)$ such that $|\sigma(s_r)| = r$. Therefore, $s_r \to T$ as $r \stackrel{>}{\to} 0$ and $u(\sigma(s_r)) \to L$ by Lemma 4.1. Together with (4.7), one gets that $u(x) \to L$ as $|x| \stackrel{>}{\to} 0$.

Let now $r \in (0,b)$ be any small real number such that $\min_{C_r} u > 0$. Since $u = 0$ on C_b and u has no critical point in $\Omega_{0,b}$, one gets that $u > 0$ in $\Omega_{r,b}$. Since $r > 0$ can be as small as wanted, one concludes that $u > 0$ in $\Omega_{0,b}$.

Similarly, if $L \in (0,+\infty)$, then for any $\varepsilon > 0$, there is $r_\varepsilon \in (0,b)$ such that $\max_{C_r} u < L + \varepsilon$ for all $r \in (0, r_\varepsilon]$. Then $u < L + \varepsilon$ in $\Omega_{r,b}$ for any such r, hence $u < L + \varepsilon$ in $\Omega_{0,b}$. Finally, since $\varepsilon > 0$ is arbitrary, one has $u \le L$ in $\Omega_{0,b}$ and since u has no critical point in $\Omega_{0,b}$, one concludes that $u < L$ in $\Omega_{0,b}$. The proof of Lemma 4.3 is thereby complete. $\square$

Lemma 4.4 *For each $x \in \Omega_{0,b}$, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic, and the streamline $\Xi_x = \xi_x(\mathbb{R})$ surrounds the origin. Furthermore,*

$$\max_{\mathbb{R}} |\xi_x| \to 0 \quad as \ |x| \stackrel{>}{\to} 0.$$

Proof. Consider any $x \in \Omega_{0,b}$. Since $u = 0$ on C_b and u equal to the constant $u(x) \in (0, L)$ along Ξ_x, it follows from the continuity of u and the previous lemma that $0 < \inf_{y \in \Xi_x} |y| \leq \sup_{y \in \Xi_x} |y| < b$. Therefore, as in Lemma 2.2, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic, and the streamline $\Xi_x = \xi_x(\mathbb{R})$ surrounds the origin.

Consider now any $\varepsilon \in (0, b)$. It follows from Lemma 4.3 and the continuity of u that there are $L' \in (0, L)$ and $\rho \in (0, \varepsilon)$ such that $\max_{\overline{\Omega_{\varepsilon,b}}} u < L' < L$ and $u \geq L'$ in $\Omega_{0,\rho}$. Therefore, for any $x \in \Omega_{0,\rho}$, the function u is equal to the constant $u(x) \geq L'$ along the streamline Ξ_x, hence $\Xi_x \subset \Omega_{0,\varepsilon}$. The proof of Lemma 4.4 is thereby complete. □

Lemma 4.5 *Let $L \in (0, +\infty]$ be defined by (4.6). There is a $C^1([0, L))$ function $f : [0, L) \to \mathbb{R}$ such that*

$$\Delta u + f(u) = 0 \quad in \ \overline{\Omega_{0,b}} \backslash \{0\}.$$

Proof. The $C^1([0, T))$ function $g := u \circ \sigma$ satisfies $g'(t) = |\nabla u(\sigma(t))|^2 = |v(\sigma(t))|^2 > 0$ for all $t \in [0, T)$, due to (4.1)-(4.2). Furthermore, $g(0) = u(\sigma(0)) = u(B) = 0$. With (4.6), the function g is then a C^1 diffeomorphism from $[0, T)$ onto $[0, L)$. Denote $g^{-1} : [0, L) \to [0, T)$ its reciprocal. From the chain rule, the function f given by

$$\begin{array}{rcl} f : [0, L) & \to & \mathbb{R} \\ s & \mapsto & f(s) := -\Delta u(\sigma(g^{-1}(s))) \end{array} \tag{4.8}$$

is of class $C^1([0, L))$, and $\Delta u(\sigma(t)) + f(u(\sigma(t))) = 0$ for all $t \in [0, T)$. Consider finally any point $x \in \Omega_{0,b}$. It follows from Lemmas 4.1 and 4.4 that $\sigma(s') \in \Xi_x$ for some $s' \in [0, T)$. Since both u and Δu are constant along Ξ_x, one gets that $\Delta u(x) + f(u(x)) = \Delta u(\sigma(s')) + f(u(\sigma(s'))) = 0$. The equation $\Delta u + f(u) = 0$ actually holds in $\overline{\Omega_{0,b}} \backslash \{0\}$ since u is at least of class $C^2(\overline{\Omega_{0,b}} \backslash \{0\})$ and $f \circ u$ is at least continuous in $\overline{\Omega_{0,b}} \backslash \{0\}$. The proof of Lemma 4.5 is thereby complete. □

With the above lemmas in hand, we shall apply Lemma 3.8 to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. One has to show that u is radially symmetric in $\overline{\Omega_{0,b}} \backslash \{0\}$. Consider any two points $x \neq y \in \overline{\Omega_{0,b}} \backslash \{0\}$ with $|x| = |y|$. As in the proof of Theorems 1.3 and 1.4, denote

$$e = \frac{y - x}{|y - x|} \in \mathbb{S}^1$$

and consider an arbitrary real number ε such that

$$0 < \varepsilon < |x| = |y| \leq b.$$

By Lemma 4.4, there is a point $x_\varepsilon \in \Omega_{0,b}$ such that $\max_{\mathbb{R}} |\xi_{x_\varepsilon}| < \varepsilon < b$. One knows that the streamline Ξ_{x_ε} surrounds the origin and that $u = u(x_\varepsilon) > 0$ along Ξ_{x_ε}. Since $u = 0$ on C_b

and u has no critical point in $\Omega_{0,b}$, one infers that $0 < u < u(x_\varepsilon)$ in the domain ω between Ξ_{x_ε} and C_b. Denote $\Xi = C_b$, $\Xi' = \Xi_{x_\varepsilon}$, $\Omega = B_b$, let $\Omega' = \Omega_{x_\varepsilon}$ be the bounded connected component of $\mathbb{R}^2 \setminus \Xi_{x_\varepsilon}$, and notice that

$$\omega = \Omega \setminus \overline{\Omega'}.$$

Set

$$R = b, \quad 0 < R' = \min_{x \in \Xi'} |x| < \varepsilon < R \quad \text{and} \quad \overline{\lambda} = \max_{x \in \Xi} x \cdot e = b > 0.$$

One has $\varepsilon \in (0, \overline{\lambda})$. The function $\varphi = u$ is of class $C^3(\overline{\omega})$ with

$$\varphi = c_1 = 0 \text{ on } \Xi = C_b, \quad \varphi = c_2 = u(x_\varepsilon) > 0 \text{ on } \Xi', \quad \text{and} \quad 0 < \varphi < u(x_\varepsilon) \text{ in } \omega.$$

Furthermore, φ satisfies $\Delta\varphi + F(\varphi) = 0$ in $\overline{\omega}$, with $F : [R', b] \times [0, u(x_\varepsilon)] \ni (r, s) \mapsto F(r, s) = f(s)$ satisfying all conditions of Lemma 3.8. Lastly, the condition (3.19) is immediately satisfied since $\Omega = B_b$ and the condition (3.20) also holds since $H_{e,\lambda} \cap \Xi' = \emptyset$ for all $\lambda > \varepsilon$ (because $\Xi' \subset B_\varepsilon$).

To sum up, all assumptions of Lemma 3.8 are fulfilled. Its conclusion with $\lambda = \varepsilon$ yields $u \leq u_{e,\varepsilon}$ in $\overline{\omega_{e,\varepsilon}}$, with

$$\omega_{e,\varepsilon} = \big(H_{e,\varepsilon} \cap (B_b \setminus \overline{\Omega'})\big) \setminus R_{e,\varepsilon}(\overline{\Omega'}).$$

Since $y \cdot e > 0$ and since $\overline{\Omega'} \subset B_\varepsilon$ and $R_{e,\varepsilon}(\overline{\Omega'}) \subset B_{3\varepsilon}$, it follows that $y \in \omega_{e,\varepsilon}$ for all $\varepsilon > 0$ small enough. As a consequence, $u(y) \leq u(y_{e,\varepsilon}) = u(y - 2(y \cdot e - \varepsilon)e)$ for all $\varepsilon > 0$ small enough and the passage to the limit as $\varepsilon \stackrel{>}{\to} 0$ yields

$$u(y) \leq u(y - 2(y \cdot e)e) = u(x)$$

by definition of e. Since this holds for all $x \neq y \in \overline{\Omega_{0,b}} \setminus \{0\}$ with $|x| = |y|$, this means that u is radially symmetric in $\overline{\Omega_{0,b}} \setminus \{0\}$. Together with (4.1) and Lemma 4.3, there is then a $C^3((0, b])$ function U such that $u(x) = U(|x|)$ and $U' < 0$ in $(0, b]$. Hence, $v(x) = U'(|x|)e_\theta(x)$ for all $x \in \overline{\Omega_{0,b}} \setminus \{0\}$, and the proof of Theorem 1.5 is thereby complete. □

5 The case of the punctured plane $\Omega_{0,\infty} = \mathbb{R}^2 \setminus \{0\}$: proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. Let v be a $C^2(\Omega_{0,\infty})$ flow solving (1.1) and such that $|v| > 0$ in $\Omega_{0,\infty}$ and $\liminf_{|x| \to +\infty} |v(x)| > 0$. One assumes that (1.13) holds and that there is $X \in \Omega_{0,\infty}$ such that the streamline Ξ_X is a Jordan curve surrounding the origin. Let Ω_X be the bounded connected component of $\mathbb{R}^2 \setminus \Xi_X$.

Thanks to the second part of assumption (1.13), there is a $C^3(\Omega_{0,\infty})$ function u such that $\nabla^\perp u = v$ in $\Omega_{0,\infty}$. Up to normalization, one can assume that $u = 0$ on Ξ_X. Furthermore, since $|\nabla u(X)| = |v(X)| > 0$ and $\nabla u(X)$ is orthogonal to Ξ_X at X, one can assume without loss of generality, up to changing v into $-v$ and u into $-u$, that $\nabla u(X)$ points in the direction of Ω_X at X.

Let then σ be the solution of (2.3) with $x = X$. Since ∇u is at least of class $C^1(\Omega_{0,\infty})$, the function σ is defined in a neighborhood of 0 and there are $-\infty \leq T^- < 0 < T^+ \leq +\infty$

such that (T^-, T^+) is the maximal interval in which σ is of class C^1 and $\sigma((T^-, T^+)) \subset \Omega_{0,\infty}$. Furthermore, because of the normalization of the previous paragraph and since $u \circ \sigma$ is increasing in (T^-, T^+) and u is constant along Ξ_X, one has

$$\sigma(t) \in \Omega_X \text{ for all } t \in (0, T^+), \quad \text{and} \quad \sigma(t) \in \mathbb{R}^2 \setminus \overline{\Omega_X} \text{ for all } t \in (T^-, 0).$$

Using that $|v| > 0$ in $\Omega_{0,\infty}$ and $\liminf_{|x|\to+\infty} |v(x)| > 0$, it follows as in the proofs of Lemma 3.1 and 4.1 that $|\sigma(t)| \to 0$ and $u(\sigma(t)) \to L$ (for some $L \in (0, +\infty]$) as $t \stackrel{<}{\to} T^+$, while $|\sigma(t)| \to +\infty$ and $u(\sigma(t)) \to -\infty$ as $t \stackrel{>}{\to} T^-$. Secondly, as in the proofs of Lemma 3.3 and 4.3, using (1.13) and the fact that u has no critical point in $\Omega_{0,\infty}$, there holds $u(x) \to L$ as $|x| \stackrel{>}{\to} 0$ and $u(x) \to -\infty$ as $|x| \to +\infty$, together with

$$0 < u < L \text{ in } \Omega_X \setminus \{0\}, \quad \text{and} \quad u < 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega_X}.$$

Thirdly, as in the proofs of Lemma 3.4 and 4.4, it follows that, for the each $x \in \Omega_{0,\infty}$, the solution ξ_x of (1.5) is defined in $\mathbb{R}$ and periodic, and the streamline $\Xi_x = \xi_x(\mathbb{R})$ surrounds the origin. Furthermore, $\max_{\mathbb{R}} |\xi_x| - \min_{\mathbb{R}} |\xi_x| \to 0$ as $|x| \to +\infty$ and $\max_{\mathbb{R}} |\xi_x| \to 0$ as $|x| \stackrel{>}{\to} 0$. Lastly, as in the proofs of Lemma 3.6 and 4.5, the function $g := u \circ \sigma$ is a C^1 diffeomorphism from (T^-, T^+) onto $(-\infty, L)$ and the function f defined by

$$\begin{array}{rcl} f : (-\infty, L) & \to & \mathbb{R} \\ s & \mapsto & f(s) := -\Delta(\sigma(g^{-1}(s))) \end{array}$$

is of class $C^1((-\infty, L))$, with $\Delta u(\sigma(t)) + f(u(\sigma(t))) = 0$ for all $t \in (T^-, T^+)$ and finally

$$\Delta u + f(u) = 0 \text{ in } \Omega_{0,\infty}.$$

On the other hand, still using the notations (3.14)-(3.15), it follows as in Lemma 3.7 from the assumption $\liminf_{|x|\to+\infty} |v(x)| > 0$ and the first condition in (1.13) that, for every $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that

$$R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega_x}) \subset \Omega_x$$

for all $e \in \mathbb{S}^1$, $\lambda > \varepsilon$ and $|x| > R_\varepsilon$.

Lastly, consider two points $x \neq y \in \Omega_{0,\infty}$ such that $|x| = |y|$. Let $e \in \mathbb{S}^1$ be defined as in (3.22). Consider an arbitrary real number ε such that $0 < \varepsilon < |x| = |y|$. As in the proofs of Theorems 1.3 and 1.5, there are two points $x_\varepsilon \in \mathbb{R}^2 \setminus \overline{\Omega_X}$ and $x'_\varepsilon \in \Omega_X$ such that $\min_{\mathbb{R}} |\xi_{x_\varepsilon}| > |x| = |y| > \varepsilon$,

$$R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega_{x_\varepsilon}}) \subset \Omega_{x_\varepsilon} \text{ for all } \lambda > \varepsilon, \tag{5.1}$$

and $\max_{\mathbb{R}} |\xi_{x'_\varepsilon}| < \varepsilon < |x| = |y|$. The streamlines $\Xi = \Xi_{x_\varepsilon}$ and $\Xi' = \Xi_{x'_\varepsilon}$ surround the origin, and u is equal to $c_1 = u(x_\varepsilon) < 0$ along Ξ and to $c_2 = u(x'_\varepsilon) > 0$ along Ξ'. Furthermore, $u(x_\varepsilon) < u < u(x'_\varepsilon)$ in the domain

$$\omega = \Omega_{x_\varepsilon} \setminus \overline{\Omega_{x'_\varepsilon}}$$

located between Ξ_{x_ε} and $\Xi_{x'_\varepsilon}$. Denote $R' = \min_{z\in\Xi'} |z| = \min_{\mathbb{R}} |\xi_{x'_\varepsilon}| \in (0, \varepsilon)$, $R = \max_{z\in\Xi} |z| = \max_{\mathbb{R}} |\xi_{x_\varepsilon}| > |x| = |y| > \varepsilon > R'$, and $\overline{\lambda} = \max_{z\in\Xi} z \cdot e > \min_{\mathbb{R}} |\xi_{x_\varepsilon}| > |x| = |y| > \varepsilon > 0$.

The $C^3(\overline{\omega})$ function $\varphi = u$ satisfies (3.18) with $[R', R] \times [c_1, c_2] \ni (r, s) \mapsto F(r, s) = f(s)$ satisfying the assumptions of Lemma 3.8 since f is of class $C^1((-\infty, L))$. Together with (5.1) and the fact that $H_{e,\lambda} \cap \Xi' = \emptyset$ for all $\lambda > \varepsilon$ (since $\Xi' \subset B_\varepsilon$), the assumptions (3.19)-(3.20) are satisfied. All assumptions of Lemma 3.8 are therefore fulfilled.

Lemma 3.8 applied with $\lambda = \varepsilon$ then implies that $u \leq u_{e,\varepsilon}$ in $\overline{\omega_{e,\varepsilon}}$ with

$$\omega_{e,\varepsilon} = \big(H_{e,\varepsilon} \cap (\Omega_{x_\varepsilon} \setminus \overline{\Omega_{x'_\varepsilon}})\big) \setminus R_{e,\varepsilon}(\overline{\Omega_{x'_\varepsilon}}).$$

As in the proof of Theorem 1.5, one has $y \in \omega_{e,\varepsilon}$ for all $\varepsilon > 0$ small enough, hence

$$u(y) \leq u_{e,\varepsilon}(y) = u(y_{e,\varepsilon}) = u(y - 2(y \cdot e - \varepsilon)e)$$

for all $\varepsilon > 0$ small enough. By passing to the limit as $\varepsilon \stackrel{>}{\to} 0$ and using the definition of e and the assumption $|x| = |y|$, one infers that $u(y) \leq u(y - 2(y \cdot e)e) = u(x)$. Since the last inequality holds for any $x \neq y \in \Omega_{0,\infty}$ such that $|x| = |y|$, the $C^3(\Omega_{0,\infty})$ function u is radially symmetric in $\Omega_{0,\infty}$. Together with the fact that $|\nabla u| = |v| > 0$ in $\Omega_{0,\infty}$ and $\lim_{|x| \to +\infty} u(x) = -\infty$, there is then a $C^3((0, +\infty))$ function U such that $U' < 0$ in $(0, +\infty)$ and $u(x) = U(|x|)$ for all $x \in \Omega_{0,\infty}$. This means that $v(x) = V(|x|)\, e_\theta(x)$ for all $x \in \Omega_{0,\infty}$ with $V = U' \in C^2((0, +\infty))$ and $V < 0$ in $(0, +\infty)$. The proof of Theorem 1.7 is thereby complete. □

6 Proof of the Serrin-type Theorems 1.9 and 1.11

We start in Section 6.1 with the proof of Theorem 1.11 dealing with the case of doubly connected bounded domains, since the proof follows easily from the arguments used in the proof of Theorems 1.1 and 1.2 and on some known results of Reichel [18] and Sirakov [24] on elliptic overdetermined boundary value problems. Section 6.2 is then devoted to the proof of Theorem 1.9

6.1 Proof of Theorem 1.11

Let ω_1, ω_2, $\Omega = \omega_2 \setminus \overline{\omega_1}$ and v be as in Theorem 1.11. Since v is divergence free and $v \cdot n = 0$ on $\partial\omega_1$, the $C^3(\overline{\Omega})$ stream function u given by (1.4) is well defined and is unique up to additive constant. Furthermore, since $v \cdot n = 0$ on $\partial\Omega = \partial\omega_1 \cup \partial\omega_2$, there are two real numbers c_1 and c_2 such that

$$u = c_1 \text{ on } \partial\omega_1 \quad \text{and} \quad u = c_2 \text{ on } \partial\omega_2.$$

As in the proof of Lemma 2.1, one can show that, for each $x \in \Omega$, the solution σ_x of (2.3) is defined in an interval (t_x^-, t_x^+) such that $t_x^- < 0 < t_x^+$, $\sigma_x((t_x^-, t_x^+)) \subset \Omega$ and $\mathrm{dist}(\sigma_x(t), \partial\Omega) \to 0$ as $t \to t_x^\pm$. Since $u \circ \sigma_x$ is increasing in (t_x^-, t_x^+), it follows that $c_1 \neq c_2$ and $\min(c_1, c_2) < u < \max(c_1, c_2)$ in Ω. Up to changing v into $-v$ and u into $-u$, one can assume without loss of generality that $c_1 < c_2$, hence

$$c_1 < u < c_2 \text{ in } \Omega.$$

Since $|\nabla u| = |v| > 0$ in $\overline{\Omega}$, this normalization implies that $\frac{\partial u}{\partial n} < 0$ on $\partial\omega_1$ and $\frac{\partial u}{\partial n} > 0$ on $\partial\omega_2$, where n denotes the outward unit normal of Ω on $\partial\Omega$.

Let then X be any point in ω_1. As in the proof of Lemma 2.2, each streamline Ξ_x, with $x \in \Omega$, surrounds the point X. Notice that, here, since $v \cdot n = 0$ on $\partial\omega_1 \cup \partial\omega_2$ and v has no stagnation point in $\overline{\Omega}$, both Jordan curves $\partial\omega_1$ and $\partial\omega_2$ are streamlines of the flow. Moreover, for an arbitrarily fixed point $A \in \partial\omega_1$, the same arguments as in the proof of Lemma 2.4 imply that the trajectory σ_A of the gradient flow is defined in an interval $[0, t_A^+]$ with $t_A^+ \in (0, +\infty)$, $\sigma_A((0, t_A^+)) \subset \Omega$ and $\sigma_A(t_A^+) \in \partial\omega_2$. The function $g := u \circ \sigma_A$ is then a C^1 diffeomorphism from $[0, t_A^+]$ onto $[c_1, c_2]$ and the $C^1([c_1, c_2])$ function f defined by (2.9) is such that

$$\Delta u + f(u) = 0$$

along the curve $\sigma_A([0, t_A^+])$ and finally in the whole set $\overline{\Omega}$ since each streamline of the flow intersects the curve $\sigma_A([0, t_A^+])$.

Since $|\frac{\partial u}{\partial n}| = |\nabla u| = |v|$ along $\partial\Omega$ and since $|v|$ is constant along $\partial\omega_1$ and along $\partial\omega_2$, it follows that $\frac{\partial u}{\partial n}$ is constant too along $\partial\omega_1$ and along $\partial\omega_2$. One concludes from [18, 24] (see also [2, 25]) that, up to shift, $\Omega = \Omega_{a,b}$ for some $0 < a < b < \infty$ and u is radially symmetric and increasing with respect to $|x|$ in $\overline{\Omega} = \overline{\Omega_{a,b}}$. The assumptions and the conclusion of Theorem 1.1 are then satisfied and the proof of Theorem 1.11 is thereby complete. □

6.2 Proof of Theorem 1.9

Let Ω be a C^2 non-empty simply connected bounded domain of $\mathbb{R}^2$. Let $v \in C^2(\overline{\Omega})$ satisfy the Euler equations (1.1). We assume that $v \cdot n = 0$ and $|v|$ is constant on $\partial\Omega$, where n denotes the outward unit normal on $\partial\Omega$, and that v has a unique stagnation point in $\overline{\Omega}$. Since Ω is simply connected and v is divergence free, there is a $C^3(\overline{\Omega})$ stream function u satisfying (1.4). Furthermore, u is constant along $\partial\Omega$ since $v \cdot n = 0$ on $\partial\Omega$. Up to normalization, one can assume without loss of generality that

$$u = 0 \ \text{ on } \partial\Omega. \tag{6.1}$$

By assumption, the function u has a unique critical point in $\overline{\Omega}$, and $|\nabla u| = |v|$ is constant along $\partial\Omega$. Then $|\frac{\partial u}{\partial n}| = |\nabla u| = |v| > 0$ on $\partial\Omega$. Up to changing v into $-v$ and u into $-u$, one can assume without loss of generality that

$$\frac{\partial u}{\partial n} = \gamma < 0 \text{ on } \partial\Omega \tag{6.2}$$

for some negative constant γ. Hence, u has a unique maximum point in $\overline{\Omega}$ (which is actually in Ω) and this point is the unique critical point of u in $\overline{\Omega}$. Up to shift, one can assume without loss of generality that this critical point is the origin 0. One also infers from the uniqueness of the critical point of u that

$$0 < u < u(0) \text{ for all } x \in \Omega \setminus \{0\}. \tag{6.3}$$

Our goal is to show that Ω is then a ball centered at the origin and that u is radially symmetric and decreasing with respect to $|x|$ in $\overline{\Omega}$. To do so, we first follow some steps of the proof of Theorem 1.5.

So, let B be any point on $\partial\Omega$. Since $\nabla u(B)\cdot n(B)<0$ by (6.2) and since 0 is the unique critical point of u, it follows as in the proof of Lemma 4.1 that the solution σ of $\dot\sigma(t)=\nabla u(\sigma(t))$ with $\sigma(0)=B$ is defined in an interval $[0,T)$ with $T\in(0,+\infty]$, and $\sigma((0,T))\subset\Omega\setminus\{0\}$ together with $|\sigma(t)|\to 0$ as $t\stackrel{<}{\to}T$. Furthermore, since ∇u is at least Lipschitz continuous in $\overline{\Omega}$ and $|\nabla u(0)|=0$, one necessarily has $T=+\infty$. Using (6.3) and the fact that Ω is simply connected, it follows as in the proof of Lemma 4.4 that, for each $x\in\Omega\setminus\{0\}$, the streamline Ξ_x of the flow containing x surrounds the origin, and that $\max_{\mathbb{R}}|\xi_x|\to 0$ as $|x|\stackrel{>}{\to}0$. Notice that, since $v\cdot n=0$ and $|v|>0$ on $\partial\Omega$, the Jordan curve $\partial\Omega$ is also a streamline of the flow, surrounding the origin.

The function $g:=u\circ\sigma$ is of class $C^1([0,+\infty))$ with $g'>0$ in $[0,+\infty)$. It is a C^1 diffeomorphism from $[0,+\infty)$ onto $[0,L)$, with $L=u(0)$. Therefore, as in the proof of Lemma 4.5, the $C^1([0,L))$ function f defined as in (4.8) by $f(s)=-\Delta u(\sigma(g^{-1}(s)))$ in $[0,L)$ is such that $\Delta u+f(u)=0$ along the curve $\sigma([0,+\infty))$ and then in $\overline{\Omega}\setminus\{0\}$ (since each streamline of the flow in $\overline{\Omega}\setminus\{0\}$ intersects $\sigma([0,+\infty))$). By setting $f(L)=-\Delta u(0)$, the function f is then continuous in $[0,L]$ and the equation

$$\Delta u+f(u)=0$$

holds in the whole closed set $\overline{\Omega}$.

Remembering that u satisfies (6.1)-(6.3), it would then follow from [22] that $\Omega=B_R$ for some $R>0$ and u is radially symmetric and decreasing with respect to $|x|$ in $\overline{\Omega}$, if the function f were known to be Lipschitz continuous in $[0,L]$. However, by using the same ideas as in Remark 2.5, it is not clear that the function f' is bounded in a neighborhood of L and thus the function f may not be Lipschitz continuous in the whole interval $[0,L]$. We will however still be able to show the desired symmetry of Ω and of u by taking off from Ω small neighborhoods of 0 and applying Serrin's strategy and the method of moving planes in punctured domains. The images by u of the closure of these punctured domains are intervals of the type $[0,L']$, with $L'\in(0,L)$, and thus f is Lipschitz continuous in $[0,L']$.

More precisely, let first $\rho>0$ be such that

$$\overline{B_\rho}\subset\Omega$$

and let e be any unit vector. Let η be any real number in $(0,\rho)$, and denote

$$\overline{\lambda}_e=\max_{x\in\partial\Omega}x\cdot e>\rho>\eta.$$

Using the same notations $T_{e,\lambda}$, $H_{e,\lambda}$ and $R_{e,\lambda}$ as in (3.14)-(3.15), it follows from [4] that there is $\widetilde{\lambda}\in(\rho,\overline{\lambda}_e)$ such that

$$R_{e,\lambda}(H_{e,\lambda}\cap\overline{\Omega})\subset\Omega\ \text{ for all }\lambda\in(\widetilde{\lambda},\overline{\lambda}_e). \tag{6.4}$$

Since $\max_{\mathbb{R}}|\xi_x|\to 0$ as $|x|\stackrel{>}{\to}0$, there is $x_\eta\in\Omega\setminus\{0\}$ such that $\Xi_{x_\eta}\subset B_\eta$. Let then Ω' be the bounded connected component of $\mathbb{R}^2\setminus\Xi_{x_\eta}$ (notice that $\overline{\Omega'}\subset B_\eta\subset\Omega$) and let

$$\omega=\Omega\setminus\overline{\Omega'}$$

be the doubly connected bounded domain located between Ξ_{x_η} and $\partial\Omega$. Notice that $\partial\omega=\Xi_{x_\eta}\cup\partial\Omega$, that $0\notin\overline{\omega}$ and that

$$0<u<u(x_\eta)\ \text{ in }\omega \tag{6.5}$$

since u has no critical point in ω.

From (6.4), two cases can occur: either

- (case a)
$$R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega}) \subset \Omega \ \text{ for all } \lambda \in [\eta, \overline{\lambda}_e), \tag{6.6}$$
- or (case b) there is $\lambda^* \in [\eta, \widetilde{\lambda}]$ such that $R_{e,\lambda}(H_{e,\lambda} \cap \overline{\Omega}) \subset \Omega$ for all $\lambda \in (\lambda^*, \overline{\lambda}_e)$, and either
  - (internal tangency) there is a point $x^* \in H_{e,\lambda^*} \cap \partial\Omega$ such that $x^*_{e,\lambda^*} = R_{e,\lambda^*}(x^*) \in \partial\Omega$,
  - or (orthogonality) T_{e,λ^*} meets $\partial\Omega$ orthogonally, at some point p^*.

We will prove that only case a occurs.

Assume by way of contradiction that case b occurs. Denote

$$\begin{cases} \Xi = \partial\Omega,\ \Xi' = \Xi_{x_\eta},\ R' = \min\limits_{x \in \Xi'} |x| \in (0, \rho),\ R = \max\limits_{x \in \Xi} |x| > \rho > R',\ \varepsilon = \lambda^* \in [\eta, \widetilde{\lambda}] \subset [0, \overline{\lambda}_e), \\ c_1 = 0 = u_{|\partial\Omega},\ c_2 = u(x_\eta) = u_{|\Xi_{x_\eta}} \in (0, L). \end{cases}$$

The $C^3(\overline{\omega})$ function $\varphi = u$ satisfies $c_1 < \varphi < c_2$ in ω by (6.5) and $\Delta\varphi + F(\varphi) = 0$ in $\overline{\omega}$, where $F(s) = f(s)$ for $s \in [c_1, c_2] \subset [0, L)$. The function F is therefore C^1 in $[c_1, c_2]$. The condition (3.19) holds by definition of ε, λ^* and $\overline{\lambda}_e$, and the condition (3.20) is automatically fulfilled since $\Xi' \subset B_\eta$ and $\varepsilon = \lambda^* \geq \eta$. Therefore, all assumptions of Lemma 3.8 are satisfied and it follows from the conclusion applied with $\lambda = \varepsilon = \lambda^*$ that

$$u \leq u_{e,\lambda^*} \ \text{ in } \overline{\omega_{e,\lambda^*}},$$

with

$$\omega_{e,\lambda^*} = (H_{e,\lambda^*} \cap \omega) \setminus R_{e,\lambda^*}(\overline{\Omega'}).$$

Denote

$$w = u_{e,\lambda^*} - u.$$

Since $F = f$ is of class C^1 in $[c_1, c_2] \subset [0, L)$, the nonnegative $C^3(\overline{\omega_{e,\lambda^*}})$ function w satisfies an equation of the type $\Delta w + cw = 0$ in $\overline{\omega_{e,\lambda^*}}$, for some function $c \in C(\overline{\omega_{e,\lambda^*}})$. Thus, the strong maximum principle implies that, for each connected component ω' of ω_{e,λ^*}, either $w > 0$ in ω', or $w \equiv 0$ in $\overline{\omega'}$. We shall now consider separately the internal tangency case and the orthogonality case.

Consider first the case of internal tangency: there is a point $x^* \in H_{e,\lambda^*} \cap \partial\Omega$ such that $x^*_{e,\lambda^*} = R_{e,\lambda^*}(x^*) \in \partial\Omega$, hence

$$u(x^*) = u(x^*_{e,\lambda^*}) = 0 \ \text{ and } \ w(x^*) = 0.$$

Since $\overline{\Omega'} \cap \partial\Omega = \emptyset$, one has $x^* \notin \overline{\Omega'} \cup R_{e,\lambda^*}(\overline{\Omega'})$. There is a connected component ω^* of ω_{e,λ^*} such that $x^* \in \partial\omega^*$, and $B(x^*, r) \cap \Omega = B(x^*, r) \cap \omega^*$ for all $r > 0$ small enough (in particular, the interior sphere condition in ω^* is satisfied at the point $x^* \in \partial\omega^*$). Let $n(\zeta)$ be the generic notation for the outward normal to Ω at a point $\zeta \in \partial\Omega$. Owing to the definitions of λ^* and x^*,

one has $R_{e,\lambda^*}(n(x^*)) = n(x^*_{e,\lambda^*})$, while $\nabla u_{e,\lambda^*}(x^*) = R_{e,\lambda^*}(\nabla u(x^*_{e,\lambda^*}))$ owing to the definition of u_{e,λ^*}. Hence,

$$\begin{aligned}\nabla w(x^*)\cdot n(x^*) &= \nabla u_{e,\lambda^*}(x^*)\cdot n(x^*) - \nabla u(x^*)\cdot n(x^*)\\ &= R_{e,\lambda^*}(\nabla u(x^*_{e,\lambda^*}))\cdot R_{e,\lambda^*}(n(x^*_{e,\lambda^*})) - \nabla u(x^*)\cdot n(x^*)\\ &= \nabla u(x^*_{e,\lambda^*})\cdot n(x^*_{e,\lambda^*}) - \nabla u(x^*)\cdot n(x^*) = 0\end{aligned}$$

since $\nabla u\cdot n$ is equal to the constant γ on $\partial\Omega$ by (6.2). It then follows from Hopf lemma applied to the function w at the point x^*, together with the strong maximum principle, that

$$w \equiv 0 \text{ in } \overline{\omega^*}, \text{ that is, } u \equiv u_{e,\lambda^*} \text{ in } \overline{\omega^*}. \tag{6.7}$$

On the other hand, as for formula (3.30) in the proof of Lemma 3.8, one has

$$\partial\omega^* \subset \partial\omega_{e,\lambda^*} \subset \underbrace{\big((T_{e,\lambda^*}\cap\overline{\omega})\setminus R_{e,\lambda^*}(\Omega')\big)}_{=:\partial_1\omega_{e,\lambda^*}} \cup \underbrace{\big((H_{e,\lambda^*}\cap\partial\Omega)\setminus R_{e,\lambda^*}(\Omega')\big)}_{=:\partial_2\omega_{e,\lambda^*}} \cup \underbrace{\big(H_{e,\lambda^*}\cap\omega\cap R_{e,\lambda^*}(\Xi_{x_\eta})\big)}_{=:\partial_3\omega_{e,\lambda^*}}.$$

Since $u_{e,\lambda^*} = u(x_\eta)$ on $R_{e,\lambda^*}(\Xi_{x_\eta})$ and $u < u(x_\eta)$ in ω, one has $w = u_{e,\lambda^*} - u > 0$ on $\partial_3\omega_{e,\lambda^*}$, hence $\partial_3\omega_{e,\lambda^*}\cap\partial\omega^* = \emptyset$ and

$$\partial\omega^* \ \subset\ \big((T_{e,\lambda^*}\cap\overline{\omega})\setminus R_{e,\lambda^*}(\Omega')\big)\cup\big((H_{e,\lambda^*}\cap\partial\Omega)\setminus R_{e,\lambda^*}(\Omega')\big) \ \subset\ (T_{e,\lambda^*}\cap\overline{\Omega})\cup(H_{e,\lambda^*}\cap\partial\Omega).$$

Therefore, ω^* is a connected component of $H_{e,\lambda^*}\cap\Omega$. Since $w\equiv 0$ in $\overline{\omega^*}$, the arguments of Reichel [19] (see also [1, 24]) imply that $\overline{\Omega} = \overline{\omega^*\cup R_{e,\lambda^*}(\omega^*)}$. Hence Ω symmetric with respect to the line T_{e,λ^*} and, moreover, u is itself symmetric with respect to T_{e,λ^*}, which is impossible since $0\notin T_{e,\lambda^*}$ and 0 is the only maximum point of u. As a consequence, the case of internal tangency is ruled out.

Consider now the case of orthogonality, that is, T_{e,λ^*} meets $\partial\Omega$ orthogonally, at some point p^*. By definition of u_{e,λ^*}, one has $u(p^*) = u_{e,\lambda^*}(p^*)$, thus $w(p^*) = 0$. Notice also, as in the case of internal tangency, that $p^*\notin\overline{\Omega'}\cup R_{e,\lambda^*}(\overline{\Omega'})$. There is a connected component ω^* of ω_{e,λ^*} such that $p^*\in\partial\omega^*$, and $B(p^*,r)\cap\Omega\cap H_{e,\lambda^*} = B(p^*,r)\cap\omega^*$ for all $r>0$ small enough. Since u and $\frac{\partial u}{\partial n}$ are constant on $\partial\Omega$ and since T_{e,λ^*} meets $\partial\Omega$ orthogonally at p^*, it follows as in [19] that all first and second order derivatives of w vanish at p^*. Serrin's corner lemma [22] and the strong maximum principle then yield $w\equiv 0$ in $\overline{\omega^*}$. One is then led to a contradiction as in the previous paragraph.

As a consequence, only case a occurs. Thus, (6.6) holds. By arguing as in the beginning of the study of case b and applying Lemma 3.8 with this time $\varepsilon = \eta$, one infers that

$$u \le u_{e,\lambda} \ \text{ in } \overline{\omega_{e,\lambda}} \ \text{ for all } \lambda\in[\eta,\overline{\lambda}_e), \tag{6.8}$$

with $\omega_{e,\lambda} = (H_{e,\lambda}\cap\omega)\setminus R_{e,\lambda}(\overline{\Omega'})$. Since (6.6) and (6.8) hold for every direction $e\in\mathbb{S}^1$ and for every $\eta\in(0,\rho)$, one finally concludes that

$$\Omega = B_R$$

for some $R>0$ and, as in the proof of Theorem 1.5, that u is radially symmetric in $\overline{\Omega} = \overline{B_R}$. Since 0 is the unique critical point of the $C^3(\overline{B_R})$ function u and since $u=0$ on ∂B_R with $u>0$

in B_R, there is then a $C^3([0,R])$ function $U : [0,R] \to \mathbb{R}$ such that $u(x) = U(|x|)$ in $\overline{B_R}$, with $U'(0) = 0$ and $U' < 0$ in $(0,R]$. Therefore,

$$v(x) = \nabla^\perp u(x) = U'(|x|)e_\theta(x) \text{ for all } x \in \overline{B_R}\backslash\{0\}$$

and the $C^2([0,R])$ function $V = U'$ satisfies the desired conclusion. The proof of Theorem 1.9 is thereby complete. □

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