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Hyperquaternions: An Efficient Mathematical Formalism for Geometry

Patrick R. Girard¹[0000-0001-8556-1816], Patrick Clarysse¹[0000-0002-5495-7655],
Romaric Pujol², Robert Goutte¹, and Philippe
Delachartre¹[0000-0003-0492-1846]

¹ Univ Lyon, INSA-LYON, Université Claude Bernard Lyon 1, UJM-Saint Etienne,
CNRS, Inserm, CREATIS UMR 5220, U1206, F-69621 LYON, France
`patrick.girard@creatis.insa-lyon.fr`
² Pôle de Mathématiques, INSA-Lyon
Bât. Léonard de Vinci, 21 avenue Jean Capelle
F-69621 Villeurbanne, France

Abstract. Hyperquaternions being defined as a tensor product of quaternion algebras (or a subalgebra thereof), they constitute Clifford algebras endowed with an associative exterior product providing an efficient mathematical formalism for differential geometry. The paper presents a hyperquaternion formulation of pseudo-euclidean rotations and the Poincaré groups in n dimensions (via dual hyperquaternions). A canonical decomposition of these groups is developed as an extension of an euclidean formalism and illustrated by a $5D$ example. Potential applications include in particular, moving reference frames and machine learning.

Keywords: Quaternions · Hyperquaternions · Pseudo-euclidean rotations · Poincaré groups · Canonical decomposition.

1 Introduction

Clifford algebras allow an excellent representation of pseudo-euclidean rotations which are important symmetry groups of physics [1–4]. A decomposition of these groups into orthogonal, commuting planar rotations is called a canonical decomposition. Various canonical decompositions have been developed which deal with either specific rotations or dimensions and are often expressed in terms of matrices [5, 6]. In a recent paper, we have introduced a hyperquaternion formulation of Clifford algebras and applied them to the unitary and unitary symplectic groups [7]. Here, we consider pseudo-euclidean rotations and the Poincaré groups in n dimensions (via dual hyperquaternions). A canonical decomposition of these groups is developed within that framework as an extension of an euclidean formalism introduced by Moore [8, 9]. After a short presentation of hyperquaternions and multivectors, we derive the pseudo-euclidean rotations and the canonical decomposition. Then we go on to the Poincaré groups and a $5D$ example. Potential applications are moving reference frames and machine learning [10]

Table 1. Biquaternion Multivector Structure

1	$i = e_3e_2$	$j = e_1e_3$	$k = e_2e_1$
$I = e_1e_2e_3$	$Ii = e_1$	$Ij = e_2$	$Ik = e_3$

2 Background: Quaternions, Hyperquaternions and Multivectors

In this section, we briefly introduce quaternions, hyperquaternions and multivectors [7, 11–15]. The quaternion algebra \mathbb{H} which contains \mathbb{R} and \mathbb{C} as particular cases is constituted by quaternions

$$a = a_1 + a_2i + a_3j + a_4k \quad (a_i \in \mathbb{R}) \quad (1)$$

where i, j, k multiply according to

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, \text{etc.} \quad (2)$$

The product of two quaternions a, b is given by

$$ab = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)i \quad (3)$$

$$+ (a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4)j + (a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)k. \quad (4)$$

The conjugate of a quaternion is $a_c = a_1 - a_2i - a_3j - a_4k$ with

$$aa_c = a_1^2 + a_2^2 + a_3^2 + a_4^2, (ab)_c = b_c a_c \quad (5)$$

The hyperquaternion algebra (over \mathbb{R}) is defined as the tensor product of quaternion algebras (or a subalgebra thereof). Examples of hyperquaternion algebras are the quaternions \mathbb{H} , tetraquaternions $\mathbb{H} \otimes \mathbb{H}$ and so on $\mathbb{H} \otimes \mathbb{H} \otimes \dots \otimes \mathbb{H}$; subalgebras are the complex numbers \mathbb{C} , biquaternions $\mathbb{H} \otimes \mathbb{C}$, Dirac algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$, etc..

Calling (i, j, k) the first quaternionic system, (I, J, K) the second one and (l, m, n) the third one, all systems commuting with each other, one has

$$i \otimes i \otimes i = iIl, \quad i \otimes j \otimes k = iJn, \text{ etc.} \quad (6)$$

which uniquely defines the multiplication.

Hyperquaternions having n generators e_i such that $e_i e_j + e_j e_i = 0$ ($i \neq j$), $e_i^2 = \pm 1$ constitute Clifford algebras C_n . The choice of the generators entails a multivector structure as shown, in the case of biquaternions, in Table 1. The 2^n elements of the algebra are composed of scalars, vectors e_i , bivectors $e_i e_j$, trivectors $e_i e_j e_k$ etc. yielding respectively the multivector spaces $V_0, V_1, V_2, V_3, \dots, V_n$. C^+ is the subalgebra constituted by products of an even number of e_i , C^- is the rest of the algebra. The multivector structure allows to define basic operations like conjugation, duality and the interior and exterior products.

Considering a general element A of the algebra, the conjugate A_c is obtained by replacing the e_i by their opposite $-e_i$ and reversing the order of the elements

$$(A_c)_c = A, (AB)_c = (B_c)(A_c). \quad (7)$$

The dual of A is $A^* = i_d A$ where $i_d = e_1 \wedge e_2 \dots \wedge e_n$ (to be defined below) and the commutator of two hyperquaternions is

$$[A, B] = \frac{1}{2} (AB - BA). \quad (8)$$

The interior and exterior products of two vectors a, b are obtained as follows. From the identity

$$2ab = \lambda \lambda^{-1} [(ab + ba) + (ab - ba)] \quad (9)$$

where $\lambda = \pm 1$ is a given coefficient (allowing to eventually change the sign of the metric), one defines

$$2a.b = \lambda^{-1} (ab + ba), 2a \wedge b = \lambda^{-1} (ab - ba) \quad (10)$$

which are respectively a scalar and a bivector. A multivector $A_p = a_1 \wedge a_2 \wedge \dots \wedge a_p$ ($2 \leq p < n$) where a_p are vectors, is then defined by recurrence

$$2a.A_p = \lambda^{-p} [aA_p - (-1)^p A_p a] \in V_{p-1} \quad (11)$$

$$2a \wedge A_p = \lambda^{-p} [aA_p + (-1)^p A_p a] \in V_{p+1} \quad (12)$$

By definition, we take

$$A_p.a \equiv (-1)^{p-1} a.A_p, A_p \wedge a \equiv (-1)^p a \wedge A_p. \quad (13)$$

An important property of the exterior product is its associativity.

Interior and exterior products between multivectors are defined by

$$A_p \wedge B_q = a_1 \wedge (a_2 \wedge \dots \wedge a_p \wedge B_q) \quad (14)$$

$$A_p.B_q = (a_1 \wedge \dots \wedge a_{p-1}).(a_p.B_q), \quad (p \leq q) \quad (15)$$

with $A_p.B_q = (-1)^{p(q+1)} B_q.A_p$ [16]. In particular, we have the following useful formulas where Bi are bivectors and $V_p[A]$ the multivector part V_p of A

$$B_1 B_2 = B_1.B_2 + B_1 \wedge B_2 + [B_1, B_2] \quad (16)$$

$$B_1 \wedge B_2 = V_4 [B_1 B_2] \quad (17)$$

$$B_1 \wedge B_2 \wedge B_3 = V_6 [B_1 (B_2 \wedge B_3)] \quad (18)$$

$$B_1.(B_2 \wedge B_3) = V_2 [B_1 (B_2 \wedge B_3)] \quad (19)$$

$$(B_1 \wedge B_2).(B_3 \wedge B_4 \wedge B_5) = V_2 [(B_1 \wedge B_2) (B_3 \wedge B_4 \wedge B_5)]. \quad (20)$$

Hyperquaternions yield all real, complex and quaternionic square matrices as well as the transposition, adjunction and transpose quaternion conjugate via a hyperconjugation defined as $\mathbb{H}_c \otimes \mathbb{H}_c \otimes \dots \otimes \mathbb{H}_c$ as indicated in Table 2.

Table 2. Hyperquaternions and matrices

$$\begin{array}{l|l}
\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R}) & \mathbb{H}_c \otimes \mathbb{H}_c \simeq [m(4, \mathbb{R})]^t \\
\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C} \simeq m(4, \mathbb{C}) & \mathbb{H}_c \otimes \mathbb{H}_c \otimes \mathbb{C}_c \simeq [m(4, \mathbb{C})]^\dagger \\
\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{H}) & \mathbb{H}_c \otimes \mathbb{H}_c \otimes \mathbb{H}_c \simeq [m(4, \mathbb{H})]_c^t.
\end{array} \tag{21}$$

3 Pseudo-Orthogonal Rotations

In this section, we derive a hyperquaternion formulation of pseudo-euclidean rotations and develop a canonical decomposition. Historically, the formula of n dimensional euclidean rotations $x' = axa^{-1}$ ($a \in C_n^+$) was given by Lipschitz [17] and Moore developed a canonical decomposition thereof [8, 9]. In this section we introduce, as an extension of Moore's method, within the hyperquaternion Clifford algebra framework, a canonical decomposition of pseudo-euclidean rotations and the Poincaré groups. After a brief review of the basic definitions and the Cartan theorem, we develop the canonical decomposition.

3.1 Definitions and Theorem

Let $C_{p,q}$ be a hyperquaternion algebra having $n = p + q$ generators e_i and the quadratic form

$$x.y = x_1y_1 + \dots + x_p y_p - (x_{p+1}y_{p+1} \dots - x_{p+q}y_{p+q}) \tag{22}$$

$$= \lambda^{-1} (xy + yx) / 2 \tag{23}$$

where x, y are vectors ($x = x_i e_i$). A vector x is timelike if $x.x > 0$, spacelike if $x.x < 0$ and isotropic if $x.x = 0$.

An orthogonal symmetry with respect to a plane going through the origin and perpendicular to a unit vector a ($a^2 = \pm 1$) is given by [12, 13]

$$x' = \pm axa \tag{24}$$

with $x'x' = (\pm axa)(\pm axa) = xx$.

Definition 1. *The pseudo-orthogonal group $O(p, q)$ is the group of linear operators which leave invariant the form $x \cdot y$.*

Theorem 1. *Every rotation of $O(p, q)$ is the product of an even number $2m \leq n$ of symmetries.*

Definition 2. *The special orthogonal group $SO^+(p, q)$ is constituted by rotations which preserve the orientation of the space of positive norm vectors and the space of negative norm vectors.*

A rotation of $SO^+(p, q)$ can thus be expressed as

$$x' = axa_c \quad (aa_c = 1) \quad (25)$$

with $a = a_1 a_2 \dots a_{2m} \in C^+$, where a_i are unit vectors (with an even number of timelike and spacelike vectors). Developing the product (with $\lambda = 1$)

$$a_i a_j = a_i \cdot a_j + a_i \wedge a_j \quad (26)$$

one sees that it contains a simple plane $B = a_i \wedge a_j$ such that $B^2 = B \cdot B + B \wedge B$ is a scalar since $B \wedge B = 0$. Hence, a rotation involves at most $m \leq n/2$ simple planes. A canonical decomposition of rotations is obtained by choosing these simple planes to be orthogonal.

3.2 Canonical Decomposition

A rotation of $SO^+(p, q)$ can be decomposed as

$$a = e^{\frac{\Phi_1}{2} B_1} e^{\frac{\Phi_2}{2} B_2} \dots e^{\frac{\Phi_m}{2} B_m} \quad (aa_c = 1) \quad (27)$$

where B_i are m simple orthogonal commuting planes such that $B_i^2 = \pm 1$ together for $i \neq j$

$$B_i \cdot B_j = 0, B_i B_j = B_j B_i, B_i B_j = B_i \wedge B_j; \quad (28)$$

Φ_i are the angles of rotation within the planes B_i . According to whether $B_i^2 = -1$ or $B_i^2 = 1$, one has respectively

$$e^{\frac{\Phi_i}{2} B_i} = \cos \frac{\Phi_i}{2} + \sin \frac{\Phi_i}{2} B_i, e^{\frac{\Phi_i}{2} B_i} = \cosh \frac{\Phi_i}{2} + \sinh \frac{\Phi_i}{2} B_i. \quad (29)$$

The rotation can be developed as

$$a = S (1 + b_1 B_1) (1 + b_2 B_2) \dots (1 + b_m B_m) \quad (30)$$

with $b_i = \tan \frac{\Phi_i}{2}$ (or $\tanh \frac{\Phi_i}{2}$). Since $aa_c = 1$ one has

$$S^2 (1 + b_1^2) (1 - b_2^2) (1 - b_3^2) = 1 \quad (31)$$

$$S = \frac{1}{\sqrt{(1 \pm b_1^2) \dots (1 \pm b_m^2)}} \quad (32)$$

which shows that S is determined by the b_i . Writing

$$B = b_1 B_1 + b_2 B_2 + b_3 B_3 \quad (33)$$

one can express a as

$$a = S \left(1 + B + \frac{B \wedge B}{2! S^2} + \dots \frac{B \wedge B \wedge B \wedge \dots (m \text{ terms})}{m! S^m} \right) \quad (34)$$

which shows that the bivector B determines completely the rotation.

If the scalar is nil, for example if $(\Phi_1 = \pm\pi, B_1^2 = -1)$, then a is proportional to B_1

$$a = B_1 e^{\frac{\Phi_2}{2} B_2} e^{\frac{\Phi_3}{2} B_3}; \quad (35)$$

one then computes $B_1^{-1}a$ and comes back to the general expression to evaluate the remaining b_i and B_i .

To determine the b_i and B_i , one makes a change of variable $X_i = b_i B_i, x_i = X_i^2 = \pm b_i^2$ and considers the linear system of equations in X_i [9]

$$P_1 = B = \sum_{i=1}^m X_i \quad (36)$$

$$P_2 = (B \wedge B) \cdot B = 2 \sum_{i,j=1}^m X_i x_j \quad (i \neq j) \quad (37)$$

$$P_3 = (B \wedge B \wedge B) \cdot (B \wedge B) = 3!2! \sum_{i,j,k=1}^m X_i x_j x_k \quad (i \neq j, j < k) \quad (38)$$

$$\dots\dots \quad (39)$$

$$P_m = (B \wedge B \wedge \dots m \text{ factors}) \cdot (B \wedge B \dots (m-1) \text{ factors}) \quad (40)$$

$$= m! (m-1)! \sum_{i=1}^m x_1 x_2 \dots x_{i-1} x_{i+1} \dots x_m X_i. \quad (41)$$

The determinant Δ is the product

$$\Delta = \left\{ m! [(m-1)!]^2 [(m-2)!]^2 \dots 1 \right\} \prod_{i,j=1}^m (x_i - x_j) \quad (i \neq j, i < j). \quad (42)$$

If $\Delta \neq 0$, one obtains the bivectors X_i as a function of P_m and x_i . To determine the x_i , one writes the equations

$$S_1 = P_1 \cdot P_1 = \sum_{i=1}^m x_i \quad (43)$$

$$S_2 = P_2 \cdot P_1 = 2! \sum_{i,j=1}^m x_i x_j \quad (i \neq j) \quad (44)$$

$$S_3 = P_3 \cdot P_1 = (3!)^2 \sum_{i,j,k=1}^m x_i x_j x_k \quad (i \neq j, j < k) \quad (45)$$

$$\dots\dots \quad (46)$$

$$S_m = P_m \cdot P_1 = (m!)^2 (x_1 x_2 \dots x_m). \quad (47)$$

The solutions yield $x_i = \pm b_i^2$, thus one obtains b_i and B_i

$$b_i = \sqrt{|x_i|}, B_i = \frac{X_i}{b_i}. \quad (48)$$

If $\Delta = 0$, the equations (36-41) are not independent, the B bivector can nevertheless be decomposed in m mutually orthogonal simple planes but this decomposition is not unique.

4 Poincaré group in n dimensions (via dual hyperquaternions)

Much of physics being covariant with respect to the $4D$ Poincaré group, we provide here a hyperquaternion representation of the nD Poincaré groups in terms of dual hyperquaternions. Thereby one comes back to a $(n+1)D$ rotation which one can be decomposed canonically. The procedure is illustrated by a $5D$ case (for example a color image with 2 spatial and 3 color dimensions) which might be of interest in machine learning [10]).

4.1 General formalism

The Poincaré group of the pseudo-euclidean space associated with the Clifford algebra $C_{p,q}$ ($n = p + q$) is constituted by the isometries of the metric

$$ds^2 = (dx_1^2 + \dots + dx_p^2) - (dx_{p+1}^2 + \dots + dx_{p+q}^2). \quad (49)$$

It includes the rotations $SO^+(p, q)$, translations and reflections (time or space-like). The reflections having already been dealt with above, we shall focus on the rotations and translations.

Consider a hyperquaternion algebra $\mathbb{H} \otimes \mathbb{H} \dots \otimes \mathbb{H}$ (or a subalgebra thereof) with $n+1$ generators $e_1, e_2, \dots, e_n, e_{n+1}$ and let X be a dual vector such that

$$X = e_{n+1} + \varepsilon x \quad (50)$$

where x belongs to the vector space V_1 with $x = \sum_{i=1}^n e_i x_i$ ($x_i \in \mathbb{R}$) and $\varepsilon^2 = 0$ (ε commuting with e_i). An nD hyperbolic rotation in V_1 leaves the last variable unchanged. Hence,

$$X' = aXa_c = e_{n+1} + \varepsilon x' \quad (51)$$

with $x' = axa_c, x'x'_c = xx_c, aa_c = 1$. A translation in V_1 can be expressed as

$$X' = bXb_c \quad (52)$$

with

$$b = e^{\varepsilon e_{n+1} \frac{t}{2}} = 1 + \varepsilon e_{n+1} \frac{t}{2}, \quad (t = \sum_{i=1}^n e_i t_i, t_i \in \mathbb{R}) \quad (53)$$

and $bb_c = 1$. Developing Eq. (52), one obtains, assuming $e_{n+1}^2 = -1$

$$X' = \left(1 + \varepsilon e_{n+1} \frac{t}{2}\right) (e_{n+1} + \varepsilon x) \left(1 - \varepsilon e_{n+1} \frac{t}{2}\right) \quad (54)$$

$$= e_{n+1} + \varepsilon x - \varepsilon e_{n+1} e_{n+1} \frac{t}{2} - \varepsilon e_{n+1} e_{n+1} \frac{t}{2} \quad (55)$$

$$= e_{n+1} + \varepsilon (x + t) \quad (56)$$

which is a translation on the variables $1\dots n$ (if $e_{n+1}^2 = 1$, one simply takes $b = e^{\varepsilon \frac{t}{2} e_{n+1}}$). A combination of an nD rotation and translation gives with $f = ab$ (or ba)

$$X' = fXf_c \quad (ff_c = 1, f \in C^+) \quad (57)$$

which can be viewed as a particular $(n+1)D$ rotation. One thus obtains a hyperquaternion representation of the Poincaré groups, distinct from the matrix one. A canonical decomposition leads to simple dual planes as will be illustrated in the following example.

4.2 Example: 5D Poincaré group

As application consider a $5D$ -space (for example a $2D$ color image) imbedded in the $6D$ hyperquaternion algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ having six generators (see Appendix)

$$e_1 = kI, e_2 = kJ, e_3 = kKl, e_4 = kKm, e_5 = kKn, e_6 = j \quad (58)$$

with the generic vector $X = e_6 + \varepsilon x$ ($x = \sum_{i=1}^5 e_i x_i$). The transformation $X' = fXf_c$ with

$$f = e^{\frac{\Phi_2}{2} J l} e^{\varepsilon i (2I + Kn)} e^{\frac{\Phi_1}{2} I (m+n)} \quad (59)$$

$$= \left(2 + \sqrt{3} J l \right) [1 + \varepsilon i (2I + Kn)] \left[\sqrt{3} + \sqrt{2} I \left(\frac{m}{\sqrt{2}} + \frac{n}{\sqrt{2}} \right) \right] \quad (60)$$

and $\tanh \frac{\Phi_1}{2} = \sqrt{\frac{2}{3}} (= b_1)$, $\tanh \frac{\Phi_2}{2} = \frac{\sqrt{3}}{2} (= b_2)$ is a $5D$ -Poincaré transform. Applying the canonical decomposition presented above, one obtains

$$f = e^{\frac{\Phi_2}{2} B_2} e^{X_3} e^{\frac{\Phi_1}{2} B_1} \quad (61)$$

with the same values of Φ_1, Φ_2 as above and the following simple commuting orthogonal dual planes B_1, B_2, X_3

$$B_1 = \frac{1}{\sqrt{2}} I (m+n) + \varepsilon \frac{1}{\sqrt{2}} \left[\frac{\sqrt{3}}{2} K (m+n) - iJ \right] \quad (62)$$

$$B_2 = J l + 2\varepsilon i \left(\frac{2}{\sqrt{3}} I - Kl \right) \quad (63)$$

$$X_3 = \frac{\varepsilon}{2} iK (-m+n). \quad (64)$$

with $(B_1)^2 = (B_2)^2 = 1, (X_3)^2 = 0$.

5 Conclusion

The paper has given a hyperquaternion representation of pseudo-euclidean rotations and the Poincaré groups in n dimensions, distinct from the matrix one. A canonical decomposition of these groups was introduced, as an extension of an euclidean formalism, within a hyperquaternion Clifford algebra framework and illustrated by a $5D$ example. Potential geometric applications include in particular, moving reference frames and machine learning.

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A Multivector structure of $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$

$$\begin{aligned}
 & \left[\begin{array}{cccc} 1 & l = e_4e_5 & m = e_5e_3 & n = e_3e_4 \\ I = e_2e_3e_4e_5 & I l = e_3e_2 & I m = e_4e_2 & I n = e_5e_2 \\ J = e_3e_1e_4e_5 & J l = e_1e_3 & J m = e_1e_4 & J n = e_1e_5 \\ K = e_2e_1 & Kl = e_2e_1e_4e_5 & Km = e_1e_2e_3e_5 & Kn = e_2e_1e_3e_4 \end{array} \right] \\
 +i & \left[\begin{array}{cccc} 1 = e_1e_2e_3e_4e_5e_6 & l = e_2e_1e_3e_6 & m = e_2e_1e_4e_6 & n = e_2e_1e_5e_6 \\ I = e_6e_1 & I l = e_4e_1e_5e_6 & I m = e_5e_1e_3e_6 & I n = e_3e_1e_4e_6 \\ J = e_6e_2 & J l = e_4e_2e_5e_6 & J m = e_5e_2e_3e_6 & J n = e_3e_2e_4e_6 \\ K = e_3e_4e_5e_6 & Kl = e_6e_3 & Km = e_6e_4 & Kn = e_6e_5 \end{array} \right] \\
 +j & \left[\begin{array}{cccc} 1 = e_6 & l = e_4e_5e_6 & m = e_6e_5e_3 & n = e_3e_4e_6 \\ I = e_2e_3e_4e_5e_6 & I l = e_3e_2e_6 & I m = e_6e_4e_2 & I n = e_6e_5e_2 \\ J = e_4e_3e_5e_6e_1 & J l = e_1e_3e_6 & J m = e_1e_4e_6 & J n = e_1e_5e_6 \\ K = e_2e_1e_6 & Kl = e_2e_1e_4e_5e_6 & Km = e_1e_2e_3e_5e_6 & Kn = e_2e_1e_3e_4e_6 \end{array} \right] \\
 +k & \left[\begin{array}{cccc} 1 = e_2e_1e_3e_4e_5 & l = e_1e_2e_3 & m = e_1e_2e_4 & n = e_1e_2e_5 \\ I = e_1 & I l = e_1e_4e_5 & I m = e_3e_1e_5 & I n = e_1e_3e_4 \\ J = e_2 & J l = e_2e_4e_5 & J m = e_3e_2e_5 & J n = e_2e_3e_4 \\ K = e_4e_3e_5 & Kl = e_3 & Km = e_4 & Kn = e_5 \end{array} \right]
 \end{aligned}$$

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