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# Exponential inequalities for the supremum of some counting processes and their square martingales

R. Le Guével

*Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France*  
*ronan.legevel@univ-rennes2.fr*

## Abstract

We establish exponential inequalities for the supremum of martingales obtained from counting processes as well as the supremum of their square martingales. Exponential inequalities are also provided for the oscillation modulus of these martingales.

**Keywords:** Counting processes, concentration inequalities, exponential martingales, U-statistics.

## 1 Introduction

The counting processes naturally arise in a lot of applied circumstances, and the understanding of their evolution is the object of a lot of modelization problems. To this end, inequalities of concentration are of great interest, particularly because they play a decisive role in the control of errors in statistics. The exponential inequalities for the distribution of random variables have been of interest for many years (see Hoeffding [1963] for instance for one of the first result about this issue), and it is still a very active domain of research for various types of processes. For example, in the chaos environment, such inequalities have been obtained for the supremum of a countable family of empirical processes in Talagrand [1996], Ledoux [1997], Massart [2000], Rio [2002] or Bousquet [2002].

In this paper we focus on counting processes and its associated square martingales. Even though some results already exists for martingales in continuous time (one may cite for example Van De Geer [1995] or Theorem 23.17 of Kallenberg [1997]) or in discrete time (see Chen et al. [2014]), we exhibit here the exponential martingales of some counting processes and their square martingales, providing also exponential inequalities for the supremum of some U-statistics of order two. While it is closely related to the work of Reynaud-Bouret [2006], where the supremum of a countable family of martingales of counting processes is considered, we use quite different proofs and we obtain results with bounds which are sharper than those in Reynaud-Bouret [2006].

Inequalities for U-statistics is a well-studied issue too. One may refer to Giné and Zinn [1992] for U-statistics of order  $m$ , Klass and Nowicki [1997] and Arcones and Giné [1993] for the order two, and Bretagnolle [1999], Giné et al. [2000] or Hanson and Wright [1971] in the sub-gaussian framework. In the specific case of the Poisson process, some exponential inequalities with explicit constants are obtained for U-statistics of order two in Houdré and Reynaud-Bouret [2003]. Once again, we obtain here sharper bounds with quite different proofs, which may be very useful for some statistical applications. Indeed the exponential inequalities are often used for statistical problems in a non-asymptotic framework, like estimation problems (see for instance Laurent [2005] for the estimation of a quadratic functional of a density) or testing problems, as in Fromont and Laurent [2006] for a goodness-of-fit test in density, or Fromont et al. [2011] for an adaptive test of homogeneity of a Poisson process.

The remainder if this article is organized as follows: in the next section, we recall some general notations, while Section 3 is devoted to the exponential martingales of the counting processes. The exponential inequalities of our martingales and their associated square martingales are presented in

Section 4. We provide some applications of the inequalities for U-statistics of order two in Section 5. We compute also the oscillation modulus of the martingales in this section. Finally, we have gathered all the proofs in Section 6.

## 2 Notations

Let  $T \geq 0$  and  $N = (N_t)_{t \geq 0}$  be a counting process with a continuous compensator  $\Lambda = (\Lambda_t)_{t \geq 0}$ , and  $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$  be the completed right-extension of its natural filtration, that is

$$\mathcal{F}_t = \bigcap_{s > t} \sigma(N_u, u \leq s) \cup \mathcal{N}$$

where  $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$ . We assume that the jumps of  $N$  are totally inaccessible, and that  $N - \Lambda$  is a martingale (not only a local martingale).

We consider also  $H = (H_s)_{s \geq 0}$ , a predictable adapted process of bounded variations, bounded by the non-random real number  $\|H\|_{\infty, [c, d]}$  on the interval  $[c, d]$ , that is  $\sup_{s \in [c, d]} |H_s| \leq \|H\|_{\infty, [c, d]}$  almost surely. If  $c = 0$ ,  $\|H\|_{\infty, [0, T]}$  will be written  $\|H\|_{\infty, T}$  for short. The non-random real number  $\|H\|_{2, [c, d]}$  will stand for a bound of the  $L^2$  norm of  $H$  in  $L^2(\Lambda([c, d]))$ , that is  $\int_c^d |H_u|^2 d\Lambda_u \leq \|H\|_{2, [c, d]}^2 < +\infty$  almost surely.

Recall that for a stochastic process  $X$ , we define  $[X]_t$  by

$$[X]_t = \langle X^c \rangle_t + \sum_{s \leq t} |\Delta X_s|^2$$

where  $\langle X^c \rangle$  is the quadratic variation of the continuous part of  $X$  and  $\Delta X_s = X_s - X_{s-}$  is the jump of  $X$  at  $s$ . We will use the fact that if  $X$  is a martingale and  $H$  is a predictable process satisfying  $\mathbb{E}[\int_0^\infty H_s^2 d[X]_s] < +\infty$ , then  $(\int_0^t H_s dX_s)_{t \geq 0}$  is a martingale (see [Bass, 2011, p.134]). Recall also that for a  $\mathcal{C}^2$  function  $f$  and a semi-martingale  $(X_t)_{t \geq 0}$ , the Itô formula ([Bass, 2011, Theorem 17.10]) entails

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-}) dX_s + \frac{1}{2} \int_0^t f''(X_{s-}) d\langle X^c \rangle_s + \sum_{s \leq t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s].$$

For  $f = \exp$  and a semi-martingale  $X$  satisfying  $\langle X^c \rangle \equiv 0$  and  $X_0 = 0$ , this leads to

$$e^{X_t} = 1 + \int_0^t e^{X_{s-}} dX_s + \sum_{s \leq t} e^{X_{s-}} [e^{\Delta X_s} - 1 - \Delta X_s]. \quad (1)$$

## 3 Martingale properties

We consider in this section the three martingales  $M = (M_t)_{t \leq T}$ ,  $\tilde{M} = (\tilde{M}_t)_{t \leq T}$  and  $\tilde{\tilde{M}} = (\tilde{\tilde{M}}_t)_{t \leq T}$  defined for  $t \leq T$  by

$$M_t = \int_0^t H_s d(N_s - \Lambda_s),$$

$$\tilde{M}_t = \left( \int_0^t H_s d(N_s - \Lambda_s) \right)^2 - \int_0^t H_s^2 dN_s = M_t^2 - \int_0^t H_s^2 dN_s$$

and

$$\tilde{\tilde{M}}_t = \left( \int_0^t H_s d(N_s - \Lambda_s) \right)^2 - \int_0^t H_s^2 d\Lambda_s = M_t^2 - \int_0^t H_s^2 d\Lambda_s.$$

Our main goal is to establish in the next section some concentration inequalities for these three martingales. We will use Chernoff bounds in order to do that, so we are first interested by the exponential martingales associated with the three processes  $M$ ,  $\tilde{M}$  and  $\tilde{\tilde{M}}$ . We start first with the process  $M$  in the following lemma.

**Lemma 1.** *Let  $Z$  be the process defined for a fixed real number  $\lambda$  and all  $t \leq T$  by*

$$Z_t = \lambda M_t - \int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s.$$

*The process  $(e^{Z_t})_{t \leq T}$  is a martingale.*

Let us define now for  $a > 0$

$$T_a = \inf_{0 < t \leq T} \{|M_t| > a\}.$$

Since the jumps of  $N$  are totally inaccessible,  $T_a$  is a stopping time ([Bass, 2011, Proposition 16.3]). The next lemma sets out a stopped exponential martingale associated with the martingale  $\tilde{M}$ .

**Lemma 2.** *Let  $\tilde{Z}$  be the process defined for a fixed real number  $\lambda$  and all  $t \leq T$  by*

$$\tilde{Z}_t = \lambda \tilde{M}_t - \int_0^t (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d\Lambda_s.$$

*For every positive  $a$ , the process  $(e^{\tilde{Z}_{t \wedge T_a}})_{t \leq T}$  is a martingale.*

Finally we present the analogue of Lemma 2 for the martingale  $\tilde{\tilde{M}}$ .

**Lemma 3.** *Let  $\tilde{\tilde{Z}}$  be the process defined for a fixed real number  $\lambda$  and all  $t \leq T$  by*

$$\tilde{\tilde{Z}}_t = \lambda \tilde{\tilde{M}}_t - \int_0^t (e^{\lambda H_s (H_s + 2M_s)} - 1 - \lambda H_s (H_s + 2M_s)) d\Lambda_s.$$

*For every positive  $a$ , the process  $(e^{\tilde{\tilde{Z}}_{t \wedge T_a}})_{t \leq T}$  is a martingale.*

## 4 Exponential inequalities

We have gathered in this section our main results, that is the exponential inequalities for the three martingales  $M$ ,  $\tilde{M}$  and  $\tilde{\tilde{M}}$ . The rates that appear in these inequalities are governed by the rate function  $I$  defined for  $x \geq 0$  by

$$I(x) = (1+x) \log(1+x) - x.$$

We start with a technical lemma that provides a useful inequality for the proofs of the main theorems.

**Lemma 4.** *Let  $I_t(H, \lambda)$  be defined for  $t \geq 0$  by  $\int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$ . For  $t \leq T$  and every real  $\lambda$ ,*

$$|I_t(H, \lambda)| \leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(|\lambda| \|H\|_{\infty,T}) \quad (2)$$

where  $g(x) = e^x - 1 - x$ .

We present now in Theorem 1 an inequality for the martingale  $M$ , with its two-sided version.

**Theorem 1.** *For every positive  $x$  and  $T$ , we have the following inequalities:*

$$\mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq x) \leq \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right) \quad (3)$$

and

$$\mathbb{P}(\sup_{0 \leq t \leq T} |M_t| \geq x) \leq 2 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right). \quad (4)$$

Such exponential inequalities have already been obtained for martingales with bounded jumps in Kallenberg [1997] or Van De Geer [1995] for instance. However in Theorem 1, we significantly improve the constants in the inequality. The bounds of the inequalities are also sharper than those in Reynaud-Bouret [2006], where a countable family of processes of the type  $(M^a)_a$  is considered therein. We obtain a term with a logarithm in the case of the large deviations (that is when  $x$  tends to infinity), and in the case of the small deviations (when  $x$  tends to zero), the bounds are of the same order of magnitude with more precise constants through the function  $I$ .

The next Theorem deals with the square martingale  $\tilde{M}$ . The same inequality is obtained for  $-\tilde{M}$ , leading to a two-sided concentration inequality.

**Theorem 2.** *For every positive  $x$  and  $T$ , we have the following inequalities:*

$$\mathbb{P}(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x) \leq 3 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right) \quad (5)$$

and

$$\mathbb{P}(\sup_{0 \leq t \leq T} -\tilde{M}_t \geq x) \leq 3 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right), \quad (6)$$

thereby we have the following two-sided concentration inequality:

$$\mathbb{P}(\sup_{0 \leq t \leq T} |\tilde{M}_t| \geq x) \leq 6 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} \sqrt{\frac{x}{2}}\right)\right). \quad (7)$$

Finally Theorem 3 is the analogue of Theorem 2 for the martingale  $\tilde{\tilde{M}}$ .

**Theorem 3.** *For every positive  $x$  and  $T$ , we have the following inequalities:*

$$\mathbb{P}(\sup_{0 \leq t \leq T} \tilde{\tilde{M}}_t \geq x) \leq 3 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2} \frac{\sqrt{1 + 8x/\|H\|_{\infty,T}^2} - 1}{4}\right)\right) \quad (8)$$

and

$$\mathbb{P}(\sup_{0 \leq t \leq T} -\tilde{\tilde{M}}_t \geq x) \leq 3 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2} \frac{\sqrt{1 + 8x/\|H\|_{\infty,T}^2} - 1}{4}\right)\right), \quad (9)$$

thereby we have the following two-sided concentration inequality:

$$\mathbb{P}(\sup_{0 \leq t \leq T} |\tilde{\tilde{M}}_t| \geq x) \leq 6 \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}^2}{\|H\|_{2,T}^2} \frac{\sqrt{1 + 8x/\|H\|_{\infty,T}^2} - 1}{4}\right)\right). \quad (10)$$

If we compare (4), (7) and (10), we can notice that the rates are similar for  $\tilde{M}$  and  $\tilde{\tilde{M}}$  (which is  $\mathcal{O}(e^{-a_1\sqrt{x}\log(x)})$ ) when  $x$  tends to infinity, and slower than the rate of  $M$  ( $\mathcal{O}(e^{-a_2x\log(x)})$ ). Since  $\tilde{\tilde{M}} = \tilde{M} + \int H^2 d(N - \Lambda)$ , this result is not very surprising and (10) does not significantly improve the inequality obtained using (4) with  $H^2$ , (7) and  $\frac{x}{2}$ . The situation is rather different when  $x$  tends to zero.  $M$  and  $\tilde{M}$  have the same rate ( $\mathcal{O}(e^{-c_1x^2})$  and  $\mathcal{O}(e^{-c_2x^2})$ ), slower than the one of  $\tilde{\tilde{M}}$  ( $\mathcal{O}(e^{-c_3x})$ ). However, for the term  $\int H^2 d(N - \Lambda)$  the constant  $c_1$  involves  $\frac{1}{\|H^2\|_{2,T}^2}$ , and  $c_2$  involves  $\frac{1}{\|H\|_{2,T}^4}$ , so it is better to use (10) instead of the relationship  $\tilde{\tilde{M}} = \tilde{M} + \int H^2 d(N - \Lambda)$  in that case because  $\|H\|_{2,T}^4 \leq \|H^2\|_{2,T}^2$ .

## 5 Examples of applications

### 5.1 U-statistics of order two

The main hypothesis of the previous theorems is to suppose that the counting process  $N$  has a continuous compensator  $\Lambda$ , which is bounded in some spaces (as well as  $H$ ) through the assumption  $\|H\|_{2,T} < +\infty$ . If the process  $N$  admits an intensity  $\lambda$ , some mild assumptions on  $\lambda$  ensure the continuity of the compensator  $\Lambda = \int \lambda(s) ds$ . This allows us to consider for instance Poisson, Cox or Hawkes processes with a bounded intensity such that  $N - \Lambda$  is a martingale. As an example, if the process  $(\frac{1}{h}\mathbb{E}[N_{t+h} - N_t | \mathcal{F}_t])_{h,t}$  is uniformly bounded for  $h$  small enough, we know that the  $\mathcal{F}$ -intensity of  $N$  is bounded and  $N - \Lambda$  is a martingale because the intensity is obtained by  $\lambda(t) = \lim_{h \rightarrow 0^+} \frac{1}{h}\mathbb{E}[N_{t+h} - N_t | \mathcal{F}_t]$  almost surely (see formula (3.5) in Chapter 2 of Brémaud [1981]).

If  $N$  is a Poisson process, some exponential inequalities have already been obtained in Houdré and Reynaud-Bouret [2003] for U-statistics of order two of the form  $Z_t = \int_0^t \int_0^{y^-} h(x, y) d(N_x - \Lambda_x) d(N_y - \Lambda_y)$  where  $h$  is a bounded Borel function. In the particular case where  $h$  is of the form  $h(x, y) = H(x)H(y)$  for some bounded Borel function  $H$ , the U-statistics of order two  $Z$  may be written  $Z_t = \frac{1}{2}\tilde{M}_t$ . Since the jumps of a Poisson process are totally inaccessible by the Meyer theorem (see [Protter, 2005, page 104]), we may also apply (5). Comparing to Houdré and Reynaud-Bouret [2003] where the supremum of  $(Z_t)_{t \geq 0}$  is not considered, the inequality (5) appears to be simpler, with a more precise rate for the small deviations and a better one for the large deviations.

### 5.2 Oscillation modulus control

The main theorems of the previous section provide also an upper bound for the oscillation modulus of the three martingales. We consider then  $c, d$  and  $x$  three non-negative real numbers, and the counting process  $N_t^c = N_{t+c} - N_c$  whose compensator is  $\Lambda_{t+c} - \Lambda_c$ .

In the case of the martingale  $M$ , we may use the relationship  $M_t - M_s = \int_c^t H_u(dN_u - \Lambda_u) - \int_c^s H_u(dN_u - \Lambda_u)$  to get

$$\begin{aligned} \sup_{(s,t) \in [c,d]} |M_t - M_s| &\leq 2 \sup_{t \in [c,d]} \left| \int_c^t H_u(dN_u - \Lambda_u) \right| \\ &= 2 \sup_{t \in [0, d-c]} \left| \int_0^t H_{u+c}(dN_u^c - d\Lambda_u^c) \right|. \end{aligned}$$

Since  $N^c$  satisfies the same assumptions than  $N$ , we may apply (4) with  $N^c$ ,  $\Lambda^c$  and the process  $u \mapsto H_{u+c}$  in order to obtain

$$\mathbb{P} \left( \sup_{(s,t) \in [c,d]^2} |M_t - M_s| \geq x \right) \leq 2 \exp \left( - \frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I \left( \frac{\|H\|_{\infty,[c,d]} x}{\|H\|_{2,[c,d]}^2} \right) \right). \quad (11)$$

For the martingale  $\tilde{M}$ , we first give a bound for  $\mathbb{P}(\sup_{(s,t) \in [c,d]^2} |(\int_s^t H_u d(N_u - \Lambda_u))^2 - \int_s^t H_u^2 dN_u| \geq x)$ .

We shall consider the following relationship

$$\left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u = \tilde{M}_t^c - \tilde{M}_s^c - 2(M_t - M_s) \int_c^s H_u(N_u - \Lambda_u)$$

where  $\tilde{M}_t^c = (\int_c^t H_u d(N_u - \Lambda_u))^2 - \int_c^t H_u^2 d\Lambda_u = (\int_0^{t-c} H_{u+c} d(N_u^c - \Lambda_u^c))^2 - \int_0^{t-c} H_{u+c}^2 d\Lambda_u^c$ . This yields for  $a > 0$

$$\begin{aligned} & \mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} \left| \left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u \right| \geq x\right) \\ & \leq \mathbb{P}\left(2 \sup_{t \in [c,d]} |\tilde{M}_t^c| \geq \frac{x}{2}\right) + \mathbb{P}\left(\sup_{s \in [c,d]} \left| \int_c^s H_u(N_u - \Lambda_u) \right| \geq a\right) + \mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \geq \frac{x}{4a}\right). \end{aligned}$$

We get then from (7), (4) and (11)

$$\begin{aligned} \mathbb{P}\left(2 \sup_{t \in [c,d]} |\tilde{M}_t^c| \geq \frac{x}{2}\right) & \leq 6 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \sqrt{\frac{x}{8}}\right)\right), \\ \mathbb{P}\left(\sup_{s \in [c,d]} \left| \int_c^s H_u(N_u - \Lambda_u) \right| \geq a\right) & \leq 2 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} a\right)\right) \end{aligned}$$

and

$$\mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \geq \frac{x}{4a}\right) \leq 2 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \frac{x}{8a}\right)\right).$$

If we choose  $a = \sqrt{\frac{x}{8}}$ , we obtain

$$\mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} \left| \left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u \right| \geq x\right) \leq 10 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \sqrt{\frac{x}{8}}\right)\right). \quad (12)$$

For the oscillation modulus of  $\tilde{M}$ , we may use similarly

$$\tilde{M}_t - \tilde{M}_s = \left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u + 2(M_t - M_s)M_s,$$

$$\begin{aligned} \mathbb{P}\left(\sup_{(s,t) \in [c,d]} |\tilde{M}_t - \tilde{M}_s| \geq x\right) & \leq \mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} \left| \left(\int_s^t H_u d(N_u - \Lambda_u)\right)^2 - \int_s^t H_u^2 dN_u \right| \geq \frac{x}{2}\right) \\ & + \mathbb{P}\left(\sup_{s \in [0,d]} |M_s| \geq a\right) + \mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \geq \frac{x}{4a}\right). \end{aligned}$$

Using (12), (4), (11) and choosing  $a = \sqrt{\frac{x}{8} \frac{\|H\|_{\infty,[c,d]}}{\|H\|_{\infty,d}} \frac{\|H\|_{2,d}}{\|H\|_{2,[c,d]}}}$  we obtain

$$\begin{aligned} \mathbb{P}\left(\sup_{(s,t) \in [c,d]} |\tilde{M}_t - \tilde{M}_s| \geq x\right) & \leq 10 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \sqrt{\frac{x}{16}}\right)\right) \\ & + 2 \exp\left(-\frac{\|H\|_{2,d}^2}{\|H\|_{\infty,d}^2} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}} \sqrt{\frac{x}{8}}\right)\right) \\ & + 2 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}} \sqrt{\frac{x}{8}}\right)\right). \end{aligned}$$

Gathering all these results, we obtain the following Theorem:

**Theorem 4.** For every non-negative  $x$ ,  $c$  and  $d$ , we have the following inequalities:

$$\mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} \left| \left( \int_s^t H_u d(N_u - \Lambda_u) \right)^2 - \int_s^t H_u^2 dN_u \right| \geq x\right) \leq 10 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \sqrt{\frac{x}{8}}\right)\right),$$

$$\mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} |M_t - M_s| \geq x\right) \leq 2 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \frac{x}{2}\right)\right)$$

and

$$\begin{aligned} \mathbb{P}\left(\sup_{(s,t) \in [c,d]^2} |\tilde{M}_t - \tilde{M}_s| \geq x\right) &\leq 10 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\|H\|_{\infty,[c,d]}}{\|H\|_{2,[c,d]}^2} \sqrt{\frac{x}{16}}\right)\right) \\ &+ 2 \exp\left(-\frac{\|H\|_{2,d}^2}{\|H\|_{\infty,d}^2} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}} \sqrt{\frac{x}{8}}\right)\right) \\ &+ 2 \exp\left(-\frac{\|H\|_{2,[c,d]}^2}{\|H\|_{\infty,[c,d]}^2} I\left(\frac{\sqrt{\|H\|_{\infty,[c,d]}\|H\|_{\infty,d}}}{\|H\|_{2,d}\|H\|_{2,[c,d]}} \sqrt{\frac{x}{8}}\right)\right). \end{aligned}$$

## 6 Proofs

**Proof of Lemma 1** The process  $Z$  is defined as  $\lambda M_t - \int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$  where  $\lambda$  is a fixed real number.  $Z$  is of bounded variations because  $H$  and  $M$  are of bounded variations, and the continuity of  $\Lambda$  entails the equality  $\Delta Z_s = \lambda H_s \Delta N_s$ . We get then from (1) that

$$\begin{aligned} e^{Z_t} &= 1 + \int_0^t e^{Z_{s-}} dZ_s + \sum_{s \leq t} e^{Z_{s-}} [e^{\lambda H_s \Delta N_s} - 1 - \lambda H_s \Delta N_s] \\ &= 1 + \int_0^t e^{Z_{s-}} [\lambda dM_s - (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s] + \int_0^t e^{Z_{s-}} (e^{\lambda H_s} - 1 - \lambda H_s) dN_s \\ &= 1 + \int_0^t e^{Z_{s-}} (e^{\lambda H_s} - 1) d(N_s - \Lambda_s). \end{aligned}$$

Then, for  $s \leq T$ ,

$$|e^{\lambda H_s} - 1| \leq e^{|\lambda| \|H\|_{\infty, T}} + 1$$

and

$$|Z_{s-}| \leq |\lambda| \|H\|_{\infty, T} N_T + (e^{|\lambda| \|H\|_{\infty, T}} + 1 + 2|\lambda| \|H\|_{\infty, T}) \Lambda_T.$$

Therefore we obtain

$$\mathbb{E}\left[\int_0^{+\infty} e^{2Z_{s-}} (e^{\lambda H_s} - 1)^2 \mathbf{1}_{s \leq T} dN_s\right] \leq C(\lambda, \|H\|_{\infty, T}, \Lambda_T) \mathbb{E}[e^{2|\lambda| \|H\|_{\infty, T} N_T} N_T] < +\infty$$

where  $C(\lambda, \|H\|_{\infty, T}, \Lambda_T) = (e^{|\lambda| \|H\|_{\infty, T}} + 1)^2 \exp(2(e^{|\lambda| \|H\|_{\infty, T}} + 1 + 2|\lambda| \|H\|_{\infty, T}) \Lambda_T)$  ■



**Proof of Lemma 2** The process  $\tilde{Z}$  is defined as  $\lambda\tilde{M}_t - \int_0^t (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d\Lambda_s$  for a fixed real  $\lambda$ .  $\tilde{Z}$  is of bounded variations because  $H$  and  $M$  are of bounded variations too, and since  $\Lambda$  is continuous, we may compute  $\Delta\tilde{Z}_s = 2\lambda H_s M_s \Delta N_s$ . We get from (1) that

$$\begin{aligned} e^{\tilde{Z}_t} &= 1 + \int_0^t e^{\tilde{Z}_{s-}} d\tilde{Z}_s + \sum_{s \leq t} e^{\tilde{Z}_{s-}} [e^{2\lambda H_s M_s \Delta N_s} - 1 - 2\lambda H_s M_s \Delta N_s] \\ &= 1 + \lambda \int_0^t e^{\tilde{Z}_{s-}} d\tilde{M}_s + \int_0^t e^{\tilde{Z}_{s-}} (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d(N_s - \Lambda_s) \\ &= 1 + \int_0^t e^{\tilde{Z}_{s-}} (e^{2\lambda H_s M_s} - 1) d(N_s - \Lambda_s) \end{aligned}$$

and

$$e^{\tilde{Z}_{t \wedge T_a}} = 1 + \int_0^t e^{\tilde{Z}_{s-}} (e^{2\lambda H_s M_s} - 1) \mathbf{1}_{s \leq T_a} d(N_s - \Lambda_s).$$

It remains to show that  $\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s-}} (e^{2\lambda H_s M_s} - 1)^2 \mathbf{1}_{s \leq T \wedge T_a} dN_s] < +\infty$ . For all  $s \leq T \wedge T_a$ ,

$$|2\lambda H_s M_s| \leq 2|\lambda| \|H\|_{\infty, T} |M_s| \leq 2|\lambda|(a + \|H\|_{\infty, T}) \|H\|_{\infty, T}$$

and

$$(e^{2\lambda H_s M_s} - 1)^2 \leq (e^{2|\lambda|(a + \|H\|_{\infty, T}) \|H\|_{\infty, T}} + 1)^2.$$

It is then enough to show that

$$\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s-}} \mathbf{1}_{s \leq T \wedge T_a} dN_s] < +\infty.$$

This last inequality comes from the fact that for  $s \leq T \wedge T_a$

$$\begin{aligned} 2\tilde{Z}_{s-} &= 2\lambda\tilde{M}_{s-} - 2 \int_0^{s-} (e^{2\lambda H_u M_u} - 1 - 2\lambda H_u M_u) d\Lambda_u \\ &\leq 2\lambda\tilde{M}_{s-} \\ &= 4\lambda \int_0^{s-} M_u - H_u d(N_u - \Lambda_u) \\ &\leq 4|\lambda|a \|H\|_{\infty, T} (N_T + \Lambda_T). \end{aligned}$$

Then  $e^{2\tilde{Z}_{s-}} \leq e^{4|\lambda|a \|H\|_{\infty, T} (N_T + \Lambda_T)}$  and

$$\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s-}} \mathbf{1}_{s \leq T \wedge T_a} dN_s] \leq \mathbb{E}[N_T e^{2|\lambda|a \|H\|_{\infty, T} (N_T + \Lambda_T)}] < +\infty \quad \blacksquare$$

**Proof of Lemma 3** We follow the steps of the proof of Lemma 2, adapting the computations to this case. The process  $\tilde{\tilde{Z}}$  is defined as  $\lambda\tilde{\tilde{M}}_t - \int_0^t (e^{\lambda H_s (H_s + 2M_s)} - 1 - \lambda H_s (H_s + 2M_s)) d\Lambda_s$  for a fixed real  $\lambda$ . The process  $\tilde{\tilde{Z}}$  is again of bounded variations because  $H$  and  $M$  are of bounded variations, and the continuity of  $\Lambda$  entails the equality  $\Delta\tilde{\tilde{Z}}_s = \lambda H_s (H_s + 2M_s) \Delta N_s$ . Then (1) yields

$$\begin{aligned} e^{\tilde{\tilde{Z}}_t} &= 1 + \int_0^t e^{\tilde{\tilde{Z}}_{s-}} d\tilde{\tilde{Z}}_s + \sum_{s \leq t} e^{\tilde{\tilde{Z}}_{s-}} [e^{\lambda H_s (H_s + 2M_s) \Delta N_s} - 1 - \lambda H_s (H_s + 2M_s) \Delta N_s] \\ &= 1 + \lambda \int_0^t e^{\tilde{\tilde{Z}}_{s-}} d\tilde{\tilde{M}}_s + \int_0^t e^{\tilde{\tilde{Z}}_{s-}} (e^{\lambda H_s (H_s + 2M_s)} - 1 - \lambda H_s (H_s + 2M_s)) d(N_s - \Lambda_s) \\ &= 1 + \int_0^t e^{\tilde{\tilde{Z}}_{s-}} (e^{\lambda H_s (H_s + 2M_s)} - 1) d(N_s - \Lambda_s) \end{aligned}$$

and

$$e^{\tilde{Z}_{t \wedge T_a}} = 1 + \int_0^t e^{\tilde{Z}_{s^-}} (e^{\lambda H_s(H_s + 2M_s)} - 1) \mathbf{1}_{s \leq T_a} d(N_s - \Lambda_s).$$

The proof is complete showing that  $\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s^-}} (e^{\lambda H_s(H_s + 2M_s)} - 1)^2 \mathbf{1}_{s \leq T \wedge T_a} dN_s] < +\infty$ . For all  $s \leq T \wedge T_a$ ,

$$|H_s(H_s + 2M_s)| \leq \|H\|_{\infty, T}^2 + 2\|H\|_{\infty, T}|M_s| \leq \|H\|_{\infty, T}^2 + 2(a + \|H\|_{\infty, T})\|H\|_{\infty, T}$$

and

$$(e^{\lambda H_s(H_s + 2M_s)} - 1)^2 \leq (e^{|\lambda|(\|H\|_{\infty, T}^2 + 2(a + \|H\|_{\infty, T})\|H\|_{\infty, T})} + 1)^2.$$

It is thus enough to show that

$$\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s^-}} \mathbf{1}_{s \leq T \wedge T_a} dN_s] < +\infty.$$

This last inequality comes from the fact that for  $s \leq T \wedge T_a$

$$\begin{aligned} 2\tilde{Z}_{s^-} &= 2\lambda\tilde{M}_{s^-} - 2\int_0^{s^-} (e^{\lambda H_u(H_u + 2M_u)} - 1 - \lambda H_u(H_u + 2M_u)) d\Lambda_u \\ &\leq 2\lambda\tilde{M}_{s^-} \\ &= \int_0^{s^-} (4\lambda M_{u^-} + 2\lambda H_u) dM_u \\ &\leq (4|\lambda|a + 2|\lambda|\|H\|_{\infty, T})\|H\|_{\infty, T}(N_T + \Lambda_T). \end{aligned}$$

Then  $e^{2\tilde{Z}_{s^-}} \leq e^{(4|\lambda|a + 2|\lambda|\|H\|_{\infty, T})\|H\|_{\infty, T}(N_T + \Lambda_T)}$  and

$$\mathbb{E}[\int_0^{+\infty} e^{2\tilde{Z}_{s^-}} \mathbf{1}_{s \leq T \wedge T_a} dN_s] \leq \mathbb{E}[N_T e^{(4|\lambda|a + 2|\lambda|\|H\|_{\infty, T})\|H\|_{\infty, T}(N_T + \Lambda_T)}] < +\infty \quad \blacksquare$$

**Proof of Lemma 4** Let  $s \leq t \leq T$  and  $\lambda \in \mathbb{R}$ . We use the following inequality:

$$\begin{aligned} |e^{\lambda H_s} - 1 - \lambda H_s| &= \left| \sum_{j \geq 2} \frac{(\lambda H_s)^j}{j!} \right| \\ &= \left| \frac{(\lambda H_s)^2}{2!} + H_s^2 \sum_{j \geq 3} \frac{\lambda^j H_s^{j-2}}{j!} \right| \\ &\leq \frac{(|\lambda|H_s)^2}{2!} + H_s^2 \sum_{j \geq 3} \frac{|\lambda|^j \|H\|_{\infty, T}^{j-2}}{j!} \\ &= H_s^2 \left( \frac{\lambda^2}{2} + \frac{1}{\|H\|_{\infty, T}^2} \sum_{j \geq 3} \frac{|\lambda|^j \|H\|_{\infty, T}^j}{j!} \right), \end{aligned}$$

that is

$$|e^{\lambda H_s} - 1 - \lambda H_s| \leq \frac{H_s^2}{\|H\|_{\infty, T}^2} \sum_{j \geq 2} \frac{|\lambda|^j \|H\|_{\infty, T}^j}{j!}. \quad (13)$$

Integrating with respect to  $d\Lambda_s$  we obtain

$$|I_t(H, \lambda)| \leq \frac{\|H\|_{2, T}^2}{\|H\|_{\infty, T}^2} g(|\lambda|\|H\|_{\infty, T})$$

where  $g(x) = e^x - 1 - x \quad \blacksquare$

**Proof of Theorem 1** Recall that  $I_t(H, \lambda)$  is defined by  $\int_0^t (e^{\lambda H_s} - 1 - \lambda H_s) d\Lambda_s$ . Using Lemma 4, we obtain for all  $\lambda > 0$ ,  $x > 0$  and  $T > 0$ ,

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq x) &= \mathbb{P}(\sup_{0 \leq t \leq T} e^{\lambda M_t - I_t(H, \lambda) + I_t(H, \lambda)} \geq e^{\lambda x}) \\ &\leq \mathbb{P}(e^{\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T})} \sup_{0 \leq t \leq T} e^{\lambda M_t - I_t(H, \lambda)} \geq e^{\lambda x}). \end{aligned}$$

Doob's maximal inequality and Lemma 1 then lead to

$$\mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq x) \leq \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T}) - \lambda x\right)$$

for every  $\lambda > 0$  with  $g(x) = e^x - 1 - x$ , so we get

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq x) &\leq \inf_{\lambda > 0} \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T}) - \lambda x\right) \\ &= \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right) \end{aligned}$$

that is (3). Applying this inequality with  $-H$  instead of  $H$ , we obtain also

$$\mathbb{P}(\sup_{0 \leq t \leq T} -M_t \geq x) \leq \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} x\right)\right).$$

Then (4) follows from the inequality

$$\mathbb{P}(\sup_{0 \leq t \leq T} |M_t| \geq x) \leq \mathbb{P}(\sup_{0 \leq t \leq T} M_t \geq x) + \mathbb{P}(\sup_{0 \leq t \leq T} -M_t \geq x) \quad \blacksquare$$

**Proof of Theorem 2** Let us begin with the proof of (5). For all  $T > 0$ ,  $\lambda > 0$  and  $x > 0$ ,

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x) &\leq \mathbb{P}(T_a < T) + \mathbb{P}(\sup_{0 \leq t \leq T} \tilde{M}_{t \wedge T_a} \geq x \cap T_a \geq T) \\ &\leq \mathbb{P}(\sup_{0 \leq t \leq T} |M_t| \geq a) + \mathbb{P}(\sup_{0 \leq t \leq T} e^{\lambda \tilde{M}_{t \wedge T_a}} \geq e^{\lambda x}). \end{aligned} \tag{14}$$

Using the inequality (13), we obtain for  $t \leq T$  and  $\lambda > 0$

$$\int_0^{t \wedge T_a} (e^{2\lambda H_s M_s} - 1 - 2\lambda H_s M_s) d\Lambda_s \leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T}).$$

Then Lemma 2 and Doob's maximal inequality yield for every  $\lambda > 0$

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq T} e^{\lambda \tilde{M}_{t \wedge T_a}} \geq e^{\lambda x}) &\leq \mathbb{P}(\sup_{0 \leq t \leq T} e^{\lambda \tilde{Z}_{t \wedge T_a}} \geq e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T})}) \\ &\leq \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T}) - \lambda x\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{\tilde{M}_t \wedge T_a} \geq e^{\lambda x}\right) &\leq \inf_{\lambda > 0} \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T}) - \lambda x\right) \\ &= \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{2a\|H\|_{2,T}^2} x\right)\right). \end{aligned}$$

The inequality (14) and Lemma 1 then entail for every  $a > 0$ ,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x\right) \leq 2e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2} a\right)} + e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{2a\|H\|_{2,T}^2} x\right)}.$$

We choose  $a = \sqrt{\frac{x}{2}}$  in order to obtain (5). For the proof of (6), notice that for  $\lambda > 0$  Lemma 4 and Lemma 2 yield

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} e^{-\lambda \tilde{M}_t \wedge T_a} \geq e^{\lambda x}\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{-\lambda \tilde{Z}_t \wedge T_a} \geq e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T})}\right) \leq e^{\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(2\lambda a \|H\|_{\infty,T}) - \lambda x}$$

and the end of the proof is similar to the one of (5). In order to obtain (7), we use the inequality

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |\tilde{M}_t| \geq x\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x\right) + \mathbb{P}\left(\sup_{0 \leq t \leq T} -\tilde{M}_t \geq x\right) \quad \blacksquare$$

**Proof of Theorem 3** This proof is similar to the one of Theorem 2. Let us begin showing the inequality (8). For all  $T > 0$ ,  $\lambda > 0$  and  $x > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x\right) &\leq \mathbb{P}(T_a < T) + \mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \wedge T_a \geq x \cap T_a \geq T\right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| \geq a\right) + \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{\lambda \tilde{M}_t \wedge T_a} \geq e^{\lambda x}\right). \end{aligned} \quad (15)$$

Using the inequality (13), we obtain for  $t \leq T$  and  $\lambda > 0$

$$\int_0^{t \wedge T_a} (e^{\lambda H_s(H_s + 2M_s)} - 1 - \lambda H_s(H_s + 2M_s)) d\Lambda_s \leq \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a)).$$

Then Lemma 3 and Doob's maximal inequality yield for every  $\lambda > 0$

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{\lambda \tilde{M}_t \wedge T_a} \geq e^{\lambda x}\right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{\lambda \tilde{Z}_t \wedge T_a} \geq e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a))}\right) \\ &\leq \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a)) - \lambda x\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{\tilde{M}_t \wedge T_a} \geq e^{\lambda x}\right) &\leq \inf_{\lambda > 0} \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a)) - \lambda x\right) \\ &= \exp\left(-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2 (2a + \|H\|_{\infty,T})} x\right)\right). \end{aligned}$$

The inequality (15) and Lemma 2 then entail for every  $a > 0$ ,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x\right) \leq 2e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_2^2} a\right)} + e^{-\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} I\left(\frac{\|H\|_{\infty,T}}{\|H\|_{2,T}^2(2a+\|H\|_{\infty,T})} x\right)}.$$

We choose  $a = \frac{x}{2a+\|H\|_{\infty,T}}$  i.e.  $a = \frac{-\|H\|_{\infty,T} + \sqrt{\|H\|_{\infty,T}^2 + 8x}}{4}$  in order to get (8). For the proof of (9), notice that for  $\lambda > 0$ , we obtain with Lemma 4 and Lemma 3

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{-\lambda \tilde{M}_t \wedge T a} \geq e^{\lambda x}\right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} e^{-\lambda \tilde{Z}_t \wedge T a} \geq e^{\lambda x - \frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a))}\right) \\ &\leq \exp\left(\frac{\|H\|_{2,T}^2}{\|H\|_{\infty,T}^2} g(\lambda \|H\|_{\infty,T} (\|H\|_{\infty,T} + 2a)) - \lambda x\right) \end{aligned}$$

and the end of the proof is similar to the one of (8). To conclude, (10) follows also from the inequality

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |\tilde{M}_t| \geq x\right) \leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \tilde{M}_t \geq x\right) + \mathbb{P}\left(\sup_{0 \leq t \leq T} -\tilde{M}_t \geq x\right) \quad \blacksquare$$

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