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ASYMPTOTIC DOMINATION OF SAMPLE MAXIMA

ENKELEJD HASHORVA AND DIDIER RULLIÈRE

Abstract: For a given random sample from some underlying multivariate distribution F we consider the domination of the component-wise maxima by some independent random vector \mathbf{W} with underlying distribution function G . We show that the probability that certain components of the sample maxima are dominated by the corresponding components of \mathbf{W} can be approximated under the assumptions that both F and G are in the max-domain of attraction of some max-stable distribution function \mathcal{F} and \mathcal{G} , respectively. We study further some basic properties of the dominated components of sample maxima by \mathbf{W} .

Key Words: Max-stable distributions; domination of sample maxima; extremal dependence; inf-argmax formula; de Haan representation; records.

AMS Classification: Primary 60G15; secondary 60G70

1. INTRODUCTION

Let $\mathbf{Z}_i, i \leq n$ be independent d -dimensional random vectors with common continuous distribution function (df) F and denote by \mathbf{M}_n their component-wise maxima, i.e., $M_{nj} = \max_{1 \leq k \leq n} Z_{kj}, j \leq d$. If \mathbf{W} is another d -dimensional random vector with continuous df G being further independent of \mathbf{M}_n the approximation of the probability that at least one component of \mathbf{W} dominates the corresponding component of \mathbf{M}_n is of interest since it is related to the dependence of the components of \mathbf{M}_n , see e.g., [1]. In the special case that \mathbf{W} has a max-stable df with unit Fréchet marginal df's $\Phi(x) = e^{-1/x}, x > 0$ and \mathbf{M}_n has almost surely positive components, then we simply have

$$\mathbb{P}\{\exists i \leq d : W_i > M_{ni}\} = 1 - \mathbb{P}\{\forall i, 1 \leq i \leq d : M_{ni} \geq W_i\} = 1 - \mathbb{E}_{\mathbf{M}_n} \left\{ \exp \left(-\mathbb{E}_{\mathbf{W}} \left\{ \max_{1 \leq i \leq d} \frac{W_i}{M_{ni}} \right\} \right) \right\},$$

where $\mathbf{W} = (W_1, \dots, W_d)$ being independent of \mathbf{M}_n is a spectral random vector of G which exists in view of the well-known de Haan representation, see e.g., [2] and (2.1) below. Note that the assumption that W_i has unit Fréchet df implies that $\mathbb{E}\{W_i\} = 1$.

The above probability is referred to as the marginal domination probability of the sample maxima. If F is also a max-stable df with unit Fréchet marginals, then by definition \mathbf{M}_n/n has for any $n > 0$ df F and since consequently

$$(1.1) \quad n[1 - \mathbb{P}\{\forall i, 1 \leq i \leq d : M_{ni} \geq W_i\}] = n \left[1 - \mathbb{E}_{\mathbf{Z}} \left\{ \exp \left(-\frac{1}{n} \mathbb{E}_{\mathbf{W}} \left\{ \max_{1 \leq i \leq d} \frac{W_i}{Z_i} \right\} \right) \right\} \right] \sim \mathbb{E} \left\{ \max_{1 \leq i \leq d} \frac{W_i}{Z_i} \right\},$$

where \sim means asymptotic equivalence as $n \rightarrow \infty$ and $\mathbf{Z} = (Z_1, \dots, Z_d)$ has df F being further independent of \mathbf{W} . Under the above assumptions, we have (set below $F_n = F^n$)

$$(1.2) \quad p_{n,T}(F_n, G) = \mathbb{P}\{\forall i, 1 \leq i \leq d : W_i > M_{ni}\} \sim \frac{1}{n} \mathbb{E} \left\{ \min_{1 \leq i \leq d} \frac{W_i}{Z_i} \right\}, \quad T = \{1, \dots, d\}$$

as $n \rightarrow \infty$, which follows by (1.1) and the inclusion-exclusion formula or directly by [1][Thm 2.5 and Prop 4.2].

Here $p_{n,T}(F_n, G)$ is referred to as the probability of the complete domination of sample maxima by \mathbf{W} . In the particular case that $F = G$ it is related to the probability of observing a multiple maxima, see [3–8].

Between these to extreme cases of interest is also to consider the partial domination of the sample maxima. Let therefore let below $T \subset \{1, \dots, d\}$ be non-empty and consider the probability that only the components of \mathbf{W} with indices in T dominate \mathbf{M}_n , i.e.,

$$\mathbb{P}\{\forall i \in T : W_i > M_{ni}, \forall i \in \bar{T} : W_i \leq M_{ni}\} =: p_{n,T}(F_n, G),$$

where $\bar{T} = \{1, \dots, d\} \setminus T$. Note that $p_{n,T}(F_n, F)$ relates to the probability of observing a T -record, see [9]. By the continuity of F and G we simply have

$$p_{n,T}(F_n, G) = \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in T : W_i > y_i, \forall i \in \bar{T} : W_i \leq y_i\} dF_n(\mathbf{y}),$$

which cannot be evaluated without knowledge of both F and G . In the particular case that F and G are max-stable df's as above, using (1.1) and the inclusion-exclusion formula we have

$$(1.3) \quad \lim_{n \rightarrow +\infty} np_{n,T}(F_n, G) = \mathbb{E} \left\{ \left(\min_{i \in T} \frac{W_i}{Z_i} - \max_{i \in \bar{T}} \frac{W_i}{Z_i} \right)_+ \right\}.$$

In case $H = Q$ the above result is known from [9][Prop 2.2]. Moreover, in the special case that T consist of one element, then the right-hand side of (1.3) is equal to $\mathbb{P}\{C(T) \subset \bar{T}\}$, where $C(T)$ is the tessellation as determined in [10]. If we are not interested on a particular index set T where the domination of sample maxima by \mathbf{W} occurs but simply on the number of components being dominated, i.e., on the random variable (rv)

$$N_n = \sum_{i=1}^d \mathbb{1}_{\{W_i > M_{ni}\}}$$

again a question of interest is if N_n can be approximated as $n \rightarrow \infty$. We have that N_n has the same distribution as

$$\sum_{i=1}^d \mathbb{1}_{\{W_i/n > Z_i\}},$$

provided that F is max-stable as above and \mathbf{Z} has df F being further independent of \mathbf{W} . Hence if W_i 's are unit Fréchet rv's, then

$$\lim_{n \rightarrow +\infty} n \mathbb{E}\{N_n\} = \sum_{i=1}^d \lim_{n \rightarrow +\infty} n \mathbb{P}\{W_i > n Z_i\} = \sum_{i=1}^d \lim_{n \rightarrow +\infty} n \left[1 - e^{-\mathbb{E}\left\{\frac{1}{n Z_i}\right\}} \right] = d.$$

Consequently, the expected number of components of sample maxima being dominated by the components of \mathbf{W} decreases as d/n when n goes to infinity. Moreover, the dependence of both \mathbf{W} and \mathbf{M}_n does not play any role. This is however not the case for the expectation of $f(N_n)$ where f is some real-valued function, since the dependence of both \mathbf{M}_n and \mathbf{W} influence the approximation as we shall show in the next section.

From our discussion above the assumptions that F and G are max-stable df's with unit Fréchet marginals lead to tractable asymptotic formulas for the approximation of various quantities related to the domination of sample

maxima M_n by \mathbf{W} .

In view of [1] we know that both (1.1) and (1.2) are valid in the more general setup that both F and G are in the max-domain of attraction of some max-stable df's (see next section for details). We shall show in this paper that the same assumptions lead to tractable approximations of both $p_{n,T}(F_n, G)$ and $\mathbb{E}\{f(N_n)\}$ as $n \rightarrow \infty$.

Brief organisation of the paper: Section 2 gives the main results concerning the approximations of the marginal domination probabilities and the expectation of $f(N_n)$. Section 3 is dedicated to properties of \mathbf{W}/\mathbf{Z} which we call the domination spectral vector, whereas in Section 4 we give some combinatorial results. All the proofs are relegated to Section 5.

2. MAIN RESULTS

We shall recall first some basic properties of max-stable df's, see [2, 11–13] for details. A d -dimensional df \mathcal{G} is max-stable with unit Fréchet marginals if

$$\mathcal{G}^t(tx_1, \dots, tx_d) = \mathcal{G}(x_1, \dots, x_d)$$

for any $t > 0, x_i \in (0, \infty), 1 \leq i \leq d$. In the light of the well-known De Haan representation

$$(2.1) \quad \mathcal{G}(\mathbf{x}) = \exp\left(-\mathbb{E}\left\{\max_{1 \leq j \leq d} \mathcal{W}_j/x_j\right\}\right), \quad \mathbf{x} = (x_1, \dots, x_d) \in (0, \infty)^d,$$

where \mathcal{W}_j 's are non-negative rv's with $\mathbb{E}\{\mathcal{W}_j\} = 1, j \leq d$ and $\mathbf{W} = (\mathcal{W}_1, \dots, \mathcal{W}_d)$ is a spectral vector for \mathcal{G} (which is not unique).

In view of multivariate extreme value theory, see e.g., [13] d -dimensional max-stable df's \mathcal{F} are limiting df's of the component-wise maxima of d -dimensional iid random vectors with some df F . In that case, F is said to be in the max-domain of attraction (MDA) of \mathcal{F} , abbreviated $F \in MDA(\mathcal{F})$. For simplicity we shall assume throughout in the following that F has marginal df's F_i 's such that

$$(2.2) \quad \lim_{n \rightarrow +\infty} F_i^n(nx) = \Phi(x), \quad x \in \mathbb{R}$$

for all $i \leq d$, where we set $\Phi(x) = 0$ if $x \leq 0$. We have thus that $F \in MDA(\mathcal{F})$ if further

$$(2.3) \quad \lim_{n \rightarrow +\infty} \sup_{x_i \in \mathbb{R}, 1 \leq i \leq d} \left| F^n(nx_1, \dots, nx_d) - \mathcal{F}(x_1, \dots, x_d) \right| = 0.$$

In the following \mathcal{F} is a d -dimensional max-stable df of some random vector \mathbf{Z} with unit Fréchet marginals and \mathcal{G} is another max-stable df with unit Fréchet marginals and spectral random vector \mathbf{W} independent of \mathbf{Z} . Further both F and G are as in the Introduction. Since our limiting result depends only on the ratio of \mathbf{W} and \mathbf{Z} we shall set below

$$\mathcal{V}_i = \frac{\mathcal{W}_i}{\mathcal{Z}_i}, \quad 1 \leq i \leq d.$$

Below we extend [14][Prop 1] which considers the case $F = G$.

Proposition 2.1. *If $F \in MDA(\mathcal{F})$ and $G \in MDA(\mathcal{G})$, then for any non-empty $T \subset \{1, \dots, d\}$ we have*

$$(2.4) \quad \lim_{n \rightarrow +\infty} np_{n,T}(F_n, G) = \mathbb{E}\left\{\left(\min_{i \in T} \mathcal{V}_i - \max_{i \in \bar{T}} \mathcal{V}_i\right)_+\right\} =: \lambda_T(\mathcal{F}, \mathcal{G}).$$

Remark 2.2. Define for a non-empty index set T the rv $K_n = \sum_{j=1}^n \mathbb{1}_{\{\forall i \in T: W_i > M_{j,i}, \forall i \in \bar{T}: W_i \leq M_{j,i}\}}$. Under the assumptions of Proposition 2.1 we have (see also [15][Corr 3.2]) that

$$(2.5) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E}\{K_n\}}{\ln n} = \lambda_T(\mathcal{F}, \mathcal{G}).$$

Example 2.3 (\mathcal{F} comonotonic and \mathcal{G} a product df). Suppose that \mathcal{F} is comonotonic, i.e., $Z_1 = \dots = Z_d$ almost surely and let \mathcal{G} be a product df with unit Fréchet marginals df's and let N be rv on $\{1, \dots, d\}$ with $\mathbb{P}\{N = i\} = 1/d, i \leq d$. A spectral vector \mathcal{W} for \mathcal{G} can be defined as follows

$$(\mathcal{W}_1, \dots, \mathcal{W}_d) = (d\mathbb{1}_{\{N=1\}}, \dots, d\mathbb{1}_{\{N=d\}}).$$

Indeed $\mathbb{E}\{Y_k\} = d\mathbb{P}\{N = k\} = 1$ for any $k \leq d$ and

$$\mathbb{E}\left\{\max_{1 \leq i \leq d} \mathcal{W}_i / x_i\right\} = d \sum_{k=1}^d \mathbb{E}\left\{\max_{1 \leq i \leq d} \mathcal{W}_i / x_i \mathbb{1}_{\{N=k\}}\right\} = d \sum_{k=1}^d \mathbb{E}\left\{\mathbb{1}_{\{N=k\}} / x_k\right\} = \sum_{k=1}^d 1/x_k$$

for any x_1, \dots, x_d positive. In particular, for a non-empty index set $K \subset \{1, \dots, d\}$ with m elements we have

$$\mathbb{E}\left\{\max_{i \in K} \mathcal{W}_i\right\} = d \sum_{k \in K} \mathbb{E}\left\{\mathbb{1}_{\{N=k\}}\right\} = m.$$

Consequently, using further that (see the proof of Proposition 2.1)

$$\lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \mathbb{E}\left\{\max_{i \in J \cup \bar{T}} \frac{\mathcal{W}_i}{Z_i}\right\}$$

we have

$$\lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \mathbb{E}\left\{\max_{i \in J \cup \bar{T}} \mathcal{W}_i\right\} = \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} (j + d - k).$$

If $k = d$, then from above e

$$(2.6) \quad \lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^d (-1)^{j+1} \sum_{J \subset T: |J|=j} j = d(1-1)^{d-1} = 0.$$

A direct probabilistic proof of (2.6) follows by the properties of \mathcal{W}_i 's, namely when $k = d \geq 2$

$$\lambda_T(\mathcal{F}, \mathcal{G}) = \mathbb{E}\left\{\min_{1 \leq i \leq d} \mathcal{W}_i / Z_i\right\} = \mathbb{E}\left\{\min_{1 \leq i \leq d} \mathcal{W}_i\right\} = d\mathbb{E}\left\{\min_{1 \leq i \leq d} \mathbb{1}_{\{N=i\}}\right\} = 0.$$

Now, let us investigate the number N_n of dominations defined as in Introduction by $\sum_{i=1}^d \mathbb{1}_{\{W_i/n > Z_i\}}$. For a given function $f: \{0, \dots, d\} \rightarrow \mathbb{R}$ we shall be concerned with the behaviour of

$$\mathbb{E}\{f(N_n)\} = \sum_{k=0}^d f(k) \mathbb{P}\{N_n = k\}$$

when n tends to $+\infty$. Throughout in the sequel we set

$$\mathcal{D} = \{1, \dots, d\}.$$

In Proposition 2.4 below, we first express this expectation as a function of minima or maxima of \mathcal{V}_i 's.

Proposition 2.4 (Limit law of N_n). *If $F \in MDA(\mathcal{F})$, $G \in MDA(\mathcal{G})$ and $e_{n,f} := \mathbb{E}\{f(N_n)\} - f(0)$, then we have*

$$(2.7) \quad \lim_{n \rightarrow +\infty} ne_{n,f} = \sum_{k=1}^d \Delta^k f(0) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{E} \left\{ \min_{i \in K} \mathcal{V}_i \right\}$$

or alternatively

$$(2.8) \quad \lim_{n \rightarrow +\infty} ne_{n,f} = \sum_{k=1}^d (-1)^{k+1} \Delta^k f(d-k) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{E} \left\{ \max_{i \in K} \mathcal{V}_i \right\},$$

where Δ is the difference operator, $\Delta f(x) = f(x+1) - f(x)$.

Proposition 2.5 (Link with order statistics). *Under the assumptions and the notation of Proposition 2.4 we have*

$$(2.9) \quad \lim_{n \rightarrow +\infty} ne_{n,f} = \sum_{k=1}^d g(k) \mathbb{E} \{ \mathcal{V}_{(k)} \},$$

where $\mathcal{V}_{(1)} \leq \dots \leq \mathcal{V}_{(d)}$ are the order statistics of $\mathcal{V}_1, \dots, \mathcal{V}_d$ and $g(k) = f(d-k+1) - f(d-k)$.

Remark 2.6 (retrieving simple cases). *For particular cases of f we have:*

- From Proposition 2.4, setting $f(x) = \mathbb{1}_{\{x=d\}}$, one can check that $\Delta^k f(0) = 0$ when $k < d$ and $\Delta^d f(0) = 1$, so that Equation (2.7) implies (1.2). Alternatively, by Proposition 2.5 since $g(1) = f(d) - f(d-1) = 1$ and $g(k) = f(d-k+1) - f(d-k) = 0 - 0 = 0$ if $k > 1$ we have that $\lim_{n \rightarrow +\infty} ne_{n,f} = \mathbb{E} \{ \mathcal{V}_{(1)} \}$.
- In view of Proposition 2.4, setting $f(x) = \mathbb{1}_{\{x \geq 1\}}$, $\Delta^k f(d-k) = \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(d-k+i)$. Thus $\Delta^k f(d-k) = 0$ if $k < d$. If $k = d$, then

$$\Delta^d f(d-d) = \Delta^d f(0) = (1-1)^d - (-1)^d = (-1)^{d+1}$$

and Equation (2.8) implies (1.1). Alternatively, by Proposition 2.5 since if $k < d$, $g(k) = f(d-k+1) - f(d-k) = 1 - 1 = 0$ and $g(d) = f(1) - f(0) = 1$ we obtain $\lim_{n \rightarrow +\infty} ne_{n,f} = \mathbb{E} \{ \mathcal{V}_{(d)} \}$.

Remark 2.7 (Interpretation of $\mathcal{V}_{(j)}$). *Let $f(k) = \mathbb{1}_{\{k \geq d-j+1\}}$, for any $j, k \in \mathcal{D}$. Then $g(k) = f(d-k+1) - f(d-k) = \mathbb{1}_{\{k=j\}}$. In this case, $f(0) = 0$ and $e_{n,f} = \mathbb{P} \{ N \geq d-j+1 \}$, thus*

$$\mathbb{E} \{ \mathcal{V}_{(j)} \} = \lim_{n \rightarrow +\infty} n \mathbb{P} \{ N_n \geq d-j+1 \}.$$

3. DOMINATION SPECTRUM

In the previous results, we have considered a particular setting, and we have expressed the domination probability and some expectations relying on number of dominations (see Section 2). We have seen that all these results were expressed as a function of

$$\mathbf{v} = \left(\frac{\mathcal{W}_i}{\mathcal{Z}_i} \right)_{i \in \mathcal{D}}.$$

By the definition all components $\mathcal{V}_i = \mathcal{W}_i / \mathcal{Z}_i$ are nonnegative, and are such that, by independence, $\mathbb{E} \{ \mathcal{V}_i \} = \mathbb{E} \{ \mathcal{W}_i \} \mathbb{E} \left\{ \frac{1}{\mathcal{Z}_i} \right\} = 1$. Thus in view of the De Haan representation \mathbf{v} can be viewed as the spectral random vector of some max-stable d -dimensional distribution. Since \mathbf{v} is related to the domination of \mathbf{M}_n by \mathbf{W} , we will refer to it by the term *domination spectrum*. In this section we shall explore some basic properties of \mathbf{w} .

Next, assume that \mathcal{W} has a copula $C_{\mathcal{W}}$ and suppose further that \mathcal{Z} has a copula $C_{\mathcal{Z}}$. Note in passing that the latter copula is unique since the marginals of \mathcal{Z} have continuous df. As each \mathcal{V}_i is a ratio between \mathcal{W}_i and \mathcal{Z}_i , we first study the link between the diagonal sections of both copulas $C_{\mathcal{W}}$ and $C_{\mathcal{Z}}$, defined for all $u \in [0, 1]$ by

$$\delta_{\mathcal{W}}(u) = C_{\mathcal{W}}(u, \dots, u) \quad \text{and} \quad \delta_{\mathcal{Z}}(u) = C_{\mathcal{Z}}(u, \dots, u).$$

We recall that the diagonal section characterizes uniquely many Archimedean copulas (under a condition that is called Frank's condition, see e.g., [16]), some non-parametric estimators of the generator of an Archimedean copulas directly rely on this diagonal section. We consider here the case where the df of \mathcal{Z} has spectral random vector \mathcal{W} . Notice that the upper tail dependence coefficients can be deduced from the regular variation properties of $\delta_{\mathcal{Z}}$ and $\delta_{\mathcal{W}}$, which is straightforward for $\delta_{\mathcal{Z}}$ in the following result.

Proposition 3.1 (Limit diagonal section of \mathcal{Z}). *Consider a d -dimensional random vector \mathcal{Z} having max-stable df with Fréchet unit marginals and with copula $C_{\mathcal{Z}}$. If the random vector \mathcal{Z} has df $H(\mathbf{y}) = \exp(-\mathbb{E}\{\max_{1 \leq j \leq d} \frac{\mathcal{W}_j}{y_j}\})$, where all \mathcal{W}_j are nonnegative rv's with mean 1, then*

$$\delta_{\mathcal{Z}}(u) = u^{r_{\mathcal{W}}} \quad \text{with} \quad r_{\mathcal{W}} = \mathbb{E} \left\{ \max_{j \in \mathcal{D}} \mathcal{W}_j \right\}.$$

In particular, when $r_{\mathcal{W}} > 1$, this diagonal section $\delta_{\mathcal{Z}}(u)$ is the one of a Gumbel copula with parameter

$$(3.1) \quad \theta = \frac{\ln d}{\ln r_{\mathcal{W}}}.$$

Furthermore, if the components of \mathcal{W} are identically distributed and if $F_{\mathcal{W}}$ is invertible, then we have

$$r_{\mathcal{W}} = \int_0^1 F_{\mathcal{W}_1}^{-1}(s) d\delta_{\mathcal{W}}(s).$$

Example 3.2 (From independence to comonotonicity). *Let $\mathcal{W}_j = B d \mathbb{1}_{\{I=j\}} + (1-B)\delta_1$, for all $j \in \mathcal{D}$, where I is a uniformly distributed rv's on \mathcal{D} , B is a Bernoulli rv with $\mathbb{E}\{B\} = \alpha \in (0, 1]$ and δ_1 is a Dirac mass at 1, all these rv's being mutually independent. In this case*

$$r_{\mathcal{W}} = \mathbb{E} \left\{ \max_{j \in \mathcal{D}} \mathcal{W}_j \right\} = \alpha d + 1 - \alpha, \quad \theta = \frac{\ln d}{\ln(1 + \alpha(d-1))}.$$

Proposition 3.3 (Distribution of $\max_{j \in K} \mathcal{V}_j$ or $\min_{j \in K} \mathcal{V}_j$). *Under the assumption that \mathcal{W}' is a spectral vector of \mathcal{Z} , with \mathcal{Z} max-stable with unit Fréchet marginals and \mathcal{Z} , \mathcal{W} , \mathcal{W}' are mutually independent, then for any $t > 0$*

$$\mathbb{P} \left\{ \max_{i \in \mathcal{D}} \mathcal{V}_i \leq t \right\} = 1 - \sum_{k=1}^d (-1)^{k+1} \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{P} \left\{ \min_{i \in K} \mathcal{V}_i > t \right\},$$

and

$$\mathbb{P} \left\{ \min_{i \in K} \mathcal{V}_i > t \right\} = \mathbb{E}_{\mathcal{W}} \left\{ \mathbb{1}_{\{\min_{i \in K} \mathcal{W}_i > 0\}} \exp \left(-t \mathbb{E}_{\mathcal{W}'} \left\{ \max_{i \in K} \frac{\mathcal{W}'_i}{\mathcal{W}_i} \right\} \right) \right\}.$$

Example 3.4 (From independence to comonotonicity, continued). *Under the settings of Example 3.2 we have when all $t_j > 0$*

$$\mathbb{E} \left\{ \max_{j \in K} \frac{\mathcal{W}_j}{t_j} \right\} = \alpha \sum_{j \in K} \frac{1}{t_j} + (1-\alpha) \frac{1}{\min_{j \in K} t_j}.$$

Let $t > 0$ and suppose that K has $m > 1$ elements. By conditioning over B , we get

$$\mathbb{P} \{ \forall i \in K, \mathcal{V}_i > t \} = (1-\alpha) \mathbb{P} \left\{ \forall i \in K, \mathcal{Z}_i < 1/t \mid B = 0 \right\}$$

since $\mathbb{P}\left\{\forall i \in K, \mathcal{V}_i > t \mid B = 1\right\} = 0$ when $m > 1$, because at least one component \mathcal{W}_i , $i \in K$, is zero when $B = 1$ and $m > 1$. Recall that \mathcal{Z} is independent from \mathcal{W} and B , thus for $t > 0$ and $|K| > 1$

$$\mathbb{P}\{\forall i \in K, \mathcal{V}_i > t\} = (1 - \alpha) \exp\left(\mathbb{E}\left\{-\max_{j \in K} \frac{\mathcal{W}'_j}{(1/t)}\right\}\right) = (1 - \alpha) \exp(-t(1 + \alpha(m - 1))).$$

This can be retrieved directly from Proposition 3.3, using the fact that $\min_{i \in K} \mathcal{W}_i > 0$ implies $B = 0$ when $|K| > 1$. A consequence of this is that one can write

$$\begin{cases} \min_{i \in K} \mathcal{V}_i &= (1 - B)\epsilon_{1+\alpha(|K|-1)} + B\mathbb{1}_{\{|K|=1\}}\mathbb{1}_{\{I=1\}}\delta_d \\ \mathbb{E}\left\{\min_{i \in K} \mathcal{V}_i\right\} &= \frac{1-\alpha}{1+\alpha(|K|-1)} + \mathbb{1}_{\{|K|=1\}}\alpha, \end{cases}$$

where B is a Bernoulli r.v. of parameter α , $\epsilon_{1+\alpha(|K|-1)}$ is an exponentially distributed r.v. with parameter $1 + \alpha(|K| - 1)$, I an uniformly distributed r.v. over \mathcal{D} , and δ_d a Dirac mass at d , all being mutually independent (for simplicity, we denote $\mathbb{1}_{\{|K|=1\}}$ the variable whose value is 1 if $|K| = 1$ or 0 otherwise). Then all results about the limit law of N_n derive immediately, using Equation (2.7) in Proposition 2.4. Notice that one could also determine $r_{\mathcal{V}}$ from this, and by application of Proposition 3.1, assess the dependence structure of the random vector whose spectrum is \mathcal{V} .

4. COMBINATORIAL RESULTS

We give hereafter some combinatoric results that show how quantities depending on a number of events can be related to quantities involving only intersections or unions of events. This generalizes inclusion-exclusion formulas that will correspond to very specific functions f and g .

Lemma 4.1 (Inclusion-exclusion relations). *Let $\mathcal{D} = \{1, \dots, d\}$ and let B_i , $i \in \mathcal{D}$ be events. Consider the number of realized events $N = \sum_{i \in \mathcal{D}} \mathbb{1}_{\{B_i\}}$. Then for any function $f : \{0, \dots, d\} \rightarrow \mathbb{R}$*

$$(4.1) \quad \sum_{k=0}^d f(k) \mathbb{P}\{N = k\} = f(0) + \sum_{j=1}^d S_j \Delta^j f(0) = f(0) + \sum_{j=1}^d \bar{S}_j (-1)^{j+1} \Delta^j f(d - j),$$

and similarly for any function $g : \mathcal{D} \rightarrow \mathbb{R}$

$$(4.2) \quad \sum_{k=0}^d g(k) \mathbb{P}\{N \geq k\} = \sum_{j=1}^d S_j \Delta^{j-1} g(1) = \sum_{j=1}^d \bar{S}_j (-1)^{j+1} \Delta^{j-1} g(d - j + 1),$$

where $S_j = \sum_{J \subset \mathcal{D}, |J|=j} \mathbb{P}\left\{\bigcap_{i \in J} B_i\right\}$ and $\bar{S}_j = \sum_{J \subset \mathcal{D}, |J|=j} \mathbb{P}\left\{\bigcup_{i \in J} B_i\right\}$.

Proof of Lemma 4.1. Let us denote $p_J = \mathbb{P}\{\bigcap_{i \in J} B_i\}$ and $\bar{p}_J = \mathbb{P}\{\bigcup_{i \in J} B_i\}$. By inclusion-exclusion principle, we get

$$(4.3) \quad S_k = \sum_{K \subset \mathcal{D}, |K|=k} \sum_{j=1}^k (-1)^{j+1} \sum_{J \subset K, |J|=j} p_J = \sum_{j=1}^k (-1)^{j+1} \binom{d-j}{k-j} \bar{S}_j.$$

The first equality in Equation (4.1) is known in actuarial sciences under the name of Schuette-Nesbitt formula, see [17, section 8.5]. This formula does not require any independence assumption, it is a simple development of $f(N) = (I + \mathbb{1}_{\{B_1\}}\Delta) \cdots (I + \mathbb{1}_{\{B_d\}}\Delta)f(0)$ where I and Δ are the identity and the difference operators respectively.

This formula generalizes a very old formula of Waring and generalizes also the inclusion exclusion formula which can be retrieved by setting $f(k) = 1$ if $k \geq 1$, and $f(k) = 0$ otherwise. Now using Equation (4.3)

$$\sum_{k=1}^d \Delta^k f(0) S_k = \sum_{k=1}^d \Delta^k f(0) \sum_{j=1}^k (-1)^{j+1} \binom{d-j}{k-j} \bar{S}_j = \sum_{j=1}^d \bar{S}_j (-1)^{j+1} \Delta^j (I + \Delta)^{d-j} f(0)$$

and since $(I + \Delta)^{d-j} f(0) = f(d-j)$, the second equality in Equation (4.1) holds. Similarly, the first equality in Equation (4.2) is a known Schuette-Nesbitt formula, see [17, Section 8.5], and one can retrieve the second equality by using Equation (4.3). Alternatively, one can also deduce (4.2) from (4.1) by setting $f(0) = 0$ and $g(k) = \Delta f(k-1)$ for all $k \in \mathcal{D}$. \square

5. PROOFS

Proof of Proposition 2.1. By inclusion-exclusion formula for a given index set $T \subset \{1, \dots, d\}$ with $k = |T|$ elements we have

$$\begin{aligned} \mathbb{P}\{\forall i \in \bar{T} : W_i \leq y_i, \exists i \in T : W_i \leq y_i\} &= \sum_{j=1}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \mathbb{P}\{\forall i \in (J \cup \bar{T}) : W_i \leq y_i\} \\ &= \sum_{j=1}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} G_{J \cup \bar{T}}(\mathbf{y}), \end{aligned}$$

where $G_L(\mathbf{y}) = \mathbb{P}\{\forall i \in L : W_i \leq y_i\}$ is the L -th marginal df of G . In particular, letting $W_i \rightarrow \infty, i \leq d$ we have

$$1 = \sum_{j=1}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} 1.$$

Consequently, for all $n > 1$

$$\begin{aligned} p_{n,T}(F_n, G) &= \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in T : W_i \geq y_i, \forall i \in \bar{T} : W_i < y_i\} dF_n(\mathbf{y}) \\ &= \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in \bar{T} : W_i \leq y_i\} dF_n(\mathbf{y}) - \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in \bar{T} : W_i \leq y_i, \exists i \in T : W_i \leq y_i\} dF_n(\mathbf{y}) \\ &= 1 - \int_{\mathbb{R}^d} \sum_{j=1}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} G_{J \cup \bar{T}}(\mathbf{y}) dF_n(\mathbf{y}) - \left(1 - \int_{\mathbb{R}^d} G_{\bar{T}}(\mathbf{y}) dF_n(\mathbf{y})\right) \\ &= \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \int_{\mathbb{R}^d} [1 - G_{J \cup \bar{T}}(\mathbf{y})] dF_n(\mathbf{y}) \\ &= \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \int_{\mathbb{R}^{m+j}} [1 - G_{J \cup \bar{T}}(\mathbf{y})] dF_{n, J \cup \bar{T}}(\mathbf{y}). \end{aligned}$$

In view of [1][Prop 4.2] we have

$$\lim_{n \rightarrow +\infty} n \int_{\mathbb{R}^{m+|J|}} [1 - G_{J \cup \bar{T}}(\mathbf{y})] dF_{n, J \cup \bar{T}}(\mathbf{y}) = - \int_{\mathbb{R}^{m+|J|}} \ln Q_{J \cup \bar{T}}(\mathbf{y}) dH_{J \cup \bar{T}}(\mathbf{y}).$$

Further by [1][Thm 2.5 and Prop 4.2]

$$- \int_{\mathbb{R}^{m+|J|}} \ln Q_{J \cup \bar{T}}(\mathbf{y}) dH_{J \cup \bar{T}}(\mathbf{y}) = \mathbb{E} \left\{ \max_{i \in J \cup \bar{T}} \frac{W_i}{Z_i} \right\}.$$

Consequently, we have

$$\lim_{n \rightarrow +\infty} np_{n,T}(F_n, G) = \sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \mathbb{E} \left\{ \max_{i \in J \cup \bar{T}} \frac{W_i}{Z_i} \right\}.$$

In view of [9][Lem 1] for given constants c_1, \dots, c_d

$$\sum_{j=0}^k (-1)^{j+1} \sum_{J \subset T: |J|=j} \max_{i \in J \cup \bar{T}} c_i = \max \left(\max_{i \in \bar{T}} c_i, \min_{i \in T} c_i \right) - \max_{i \in \bar{T}} c_i = \left(\min_{i \in T} c_i - \max_{i \in \bar{T}} c_i \right)_+$$

implying the claim.

Alternatively, we have using again inclusion-exclusion formula

$$\begin{aligned} p_{n,T}(F_n, G) &= \int_{\mathbb{R}^d} \mathbb{P}\{w_i \geq \bar{M}_i, i \in T, w_i < M_i, i \in \bar{T}\} dG(\mathbf{w}) \\ &= \int_{\mathbb{R}^d} \mathbb{P}\{M_i \leq w_i, i \in T\} dG(\mathbf{w}) - \int_{\mathbb{R}^d} \mathbb{P}\{M_i \leq w_i, i \in T, \exists i \in \bar{T} : M_i \leq w_i\} dG(\mathbf{w}) \\ &= \int_{\mathbb{R}^d} F_{n,T}(\mathbf{w}) dG_T(\mathbf{w}) - \int_{\mathbb{R}^d} \sum_{j=1}^m (-1)^{j+1} \sum_{J \subset \bar{T}: |J|=j} F_{J \cup T}^n(\mathbf{w}) dG(\mathbf{w}) \\ &= \sum_{j=0}^{d-k} (-1)^j \sum_{J \subset \bar{T}: |J|=j} \int_{\mathbb{R}^{k+j}} F_{n, J \cup T}(\mathbf{w}) dG_{J \cup T}(\mathbf{w}). \end{aligned}$$

Applying [1][Thm 2.5 and Prop 4.2] we obtain

$$\lim_{n \rightarrow +\infty} n \int_{\mathbb{R}^{k+i}} F_{n, J \cup T}(\mathbf{y}) dG_{J \cup T}(\mathbf{y}) = \mathbb{E} \left\{ \min_{i \in J \cup T} \frac{W_i}{Z_i} \right\}$$

and thus

$$(5.1) \quad \mu_T(H, Q) = \sum_{j=0}^{d-k} (-1)^j \sum_{J \subset \bar{T}: |J|=j} \mathbb{E} \left\{ \min_{i \in J \cup T} \frac{W_i}{Z_i} \right\}.$$

By [9][Lem 1] we obtain further

$$\mu_T(H, Q) = \mathbb{E} \left\{ \min_{i \in T} \frac{W_i}{Z_i} - \min \left(\min_{i \in T} \frac{W_i}{Z_i}, \max_{i \in \bar{T}} \frac{W_i}{Z_i} \right) \right\},$$

hence the proof is complete. \square

Proof of Proposition 2.4. By combinatorial arguments or utilising the classical actuarial Schuette-Nesbitt formula, see first equality in (4.1) and [17, Section 8.5], we get

$$\mathbb{E} \{f(N_n)\} = f(0) + \sum_{k=1}^d \Delta^k f(0) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{P} \{ \forall i \in K, W_i \geq M_{ni} \}.$$

Alternatively, using Equation (4.1) in Lemma 4.1 below,

$$\mathbb{E} \{f(N_n)\} = f(0) + \sum_{k=1}^d (-1)^{k+1} \Delta^k f(d-k) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{P} \{ \exists i \in K, W_i \geq M_{ni} \}.$$

Thus, using Equation (1.1), the result holds. \square

Proof of Proposition 2.5. Let us consider $\mathbb{P}\{\mathcal{V}_{(k)} \leq x\} = \mathbb{P}\{\text{at least } k \text{ events } [\mathcal{V}_i \leq x] \text{ are realized, } i \in \mathcal{D}\}$. Using the first equality in Equation (4.2), we get for any function $g : \{1, \dots, d\} \rightarrow \mathbb{R}$

$$\sum_{k=1}^d g(k) \mathbb{P}\{\mathcal{V}_{(k)} \leq x\} = \sum_{k=1}^d \Delta^{k-1} g(1) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{P}\{\forall i \in K, \mathcal{V}_i \leq x\}$$

and hence letting $x \rightarrow \infty$ we have

$$\sum_{k=1}^d g(k) = \sum_{k=1}^d \Delta^{k-1} g(1) \sum_{K \subset \mathcal{D}, |K|=k} 1.$$

Consequently, for any real x

$$\sum_{k=1}^d g(k) \mathbb{P}\{\mathcal{V}_{(k)} > x\} = \sum_{k=1}^d \Delta^{k-1} g(1) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{P}\left\{\max_{i \in K} \mathcal{V}_i > x\right\}.$$

By the assumptions

$$\mathbb{E}\left\{\max_{1 \leq i \leq d} \mathcal{V}_i\right\} \leq \sum_{i=1}^d \mathbb{E}\{\mathcal{V}_i\} = d,$$

hence since \mathcal{V}_i 's are non-negative it follows that

$$\sum_{k=1}^d g(k) \mathbb{E}\{\mathcal{V}_{(k)}\} = \sum_{k=1}^d \Delta^{k-1} g(1) \sum_{K \subset \mathcal{D}, |K|=k} \mathbb{E}\left\{\max_{i \in K} \mathcal{V}_i\right\}.$$

Finally, in order to retrieve Equation (2.8), we must have for any $k \in \{1, \dots, d\}$

$$\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k).$$

Now, assuming that for all $k \in \{1, \dots, d\}$, $g(k) = f(d-k+1) - f(d-k) = \Delta f(d-k)$, then denoting by $T = \Delta + I$ the translation operator

$$\Delta^{k-1} g(1) = \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} T^{-i} \Delta f(d-1).$$

This implies

$$\Delta^{k-1} g(1) = (-I + T^{-1})^{k-1} \Delta f(d-1) = (-1)^{k-1} (T^{-1}(T-I))^{k-1} \Delta f(d-1).$$

Thus, for all $k \in \{1, \dots, d\}$ we have

$$\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k)$$

and hence the claim follows. \square

Proof of Proposition 3.1. For the first equality, since \mathbf{Z} has unit Fréchet marginals for any $u > 0$ we have

$$C_{\mathbf{Z}}(u, \dots, u) = H\left(\frac{1}{-\ln u}, \dots, \frac{1}{-\ln u}\right) = \exp\left(\mathbb{E}\left\{\max_{1 \leq j \leq d} \ln(u) \mathcal{W}_j\right\}\right) = u^{\mathbb{E}\left\{\max_{j \in \mathcal{D}} \mathcal{W}_j\right\}}$$

and thus $\delta_{\mathbf{Z}}(u) = u^{r_{\mathbf{Z}}}$. Since the diagonal section of a d -dimensional Archimedean copula with parameter θ is $u^{d^{1/\theta}}$ we obtain the formula for θ . This is consistent with the fact that the Gumbel copula is an Extreme Value Copula (the only Archimedean one, see [18]).

For the last equality, setting $\mathcal{W}_j = F_{\mathcal{W}_1}^{-1}(U_j)$, we get $\max_{j \in \mathcal{D}} \mathcal{W}_j = \max_{j \in \mathcal{D}} F_{\mathcal{W}_1}^{-1}(U_j)$. Assuming further that all \mathcal{W}_i 's have a common df $F_{\mathcal{W}_1}$, then $\max_{j \in \mathcal{D}} F_{\mathcal{W}_1}^{-1}(U_j) = F_{\mathcal{W}_1}^{-1}(\max_{j \in \mathcal{D}}(U_j))$. Using

$$\mathbb{P} \left\{ \max_{j \in \mathcal{D}} U_j \leq u \right\} = \mathbb{P} \{U_1 \leq u, \dots, U_d \leq u\} = C_{\mathbf{Y}}(u, \dots, u) = \delta_{\mathbf{Y}}(u)$$

we get $\mathbb{E} \{ \max_{j \in \mathcal{D}} \mathcal{W}_j \} = \int_0^1 F_{\mathcal{W}_1}^{-1}(s) d\delta_{\mathbf{Y}}(s)$. \square

Proof of Proposition 3.3. This derives from inclusion-exclusion formula, as $\mathbb{P} \{ \forall i \in \mathcal{D}, \mathcal{V}_i \leq t \} = 1 - \mathbb{P} \{ \cup_{i \in \mathcal{D}} \mathcal{V}_i > t \}$ for any $t > 0$ and from

$$\begin{aligned} \mathbb{P} \{ \forall i \in K, \mathcal{V}_i > t \} &= \mathbb{P} \{ \forall i \in K, \mathcal{W}_i / \mathcal{Z}_i > t \} \\ &= \mathbb{E}_{\mathcal{W}} \left\{ \mathbb{P}_{\mathcal{Z}} \left\{ \forall i \in K, \mathcal{Z}_i < \mathcal{W}_i / t, \min_{i \in K} \mathcal{W}_i > 0 \right\} \right\} \\ &= \mathbb{E}_{\mathcal{W}} \left\{ \mathbb{1}_{\{ \min_{i \in K} \mathcal{W}_i > 0 \}} \exp \left(-t \mathbb{E}_{\mathcal{W}'} \left\{ \max_{i \in K} \frac{\mathcal{W}'_i}{\mathcal{W}_i} \right\} \right) \right\} \end{aligned}$$

establishing the proof. \square

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ENKELEJD HASHORVA, DEPARTMENT OF ACTUARIAL SCIENCE UNIVERSITY OF LAUSANNE,, UNIL-DORIGNY, 1015 LAUSANNE, SWITZERLAND
E-mail address: Enkelejd.Hashorva@unil.ch

DIDIER RULLIÈRE, ECOLE ISFA, LSAF, UNIVERSITÉ LYON 1, 69366 LYON, FRANCE
E-mail address: didier.rulliere@univ-lyon1.fr