Book review: Input-to-State Stability for PDEs, Iasson Karafyllis, Miroslav Krstic. Communications and Control Engineering, Springer, (2018). ISBN: 978-3-319-91011-6

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"Input-to-State Stability for PDEs" by Iasson Karafyllis and Miroslav Krstic, ISBN 978-3-319-91011-6, Springer, 2018

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1 Introduction

Partial Differential Equations (PDEs) are the key of many sciences because it is very often used to model dynamics in the physical world and in social networks. This book develops the Input-to-State Stability (ISS) theory for such infinite-dimensional models, and derive some sufficient conditions to check the ISS property. Many results can be seen as design methods for the synthesis of control laws providing such ISS property to the closed-loop system. Various types of disturbances are considered, such as boundary disturbances (affecting the boundary conditions), in-domain perturbations (e.g., to model dynamics uncertainties) and non-local terms. This book provides also estimations of the disturbance gains, and succeeds to develop small-gain analysis that is useful when interconnecting separate subsystems. The authors tackle a large class of nonlinear PDEs, such as hyperbolic and parabolic PDEs with Lipschitz and non-local nonlinearities, together with their interconnection with other PDEs or Ordinary Differential Equations (ODEs).

The results are constructive providing effective methods for the ISS analysis and the control design. Many techniques are based on dedicated Lyapunov functions. Moreover, most of the results are new, and they improve recent results of the authors. Many examples successively illustrate the results and the constructive approaches, some of them are used several times along the textbook. All results are proved with clear and precise calculations, so that this textbook is self-contained. This book can be very useful as materials for teachers of a course on hyperbolic (or parabolic) PDEs with their ISS

properties. The running examples can be nice supports for labs and for exercises for the students.

This book provides a sufficient background as the foundation of PDEs and of ISS. It is also accompanied with an extensive bibliography and some notes on related results. This book is complete and exhaustive on the ISS for this class of PDEs. The style is clear and some ideas are powerful, and, as noted as the end of the monograph, the presented techniques may be useful for other infinite-dimensional problems. Therefore this book would be a good introductory monograph for researchers just getting into these areas, but it is also useful for experts on distributed parameter systems to get an up-to-date picture on this subject, and to foresee research lines on other questions.

2 Book description and analysis

The first part of the book is devoted to the study of first-order hyperbolic PDEs. Chapter 2 contains all existence and uniqueness results of the considered class of nonlinear hyperbolic systems. The results are illustrated with two examples containing a non-local term: a chemical reactor with an exothermic reaction, and a dynamical age distribution in a population. Adequate normed spaces are introduced and the regularity properties are precisely stated. The topology needs to be precisely studied because trace terms in the boundary conditions are considered in the subsequent chapters.

Chapter 3 deals with nonlinear hyperbolic PDEs with a constant transport velocity. Two different methodologies that allow the derivation of ISS estimates for hyperbolic PDEs are presented: the ISS-Lyapunov functional for the PDE model and an equivalent model written by Integral Delay Equations (IDEs). First ISS-Lyapunov functionals (ISS-LFs) are derived providing estimates written in terms of the spatial L^p-norm of the state (with $p \in (1, \infty)$). Then ISS properties are derived for hyperbolic systems given as IDEs. ISS properties are derived for this class of delay systems and Lyapunov-like functionals are provided. This chapter and Part I

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end with some bibliographical comments. Some links to other works and other notions of solutions and ISS properties are also given.

Part II of this book deals with parabolic PDEs. The organization of this part parallels the one of the previous part. First in Chapter 4, existence and uniqueness results are precisely stated for parabolic PDEs. Such infinite-dimensional systems are first written in terms of the Sturm-Liouville operator, and then interconnected with ODEs, globally Lipschitz nonlinearities and nonlocal terms. For such systems, global results for the well-posedness are stated and proven. Then two examples are studied to illustrate the results of this chapter: a chemical reactor with a cooling jacket (this is a modification of one example considered in Chapter 2), and a water tank that could be useful for engineers working on fluid networks, and also for those working on thermal storage equipments. To conclude this chapter, boundary inputs and distributed inputs are introduced and the class of admissible inputs is defined, so that the existence and uniqueness of solutions still hold.

Chapter 5 focuses on derivations of ISS estimates for both the spatial L^2 and H^1-norms. Two different methodologies are given: one based on the eigenfunction expansion and the other exploiting ISS Lyapunov functionals. Some ISS estimates in the spatial L^2-norms are first provided, allowing less regular inputs for ISS than with classical solutions. Both internal and boundary perturbations are tackled in the ISS estimates, assuming a lower bound of the principal eigenvalue of the Sturm-Liouville operator (but without the knowledge of all the set of eigenvalues). As far as the H^1-norm is concerned, estimates are proven with different boundary conditions and boundary disturbances, exploiting computations on the eigenvalue series. Then ISS Lyapunov functionals are computed providing ISS estimates in L^2-norms. The boundary conditions could be of different types, as the Robin type or Dirichlet type with or without any disturbance. The sufficient conditions for the design of ISS Lyapunov functionals can be easily tested in applications as illustrated in an example studying the sensitivity of the temperature distribution of a solid bar with respect to variations of the surrounding air temperature. Finally a reaction-advection-diffusion PDE is considered at the end of Chapter 5.

Chapter 6 deals also with ISS properties for parabolic PDEs, but the L^p-norm (with $p \in [1, \infty]$) is considered in this chapter (without Hilbert structure as in the previous chapter). For this topology, a completely different method is introduced: the computation of ISS Lyapunov Functionals Under Discretization (ISS-LFUD). Roughly speaking such functionals are less regular than previously introduced ISS Lyapunov functionals. They exploit calculus that are quite usual for finite-dimensional systems, and are based on a family of finite-dimensional discrete-time systems obtained

by a discretization procedure. Such ISS-LFUD are the "limits" of classical discrete-time Lyapunov functions, and allow the derivations of ISS estimates, in the sup-norm and in the L^p-norm for $1 \le p < \infty$, for parabolic Stum-Liouville operators. Two applications are also presented. First a very interesting robustness property of boundary feedback laws designed by the usual backstepping techniques. This application is very useful by itself and it is natural to conjecture a robustness issue for other parabolic PDEs in closed-loop with other backstepping controllers. The second application is no less important since it deals with ISS in Taylor-Couette flow of a viscous fluid inside of two rotating infinitely long cylinders. This chapter and Part II also end with a long series of comments and notes allowing to compare the results of this part with other approaches and to insist on the novelty of the approach based on ISS-LFUD. The reviewer is confident that this approach could be applied to other PDEs. Moreover a possible extension of this part is to deal with robustness properties of controllers that are designed by backstepping methods or by Control Lyapunov Functionals (as done in e.g., [3]).

Part III deals with small-gain analysis and feedback interconnections of dynamics. Different possible interconnections are possible such as two PDEs (of different nature) or one PDE and one ODE. Chapter 7 is a short chapter containing two technical lemmas dealing with ISS estimates where past input values have less effect than present input values. These estimates are thus named "fading-memory" estimates, and these lemmas are used in the following chapter. They may be also useful in other contexts than the one considered in this monograph, and for other works dealing with ISS estimations.

Chapter 8 deals with the interconnection of a system of ODEs with either a hyperbolic PDE or a parabolic one. First, under appropriate assumptions and under a suitable small-gain condition, the case of a hyperbolic PDE in feedback with a system of ODEs is studied, and an ISS property is proven for this interconnection. A simple transport equation in feedback with a first-order ODE controller illustrates this result. Then the interconnection of a parabolic PDE (written again as a Sturm-Liouville operator) and of a system of ODEs is considered. Under some assumptions and a small-gain condition, such feedback loop is shown to be ISS as well. Both these results could be seen as design methods of finite-dimensional boundary controllers yielding an ISS property of the infinite-dimensional closed-loop system. Nice applications illustrate these results. More precisely, the chemical reactor in a cooling jacket is again considered, together with the water tank, allowing to emphasize the importance of the small-gain methodology for real applications.

Chapter 9 studies the case where two first-order hyperbolic PDEs are interconnected in feedback form. As re-

marked by the authors, there is no need to assume that the transport velocities have all the same sign. First the existence and uniqueness of solution (in an appropriate class of functions) are proved, even in presence of non-linear and non-local terms. Then small-gain methodology is presented for the interconnection of two hyperbolic PDEs, giving a sufficient condition for the ISS of the closed-loop system. A comparison of this sufficient condition with the classical Bastin-Coron condition [1, Corollary 5.5, Section 5.3, Page 177] for the stability of balance laws is provided. This interesting discussion shows that both conditions are equivalent for linear systems of conservation laws, when the sup-norm is employed. Moreover, as far as systems of balance laws are concerned, when the L^2-norm is considered, the sufficient condition of Chapter 9 is not necessarily more demanding than the Bastin-Coron condition. Finally it is shown that the sufficient condition of Chapter 9 is the first sufficient condition for the exponential stability of linear systems of balance laws, in the sup-norm. Such results exist for quasilinear system of conservation (see [2]) but no result exists for exponential stability of linear systems of balance laws, in the sup-norm, prior to this chapter.

Chapter 10 is concerned with the interconnection of two parabolic PDEs with non-local terms. First well-posedness results are proven for generalized solutions and mild solutions. Then small-gain analysis is provided when the interconnection is either in the domain or at the boundary. ISS estimates are obtained by using some results of Chapter 6. A model of the flow of a liquid in a fissured porous medium illustrates the small-gain approach, explicit computations are performed for this example of two parabolic PDEs connected at the boundary. This last study could be seen as application techniques of Chapters 6 and 10.

Chapter 11 studies a case that is not so often considered in the literature: the interconnection of one hyperbolic PDE and another parabolic PDE. First a limit case is studied; that is, when the transport velocity is zero, with a diffusion phenomenon. Such a model is useful for dynamics of chemicals in underground water, and can be related to the wave equation with Kevin-Voigt damping. Then a general interconnection of a hyperbolic PDE and a parabolic PDE is studied. Both cases of in-domain and boundary interconnections are studied together. Firstly an existence and uniqueness result is proven, and secondly a sufficient condition is given for the exponential stability of the interconnected system, in sup-norm. Interestingly enough, the authors conclude this chapter with an example and a nice robustness result: the exponential stability of a system of two conservation laws implies the exponential stability when one transport equation is equipped with a small diffusion. This robustness result is a nice and elegant consequence of the last main result. It follows from basic and direct computations, exploiting the small-gain methodology of Part III. Some concluding notes and possible research lines are collected at the end of this part.

3 Summary

This book is very clear and can be seen as a complete study of the ISS property for hyperbolic PDEs and parabolic PDEs, with non-local terms, global-Lipchitz nonlinearities, and linear differential parts. All kinds of interconnections in the domain or at the boundaries are considered by mixing a hyperbolic PDE, a parabolic PDE or a systems of ODEs. All results are rigorously proven, commented, compared with other results and illustrated by some applicative examples. The reader can note a methodological pattern, which can be used for other cases of PDEs and interconnections of PDEs, as maybe for infinite-dimensional systems with more than just one spatial dimension, or quasilinear systems (that is with a nonlinear term of the differential part), or interconnection to more than two PDEs. In a conclusion, this book is recommended for graduate and PhD students as a complete and self-contained reference on ISS of PDEs. Moreover this monograph is not only useful for lecturers interested in teaching, e.g., control theory of PDEs, methods for designing stabilizing boundary controllers, or applications of small-gains methods to ISS property. But this book is also recommended for researchers working on distributed parameters systems, and their applications.

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Brief bio of the author

Christophe Prieur graduated in Mathematics from the Ecole Normale Supérieure de Cachan, France in 2000. He received the Ph.D degree in 2001 in Applied Mathematics from Paris-Sud University, France, and the "Habilitation à Diriger des Recherches" (HDR degree) in 2009 from Toulouse University. Currently he is a CNRS senior researcher at Gipsa-lab, Grenoble, France. He is an IMA Fellow, and an IEEE Senior Member. His current research interests include nonlinear control theory, hybrid systems, and control of partial differential equations.