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GLOBAL ANALYSIS OF A MODEL OF COMPETITION IN CHEMOSTAT WITH INTERNAL INHIBITOR

MOHAMED DELLAL*

Université Ibn Khaldoun, 14000 Tiaret, Algérie

BACHIR BAR

Université AbouBakr Belkaid, LSDA, Tlemcen, Algérie

ABSTRACT. A model of two microbial species in a chemostat competing for a single resource in the presence of an internal inhibitor is considered. The model is a four-dimensional system of ordinary differential equations. Using general growth rate functions of the species, we give a complete analysis for the existence and local stability of all steady states. We describe the behavior of the system with respect to the operating parameters represented by the dilution rate and the input concentrations of the substrate. The operating diagram has the operating parameters as its coordinates and the various regions defined in it correspond to qualitatively different asymptotic behavior: washout, competitive exclusion of one species, coexistence of the species, bistability, multiplicity of positive steady states. This bifurcation diagram which determines the effect of the operating parameters, is very useful to understand the model from both the mathematical and biological points of view, and is often constructed in the mathematical and biological literature.

1. Introduction. The chemostat is an important laboratory apparatus used for the continuous culture of micro-organisms. Competition for single and multiple resources, evolution of resource acquisition, and competition among micro-organisms have been investigated in ecology and biology using chemostats [11, 21, 22, 28]. A detailed mathematical description of competition in the chemostat may be found in [10, 26].

The basic chemostat model predicts that coexistence of two or more microbial populations competing for a single non-reproducing nutrient is not possible. Only the species with the lowest ‘break-even’ concentration survives, this is the species which consumes less substrate to attain its steady state [13]. This result, known as the Competitive Exclusion Principle [9], was established under various hypotheses [4, 12, 24, 30]. The reader may consult [18, 20, 25] for a thorough account on the contributions of diverse authors.

This theoretical prediction has been corroborated by the experiences of Hansen and Hubell [8], but the biodiversity found in nature as well as in waste-water treatment processes and biological reactors are exceptions to this principle. Several authors [3, 5, 14, 15, 16, 19, 29], and recently [2, 6] studied the inhibition as a factor in the maintenance of the diversity of microbial ecosystems: Can the production

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* Corresponding author.

of internal inhibitors or the introduction of external inhibitors induce the stable coexistence of competitors in a chemostat-like environment?

In this paper we consider the model introduced by De Freitas and Fredrickson [5]. In this model, two species x and y compete for a single limiting resource S in presence of an internal inhibitor p , where both competitors produce a toxin that inhibits their growth rates. Let $S(t)$ denote the concentration of the substrate at time t ; let $x(t)$, $y(t)$ denote the concentrations of the competitors and $p(t)$ is the concentration of the internal inhibitor. The model takes the form

$$\begin{cases} S' &= (S^0 - S)D - \mu_1(S, p)\frac{x}{\gamma_1} - \mu_2(S, p)\frac{y}{\gamma_2} \\ x' &= [\mu_1(S, p) - D]x \\ y' &= [\mu_2(S, p) - D]y \\ p' &= \alpha_1\mu_1(S, p)x + \alpha_2\mu_2(S, p)y - Dp \end{cases} \quad (1)$$

with initial condition $S(0) \geq 0$, $x(0) > 0$, $y(0) > 0$ and $p(0) \geq 0$. S^0 and D denote, respectively, the concentration of the nutrient in the feed bottle and the dilution rate of the chemostat, all of which are assumed to be constant and are under the control of the experimenter. The parameters $\gamma_i > 0$, $i = 1, 2$, are the growth yield coefficients. The product $\alpha_i\gamma_i > 0$ is the yield of inhibitor by the i th population. The growth function $\mu_i(S, p)$, $i = 1, 2$ can depend not only on the substrate S but also on the inhibitor concentration p . The function μ_i is assumed to be increasing in the variable S and decreasing in the variable p . This model was considered by De Freitas and Fredrickson [5] when

$$\mu_i(S, p) = \frac{m_i S}{(a_i + S) \left(1 + \frac{p}{K_i}\right)}, \quad i = 1, 2 \quad (2)$$

where m_i , a_i , K_i , $i = 1, 2$ are some positive constant parameters. Here, except the two variable operating (or control) parameters, which are the dilution rate D and the inflowing substrate S^0 , all the other parameters are biological parameters which depend on the organisms and substrate considered. The approach in [5] was to fix the biological parameters of the model, and discuss the behavior of the model with respect to the input concentrations of the limiting nutrient and the dilution rate, which are operating parameters of the model. By numerical computation, these authors established the ‘operating diagram’ of the model: six possible outcomes were shown, corresponding to six regions of the operating diagram, see [5], Fig. 1. Stability considerations were all local.

Hsu and Waltman [17] considered the case where $\alpha_2 = 0$ and the growth rate $\mu_1(S, p)$ depends only on the concentration of substrate S . These authors considered a particular situation where the growth functions are of the form

$$\mu_1(S) = \frac{m_1 S}{a_1 + S} \quad \text{and} \quad \mu_2(S, p) = \frac{m_2 S}{a_2 + S} e^{-K_2 p}. \quad (3)$$

The authors rescaled the biological and operating parameters of the model, creating a ‘standard’ environment in which the operating parameters are fixed to the value 1. This rescaling is often used in the mathematical literature on the chemostat [26]. The authors established global results and shown that system (3) has a unique positive equilibrium of coexistence, but which is unstable. However the operating diagram was not presented in [17].

The operating diagram has the operating parameters as its coordinates and the various regions defined in it correspond to qualitatively different dynamics. This bifurcation diagram which determines the effect of the operating parameters, that are controlled by the operator and which are the dilution rate and the input concentrations, is very useful to understand the model from both the mathematical and biological points of view, and is often constructed in the mathematical and biological literature [1, 7, 23].

It is more convenient to develop the theory for a general model. In this paper we extend [5, 17] by considering general growth functions and by describing the operating diagram. Using the concept of steady-state characteristic we present a geometric characterization of the existence of all equilibria of the model and their stability. We give a necessary and sufficient condition for the existence of a positive equilibrium, not only a sufficient condition as it is done in [5]. We show the existence of one or multiple positive locally exponentially stable equilibria. In the case of multiplicity, bistability can occur, for certain values of the operating parameters. We also extend [5] by describing theoretically the various regions of the operating diagram.

The organization of this paper is as follows. In Section 2, we present the assumptions on general model and some properties of its solutions. In Section 3, we discuss the existence and the local asymptotic stability of equilibria, and some global results. In Section 4, we present the operating diagrams. In Section 5, we consider examples and we give numerical simulations. A discussion follows in Section 6.

2. Assumptions on the model. We consider the general model (1) without restricting ourselves to the particular case where growth rates μ_i are of the form (2). We suppose only that μ_i , $i = 1, 2$, in system (1) are C^1 -functions satisfying the following conditions:

- (H1) $\mu_i(0, p) = 0$ and $\mu_i(S, p) > 0$ for all $S > 0$ and $p \geq 0$.
- (H2) $\frac{\partial \mu_i}{\partial S}(S, p) > 0$ and $\frac{\partial \mu_i}{\partial p}(S, p) < 0$ for all $S \geq 0$ and $p > 0$.
- (H3) There is $\lambda_i > 0$ such that $\mu_i(\lambda_i, 0) = D$.

(H1) means that the growth can take place if and only if the substrate is present. (H2) means that the growth rate of each species increases with the concentration of substrate and decreases with the inhibitor. (H3) means that in the absence of the inhibitor, the two species are not washed out. We have the following result:

Proposition 1. *For non-negative initial conditions, all solutions of system (1) are bounded and remain non-negative for all $t > 0$. Moreover, the set*

$$\Omega = \{(S, x, y, p) \in \mathbb{R}_+^4 : p = \alpha_1 x + \alpha_2 y, \quad S + x/\gamma_1 + y/\gamma_2 = S^0\}$$

is positively invariant and is a global attractor for system (1).

Proof. The invariance of \mathbb{R}_+^4 is guaranteed by the fact that:

$$\begin{aligned} S = 0 &\implies S' = DS^0 > 0, \\ x = 0 &\implies x' = 0, \\ y = 0 &\implies y' = 0, \\ p = 0 &\implies p' = \alpha_1 \mu_1(S, 0)x + \alpha_2 \mu_2(S, 0)y > 0. \end{aligned}$$

Let $\Sigma = S + x/\gamma_1 + y/\gamma_2$. So, we have $\Sigma = -D(\Sigma - S^0)$, and thus the explicit solution,

$$S(t) + x(t)/\gamma_1 + y(t)/\gamma_2 = S^0 + (S(0) + x(0)/\gamma_1 + y(0)/\gamma_2 - S^0)e^{-Dt}. \quad (4)$$

Thus, we obtain

$$S(t) + x(t)/\gamma_1 + y(t)/\gamma_2 \leq \max(S^0, S(0) + x(0)/\gamma_1 + y(0)/\gamma_2) \quad \text{for all } t \geq 0.$$

So $S(t)$, $x(t)$ and $y(t)$ are positively bounded. Let $\Gamma = p - \alpha_1 x - \alpha_2 y$, then $\Gamma' = -D\Gamma$, and thus the explicit solution,

$$p(t) - \alpha_1 x(t) - \alpha_2 y(t) = (p(0) - \alpha_1 x(0) - \alpha_2 y(0))e^{-Dt}. \quad (5)$$

One can write,

$$p(t) - \alpha_1 x(t) - \alpha_2 y(t) \leq p(0) - \alpha_1 x(0) - \alpha_2 y(0) \quad \text{for all } t \geq 0.$$

So $p(t)$ is positively bounded. Therefore, the solutions of (1) are positively bounded and are defined for all $t \geq 0$. From (4) and (5), it can be deduced that the set Ω is positively invariant and is a global attractor for (1). \square

3. Existence and local stability of equilibria. Hereafter we use the following conditions and notations:

$$f_i(S) = \mu_i(S, 0) \quad \text{and} \quad g_i(p) = \mu_i(+\infty, p) \quad i = 1, 2.$$

The functions $f_i(\cdot)$, $i = 1, 2$ are strictly increasing and positive for all $S > 0$. When equations $f_1(S) = D$ and $f_2(S) = D$ have solutions, they are unique and then we define the break-even concentrations as:

$$\lambda_1 = f_1^{-1}(D) \quad \text{and} \quad \lambda_2 = f_2^{-1}(D). \quad (6)$$

Otherwise, we put $\lambda_1 = +\infty$ and $\lambda_2 = +\infty$. The functions $g_i(\cdot)$, $i = 1, 2$ are strictly decreasing and positive for all $p \geq 0$, and we have

$$g_i(0) = \mu_i(+\infty, 0) = f_i(+\infty) \quad i = 1, 2.$$

When equations $g_1(p) = D$ and $g_2(p) = D$ have solutions, they are unique and then we set

$$p_1^* = g_1^{-1}(D), \quad \text{and} \quad p_2^* = g_2^{-1}(D). \quad (7)$$

Otherwise, we put $p_1^* = +\infty$, $p_2^* = +\infty$.

3.1. Existence of equilibria. By the implicit function theorem, we show the following lemma

Lemma 1. *Assume that $\lambda_i < D$ holds for $i = 1, 2$. The equation $\mu_i(S, p) - D = 0$ defines a smooth increasing function*

$$F_i : [\lambda_i, +\infty) \longrightarrow [0, g_i^{-1}(D)[\\ S \longrightarrow F_i(S) = p \quad (8)$$

such that $F_i(\lambda_i) = 0$, and $\lim_{S \rightarrow +\infty} F_i(S) = g_i^{-1}(D)$.

Proof. Since the function μ_i is of class \mathcal{C}^1 on \mathbb{R}_+^2 , then according to the implicit function theorem, the equation $\mu_i(S, p) - D = 0$, defines a smooth increasing function $p = F_i(S)$ of class \mathcal{C}^1 on $[\lambda_i, +\infty)$ with

$$F_i'(S) = \frac{-\frac{\partial \mu_i}{\partial S}(S, F_i(S))}{\frac{\partial \mu_i}{\partial p}(S, F_i(S))} \quad (9)$$

Indeed, using (H2), we have $\frac{\partial \mu_1}{\partial S}(S, p) > 0$ and $\frac{\partial \mu_1}{\partial p}(S, p) < 0$, Thus $F_1'(S) > 0$, which means that the function F_1 is increasing. Moreover, if $S = \lambda_1$, then

$$\begin{aligned} F_1(\lambda_1) = p &\iff \mu_1(\lambda_1, p) - D = 0 &\iff \mu_1(\lambda_1, p) = f_1(\lambda_1) \\ &\iff \mu_1(\lambda_1, p) = \mu_1(\lambda_1, 0) &\iff p = 0. \end{aligned}$$

If $S = +\infty$, then we have

$$F_1(+\infty) = p \iff \mu_1(+\infty, p) - D = 0 \iff g_1(p) = D \iff p = g_1^{-1}(D).$$

□

The equilibria of (1) are the solutions of the nonlinear algebraic system

$$\begin{cases} (S^0 - S)D = \mu_1(S, p)\frac{x}{\gamma_1} + \mu_2(S, p)\frac{y}{\gamma_2} \\ x[\mu_1(S, p) - D] = 0 \\ y[\mu_2(S, p) - D] = 0 \\ \alpha_1\mu_1(S, p)x + \alpha_2\mu_2(S, p)y = Dp \end{cases} \quad (10)$$

• If $x = y = 0$, then from the first equation of (10), we have $S = S^0$, and the fourth equation we have $p = 0$. It's the washout equilibrium

$$E_0(S^0, 0, 0, 0).$$

This equilibrium always exists.

• If $y = 0$ and $x > 0$, then from the second equation of (10), we have $\mu_1(S, p) = D$, and from the first equation, we get

$$x = \gamma_1(S^0 - S), \quad (11)$$

while the fourth equation gives us $p = \alpha_1 x$. Therefore,

$$p = L_1(S) := \alpha_1 \gamma_1 (S^0 - S). \quad (12)$$

The function $L_1(S)$ is strictly decreasing, where $L_1(0) = \alpha_1 \gamma_1 S^0$ and $L_1(S^0) = 0$. Now, using Lemma 1, for all $S \geq 0$ and $p > 0$, the equation $\mu_1(S, p) - D = 0$ defines a smooth increasing function

$$p = F_1(S) \quad (13)$$

such that $F_1(\lambda_1) = 0$ and $\lim_{S \rightarrow \infty} F_1(S) = g_1^{-1}(D)$. The equilibria are the points of intersection of the graphs of functions

$$p = L_1(S) \quad \text{and} \quad p = F_1(S).$$

Since the function L_1 is strictly decreasing and the function F_1 is strictly increasing, then there exists a unique positive solution S_1 if $\lambda_1 < S^0$ and no solution if $\lambda_1 > S^0$ (see Fig.1). Replacing S by S_1 in (11) and (12) we obtain $x_1 = \gamma_1(S^0 - S_1)$ and $p_1 = \alpha_1 \gamma_1(S^0 - S_1)$. Which is the extinction equilibrium of y ,

$$E_1(S_1, x_1, 0, p_1).$$

• If $y \neq 0$ and $x = 0$, then from the third equation of (10), we have $\mu_2(S, p) = D$, and from the first equation, we get

$$y = \gamma_2(S^0 - S), \quad (14)$$

while the fourth equation gives us $p = \alpha_2 y$. Therefore,

$$p = L_2(S) := \alpha_2 \gamma_2 (S^0 - S). \quad (15)$$

The function $L_2(S)$ is strictly decreasing, where $L_2(0) = \alpha_2\gamma_2S^0$ and $L_2(S^0) = 0$. Now, using Lemma 1, for all $S \geq 0$ and $p > 0$, the equation $\mu_2(S, p) - D = 0$ defines a smooth increasing function

$$p = F_2(S) \quad (16)$$

such that $F_2(\lambda_2) = 0$ and $\lim_{S \rightarrow \infty} F_2(S) = g_2^{-1}(D)$. The equilibria are the points of intersection of the graphs of functions

$$p = L_2(S) \quad \text{and} \quad p = F_2(S).$$

Since the function L_2 is strictly decreasing and the function F_2 is strictly increasing, then there exists a unique positive solution S_2 if $\lambda_2 < S^0$ and no solution if $\lambda_2 > S^0$ (see Fig.1). Replacing S by S_2 in (14) and (15) we obtain $x_2 = \gamma_2(S^0 - S_2)$ and $p_2 = \alpha_2\gamma_2(S^0 - S_2)$. Which is the extinction equilibrium of x ,

$$E_2(S_2, 0, y_2, p_2).$$

We can state now the following result

Proposition 2. *Assume that (H1), (H2) and (H3) are satisfied. System (1) has three boundary equilibrium:*

- *The washout equilibrium $E_0(S^0, 0, 0, 0)$, that always exists.*
- *The equilibrium $E_1(S_1, x_1, 0, p_1)$ of extinction of species y , It exists if and only if $\lambda_1 < S^0$, where S_1 is the solution of the equation*

$$F_1(S) = L_1(S), \quad x_1 = \gamma_1(S^0 - S_1), \quad \text{and} \quad p_1 = \alpha_1\gamma_1(S^0 - S_1).$$

- *The equilibrium $E_2(S_2, 0, y_2, p_2)$ of extinction of species x , It exists if and only if $\lambda_2 < S^0$, where S_2 is the solution of the equation*

$$F_2(S) = L_2(S), \quad y_2 = \gamma_2(S^0 - S_2), \quad \text{and} \quad p_2 = \alpha_2\gamma_2(S^0 - S_2).$$

3.2. Existence of the positive equilibrium. A positive equilibrium E_c is determined by the solutions of equations

$$\begin{cases} (S^0 - S)D = \mu_1(S, p) \frac{x}{\gamma_1} + \mu_2(S, p) \frac{y}{\gamma_2} \\ \mu_1(S, p) - D = 0 \\ \mu_2(S, p) - D = 0 \\ \alpha_1\mu_1(S, p)x + \alpha_2\mu_2(S, p)y = Dp \end{cases} \quad (17)$$

Let us denote by h_i , $i = 1, 2$, the following functions of two variables:

$$h_1(S, p) := \mu_1(S, p) - D \quad \text{and} \quad h_2(S, p) := \mu_2(S, p) - D.$$

A positive equilibrium E_c is given by the intersection of the graphs of functions h_i , $i = 1, 2$. Our aim is to show the following result:

Proposition 3. *If $[F_1(S_1) - F_2(S_1)][F_1(S_2) - F_2(S_2)] < 0$, then there is at least one positive equilibrium $E_c(S_c, x_c, y_c, p_c)$ where S_c is the solution of the equation*

$$F_1(S) = F_2(S), \quad p_c = F_1(S_c), \quad x_c = \frac{\gamma_1[p_c - L_2(S_c)]}{\gamma_1\alpha_1 - \gamma_2\alpha_2} \quad \text{and} \quad y_c = \frac{\gamma_2[L_1(S_c) - p_c]}{\gamma_1\alpha_1 - \gamma_2\alpha_2}.$$

Proof. Without loss of generality, we assume in the first case $S_1 < S_2$. A positive equilibrium is given by the solutions of equations

$$h_1(S, p) = 0 \quad \text{and} \quad h_2(S, p) = 0.$$

according to Lemma 1, these equations define two functions $p = F_1(S)$ and $p = F_2(S)$ in $[S_1, S_2]$.

$$\begin{aligned} F_1 : [S_1, S_2] &\longrightarrow [p_1, \bar{p}_1] \\ S &\longrightarrow F_1(S) = p \end{aligned} \quad (18)$$

such that $F_1(S_1) = p_1$ and $F_1(S_2) = \bar{p}_1$.

$$\begin{aligned} F_2 : [S_1, S_2] &\longrightarrow [\bar{p}_2, p_2] \\ S &\longrightarrow F_2(S) = p \end{aligned} \quad (19)$$

such that $F_2(S_2) = p_2$ and $F_2(S_1) = \bar{p}_2$.

Thus, the positive equilibrium is given by the intersection of two isoclines corresponding to these functions (see Fig.1). For that we consider the function

$$F(S) = F_1(S) - F_2(S).$$

So, we have

$$F(S_1) = F_1(S_1) - F_2(S_1) \quad \text{and} \quad F(S_2) = F_1(S_2) - F_2(S_2).$$

According to the intermediate value theorem, there is at least one solution $S_c \in]S_1, S_2[$ such that $F_1(S_c) = F_2(S_c) = p_c$ if and only if

$$[F_1(S_1) - F_2(S_1)][F_1(S_2) - F_2(S_2)] < 0. \quad (20)$$

Thereafter, x_c and y_c are the solutions of the system

$$\begin{cases} x/\gamma_1 + y/\gamma_2 = S^0 - S_c \\ \alpha_1 x + \alpha_2 y = p_c \end{cases}$$

This system has a unique solution (x_c, y_c) if and only if $\gamma_1\alpha_1 \neq \gamma_2\alpha_2$, where

$$x_c = \frac{\gamma_1[p_c - \gamma_2\alpha_2(S^0 - S_c)]}{\gamma_1\alpha_1 - \gamma_2\alpha_2} = \frac{\gamma_1[p_c - L_2(S_c)]}{\gamma_1\alpha_1 - \gamma_2\alpha_2}, \quad (21)$$

$$y_c = \frac{\gamma_2[\gamma_1\alpha_1(S^0 - S_c) - p_c]}{\gamma_1\alpha_1 - \gamma_2\alpha_2} = \frac{\gamma_2[L_1(S_c) - p_c]}{\gamma_1\alpha_1 - \gamma_2\alpha_2}. \quad (22)$$

Now, using the fact that the functions L_i , $i = 1, 2$ defined by (12), (15), are decreasing, and the functions F_i , $i = 1, 2$ defined by (13), (16), are increasing, and as $S_c \in]S_1, S_2[$ we get

$$S_1 < S_c < S_2 \iff \begin{cases} L_1(S_c) < L_1(S_1) \text{ and } F_1(S_1) < F_1(S_c) \\ L_2(S_c) > L_2(S_2) \text{ and } F_2(S_2) > F_2(S_c) \end{cases} \iff \begin{cases} L_1(S_c) < p_c \\ L_2(S_c) > p_c \end{cases}$$

and as $S_1 < S_c < S_2$, one can easily verify that $\gamma_1\alpha_1 < \gamma_2\alpha_2$.

$$\begin{aligned} S_1 < S_c < S_2 &\implies L_1(S_1) = F_1(S_1) < F_1(S_c) = F_2(S_c) < F_2(S_2) = L_2(S_2) \\ &\implies \gamma_1\alpha_1(S^0 - S_1) < \gamma_2\alpha_2(S^0 - S_2) \implies \gamma_1\alpha_1 < \gamma_2\alpha_2 \end{aligned}$$

So, we deduce that

$$x_c = \frac{\gamma_1[p_c - L_2(S_c)]}{\gamma_1\alpha_1 - \gamma_2\alpha_2} > 0 \quad \text{and} \quad y_c = \frac{\gamma_2[L_1(S_c) - p_c]}{\gamma_1\alpha_1 - \gamma_2\alpha_2} > 0.$$

This ends the proof of the proposition. \square

Remark 1. The condition $[F_1(S_1) - F_2(S_1)][F_1(S_2) - F_2(S_2)] < 0$ of existence of the positive equilibrium E_c is not necessary, and if it exists the positive equilibrium is not always unique.

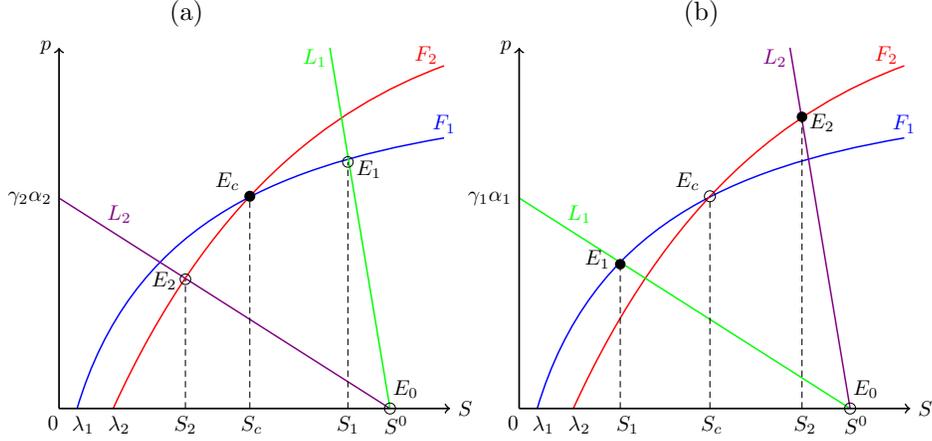


FIGURE 1. Projection of equilibria E_0, E_1, E_2 and E_c in the plane (S, p) and existence and stability conditions of these points. Where stable equilibrium points are denoted by filled circles and unstable equilibrium points are denoted by empty circles.

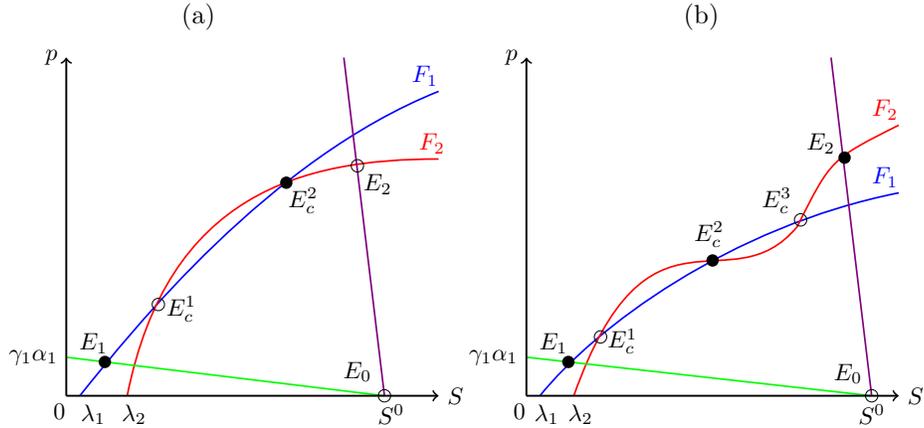


FIGURE 2. The existence of multiple positive equilibria when the condition (20) (a): is satisfied; (b): is not satisfied.

The fact that this sufficient condition of existence of the positive equilibrium is not necessary is illustrated by Fig. 2(a). The fact that the positive equilibrium is not unique, if it exists, is illustrated in Fig. 2(b).

Using similar arguments to those from Proposition 3, we show the following result:

Proposition 4. *A positive equilibrium $E_c(S_c, x_c, y_c, p_c)$ of (1) exists if and only if the graphs of functions F_1 and F_2 have a positive intersection between the lines L_1 and L_2 , that is, (S_c, p_c) is a positive solution of equations*

$$p = F_1(S) \quad \text{and} \quad p = F_2(S) \quad \text{with} \quad \min(S_1, S_2) < S_c < \max(S_1, S_2). \quad (23)$$

x_c, y_c are given by (21), (22), respectively.

3.3. Local asymptotic stability of equilibria. For the study of the stability of each equilibrium point of dynamics (1), it is convenient to use the change of variable

$$\Sigma = S^0 - S - x/\gamma_1 - y/\gamma_2$$

in system (1) that reveals the cascade structure of the system. Since $\Sigma = -D\Sigma$, the system (1) may then be replaced by

$$\begin{cases} \Sigma' = -D\Sigma \\ x' = [\mu_1(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, p) - D]x \\ y' = [\mu_2(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, p) - D]y \\ p' = \alpha_1\mu_1(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, p)x \\ \quad + \alpha_2\mu_2(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, p)y - Dp \end{cases} \quad (24)$$

Now we make the change of variable $\Gamma = p - \alpha_1x - \alpha_2y$ in (24). Since $\Gamma' = -D\Gamma$, the system (24) may then be replaced by

$$\begin{cases} \Gamma' = -D\Gamma \\ \Sigma' = -D\Sigma \\ x' = [\mu_1(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, \Gamma + \alpha_1x + \alpha_2y) - D]x \\ y' = [\mu_2(S^0 - \Sigma - x/\gamma_1 - y/\gamma_2, \Gamma + \alpha_1x + \alpha_2y) - D]y \end{cases} \quad (25)$$

The Jacobian matrix for the linearization of (25) at an equilibrium point $E^* = (0, 0, x^*, y^*)$ takes the triangular form

$$J = \begin{bmatrix} -D & 0 & 0 \\ 0 & -D & 0 \\ A & B & M \end{bmatrix}$$

where M is the square matrix

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad (26)$$

with

$$\begin{aligned} m_{11} &= \left(\frac{-1}{\gamma_1} \frac{\partial \mu_1}{\partial S} + \alpha_1 \frac{\partial \mu_1}{\partial p} \right) x^* + \mu_1(S^0 - x^*/\gamma_1 - y^*/\gamma_2, \alpha_1 x^* + \alpha_2 y^*) - D, \\ m_{12} &= \left(\frac{-1}{\gamma_2} \frac{\partial \mu_1}{\partial S} + \alpha_2 \frac{\partial \mu_1}{\partial p} \right) x^*, \quad m_{21} = \left(\frac{-1}{\gamma_1} \frac{\partial \mu_2}{\partial S} + \alpha_1 \frac{\partial \mu_2}{\partial p} \right) y^*, \\ m_{22} &= \left(\frac{-1}{\gamma_2} \frac{\partial \mu_2}{\partial S} + \alpha_2 \frac{\partial \mu_2}{\partial p} \right) y^* + \mu_2(S^0 - x^*/\gamma_1 - y^*/\gamma_2, \alpha_1 x^* + \alpha_2 y^*) - D. \end{aligned}$$

Therefore, J has a double eigenvalue $-D$, together with the eigenvalues of matrix M . Hence the equilibrium point E^* is locally exponentially stable (LES) if and only if the eigenvalues of M are of negative real parts. The local stability of equilibria of system (25) is given by the following result.

Proposition 5. *Assume that (H1), (H2) and (H3) are satisfied. The stability of equilibria of (1) is as follows:*

- The equilibrium E_0 is LES if and only if $\lambda_1 > S^0$ and $\lambda_2 > S^0$.
- The equilibrium E_1 , if it exists, has at least three dimensional stable manifold and is LES if and only if $F_1(S_1) > F_2(S_1)$.
- The equilibrium E_2 , if it exists, has at least three dimensional stable manifold and is LES if and only if $F_2(S_2) > F_1(S_2)$.
- The equilibrium E_c , if it exists, is LES if and only if

$$(\alpha_1\gamma_1 - \alpha_2\gamma_2)[F_2'(S_c) - F_1'(S_c)] > 0. \quad (27)$$

The following table summarizes all the results on the existence and the local stability of the equilibria of (1),

Equilibria	Existence	Local exponential stability
E_0	Always	$\lambda_1 > S^0$ and $\lambda_2 > S^0$
E_1	$\lambda_1 < S^0$	$F_1(S_1) > F_2(S_1)$
E_2	$\lambda_2 < S^0$	$F_2(S_2) > F_1(S_2)$
E_c	(23) has a solution	$(\alpha_1\gamma_1 - \alpha_2\gamma_2)[F_2'(S_c) - F_1'(S_c)] > 0$

TABLE 1. Existence and local asymptotic stability of equilibria of system (1).

In the following, without loss of generality, we suppose that $S_1 < S_2$, we study the local stability of the points of equilibrium of the system (1), according to the number n of solutions of the equation $F_1(S) = F_2(S)$ for $S_1 < S < S_2$. We restrict our attention to the generic situation, where all intersections are transversal. Note by E_c^i the positive equilibria for $i = 1, 2, \dots, n$.

Proposition 6. *Assume that $S_1 < S_2$, we distinguish the two following cases:*

- 1) $F_1(S_1) > F_2(S_1)$, in this case, E_1 is stable and E_0 is unstable. Moreover,
 - If $n = 2p$, $p \in \mathbb{N}^*$, then E_2 is unstable
 - If $n = 2p + 1$, $p \in \mathbb{N}$, then E_c^{2p+1} is unstable and E_2 is stable where E_c^{2i-1} is unstable and E_c^{2i} is stable for $1 \leq i \leq p$.
- 2) $F_1(S_1) < F_2(S_1)$, in this case, E_1 and E_0 are unstable. Moreover,
 - If $n = 2p$, $p \in \mathbb{N}^*$, then E_2 is stable
 - If $n = 2p + 1$, $p \in \mathbb{N}$, then E_c^{2p+1} is stable and E_2 is unstable where E_c^{2i-1} is stable and E_c^{2i} is unstable for $1 \leq i \leq p$.

Proof. Assume that in the first case that $F_1(S_1) > F_2(S_1)$, according to Lemma 5, we have E_1 is stable and E_0 is unstable. We now assume that the equation $F_1(S) = F_2(S)$ admits $n = 2p + 1$ solutions S_c^j , $j = 1, 2, \dots, n$. As the intersections are transversal (see Fig. 3), then for each $1 \leq i \leq p$ we have:

$$F_1'(S_c^{2i+1}) < F_2'(S_c^{2i+1}), \quad F_1'(S_c^{2i}) > F_2'(S_c^{2i}) \quad \text{and} \quad F_1'(S_c^1) < F_2'(S_c^1).$$

By Lemma 5, we deduce that the equilibria E_c^{2i+1} are unstable and the equilibria E_c^{2i} are stable.

If $n = 2p$, then $F_1(S_2) > F_2(S_2)$, that is, the equilibrium E_2 is unstable. If $n = 2p + 1$, then $F_1(S_2) < F_2(S_2)$ and the equilibrium E_2 is stable. Similarly, the second case could be checked easily. \square

3.4. Global asymptotic stability of the positive equilibrium. The aim of this section is to study the global asymptotic stability of the positive equilibrium of (1) when it exists and is unique.

Let $\Sigma = S + x/\gamma_1 + y/\gamma_2$ and $\Gamma = p - \alpha_1x - \alpha_2y$, by Proposition 1, for non-negative initial conditions, all solutions of system (1) are bounded. Moreover, the set Ω attracts all the trajectories. Using (5) and (4), system (1) is equivalent to non-autonomous system of two differential equations

$$\begin{cases} x' &= [\mu_1(\Sigma(t) - x/\gamma_1 - y/\gamma_2, \Gamma(t) + \alpha_1x + \alpha_2y) - D]x \\ y' &= [\mu_2(\Sigma(t) - x/\gamma_1 - y/\gamma_2, \Gamma(t) + \alpha_1x + \alpha_2y) - D]y \end{cases}$$

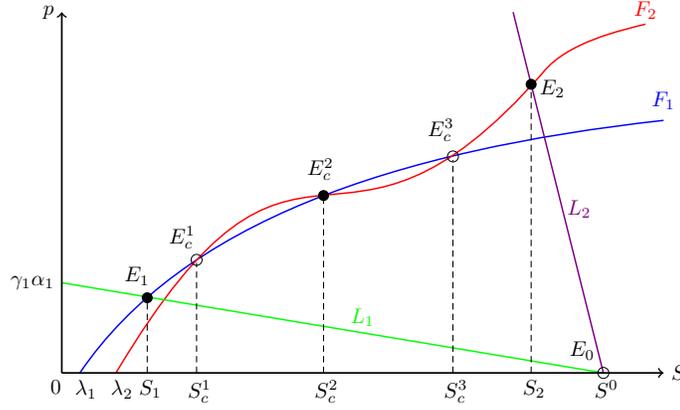


FIGURE 3. Case $n = 3$: Local stability of E_1 , E_2 , E_c^2 and instability of E_c^1 , E_c^2 , E_0 .

This is an asymptotically autonomous differential system which converges to the autonomous system

$$\begin{cases} x' &= [\mu_1(S^0 - x/\gamma_1 - y/\gamma_2, \alpha_1 x + \alpha_2 y) - D]x \\ y' &= [\mu_2(S^0 - x/\gamma_1 - y/\gamma_2, \alpha_1 x + \alpha_2 y) - D]y \end{cases} \quad (28)$$

The system (28) is defined on the set $\Delta = \{(x, y) \in \mathbb{R}_+^2 : x/\gamma_1 + y/\gamma_2 \leq S^0\}$. The set Δ is bounded and positively invariant, that is, system (28) is dissipative. Notice that the equilibria of (28)

$$\mathbb{E}_0 = (0, 0), \quad \mathbb{E}_1 = (x_1, 0), \quad \mathbb{E}_2 = (0, y_2), \quad \mathbb{E}_c = (x_c, y_c)$$

correspond to the equilibria of (1), respectively,

$$E_0(S^0, 0, 0, 0), \quad E_1(S_1, x_1, 0, p_1), \quad E_2(S_2, 0, y_2, p_2), \quad E_c(S_c, x_c, y_c, p_c)$$

where $x_1 = \gamma_1(S^0 - S_1)$, $y_2 = \gamma_2(S^0 - S_2)$ and x_c , y_c are given by (21), (22), respectively.

By Proposition 5 the study of phase portrait of (28) on Δ , shows that there are only stable nodes, unstable nodes and saddle points. For any initial condition on the x axis [resp. y axis], the solution cannot converge to \mathbb{E}_2 [resp. \mathbb{E}_1] since \mathbb{E}_0 is an unstable node. Hence, there is no trajectory joining two saddle points. Thus, according to Thieme [27], we thereof deduce the asymptotic behavior of the solution of (1) from the asymptotic behavior of the autonomous system (28) which is called the reduced model.

The following lemma shows that the system (28) cannot have periodic orbits inside Δ .

Lemma 2. *The system (28) has no periodic orbit inside Δ .*

Proof. By the change of variable $\xi_1 = \ln(x)$ and $\xi_2 = \ln(y)$, whose derivatives with respect to time $\xi_1' = x'/x$ and $\xi_2' = y'/y$, respectively. We then obtain the following system:

$$\begin{cases} \xi_1' = k_1(\xi_1, \xi_2) & := \mu_1(S^0 - e^{\xi_1}/\gamma_1 - e^{\xi_2}/\gamma_2, \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2}) - D \\ \xi_2' = k_2(\xi_1, \xi_2) & := \mu_2(S^0 - e^{\xi_1}/\gamma_1 - e^{\xi_2}/\gamma_2, \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2}) - D \end{cases} \quad (29)$$

We have

$$\frac{\partial k_1}{\partial \xi_1} + \frac{\partial k_2}{\partial \xi_2} = -\frac{e^{\xi_1}}{\gamma_1} \frac{\partial \mu_1}{\partial S} + \alpha_1 e^{\xi_1} \frac{\partial \mu_1}{\partial p} - \frac{e^{\xi_2}}{\gamma_2} \frac{\partial \mu_2}{\partial S} + \alpha_2 e^{\xi_2} \frac{\partial \mu_2}{\partial p} < 0,$$

according to the Dulac criterion, we deduce that the system (29) has no periodic trajectory. Therefore (28) has no periodic orbit in Δ . \square

Theorem 3.1. *Assume that the equation $F_1(S) = F_2(S)$ has a unique solution S_c , such that $S_1 < S_c < S_2$. If $F_1(S_1) < F_2(S_1)$, then the positive equilibrium E_c is globally asymptotically stable with respect to solutions with positive initial conditions.*

Proof. The conditions of existence and stability of $\mathbb{E}_0, \mathbb{E}_1, \mathbb{E}_2$ and \mathbb{E}_c are those of E_0, E_1, E_2 and E_c , respectively, given in Table 1. So, if $F_1(S_1) < F_2(S_1)$, then from Proposition 6, the equilibria $\mathbb{E}_0, \mathbb{E}_1$ and \mathbb{E}_2 are unstable, the equilibrium \mathbb{E}_c is stable. To prove the theorem it remains only to show that it is a global attractor.

We have $x/\gamma_1 + y/\gamma_2 \leq \max(S^0, x(0)/\gamma_1 + y(0)/\gamma_2)$ for all $(x, y) \in \mathbb{R}_+^2$ then for any non-negative initial condition, the solutions of the system (29) are positively bounded and consequently the ω -limit sets are compact non-empty. Using Poincaré-Bendixson Theorem, these limit sets are either equilibrium points or periodic orbits or polycycles. According to Lemma 2, we can not have in the set ω -limit a periodic orbit.

We show now that the ω -limit set (It will be denoted by ω) cannot contain any point of the axes $x = 0$ or $y = 0$. The stable manifold of \mathbb{E}_0 is the origin $(0, 0)$. The stable manifolds of \mathbb{E}_1 and \mathbb{E}_2 are the x -axis and y -axis, respectively. Since the initial condition of the solution $(x(t), y(t))$ does not belong to any of these stable manifolds, then ω cannot be any of the three equilibria. Moreover, ω cannot contain any of these equilibria by the Butler-McGehee theorem [10, Lemma A1]. It cannot contain \mathbb{E}_0 since it is a repeller. Suppose then that $\mathbb{E}_1 \in \omega$ [resp. $\mathbb{E}_2 \in \omega$]. As it has been shown that $\omega \neq \{\mathbb{E}_1\}$ [resp. $\omega \neq \{\mathbb{E}_2\}$], then it must also contain an entire orbit different from \mathbb{E}_1 [resp. \mathbb{E}_2] belonging to the x -axis [resp. y -axis]. There are only two possible orbits; one is unbounded, thus it cannot be included in the compact set ω , and the other has α -limit set \mathbb{E}_0 , thus it cannot contain \mathbb{E}_0 since it is a reseller. Therefore, \mathbb{E}_1 [resp. \mathbb{E}_2] cannot be included in ω . Now, if ω contains a point of the x -axis or y -axis then, by the invariance of ω , it must contain one of the equilibria $\mathbb{E}_0, \mathbb{E}_1, \mathbb{E}_2$ or an unbounded orbit, thus it cannot be included in the compact set ω . Since none of these alternatives are possible, then ω cannot contain any point of the axes $x = 0$ or $y = 0$. If it does not contain the point \mathbb{E}_c then, according to the Poincaré-Bendixson theorem, solutions converge toward a cycle, and according to Lemma 2, this system cannot have a periodic orbit inside Δ . Therefore, it contains the point \mathbb{E}_c . It is equal to it because \mathbb{E}_c is LES. This shows that \mathbb{E}_c is GAS. Finally, applying the result of Thieme [27] and concluding that E_c is a globally asymptotically stable equilibrium point for the system (1). \square

4. Operating diagrams. The operating diagram has the operating parameters D and S^0 as its coordinates and the various regions defined in it correspond to qualitatively different dynamics. This bifurcation diagram which determines the effect of the operating parameters, that are controlled by the operator and which are the dilution rate and the the inflowing substrate, is very useful to understand the model from both the mathematical and biological points of view. We assume

that the functions μ_i , $i = 1, 2$ are fixed. The following change of dependent variables

$$\hat{x} = \frac{x}{\gamma_1}, \quad \hat{y} = \frac{y}{\gamma_2},$$

which reduces (1) to

$$\begin{cases} S' &= (S^0 - S)D - \mu_1(S, p)\hat{x} - \mu_2(S, p)\hat{y} \\ \hat{x}' &= [\mu_1(S, p) - D]\hat{x} \\ \hat{y}' &= [\mu_2(S, p) - D]\hat{y} \\ p' &= \hat{\alpha}_1\mu_1(S, p)\hat{x} + \hat{\alpha}_2\mu_2(S, p)\hat{y} - Dp \end{cases} \quad (30)$$

where $\hat{\alpha}_1 = \alpha_1\gamma_1$ and $\hat{\alpha}_2 = \alpha_2\gamma_2$. Therefore, without loss of generality we can assume that the yields in (1) are equal to 1 ($\gamma_1 = \gamma_2 = 1$). So, we consider the system

$$\begin{cases} S' &= (S^0 - S)D - \mu_1(S, p)x - \mu_2(S, p)y \\ x' &= [\mu_1(S, p) - D]x \\ y' &= [\mu_2(S, p) - D]y \\ p' &= \alpha_1\mu_1(S, p)x + \alpha_2\mu_2(S, p)y - Dp \end{cases} \quad (31)$$

Our aim now is to describe operating diagram. The operating diagram shows how the system behaves when we vary the two control parameters D and S^0 in the system (31).

Let Γ_i be the curve of equation $S^0 = \lambda_i(D)$, $i = 1, 2$,

$$\Gamma_i = \{(D, S^0) : S^0 = \lambda_i(D)\}. \quad (32)$$

According to Table 1, the curve Γ_1 is the border to which E_1 exists (the blue curve in Figs. 5 and 6), the curve Γ_2 is the border to which E_2 exists (the black curve in Figs. 5 and 6).

Recall that S_i , $i = 1, 2$ are defined as the solutions of $F_i(S, D) = L_i(S, S^0)$, $i = 1, 2$, respectively. Therefore, S_i , $i = 1, 2$ depend on the operating parameters D and S^0 . We note them by $S_i(D, S^0)$, $i = 1, 2$. We define the sets

$$\begin{aligned} \Gamma_1^c &= \{(D, S^0) : F_1(S_1(D, S^0), D) = F_2(S_1(D, S^0), D)\}, \\ \Gamma_2^c &= \{(D, S^0) : F_1(S_2(D, S^0), D) = F_2(S_2(D, S^0), D)\}, \end{aligned} \quad (33)$$

which are curves in the generic case. If the curves Γ_i , $i = 1, 2$ intersect in a point (D_*, S_*^0) of the plane, then one has

$$\mu_1(S_*^0, 0) = \mu_2(S_*^0, 0) = D_*, \quad (34)$$

using Lemma 1, equation (34) can be written as follows

$$F_1(S_*^0, D_*) = F_2(S_*^0, D_*) = 0 \quad (35)$$

From (35), we deduce that

$$\begin{aligned} F_1(S_1(D_*, S_*^0), D_*) &= F_2(S_1(D_*, S_*^0), D_*) \\ F_1(S_2(D_*, S_*^0), D_*) &= F_2(S_2(D_*, S_*^0), D_*). \end{aligned}$$

Therefore, the point (D_*, S_*^0) belongs to the sets Γ_i^c , $i = 1, 2$. Hence, if Γ_i , $i = 1, 2$ intersect in a point of the plan (D, S^0) , then the curves Γ_i^c intersect in the same point

The curves Γ_1^c and Γ_2^c are the boundaries for which E_1 and E_2 unstable and at the same time E_c exists and unstable (the red and green curves in Figs. 5 and 6, respectively).

5. Illustrative examples. In this section, we consider the model (31) with μ_i , $i = 1, 2$ given by (2). Let us show the usefulness of our results on the construction of the operating diagram corresponding to various set of biological parameters encountered in the literature, in particular those considered in [5], see Cases (a) and (b), in Table 2.

We consider also set of biological parameters which are not taken from the existing literature, and are chosen due to their interesting properties, see Case (c), in Table 2.

We restrict our attention to (31) where μ_i , $i = 1, 2$ are given by (2). In this case, the functions $\lambda_1(D)$ and $\lambda_2(D)$ defined by (6) are given explicitly:

$$\lambda_1(D) = \frac{a_1 D}{m_1 - D} \quad \text{and} \quad \lambda_2(D) = \frac{a_2 D}{m_2 - D}. \quad (36)$$

The expressions of functions F_i , $i = 1, 2$ and L_i , $i = 1, 2$ defined by (13), (16), (12) and (15), respectively, can be calculated explicitly:

$$F_i(S, D) = \frac{K_i[(m_i - D)S - a_i D]}{D(a_i + S)} \quad \text{and} \quad L_i(S, S^0) = \alpha_i(S^0 - S), \quad i = 1, 2. \quad (37)$$

On the other hand, the solutions $S_i(D, S^0)$, $i = 1, 2$ of equations $F_i(S_i, D) = L_i(S_i, S^0)$, $i = 1, 2$ which are used in E_i , $i = 1, 2$, respectively, are simply the positive solutions of the quadratic equations:

$$K_i[(m_i - D)S_i - a_i D] = \alpha_i D(S^0 - S_i)(a_i + S_i), \quad i = 1, 2. \quad (38)$$

The solution $S_c(D)$ of the equation $F_1(S_c, D) = F_2(S_c, D)$ which is used in E_c is simply the positive solution of the quadratic equation:

$$K_1 D(a_2 + S_c)[(m_1 - D)S_c - a_1 D] = K_2 D(a_1 + S_c)[(m_2 - D)S_c - a_2 D]. \quad (39)$$

Note that the equation (39) can not have more than two solutions.

If we assume that equation $F_1(S_c, D) = F_2(S_c, D)$ has at most two positive solutions $S_c^1(D)$ and $S_c^2(D)$ as in the preceding case, then Γ_1^c [resp. Γ_2^c] given by (33) is simply the union of the graphs of functions $S_1(D, S^0) = S_c^1(D)$ and $S_1(D, S^0) = S_c^2(D)$ [resp. $S_2(D, S^0) = S_c^1(D)$ and $S_2(D, S^0) = S_c^2(D)$], see Fig.4.

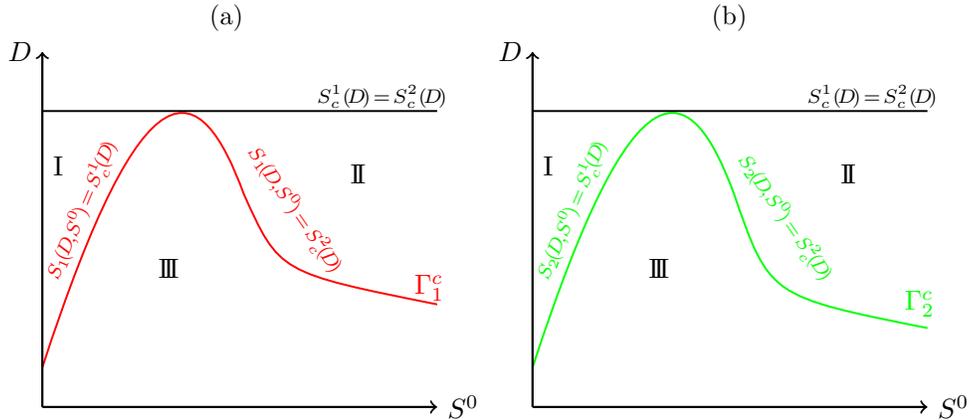


FIGURE 4. Illustrative graph of Γ_i^c , $i = 1, 2$ defined by (33), showing the relative positions of the roots $S_c^1(D)$ and $S_c^2(D)$ of (39) with respect to the root $S_i(D, S^0)$ of (38). Region I: $S_i < S_c^1 < S_c^2$; region II: $S_c^1 < S_c^2 < S_i$; region III: $S_c^1 < S_i < S_c^2$.

Parameters Units	m_1 h^{-1}	m_2 h^{-1}	a_1 gl^{-1}	a_2 gl^{-1}	K_1 gl^{-1}	K_2 gl^{-1}	α_1	α_2	Figures
Case (a)	1.0	2.0	0.01	0.04	0.01	0.006	0.1	4.0	5(a)
Case (b)	1.0	2.0	0.01	0.04	0.01	0.006	4.0	0.1	5(b)
Case (c)	2.0	9.0	0.006	0.04	0.005	0.001	0.005	0.4	6(a), 7
Case (d)	2.0	9.0	0.006	0.04	0.005	0.001	0.4	0.005	6(b)

TABLE 2. Parameter values used in Section 5 where μ_i are given by (2).

5.1. Operating diagram: the curves Γ_1^c and Γ_2^c do not intersect. This case corresponds to the parameter values used by [5] given in Table 2(a,b). In this case, the curves Γ_1^c and Γ_2^c do not intersect. Notice that Γ_1^c and Γ_2^c are simply the graphs of functions $S_1(D, S^0) = S_c^2(D)$ and $S_2(D, S^0) = S_c^2(D)$, respectively. We see from Table 1 that the curve Γ_1 and Γ_2 of the operating diagram, given by (32) are the border to which E_1 and E_2 exist, respectively. Beside these curves, we plot also on the operating diagram of Fig. 5, the curves Γ_1^c and Γ_2^c . According to Proposition 5, these curves separate the region of existence of E_c into two subregions labelled \mathcal{J}_6 and \mathcal{J}_7 , such that E_c is stable in \mathcal{J}_6 (see Fig. 5(b)), and unstable in \mathcal{J}_7 (see Fig. 5(a)). In the region \mathcal{J}_7 , the system exhibits bistability of E_1 and E_2 . Therefore, the curves Γ_i and Γ_i^c , $i = 1, 2$ separate the operating plane (D, S^0) into at most seven regions, as illustrated by Fig. 5, labelled $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6$ and \mathcal{J}_7 . Some of these regions may be empty as shown on Fig. 5. The region \mathcal{J}_6 is empty in case (a) and the region \mathcal{J}_7 is empty in case (b). The operating diagram is illustrated in Fig. 5 (Our diagram is similar to the diagram in [5], Fig. 1). From Table 1, we deduce the following result:

Proposition 7. *The results are summarised in Table 3, which shows the existence and local stability of the equilibria E_0, E_1, E_2 and E_c in the regions of the operating diagram in Fig. 5.*

Region	The relative positions of S_i and S_c^i	E_0	E_1	E_2	E_c^1	E_c^2
$(S^0, D) \in \mathcal{J}_1$	S_1 and S_2 do not exist	S				
$(S^0, D) \in \mathcal{J}_2$	S_1 does not exist	U		S		
$(S^0, D) \in \mathcal{J}_3$	$S_c^1 < S_c^2 < S_i, i = 1, 2$ **	U	U	S		
$(S^0, D) \in \mathcal{J}_4$	$S_c^1 < S_i < S_c^2, i = 1, 2$	U	S	U		
$(S^0, D) \in \mathcal{J}_5$	S_2 does not exist	U	S			
$(S^0, D) \in \mathcal{J}_6$	$S_c^1 < S_2 < S_c^2 < S_1$	U	U	U		S
$(S^0, D) \in \mathcal{J}_7$	$S_c^1 < S_1 < S_c^2 < S_2$	U	S	S		U

** When S_c^1 and S_c^2 do not exist, the condition reduces to $S_i, i = 1, 2$ exist.

TABLE 3. Existence and stability of equilibria in the regions of the operating diagrams of Fig. 5, when the curves Γ_i^c do not intersect. The letter S (resp. U) means stable (resp. unstable) and empty if that equilibrium does not exist.

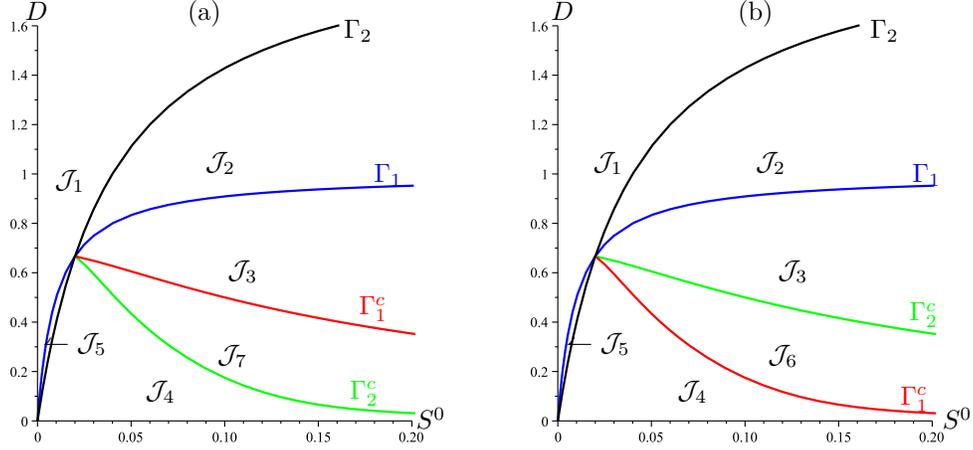


FIGURE 5. The operating diagram of the system (1) where μ_i are given by (2) and the curves Γ_i^c do not intersect. (a): The occurrence of the bistability region \mathcal{J}_7 , where the coexistence region \mathcal{J}_6 does not exist. (b): The occurrence of the coexistence region \mathcal{J}_6 . The biological parameters used to construct Figs. 5(a,b) are exactly the same except that the values of α_i have been inverted.

5.2. Operating diagram: the curves Γ_1^c and Γ_2^c intersect. This case corresponds to the parameter values given in Table 2(c,d). In this case, the curves Γ_1^c and Γ_2^c intersect. When $D < 1.1769$, equation (39) has exactly two solutions denoted by $S_c^1(D)$ and $S_c^2(D)$, corresponding to two equilibria E_c^1 and E_c^2 , respectively. Notice that Γ_1^c [resp. Γ_2^c] is simply the union of the graphs of functions $S_1(D, S^0) = S_c^1(D)$ and $S_1(D, S^0) = S_c^2(D)$ [resp. $S_2(D, S^0) = S_c^1(D)$ and $S_2(D, S^0) = S_c^2(D)$]. We plot also on the operating diagram of Fig. 6, the horizontal line Γ_3 defined by

$$\Gamma_3 = \{(D, S^0) : D \approx 1.1769, S^0 > 0.0207\},$$

where $(S^0 \approx 0.0207, D \approx 1.1769)$ are maximum point of the curve Γ_2^c of Fig. 6(a) [resp. Γ_1^c of Fig. 6(b)].

Therefore, the curves Γ_i , $i = 1, 3$ and Γ_i^c , $i = 1, 2$ separate the operating plane (D, S^0) into at most eleven regions, as illustrated by Fig. 6, labelled \mathcal{J}_i , $i = 1..5$ and \mathcal{J}_i^a , $i = 6..8$ and \mathcal{J}_i^b , $i = 6..8$. The operating diagram is illustrated in Fig. 6. From Table 1, we deduce the following result:

Proposition 8. *The results are summarised in Table 4, which shows the existence and local stability of the equilibria E_0 , E_1 , E_2 , E_c^1 and E_c^2 in the regions of the operating diagram in Fig. 6.*

In order to illustrate and validate the results of this section, we perform the simulations for the same biological parameter values that were considered for the plot of operating diagram in Fig. 6(a).

Fig. 7(a) shows the bi-stability of $\mathbb{E}_1 \approx (0.094, 0)$ and $\mathbb{E}_2 \approx (0, 0.014)$ which are LES while $\mathbb{E}_c^2 \approx (0.023, 0.013)$ and $\mathbb{E}_0 = (0, 0)$ are unstable when

$$(S^0, D) = (0.1, 0.9) \in \mathcal{J}_7^a \quad \text{or whenever} \quad S_c^1 < S_1 < S_c^2 < S_2.$$

Region	The relative positions of S_i and S_c^i	E_0	E_1	E_2	E_c^1	E_c^2
$(S^0, D) \in \mathcal{J}_1$	S_1 and S_2 do not exist	S				
$(S^0, D) \in \mathcal{J}_2$	S_1 does not exist	U		S		
$(S^0, D) \in \mathcal{J}_3$	$S_i < S_c^1 < S_c^2, i = 1, 2$ **	U	U	S		
$(S^0, D) \in \mathcal{J}_4$	$S_c^1 < S_i < S_c^2, i = 1, 2$	U	S	U		
$(S^0, D) \in \mathcal{J}_5$	S_2 does not exist	U	S			
$(S^0, D) \in \mathcal{J}_6^a$	$S_1 < S_c^1 < S_2 < S_c^2$	U	U	U	S	
$(S^0, D) \in \mathcal{J}_7^a$	$S_c^1 < S_1 < S_c^2 < S_2$	U	S	S		U
$(S^0, D) \in \mathcal{J}_8^a$	$S_1 < S_c^1 < S_c^2 < S_2$	U	U	S	S	U
$(S^0, D) \in \mathcal{J}_6^b$	$S_c^1 < S_2 < S_c^2 < S_1$	U	U	U		S
$(S^0, D) \in \mathcal{J}_7^b$	$S_2 < S_c^1 < S_1 < S_c^2$	U	S	S	U	
$(S^0, D) \in \mathcal{J}_8^b$	$S_2 < S_c^1 < S_c^2 < S_1$	U	U	S	U	S

** When S_c^1 and S_c^2 do not exist, the condition reduces to $S_i, i = 1, 2$ exist.

TABLE 4. Existence and stability of equilibria in the regions of the operating diagrams of Fig. 6, when the curves Γ_i^c intersect.

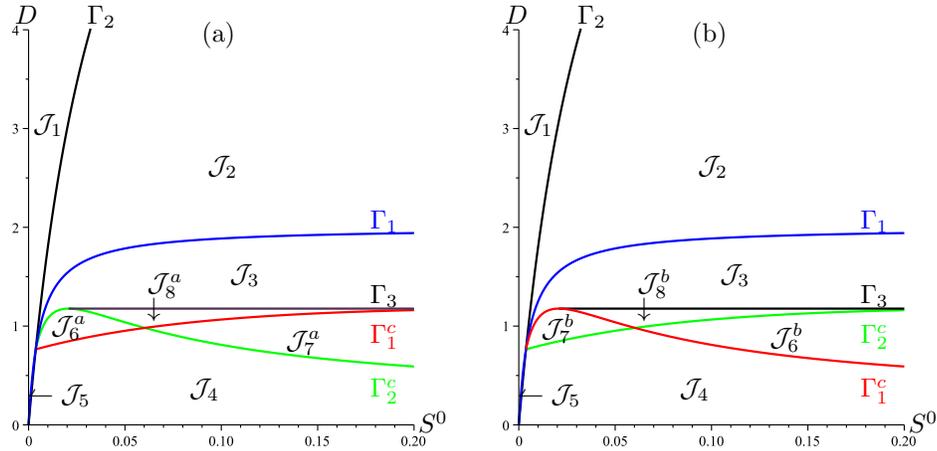


FIGURE 6. The operating diagram of the system (1) where μ_i are given by (2) and the curves Γ_i^c intersect. The pictures show the occurrence of the bistability region \mathcal{J}_8 of E_2 and E_c^i . The biological parameters used to construct Figs. 6(a,b) are exactly the same except that the values of α_i have been inverted.

(S^0, D)	Regions	S_1	S_2	S_c^1	S_c^2	Figures
(0.1, 0.9)	\mathcal{J}_7^a	0.006	0.085	0.005	0.064	7(a)
(0.05, 1.15)	\mathcal{J}_8^a	0.009	0.042	0.012	0.025	7(b)
(0.02, 1.15)	\mathcal{J}_6^a	0.008	0.017	0.012	0.025	7(c)

TABLE 5. Parameter values of S_i and S_c^i used in Fig. 7.

Here \mathbb{E}_1 and \mathbb{E}_2 are both stable and E_c^1 is a saddle point whose separatrix separates the phase plane into the basins of attraction of \mathbb{E}_1 and \mathbb{E}_2 .

Fig. 7(b) shows the bi-stability of $\mathbb{E}_2 \approx (0, 0.008)$ and $\mathbb{E}_c^1 \approx (0.036, 0.002)$ which are LES while $\mathbb{E}_c^2 \approx (0.02, 0.005)$, $\mathbb{E}_1 \approx (0.041, 0)$ and \mathbb{E}_0 are unstable when

$$(S^0, D) = (0.05, 1.15) \in \mathcal{J}_8^a \quad \text{or whenever} \quad S_1 < S_c^1 < S_c^2 < S_2.$$

Here \mathbb{E}_1 and \mathbb{E}_c^2 are both stable and \mathbb{E}_c^1 is a saddle point whose separatrix separates the phase plane into the basins of attraction of \mathbb{E}_1 and \mathbb{E}_c^2 .

In this case, Fig. 7(c) shows the coexistence of the two species and the global convergence to the unique positive equilibrium $\mathbb{E}_c^1 \approx (0.006, 0.002)$ when

$$(S^0, D) = (0.02, 1.15) \in \mathcal{J}_6^a \quad \text{or whenever} \quad S_1 < S_c^1 < S_2 < S_c^2.$$

In each case, the parameter values of S_i , $i = 1, 2$ and S_c^i , $i = 1, 2$ is given in Table 5, where the equilibria in green are stable, the blue ones are unstable.

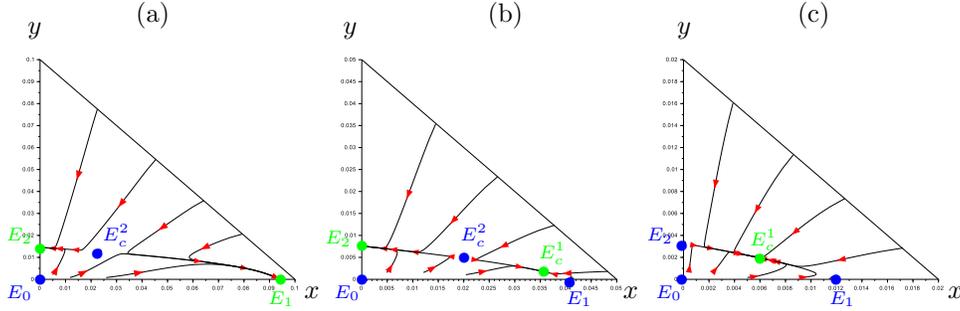


FIGURE 7. The trajectories of the reduced model (28) where (S^0, D) are chosen in regions of Fig. 6(a). (a): Bistability of E_1 and E_2 when $(S^0, D) = (0.1, 0.9) \in \mathcal{J}_7^a$. (b): Bistability of E_c^1 and E_2 when $(S^0, D) = (0.05, 1.15) \in \mathcal{J}_8^a$. (c): Global stability of E_c^1 when $(S^0, D) = (0.02, 1.15) \in \mathcal{J}_6^a$.

6. Conclusion. In this work we have extended the model (1) with (2) of competition in the chemostat with an internal inhibitor proposed by [5] by considering the model (1) with general growth rate functions of competitors, the functions are assumed to be increasing in the substrate S and decreasing in inhibitor p , it also includes the model considered in [16] when only one of the competitors produces an inhibitor which inhibits the other specie (i.e (1) together with (3)).

We give the conditions of existence and local stability of all corresponding steady states of (1). We define the functions L_1 , F_1 , L_2 , and F_2 , and the numbers S_1, S_2 , and illustrate the relative positions of the different equilibria geometrically (see Figs.1, 2, and 3), Props.3 and 4 give sufficient and necessary conditions for the existence of the positive equilibria, specifically our mathematical analysis of the model has revealed the possibility of multiple positive steady states, we show that the system may exhibit bistability, which could occur in the classic chemostat model only when the growth rate is non-monotonic [10]. By reducing the model (1) to planer model (28), and using Dulac's criterion, we show the nonexistence of periodic orbits for the system, and due to Thieme's results [27], the global asymptotic stability of the positive steady state of (1) when it's unique is obtained.

The operating diagrams show the effect of the control parameter S^0 and D on the asymptotic behavior of the system, to have the coexistence of the two microbial

species in the process, the operating parameters values should be chosen in the region \mathcal{J}_6 which is more preferable because we have a globally asymptotically stable equilibrium, whereas \mathcal{J}_8 there is bistability between coexistence and exclusion of the first species. Furthermore, the operating diagrams show that the system can have a unique steady state of washout \mathcal{J}_1 , or exclusion of one of two species ($\mathcal{J}_i, i = 2..5$), or bistability between exclusion of the first species and the second species \mathcal{J}_7 .

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E-mail address: dellal.m48@univ-tiaret.dz

E-mail address: bachir.bar@mail.univ-tlemcen.dz