The First Eigenvalue of the Dirac Operator on Compact Outer Spin Symmetric Spaces

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Abstract. In two previous papers, we started a study of the first eigenvalue of the Dirac operator on compact spin symmetric spaces, providing, for symmetric spaces of "inner" type, a formula giving this first eigenvalue in terms of the algebraic data of the groups involved. We conclude here that study by giving the explicit expression of the first eigenvalue for "outer" compact spin symmetric spaces.

1. Introduction

It is well-known that symmetric spaces provide examples where the spectrum of Laplace or Dirac operators can be (theoretically) explicitly computed. However this explicit computation is far from being simple in general and only a few examples are known. On the other hand, several classical results in geometry involve the first (nonzero) eigenvalue of those spectra, so it seems interesting to get this eigenvalue without computing all the spectrum. In two previous papers (see [Mil05] and [Mil06]), we stated a formula giving the square of the first eigenvalue of the Dirac operator of a spin compact symmetric space in terms of the algebraic data of the groups involved. However, this formula was based on a result of R. Parthasarathy, [Par71], only valid for symmetric spaces of inner type. Recall that a symmetric space G/K is said to be of inner type if the involution characterizing it is given by an inner conjugation in the group G, or alternatively, by the fact that the groups G and K have same rank, i.e. own a common maximal torus. Otherwise, it is said to be outer, (see for instance Sec. 8.6 in [Wol72] for details). The study of subgroups K of maximal rank in a compact Lie group was initiated by A. Borel and J. De Siebenthal in [BDS49], with an explicit description for compact simple groups. In [Mur52] (see also [Mur65]), S. Murakami gave a general method to study outer involutive automorphisms of compact simple Lie algebras. Using those results, the following complete list of irreducible compact simply-connected Riemannian symmetric spaces G/K of type I with $\operatorname{rank} K < \operatorname{rank} G$, can be obtained (see [Mur65] or Sec. 8.12 in J. A. Wolf's book [Wol72]) :

$$\mathrm{SU}(2m)/\mathrm{SO}(2m)\ ;\ \mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)\ ;\ \mathrm{SU}(2m)/\mathrm{Sp}(m)\ ;$$
$$\mathrm{SO}(2p+2q+2)/\mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1)\ ;\ \mathrm{E}_6/\mathrm{F}_4\ ;\ \mathrm{E}_6/\mathrm{Sp}(4)\,.$$

It was proven by M. Cahen and S. Gutt in [CG88], that all symmetric spaces in that list, except $\mathrm{SU}(2m+1)/\mathrm{SO}(2m+1)$, are spin.

In the present paper, the following explicit value for the square of the first eigenvalue of the Dirac operator is obtained for all those symmetric spaces G/K, endowed with the Riemannian metric induced by the Killing form of G sign-changed.

Symmetric space	Square of the first eigenvalue of the Dirac operator
$\frac{\mathrm{SU}(2m)}{\mathrm{SO}(2m)}$	m even, $\frac{1}{12}(m+1)(4m-1)+\frac{1}{32}$ m odd, $\frac{1}{12}(m+1)(4m-1)+\frac{1}{32}\left(1-\frac{1}{m^2}\right)$
$\frac{\mathrm{SU}(2m)}{\mathrm{Sp}(m)}$	m even, $\frac{1}{12}(m-1)(4m+1)+\frac{1}{32}$ m odd ≥ 3, $\frac{1}{12}(m-1)(4m+1)+\frac{1}{32}\left(1-\frac{1}{m^2}\right)$
$\frac{\mathrm{SO}(2p+2q+2)}{\mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1)}$ $p\leq q$	$\frac{1}{16\,(p+q)}\left(8pq\,(2p+q+1)+4p\,(p+1)+4q\,(q+1)+1\right)$
$\frac{\mathrm{E}_6}{\mathrm{F}_4}$	$\frac{277}{72}$
$\frac{\mathrm{E}_6}{\mathrm{Sp}_4}$	$\frac{529}{72}$

2. Preliminaries for the proof

2.1. **Spectrum of the Dirac operator on spin compact irreducible symmetric spaces.** From now on, we consider a spin compact simply connected irreducible symmetric space G/K of "type I", where G is a simple compact and simply-connected Lie group and K is the connected subgroup formed by the fixed elements of an involution σ of G. This involution induces the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G into

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}\,,$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the vector space $\{X\in\mathfrak{g}\,;\,\sigma_*\cdot X=-X\}$. The symmetric space G/K is endowed with the Riemannian metric induced by the restriction to $\mathfrak{p}$ of the Killing form B_G of G sign-changed.
The spin condition implies that the homomorphism

$$\alpha : h\in K\longmapsto \mathrm{Ad}_G(h)_{|\mathfrak{p}}\in\mathrm{SO}(\mathfrak{p})$$

lifts to a homomorphism $\widetilde{\alpha}:K\longrightarrow\mathrm{Spin}(\mathfrak{p})$ such that $\xi\circ\widetilde{\alpha}=\alpha$ where ξ is the two-fold covering $\mathrm{Spin}(\mathfrak{p})\to\mathrm{SO}(\mathfrak{p})$, [CG88].
Then the group K inherits a spin representation given by

$$\widetilde{\rho_K}:K\xrightarrow{\widetilde{\alpha}}\mathrm{Spin}(\mathfrak{p})\xrightarrow{\rho}\mathrm{GL}_{\mathbb{C}}(\Sigma)\,,$$

where ρ is the spinor representation in the complex spinor space Σ.

The Dirac operator has a real discrete spectrum, symmetric with respect to the origin. A real number λ belongs to the spectrum if and only if there exists an irreducible representation $\gamma : G \to \mathrm{GL}_{\mathbb{C}}(V_\gamma)$ whose restriction $\mathrm{Res}^G_K(\gamma)$ to the subgroup K, contains in its decomposition into irreducible parts, a representation equivalent to some irreducible component of the decomposition of the spin representation $\widetilde{\rho_K}$ of K. Then

$$\lambda^2 = c_\gamma + n/16\,, \tag{1}$$

where c_γ is the Casimir eigenvalue of the irreducible representation γ (which only depends on the equivalence class of γ) and where $n = \dim(G/K)$, $n/16$ being Scal/8 for the choice of the metric (cf. [HBM+15] or [Gin09] for details).
Hence the first eigenvalue of the Dirac operator is given by the lowest c_γ, among the irreducible representations $\gamma' s$ of G such that $\mathrm{Res}^G_K(\gamma)$ contains an irreducible component of the spin representation $\widetilde{\rho_K}$ of K. We will say for short that such an irreducible representation γ (or its highest weight) verifies the "spin condition".

2.2. **Outer symmetric spaces.** Let T be a maximal torus of G. Then $T_K := T \cap K$ is a maximal torus of K.
As it was already mentioned, the symmetric space G/K is said to be outer if the involution $\sigma : G \to G$ is not a conjugation in the group, and this is equivalent to the condition $\dim(T_K) < \dim(T)$.
Let $\mathfrak{t}$ and $\mathfrak{t}_K$ be the Lie algebras of T and T_K, and let $\mathfrak{t}_0 := \mathfrak{t} \cap \mathfrak{p}$. Note that $\mathfrak{t}_0$ is the orthogonal complement of $\mathfrak{t}_K$ in $\mathfrak{t}$ for the scalar product $-\mathrm{B}_G$.
Let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ be the complexifications of the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, and $\langle\ ,\ \rangle$ the $\mathbb{C}$-extension of $-\mathrm{B}_G$ to $\mathfrak{g}_{\mathbb{C}}$. The root decompositions of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ under the respective actions of T and T_K are given by

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\bigoplus_{i=1}^{N} \mathfrak{g}_{\pm\theta_i} \right)\,, \tag{2}$$

where $\pm\theta_i$ are the G-roots, and

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{K,\mathbb{C}} \oplus \left(\bigoplus_{i=1}^{M} \mathfrak{k}_{\pm\theta'_i} \right)\,,$$

where $\pm\theta'_i$ are the K-roots.
The set Φ of G-roots decomposes into the disjoint union

$$\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3\,,$$

where

$$\begin{aligned}
\Phi_1 &= \{\theta \in \Phi, \mathfrak{g}_\theta \subset \mathfrak{k}_{\mathbb{C}}\}\,,\\
\Phi_2 &= \{\theta \in \Phi, \mathfrak{g}_\theta \subset \mathfrak{p}_{\mathbb{C}}\}\,,\\
\Phi_3 &= \Phi \backslash (\Phi_1 \cup \Phi_2)\,.
\end{aligned}$$

Note that if $\theta \in \Phi_i$, $i = 1, 2, 3$, then $-\theta \in \Phi_i$, since $\mathfrak{g}_{-\theta} = \overline{\mathfrak{g}_\theta}$.
We use the same notation for the involution $\sigma_* : \mathfrak{g} \to \mathfrak{g}$ and its $\mathbb{C}$-linear extension to $\mathfrak{g}_{\mathbb{C}}$. As $\sigma_*(\mathfrak{t}) = \mathfrak{t}$, by means of the scalar product $-\mathrm{B}_G$, $\sigma_{*|\mathfrak{t}}$ induces an involution $\sigma^* : i\,\mathfrak{t}^* \to i\,\mathfrak{t}^*$. The scalar product on $i\,\mathfrak{t}^*$ induced by $-\mathrm{B}_G$ is denoted by $\langle\ ,\ \rangle$.
We consider for any G-root θ_i, a basis E_{θ_i} of the one-dimensional space $\mathfrak{g}_{\theta_i}$, and set

$$U_{\theta_i} := \frac{1}{2}\,\left(E_{\theta_i} + \sigma_* E_{\theta_i}\right) \text{ and } V_{\theta_i} := \frac{1}{2}\,\left(E_{\theta_i} - \sigma_* E_{\theta_i}\right)\,.$$

Then $U_{\theta_i} \in \mathfrak{k}$, $V_{\theta_i} \in \mathfrak{p}$, $E_{\theta_i} = U_{\theta_i} + V_{\theta_i}$, and

$$\theta_i \in \Phi_3 \Longleftrightarrow (U_{\theta_i} \neq 0 \text{ and } V_{\theta_i} \neq 0)\,.$$

In order to determine an expression of the weights of the spin representation $\widetilde{\rho_K} : K \to \mathrm{GL}_{\mathbb{C}}(\Sigma)$, we will have to use the following properties of G-roots. For any root $\theta \in \Phi$, the restriction $\theta_{|\mathfrak{t}_K}$ is denoted θ' for short.

Lemma 2.1. **R1:** *$\forall \alpha \in \Phi_3$, $\exists X \in \mathfrak{t}_K$ such that $\alpha(X) \neq 0$, hence α' is a (non zero) K-root.*

R2: *$\forall \alpha \in \Phi$, $\alpha \in \Phi_1 \cup \Phi_2 \Longleftrightarrow \alpha_{|\mathfrak{t}_0} = 0$. Hence if α and β are two roots in Φ_1 such that $\beta \neq \pm\alpha$, then α' and β' are two K-roots such that $\beta' \neq \pm\alpha'$.*

R3: *If $\alpha \in \Phi_1$ and $\beta \in \Phi_3$, then α' and β' are two K-roots such that $\beta' \neq \pm\alpha'$.*

R4: *For all $\alpha \in \Phi$, $\sigma^*(\alpha)$ is a root such that*

(1) *if $\alpha \in \Phi_1 \cup \Phi_2$, then $\sigma^*(\alpha) = \alpha$,*

(2) *if $\alpha \in \Phi_3$, then $\sigma^*(\alpha) \neq \pm\alpha$,*

R5: *For all α and $\beta \in \Phi_3$, $\beta' = \alpha' \Longleftrightarrow \beta = \alpha$ or $\beta = \sigma^*(\alpha)$.*

Proof. The proofs are given in Appendix. Some of those results appear in some way in Chapter 3 of [BR90]. □

According to **R4**, we consider $\Phi_3' = \{\gamma_1, \dots, \gamma_p\} \subset \Phi_3$ such that Φ_3 is the disjoint union

$$\Phi_3 = \{\gamma_1, \dots, \gamma_p\} \cup \{\sigma^*(\gamma_1), \dots, \sigma^*(\gamma_p)\}\,.$$

Then

Lemma 2.2. *The set Φ_K of K-roots under the action of T_K is given by*

$$\Phi_K = \{\theta_{|\mathfrak{t}_K};\, \theta \in \Phi_1 \cup \Phi_3'\}\,.$$

Proof. By definition, the restriction to $\mathfrak{t}_K$ of a root in Φ_1 is a K-root, and by **R1**, the restriction to $\mathfrak{t}_K$ of a root in Φ_3' is also a K-root. Conversely, let $E'_\varrho \in \mathfrak{k}_{\mathbb{C}}$ be a root-vector for a K-root ϱ. As for any $\alpha \in \Phi_3$, $\sigma_*(E_\alpha)$ is a root-vector for the root $\sigma^*(\alpha)$ by (11) and (12), we may write, according to (2),

$$\begin{aligned} E'_\varrho &= H + \sum_{\alpha\in\Phi_1} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_2} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_3'} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_3'} \lambda_{\sigma^*(\alpha)}\, \sigma_*(E_\alpha)\,, \\ &\qquad\qquad \text{where } H \in \mathfrak{t} \text{ and } \lambda_\alpha \in \mathbb{C}, \\ &= H + \sum_{\alpha\in\Phi_1} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_2} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_3'} \big(\lambda_\alpha + \lambda_{\sigma^*(\alpha)}\big)\, U_\alpha \\ &\qquad\qquad + \sum_{\alpha\in\Phi_3'} \big(\lambda_\alpha - \lambda_{\sigma^*(\alpha)}\big)\, V_\alpha\,. \end{aligned}$$

Since $\sigma_*(E'_\varrho) = E'_\varrho$, one gets $H \in \mathfrak{t}_K$, $\forall \alpha \in \Phi_2$, $\lambda_\alpha = 0$, and $\forall \alpha \in \Phi_3'$, $\lambda_{\sigma^*(\alpha)} = \lambda_\alpha$, hence

$$E'_\varrho = H + \sum_{\alpha\in\Phi_1} \lambda_\alpha\, E_\alpha + 2 \sum_{\alpha\in\Phi_3'} \lambda_\alpha\, U_\alpha\,.$$

Now, as for any $X \in \mathfrak{t}_K$, $[X, E'_\varrho] = \varrho(X)\, E'_\varrho$, one gets

$$\begin{aligned} \forall X \in \mathfrak{t}_K\,,\ -\varrho(X)\, H + \sum_{\alpha\in\Phi_1} \lambda_\alpha\, (\alpha(X) - \rho(X))\, E_\alpha \\ + 2 \sum_{\alpha\in\Phi_3'} \lambda_\alpha\, (\alpha(X) - \rho(X))\, U_\alpha = 0\,. \end{aligned}$$

Hence $H = 0$, and, as there exists at least one $\lambda_\alpha \neq 0$, $\varrho = \alpha'$. Note furthermore that by **R2**, **R3** and **R5**, there exists only one such a $\lambda_\alpha \neq 0$. □

By the results **R1** and **R2**, the restriction to $\mathfrak{t}_K$ of any G-root is nonzero. As $\mathfrak{t}_K$ can not be the finite union of the hyperplanes $\mathrm{Ker}(\alpha')$ for $\alpha \in \Phi$, there exists $X \in \mathfrak{t}_K$ such that for any $\alpha \in \Phi$, $\alpha(X) \neq 0$. This implies that X is regular in $\mathfrak{t}$, and also regular in $\mathfrak{t}_K$, by the description of K-roots given in the above lemma. We define the set Φ^+ (resp. Φ_K^+) of positive roots of G (resp. K) by the condition

$$\theta \in \Phi^+ \text{ (resp. } \Phi_K^+) \iff \theta(X) > 0\,.$$

Note that as $X \in \mathfrak{t}_K$, one has $\theta \in \Phi^+ \iff \sigma^*(\theta) \in \Phi^+$, and that by the above considerations

$$\Phi_K^+ = \{\theta_{|\mathfrak{t}_K};\, \theta \in \Phi_1^+ \cup \Phi_3'^+\}\,. \tag{3}$$

Note furthermore that, by the description of positive G-roots, the half-sum δ_G of the positive roots verifies

$$\sigma^*(\delta_G) = \delta_G\,.$$

2.3. **Weights of the spin representation of K.** The decomposition of the spin representation depends on the parity of $\dim \mathfrak{p}_{\mathbb{C}}$, so we begin by considering the even dimensional case.

2.3.1. *First case :* $\dim \mathfrak{p}_{\mathbb{C}}$ *even.* As it will be seen below, this amounts to suppose that $\dim \mathfrak{t}_0$ is even. Setting $\dim \mathfrak{t}_0 = 2r_0$, we consider an orthonormal basis (T_k), $k = 1, \dots 2r_0$, of $\mathfrak{t}_0$, and construct the Witt basis $(Z_k, \overline{Z_k})$, $k = 1, \dots r_0$, defined by

$$Z_k := \frac{1}{2}\,(T_{2k-1} + i\,T_{2k})\,, \quad \text{and} \quad \overline{Z_k} := \frac{1}{2}\,(T_{2k-1} - i\,T_{2k})\,.$$

For the $\mathbb{C}$-linear extension $\langle\ ,\ \rangle$ of the scalar product on $\mathfrak{g}$, one has

$$\langle Z_i, Z_j\rangle = \langle \overline{Z_i}, \overline{Z_j}\rangle = 0\,, \quad \text{and} \quad \langle Z_i, \overline{Z_j}\rangle = \frac{1}{2}\,\delta_{ij}\,.$$

Let $\Phi_1^+ = \{\alpha_1, \dots, \alpha_\ell\}$, $\Phi_2^+ = \{\beta_1, \dots, \beta_p\}$ and $\Phi_3'^+ = \{\gamma_1, \dots, \gamma_q\}$.

Lemma 2.3. *Vectors E_θ, $E_{-\theta}$, $\theta = \beta_1, \dots, \beta_p$, V_θ, $V_{-\theta}$, $\theta = \gamma_1 \dots, \gamma_q$, may be choosen such that, with Z_i, $\overline{Z_i}$, $i = 1, \dots, r_0$, they define a Witt basis of $\mathfrak{p}_{\mathbb{C}}$, in the sense that*

$$\begin{aligned}\langle E_\alpha, E_\beta\rangle = \langle E_{-\alpha}, E_{-\beta}\rangle = \langle E_\alpha, V_\beta\rangle = \langle E_{-\alpha}, V_{-\beta}\rangle = \langle V_\alpha, V_\beta\rangle = \langle V_{-\alpha}, V_{-\beta}\rangle\,,\\ = \langle Z_i, Z_j\rangle = \langle \overline{Z_i}, \overline{Z_j}\rangle = 0\,,\end{aligned}$$

and

$$\langle E_\alpha, E_{-\beta}\rangle = \frac{1}{2}\,\delta_{\alpha\beta}\,, \quad \langle V_\alpha, V_{-\beta}\rangle = \frac{1}{2}\,\delta_{\alpha\beta}\,, \quad \langle Z_i, \overline{Z_j}\rangle = \frac{1}{2}\,\delta_{ij}\,. \tag{4}$$

Proof. Clearly, any such vectors span $\mathfrak{p}_{\mathbb{C}}$: any $X \in \mathfrak{p}_{\mathbb{C}}$ may be written as

$$\begin{aligned}H + \sum_{\alpha\in\Phi_1} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_2} \lambda_\alpha\, E_\alpha + \sum_{\alpha\in\Phi_3'} \big(\lambda_\alpha + \lambda_{\sigma^*(\alpha)}\big)\, U_\alpha\\ + \sum_{\alpha\in\Phi_3'} \big(\lambda_\alpha - \lambda_{\sigma^*(\alpha)}\big)\, V_\alpha\,,\end{aligned}$$

with $H \in \mathfrak{t}_{\mathbb{C}}$ and $\lambda_\alpha \in \mathbb{C}$. The condition $\sigma_*(X) = -X$ implies $H \in \mathfrak{t}_{0,\mathbb{C}}$ and $\forall \alpha \in \Phi_1$, $\lambda_\alpha = 0$, and $\forall \alpha \in \Phi_3'$, $\lambda_\alpha = -\lambda_{\sigma^*(\alpha)}$, hence

$$X = H + \sum_{\alpha\in\Phi_2^+} \lambda_\alpha E_\alpha + \sum_{\alpha\in\Phi_2^+} \lambda_{-\alpha} E_{-\alpha} + 2\sum_{\alpha\in\Phi_3'^+} \lambda_\alpha V_\alpha + 2\sum_{\alpha\in\Phi_3'^+} \lambda_{-\alpha} V_{-\alpha}\,,$$

so X is a linear combination of the considered vectors.
On the other hand, it is well-known that, for any couple of roots (α, β) such that $\alpha + \beta \neq 0$, $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = 0$. Hence

- if α and $\beta \in \Phi_2$ are such that $\alpha + \beta \neq 0$, then $\langle E_\alpha, E_\beta\rangle = \langle E_{-\alpha}, E_{-\beta}\rangle = 0$.
- if $\alpha \in \Phi_2$ and $\beta \in \Phi_3$, then from $\langle E_\alpha, E_\beta\rangle = 0$, one obtains as $\langle E_\alpha, U_\beta\rangle = 0$, $\langle E_\alpha, V_\beta\rangle = 0$, and then $\langle E_{-\alpha}, V_{-\beta}\rangle = 0$.
- if $\alpha \in \Phi_3'$ and $\beta \in \Phi_3'$ are such that $\alpha + \beta \neq 0$, then $\alpha' + \beta' \neq 0$ (otherwise by **R5**, $\beta = \sigma^*(-\alpha)$, which is impossible), so $\langle U_\alpha, U_\beta\rangle = 0$, and then $\langle V_\alpha, V_\beta\rangle = 0$, (so $\langle V_{-\alpha}, V_{-\beta}\rangle = 0$ also).

Finally, as $\langle\ ,\ \rangle$ is non-degenerate, $(E_\alpha, E_{-\alpha}) \neq 0$, $\alpha \in \Phi_2^+$, and $(V_\alpha, V_{-\alpha}) \neq 0$, $\alpha \in \Phi_3'^+$, and we may choose the basis (E_α) such that (4) is verified. All of those orthogonality relations imply that the considered vectors are linearly independent. □

Since $E_{-\alpha}$ belongs to $\overline{\mathfrak{g}_\alpha}$, we may suppose that $E_{-\beta_j} = \overline{E_{\beta_j}}$ and $V_{-\gamma_j} = \overline{V_{\gamma_j}}$.
In the Clifford algebra $\mathbb{C}\ell(\mathfrak{p})$, let

$$\overline{w} := \overline{E_{\beta_1}}\cdots\overline{E_{\beta_p}}\cdot\overline{V_{\gamma_1}}\cdots\overline{V_{\gamma_q}}\cdot\overline{Z_1}\cdots\overline{Z_{r_0}}\,.$$

For any $I = \{i_1, \dots, i_a\} \subset \{1, \dots, p\}$, $i_1 < i_2 < \cdots < i_a$, (resp. $J = \{j_1, \dots, j_b\} \subset \{1, \dots, q\}$, $j_1 < j_2 < \cdots < j_b$; resp. $K = \{k_1, \dots, k_c\} \subset \{1, \dots, r_0\}$, $k_1 < k_2 < \cdots < k_c$), we introduce the notation

$$E_I \cdot V_J \cdot Z_K := E_{\beta_{i_1}} \cdots E_{\beta_{i_a}} \cdot V_{\gamma_{j_1}} \cdots V_{\gamma_{j_b}} \cdot Z_{k_1} \cdots Z_{k_c}\,,$$

setting E_I (resp. V_J, resp. Z_K) $= 1$, if I (resp. J, resp. K) $= \emptyset$. As I (resp. J, resp. K) runs through the set of subsets of $\{1, \dots, p\}$, (resp. $\{1, \dots, q\}$, resp. $\{1, \dots, r_0\}$), the vectors $E_I \cdot V_J \cdot Z_K \cdot \overline{w}$ define a basis of the spinor space $\Sigma := \mathbb{C}\ell(\mathfrak{p})\cdot\overline{w}$ (cf. [HBM+15]).
Considering

$$(5)\qquad \begin{cases} X_j &= E_{\beta_j} + \overline{E_{\beta_j}}\,, \\ Y_j &= i\,\left(E_{\beta_j} - \overline{E_{\beta_j}}\right)\,, \\ & 1 \le j \le p\,, \end{cases} \quad \text{and} \quad \begin{cases} F_j &= V_{\gamma_j} + \overline{V_{\gamma_j}}\,, \\ G_j &= i\,\left(V_{\gamma_j} - \overline{V_{\gamma_j}}\right)\,, \\ & 1 \le j \le q\,, \end{cases}$$

one obtains an orthonormal basis $(X_i, Y_i, F_j, Y_j, T_k)_{\substack{1\le i\le p,\\ 1\le j\le q,\\ 1\le k\le 2r_0}}$ of $\mathfrak{p}$.

Then for any $X \in \mathfrak{t}_K$, one has

$$\begin{cases} [X, X_j] &= -i\,\beta_j(X)\,Y_j\,, \\ [X, Y_j] &= i\,\beta_j(X)\,X_j\,, \\ & 1 \le j \le p\,, \end{cases} ; \begin{cases} [X, F_j] &= -i\,\gamma_j(X)\,G_j\,, \\ [X, G_j] &= i\,\gamma_j(X)\,F_j\,, \\ & 1 \le j \le q\,, \end{cases} \text{ and } \begin{cases} [X, T_j] = 0\,, \\ 1 \le j \le 2r_0\,. \end{cases}$$

So

$$\forall X \in \mathfrak{t}_K\,,\quad \alpha_*(X) = -i\sum_{j=1}^{p} \beta_j(X)\,X_j \wedge Y_j - i\sum_{j=1}^{q} \gamma_j(X)\,F_j \wedge G_j\,,$$

hence

$$\forall X \in \mathfrak{t}_K\,, \quad \widetilde{\alpha}_*(X) = -\frac{1}{2}\, i \sum_{j=1}^{p} \beta_j(X)\, X_j \cdot Y_j - \frac{1}{2}\, i \sum_{j=1}^{q} \gamma_j(X)\, F_j \cdot G_j\,.$$

Now it is easy to verify that

$$X_k \cdot Y_k \cdot E_I \cdot V_J \cdot Z_K \cdot \overline{w} = \begin{cases} -i\, E_I \cdot V_J \cdot Z_K \cdot \overline{w} & \text{if } k \notin I, \\ i\, E_I \cdot V_J \cdot Z_K \cdot \overline{w} & \text{if } k \in I, \end{cases}$$

$$F_k \cdot G_k \cdot E_I \cdot V_J \cdot Z_K \cdot \overline{w} = \begin{cases} -i\, E_I \cdot V_J \cdot Z_K \cdot \overline{w} & \text{if } k \notin J, \\ i\, E_I \cdot V_J \cdot Z_K \cdot \overline{w} & \text{if } k \in J. \end{cases}$$

Hence

$$\forall X \in \mathfrak{t}_K\,, \quad \widetilde{\rho_K}_*(X)(E_I \cdot V_J \cdot Z_K \cdot \overline{w}) = \\ \frac{1}{2}\left(\sum_{i\in I} \beta_i(X) - \sum_{i\notin I} \beta_i(X) + \sum_{j\in J} \gamma_j(X) - \sum_{j\notin J} \gamma_j(X)\right)\, E_I \cdot V_J \cdot Z_K \cdot \overline{w}\,.$$

So the $E_I \cdot V_J \cdot Z_K \cdot \overline{w}$ are weight-vectors, and a $\mu \in i\,\mathfrak{t}_K^*$ is a weight if and only if it can be expressed as

$$\frac{1}{2}\left(\pm\beta'_1 \pm \cdots \pm \beta'_p \pm \gamma'_1 \pm \cdots \pm \gamma'_q\right)\,. \tag{6}$$

Note that as K runs through the set of subsets of $\{1, \ldots, r_0\}$, all of the $E_I \cdot V_J \cdot Z_K \cdot \overline{w}$ for a given I and J, belong to same weight-space, hence the multiplicity of a weight of the form (6) is 2^{r_0} times the number of ways in which it can be expressed in the given form.
Considering the volume element

$$\omega := i^{p+q+r_0}\, X_1 \cdot Y_1 \cdots X_p \cdot Y_p \cdot F_1 \cdot G_1 \cdots F_q \cdot G_q \cdot T_1 \cdot T_2 \cdots T_{2r_0-1} \cdot T_{2r_0}\,,$$

one has

$$\omega \cdot E_I \cdot V_J \cdot Z_K \cdot \overline{w} = (-1)^{\#I+\#J+\#K}\, E_I \cdot V_J \cdot Z_K \cdot \overline{w}\,,$$

hence the spinor space Σ decomposes into two irreducible components

$$\Sigma = \Sigma^+ \oplus \Sigma^-\,,$$

where Σ^+ (resp. Σ^-) is the linear span of the $E_I \cdot V_J \cdot Z_K \cdot \overline{w}$'s such that $\#I + \#J + \#K$ is even (resp. odd).

2.3.2. *Second case :* $\dim \mathfrak{p}_{\mathbb{C}}$ *odd.* This corresponds to the case $\dim \mathfrak{t}_0$ odd. Setting $n = \dim \mathfrak{p}$, by the choice of an orthonormal basis such as (5), $\mathrm{SO}(\mathfrak{p})$ is identified with $\mathrm{SO}(n)$, which is itself embedded in $\mathrm{SO}(n+1)$ in such a way that $\mathrm{SO}(n)$ acts trivially on the last vector e_{n+1} of the standard basis of $\mathbb{R}^{n+1}$.
Setting $\dim \mathfrak{t}_0 = 2r_0 - 1$, $r_0 \geq 1$, the spinor space can be described (cf. [HBM+15]), as the space of positive spinors introduced above in the even-dimensional case : $\Sigma = \mathrm{Span}\{E_I \cdot V_J \cdot Z_K \cdot \overline{w},\ \#I + \#J + \#K \text{ even}\}$. By the result above, the weights may be expressed as (6), but now, the multiplicity of such a weight is 2^{r_0-1} times the number of ways in which it can be expressed in the given form.

2.3.3. *A remark on highest weights.*

Lemma 2.4. *Any highest weight of the spinor representation of K has necessarily the form*

$$\frac{1}{2}\left(\pm\beta'_1 \pm \cdots \pm \beta'_p + \gamma'_1 + \cdots + \gamma'_q\right) . \tag{7}$$

Proof. Let μ be a weight such that for a $j_0 \in \{1, \dots, q\}$, γ'_{j_0} appears with a minus sign in the expression of μ. Then $\mu + \gamma'_{j_0}$ is also a weight, hence μ can not be dominant, since γ'_{j_0} is a positive K-root. □

3. The symmetric space $\dfrac{\mathrm{SU}(2m)}{\mathrm{SO}(2m)}$.

In the following, $M_{2m}(\mathbb{C})$ denotes the space of $2m \times 2m$ matrices with complex coefficients, and (E_{ij}) its standard basis. Here

$$G = \mathrm{SU}(2m) = \{A \in M_{2m}(\mathbb{C}), {}^t\overline{A}A = I_{2m}\} ,$$

and

$$K = \{A \in \mathrm{SU}(2m) \; ; \; \overline{A} = A\} = \mathrm{SO}(2m) .$$

The symmetric space structure is given by the involution $\sigma : G \to G$, $A \mapsto {}^tA^{-1}$. This involution induces the decomposition of the Lie algebra $\mathfrak{su}_{2m}$ of $\mathrm{SU}(2m)$ into:

$$\mathfrak{su}_{2m} = \mathfrak{so}_{2m} \oplus \mathfrak{p} ,$$

where $\mathfrak{so}_{2m}$ is the Lie algebra $\{X \in \mathfrak{su}_{2m} \; ; \; {}^tX = -X\}$ of $\mathrm{SO}(2m)$, and

$$\mathfrak{p} = \{X \in \mathfrak{su}_{2m} \; ; \; {}^tX = X\} .$$

Elements of $\mathfrak{p}$ are traceless symmetric $2m \times 2m$ matrices with coefficients in $i\,\mathbb{R}$, hence $\dim(\mathfrak{p}) = 2m^2 + m - 1$.

Let $T_s = \left\{ \begin{pmatrix} \mathrm{e}^{i\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\theta_{2m}} \end{pmatrix} \; ; \; \theta_i \in \mathbb{R} \; , \; \sum_{i=1}^{2m} \theta_i = 0 \right\}$ be the standard maximal torus of $\mathrm{SU}(2m)$. We consider a conjugate T of T_s in G in such a way that $T \cap K$ is the standard maximal torus of K. For that, let $A_0 \in \mathrm{SU}(2m)$ be the 2×2 block diagonal matrix defined by

$$A_0 = \frac{\sqrt{2}}{2} \begin{pmatrix} \square & & \\ & \ddots & \\ & & \square \end{pmatrix} , \quad \text{where } \square = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} .$$

Then for any $H = \begin{pmatrix} \mathrm{e}^{i\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{i\theta_{2m}} \end{pmatrix} \in T_s$, $A_0 H A_0^{-1} = \begin{pmatrix} \square_1 & & \\ & \ddots & \\ & & \square_m \end{pmatrix}$, where

$$\square_j = \mathrm{e}^{\frac{1}{2}i\,(\theta_{2j-1}+\theta_{2j})} \begin{pmatrix} \cos\left(\frac{1}{2}(\theta_{2j-1} - \theta_{2j})\right) & \sin\left(\frac{1}{2}(\theta_{2j-1} - \theta_{2j})\right) \\ -\sin\left(\frac{1}{2}(\theta_{2j-1} - \theta_{2j})\right) & \cos\left(\frac{1}{2}(\theta_{2j-1} - \theta_{2j})\right) \end{pmatrix} .$$

Hence $A_0 H A_0^{-1} \in K \Longleftrightarrow A_0 H A_0^{-1} = \begin{pmatrix} \square'_1 & & \\ & \ddots & \\ & & \square'_m \end{pmatrix}$, where

$$\square'_j = \begin{pmatrix} \cos(\theta_{2j}) & -\sin(\theta_{2j}) \\ \sin(\theta_{2j}) & \cos(\theta_{2j}) \end{pmatrix},$$

so $A_0 H A_0^{-1} \in K$ if and only if it belongs to the standard maximal torus T_K of K. The Lie algebra $\mathfrak{t}$ of T is defined by

$$\mathfrak{t} = \Big\{ \sum_{j=1}^{m} i\gamma_{2j-1} \left(E_{2j-1\,2j-1} + E_{2j\,2j}\right) + \gamma_{2j} \left(E_{2j\,2j-1} - E_{2j-1\,2j}\right) ; \\ \gamma_j \in \mathbb{R}, \sum_{j=1}^{m} \gamma_{2j-1} = 0 \Big\},$$

and

$$\mathfrak{t}_K = \left\{ \sum_{j=1}^{m} \gamma_{2j} \left(E_{2j\,2j-1} - E_{2j-1\,2j}\right) ; \gamma_{2j} \in \mathbb{R} \right\},$$

$$\mathfrak{t}_0 = \left\{ \sum_{j=1}^{m} i\gamma_{2j-1} \left(E_{2j-1\,2j-1} + E_{2j\,2j}\right), \gamma_{2j-1} \in \mathbb{R}, \sum_{j=1}^{m} \gamma_{2j-1} = 0 \right\}.$$

It is useful to consider the elements $\widehat{x}_j$, $j = 1, \ldots, 2m$ of $i\,\mathfrak{t}^*$, defined by considering the restriction to $\mathfrak{t}$ of the dual basis of the family of vectors $i\,(E_{2j-1\,2j-1} + E_{2j\,2j})$, $(E_{2j\,2j-1} - E_{2j-1\,2j})$, $j = 1, \ldots, m$:

$$\forall H = \sum_{j=1}^{m} i\gamma_{2j-1} \left(E_{2j-1\,2j-1} + E_{2j\,2j}\right) + \gamma_{2j} \left(E_{2j\,2j-1} - E_{2j-1\,2j}\right) \in \mathfrak{t},$$
$$\widehat{x}_{2j-1}(H) = i\,\gamma_{2j-1} \quad \text{and} \quad \widehat{x}_{2j}(H) = i\,\gamma_{2j}\,.$$

The scalar product on $i\,\mathfrak{t}^*$ is given by the scalar product on $\mathfrak{su}_{2m}$ defined by

$$\langle X, Y \rangle = -\frac{1}{2}\,\Re\big(\mathrm{Tr}(XY)\big) = -\frac{1}{8m}\mathrm{B}(X, Y), \quad X, Y \in \mathfrak{su}_{2m}\,.$$

Lemma 3.1. *Each $\mu \in i\,\mathfrak{t}^*$ may be uniquely written as*

$$\mu = \sum_{i=1}^{2m} \mu_i\,\widehat{x}_i\,, \text{ where } \mu_i \in \mathbb{R} \text{ verify } \sum_{i=1}^{m} \mu_{2i-1} = 0, \tag{8}$$

and, for any $\mu = \sum_{i=1}^{2m} \mu_i\,\widehat{x}_i$ and $\mu' = \sum_{i=1}^{2m} \mu'_i\,\widehat{x}_i \in i\,\mathfrak{t}^*$,

$$\langle \mu, \mu' \rangle = \sum_{i=1}^{2m} \mu_i\,\mu'_i\,.$$

Proof. As the $\widehat{x}_{2j-1}$, $j = 1, \ldots, m-1$ and $\widehat{x}_{2j}$, $1, \ldots, m$ define a basis of $\mathfrak{t}^*$, any $\mu \in \mathfrak{t}^*$ is uniquely written as $\mu = \sum_{i=1}^{m-1} \mu_{2i-1}\,\widehat{x}_{2i-1} + \sum_{i=1}^{m} \mu_{2i}\,\widehat{x}_{2j}$, $\mu_i \in \mathbb{R}$. Now since $\widehat{x}_{2m-1} = -\sum_{i=1}^{m-1} \widehat{x}_{2i-1}$, setting $\mu_{2m-1} = -\frac{1}{m}\,\sum_{i=1}^{m-1} \mu_{2j-1}$, one gets

$$\mu = \sum_{i=1}^{m-1} (\mu_{2i-1} + \mu_{2m-1})\,\widehat{x}_{2i-1} + \mu_{2m-1}\,\widehat{x}_{2m-1} + \sum_{i=1}^{m} \mu_{2i}\,\widehat{x}_{2i}\,,$$

so μ may be written as (8). Now if $\mu = \sum_{i=1}^{2m} \lambda_i \, \widehat{x}_i$, where $\lambda_i \in \mathbb{R}$ verify $\sum_{i=1}^{m} \lambda_{2i-1} = 0$, then $\mu = \sum_{i=1}^{m-1} (\lambda_{2i-1} - \lambda_{2m-1}) \, \widehat{x}_{2i-1} + \sum_{i=1}^{m} \lambda_{2i} \, \widehat{x}_{2i}$, and so

$$\mu_{2i-1} = \lambda_{2i-1} - \lambda_{2m-1} \, , \; i = 1, \dots, m-1 \, , \quad \text{and} \quad \mu_{2i} = \lambda_{2i} \, , \; i = 1, \dots, m \, .$$

Then $\mu_{2m-1} = -\frac{1}{m} \sum_{i=1}^{m-1} \mu_{2i-1} = \lambda_{2m-1}$, hence $\mu_{2i-1} + \mu_{2m-1} = \mu_{2i-1} + \lambda_{2m-1} = \lambda_{2i-1}$, hence the unicity of the writting (8).
The last result follows from the fact that the vectors $E_{2j-1\,2j-1} + E_{2j\,2j}$, $E_{2j\,2j-1} - E_{2j-1\,2j}$, $j = 1, \dots, m$, define an orthonormal set for the scalar product $\langle \, , \, \rangle$. $\square$
In the following, any $\mu \in i\,\mathfrak{t}^*$, of the form (8) will be denoted

$$\mu = [\mu_1, \mu_3, \dots, \mu_{2m-1}; \mu_2, \mu_4, \dots, \mu_{2m}] \, , \quad \sum_{i=1}^{m} \mu_{2i-1} = 0 \, .$$

Note that the $\widehat{x}'_{2i}$, $1 \leq i \leq m$, define a basis of $i\,\mathfrak{t}_K^*$. Any $\lambda \in i\,\mathfrak{t}_K^*$ of the form $\lambda = \sum_{i=1}^{m} \lambda_{2i} \, \widehat{x}'_{2i}$, $\lambda_{2i} \in \mathbb{R}$, will be denoted

$$\lambda = (\lambda_2, \lambda_4, \dots, \lambda_{2m}) \, .$$

The involution σ^* of $i\mathfrak{t}^*$ induced by σ is defined by

$$[\mu_1, \mu_3, \dots, \mu_{2m-1}; \mu_2, \mu_4, \dots, \mu_{2m}] \xrightarrow{\sigma^*} [-\mu_1, -\mu_3, \dots, -\mu_{2m-1}; \mu_2, \mu_4, \dots, \mu_{2m}] \, .$$

3.1. **Sets of roots.** Since the root-vectors relative to the standard torus T_s are the E_{ij}, the root-vectors relative to the torus T are given by the $A_0 E_{ij} A_0^{-1}$. Explicitly, the roots are

$$\begin{aligned}
&(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} - \widehat{x}_{2j}) \, , \; 1 \leq i \neq j \leq m \, , \quad \text{root-space} : \mathbb{C}\, A_0 E_{2i-1\,2j-1} A_0^{-1}, \\
&(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) - (\widehat{x}_{2i} - \widehat{x}_{2j}) \, , \; 1 \leq i \neq j \leq m \, , \quad \text{root-space} : \mathbb{C}\, A_0 E_{2i\,2j} A_0^{-1}, \\
&(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} + \widehat{x}_{2j}) \, , \; 1 \leq i, j \leq m \, , \quad \text{root-space} : \mathbb{C}\, A_0 E_{2i-1\,2j} A_0^{-1}, \\
&(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) - (\widehat{x}_{2i} + \widehat{x}_{2j}) \, , \; 1 \leq i, j \leq m \, , \quad \text{root-space} : \mathbb{C}\, A_0 E_{2i\,2j-1} A_0^{-1}.
\end{aligned}$$

Note that the only roots that verify $\sigma^*(\theta) = \theta$ are $\pm 2\, \widehat{x}_{2i}$, and, as they are not K-roots, one may conclude from lemma 2.1 that

$$\Phi_1 = \emptyset \, , \quad \Phi_2 = \{ \pm 2\, \widehat{x}_{2i} \, , \, i = 1, \dots, m \} \, ,$$

and all other roots belong to Φ_3.
We choose positive roots such that

$$\begin{aligned}
\Phi_2^+ &= \{ 2\, \widehat{x}_{2i} \, , \; i = 1, \dots, m \} \, , \\
\Phi_3'^+ &= \left\{ \begin{array}{l} (\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} - \widehat{x}_{2j}) \, , \quad 1 \leq i < j \leq m \, , \\ (\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} + \widehat{x}_{2j}) \, , \quad 1 \leq i < j \leq m \, , \end{array} \right\} , \\
\sigma^*(\Phi_3'^+) &= \left\{ \begin{array}{l} -(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} - \widehat{x}_{2j}) \, , \quad 1 \leq i < j \leq m \, , \\ -(\widehat{x}_{2i-1} - \widehat{x}_{2j-1}) + (\widehat{x}_{2i} + \widehat{x}_{2j}) \, , \quad 1 \leq i < j \leq m \end{array} \right\} .
\end{aligned}$$

It is easy to verify that a system of simple roots is given by

$$\begin{aligned}
\beta &:= 2\, \widehat{x}_{2m} \, , \\
\gamma_i &:= (\widehat{x}_{2i-1} - \widehat{x}_{2i+1}) + (\widehat{x}_{2i} - \widehat{x}_{2(i+1)}) \, , \quad i = 1, \dots, m-1 \, , \\
\sigma^*(\gamma_i) &= -(\widehat{x}_{2i-1} - \widehat{x}_{2i+1}) + (\widehat{x}_{2i} - \widehat{x}_{2(i+1)}) \, , \quad i = 1, \dots, m-1 \, .
\end{aligned}$$

Note that

$$\begin{aligned}\langle\gamma_i,\gamma_i\rangle &= \langle\sigma^*(\gamma_i),\sigma^*(\gamma_i)\rangle = \langle\beta,\beta\rangle = 4\,,\quad i=1,\dots,m-1\,,\\ \langle\gamma_i,\gamma_{i+1}\rangle &= \langle\sigma^*(\gamma_i),\sigma^*(\gamma_{i+1})\rangle = -2\,,\ i=1,\dots,m-2\,,\\ \langle\gamma_{m-1},\beta\rangle &= \langle\sigma^*(\gamma_{m-1}),\beta\rangle = -2,\ ,\end{aligned}$$

all the other scalar products being zero. Hence

Lemma 3.2. *A vector $\mu=[\mu_1,\mu_3,\dots,\mu_{2m-1};\mu_2,\mu_4,\dots,\mu_{2m}]\in i\,\mathfrak{t}^*$, $\sum_{i=1}^m\mu_{2i-1}=0$, is a dominant weight if and only if*

$$\left.\begin{array}{l}\mu_{2i}-\mu_{2(i+1)}\in\mathbb{N}\,,\\ \mu_{2i-1}-\mu_{2i+1}\in\mathbb{Z}\,,\end{array}\right\}\ \textit{both simultaneously odd or even,}$$
$$|\mu_{2i-1}-\mu_{2i+1}|\le\mu_{2i}-\mu_{2(i+1)}\,,\quad 1\le i\le m-1\,,$$
$$\mu_{2m}\in\mathbb{N}\,.$$

Proof. Note just that the conditions $2\,\frac{\langle\gamma_i,\mu\rangle}{\langle\gamma_i,\gamma_i\rangle}\in\mathbb{N}$, $2\,\frac{\langle\sigma^*(\gamma_i),\mu\rangle}{\langle\gamma_i,\gamma_i\rangle}\in\mathbb{N}$, $i=1,\dots m-1$, are equivalent to

$$\frac{(\mu_{2i-1}-\mu_{2i+1})+(\mu_{2i}-\mu_{2(i+1)})}{2}\in\mathbb{N}\,,$$

and

$$\frac{-(\mu_{2i-1}-\mu_{2i+1})+(\mu_{2i}-\mu_{2(i+1)})}{2}\in\mathbb{N}\,,$$

which imply (and are equivalent to) the first three conditions of the lemma. □

As it was remarked before, the positive K-roots are given by considering the restrictions to $\mathfrak{t}_K$ of the positive roots in Φ_3', so

$$\Phi_K^+=\left\{\widehat{x}_{2i}'-\widehat{x}_{2j}'\,,\,\widehat{x}_{2i}'+\widehat{x}_{2j}'\,,\,1\le i<j\le m\right\}\,.$$

A set of simple roots is given by

$$\begin{aligned}\theta_i' &= \widehat{x}_{2i}'-\widehat{x}_{2(i+1)}'\,,\quad i=1,\dots,m-1\,,\\ \theta_m' &= \widehat{x}_{2(m-1)}'-\widehat{x}_{2m}'\,.\end{aligned}$$

Note that

$$\begin{aligned}\langle\theta_i',\theta_i'\rangle &= 2\,,\quad \langle\theta_i',\theta_{i+1}'\rangle=-1\,,\quad 1\le i\le m-2\,,\\ \langle\theta_{m-2}',\theta_m'\rangle &= -1\,,\quad \langle\theta_{m-1}',\theta_m'\rangle=0\,,\end{aligned}$$

hence the following "classical" characterization :

Lemma 3.3. *A vector $\lambda=(\lambda_2,\lambda_4,\dots,\lambda_{2m})\in i\,\mathfrak{t}_K^*$, is a dominant weight if and only if*

$$\lambda_2\ge\lambda_4\ge\cdots\ge\lambda_{2(m-1)}\ge|\lambda_{2m}|\,,$$

and the λ_{2i} are all simultaneously integers or half-integers.

3.2. **Highest weights of the spin representation of K.**

Proposition 3.4. *The spin representation of K has two highest weights:*

$$(m, m-1, \dots, 2, \pm 1)\,,$$

both with multiplicity $2^{[\frac{m-1}{2}]}$.

Proof. One has $\frac{1}{2}\sum_{\gamma\in\Phi_3'^+}\gamma' = \sum_{1\leq i<j\leq m}\widehat{x}'_{2i} = \sum_{i=1}^{m-1}(m-i)\,\widehat{x}'_{2j}$. Hence by the result of lemma 2.4, any highest weight λ has necessarily the form

$$\lambda = \sum_{i=1}^{m}(m-i\pm 1)\,\widehat{x}'_{2i}\,.$$

But the dominance condition of lemma 3.3 implies then

$$\lambda = \sum_{i=1}^{m-1}(m-i+1)\,\widehat{x}'_{2i} \pm \widehat{x}'_{2m} = (m, m-1, \dots, 2, \pm 1)\,.$$

Denote by $\lambda_\pm$ those two dominant weights. Let δ_K be the half-sum of the positive K-roots. One has $\delta_K = (m-1, m-2, \dots, 1, 0)$, hence

$$\langle \lambda_+ + \delta_K, \lambda_+ + \delta_K\rangle = \langle \lambda_- + \delta_K, \lambda_- + \delta_K\rangle\,.$$

Hence the two weights are both highest weights, since otherwise, one of the two should be contained in the set of weights of an irreducible representation having the other one as highest weight, and the above equality should be impossible (cf. lemma C, 13.4 in [Hum72]).
With the help of the Weyl dimension formula, it may be checked that any irreducible module with highest weight $\lambda_\pm$ has dimension $2^{(m-1)(m+1)}$. As we noticed it before, the multiplicity of each one of those two weights is at least $2^{\frac{m-1}{2}}$ if m is odd, and $2^{\frac{m-2}{2}}$ if m is even. Since if m is odd,

$$2\times 2^{\frac{m-1}{2}}\times 2^{(m-1)(m+1)} = 2^{\frac{2m^2+m-1}{2}} = \dim(\Sigma)\,,$$

and if m is even,

$$2\times 2^{\frac{m-2}{2}}\times 2^{(m-1)(m+1)} = 2^{\frac{2m^2+m-2}{2}} = \dim(\Sigma)\,,$$

one concludes that the multiplicity of each weight is exactly $2^{\left[\frac{m-1}{2}\right]}$. □

3.3. **The first eigenvalue of the Dirac operator.** Recall that the first eigenvalue of the Dirac operator is given by the lowest Casimir eigenvalue c_γ, among the irreducible representations γ of G verifying the "spin condition" : $\mathrm{Res}^G_K(\gamma)$ contains an irreducible component of the spin representation $\widetilde{\rho_K}$ of K. By the Freudenthal formula, the Casimir eigenvalue of an irreducible representation with highest weight μ_γ is given by

$$c_\gamma = \langle \mu_\gamma, \mu_\gamma + 2\,\delta_G\rangle = \|\mu_\gamma + \delta_G\|^2 - \|\delta_G\|^2\,.$$

Hence we look for G-dominant weights μ_γ, verifying the spin condition and such that $\|\mu_\gamma + \delta_G\|^2$ is minimal.
We first determine the G-weights μ (non necessarily dominant) for which $\mu_{|\mathfrak{t}_K} = \lambda_\pm$ and $\|\mu + \delta_G\|^2$ is minimal.
Note first that $\mu = [\mu_1, \mu_3, \dots, \mu_{2m-1}; \mu_2, \mu_4, \dots, \mu_{2m}]$, $\sum_{i=1}^m \mu_{2i-1} = 0$, is a G-weight if and only if $\mu_{2i-1} - \mu_{2i+1} \in \mathbb{Z}$, $i = 1, \dots m-1$, $\mu_{2i} \in \mathbb{Z}$, $i = 1, \dots, m$, and $\mu_{2i-1} - \mu_{2i+1}$, $\mu_{2i} - \mu_{2(i+1)}$ are both simultaneously even or odd, $i = 1, \dots m-1$.

Let μ be such a G-weight. It verifies $\mu_{|\mathfrak{t}_K} = \lambda_\pm$ if and only if it has the form

$$\mu = [\mu_1, \mu_3, \dots, \mu_{2m-1}; m, m-1, \dots, 2, \pm 1]\,,$$

where $\sum_{i=1}^m \mu_{2i-1} = 0$, and $\mu_{2i-1} - \mu_{2i+1}$ are odd integers, $i = 1, \dots m-1$.
The half-sum of the positive G-roots δ_G is given by

$$\begin{aligned}\delta_G &= 2 \sum_{1\le k<l\le m} \widehat{x}_{2k} + \sum_{k=1}^m \widehat{x}_{2k} = 2 \sum_{k=1}^m \left(m-k+\frac{1}{2}\right) \widehat{x}_{2k}\,,\\ &= [0, 0, \dots, 0; (2m-1), (2m-3), \dots, 1]\,.\end{aligned}$$

Hence $\|\mu + \delta_G\|^2$ is minimal if and only if $\sum_{i=1}^m \mu_{2i-1}^2$ is minimal.
Set $\mu_{2i-1} - \mu_{2i+1} = k_i$, $i = 1, \dots, m-1$. For any $j = 1, \dots, m-1$, one has

$$\begin{aligned}p_j := \sum_{i=1}^j k_i &= \mu_1 + \sum_{i=1}^{j-1} \mu_{2i+1} - \sum_{i=1}^{j-1} \mu_{2i+1} - \mu_{2j+1}\,,\\ &= \mu_1 - \mu_{2j+1}\,.\end{aligned}$$

Using $\sum_{i=1}^m \mu_{2i-1} = 0$, one then gets

$$\sum_{j=1}^{m-1} p_j = (m-1)\,\mu_1 + \mu_1 = m\,\mu_1\,,$$

hence

$$\mu_1 = \frac{1}{m} \sum_{i=1}^{m-1} p_i\,,$$

and

$$\mu_{2j+1} = \mu_1 - p_j = \frac{1}{m} \left(\sum_{i=1}^{m-1} p_i - m\,p_j \right)\,.$$

The expression $\sum_{i=1}^m \mu_{2i-1}^2$ is a polynomial $F(p_1, \dots, p_{m-1})$ of the variables p_1, $p_2 \dots, p_{m-1}$. With the notation $p = (p_1, \dots, p_{m-1})$, one has

$$\frac{\partial F}{\partial p_i}(p) = -2\,\mu_{2i+1}\,.$$

With no surprise, F as only one critical point at $(0, \dots, 0)$.
Now

$$\frac{\partial^2 F}{\partial p_i \partial p_j}(0) = -\frac{2}{m}\,(1 - m\,\delta_{ij})\,.$$

Denote by H the Hessian matrix of F at 0. It has for eigenvalues $\frac{2}{m}$ with multiplicity one and 2 with multiplicity $m-2$, and the following vectors define an orthogonal basis of eigenvectors.

$$\begin{aligned}v_1 &= (1, 1, \dots, 1)\,,\\ v_i &= (0, 0, \dots, 0, m-i, \underbrace{-1, -1, \dots, -1}_{m-i})\,,\ 2 \le i \le m-1\,.\end{aligned}$$

Considering the orthonormal basis $(\epsilon_i\, v_i)_{1\leq i\leq m-1}$, $\epsilon_i = 1/\|v_i\|$, and denoting by Q the orthogonal matrix of the change of basis, one gets

$$\begin{aligned}
F(p) &= \frac{1}{2}\,{}^t pHp\,,\\
&= \frac{1}{m}\,{}^t p\,Q\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & m & 0 & 0\\ & & \ddots & \\ 0 & 0 & 0 & m\end{pmatrix}{}^t Q\,p\,,\\
&= \frac{1}{m}\,\epsilon_1\left((m-1)k_1+(m-2)k_2+\cdots+k_{m-1}\right)^2\\
&\quad+\epsilon_2\left((m-2)k_2+(m-3)k_3\cdots+k_{m-1}\right)^2\\
&\quad+\cdots\\
&\quad+\epsilon_{m-i}\left(i\,k_{m-i}+\cdots+k_{m-1}\right)^2\\
&\quad+\cdots\\
&\quad+\epsilon_{m-2}\left(2k_{m-2}+k_{m-1}\right)^2\\
&\quad+\epsilon_{m-1}\left(k_{m-1}\right)^2.
\end{aligned}$$

The minimum is obtained if and only if all the squares are minimal. Hence, since the k_i have to be odd integers,

Lemma 3.5. *The minimum is obtained only when*

$$(k_1,k_2,\ldots,k_{m-1}) = (1,-1,1,-1,\ldots) \text{ or } (-1,1,-1,1,\ldots)\,.$$

Proof. Assume that all the squares are minimal. First, since k_{m-1} is an odd integer, k_{m-1}^2 is minimum only when $k_{m-1}=\pm 1$. Now, let us show by induction that $k_{m-i} = (-1)^{i+1}\,k_{m-1}$, $i=1,\ldots,m-1$. Assuming that this is true for $1\leq i\leq 2j-1$, $j\geq 1$, one gets

$$\begin{aligned}
\left(2j\,k_{m-2j}+(2j-1)\,k_{m-(2j-1)}+\cdots+k_{m-1}\right)^2 &= \left(2j\,k_{m-2j}+j\,k_{m-1}\right)^2,\\
&= j^2\left(2\,k_{m-2j}+k_{m-1}\right)^2.
\end{aligned}$$

Since $2\,k_{m-2j}+k_{m-1}$ is an odd integer, the above square is minimal if and only if $\left(2\,k_{m-2j}+k_{m-1}\right)^2 = 1$, which implies

$$4\,k_{m-2j}\left(k_{m-2j}+k_{m-1}\right) = 0\,,$$

hence $k_{m-2j} = -k_{m+1} = (-1)^{2j+1}\,k_{m-1}$. Thus the result is also true for $i=2j$, and one gets

$$\left((2j+1)\,k_{m-(2j+1)}+2j\,k_{m-(2j-1)}+\cdots+k_{m-1}\right)^2 = \left((2j+1)\,k_{m-(2j+1)}-j\,k_{m-1}\right)^2.$$

If $k_{m-1}=1$, then $(2j+1)\,k_{m-(2j+1)}-j\,k_{m-1}\geq j+1$ if $k_{m-(2j+1)}\geq 1$ and $(2j+1)\,k_{m-(2j+1)}-j\,k_{m-1}\leq -(3j+1)$ if $k_{m-(2j+1)}\leq -1$, thus the above square is minimal if and only if $k_{m-(2j+1)} = 1 = k_{m-1} = (-1)^{2j+2}\,k_{m-1}$.
If $k_{m-1}=-1$, then $(2j+1)\,k_{m-(2j+1)}-j\,k_{m-1}\geq 3j+1$ if $k_{m-(2j+1)}\geq 1$ and $(2j+1)\,k_{m-(2j+1)}-j\,k_{m-1}\leq -(j+1)$ if $k_{m-(2j+1)}\leq -1$, thus the above square is minimal if and only if $k_{m-(2j+1)} = -1 = k_{m-1} = (-1)^{2j+2}\,k_{m-1}$. □

Hence the G-weights μ for which $\mu_{|\mathfrak{t}_K} = \lambda_\pm$ and $\|\mu+\delta_G\|^2$ is minimal are given by

(1) If m is even $m=2p$,

(a) if $(k_1, k_2, \dots, k_{m-1}) = (1, -1, 1, -1, \dots, -1, 1)$, then since $p_{2i-1} = 1$, $i = 1, \dots, p$, and $p_{2i} = 0$, $i = 1, \dots, p-1$,

$$\mu_\pm = \left[\frac{1}{2}, -\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2}; m, m-1, \dots, 2, \pm 1\right],$$

(b) if $(k_1, k_2, \dots, k_{m-1}) = (-1, 1, -1, 1, \dots, 1, -1)$, then since $p_{2i-1} = -1$, $i = 1, \dots, p$, and $p_{2i} = 0$, $i = 1, \dots, p-1$,

$$\mu'_\pm = \left[-\frac{1}{2}, \frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}; m, m-1, \dots, 2, \pm 1\right].$$

Note that $\mu'_\pm = \sigma^*(\mu_\pm)$ and $\|\mu_\pm + \delta_G\|^2 = \|\mu'_\pm + \delta_G\|^2$.

(2) If m is odd, $m = 2p+1$,

(a) if $(k_1, k_2, \dots, k_{m-1}) = (1, -1, 1, -1, \dots, 1, -1)$, then since $p_{2i-1} = 1$ and $p_{2i} = 0$, $i = 1, \dots, p$,

$$\mu_\pm = \left[\frac{1}{2} - \frac{1}{2m}, -\frac{1}{2} - \frac{1}{2m} \dots, -\frac{1}{2} - \frac{1}{2m}, \frac{1}{2} - \frac{1}{2m}; m, m-1, \dots, 2, \pm 1\right],$$

(b) if $(k_1, k_2, \dots, k_{m-1}) = (-1, 1, -1, 1, \dots, -1, 1)$, then since $p_{2i-1} = -1$ and $p_{2i} = 0$, $i = 1, \dots, p$,

$$\mu'_\pm = \left[-\frac{1}{2} + \frac{1}{2m}, \frac{1}{2} + \frac{1}{2m}, \dots, \frac{1}{2} + \frac{1}{2m}, -\frac{1}{2} + \frac{1}{2m}; m, m-1, \dots, 2, \pm 1\right],$$

Here also note that $\mu'_\pm = \sigma^*(\mu_\pm)$ and $\|\mu_\pm + \delta_G\|^2 = \|\mu'_\pm + \delta_G\|^2$.

Note that, by lemma 3.2, the weights μ_+ and μ'_+ are G-dominant, whereas μ_- (resp. μ'_-) belongs to the orbit of μ_+ (resp. μ'_+) under the Weyl group, since $\mu_- = \sigma_\beta(\mu_+)$, (resp. $\mu'_- = \sigma_\beta(\mu'_+)$), where σ_β is the reflexion across $\beta^\perp$, β being the simple root $2\,\widehat{x}_{2m}$.

In order to conclude, we first remark that the G-dominant weights μ_+ and μ_- verify the spin condition. That follows from the following general result.

Lemma 3.6. *Let μ_γ be the highest weight of an irreducible representation γ of G. Then $\mu_{\gamma|\mathfrak{t}_K}$ is a K-dominant weight, and any irreducible representation of K having $\mu_{\gamma|\mathfrak{t}_K}$ as a highest weight is contained in the restriction $\mathrm{Res}^G_K(\gamma)$ of γ to K.*

Proof. By the result of lemma 3.2, a dominant G-weight

$$\mu = [\mu_1, \mu_3, \dots, \mu_{2m-1}; \mu_2, \mu_4, \dots, \mu_{2m}], \quad \sum_{i=1}^m \mu_{2i-1} = 0,$$

restricts to $\mathfrak{t}_K$ as $\mu_{|\mathfrak{t}_K} = \sum_{i=1}^m \mu_{2i}\,\widehat{x}_{2i} = (\mu_2, \mu_4, \dots, \mu_{2m})$, where the μ_{2i} are non-negative integers verifying the condition $\mu_2 \geq \mu_4 \geq \cdots \geq \mu_{2m}$, hence, by the result of lemma 3.3, $\mu_{|\mathfrak{t}_K}$ is a K-dominant weight.

Now let v_γ be the[1] maximal vector of the representation γ. Since it is killed by the action of root-vectors corresponding to positive roots, it is in particular killed by the action of the E_α and $E_{\sigma^*(\alpha)}$, $\alpha \in {\Phi'_3}^+$, hence by the action of the $U_\alpha = \frac{1}{2}(E_\alpha + \sigma_*(E_\alpha))$, $\alpha \in {\Phi'_3}^+$. Since the U_α are root-vectors for the positive K-roots α', $\alpha \in {\Phi'_3}^+$, (see (3)), v_γ is a maximal vector of $\mathrm{Res}^G_K(\gamma)$ for the weight $\mu_{\gamma|\mathfrak{t}_K}$. □

The conclusion now results from the following remark.

[1] it is unique up to a scalar multiple.

Lemma 3.7. *Let μ be a G-dominant weight. Let Π_μ be the set of weights of any irreducible G-representation with highest weight μ.*
If $\lambda_\pm$ is the restriction to $\mathfrak{t}_K$ of a weight $\lambda \in \Pi_\gamma$ and if $\|\mu + \delta_G\|^2$ is minimal, then λ belongs to the orbit of μ under the Weyl group W_G.

Proof. The weight λ lies in the orbit under the Weyl group W_G of G of a dominant weight $\varrho \in \Pi_\gamma$. By the result of lemma B, 13.3 in [Hum72], $\|\lambda + \delta_G\|^2 \leq \|\varrho + \delta_G\|^2$, with equality only if $\lambda = \varrho$, and by lemma C, 13.4 in [Hum72], $\|\varrho + \delta_G\|^2 \leq \|\mu + \delta_G\|^2$, with equality only if $\varrho = \mu$. Now as ϱ is a G-dominant weight which posses in its orbit a weight λ whose restriction to $\mathfrak{t}_K$ is $\lambda_\pm$, the minimality condition verified by μ on that sort of weights implies $\|\varrho + \delta_G\|^2 = \|\mu + \delta_G\|^2$, hence $\varrho = \mu$, and λ belongs to the orbit of μ. □

Lemma 3.8. *If μ_γ is a G-dominant weight verifying the spin condition, then*

$$\|\mu_\gamma + \delta_G\|^2 \geq \|\mu_+ + \delta_G\|^2 .$$

Proof. As μ_γ verifies the spin condition, there exists a weight $\lambda \in \Pi_\gamma$ such that $\lambda_{|\mathfrak{t}_K} = \lambda_\pm$. By the result of the above lemma, λ belongs to the orbit of μ_γ under the Weyl group W_G, so $\|\lambda + \delta_G\|^2 \leq \|\mu_\gamma + \delta_G\|^2$, with equality if and only if $\lambda = \mu_\gamma$. If $\lambda = \mu_\gamma$, then, by the above considerations $\mu_\gamma = \mu_+$ or μ'_+, and the result follows since $\|\mu_+ + \delta_G\|^2 = \|\mu'_+ + \delta_G\|^2$. If $\lambda \neq \mu_\gamma$, then by replacing λ by $\sigma_\beta(\lambda)$ if necessary, we may suppose that $\lambda_{|\mathfrak{t}_K} = \lambda_+$. But then, by the above considerations, $\|\mu_+ + \delta_G\|^2 \leq \|\lambda + \delta_G\|^2$, and so $\|\mu_+ + \delta_G\|^2 < \|\mu_\gamma + \delta_G\|^2$. □
Finally, we may conclude that the square of the first eigenvalue of the Dirac operator is given by

$$\frac{1}{8m} \langle \mu_+, \mu_+ + 2\delta_G \rangle + \frac{2m^2 + m - 1}{16} ,$$

hence the result.

4. The symmetric space $\dfrac{\mathrm{SU}(2m)}{\mathrm{Sp}(m)}$.

Let J be the matrix in $\mathrm{SU}(2m)$ defined by

$$J := \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} .$$

Note that $J^2 = -I_{2m}$ and $J^{-1} = -J = {}^tJ$.
The group $\mathrm{Sp}(m)$ is identified with the subgroup of $\mathrm{SU}(2m)$

$$\mathrm{Sp}(m) := \{A \in \mathrm{SU}(2m) \,;\, {}^tA\, J\, A = J\} .$$

The symmetric space structure is given by the involution

$$\sigma : G \to G , \quad A \mapsto {}^tJ\, {}^tA^{-1}\, J .$$

This involution induces the decomposition of the Lie algebra $\mathfrak{su}_{2m}$ into:

$$\mathfrak{su}_{2m} = \mathfrak{sp}_m \oplus \mathfrak{p} ,$$

where $\mathfrak{sp}_{2m}$ is the Lie algebra

$$\{X \in \mathfrak{su}_{2m} \,;\, {}^tXJ = -JX\} = \{X \in \mathfrak{su}_{2m} \,;\, JXJ = {}^tX\} ,$$

of $\mathrm{Sp}(m)$, and

$$\mathfrak{p} = \{X \in \mathfrak{su}_{2m} \,;\, {}^tXJ = JX\} = \{X \in \mathfrak{su}_{2m} \,;\, JXJ = -{}^tX\} .$$

Elements of $\mathfrak{p}$ are matrices X of the form $X = \begin{pmatrix} X_1 & \overline{X_2} \\ X_2 & -\overline{X_1} \end{pmatrix}$ where X_1 and X_2 are $m \times m$ matrices verifying ${}^t\overline{X_1} = -X_1$ and ${}^t X_2 = -X_2$. hence $\dim(\mathfrak{p}) = 2m^2 - m - 1$. Denoting by T the maximal standard torus of $\mathrm{SU}(2m)$, $T \cap K$ is the standard maximal torus of $\mathrm{Sp}(m)$:

$$T_K := T \cap K = \left\{ \begin{pmatrix} \mathrm{e}^{i\beta_1} & & & & & & \\ & \ddots & & & & & \\ & & \mathrm{e}^{i\beta_m} & & & & \\ & & & \mathrm{e}^{-i\beta_1} & & & \\ & & & & \ddots & & \\ & & & & & \mathrm{e}^{-i\beta_m} \end{pmatrix} \; ; \; \beta_j \in \mathbb{R} \right\} .$$

The Lie algebra $\mathfrak{t}$ of T is defined by

$$\mathfrak{t} = \left\{ \sum_{j=1}^{2m} i\beta_j \, E_{jj} \; ; \; \beta_j \in \mathbb{R} \, , \; \sum_{j=1}^{2m} \beta_j = 0 \right\} ,$$

and

$$\mathfrak{t}_K = \left\{ \sum_{j=1}^{m} i\beta_j \, (E_{jj} - E_{m+j\, m+j}) \; ; \; \beta_j \in \mathbb{R} \right\} ,$$

$$\mathfrak{t}_0 = \left\{ \sum_{j=1}^{m} i\gamma_j \, (E_{jj} + E_{m+j\, m+j}) \; ; \; \gamma_j \in \mathbb{R} \, , \; \sum_{j=1}^{m} \gamma_j = 0 \right\} .$$

Let $\widehat{y}_j$, $1 \leq j \leq 2m$, be the vectors of $i\,\mathfrak{t}^*$ defined by considering the restriction to $\mathfrak{t}$ of the dual basis of the family of vectors $i\,E_{jj}$, $1 \leq j \leq 2m$:

$$\forall H = \sum_{j=1}^{2m} i\beta_j \, E_{jj} \in \mathfrak{t} \, , \quad \widehat{y}_j(H) = \beta_j \, .$$

Any element $\mu \in i\,\mathfrak{t}^*$ may be uniquely written as

$$\mu = \sum_{j=1}^{2m} \mu_j \, \widehat{y}_j \, , \quad \sum_{j=1}^{2m} \mu_j = 0 \, ,$$

and is denoted

$$\mu = [\mu_1, \ldots, \mu_{2m}] \, .$$

The scalar product on $i\,\mathfrak{t}^*$ considered here is given by the scalar product on $\mathfrak{su}_{2m}$ defined by

$$\langle X, Y \rangle = -\Re\big(\mathrm{Tr}(XY)\big) = -\frac{1}{4m} \mathrm{B}(X, Y) \, , \quad X, Y \in \mathfrak{su}_{2m} \, .$$

For $\mu = [\mu_1, \ldots, \mu_{2m}]$ and $\mu' = [\mu'_1, \ldots, \mu'_{2m}] \in i\,\mathfrak{t}^*$,

$$\langle \mu, \mu' \rangle = \sum_{j=1}^{2m} \mu_j \, \mu'_j \, .$$

The involution σ^* of $i\mathfrak{t}^*$ induced by σ is defined by

$$\mu = [\mu_1, \dots, \mu_{2m}] \xrightarrow{\sigma^*} [-\mu_{m+1}, -\mu_{m+2}, \dots, -\mu_{2m}, -\mu_1, -\mu_2, \dots, -\mu_m]\,.$$

Note that the vectors $\widehat{x}_j = \widehat{y}'_j := \widehat{y}_{j|\mathfrak{t}_K}$, $j = 1, \dots, m$, define a basis of $i\mathfrak{t}_K^*$. Any $\lambda \in i\mathfrak{t}_K^*$ of the form $\lambda = \sum_{j=1}^m \lambda_j\, \widehat{x}_j$ is denoted

$$\lambda = (\lambda_1, \dots, \lambda_m)\,.$$

4.1. **Sets of roots.** The root-vectors of $\mathrm{SU}(2m)$ relative to the standard torus are the E_{ij}. Explicitly the roots are

$$\begin{aligned}
\widehat{y}_k - \widehat{y}_l\,, &\quad 1 \le k \ne l \le m\,, \quad \text{with root-vector } E_{kl},\\
\widehat{y}_{m+k} - \widehat{y}_{m+l}\,, &\quad 1 \le k \ne l \le m\,, \quad \text{with root-vector } E_{m+k\,m+l},\\
\pm(\widehat{y}_k - \widehat{y}_{m+l})\,, &\quad 1 \le k, l \le m\,, \quad \text{with root-vector } E_{k\,m+l} \text{ (resp. } E_{m+l\,k}).
\end{aligned}$$

Note that the only roots θ that verify $\sigma^*(\theta) = \theta$ are $\pm(\widehat{y}_k - \widehat{y}_{m+k})$, and as $E_{k\,m+k}$ and $E_{m+k\,k}$ belong to $\mathfrak{sp}_m \otimes \mathbb{C}$, those roots belong to Φ_1. Hence

$$\Phi_1 = \{\pm(\widehat{y}_k - \widehat{y}_{m+k})\,,\ 1 \le k \le m\}\,, \quad \Phi_2 = \emptyset\,,$$

and all other roots belong to Φ_3.
We choose positive roots such that

$$\begin{aligned}
\Phi_1^+ &= \{\widehat{y}_k - \widehat{y}_{m+k}\,,\ 1 \le k \le m\}\,,\\
\Phi_3'^+ &= \left\{ \begin{array}{l} \widehat{y}_k - \widehat{y}_l \\ \widehat{y}_k - \widehat{y}_{m+l} \end{array} \ ,\ 1 \le k < l \le m \right\}\,,\\
\sigma^*\big(\Phi_3'^+\big) &= \left\{ \begin{array}{l} \sigma^*(\widehat{y}_k - \widehat{y}_l) = -\widehat{y}_{m+k} + \widehat{y}_{m+l} \\ \sigma^*(\widehat{y}_k - \widehat{y}_{m+l}) = -\widehat{y}_{m+k} + \widehat{y}_l \end{array} \ ,\ 1 \le k < l \le m \right\}\,.
\end{aligned}$$

It is easy to verify that a system of simple roots is given by

$$\begin{aligned}
\alpha &:= \widehat{y}_m - \widehat{y}_{2m}\,,\\
\gamma_k &:= \widehat{y}_k - \widehat{y}_{k+1}\,, \quad 1 \le k \le m-1\,,\\
\sigma^*(\gamma_k) &= -\widehat{y}_{m+k} + \widehat{y}_{m+k+1}\,, \quad 1 \le k \le m-1\,.
\end{aligned}$$

Note that

$$\begin{aligned}
\langle \alpha, \alpha\rangle = \langle \gamma_k, \gamma_k\rangle = \langle \sigma^*(\gamma_k), \sigma^*(\gamma_k)\rangle &= 2\,, \quad k = 1, \dots, m-1\,,\\
\langle \gamma_k, \gamma_{k+1}\rangle = \langle \sigma^*(\gamma_k), \sigma^*(\gamma_{k+1})\rangle &= -1\,,\ k = 1, \dots, m-2\,,\\
\langle \gamma_{m-1}, \alpha\rangle = \langle \sigma^*(\gamma_{m-1}), \alpha\rangle &= -1\,,
\end{aligned}$$

all the other scalar products being zero. Hence

Lemma 4.1. *A vector $\mu = [\mu_1, \dots, \mu_{2m}] \in i\,\mathfrak{t}^*$, $\sum_{j=1}^{2m} \mu_j = 0$, is a dominant weight if and only if*

$$\left.\begin{array}{l} \mu_k - \mu_{k+1} \in \mathbb{N}\,, \\ \mu_{m+k+1} - \mu_{m+k} \in \mathbb{N}\,, \\ \mu_m - \mu_{2m} \in \mathbb{N}\,. \end{array}\right\}\ k = 1, \dots, m-1\,,$$

By (3), the positive K-roots are the restrictions to $\mathfrak{t}_K$ of the positive G-roots in Φ_1 and Φ'_3, hence, as $\widehat{y}_{m+k|\mathfrak{t}_K} = -\widehat{y}_{k|\mathfrak{t}_K}$,

$$\Phi_K^+ = \left\{ \begin{array}{ll} 2\,\widehat{x}_k\,, & 1 \le k \le m \\ \widehat{x}_k - \widehat{x}_l\,, & 1 \le k < l \le m \\ \widehat{x}_k + \widehat{x}_l\,, & 1 \le k < l \le m \end{array} \right\}\,.$$

A system of simple roots is given by

$$\theta'_k := \widehat{x}_k - \widehat{x}_{k+1}\,,\ k = 1, \dots, m-1\,, \quad \text{and} \quad \theta'_m := 2\,\widehat{x}_m\,.$$

Note that, as $\widehat{x}_k = \frac{1}{2}(\widehat{y}_k - \widehat{y}_{m+k})$,

$$\langle \theta'_k, \theta'_k \rangle = 1\,, \quad k = 1, \dots, m-1\,, \quad \langle \theta'_m, \theta'_m \rangle = 2\,,$$
$$\langle \theta'_k, \theta'_{k+1} \rangle = -1/2\,, \quad k = 1, \dots, m-2\,, \quad \langle \theta'_{m-1}, \theta'_m \rangle = -1\,,$$

all the other scalar products being zero. Hence

Lemma 4.2. *A vector $\lambda = (\lambda_1, \dots \lambda_m) \in i\,\mathfrak{t}_K^*$, is a dominant weight if and only if*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0\,,$$

and all the λ_i are integers.

4.2. **Highest weights of the spin representation of K.** Since $\Phi_2 = \emptyset$, we may conclude from lemma 2.4 :

Lemma 4.3. *The spin representation of K has only one highest weight*

$$\lambda_0 := (m-1, m-2, \dots, 1, 0)\,,$$

with multiplicity $2^{[\frac{m-1}{2}]}$

Proof. As we saw it before the multiplicity of λ_0 is at least $2^{[\frac{m-1}{2}]}$. But it may be checked, using the Weyl dimension formula, that any irreducible K-module with highest weight λ_0 has dimension $2^{m(m-1)}$, and

$$2^{[\frac{m-1}{2}]} \times 2^{m(m-1)} = \dim(\Sigma)\,,$$

hence the result. □

4.3. **The first eigenvalue of the Dirac operator.** We apply the same method as above, and begin by determining the G-weights μ such that $\mu_{|\mathfrak{t}_K} = \lambda_0$ and $\|\mu + \delta_G\|^2$ is minimal.
First a vector $\mu = [\mu_1, \dots, \mu_{2m}] \in i\mathfrak{t}^*$, $\sum_{k=1}^{2m} \mu_k = 0$, is a G-weight if and only if $\mu_k - \mu_{k+1} \in \mathbb{Z}$, $k = 1, \dots, 2m-1$.
Such a vector μ verifies $\mu_{|\mathfrak{t}_K} = \lambda_0$ if and only if

$$\mu_k - \mu_{m+k} = m - k\,, \quad k = 1, \dots, m\,.$$

The condition $\sum_{k=1}^{2m} \mu_k = 0$ then implies that

$$\sum_{k=1}^{m} \mu_k = \frac{m(m-1)}{4}\,.$$

The half-sum of the positive G-roots is given by

$$\delta_G = \sum_{k=1}^{m} \left(m - k + \frac{1}{2} \right) (\widehat{y}_k - \widehat{y}_{m+k})\,.$$

(Note that $\sigma^*(\delta_G) = \delta_G$). Hence

$$\mu + \delta_G = \sum_{k=1}^{m} \left(\mu_k + m - k + \frac{1}{2} \right) \widehat{y}_k + \sum_{k=1}^{m} \left(\mu_k - 2(m-k) - \frac{1}{2} \right) \widehat{y}_{m+k}\,.$$

As $\sum_{k=1}^{m} \mu_k = \frac{m(m-1)}{4}$, $\|\mu + \delta_G\|^2$ is minimal if and only if $\sum_{k=1}^{m} (\mu_k^2 + k\,\mu_k)$ is minimal.

Set $\mu_i - \mu_{i+1} = k_i$, $i = 1, \dots, m-1$. For any $j = 1, \dots, m-1$, one has

$$p_j := \sum_{i=1}^{j} k_i = \mu_1 - \mu_{j+1}\,.$$

Using $\sum_{k=1}^{m} \mu_k = \frac{m(m-1)}{4}$, one then gets

$$\sum_{j=1}^{m-1} p_j = (m-1)\,\mu_1 + \mu_1 - \frac{m(m-1)}{4} = m\,\mu_1 - \frac{m(m-1)}{4}\,,$$

hence

$$\mu_1 = \frac{m-1}{4} + \frac{1}{m} \sum_{i=1}^{m-1} p_i\,,$$

and

$$\mu_{j+1} = \mu_1 - p_j = \frac{m-1}{4} + \frac{1}{m} \left(\sum_{i=1}^{m-1} p_i - m\,p_j \right)\,.$$

The expression $\sum_{k=1}^{m} (\mu_k^2 + k\,\mu_k)$ is a polynomial $F(p_1, \dots, p_{m-1})$ of the variables p_1, $p_2 \dots, p_{m-1}$. With the notation $p = (p_1, \dots, p_{m-1})$, one has

$$\frac{\partial F}{\partial p_i}(p) = -2\mu_{i+1} + m - (i+1)\,.$$

The function F has a unique critical point when $\mu_i = \frac{m-i}{2}$, $i = 1, \dots, m$, which is equivalent to $p_i = \frac{i}{2}$, or $k_i = \frac{1}{2}$, $i = 1, \dots, m-1$.
Let $p_0 = \left(\frac{1}{2}, 1, \frac{3}{2}, \dots \frac{m-1}{2}\right)$. As

$$\frac{\partial^2 F}{\partial p_i \partial p_j}(p_0) = -\frac{2}{m}\,(1 - m\,\delta_{ij})\,,$$

we may apply the result of the previous example. With the same notations,

$$\begin{aligned}
F(p)-F(p_0) &= \frac{1}{2}\,{}^t(p-p_0)H(p-p_0)\,,\\
&= \frac{1}{m}\,{}^t(p-p_0)\,Q\begin{pmatrix}1&0&0&0\\0&m&0&0\\&&\ddots&\\0&0&0&m\end{pmatrix}{}^tQ\,(p-p_0)\,,\\
&= \frac{1}{m}\,\epsilon_1\left((m-1)k_1+(m-2)k_2+\cdots+k_{m-1}-\frac{m(m-1)}{4}\right)^2\\
&\quad+\epsilon_2\left((m-2)k_2+(m-3)k_3\cdots+k_{m-1}-\frac{(m-1)(m-2)}{4}\right)^2\\
&\quad+\cdots\\
&\quad+\epsilon_{m-i}\left(i\,k_{m-i}+\cdots+k_{m-1}-\frac{i(i+1)}{4}\right)^2\\
&\quad+\cdots\\
&\quad+\epsilon_{m-2}\left(2k_{m-2}+k_{m-1}-\frac{3}{2}\right)^2\\
&\quad+\epsilon_{m-1}\left(k_{m-1}-\frac{1}{2}\right)^2.
\end{aligned}$$

The minimum is obtained if and only if all the squares are minimal. Hence, since the k_i have to be integers,

Lemma 4.4. *The minimum is obtained only when*

$$(k_1,k_2,\ldots,k_{m-1})=(1,0,1,0,\ldots)\ \textit{or}\ (0,1,0,1,\ldots)\,.$$

Proof. Note that

$$\begin{aligned}
F(p)-F(p_0) &= \frac{1}{4m}\,\epsilon_1\,\left((m-1)(2k_1-1)+(m-2)(2k_2-1)+\cdots+(2k_{m-1}-1)\right)^2\\
&\quad+\frac{1}{4}\,\epsilon_2\,\left((m-2)(2k_2-1)+(m-3)(2k_3-1)\cdots+(2k_{m-1}-1)\right)^2\\
&\quad+\cdots\\
&\quad+\frac{1}{4}\,\epsilon_{m-i}\,\left(i\,(2k_{m-i}-1)+\cdots+(2k_{m-1}-1)\right)^2\\
&\quad+\cdots\\
&\quad+\frac{1}{4}\,\epsilon_{m-2}\,\left(2(2k_{m-2}-1)+(2k_{m-1}-1)\right)^2\\
&\quad+\frac{1}{4}\,\epsilon_{m-1}\,\left(2k_{m-1}-1\right)^2\,.
\end{aligned}$$

Since the $2k_i-1$ are odd integers, the result of lemma 3.5 shows that the minimum is obtained if and only if $(2k_1-1,2k_2-1,\ldots,2k_{m-1}-1)=(1,-1,1,-1,\ldots)$ or $(-1,1,-1,1,\ldots)$, hence if and only if $(k_1,k_2,\ldots,k_{m-1})=(1,0,1,0,\ldots)$ or $(0,1,0,1,\ldots)$. □

Hence the G-weights μ for which $\mu_{|\mathfrak{t}_K}=\lambda_0$ and $\|\mu+\delta_G\|^2$ is minimal are given by

(1) If m is even, $m=2p$,

(a) if $(k_1, k_2, \dots, k_{m-1}) = (1, 0, 1, 0, \dots, 1)$, since $p_{2i} = i$, and $p_{2i+1} = i+1$,

$$\mu_0 := \sum_{i=1}^{p} \left(p + \frac{3}{4} - i\right) \widehat{y}_{2i-1} + \sum_{i=1}^{p} \left(p - \frac{1}{4} - i\right) \widehat{y}_{2i} - \sum_{i=1}^{p} \left(p + \frac{1}{4} - i\right) \widehat{y}_{m+2i-1} - \sum_{i=1}^{p} \left(p + \frac{1}{4} - i\right) \widehat{y}_{m+2i}\,.$$

(b) if $(k_1, k_2, \dots, k_{m-1}) = (0, 1, 0, 1, \dots, 0)$, since $p_{2i} = p_{2i+1} = i$,

$$\mu_0' := \sum_{i=1}^{p} \left(p + \frac{1}{4} - i\right) \widehat{y}_{2i-1} + \sum_{i=1}^{p} \left(p + \frac{1}{4} - i\right) \widehat{y}_{2i} - \sum_{i=1}^{p} \left(p + \frac{3}{4} - i\right) \widehat{y}_{m+2i-1} - \sum_{i=1}^{p} \left(p - \frac{1}{4} - i\right) \widehat{y}_{m+2i}\,.$$

Note that $\mu_0' = \sigma^*(\mu_0)$ and $\|\mu_0 + \delta_G\|^2 = \|\mu_0' + \delta_G\|^2$.

(2) If m is odd, $m = 2p + 1$,

(a) if $(k_1, k_2, \dots, k_{m-1}) = (1, 0, 1, 0, \dots, 0)$, since $p_{2i} = i$, and $p_{2i+1} = i+1$,

$$\mu_0 := \sum_{i=1}^{p+1} \left(p + \frac{5}{4} - i - \frac{1}{4m}\right) \widehat{y}_{2i-1} + \sum_{i=1}^{p} \left(p + \frac{1}{4} - i - \frac{1}{4m}\right) \widehat{y}_{2i} - \sum_{i=1}^{p+1} \left(p + \frac{3}{4} - i + \frac{1}{4m}\right) \widehat{y}_{m+2i-1} - \sum_{i=1}^{p} \left(p + \frac{3}{4} - i + \frac{1}{4m}\right) \widehat{y}_{m+2i}\,.$$

(b) if $(k_1, k_2, \dots, k_{m-1}) = (0, 1, 0, 1, \dots, 1)$, since $p_{2i} = p_{2i+1} = i$,

$$\mu_0' := \sum_{i=1}^{p+1} \left(p + \frac{3}{4} - i + \frac{1}{4m}\right) \widehat{y}_{2i-1} + \sum_{i=1}^{p} \left(p + \frac{3}{4} - i + \frac{1}{4m}\right) \widehat{y}_{2i} - \sum_{i=1}^{p+1} \left(p + \frac{5}{4} - i - \frac{1}{4m}\right) \widehat{y}_{m+2i-1} - \sum_{i=1}^{p} \left(p + \frac{1}{4} - i - \frac{1}{4m}\right) \widehat{y}_{m+2i}\,.$$

Note that $\mu_0' = \sigma^*(\mu_0)$ and $\|\mu_0 + \delta_G\|^2 = \|\mu_0' + \delta_G\|^2$.

Note that μ_0 and μ_0' are G-dominant, hence we may conclude exactly as in the above case with the result of lemma 3.8 : the square of the first eigenvalue of the Dirac operator is given by

$$\frac{1}{4m} \langle \mu_0, \mu_0 + 2\delta_G \rangle + \frac{2m^2 - m - 1}{16}\,,$$

hence the result.

5. The symmetric space $\dfrac{\mathrm{SO}(2p+2q+2)}{\mathrm{SO}(2p+1) \times \mathrm{SO}(2q+1)}$, $p \leq q$, $p + q \geq 1$.

Let $(e_i)_{1 \leq i \leq 2p+2q+2}$ be the standard basis of $\mathbb{R}^{2p+2q+2}$. Let J be the diagonal matrix

$$J = \begin{pmatrix} -I_{2p} & 0 & 0 & 0 \\ 0 & I_{2q} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider the involution σ of the group $\mathrm{SO}(2p+2q+2)$ defined by

$$A \longmapsto JAJ^{-1}\,.$$

Note that J is orthogonal but $\det(J) = -1$ hence σ is not a conjugation in the group. The connected component of the subgroup of fixed points is isomorphic to $\mathrm{SO}(2p+1)\times\mathrm{SO}(2q+1)$.
We choose to consider $G = \mathrm{Spin}(2p+2q+2)$ instead. We view $\mathrm{Spin}(2p+1)$ and $\mathrm{Spin}(2q+1)$ as subgroups of G by considering

$$\mathrm{Spin}(2p+1) = \{v_1\cdots v_{2k}\,;\, v_i \in \mathrm{span}\{e_1,\ldots,e_{2p},e_{2p+2q+1}\}\,;\, \|v_i\|=1\}\,,$$

and

$$\mathrm{Spin}(2q+1) = \{v_1\cdots v_{2\ell}\,;\, v_i \in \mathrm{span}\{e_{2p+1},\ldots,e_{2p+2q},e_{2p+2q+2}\}\,;\, \|v_i\|=1\}\,.$$

Let K be the connected subgroup of G defined by the image of the morphism

$$\mathrm{Spin}(2p+1)\times\mathrm{Spin}(2q+1)\longrightarrow \mathrm{Spin}(2p+2q+2)\,;\,(\varphi,\psi)\longmapsto\varphi\cdot\psi\,.$$

This group K is the connected component of the subgroup of fixed elements of the (outer) involution

$$\sigma : G\longrightarrow G\,;\,\psi\longmapsto\varphi_0\cdot\psi\cdot\varphi_0^{-1}\,,$$

where

$$\varphi_0 := e_{2p+1}\cdot e_{2p+2}\cdots e_{2p+2q}\cdot e_{2p+2q+2}\,.$$

The decomposition of the Lie algebra $\mathfrak{spin}_{2p+2q+2}$ induced by the involution σ is given by

$$\mathfrak{spin}_{2p+2q+2} = (\mathfrak{spin}_{2p+1}\oplus\mathfrak{spin}_{2q+1})\oplus\mathfrak{p}\,,$$

where

$$\mathfrak{p} = \left\{ \sum_{\substack{i\in\{1,\ldots,2p,2p+2q+1\}\\ j\in\{2p+1,\ldots,2p+2q,2p+2q+2\}}} \beta_{ij}\, e_i\cdot e_j\,;\,\beta_{ij}\in\mathbb{R}\right\}.$$

Thus

$$\dim(\mathfrak{p}) = (2p+1)(2q+1)\,.$$

Let T be the standard maximal torus of G :

$$T = \left\{ \prod_{k=1}^{p+q+1} (\cos\,\beta_k + \sin\,\beta_k\, e_{2k-1}\cdot e_{2k})\;;\;\beta_k\in\mathbb{R}\right\}.$$

Then

$$\mathfrak{t} = \left\{ \sum_{k=1}^{p+q+1}\beta_k\, e_{2k-1}\cdot e_{2k}\;;\;\beta_k\in\mathbb{R}\right\},$$

and

$$\mathfrak{t}_K = \left\{ \sum_{k=1}^{p}\beta_k\, e_{2k-1}\cdot e_{2k} + \sum_{k=p+1}^{p+q}\beta_k\, e_{2k-1}\cdot e_{2k}\;;\;\beta_k\in\mathbb{R}\right\},$$
$$\mathfrak{t}_0 = \mathrm{span}\{e_{2p+2q+1}\cdot e_{2p+2q+2}\}\,.$$

Let $\widehat{x}_k$, $k=1,\dots,p+q+1$, be the basis of $i\,\mathfrak{t}^*$ defined by

$$\widehat{x}_k(H) = 2i\,\beta_k\,,\ \text{for } H = \sum_{k=1}^{p+q+1} \beta_k\, e_{2k-1}\cdot e_{2k} \in \mathfrak{t}\,.$$

Any element μ in $i\,\mathfrak{t}^*$ of the form

$$\mu = \sum_{k=1}^{p+q+1} \mu_k\,\widehat{x}_k\,,\ \mu_k\in\mathbb{R}\,,$$

is denoted

$$\mu = (\mu_1,\dots,\mu_{p+q+1})\,.$$

The scalar product on $i\,\mathfrak{t}^*$ considered here is given by the scalar product on the Lie algebra $\mathfrak{spin}_{2p+2q+2}$ defined by

$$\langle X,Y\rangle = -\frac{1}{2}\,\mathrm{Tr}\,(\xi_*(X)\xi_*(Y)) = -\frac{1}{4(p+q)}\,\mathrm{B}(X,Y)\,,\ X,Y\in\mathfrak{spin}_{2p+2q+2}\,,$$

where ξ is the covering $\mathrm{Spin}(2p+2q+2)\to\mathrm{SO}(2p+2q+2)$. For any $\mu=(\mu_1,\dots,\mu_{p+q+1})$ and any $\mu'=(\mu'_1,\dots,\mu'_{p+q+1})$

$$\langle\mu,\mu'\rangle = \sum_{k=1}^{p+q+1}\mu_k\,\mu'_k\,.$$

The involution σ^* of $i\,\mathfrak{t}^*$ induced by σ is defined by

$$\mu=(\mu_1,\dots,\mu_{p+q},\mu_{p+q+1}) \xrightarrow{\sigma^*} (\mu_1,\dots,\mu_{p+q},-\mu_{p+q+1})\,.$$

Note that the vectors $\widehat{x}'_k := \widehat{x}_{k|\mathfrak{t}_K}$, $k=1,\dots,p+q$, define a basis of $i\,\mathfrak{t}_K^*$. Any $\lambda\in i\,\mathfrak{t}_K^*$ of the form $\lambda=\sum_{k=1}^{p+q}\lambda_k\,\widehat{x}'_k$ is denoted

$$\lambda=(\lambda_1,\dots,\lambda_{p+q})\,.$$

5.1. **Sets of roots.** Let

$$u_k=\frac{1}{2}(e_{2k-1}-i\,e_{2k})\quad\text{and}\quad v_k=\frac{1}{2}(e_{2k-1}+i\,e_{2k})\,,\ k=1,\dots,p+q+1\,.$$

The G-roots are

$$\begin{aligned}
&\widehat{x}_i+\widehat{x}_j\,,\quad 1\le i<j\le p+q+1\,,\quad \text{with root-vector space } \mathbb{C}\,u_i\cdot u_j,\\
&-(\widehat{x}_i+\widehat{x}_j)\,,\quad 1\le i<j\le p+q+1\,,\quad \text{with root-vector space } \mathbb{C}\,v_i\cdot v_j,\\
&\widehat{x}_i-\widehat{x}_j\,,\quad 1\le i<j\le p+q+1\,,\quad \text{with root-vector space } \mathbb{C}\,u_i\cdot v_j,\\
&-(\widehat{x}_i-\widehat{x}_j)\,,\quad 1\le i<j\le p+q+1\,,\quad \text{with root-vector space } \mathbb{C}\,v_i\cdot u_j.
\end{aligned}$$

Note that

$$\begin{aligned}
\Phi_1&=\{\pm(\widehat{x}_i\pm\widehat{x}_j)\,;\ 1\le i<j\le p\,,\ p+1\le i<j\le p+q\}\,,\\
\Phi_2&=\{\pm(\widehat{x}_i\pm\widehat{x}_j)\,;\ 1\le i\le p\,,\ p+1\le j\le p+q\}\,,\\
\Phi_3&=\{\pm(\widehat{x}_i\pm\widehat{x}_{p+q+1})\,;\ 1\le i\le p+q\}\,.
\end{aligned}$$

We choose positive roots such that

$$\Phi_1^+ = \left\{ \begin{array}{l} \widehat{x}_i - \widehat{x}_j \\ \widehat{x}_i + \widehat{x}_j \end{array} \; ; \; 1 \le i < j \le p \,,\; p+1 \le i < j \le p+q \right\},$$
$$\Phi_2^+ = \left\{ \begin{array}{l} \widehat{x}_i - \widehat{x}_j \\ \widehat{x}_i + \widehat{x}_j \end{array} \; ; \; 1 \le i \le p \,,\; p+1 \le j \le p+q \right\},$$
$$\Phi_3'^+ = \{\widehat{x}_i - \widehat{x}_{p+q+1} \; ; \; 1 \le i \le p+q\} \,,$$
$$\sigma^*(\Phi_3'^+) = \{\widehat{x}_i + \widehat{x}_{p+q+1} \; ; \; 1 \le i \le p+q\} \,.$$

It is easy to see that a system of simple roots is given by

$$\alpha_k := \widehat{x}_k - \widehat{x}_{k+1} \,,\; k = 1, \dots, p-1 \,,\; k = p+1, \dots, p+q-1 \,, (\in \Phi_1^+) \,,$$
$$\beta := \widehat{x}_p - \widehat{x}_{p+1} \,, (\in \Phi_2^+) \,,$$
$$\gamma := \widehat{x}_{p+q} - \widehat{x}_{p+q+1} \,, (\in \Phi_3'^+) \,,$$
$$\sigma^*(\gamma) := \widehat{x}_{p+q} + \widehat{x}_{p+q+1} \,, (\in \sigma^*(\Phi_3'^+)) \,.$$

Note that

$$\langle \alpha_k, \alpha_k \rangle = \langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = \langle \sigma^*(\gamma), \sigma^*(\gamma) \rangle = 2 \,,$$
$$k = 1, \dots, p-1, p+1, \dots, p+q-1 \,,$$
$$\langle \alpha_k, \alpha_{k+1} \rangle = \langle \alpha_{p-1}, \beta \rangle = \langle \alpha_{p+1}, \beta \rangle = \langle \alpha_{p+q-1}, \gamma \rangle = \langle \alpha_{p+q-1}, \sigma^* \gamma \rangle = -1 \,,$$
$$k = 1, \dots, p-2, p+1, \dots, p+q-2 \,,$$

all the other scalar products being zero. Hence the following (classical) characterization.

Lemma 5.1. *A vector $\mu = (\mu_1, \dots, \mu_{p+q+1}) \in i\,\mathfrak{t}^*$ is G-dominant if and only if*

$$\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{p+q} \ge |\mu_{p+q+1}| \,,$$

the μ_i being all simultaneously integers or half-integers.

By (3), the positive K-roots are the restrictions to $\mathfrak{t}_K$ of the positive G-roots in Φ_1 and Φ_3', hence,

$$\Phi_K^+ = \left\{ \begin{array}{l} \widehat{x}_i' - \widehat{x}_j' \\ \widehat{x}_i' + \widehat{x}_j' \end{array} \right\} \begin{array}{l} 1 \le i < j \le p \,, \\ p+1 \le i < j \le p+q \,, \end{array} \; ; \; \widehat{x}_i' \,,\; 1 \le i \le p+q \right\}.$$

A system of simple roots is given by

$$\theta_k' = \widehat{x}_k' - \widehat{x}_{k+1}' \,,\; k = 1, \dots, p-1, p+1, \dots p+q-1 \,, \quad \widehat{x}_p' \quad \text{and} \quad \widehat{x}_{p+q}' \,.$$

Note that

$$\langle \theta_k', \theta_k' \rangle = 2 \,,\; k = 1, \dots, p-1, p+1, \dots p+q-1 \,, \quad \langle \widehat{x}_p', \widehat{x}_p' \rangle = \langle \widehat{x}_{p+q}', \widehat{x}_{p+q}' \rangle = 1 \,,$$
$$\langle \theta_k', \theta_{k+1}' \rangle = \langle \theta_{p-1}', \widehat{x}_p' \rangle = \langle \theta_{p+q-1}', \widehat{x}_{p+q}' \rangle = -1 \,,\; k = \begin{cases} 1, \dots, p-2 \,, \\ p+1, \dots p+q-2 \,. \end{cases}$$

Hence

Lemma 5.2. *A vector $\lambda = (\lambda_1, \dots, \lambda_{p+q}) \in i\,\mathfrak{t}_K^*$ is K-dominant if and only if*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0 \,, \quad \textit{and} \quad \lambda_{p+1} \ge \lambda_{p+2} \ge \cdots \ge \lambda_{p+q} \ge 0 \,,$$

the λ_i for $i = 1, \dots, p$ (resp. for $i = p+1, \dots, p+q$) being all simultaneously integers or half-integers.

5.2. **A characterization of the highest weights of the spin representation of K.** The explicit determination of the highest weights is far from being simple. Some results on the decomposition of the spin representation for oriented grassmannians may be found in [Kli07].
By the result of lemma 2.4, any highest weight of the spin representation of K has the form

$$\frac{1}{2}\left(\sum_{\beta\in\Phi_2^+}\pm\beta' + \sum_{i=1}^{p+q}\widehat{x}_i'\right).$$

Now, remark that the $\alpha' s$ in Φ_1 and the $\beta' s$ in Φ_2 are respectively compact and non-compact roots relative to the maximal common torus T_K of the groups $G_1 = \mathrm{SO}(2p+2q)$ and $K_1 = \mathrm{SO}(2p)\times\mathrm{SO}(2q)$. Since G_1/K_1 is an inner symmetric space, the results of R. Parthasarathy in [Par71] may be applied here.
First the weights of the spin representation of K_1 are

$$\frac{1}{2}\left(\sum_{\beta\in\Phi_2^+}\pm\beta'\right),$$

see Remark 2.1 in [Par71].
Now from lemma 2.2 in [Par71] we may conclude

Lemma 5.3. *Any highest weight of the spin representation of K has necessarily the form*

$$w\cdot\delta_{G_1} - \delta_{K_1} + \frac{1}{2}\sum_{i=1}^{p+q}\widehat{x}_i', \quad w\in W_1,$$

where

- *δ_{G_1} is the half-sum of the positive G_1-roots, that are the $\alpha' s$ in Φ_1^+ and the $\beta' s$ in Φ_2^+, and whose set is denoted $\Phi_{G_1}^+$,*
- *δ_{K_1} is the half-sum of the positive roots of K_1, that are the $\alpha' s$ in Φ_1^+, and whose set is denoted $\Phi_{K_1}^+$,*
- *W_1 is the subset of the Weyl group W_{G_1} of G_1 defined by*

$$W_1 = \{w\in W_{G_1}\,;\, w\cdot\Phi_{G_1}^+ \supset \Phi_{K_1}^+\}.$$

Proof. Let $\lambda = \frac{1}{2}\left(\sum_{\beta\in\Phi_2^+}\varepsilon_{\lambda\beta}\,\beta' + \sum_{i=1}^{p+q}\widehat{x}_i'\right)$, $\varepsilon_{\lambda\beta}=\pm1$, be a highest weight of the spin representation of K.
For any $\alpha\in\Phi_1$,

$$\alpha' + \frac{1}{2}\sum_{\beta\in\Phi_2^+}\varepsilon_{\lambda\beta}\,\beta' \neq \frac{1}{2}\sum_{\beta\in\Phi_2^+}\pm\beta',$$

since otherwise $\lambda+\alpha'$ is a weight of the spin representation of K, contradicting the fact that λ is a highest weight.
So $\frac{1}{2}\sum_{\beta\in\Phi_2^+}\varepsilon_{\lambda\beta}\,\beta'$ is a highest weight of the spin representation of K_1, hence of the form $w\cdot\delta_{G_1}-\delta_{K_1}$, $w\in W_1$, by the result of Parthasarathy. □
Now let λ be a highest weight of that sort. One has

$$\delta_{G_1} = \sum_{i=1}^{p+q}(p+q-i)\,\widehat{x}_i,$$

and

$$\delta_{K_1} = \sum_{i=1}^{p}(p-i)\,\widehat{x}_i + \sum_{i=1}^{p+q}(p+q-i)\,\widehat{x}_i\,.$$

On the other hand, the Weyl group W_{G_1} acts on $i\,\mathfrak{t}_K^*$ as

$$(\lambda_1,\dots,\lambda_{p+q}) \mapsto (\epsilon_1\,\lambda_{\sigma(1)}, \epsilon_2\,\lambda_{\sigma(2)},\dots,\epsilon_{p+q}\,\lambda_{\sigma(p+q)})\,,$$

where $\sigma \in \mathfrak{S}_{p+q}$, $\epsilon_i = \pm 1$, $\epsilon_1\cdots\epsilon_{p+q} = 1$, see for instance [HBM+15]. So, for any $w \in W_{G_1}$,

$$\begin{aligned} w\cdot\delta_{G_1} - \delta_{K_1} = &\sum_{i=1}^{p}\big(\epsilon_{\sigma(i)}\,(p+q-\sigma(i)) - (p-i)\big)\,\widehat{x}_i \\ &+ \sum_{i=p+1}^{p+q}\big(\epsilon_{\sigma(i)}\,(p+q-\sigma(i)) - (p+q-i)\big)\,\widehat{x}_i\,, \end{aligned}$$

where $\sigma \in \mathfrak{S}_{p+q}$. But the dominance conditions verified by λ implies $\epsilon_i = 1$, $i = 1,\dots,p-1,p+1,\dots,p+q-1$, and this is also true for ϵ_p and ϵ_{p+q} since $\epsilon_p = \epsilon_{p+q} = -1$ implies $\sigma(p) = p+q = \sigma(p+q)$. Now the dominance conditions also imply that any highest weight λ of the spin representation of K has necessarily the form

$$\lambda = \sum_{i=1}^{p}\left(q+i-\sigma(i)+\frac{1}{2}\right)\widehat{x}_i + \sum_{i=p+1}^{p+q}\left(i-\sigma(i)+\frac{1}{2}\right)\widehat{x}_i\,, \tag{9}$$

where $\sigma \in \mathfrak{S}_{p+q}$ verifies $\sigma(i) \le \sigma(i+1)-1$, $1 \le i \le p-1$, $p+1 \le i \le p+q-1$. Note that this implies $\sigma(i) \ge i$, $1 \le i \le p$, and $\sigma(i) \le i$, $p+1 \le i \le p+q$.

5.3. **The first eigenvalue of the Dirac operator.** We begin by determining necessary conditions for a G-weight $\mu = (\mu_1,\dots,\mu_{p+q+1})$ in order that $\mu_{|\mathfrak{t}_K} = (\mu_1,\dots,\mu_{p+q})$ is a highest weight of the spin representation and $\|\mu+\delta_G\|^2$ is minimal.
Note first that a vector $\mu = (\mu_1,\dots,\mu_{p+q+1}) \in i\mathfrak{t}^*$, is a G-weight if and only if all the μ_i are all simultaneously integers or half-integers.
So, by (9), for such a G-weight, $\mu_{|\mathfrak{t}_K} = (\mu_1,\dots,\mu_{p+q})$ is a highest weight of the spin representation only if μ_{p+q+1} is a half-integer. Furthermore, since

$$\delta_G = (p+q, p+q-1,\dots,1,0)\,,$$

the condition that $\|\mu+\delta_G\|^2$ is minimal implies then $\mu_{p+q} = \pm\frac{1}{2}$ (with the same value in both cases).
So let μ be such a weight. We first consider the case where μ is dominant.

5.3.1. *First case: μ is G-dominant.* By (9), one has

$$\mu+\delta_G = \sum_{i=1}^{p}\left(p+2q+\frac{3}{2}-\sigma(i)\right)\widehat{x}_i + \sum_{i=p+1}^{p+q}\left(p+q+\frac{3}{2}-\sigma(i)\right)\widehat{x}_i\,,$$

where $\sigma \in \mathfrak{S}_{p+q}$ verifies $\sigma(i) \le \sigma(i+1)-1$, $1 \le i \le p-1$, $p+1 \le i \le p+q-1$, and also, as μ is G-dominant $\sigma(p) \le \sigma(p+1)+q-1$.

So

$$\begin{aligned}\|\mu+\delta_G\|^2 &= \sum_{i=1}^{p}\left(p+2q+\frac{3}{2}-\sigma(i)\right)^2+\sum_{i=p+1}^{p+q}\left(p+q+\frac{3}{2}-\sigma(i)\right)^2,\\ &= \sum_{i=1}^{p+q}\sigma(i)^2-2\left(p+q+\frac{3}{2}\right)\sum_{i=1}^{p+q}\sigma(i)-2\,q\sum_{i=1}^{p}\sigma(i)\\ &\qquad+\sum_{i=1}^{p}\left(p+2q+\frac{3}{2}\right)^2+\sum_{i=p+1}^{p+q}\left(p+q+\frac{3}{2}\right)^2,\\ &= \sum_{i=1}^{p+q}i^2-2\left(p+q+\frac{3}{2}\right)\sum_{i=1}^{p+q}i-2\,q\sum_{i=1}^{p}\sigma(i)\\ &\qquad+\sum_{i=1}^{p}\left(p+2q+\frac{3}{2}\right)^2+\sum_{i=p+1}^{p+q}\left(p+q+\frac{3}{2}\right)^2.\end{aligned}$$

Hence

$$\|\mu+\delta_G\|^2 \text{ is minimal} \Longleftrightarrow \sum_{i=1}^{p}\sigma(i) \text{ is maximal}.$$

Note that the conditions $\sigma(i)\leq\sigma(i+1)-1$, $1\leq i\leq p-1$ imply that $\sum_{i=1}^{p}\sigma(i)$ is maximal if and only if $\sigma(p)$ is maximal and $\sigma(p-1)=\sigma(p)-1$,..., $\sigma(1)=\sigma(p)-p+1$. But $\sigma(p)$ can not be $>q$ in that case. Indeed, if $\sigma(p)>q$, then, as we suppose $p\leq q$, $\sigma(1)>1$. So $\sigma(p+1),\dots\sigma(p+q)$ belong to the set $\{1,\dots,\sigma(1)-1,\sigma(p)+1,\dots,p+q\}$, and then the conditions $\sigma(i)\leq\sigma(i+1)-1$, $p+1\leq i\leq p+q-1$, imply $\sigma(p+1)=1$. But that contradicts the condition $\sigma(p)\leq\sigma(p+1)+q-1$. Hence $\|\mu+\delta_G\|^2$ is minimal if and only if

$$\begin{array}{ccc}\sigma(1) &=& q-p+1,\\ \sigma(2) &=& q-p+2,\\ \vdots &\vdots& \vdots\\ \sigma(p) &=& q,\\ \sigma(p+1) &=& 1,\\ \sigma(p+2) &=& 2,\\ \vdots &\vdots& \vdots\\ \sigma(q) &=& q-p,\\ \sigma(q+1) &=& q+1,\\ \vdots &\vdots& \vdots\\ \sigma(p+q) &=& p+q,\end{array}$$

So we may conclude

Lemma 5.4. *If μ is a G-dominant weight such that $\mu_{|\mathfrak{t}_K}$ is a highest weight λ of the spin representation and $\|\mu+\delta_G\|^2$ is minimal, then necessarily,*

$$\mu=(\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p},\pm\frac{1}{2}),$$

and

$$\lambda = (\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p}).$$

5.3.2. *Second case: μ is not G-dominant.* In that case, one obtains also that

$$\|\mu+\delta_G\|^2 \text{ is minimal} \Longleftrightarrow \sum_{i=1}^{p}\sigma(i) \text{ is maximal},$$

but now $\sigma \in \mathfrak{S}_{p+q}$ has only to verify the conditions $\sigma(i) \leq \sigma(i+1)-1$, $1 \leq i \leq p-1$, $p+1 \leq i \leq p+q-1$. In that case $\sigma(p)$ is maximal if and only if $\sigma(p) = p+q$, hence $\|\mu+\delta_G\|^2$ is minimal if and only if

$$\begin{array}{rcc}
\sigma(1) & = & q+1\,,\\
\sigma(2) & = & q+2\,,\\
\vdots & \vdots & \vdots\\
\sigma(p) & = & p+q\,,\\
\sigma(p+1) & = & 1\,,\\
\sigma(p+2) & = & 2\,,\\
\vdots & \vdots & \vdots\\
\sigma(p+q) & = & q\,,
\end{array}$$

hence

$$\mu = (\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p},\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\pm\frac{1}{2}),$$

Now, as μ is conjugate under W_G to one and only one G-dominant weight, one sees that since the Weyl group W_G acts on $i\,\mathfrak{t}^*$ as

$$(\mu_1,\dots,\mu_{p+q+1}) \mapsto (\epsilon_1\,\mu_{\sigma(1)},\epsilon_2\,\mu_{\sigma(2)},\dots,\epsilon_{p+q+1}\,\mu_{\sigma(p+q+1)})\,,$$

where $\sigma \in \mathfrak{S}_{p+q+1}$, $\epsilon_i = \pm 1$, $\epsilon_1\cdots\epsilon_{p+q+1} = 1$, μ is conjugate under the Weyl group to the G-dominant weight met in the above case

$$(\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p},\pm\frac{1}{2}).$$

Finally, we may conclude that

Lemma 5.5. *If a G-dominant weight μ verifies the spin condition and is such that $\|\mu+\delta_G\|^2$ is minimal, then necessarily,*

$$\mu = (\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p},\pm\frac{1}{2}).$$

In order to conclude, we only have to verify that

Lemma 5.6. *The vector $\lambda \in i\,\mathfrak{t}_K^*$ defined by*

$$\lambda = (\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p}),$$

is a highest weight of the spin representation of K.

Proof. As $\mathfrak{p}$ is odd-dimensional, we have to use the description of spinors given in § 2.3.2. By the choice of an orthonormal basis of $\mathfrak{p}$ such as (5), the group $\mathrm{SO}(\mathfrak{p})$ is identified with $\mathrm{SO}(4pq+2(p+q)+1)$, which is itself embedded in the group $\mathrm{SO}(4pq+2(p+q)+2)$ in such a way that it acts trivially on the last vector of the standard basis of $\mathbb{R}^{4pq+2(p+q)+2}$.
Let $(Z,\overline{Z})$ be the Witt basis defined by the two orthonormal vectors on which $\mathrm{SO}(\mathfrak{p})$ acts trivially. Using the notations of § 2.3, a basis of spinors is given by considering the vectors

$$\begin{aligned} &E_I\cdot V_J\cdot\overline{w}\,, \quad \text{if } \#I+\#J \text{ is even},\\ &E_I\cdot V_J\cdot Z\cdot\overline{w}\,, \quad \text{if } \#I+\#J \text{ is odd}.\end{aligned}$$

Denoting for short $E_{ij}=E_{\widehat{x}_i+\widehat{x}_j}$, $E'_{ij}=E_{\widehat{x}_i-\widehat{x}_j}$, $1\le i\le p$, $p+1\le j\le p+q$, and $V_j=E_{\widehat{x}_j-\widehat{x}_{p+q+1}}$, $1\le j\le p+q$, one gets from (6), that the spinor

$$\begin{aligned} v_\lambda :=& \prod_{\substack{1\le i\le p\\ p+1\le j\le p+q}} E_{ij}\cdot \prod_{\substack{1\le i\le p\\ q+1\le j\le p+q}} E'_{ij}\cdot V_1\cdots V_{p+q}\cdot\overline{w}\,, \text{ if } (p+1)(p+q) \text{ is even},\\ :=& \prod_{\substack{1\le i\le p\\ p+1\le j\le p+q}} E_{ij}\cdot \prod_{\substack{1\le i\le p\\ q+1\le j\le p+q}} E'_{ij}\cdot V_1\cdots V_{p+q}\cdot Z\cdot\overline{w}\,, \text{ if } (p+1)(p+q) \text{ is odd},\end{aligned}$$

is a weight-vector for the weight λ.
Now, it may be checked that the root-vector $u_k\cdot v_{k+1}$, associated to the simple root $\theta'_k=\widehat{x}'_k-\widehat{x}'_{k+1}$, acts on spinors by a linear combination of

$$\begin{aligned} &\sum_{j=p+1}^{p+q}\overline{E_{k+1\,j}}\cdot E_{kj}\,,\ \sum_{j=p+1}^{p+q}\overline{E'_{k+1\,j}}\cdot E'_{kj}\,, \quad \text{and} \quad \overline{V_{k+1}}\cdot V_k\,, \quad \text{if } 1\le k\le p-1,\\ &\sum_{i=1}^{p}\overline{E_{ik+1}}\cdot E_{ik}\,,\ \sum_{i=1}^{p}\overline{E'_{ik}}\cdot E'_{ik+1}\,, \quad \text{and} \quad \overline{V_{k+1}}\cdot V_k\,, \quad \text{if } p+1\le k\le p+q-1.\end{aligned}$$

So v_λ is killed by the action of $u_k\cdot v_{k+1}$ since all the E_{ij}'s and all the V_j's occur in the expression of v_λ, and either E'_{kj} (resp. E'_{ik+1}) occur in that expression or both E'_{kj} and $E'_{k+1\,j}$, (resp. E'_{ik+1} and E'_{ik}) do not occur in the expression.
In the same way, v_λ is killed by the action of the root-vector $u_p\cdot e_{2p+2q+1}$, associated to the simple root $\widehat{x}'_p$, since that root-vector acts on spinors by a linear combinations of

$$\sum_{j=p+1}^{p+q}E_{pj}\cdot\overline{V_j}\,,\ \sum_{j=p+1}^{p+q}E'_{pj}\cdot V_j\,, \quad \text{and} \quad V_p\cdot(e_{2p+2q+1}\cdot e_{2p+2q+2})\,.$$

Finally, v_λ is also killed by the action of the root-vector $u_{p+q+1}\cdot e_{2p+2q+2}$, associated to the simple root $\widehat{x}'_{p+q}$, since that root-vector acts on spinors by a linear combinations of

$$\sum_{i=1}^{p}E_{i\,p+q}\cdot\overline{V_i}\,,\ \sum_{i=1}^{p}E'_{i\,p+q}\cdot V_i\,, \quad \text{and} \quad V_{p+q}\cdot(e_{2p+2q+1}\cdot e_{2p+2q+2})\,.$$

Thus v_λ is a maximal vector, and so λ is a highest weight of the spin representation. □

So $\mu_\pm = (\underbrace{p+\frac{1}{2},\dots,p+\frac{1}{2}}_{q},\underbrace{\frac{1}{2},\dots,\frac{1}{2}}_{p},\pm\frac{1}{2})$ are two G-dominant weights verifying the spin condition and such that $\|\mu+\delta_G\|^2$ is minimal among all the G-dominant weights μ verifying the spin condition.
Hence the square of the first eigenvalue λ of the Dirac operator is given by

$$\begin{aligned}\lambda^2 &= \frac{1}{4(p+q)}\left\langle \mu_+, \mu_+ + 2\delta_G\right\rangle + \frac{(2p+1)(2q+1)}{16}\,,\\ &= \frac{1}{16\,(p+q)}\left(8pq\,(2p+q+1)+4p\,(p+1)+4q\,(q+1)+1\right).\end{aligned}$$

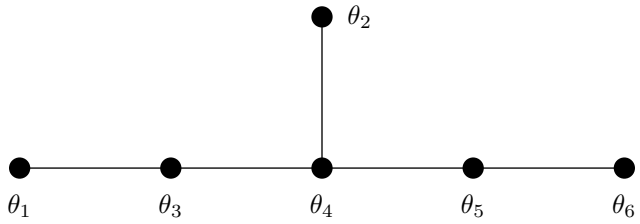
Note that for $p=0$, we retrieve[2] the value of the square of the first eigenvalue of the Dirac operator on the standard sphere S^{2q+1} :

$$\lambda^2 = \frac{1}{4q}\left(\frac{2q+1}{2}\right)^2,$$

cf. [Sul79].

6. The symmetric space $\frac{\mathrm{E}_6}{\mathrm{F}_4}$.

We use here the results of Murakami [Mur65] (for an outline, see for instance the section "non-inner involutions" in chapter 3 of [BR90]).
Explicit computations can be made (and are obtained) with the help of the programs GAP3, [S+97], and LiE, [vLCL92].
First, the existence of non-inner involutions corresponds with the existence of non-trivial symmetries of the Dynkin diagram.
The Dynkin diagram of E_6 is



There is only one non-trivial symmetry s given by

$$\begin{aligned} s(\theta_1)=\theta_6\,,\quad s(\theta_2)=\theta_2\,,\quad s(\theta_3)=\theta_5\,,\\ s(\theta_4)=\theta_4\,,\quad s(\theta_5)=\theta_3\,,\quad s(\theta_6)=\theta_1\,.\end{aligned}$$

The symmetry s is extended by linearity to an involution σ^* of $i\,\mathfrak{t}^*$, which itself induces an involution σ_* of $\mathfrak{t}$, by means of the scalar product (re-normalized here in such a way that all simple roots θ verify $\|\theta\|^2=2$).
Now, choosing a root-vector E_θ for each simple root θ, σ_* is extended to the span of these vectors by

$$\sigma_*(E_\theta) = E_{\sigma_*(\theta)}\,. \tag{10}$$

[2] in that case, $\Phi_2=\emptyset$, so the spin representation of K has only one highest weight.

Finally σ_* can be uniquely extended to a non-inner involution of $\mathfrak{g}$ (cf. § 14.2 in [Hum72]).
A first outer symmetric space structure is obtained by considering the connected subgroup K of E_6 whose Lie algebra is the set $\{X \in \mathfrak{g}\,;\, \sigma_*(X) = X\}$. It is a simple group, and a system of simple roots (relatively to the maximal torus $T_K = K \cap T$) is obtained by considering the restriction to $\mathfrak{t}_K$ of the simple roots θ_i, $1 \leq i \leq 6$, (cf. Proposition 3.20 in [BR90]).
The group E_6 has 36 positive roots. The positive roots θ such that $\sigma^*(\theta) = \theta$ all belong to Φ_1^+ by (10). The partition of the set of positive roots is given by

$$\begin{aligned}
\Phi_1^+ = \{&\theta_2, \theta_4, \theta_2+\theta_4, \theta_3+\theta_4+\theta_5, \theta_2+\theta_3+\theta_4+\theta_5, \theta_1+\theta_3+\theta_4+\theta_5+\theta_6\,,\\
&\theta_2+\theta_3+2\,\theta_4+\theta_5, \theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6\,,\\
&\theta_1+\theta_2+\theta_3+2\,\theta_4+\theta_5+\theta_6, \theta_1+\theta_2+2\,\theta_3+2\,\theta_4+2\,\theta_5+\theta_6\,,\\
&\theta_1+\theta_2+2\,\theta_3+3\,\theta_4+2\,\theta_5+\theta_6, \theta_1+2\,\theta_2+2\,\theta_3+3\,\theta_4+2\,\theta_5+\theta_6\}\,,\\
\Phi_2^+ = \emptyset\,,&\\
\Phi_3'^+ = \{&\theta_1, \theta_3, \theta_1+\theta_3, \theta_3+\theta_4, \theta_1+\theta_3+\theta_4, \theta_2+\theta_3+\theta_4, \theta_1+\theta_2+\theta_3+\theta_4\,,\\
&\theta_1+\theta_3+\theta_4+\theta_5, \theta_1+\theta_2+\theta_3+\theta_4+\theta_5, \theta_1+\theta_2+\theta_3+2\,\theta_4+\theta_5\,,\\
&\theta_1+\theta_2+2\,\theta_3+2\,\theta_4+\theta_5, \theta_1+\theta_2+2\,\theta_3+2\,\theta_4+\theta_5+\theta_6\}\,,\\
\sigma^*(\Phi_3'^+) = \{&\theta_6, \theta_5, \theta_5+\theta_6, \theta_4+\theta_5, \theta_4+\theta_5+\theta_6, \theta_2+\theta_4+\theta_5, \theta_2+\theta_4+\theta_5+\theta_6\,,\\
&\theta_3+\theta_4+\theta_5+\theta_6, \theta_2+\theta_3+\theta_4+\theta_5+\theta_6, \theta_2+\theta_3+2\,\theta_4+\theta_5+\theta_6\,,\\
&\theta_2+\theta_3+2\,\theta_4+2\,\theta_5+\theta_6, \theta_1+\theta_2+\theta_3+2\,\theta_4+2\,\theta_5+\theta_6\}\,.
\end{aligned}$$

The set of positive K-roots is

$$\Phi_K^+ = \{\theta_{|\mathfrak{t}_K}\,;\, \theta \in \Phi_1^+ \cup \Phi_3'^+\}\,,$$

and a system of simple K-roots is given by

$$\theta_1' = \theta_6' = \frac{1}{2}(\theta_1' + \theta_6')\,, \quad \theta_2'\,, \quad \theta_3' = \theta_5' = \frac{1}{2}(\theta_3' + \theta_5')\,, \quad \theta_4'\,.$$

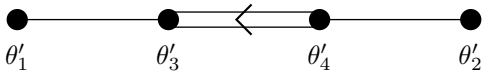
Note that

$$\begin{aligned}
\dim\mathfrak{g} = 78\,, \quad \dim\mathfrak{k} = 52\,, \quad \dim\mathfrak{p} = 26\,,\\
\dim\mathfrak{t} = 6\,, \quad \dim\mathfrak{t}_K = 4\,, \quad \dim\mathfrak{t}_0 = 2\,.
\end{aligned}$$

One has

$$\begin{aligned}
\|\theta_1'\|^2 = \|\theta_3'\|^2 = 1\,, \quad \|\theta_2'\|^2 = \|\theta_4'\|^2 = 2\,,\\
\langle\theta_1', \theta_3'\rangle = -\frac{1}{2}\,, \quad \langle\theta_2', \theta_4'\rangle = \langle\theta_3', \theta_4'\rangle = -1\,,
\end{aligned}$$

all the other scalar products being zero. Hence the Dynkin diagram of K is



Setting

$$\alpha_1 = \theta_2'\,, \quad \alpha_2 = \theta_4'\,, \quad \alpha_3 = \theta_3'\,, \quad \text{and} \quad \alpha_4 = \theta_1'\,,$$

this is the "classical" Dynkin diagram of the group F_4. In that basis, the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

In the following, $\omega_1, \omega_2, \omega_3, \omega_4$ are the fundamental weights associated with α_1, α_2, α_3, α_4.

6.1. **Highest weights of the spin representation of K.** Since $\Phi_2 = \emptyset$, and $\dim \mathfrak{t}_0 = 2$, we may conclude from lemma 2.4,

Lemma 6.1. *The spin representation of K has only one highest weight λ with multiplicity 2,*

$$\lambda = 5\,\theta'_1 + 3\,\theta'_2 + 9\,\theta'_3 + 6\,\theta'_4\,.$$

Proof. By the result of lemma 2.4,

$$\begin{aligned} \lambda = \frac{1}{2}\sum_{\gamma\in\Phi_3'^+}\gamma' &= 5\,\theta'_1 + 3\,\theta'_2 + 9\,\theta'_3 + 6\,\theta'_4\,, \\ &= 3\,\alpha_1 + 6\,\alpha_2 + 9\,\alpha_3 + 5\,\alpha_4 = \omega_3 + \omega_4\,. \end{aligned}$$

Any irreducible module with highest weight $\omega_3 + \omega_4$ has dimension 2^{12}. Since $\dim \mathfrak{t}_0 = 2$, one knows from § 2.3 that the multiplicity of the weight is at least 2. Now $2 \times 2^{12} = 2^{13} = 2^{\dim(\mathfrak{p})/2} = \dim \Sigma$, hence the result. □

6.2. **The first eigenvalue of the Dirac operator.** As we did before, we first determine G-weights μ (non necessarily dominant) such that $\mu_{|\mathfrak{t}_K} = \lambda$ and $\|\mu+\delta_G\|^2$ is minimal.
Let $\mu = \sum_{i=1}^6 \mu_i\,\theta_i \in i\,\mathfrak{t}^*$. First

$$\mu_{|\mathfrak{t}_K} = \lambda \Longleftrightarrow \begin{cases} \mu_1 + \mu_6 &= 5\,, \\ \mu_2 &= 3\,, \\ \mu_3 + \mu_5 &= 9\,, \\ \mu_4 &= 6\,. \end{cases}$$

As $\|\theta_i\|^2 = 2$, $1 \le i \le 6$, μ is a G-weight (resp. dominant G-weight) if and only if $\langle \mu, \theta_i\rangle \in \mathbb{Z}$, (resp. $\mathbb{N}$), hence $\mu_{|\mathfrak{t}_K} = \lambda$ and μ is a G-weight (resp. dominant G-weight) if and only if

$$\left.\begin{cases} \mu_1 + \mu_6 &= 5\,, \\ \mu_2 &= 3\,, \\ \mu_3 + \mu_5 &= 9\,, \\ \mu_4 &= 6\,, \end{cases} \quad \text{and} \quad \begin{aligned} 2\,\mu_1 - \mu_3 \\ -\mu_1 + 2\,\mu_3 - 6 \\ \mu_1 - 2\,\mu_3 + 7 \\ -2\,\mu_1 + \mu_3 + 1 \end{aligned} \right\} \in \mathbb{Z} \text{ (resp. } \mathbb{N}).$$

Setting

$$\left.\begin{cases} k := 2\,\mu_1 - \mu_3\,, \\ l := -\mu_1 + 2\,\mu_3 - 6\,, \end{cases} \quad \left(\Longleftrightarrow \begin{cases} \mu_1 = \frac{2k+l+6}{3}\,, \\ \mu_2 = \frac{k+2l+12}{3}\,, \end{cases}\right.\right.$$

the last condition is equivalent to

$$k \text{ and } l \in \mathbb{Z} \quad (\text{resp. } k = 0 \text{ or } 1 \text{ and } l = 0 \text{ or } 1).$$

Viewing then $\|\mu+\delta_G\|^2$ as a polynomial $f(k,l)$ of the variables k and l, one gets

$$\frac{\partial f}{\partial k}(k,l)=2\,(2\mu_1-5)\quad\text{and}\quad\frac{\partial f}{\partial l}(k,l)=2\,(2\mu_3-9)\,,$$

Thus $f(k,l)$ has only one critical point $(\frac{1}{2},\frac{1}{2})$. Now

$$\frac{\partial^2 f}{\partial k^2}(k,l)=\frac{8}{3}\,,\quad\frac{\partial^2 f}{\partial k\,\partial l}(k,l)=\frac{4}{3}\quad\text{and}\quad\frac{\partial^2 f}{\partial l^2}(k,l)=\frac{8}{3}\,,$$

So

$$f(k,l)-f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{2}{3}\,\left((k-\frac{1}{2})^2+(l-\frac{1}{2})^2+(k+l-1)^2\right)\,.$$

Hence among the G-weights μ such that $\mu_{|\mathfrak{t}_K}=\lambda$, the minimum of $\|\mu+\delta_G\|^2$ is obtained if and only if

$$(k,l)=(0,1)\,,\left(\Leftrightarrow(\mu_1,\mu_3)=\left(\frac{7}{3},\frac{14}{3}\right)\right)\,,$$

or

$$(k,l)=(1,0)\,,\left(\Leftrightarrow(\mu_1,\mu_3)=\left(\frac{8}{3},\frac{13}{3}\right)\right)\,.$$

By the above remarks, the corresponding two weights

$$\mu_1=\frac{7}{3}\,\theta_1+3\,\theta_2+\frac{14}{3}\,\theta_3+6\,\theta_4+\frac{13}{3}\,\theta_5+\frac{8}{3}\,\theta_6\,,$$

and

$$\mu_2=\frac{8}{3}\,\theta_1+3\,\theta_2+\frac{13}{3}\,\theta_3+6\,\theta_4+\frac{14}{3}\,\theta_5+\frac{7}{3}\,\theta_6\,,$$

are G-dominant. Hence we may conclude exactly as we did it in the first example with the result of lemma 3.8. We have first to note that, for the scalar product $\langle\,,\,\rangle_K$ induced by the Killing form sign-changed, the "strange formula" of Freudenthal and de Vries, [FdV69], gives

$$\|\delta_G\|_K^2=\frac{\dim(\mathfrak{g})}{24}=\frac{13}{4}\,,$$

whereas for our choice of scalar product $\|\delta_G\|^2=78$, hence $\langle\,,\,\rangle_K=\frac{1}{24}\,\langle\,,\,\rangle$.
So, finally, the square of the first eigenvalue of the Dirac operator is given by

$$\frac{1}{24}\langle\mu_1+2\,\delta_G,\mu_1\rangle+\frac{\dim(\mathfrak{p})}{16}=\frac{20}{9}+\frac{13}{8}=\frac{277}{72}\,.$$

7. The symmetric space $\frac{\mathrm{E}_6}{\mathrm{Sp}_4}$.

By the result of Murakami, there also exists a complementary non-inner involution on the Lie algebra of E_6, which is not conjugate to the above non-inner involution σ and is defined on the Lie algebra by

$$\sigma'_*=\sigma_*\circ\mathrm{Ad}_{\exp(\pi\,\xi_2)}\,,$$

where $\xi_2\in\mathfrak{t}$ is defined by $\theta_j(\xi_2)=i\,\delta_{j2}$, cf. Theorem 3.25 in [BR90].
A new outer symmetric space structure is obtained by considering the connected subgroup K' of E_6 whose Lie algebra is the set $\{X\in\mathfrak{g}\,;\,\sigma'_*(X)=X\}$.
Note that as $\xi_2\in\mathfrak{t}$, $\sigma'^*=\sigma^*$.

The partition of the set of positive roots is now given by

$$\begin{aligned}
\Phi_1^+ &= \{\theta_4, \theta_3+\theta_4+\theta_5, \theta_1+\theta_3+\theta_4+\theta_5+\theta_6,\\
&\qquad \theta_1+2\,\theta_2+2\,\theta_3+3\,\theta_4+2\,\theta_5+\theta_6\}\,,\\
\Phi_2^+ &= \{\theta_2, \theta_2+\theta_4, \theta_2+\theta_3+\theta_4+\theta_5, \theta_2+\theta_3+2\,\theta_4+\theta_5\,,\\
&\qquad \theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6, \theta_1+\theta_2+\theta_3+2\,\theta_4+\theta_5+\theta_6,,\\
&\qquad \theta_1+\theta_2+2\,\theta_3+2\,\theta_4+2\,\theta_5+\theta_6, \theta_1+\theta_2+2\,\theta_3+3\,\theta_4+2\,\theta_5+\theta_6\}\,,\\
\Phi_3'^+ &= \{\theta_1, \theta_3, \theta_1+\theta_3, \theta_3+\theta_4, \theta_1+\theta_3+\theta_4, \theta_2+\theta_3+\theta_4, \theta_1+\theta_2+\theta_3+\theta_4\,,\\
&\qquad \theta_1+\theta_3+\theta_4+\theta_5, \theta_1+\theta_2+\theta_3+\theta_4+\theta_5, \theta_1+\theta_2+\theta_3+2\,\theta_4+\theta_5\,,\\
&\qquad \theta_1+\theta_2+2\,\theta_3+2\,\theta_4+\theta_5, \theta_1+\theta_2+2\,\theta_3+2\,\theta_4+\theta_5+\theta_6\}\,,\\
\sigma^*(\Phi_3'^+) &= \{\theta_6, \theta_5, \theta_5+\theta_6, \theta_4+\theta_5, \theta_4+\theta_5+\theta_6, \theta_2+\theta_4+\theta_5, \theta_2+\theta_4+\theta_5+\theta_6\,,\\
&\qquad \theta_3+\theta_4+\theta_5+\theta_6, \theta_2+\theta_3+\theta_4+\theta_5+\theta_6, \theta_2+\theta_3+2\,\theta_4+\theta_5+\theta_6\,,\\
&\qquad \theta_2+\theta_3+2\,\theta_4+2\,\theta_5+\theta_6, \theta_1+\theta_2+\theta_3+2\,\theta_4+2\,\theta_5+\theta_6\}\,.
\end{aligned}$$

The set of positive K'-roots is

$$\Phi_K'^+ = \{\theta_{|\mathfrak{t}'_K};\, \theta \in \Phi_1^+ \cup \Phi_3'^+\}\,,$$

and a system of simple K'-roots is given by

$$\alpha_1' := \theta_2' + \theta_4' + \frac{1}{2}(\theta_3' + \theta_5')\,,\ \alpha_2' := \frac{1}{2}(\theta_1' + \theta_6')\,,\ \alpha_3' := \frac{1}{2}(\theta_3' + \theta_5')\,,\ \alpha_4' = \theta_4'\,,$$

(cf. Theorem 3.25 in [BR90]).
Note that here

$$\begin{aligned}
\dim\mathfrak{g} = 78\,, \quad \dim\mathfrak{k}' = 36\,, \quad \dim\mathfrak{p} = 42\,,\\
\dim\mathfrak{t} = 6\,, \quad \dim\mathfrak{t}_{K'} = 4\,, \quad \dim\mathfrak{t}_0 = 2\,.
\end{aligned}$$

One has

$$\begin{aligned}
\|\alpha_1'\|^2 = \|\alpha_2'\|^2 = \|\alpha_3'\|^2 = 1\,, \quad \|\alpha_4'\|^2 = 2\,,\\
\langle\alpha_1', \alpha_2'\rangle = \langle\alpha_2', \alpha_3'\rangle = -\frac{1}{2}\,, \quad \langle\alpha_3', \alpha_4'\rangle = -1\,,
\end{aligned}$$

all the other scalar products being zero. The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -2 & 2 \end{pmatrix}\,,$$

so the Dynkin diagram of K' is

This is the Dynkin diagram of Sp_4.

7.1. **Highest weights of the spin representation of K'.** By the result of lemma 2.4, any highest weight of the spin representation of K has the form

$$\frac{1}{2}\left(\sum_{\beta\in\Phi_2^+}\pm\beta'+\sum_{\gamma\in\Phi_3'^+}\gamma'\right).$$

Lemma 7.1. *The spin representation of K' has three highest weights both of multiplicity 2 :*

$$\begin{aligned}\lambda_1&:=7\,\alpha_1'+9\,\alpha_2'+10\,\alpha_3'+5\,\alpha_4',\\ \lambda_2&:=6\,\alpha_1'+9\,\alpha_2'+11\,\alpha_3'+6\,\alpha_4',\\ \lambda_3&:=5\,\alpha_1'+9\,\alpha_2'+12\,\alpha_3'+6\,\alpha_4'.\end{aligned}$$

Proof. Note first that

$$\sum_{\gamma\in\Phi_3'^+}\gamma'=5\,\alpha_1'+3\,\alpha_2'+6\,\alpha_3'+3\,\alpha_4'.$$

Now, denoting by $\beta_1,\dots,\beta_8$ the roots occurring in that order in the expression of Φ_2^+ above, it is easily checked that

$$\begin{aligned}\lambda_1&=\frac{1}{2}(\beta_1'+\beta_2'+\beta_3'+\beta_4'+\beta_5'+\beta_6'+\beta_7'+\beta_8')+\frac{1}{2}\sum_{\gamma\in\Phi_3'^+}\gamma',\\ \lambda_2&=\frac{1}{2}(-\beta_1'+\beta_2'+\beta_3'+\beta_4'+\beta_5'+\beta_6'+\beta_7'+\beta_8')+\frac{1}{2}\sum_{\gamma\in\Phi_3'^+}\gamma',\\ \lambda_3&=\frac{1}{2}(-\beta_1'-\beta_2'+\beta_3'+\beta_4'+\beta_5'+\beta_6'+\beta_7'+\beta_8')+\frac{1}{2}\sum_{\gamma\in\Phi_3'^+}\gamma',\end{aligned}$$

so λ_1, λ_2 and λ_3 are actually weights of the spin representation of K'.
It is also easily checked that they are also G-dominant. Indeed, considering the basis of fundamental weights ω_1', ω_2', ω_3', ω_4' associated with α_1', α_2', α_3', α_4', one gets[3]

$$\begin{aligned}\lambda_1&=5\,\omega_1'+\omega_2'+\omega_3',\\ \lambda_2&=3\,\omega_1'+\omega_2'+\omega_3'+\omega_4',\\ \lambda_3&=\omega_1'+\omega_2'+3\,\omega_3'.\end{aligned}$$

Note now that, for any $i,j=1,2,3$, $i\neq j$, any relation of the form $\lambda_i\prec\lambda_j$, where "$\prec$" is the standard order on weights, is impossible, so λ_i can not be an element of the set of weights of an irreducible module with highest weight λ_j. This implies that λ_1, λ_2 and λ_3 are actually highest weights of the spin representation of K'.

[3]multiplying for instance the transpose of the Cartan matrix with the vector representing the weight in the basis $(\alpha_1',\alpha_2',\alpha_3',\alpha_4')$.

Since $\dim \mathfrak{t}_0 = 2$, one knows from § 2.3 that the multiplicity of each such highest weight is at least 2. Now, denoting by n_i the dimension of any irreducible module with highest weight λ_i, $i = 1, 2, 3$, one gets with the help of the LiE program

$$n_1 = 180224\,, \quad n_2 = 524288 \quad \text{and} \quad n_3 = 344064\,.$$

Now

$$2 \times n_1 + 2 \times n_2 + 2 \times n_3 = 2^{21} = 2^{\dim(\mathfrak{p})/2} = \dim \Sigma\,,$$

hence the result. □

Remark 7.2. *Actually, the determination of the three possible highest weights was obtained by using the same argument as in § 5.2. Indeed the α's in Φ_1 and the β's in Φ_2 appear to be respectively compact and non-compact roots for the inner symmetric space* $\mathrm{SO}(8)/(\mathrm{SO}(4) \times \mathrm{SO}(4))$*, relatively to the standard torus of* $\mathrm{SO}(8)$*. Using the results of R. Parthasarathy in* [Par71]*, we obtain, with the help of the GAP3 program, the twelve highest weights of the spin representation for the symmetric space* $\mathrm{SO}(8)/(\mathrm{SO}(4) \times \mathrm{SO}(4))$*, providing a list of twelve candidates for the possible highest weights of the spin representation of K'. Finally, the dominance condition (for K') reduces that list to the three weights λ_1, λ_2 and λ_3.*

7.2. **The first eigenvalue of the Dirac operator.** We determine G-weights μ (non necessarily dominant) such that $\mu_{|\mathfrak{t}_K} = \lambda_1$, λ_2 or λ_3, and $\|\mu+\delta_G\|^2$ is minimal. Let $\mu = \sum_{i=1}^{6} \mu_i\, \theta_i \in i\,\mathfrak{t}^*$. First

$$\mu_{|\mathfrak{t}_K} = \mu_2\, \alpha'_1 + (\mu_1 + \mu_6)\, \alpha'_2 + (\mu_3 + \mu_5 - \mu_2)\, \alpha'_3 + (\mu_4 - \mu_2)\, \alpha'_4\,,$$

so $\mu_{|\mathfrak{t}_K} = \lambda_i$, $i = 1, 2, 3$ if and only if

$$\begin{aligned}
&\mu_1 + \mu_6 = 9\,, \quad \mu_2 = 7\,, \quad \mu_3 + \mu_5 = 17\,, \quad \mu_4 = 12\,,\\
&\mu_1 + \mu_6 = 9\,, \quad \mu_2 = 6\,, \quad \mu_3 + \mu_5 = 17\,, \quad \mu_4 = 12\,,\\
&\mu_1 + \mu_6 = 9\,, \quad \mu_2 = 5\,, \quad \mu_3 + \mu_5 = 17\,, \quad \mu_4 = 11\,.
\end{aligned}$$

Let μ be such a weight. We first examine the case where it is G-dominant.

7.2.1. *First case: μ is G-dominant.* The weight μ is G-dominant if and only if

$$\begin{cases}
2\,\mu_1 - \mu_3 \in \mathbb{N}\,,\\
\mu_2 = 7 \text{ and } \mu_4 = 12\,, \text{ or } \mu_2 = 6 \text{ and } \mu_4 = 12\,,\\
-\mu_1 + 2\,\mu_3 - 12 \in \mathbb{N}\,,\\
\mu_1 - 2\,\mu_3 + 13 \in \mathbb{N}\,,\\
-2\,\mu_1 + \mu_3 + 1 \in \mathbb{N}\,.
\end{cases}$$

Thus setting

$$\begin{cases} k := 2\,\mu_1 - \mu_3\,,\\ l := -\mu_1 + 2\,\mu_3 - 12\,, \end{cases} \qquad \left(\iff \begin{cases} \mu_1 = \frac{2k+l+12}{3}\,,\\ \mu_3 = \frac{k+2l+24}{3}\,, \end{cases} \right)$$

$$\mu \text{ is } G\text{-dominant} \iff \begin{cases}
k = 0 \text{ or } 1\,,\\
l = 0 \text{ or } 1\,,\\
\mu_2 = 7 \text{ and } \mu_4 = 12\,, \text{ or } \mu_2 = 6 \text{ and } \mu_4 = 12\,.
\end{cases}$$

The term involving μ_2 in the expression of $\|\mu + \delta_G\|^2$ is

$$2\big((\mu_2 + 11)^2 - (\mu_2 + 11)(\mu_4 + 21)\big)\,,$$

hence the minimum of $\|\mu+\delta_G\|^2$ is obtained only when $\mu_2 = 6$ (and then $\mu_4 = 12$).

Viewing then $\|\mu+\delta_G\|^2$ as a polynomial $f(k,l)$ of the variables k and l, one gets

$$\frac{\partial f}{\partial k}(k,l)=\frac{2}{3}\left(6\,\mu_1-27\right)\quad\text{and}\quad\frac{\partial f}{\partial l}(k,l)=\frac{2}{3}\left(6\,\mu_3-51\right).$$

Thus $f(k,l)$ has only one critical point $(\frac{1}{2},\frac{1}{2})$. Now

$$\frac{\partial^2 f}{\partial k^2}(k,l)=\frac{8}{3}\,,\quad\frac{\partial^2 f}{\partial k\,\partial l}(k,l)=\frac{4}{3}\quad\text{and}\quad\frac{\partial^2 f}{\partial l^2}(k,l)=\frac{8}{3}\,,$$

So

$$f(k,l)-f\left(\frac{1}{2},\frac{1}{2}\right)=\frac{2}{3}\,\left((k-\frac{1}{2})^2+(l-\frac{1}{2})^2+(k+l-1)^2\right).$$

Hence among the dominant G-weights μ such that $\mu_{|\mathfrak{t}_K}=\lambda_i$, $i=1,2,3$, the minimum of $\|\mu+\delta_G\|^2$ is obtained if and only if

$$(k,l)=(0,1)\,,\left(\Leftrightarrow(\mu_1,\mu_3)=\left(\frac{13}{3},\frac{26}{3}\right)\right),$$

or

$$(k,l)=(1,0)\,,\left(\Leftrightarrow(\mu_1,\mu_3)=\left(\frac{14}{3},\frac{25}{3}\right)\right),$$

hence for the two weights

$$\mu_1=\frac{13}{3}\,\theta_1+6\,\theta_2+\frac{26}{3}\,\theta_3+12\,\theta_4+\frac{25}{3}\,\theta_5+\frac{14}{3}\,\theta_6\,,$$

or

$$\mu_2=\frac{14}{3}\,\theta_1+6\,\theta_2+\frac{25}{3}\,\theta_3+12\,\theta_4+\frac{26}{3}\,\theta_5+\frac{13}{3}\,\theta_6\,,$$

and in that case $\mu_{|\mathfrak{t}_K}=\lambda_2$.
It is easily verified that

$$\langle\mu_1+2\delta_G,\mu_1\rangle=\langle\mu_2+2\delta_G,\mu_2\rangle=\frac{340}{3}\,.$$

7.2.2. *Second case: μ is not G-dominant.* In that case, k and l are arbitrary integers, and, considering the term involving μ_2 in the expression of $\|\mu+\delta_G\|^2$, the minimum of $\|\mu+\delta_G\|^2$ is obtained only when $\mu_2=5$ (and then $\mu_4=11$). Up to a constant term, $\|\mu+\delta_G\|^2$ is then the same polynomial $f(k,l)$ as above, so the minimum is obtained for $k=0$ and $l=1$, or $k=1$ and $l=0$, hence we may concude that among the G-weights μ such that $\mu_{|\mathfrak{t}_K}=\lambda_i$, $i=1,2,3$, the minimum of $\|\mu+\delta_G\|^2$ is obtained for the two weights

$$\mu_1'=\frac{13}{3}\,\theta_1+5\,\theta_2+\frac{26}{3}\,\theta_3+11\,\theta_4+\frac{25}{3}\,\theta_5+\frac{14}{3}\,\theta_6\,,$$

or

$$\mu_2'=\frac{14}{3}\,\theta_1+5\,\theta_2+\frac{25}{3}\,\theta_3+11\,\theta_4+\frac{26}{3}\,\theta_5+\frac{13}{3}\,\theta_6\,,$$

and in that case $\mu_{|\mathfrak{t}_K}=\lambda_3$.
Now each weight is conjugate under the action of the Weyl group W_G to one and only one dominant G-weight. Using the function "W_orbit" of the LiE program, it may be checked that μ_1' is conjugate to μ_1, and μ_2' to μ_2. Hence we may conclude that

Lemma 7.3. *Among the G-dominant weights μ verifying the spin condition,*

$$\|\mu+\delta_G\|^2 \text{ is minimal if and only if } \mu=\mu_1 \text{ or } \mu=\mu_2.$$

Hence the square of the first eigenvalue of the Dirac operator is

$$\frac{1}{24}\langle \mu_1+2\,\delta_G,\mu_1\rangle+\frac{\dim(\mathfrak{p})}{16}=\frac{85}{18}+\frac{21}{8}=\frac{529}{72}\,.$$

8. Appendix

8.1. Proof of lemma 2.1.

R1: For any $\theta\in\Phi_3$, one has

$$\forall X\in\mathfrak{t}_K\,,\quad [X,\sigma_*(E_\theta)]=[\sigma_*(X),\sigma_*(E_\theta)]=\sigma_*\left([X,E_\theta]\right)=\theta(X)\,\sigma_*(E_\theta)\,, \tag{11}$$

and

$$\forall Y\in\mathfrak{t}_0\,,\quad [Y,\sigma_*(E_\theta)]=-[\sigma_*(Y),\sigma_*(E_\theta)]=-\sigma_*\left([Y,E_\theta]\right)=-\theta(Y)\,\sigma_*(E_\theta)\,. \tag{12}$$

Hence

$$\forall X\in\mathfrak{t}_K\,,\ [X,U_\theta]=\theta(X)\,U_\theta\quad\text{and}\quad [X,V_\theta]=\theta(X)\,V_\theta\,. \tag{13}$$

Now, suppose that for any $X\in\mathfrak{t}_K$, $\theta(X)=0$. Then there exists $Y\in\mathfrak{t}_0$ such that $\theta(Y)\neq 0$. Set $U_\theta=A_\theta+i\,B_\theta$, where A_θ and $B_\theta\in\mathfrak{g}$. By (13), for any $X\in\mathfrak{t}_K$, one has $[X,U_\theta]=0$, which implies that both A_θ and B_θ belong to the centralizer of $\mathfrak{t}_K$ in $\mathfrak{k}$ which is equal to $\mathfrak{t}_K$, as $\mathfrak{t}_K$ is maximal. Hence $U_\theta\in\mathfrak{t}_{K,\mathbb{C}}$, so $[Y,U_\theta]=0$. But using (12), $[Y,U_\theta]=\theta(Y)\,V_\theta$, which can not be 0 as $\theta(Y)\neq 0$ and $V_\theta\neq 0$. Hence, there exists $X\in\mathfrak{t}_K$ such that $\theta(X)\neq 0$, and so by (13), θ' is a K-root. □

R2: Let $\theta\in\Phi_1\cup\Phi_2$. As $\sigma_*(E_\theta)=\pm E_\theta$, one has for any $Y\in\mathfrak{t}_0$, $[Y,\sigma_*(E_\theta)]=\theta(Y)\,\sigma_*(E_\theta)$. But (12) implies $[Y,\sigma_*(E_\theta)]=-\theta(Y)\,\sigma_*(E_\theta)$, hence $\theta(Y)=0$. Conversely, if $\theta_{|\mathfrak{t}_0}=0$, then by (11), for any $X\in\mathfrak{t}$,

$$[X,\sigma_*(E_\theta)]=\theta(X)\,\sigma_*(E_\theta)\,,$$

so $\sigma_*(E_\theta)\in g_\theta$. Hence there exists $\lambda\in\mathbb{C}$ such that $\sigma_*(E_\theta)=\lambda\,E_\theta$. As σ_* is an involution, one has $\lambda^2=1$, hence $\sigma_*(E_\theta)$ belongs to $\mathfrak{k}_{\mathbb{C}}$ or $\mathfrak{p}_{\mathbb{C}}$, so $\theta\in\Phi_1\cup\Phi_2$.

R3: By **R1**, β' is a K-root, and by **R2**, there exists $Y\in\mathfrak{t}_0$ such that $\beta(Y)\neq 0$. Suppose $\beta'=\pm\alpha'$. Then by (13)

$$\forall X\in\mathfrak{t}_K\,,\quad [X,E_\alpha]=\alpha(X)\,E_\alpha\quad\text{and}\quad [X,U_\beta]=\pm\alpha(X)\,U_\beta\,.$$

Hence $U_\beta\in\mathfrak{k}_{\pm\alpha'}$, so there exists $\lambda\in\mathbb{C}$, such that $U_\beta=\lambda\,E_{\pm\alpha}$. Now by **R2**, on has $[Y,E_{\pm\alpha}]=0$, so $[Y,U_\beta]=0$. But by (12), $[Y,U_\beta]=\beta(Y)\,V_\beta$, which is nonzero. □

R4: Let $\theta\in\Phi_1\cup\Phi_2$. For any $X\in\mathfrak{t}_K$, one has $\sigma^*(\alpha)(X)=\alpha(\sigma_*(X))=\alpha(X)$, and for any $Y\in\mathfrak{t}_0$, by **R2**, $\sigma^*(\alpha)(Y)=\alpha(\sigma_*(Y))=\alpha(-Y)=0=\alpha(Y)$, hence $\sigma^*(\alpha)=\alpha$.

Now let $\theta \in \Phi_3$. By **R2**, there exists $Y \in \mathfrak{t}_0$ such that $\theta(Y) \neq 0$. Note that

$$\begin{aligned}\forall X \in \mathfrak{t}_K\,, \quad [X, \sigma_*(E_\theta)] &= [\sigma_* X, \sigma_*(E_\theta)] = \sigma_*\left([X, E_\theta]\right)\\ &= \theta(X)\,\sigma_*(E_\theta) = \sigma^*(\theta)(X)\,\sigma_*(E_\theta)\,,\\ \forall X \in \mathfrak{t}_0\,, \quad [X, \sigma_*(E_\theta)] &= -[\sigma_* X, \sigma_*(E_\theta)] = -\sigma_*\left([X, E_\theta]\right)\\ &= -\theta(X)\,\sigma_*(E_\theta) = \sigma^*(\theta)(X)\,\sigma_*(E_\theta)\,.\end{aligned}$$

Hence $\sigma^*(\theta)$ is a root. As $\sigma^*(\theta)(Y) = -\theta(Y) \neq 0$, $\sigma^*(\theta) \in \Phi_3$ by **R2**. Furthermore, as for any $X \in \mathfrak{t}_K$, $\sigma^*(\theta)(X) = \theta(X)$, $\sigma^*(\theta) \neq -\theta$, and as $\sigma^*(\theta)(Y) = -\theta(Y) \neq \theta(Y)$, $\sigma^*(\theta) \neq \theta$.

R5: As σ^* is an involution, any root θ may be decomposed into $\theta = \theta_+ + \theta_-$, with $\sigma^*(\theta_+) = \theta_+$ and $\sigma^*(\theta_-) = -\theta_-$. Note that $\theta_{+|\mathfrak{t}_0} = 0$ and $\theta_{-|\mathfrak{t}_K} = 0$ since

$$\forall X \in \mathfrak{t}_0\,, \quad \theta_+(X) = \sigma^*(\theta_+)(X) = \theta_+(\sigma_*(X)) = -\theta_+(X)\,,$$

and

$$\forall X \in \mathfrak{t}_K\,, \quad \theta_-(X) = -\sigma^*(\theta_-)(X) = -\theta_-(\sigma_*(X)) = -\theta_-(X)\,.$$

This implies in particular that the decomposition is orthogonal, and $\theta_+ = \theta_{|\mathfrak{t}_K} = \theta'$.

Now let α and $\beta \in \Phi_3$ such that $\beta' = \alpha'$, with $\beta \neq \alpha$ and $\beta \neq \sigma^*(\alpha)$. We first claim that $\alpha + \beta$ is not a root. Otherwise

$$[E_\alpha, E_\beta] = \underbrace{[U_\alpha, U_\beta] + [V_\alpha, V_\beta]}_{\in \mathfrak{k}_\mathbb{C}} + \underbrace{[U_\alpha, V_\beta] + [V_\alpha, U_\beta]}_{\in \mathfrak{p}_\mathbb{C}}\,,$$

is a nonzero element in $\mathfrak{g}_{\alpha+\beta}$. If $[U_\alpha, U_\beta] + [V_\alpha, V_\beta] = 0$, then $[E_\alpha, E_\beta] \in \mathfrak{p}_\mathbb{C}$, so $\alpha + \beta \in \Phi_2$, hence $(\alpha+\beta)_{|\mathfrak{t}_0} = 0$, by **R2**, so $\beta_{|\mathfrak{t}_0} = -\alpha_{|\mathfrak{t}_0}$, but this is impossible since $\beta \neq \sigma^*(\alpha)$.

Hence $[U_\alpha, U_\beta] + [V_\alpha, V_\beta] \neq 0$. But then $2\alpha'$ is a K-root since, as $\beta' = \alpha'$,

$$\forall X \in \mathfrak{t}_K\,, \quad [X, [U_\alpha, U_\beta] + [V_\alpha, V_\beta]] = 2\alpha(X)\left([U_\alpha, U_\beta] + [V_\alpha, V_\beta]\right).$$

But α' is a K-root (**R1**) and it is well known that α' and $2\alpha'$ can not be both K-roots.

Thus $\alpha + \beta$ is not a root, and so $\langle \alpha, \beta \rangle \geq 0$, since otherwise $\alpha + \beta$ should be a root (see for instance 9.4. in [Hum72]).

One the other hand, $\alpha - \beta$ can not be a root by **R1** since $(\alpha - \beta)_{|\mathfrak{t}_K} = 0$. Thus $\langle \alpha, \beta \rangle \leq 0$, since otherwise $\alpha - \beta$ should be a root.

Finally, $\langle \alpha, \beta \rangle = 0$. Note that, as $\sigma^*(\alpha)_{|\mathfrak{t}_K} = \alpha_{|\mathfrak{t}_K}$, the same arguments hold for $\sigma^*(\alpha)$ instead of α. Hence $\langle \sigma^*(\alpha), \beta \rangle = 0$, and so $\langle \alpha + \sigma^*(\alpha), \beta \rangle = 0$. Now as $\alpha + \sigma^*(\alpha) = 2\,\alpha_+$, this implies $\langle \alpha_+, \beta_+ \rangle = 0$. But $\alpha_+ = \alpha' = \beta' = \beta_+$, hence $\|\alpha'\|^2 = 0$, which is impossible. □

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