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► **To cite this version:**

Benoît Perthame, Alexandre Poulain. Relaxation of the Cahn-Hilliard equation with degenerate double-well potential and degenerate mobility. 2019. hal-02274417v1

HAL Id: hal-02274417

<https://hal.science/hal-02274417v1>

Preprint submitted on 29 Aug 2019 (v1), last revised 27 Apr 2022 (v4)

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Relaxation of the Cahn-Hilliard equation with degenerate double-well potential and degenerate mobility.

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August 28, 2019

Abstract

The degenerate Cahn-Hilliard equation is a standard model to describe living tissues. It takes into account cell populations undergoing short range attraction and long range repulsion effects. In this framework, we consider the usual Cahn-Hilliard equation with a degenerate double-well potential and degenerate mobility. These degeneracies induce numerous difficulties, in particular for its numerical simulation. To overcome these issues, we propose a relaxation system formed of two second order equations which can be solved with standard packages. This system is endowed with an energy and an entropy structure compatible with the limiting equation. Here, we study the theoretical properties of this system; global existence and convergence of the relaxed system to the degenerate Cahn-Hilliard equation. We also study the long time asymptotics which interest relies on the numerous possible steady states with given mass.

2010 *Mathematics Subject Classification.* 35B40; 35G20 ; 35Q92; 92C10

Keywords and phrases. Degenerate Cahn-Hilliard equation; Relaxation method; Asymptotic analysis; Living tissues

1 Introduction

The Degenerate Cahn-Hilliard equation (DCH in short) is a standard model, widely used in the mechanics of living tissues, [7, 30, 16, 2, 4, 20]. It is usual to set this problem in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ with the zero flux boundary condition

$$\partial_t n = \nabla \cdot (b(n) \nabla (-\gamma \Delta n + \psi'(n))) \quad \text{in } \Omega \times (0, +\infty), \quad (1)$$

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[§]The authors have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 740623)

$$\frac{\partial n}{\partial \nu} = \frac{\partial (-\gamma \Delta n + \psi'(n))}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \quad (2)$$

where ν is the outward normal vector to the boundary $\partial \Omega$.

Degeneracies of both the coefficient $b(n)$ and the potential $\psi(n)$ make this problem particularly difficult to solve numerically. Motivated by the use of standard software for elliptic or parabolic equations, we propose to study the following relaxed degenerate Cahn-Hilliard equation (RDHC in short)

$$\begin{cases} \partial_t n = \nabla \cdot (b(n) \nabla (\varphi + \psi'_+(n))) & \text{in } \Omega \times (0, +\infty), \\ -\sigma \Delta \varphi + \varphi = -\gamma \Delta n + \psi'_-(n - \frac{\sigma}{\gamma} \varphi), \end{cases} \quad (3)$$

supplemented with zero-flux boundary conditions

$$\frac{\partial (\gamma n - \sigma \varphi)}{\partial \nu} = \frac{\partial (\varphi + \psi'_+(n))}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, +\infty). \quad (4)$$

Our purpose is to study existence for this system, to prove that as $\sigma \rightarrow 0$, the solution of RDCH system converges to the solution of the DCH equation and study the possible long term limits to steady states. An alternative form, with change of unknown function φ is proposed in section 6 which might be better adapted for numerical purposes

We make the following assumptions for the different inputs of the system (3). The usual assumption is that the potential ψ is concave degenerate near $n = 0$ (short range attraction) and convex for n not too small (long range repulsion) with a singularity at $n = 1$ which represents saturation by one phase. For these reasons, we call the potential *degenerate double-well* or *single-well* and we decompose it in a convex and a concave part ψ_{\pm}

$$\psi(n) = \psi_+(n) + \psi_-(n), \quad \pm \psi''_{\pm}(n) \geq 0, \quad \psi'(0) = 0. \quad (5)$$

The singularity is contained in the convex part of the potential and we assume that

$$\psi_+ \in C^2(]-\infty, 1[), \quad \psi'_+(1) = \infty, \quad \psi_- \in C^2(\mathbb{R}), \quad \psi'_- \in L^\infty(\mathbb{R}). \quad (6)$$

Typical examples of potentials are, for some $n^* \in (0, 1)$

$$\psi'(n) = \frac{n^2(n - n^*)}{1 - n} \quad \text{or} \quad \psi(n) = \frac{1}{2}n \ln n + (1 - n) \ln(1 - n) - (n - \frac{1}{2})^2. \quad (7)$$

The first of these two potentials has been described within the context of tumor growth modelling by Ambrosi and Preziosi [5] and Byrne and Preziosi [10]. Promising results have been obtained in the work of Agosti *et al.* [3] in which the Cahn-Hilliard equation featuring the above first potential has been used to model the interaction between cancer cells from a glioblastoma multiforme and healthy cells. The second potential is an adaptation of the potential proposed by Cahn and Hilliard in [12] and the resulting equation has been analyzed by Cherfilis *et al.* [14] for Neumann and dynamic boundary conditions.

We also use the degeneracy assumption on $b \in C^1([0, 1]; \mathbb{R}^+)$,

$$b(0) = b(1) = 0, \quad b(n) > 0 \text{ for } 0 < n < 1. \quad (8)$$

The typical expression in the applications we have in mind is $b(n) = n(1 - n)^2$. Consequently, when considered as transport equations, both (1) and (3) impose formally the property that $0 \leq n \leq 1$. However, we need an additional technical assumption, namely that there is some cancellation at 1 and one can define

$$b(\cdot)\psi''(\cdot) \in C([0, 1]; \mathbb{R}). \quad (9)$$

We implicitly assume (5)–(9) in this paper. Also, we always impose an initial condition satisfying

$$n^0 \in H^1(\Omega), \quad 0 \leq n^0 < 1 \quad \text{a.e. in } \Omega. \quad (10)$$

Thanks to the boundary condition (2), the system conserves the initial mass

$$\int_{\Omega} n(x, t) dx = \int_{\Omega} n^0(x) dx =: M, \quad \forall t \geq 0.$$

We denote the flux associated with the RDCH system by

$$J_{\sigma}(n, \varphi) := b(n)\nabla(\varphi + \psi'_+(n)). \quad (11)$$

The first use of the Cahn-Hilliard equation is to model the spinodal decomposition occurring in binary materials during a sudden cooling [12, 11]. The bilaplacian $-\gamma\Delta^2 n$ is used to represent surface tension and the parameter γ is the square of the width of the diffuse interface between the two phases. In both equations (1) and (3), $n = n(x, t)$ is a relative quantity: for our biological application this represents a relative cell density as derived from phase field models [10] and for this reason the property $n \in [0, 1]$ is relevant. For instance, the two phases can be the relative density of cancer cells and the other component represents the extracellular matrix, liquid and other cells. This binary mixture tends to form aggregates in which the density of one component of the binary mixture is larger than the other component. The interest of the Cahn-Hilliard equation stems from solutions that reproduce the formation of such clusters of cells *in vivo* or on petri dishes. Several variants are also used. A Cahn-Hilliard-Hele-Shaw model is proposed by Lowengrub *et al* [27] to describe the avascular, vascular and metastatic stages of solid tumor growth. They proved the existence and uniqueness of a strong solution globally for $d \leq 2$ and locally for $d = 3$ as well as the long term convergence to steady state. The case with a singular potential is treated in [25]. Variants can include the coupling with fluid equations and chemotaxis, see for instance [17] and the references therein. The case of multiphase Cahn-Hilliard systems is also very active presently, [9, 13].

The analysis of the long-time behavior of the solution of the Cahn-Hilliard equation has also attracted much attention since the seminal paper [8]. A precise description of the ω -limit set has been obtained in one dimension for the case of smooth polynomial potential and constant mobility in [29]. In this work, the effect of the different parameters of the model such as the initial mass, the width of the diffuse interface are investigated. In fact, the authors show that when γ is large, the solution converges to a constant as $t \rightarrow \infty$. The same happens when the initial mass is large. However when γ is positive

and small enough, the system admits nontrivial steady-states. For logarithmic potentials and constant mobility, Abels and Wilke [1] prove that solutions converge to a steady state as time goes to infinity using the the Lojasiewicz–Simon inequality. Other works have been made on the long term behavior of the solutions of some Cahn-Hilliard models including a source term [15], with dynamic boundary conditions [24], coupled with the Navier-Stokes equation [21], for non-local interactions and a reaction term [26].

Many difficulties, both analytical and numerical, arise in the context of Cahn-Hilliard equation and its variants. Because of the bilaplacian term, most of the numerical methods require to change the equation (1) into a system of two coupled equations

$$\begin{cases} \partial_t n = \nabla \cdot (b(n)\nabla v), \\ v = -\gamma\Delta n + \psi'(n). \end{cases} \quad (12)$$

This system (12) induces difficulties because the second equation contains all the backward diffusion and without a regularization of the unknown v , this system will lead to an ill-posed problem. However, this system of equations has been analyzed in the case where the mobility is degenerate and the potential is a logarithmic double-well functional by Elliott and Garcke [18]. They establish the existence of weak solutions of this system. Agosti *et al* [2] establish the existence of weak solutions when ψ is a single-well logarithmic potential which is more relevant for biological applications. They also prove that this system preserves the positivity of the cell density and the weak solutions belong to

$$n \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; (H^1(\Omega))'), \quad J \in L^2((0, T) \times \Omega, \mathbb{R}^d) \quad \forall T > 0,$$

Numerical simulations of the DCH system have been also performed in the context of double-well potentials in [19, 6]. To keep the energy inequality is a major concern in numerical methods and the survey paper by Shen *et al* [28] presents a general method applied to the present context.

The motivation behind the relaxation of equation (1) follows the argument above for the writing (12). We recover the system (3) which is composed of a parabolic transport equation and an elliptic equation on the new variable by introducing a new variable and regularizing the equation of the unknown v , using the decomposition of the potential (5) to keep the convex and stable part in the main equation for n , rejecting the concave and unstable part in the regularized equation. The relaxation parameter is σ and we need to verify that in the limit $\sigma \rightarrow 0$ we recover the original DCH equation (12). This is the main purpose of the present paper. Intending to design a numerical scheme that can approximate the solutions of the CH equation, we use a slight modification of the RDCH system (3) which is even more regularized when $\sigma > 0$. We refer the interested reader to the conclusion of this paper where this model is described. For the moment, we focus on the RDCH system (3).

Another standard relaxation method for the Cahn-Hilliard equation [22, 23] which, with our notations, reads

$$\partial_t n = \nabla \cdot [b(n)\nabla(K_\sigma \star n + \psi'(n))],$$

with a symmetric smooth kernel $K_\sigma \xrightarrow{\sigma \rightarrow 0} \Delta\delta$. The convergence to the DCH equation is a long standing open question in the field. Although, very similar in their form, the two relaxation models undergo different a priori estimates which allow us to study the limit $\sigma \rightarrow 0$ for (3).

As a first step towards the existence of solutions of (3), in section 2, we introduce a regularized problem which is not anymore degenerate. We show energy and entropy estimates from which we obtain a priori estimates which are used later on. At the end of this section, we prove the existence of weak solutions of the regularized-relaxed Cahn-Hilliard system. Departing from this existence result, in section 3, we can pass to the limit in the regularization parameter and show the existence of weak solutions of the RDCH system. Then, in section 4, we prove the convergence as $\sigma \rightarrow 0$ to the full DHC model. Section 5 is dedicated to the study of the long term convergence of the solutions to steady-states. We end the paper with some conclusions and perspectives.

2 Regularized problem

2.1 Regularization procedure

In order to prove that the system (3), admits solutions and to precise the functional spaces, we first define a regularized problem.

We consider a small positive parameter $0 < \epsilon \ll 1$ and define the regularized mobility

$$B_\epsilon(n) = \begin{cases} b(1 - \epsilon) & \text{for } n \geq 1 - \epsilon, \\ b(\epsilon) & \text{for } n \leq \epsilon, \\ b(n) & \text{otherwise.} \end{cases} \quad (13)$$

Then, there are two positive constants b_1 and B_1 , such that

$$b_1 < B_\epsilon(n) < B_1, \quad \forall n \in \mathbb{R}. \quad (14)$$

To define a regular potential, we smooth out the singularity located at $n = 1$ which only occurs in ψ_+ , see (6), and preserve the assumption (9) by setting

$$\psi''_{+,\epsilon}(n) = \begin{cases} \psi''_+(1 - \epsilon) & \text{for } n \geq 1 - \epsilon, \\ \psi''_+(\epsilon) & \text{for } n \leq \epsilon, \\ \psi''_+(n) & \text{otherwise.} \end{cases} \quad (15)$$

then, we set,

$$\psi_\epsilon(n) = \psi_{+,\epsilon}(n) + \psi_-(n),$$

Finally, there is a positive constant D such that

$$\psi_\epsilon(n) \in C^2(\mathbb{R}, \mathbb{R}) \text{ and } |\psi''_\epsilon(n)| \leq D, \quad \forall n \in \mathbb{R}. \quad (16)$$

We can now define the regularized problem

$$\begin{cases} \partial_t n_{\sigma,\epsilon} = \nabla \cdot [B_\epsilon(n_{\sigma,\epsilon}) \nabla (\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))], \\ -\sigma \Delta \varphi_{\sigma,\epsilon} + \varphi_{\sigma,\epsilon} = -\gamma \Delta n_{\sigma,\epsilon} + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}), \end{cases} \quad (17)$$

with zero-flux boundary conditions

$$\frac{\partial(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon})}{\partial\nu} = \frac{\partial(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))}{\partial\nu} = 0 \quad \text{on } \partial\Omega \times (0, +\infty). \quad (18)$$

It is convenient to define the flux of the regularized system as

$$J_{\sigma,\epsilon} = B_\epsilon(n_{\sigma,\epsilon})\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})).$$

2.2 Energy, entropy and a priori estimates

The relaxed and regularized system (17) comes with an energy and an entropy. These provide us with estimates which are useful to prove the existence of global weak solutions of (17) and their convergence to the weak solutions of the original DHC equation or to the RDHC as ϵ and/or $\sigma \rightarrow 0$.

Being given a smooth enough function $n(x)$, we define the energy associated with the regularized potential $\psi_{+,\epsilon}$ and relaxed system as

$$\mathcal{E}_{\sigma,\epsilon}[n] = \int_{\Omega} \left[\psi_{+,\epsilon}(n) + \frac{\gamma}{2} |\nabla(n - \frac{\sigma}{\gamma}\varphi_{\sigma})|^2 + \frac{\sigma}{2\gamma} |\varphi_{\sigma}|^2 + \psi_{-}(n - \frac{\sigma}{\gamma}\varphi_{\sigma}) \right], \quad (19)$$

where φ_{σ} is obtained from n by solving the elliptic equation in (3), or (17).

Proposition 1 (Energy) *Consider a strong solution $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ of (17)–(18), then, the energy of the system $\mathcal{E}_{\sigma,\epsilon}$ satisfies*

$$\frac{d}{dt} \mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}(t)] = - \int_{\Omega} B_\epsilon(n_{\sigma,\epsilon}) |\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2 \leq 0, \quad (20)$$

As a consequence, we obtain a first a priori estimate

$$\mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}(T)] + \int_0^T \int_{\Omega} B_\epsilon(n_{\sigma,\epsilon}) |\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0]. \quad (21)$$

Proof. To establish the energy of the regularized system, we begin with multiplying the first equation of (17) by $\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})$. Then, we integrate on the domain Ω and use the second boundary condition (18) to obtain

$$\int_{\Omega} [\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})] \partial_t n_{\sigma,\epsilon} = - \int_{\Omega} B_\epsilon(n_{\sigma,\epsilon}) |\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2.$$

Since $\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) \partial_t n_{\sigma,\epsilon} = \partial_t \psi_{+,\epsilon}(n_{\sigma,\epsilon})$, to retrieve the energy equality (20) we need to focus on the calculation of $\int_{\Omega} \varphi_{\sigma,\epsilon} \partial_t n_{\sigma,\epsilon}$. We write

$$\int_{\Omega} \varphi_{\sigma,\epsilon} \partial_t n_{\sigma,\epsilon} = \int_{\Omega} \varphi_{\sigma,\epsilon} \partial_t [n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}] + \int_{\Omega} \frac{\sigma}{2\gamma} |\varphi_{\sigma,\epsilon}|^2$$

and, using the second equation of (17), we rewrite the first term as

$$\begin{aligned} \int_{\Omega} \varphi_{\sigma,\epsilon} \partial_t [n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}] &= \int_{\Omega} [-\gamma \Delta (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})] \partial_t [n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}] \\ &= \frac{d}{dt} \int_{\Omega} \frac{\gamma}{2} |\nabla (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})|^2 + \psi_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \end{aligned}$$

where we have used the first boundary condition (18).

Altogether, we have recovered the expression (19) and the equality (20).

We can now turn to the entropy inequality. It is classical to define the mapping $\phi_{\epsilon} : [0, \infty) \mapsto [0, \infty)$

$$\phi_{\epsilon}''(n) = \frac{1}{B_{\epsilon}(n)}, \quad \phi_{\epsilon}(0) = \phi_{\epsilon}'(0) = 0, \quad (22)$$

which is well defined because $B_{\epsilon} \in C(\mathbb{R}, \mathbb{R}^+)$ from (14). For a nonnegative function $n(x)$, we define the entropy as

$$\Phi_{\epsilon}[n] = \int_{\Omega} \phi_{\epsilon}(n(x)) dx.$$

It is useful to keep in mind that, for $\epsilon = 0$, the entropy functional behaves as follows in the biophysical case $b(n) = n(1-n)^2$

$$\phi(n) = n \log(n), \quad n \approx 0^+, \quad \phi(n) = -\log(1-n), \quad n \approx 1^-.$$

Proposition 2 (Entropy) *Consider a strong solutions of (17)–(18), then the entropy of the system satisfies*

$$\begin{aligned} \frac{d\Phi_{\epsilon}[n_{\sigma,\epsilon}(t)]}{dt} &= - \int_{\Omega} \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2 + \psi''_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \left| \nabla \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 \\ &\quad + \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2. \end{aligned} \quad (23)$$

The equality (23) does not provide us with a direct a priori estimate because of the negative term ψ''_- , therefore we have to combine it with the entropy identity and write

$$\begin{aligned} \frac{d\Phi_{\epsilon}[n_{\sigma,\epsilon}(t)]}{dt} + \int_{\Omega} \left[\left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2 + \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2 \right] \\ \leq \frac{2}{\gamma} \|\psi''_-\|_{\infty} \mathcal{E}_{\sigma,\epsilon}[n_{\sigma,\epsilon}^0]. \end{aligned}$$

Proof. We compute, using the definition of ϕ_ϵ'' ,

$$\begin{aligned}
\int_{\Omega} \partial_t \phi_\epsilon(n_{\sigma,\epsilon}) &= \int_{\Omega} \partial_t n_{\sigma,\epsilon} \phi_\epsilon'(n_{\sigma,\epsilon}) \\
&= \int_{\Omega} \nabla \cdot [B_\epsilon(n_{\sigma,\epsilon}) \nabla (\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))] \phi_\epsilon'(n_{\sigma,\epsilon}) \\
&= - \int_{\Omega} B_\epsilon(n_{\sigma,\epsilon}) \nabla (\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \phi_\epsilon''(n_{\sigma,\epsilon}) \nabla n_{\sigma,\epsilon} \\
&= - \int_{\Omega} \nabla (\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \nabla n_{\sigma,\epsilon} \\
&= - \int_{\Omega} \nabla \varphi_{\sigma,\epsilon} \nabla (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) + \psi''_{+,\epsilon}(n_{\sigma,\epsilon}) |\nabla n_{\sigma,\epsilon}|^2 + \frac{\sigma}{\gamma} |\nabla \varphi_{\sigma,\epsilon}|^2.
\end{aligned} \tag{24}$$

To rewrite the term $\int_{\Omega} \nabla \varphi_{\sigma,\epsilon} \nabla (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})$, we use the second equation of the regularized system (17)

$$\varphi_{\sigma,\epsilon} = -\gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}). \tag{25}$$

Using (25) and the boundary condition (18), we can rewrite the term under consideration as

$$\begin{aligned}
\int_{\Omega} \varphi_{\sigma,\epsilon} \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) &= \int_{\Omega} -\gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \\
&= - \int_{\Omega} \gamma \left| \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2 + \psi''_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \left| \nabla \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \right|^2.
\end{aligned}$$

Injecting this equality into (24), we obtain the identity (23).

2.3 Inequalities

From the energy and entropy properties, we can conclude the following a priori bounds, where we assume that the initial data has finite energy and entropy,

$$\frac{\sigma}{2\gamma} \int_{\Omega} |\varphi_{\sigma,\epsilon}(t)|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \quad \forall t \geq 0, \tag{26}$$

$$\frac{\sigma}{\gamma} \int_0^T \int_{\Omega} |\nabla \varphi_{\sigma,\epsilon}|^2 \leq \Phi_\epsilon[n^0] + T \frac{2}{\gamma} \|\psi''_-\|_\infty \mathcal{E}_{\sigma,\epsilon}[n^0], \quad \forall T \geq 0, \tag{27}$$

$$\frac{\gamma}{2} \int_{\Omega} \left| \nabla (n_{\sigma,\epsilon}(t) - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}(t)) \right|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \quad \forall t \geq 0, \tag{28}$$

$$\int_0^T \int_{\Omega} \left| \Delta (n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \right|^2 \leq \Phi_\epsilon[n^0] + T \frac{2}{\gamma} \|\psi''_-\|_\infty \mathcal{E}_{\sigma,\epsilon}[n^0], \quad \forall T \geq 0, \tag{29}$$

$$\int_0^T \int_{\Omega} B_\epsilon(n_{\sigma,\epsilon}) |\nabla (\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon}))|^2 \leq \mathcal{E}_{\sigma,\epsilon}[n^0], \quad \forall T \geq 0. \tag{30}$$

2.4 Existence for the regularized problem

We can now state the existence theorem for the regularized problem (17).

Theorem 3 (Existence for $\varepsilon > 0$) *Assuming $n^0 \in H^1(\Omega)$, there exists a pair of functions $(n_{\sigma,\varepsilon}, \varphi_{\sigma,\varepsilon})$ such that for all $T > 0$,*

$$\begin{aligned} n_{\sigma,\varepsilon} &\in L^2(0, T; H^1(\Omega)), & \partial_t n_{\sigma,\varepsilon} &\in L^2(0, T; (H^1(\Omega))'), \\ \varphi_{\sigma,\varepsilon} &\in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \end{aligned}$$

which satisfies the Regularized-Relaxed Degenerate Cahn-Hilliard equation (17), (18) in the following weak sense, for all test function $\chi \in L^2(0, T; H^1(\Omega))$, it holds

$$\begin{aligned} \int_0^T \langle \chi, \partial_t n \rangle &= \int_{\Omega_T} B_\varepsilon(n_{\sigma,\varepsilon}) \nabla (\varphi_{\sigma,\varepsilon} + \psi'_+(n_{\sigma,\varepsilon})) \nabla \chi, \\ \sigma \int_{\Omega_T} \nabla \varphi_{\sigma,\varepsilon} \nabla \chi + \int_{\Omega_T} \varphi_{\sigma,\varepsilon} \chi &= \gamma \int_{\Omega_T} \nabla n_{\sigma,\varepsilon} \nabla \chi + \int_{\Omega_T} \psi'_-(n_{\sigma,\varepsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\varepsilon}) \chi. \end{aligned} \quad (31)$$

Proof. We adapt the proof of the theorems 2 and 3 in [18] where the authors prove the existence of solutions of the Cahn-Hilliard system with positive mobilities. Since the regularized mobility here is positive due to (14), we can apply the same theorem. To simplify this part, let us drop the notation ε and σ . The proof of existence follows the following different stages

Step 1. Galerkin approximation. Firstly, we make an approximation of the regularized problem (17). Let us define the family of basis functions $\{\phi_i\}_{i \in \mathbb{N}}$, the eigenfunctions of the Laplace operator subjected to zero Neumann boundary conditions.

$$-\Delta \phi_i = \lambda_i \phi_i \text{ in } \Omega \text{ with } \nabla \phi_i = 0 \text{ on } \partial\Omega.$$

ϕ are orthogonal functions for the $H^1(\Omega)$ and $L^2(\Omega)$ scalar products and we normalize them, i.e. $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$ to obtain an orthonormal basis. We assume without loss of generality that the first eigenvalue $\lambda_1 = 0$.

We consider the following discretization of (17)

$$n^N(t, x) = \sum_{i=1}^N c_i^N(t) \phi_i(x), \quad \varphi^N(t, x) = \sum_{i=1}^N d_i^N(t) \phi_i(x), \quad (32)$$

$$\int_{\Omega} \partial_t n^N \phi_j = - \int_{\Omega} B(n^N) \nabla (\varphi^N + \psi'_+(n^N)) \nabla \phi_j, \quad \text{for } j = 1, \dots, N, \quad (33)$$

$$\int_{\Omega} \varphi^N \phi_j = \gamma \int_{\Omega} \nabla \left(n^N - \frac{\sigma}{\gamma} \varphi^N \right) \nabla \phi_j + \int_{\Omega} \psi'_-(n^N - \frac{\sigma}{\gamma} \varphi^N) \phi_j, \quad \text{for } j = 1, \dots, N, \quad (34)$$

$$n^N(0, x) = \sum_{i=1}^N (n_0, \phi_i)_{L^2(\Omega)} \phi_i. \quad (35)$$

This gives the following initial value problem for a system of ordinary differential equations, for all $j = 1, \dots, N$,

$$\partial_t c_j^N = - \sum_{k=1}^N d_k^N \int_{\Omega} B \left(\sum_{i=1}^N c_i^N \phi_i \right) \nabla \phi_k \nabla \phi_j - \sum_{k=1}^N c_k^N \int_{\Omega} (B \psi_+'') \left(\sum_{i=1}^N c_i^N \phi_i \right) \nabla \phi_k \nabla \phi_j, \quad (36)$$

$$d_j^N = \gamma \lambda_j c_j^N - \sigma \lambda_j d_j^N + \int_{\Omega} \psi_-' \left(\sum_{k=1}^N c_k^N - \frac{\sigma}{\gamma} d_k^N \right) \phi_j, \quad (37)$$

$$c_j^N(0) = (n_0, \phi_i)_{L^2(\Omega)}. \quad (38)$$

Since the right-hand side of equation (36) depends continuously on the coefficients c_j^N , the initial value problem has a local solution.

Step 2. Energy estimate. To prove the existence of global solutions, we need to define an energy $E(t)$ and entropy $\Phi(t)$ from the semi-discrete system (32)–(35). They simply are the semi-discrete versions of the energy and entropy defined by (19) and (23). Therefore, we have

$$E(t) = \int_{\Omega} \psi_+(n^N) + \frac{\gamma}{2} |\nabla(n^N - \frac{\sigma}{\gamma} \varphi^N)|^2 + \frac{\sigma}{2\gamma} |\varphi^N|^2 + \psi_-(n^N - \frac{\sigma}{\gamma} \varphi^N),$$

$$\frac{d}{dt} E(t) = - \int_{\Omega} B(n^N) |\nabla(\varphi^N + \psi_+'(n^N))|^2.$$

From which we can gather the following energy inequality

$$E(T) + \int_0^T \int_{\Omega} B(n^N) |\nabla(\varphi^N + \psi_+'(n^N))|^2 \leq E(0). \quad (39)$$

Where $E(0)$ is integrable due to the assumptions (16). And we compute the semi-discrete entropy

$$\begin{aligned} \frac{d\Phi(t)}{dt} = & - \int_{\Omega} \left| \Delta \left(n^N - \frac{\sigma}{\gamma} \varphi^N \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi^N|^2 + \psi_-''(n^N - \frac{\sigma}{\gamma} \varphi^N) \left| \nabla \left(n^N - \frac{\sigma}{\gamma} \varphi^N \right) \right|^2 \\ & + \psi_+''(n^N) |\nabla n^N|^2. \end{aligned}$$

Form this, we can obtain the following entropy inequality

$$\frac{d\Phi(t)}{dt} + \int_{\Omega} \left[\left| \Delta \left(n^N - \frac{\sigma}{\gamma} \varphi^N \right) \right|^2 + \frac{\sigma}{\gamma} |\nabla \varphi^N|^2 + \psi_+''(n^N) |\nabla n^N|^2 \right] \leq \frac{2}{\gamma} \|\psi_-''\|_{\infty} E(t). \quad (40)$$

And we assume that the semi-discrete entropy has a finite initial data.

Step 3. Inequalities. From (39) and (40), we obtain the following inequalities

$$\frac{\gamma}{2} \int_{\Omega} |\nabla(n^N - \frac{\sigma}{\gamma}\varphi^N)|^2 \leq C, \quad (41)$$

$$\frac{\sigma}{2\gamma} \int_{\Omega} |\varphi^N|^2 \leq C, \quad \frac{\sigma}{\gamma} \int_0^T \int_{\Omega} |\nabla\varphi^N|^2 \leq \Phi(0) + T\frac{2}{\gamma}\|\psi''\|_{\infty}E(0), \quad (42)$$

$$\int_0^T \int_{\Omega} \left| \Delta(n^N - \frac{\sigma}{\gamma}\varphi^N) \right|^2 \leq \Phi(0) + T\frac{2}{\gamma}\|\psi''\|_{\infty}E(0), \quad (43)$$

$$\int_0^T \int_{\Omega} B(n^N) |\nabla(\varphi^N + \psi'_+(n^N))|^2 \leq E(0). \quad (44)$$

Which holds for small positive values of γ, σ and also for all finite time $T \geq 0$. Therefore, from these inequalities we can extract subsequences of (n^N, φ^N) such that the following convergences hold for any time $T \geq 0$ and small positive values of γ, σ . First, from (42), we deduce

$$\varphi^N \rightharpoonup \varphi \text{ weakly in } L^2(0, T; H^1(\Omega)). \quad (45)$$

Using (41), the Poincaré inequality and the convergence (45), we have

$$n^N \rightharpoonup n \text{ weakly in } L^2(0, T; H^1(\Omega)). \quad (46)$$

This result implies that the coefficients c_j^N are bounded and a global solution to (36)–(38) exists. To be able to prove some strong convergence in $L^2(0, T; L^2(\Omega))$, we need an information about the temporal derivative $\partial_t n^N$. From the first equation of the system, if we denote the projection of $L^2(\Omega)$ on the space generated by $\text{span}\{\phi_1, \dots, \phi_N\}$, we have for all test function $\phi \in L^2(0, T; H^1(\Omega))$

$$\begin{aligned} \left| \int_{\Omega_T} \partial_t n^N \phi \right| &= \left| \int_{\Omega_T} \partial_t n^N \Pi_N \phi \right| \\ &= \left| \int_{\Omega_T} b(n^N) \nabla(\varphi^N + \psi'_+(n^N)) \nabla \Pi_N \phi \right| \\ &\leq \left(B_1 \int_{\Omega_T} b(n^N) |\nabla(\varphi^N + \psi'_+(n^N))|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega_T} |\nabla \Pi_N \phi|^2 \right)^{\frac{1}{2}} \\ &\leq C \|\nabla \phi\|_{L^2(\Omega_T)}. \end{aligned}$$

Form which we can obtain

$$\partial_t n^N \rightharpoonup \partial_t n \text{ weakly in } L^2(0, T; (H^1(\Omega))'). \quad (47)$$

From (46) and (47) and using the Lions-Aubin Lemma, we obtain the strong convergence

$$n^N \rightarrow n \text{ strongly in } L^2(0, T; L^2(\Omega)).$$

Step 4. Limiting equation. From the above weak and strong convergences we can pass to the limit in the system to obtain the limiting system (31).

3 Existence: convergence as $\epsilon \rightarrow 0$

The next step is to prove existence of global weak solutions for the RDCH system (3) by letting ϵ vanish. We establish the following

Theorem 4 (Existence for $\sigma > 0, \epsilon = 0$) *Assuming an initial condition $n^0 \in H^1(\Omega)$, $0 \leq n^0 \leq 1$, there exists a pair of functions $(n_\sigma, \varphi_\sigma)$, such that*

$$n_\sigma \in L^2(0, T; H^1(\Omega)), \quad \partial_t n_\sigma \in L^2(0, T; (H^1(\Omega))'). \quad (48)$$

$$\varphi_\sigma \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \quad (49)$$

$$0 \leq n_\sigma \leq 1, \quad \text{a.e. } \in \Omega_T, \quad (50)$$

and $n_\sigma < 1$ a.e. if b vanishes fast enough at 1 so that $\phi(1) = \infty$.

Moreover, $(n_\sigma, \varphi_\sigma)$ is a global weak solution of the relaxed degenerate Cahn-Hilliard equation (3), (4) in the following weak sense: for all $\chi \in L^2(0, T; H^1(\Omega))$, it holds

$$\begin{aligned} \int_0^T \langle \chi, \partial_t n \rangle &= \int_{\Omega_T} b(n) (\nabla \varphi + \psi'_+(n) \nabla n) \nabla \chi, \\ \sigma \int_{\Omega_T} \nabla \varphi \nabla \chi + \int_{\Omega_T} \varphi \chi &= \gamma \int_{\Omega_T} \nabla n \nabla \chi + \int_{\Omega_T} \psi'_-(n - \frac{\sigma}{\gamma} \varphi) \chi. \end{aligned} \quad (51)$$

Proof. The proof relies on compactness results and the inequalities presented in section 2.3. From these inequalities, we can extract subsequences of $(n_{\sigma, \epsilon}, \varphi_{\sigma, \epsilon})$ such that the following convergences for $\epsilon \rightarrow 0$ hold for all $T > 0$. *Step 1. Weak limits.* From (26) and (27), we immediately have

$$\varphi_{\sigma, \epsilon} \rightharpoonup \varphi_\sigma \text{ in } L^2((0, T); H^1(\Omega)). \quad (52)$$

Next, from (28), and the above convergence, we conclude

$$n_{\sigma, \epsilon} \rightharpoonup n_\sigma \text{ weakly in } L^2(0, T; H^1(\Omega)), \quad (53)$$

Finally from (30) and the equation on $n_{\sigma, \epsilon}$ itself, we have

$$\partial_t n_{\sigma, \epsilon} \rightharpoonup \partial_t n_\sigma \text{ weakly in } L^2(0, T; (H^1(\Omega))'). \quad (54)$$

Step 2. Strong convergence. Therefore, from the Lions-Aubin lemma, we obtain the strong convergence

$$n_{\sigma, \epsilon} \rightarrow n_\sigma \in L^2(0, T; L^2(\Omega)). \quad (55)$$

Step 3. Bounds $0 \leq n_\sigma \leq 1$. To prove these bounds on n_σ , several authors have used the entropy relation. In the context of DCH equation with double-well potentials featuring singularities at $n = 1$ and $n = -1$, the solution lies a.e. in the interval $-1 < n < 1$. Elliott and Garcke [18] prove this result using the definition of the regularized entropy and by a contradiction argument. For single-well

potential, Agosti *et al.* [2] used a reasoning on the measure of the set of solutions outside the set $0 \leq n < 1$ and find contradictions with the boundedness of the entropy. This is the route we follow here.

We begin by the upper bound. For $\alpha > 0$, we consider the set

$$V_\alpha^\varepsilon = \{(t, x) \in \Omega_T | n_{\sigma, \varepsilon}(t, x) \geq 1 + \alpha\}.$$

Consider $A > 0$ (large), for ε small enough, we have (because $b(1) = 0$)

$$\phi_\varepsilon''(n) = \frac{1}{b(1 - \varepsilon)} \geq 2A \quad \forall n \geq 1.$$

Thus, integrating this quantity twice, we obtain

$$\phi_\varepsilon(n) \geq A(n - 1)^2 \quad \forall n \geq 1.$$

Also, from (23), we know that the entropy is uniformly bounded in ε . Therefore, we obtain

$$|V_\alpha^\varepsilon| A \alpha^2 \leq \int_{\Omega_T} \phi_\varepsilon(n_{\sigma, \varepsilon}(t, x)) \leq C(T), \quad |V_\alpha^\varepsilon| \leq \frac{C(T)}{A \alpha^2}.$$

In the limit $\varepsilon \rightarrow 0$, we conclude that

$$|\{(t, x) \in \Omega_T | n_\sigma(t, x) \geq 1 + \alpha\}| \leq \frac{C(T)}{A \alpha^2}, \quad \forall A > 0.$$

In other words $n_\sigma(t, x) \leq 1 + \alpha$ for all $\alpha > 0$, which means $n_\sigma(t, x) \leq 1$.

The same argument also gives $n_\sigma \geq 0$ and we do not repeat it.

The second statement, $n_\sigma < 1$ under the assumption $\phi(1) = +\infty$, is a consequence of the bound

$$\int_{\Omega_T} \phi(n_\sigma(t, x)) \leq C(T),$$

which holds true by strong convergence of $n_{\sigma, \varepsilon}$ and because $\phi_\varepsilon \nearrow \phi$ as $\varepsilon \searrow 0$.

Step 4. Limiting equation. Finally, it remains to show that the limit of subsequences satisfies the RDCH equation in the weak form. In other words, we need to prove the following weak convergence, recalling that (30) provides a uniform L^2 bound over Ω_T , on $J_{\sigma, \varepsilon}$

$$J_{\sigma, \varepsilon} := B_\varepsilon(n_{\sigma, \varepsilon}) \nabla(\varphi_{\sigma, \varepsilon} + \psi'_{+, \varepsilon}(n_{\sigma, \varepsilon})) \rightharpoonup b(n_\sigma) \nabla(\varphi_\sigma + \psi'_+(n_\sigma)) \text{ weakly in } L^2(\Omega). \quad (56)$$

The convergence of $B_\varepsilon(n_{\sigma, \varepsilon}) \nabla \varphi_{\sigma, \varepsilon}$ follows from the weak convergence in $L^2(\Omega_T)$ of $\nabla \varphi_{\sigma, \varepsilon}$ and the strong convergence $B_\varepsilon(n_{\sigma, \varepsilon}) \rightarrow b(n_\sigma)$ in all $L^p(\Omega_T)$, $1 \leq p < \infty$ which follows from (55) and the fact that $B_\varepsilon(\cdot) \rightarrow b(\cdot)$ uniformly.

Because of the singularity $\psi'_+(1) = \infty$, we use the assumption (9) and that $B_\varepsilon(\cdot) \psi''_{+, \varepsilon}(\cdot) \rightarrow b(\cdot) \psi''_+(\cdot)$ uniformly and thus $B_\varepsilon(n_{\sigma, \varepsilon}) \psi''_{+, \varepsilon}(n_{\sigma, \varepsilon}) \rightarrow b(n_{\sigma, \varepsilon}) \psi''_+(n_{\sigma, \varepsilon})$ a.e. in Ω . This achieves the proof.

4 Convergence as $\sigma \rightarrow 0$

We are now ready to study the limit of the relaxed solution n_σ towards a solution of the DCH equation, Our main result is as follows.

Theorem 5 (Limit $\sigma = 0$) *Let $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ be a sequence of weak solutions of the RDHC system (17) with initial conditions n^0 , $0 \leq n^0 < 1$, with finite energy and entropy. Then, as $\epsilon, \sigma \rightarrow 0$, we can extract a subsequence of $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that*

$$\varphi_{\sigma,\epsilon} \rightharpoonup -\gamma \Delta n + \psi'_-(n) \quad \text{weakly in } L^2(\Omega_T), \quad (57)$$

$$n_{\sigma,\epsilon}, \nabla n_{\sigma,\epsilon} \rightarrow n, \nabla n \quad \text{strongly in } L^2(\Omega_T), \text{ and } 0 \leq n \leq 1, \quad (58)$$

and $n_\sigma < 1$ a.e. if b vanishes fast enough at 1 so that $\phi(1) = \infty$.

$$\partial_t n_{\sigma,\epsilon} \rightharpoonup \partial_t n \quad \text{weakly in } L^2(0, T; (H^1(\Omega))'). \quad (59)$$

This limit n satisfies the DCH system (1) in the weak sense.

We recall the definition of weak solutions; for all $\chi \in L^2(0, T; H^2(\Omega))$ with $\nabla \chi \cdot \nu = 0$ on $\partial\Omega \times (0, T)$,

$$\begin{cases} \int_0^T \langle \chi, \partial_t n \rangle &= \int_{\Omega_T} J \cdot \nabla \chi, \\ \int_{\Omega_T} J \cdot \nabla \chi &= \int_{\Omega_T} \gamma \Delta n [b'(n) \nabla n \cdot \nabla \chi + b(n) \Delta \chi] + (b\psi'')(n) \nabla n \cdot \nabla \chi. \end{cases} \quad (60)$$

Proof. We gathered, from the energy and entropy estimates of section 2.2, the a priori bounds of the section 2.3.

Step 1. Weak limits. From the above mentioned inequalities, we can extract subsequences of $(n_{\sigma,\epsilon}, \varphi_{\sigma,\epsilon})$ such that the following convergences hold for all $T > 0$. From (26) and (27), we immediately have

$$\sigma \varphi_{\sigma,\epsilon} \rightarrow 0 \text{ in } L^\infty((0, T); H^1(\Omega)). \quad (61)$$

Next, from (28), and the above convergence, we conclude

$$n_{\sigma,\epsilon} \rightharpoonup n \text{ weakly in } L^2(0, T; H^1(\Omega)),$$

and (29) gives directly

$$\Delta(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \rightharpoonup \Delta n \text{ weakly in } L^2(\Omega_T). \quad (62)$$

The system equations can also be used to complement these results. We find

$$\varphi_{\sigma,\epsilon} \rightharpoonup \varphi \text{ weakly in } L^2(\Omega_T),$$

using the second equation of the system (17) and triangular inequality,

$$\|\varphi_{\sigma,\epsilon}\|_{L^2(\Omega_T)} \leq \gamma \|\Delta(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})\|_{L^2(\Omega_T)} + \|\psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon})\|_{L^2(\Omega_T)}.$$

The conclusion (57) follows. Finally from (30) and the equation on $n_{\sigma,\epsilon}$ itself, we conclude (59).

Step 2. Strong convergence. We continue with proving the strong convergences in (58). From the inequality (29), we know that $\Delta\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\right)$ is uniformly bounded in $L^2(\Omega_T)$. We also have the boundary conditions, $\nabla\left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi\right) \cdot \nu = 0$ and the conservation of both quantities. Therefore elliptic regularity theory gives us

$$\|n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi_{\sigma,\epsilon}\|_{L^2(0,T;H^2(\Omega))} \leq C.$$

Therefore strong compactness in space holds for the quantities $n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi$ and $\nabla[n_{\sigma,\epsilon} - \frac{\sigma}{\gamma}\varphi]$. Furthermore, from the limit (61), it means that both $n_{\sigma,\epsilon}$ and $\nabla n_{\sigma,\epsilon}$ are compact in space. Their compactness in time is an immediate consequence, thanks to the Lions-Aubin method, of the equation on $n_{\sigma,\epsilon}$ and of the bound (30).

The bounds $0 \leq n < 1$ can be obtained as in the case $\epsilon \rightarrow 0$, see Theorem 4 and we do not repeat the argument.

Step 3. Limiting equation. Next, we need to verify that the limit of the subsequence $n_{\sigma,\epsilon}$ satisfies the DCH equation. The argument is different from the case $\epsilon \rightarrow 0$ because we do not control $\nabla\varphi_{\sigma,\epsilon}$ in the case at hand. From the L^2 bound in (30), we need to identify the weak limit

$$J_{\sigma,\epsilon} := B_\epsilon(n_{\sigma,\epsilon})\nabla(\varphi_{\sigma,\epsilon} + \psi'_{+,\epsilon}(n_{\sigma,\epsilon})) \rightharpoonup b(n)\nabla(\varphi + \psi'_+(n)) \quad \text{weakly in } L^2(\Omega_T). \quad (63)$$

For a test function $\eta \in L^2(0, T; H^1(\Omega, \mathbb{R}^d)) \cap L^\infty(\Omega_T, \mathbb{R}^d)$ and $\eta \cdot \mu = 0$ on $\partial\Omega \times (0, T)$, we integrate the left hand side to obtain

$$\int_{\Omega_T} J_{\sigma,\epsilon} \cdot \eta = \int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta) + B_\epsilon(n_{\sigma,\epsilon}) \nabla \left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \right) \cdot \eta.$$

We have mainly two types of terms on the right-hand side $\int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta)$ and $\int_{\Omega_T} B_\epsilon(n_{\sigma,\epsilon}) \nabla \left(\psi'_{+,\epsilon}(n_{\sigma,\epsilon}) + \psi'_-(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon}) \right) \cdot \eta$. The latter can be treated as in the limit $\epsilon \rightarrow 0$ and we do not repeat the argument. Let us focus on the first term

$$\begin{aligned} \int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) \nabla \cdot (B_\epsilon(n_{\sigma,\epsilon})\eta) &= \int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) B_\epsilon(n_{\sigma,\epsilon}) \nabla \cdot \eta \\ &\quad + \int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) B'_\epsilon(n_{\sigma,\epsilon}) \nabla n_{\sigma,\epsilon} \cdot \eta. \end{aligned}$$

From the strong convergence (58) and the weak one (62) with the fact that $B_\epsilon(\cdot) \rightarrow b(\cdot)$ uniformly, we obtain the convergence of the first term of the right-hand term

$$\int_{\Omega_T} \gamma \Delta \left(n_{\sigma,\epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma,\epsilon} \right) B_\epsilon(n_{\sigma,\epsilon}) \nabla \cdot \eta \rightarrow \int_{\Omega_T} \gamma \Delta n b(n) \nabla \cdot \eta,$$

as $\sigma, \epsilon \rightarrow 0$ and thus we have passed to the limit in the first term of the right hand side. For the second term, we use that the derivative $B'_\epsilon(\cdot) \rightarrow b'(\cdot)$ uniformly. We also use the strong convergence of $\nabla n_{\sigma,\epsilon}$

from (58). From the results above and a generalized version of the Lebesgue convergence theorem we obtain

$$\int_{\Omega_T} \gamma \Delta \left(n_{\sigma, \epsilon} - \frac{\sigma}{\gamma} \varphi_{\sigma, \epsilon} \right) B'_\epsilon(n_{\sigma, \epsilon}) \nabla n_{\sigma, \epsilon} \cdot \eta \rightarrow \int_{\Omega_T} \gamma \Delta n b'(n) \nabla n \cdot \eta,$$

as $\sigma, \epsilon \rightarrow 0$.

This finishes the proof of (63), i.e., that the limit solution n satisfies the weak formulation of the DCH equation (1) and also the proof of Theorem 5.

5 Long-time behavior

To complete our study of the RDCH model, we give some insights concerning the long-time behavior and convergence to steady states, $(n_\infty, \varphi_\infty)$ determined by the steady problem

$$\begin{cases} \nabla \cdot (b(n_\infty) \nabla (\varphi_\infty + \psi'_+(n_\infty))) = 0 & \text{in } \Omega, \\ -\sigma \Delta \varphi_\infty + \varphi_\infty = -\gamma \Delta n_\infty + \psi'_-(n_\infty - \frac{\sigma}{\gamma} \varphi_\infty) & \text{in } \Omega, \\ \frac{\partial (n_\infty - \frac{\sigma}{\gamma} \varphi_\infty)}{\partial \nu} = \frac{\partial (\varphi_\infty + \psi'_+(n_\infty))}{\partial \nu} = 0 & \text{on } \partial \Omega. \end{cases} \quad (64)$$

The analysis of the steady-states is not performed in this paper, however, numerical simulations can help us to have an idea of their shape for different initial situations.

The steady-states of the RDCH model present a configuration which minimizes the energy of the system. The solution obtained at the end of the simulation depends mainly on three parameters: the initial mass M , the width of the diffuse interface $\sqrt{\gamma}$ and the relaxation parameter σ .

In fact, if the initial mass is large enough, saturated aggregates are formed and we can describe two regions in the domain: the aggregates and the absence of cells. Between these two regions, the transition is smooth and the length of this interface is $\sqrt{\gamma}$. If the initial mass is small, aggregates are still formed but they are thicker and their maximum concentration does not reach 1 or the critical value n^* as in the definition of the potential (7).

The formation of aggregates happens only if γ is small enough. If γ , the initial mass M or the relaxation parameter σ is too large, the the solution converges to the constant one

$$n_\infty = \frac{1}{|\Omega|} \int_{\Omega} n^0 dx, \quad \text{a.e. in } \Omega.$$

The interesting fact about these observations is that the long-time behavior of the solutions of the RDCH system seems to follow the analytical description of the steady-states made by Songmu [29].

Let us now state our result about the convergence of the weak solutions of the RDCH model to steady-states. In order to do so, we consider a global weak solution (n, φ) of the RDCH system with $\sigma > 0$, according to Theorem 4. The initial condition satisfies $0 \leq n^0 < 1$ and has finite energy and entropy. We recall the a priori estimates from the transport structure, the energy dissipation (20) and the entropy dissipation (23),

$$0 \leq n < 1 \text{ a.e. } (0, \infty) \times \Omega. \quad (65)$$

$$\begin{cases} \mathcal{E}[n(t)] &= \int_{\Omega} \left[\psi_+(n) + \frac{\gamma}{2} |\nabla(n - \frac{\sigma}{\gamma}\varphi)|^2 + \frac{\sigma}{2\gamma} |\varphi|^2 + \psi_-(n - \frac{\sigma}{\gamma}\varphi) \right] \leq 0, \\ \frac{d}{dt} \mathcal{E}[n(t)] &= - \int_{\Omega} b(n) |\nabla(\varphi + \psi'_+(n))|^2 \leq 0, \end{cases} \quad (66)$$

$$\begin{cases} \Phi[n(t)] = \int \phi(n(x, t)) dx, & \phi''(n) = \frac{1}{B(n)}, \\ \frac{d\Phi[n(t)]}{dt} = - \int_{\Omega} \left[\Delta \left(n - \frac{\sigma}{\gamma}\varphi \right) \right]^2 + \frac{\sigma}{\gamma} |\nabla\varphi|^2 + \psi''_-(n - \frac{\sigma}{\gamma}\varphi) \left| \nabla \left(n - \frac{\sigma}{\gamma}\varphi \right) \right|^2 + \psi''_+(n) |\nabla n|^2 \right]. \end{cases} \quad (67)$$

Based on the controls provided by these relations, and using a standard method, we are going to study the large time behavior as the limit for large k of the sequence of functions

$$n_k(t, x) = n(t + k, x), \quad \text{and} \quad \varphi_k(t, x) = \varphi(t + k, x).$$

Proposition 6 (Long term convergence along subsequences) *Let (n, φ) be a weak solution of (3) with boundary conditions (4) and initial condition n^0 with finite energy and $0 \leq n^0 < 1$. Then, we can extract a subsequence, still denoted by index k , of (n_k, φ_k) such that*

$$\lim_{k \rightarrow \infty} n_k(x, t) = n_{\infty}(x), \quad \lim_{k \rightarrow \infty} \varphi_k(x, t) = \varphi_{\infty}(x) \quad \text{strongly in } L^2((-T, T) \times \Omega), \quad \forall T > 0, \quad (68)$$

where $(n_{\infty}, \varphi_{\infty})$ are solutions of (64) satisfying

$$b(n_{\infty}) \nabla (\varphi_{\infty} + \psi'_+(n_{\infty})) = 0. \quad (69)$$

Proof. The proof uses the energy and entropy inequalities to obtain both uniform (in k) a priori bounds and zero entropy dissipation in the limit, which imply the result. We write these arguments in several steps.

1st step. A priori bounds from energy. Energy decay implies that $\mathcal{E}[n_k(t)]$ remains bounded in k for $t > -k$. As a consequence, the sequence (n_k, φ_k) satisfies

$$\frac{\sigma}{2\gamma} \int_{\Omega} |\varphi_k(t)|^2 \leq \mathcal{E}[n^0], \quad \forall t \geq 0, \quad (70)$$

$$\frac{\gamma}{2} \int_{\Omega} \left| \nabla \left(n_k(t) - \frac{\sigma}{\gamma} \varphi_k(t) \right) \right|^2 \leq \mathcal{E}[n^0], \quad \forall t \geq 0, \quad (71)$$

$$\int_{-T}^T \int_{\Omega} b(n_k) |\nabla(\varphi_k + \psi'_+(n_k))|^2 := L_k(T), \quad L_k(T) \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (72)$$

And this last line is because

$$\int_0^{\infty} \int_{\Omega} b(n) |\nabla(\varphi + \psi'_+(n))|^2 \leq \mathcal{E}[n^0], \quad L_k(T) \leq \int_{k-T}^{\infty} \int_{\Omega} b(n) |\nabla(\varphi + \psi'_+(n))|^2 \xrightarrow[k \rightarrow \infty]{} 0.$$

2nd step. A priori bounds from entropy. Because the right hand side in the entropy balance has a

positive term, it cannot be used as easily as the energy. We simply notice that the a priori bound $0 \leq n < 1$ makes that the entropy itself is bounded, therefore we can integrate (67) from $k - T$ to $k + T$. We obtain that there is a constant $K^0(T)$ only depending on the initial data, such that for all $k \geq T$,

$$\frac{\sigma}{\gamma} \int_{-T}^T \int_{\Omega} |\nabla \varphi_k|^2 \leq \Phi[n(k - T)] - \Phi[n(k + T)] + \frac{4T}{\gamma} \|\psi''\|_{\infty} \mathcal{E}[n(k - T)] \leq K^0(T), \quad (73)$$

$$\int_{-T}^T \int_{\Omega} \left| \Delta \left(n_k - \frac{\sigma}{\gamma} \varphi_k \right) \right|^2 \leq K^0(T). \quad (74)$$

3rd step. Extracting subsequences. From these inequalities, we can extract subsequences of (n_k, φ_k) such that for $k \rightarrow \infty$, the following convergences hold toward some functions $n_{\infty}(x, t)$ and $\varphi_{\infty}(x, t)$.

We can conclude from inequalities (70), (73) that, as $k \rightarrow \infty$,

$$\varphi_k \rightharpoonup \varphi_{\infty} \text{ weakly in } L^2(-T, T; H^1(\Omega)). \quad (75)$$

From (71) and the Poincaré inequality, we obtain

$$n_k - \frac{\sigma}{\gamma} \varphi_k \rightharpoonup n_{\infty} - \frac{\sigma}{\gamma} \varphi_{\infty} \text{ weakly in } L^2(0, T; H^1(\Omega)), \quad (76)$$

and thus

$$n_k \rightharpoonup n_{\infty} \text{ weakly in } L^2(0, T; H^1(\Omega)), \quad (77)$$

Finally, we obtain from (72) and the Cauchy-Schwarz inequality,

$$\partial_t n_k \rightharpoonup \partial_t n_{\infty} = 0 \text{ weakly in } L^2(0, T; (H^1(\Omega))'). \quad (78)$$

Indeed, for any test function $\phi \in C_0^{\infty}((-T, T) \times \Omega)$, it holds

$$\int_{-T}^T \int_{\Omega} \partial_t n_k \phi dx dt = - \int_{-T}^T \int_{\Omega} b(n_k) \nabla (\varphi_k + \psi'_+(n_k)) \cdot \nabla \phi,$$

$$\left| \int_{-T}^T \int_{\Omega} \partial_t n_k \phi dx dt \right|^2 \leq 2T |\Omega| \|b\|_{\infty} \|\nabla \phi\|_{\infty}^2 \int_{-T}^T \int_{\Omega} b(n_k) |\nabla (\varphi_k + \psi'_+(n_k))|^2 \rightarrow 0$$

as $k \rightarrow \infty$. This also shows that n_{∞} only depends on x .

4th step. Strong limits. The strong compactness of n_k and φ_k follows from (71)–(73). Then, time compactness of n_k , stated in (68) follows from the Lions-Aubin lemma, thanks to (78).

Then, the strong convergence of φ_k is a consequence of the elliptic equation for φ_k .

And we also have, from the strong convergence of n_k and (76), thanks to the above argument,

$$b(n_k) \nabla (\varphi_k + \psi'_+(n_k)) \rightarrow b(n_{\infty}) \nabla (\varphi_{\infty} + \psi'_+(n_{\infty})) = 0, \quad (79)$$

which establishes the zero-flux equality (69).

6 Conclusion

The proposed relaxation system of the degenerate Cahn-Hilliard equation with degenerate double-well potential reduces the model to two parabolic/elliptic equations which can be solved by standard numerical solvers. The relaxation uses a regularization in space of the new variable used to transform the original fourth-order equation into two second-order equations. This new system is a non-local relaxation of the original equation which is similar in a sense to the Cahn-Hilliard equation with a convolutional kernel proposed in [22, 23]. However, unlike this model, we have been able to prove that in the limit of vanishing relaxation, we retrieve the original weak solutions of the DCH equation. The convergence of this model was proved using compactness methods and estimates borrowed from energy and entropy functionals. The long-time behavior of the solutions of the RDCH system can also be studied along the same lines. We showed that the system converges to steady state solutions as time goes to infinity, with zero flux.

These latter exhibits some interesting properties due to the degeneracy of the mobility. These solutions are split into two distinct zones: whether the mobility is null which is possible only in the pure phases or the flux is null.

The RDCH system aims at the design of a numerical method to simulate the DCH equation using only second order elliptic problems. Such a numerical scheme, may depend on details of the relaxed model. For example, the solution represents a density and its numerical positivity is a desired property. Also the discrete stability is useful and a change of unknown in the RDCH system might be better adapted, using $U = \varphi - \frac{\gamma}{\sigma}n$,

$$\begin{aligned}\partial_t n &= \nabla \cdot \left(b(n) \nabla \left(U + \frac{\gamma}{\sigma}n + \psi'_+(n) \right) \right), \\ -\sigma \Delta U + U &= -\frac{\gamma}{\sigma}n + \psi'_-\left(-\frac{\sigma}{\gamma}U\right).\end{aligned}$$

Even though this model is also formed of a parabolic transport equation and an elliptic equation, the regularity is enhanced. On the one hand, in the first equation, the term $\frac{\gamma}{\sigma}n$ increases the diffusion for n . On the other hand, the second equation becomes a more regular elliptic equation for the new variable U because it depends on n rather than Δn . Altogether, this model improves the regularity of the two unknowns and might be interesting for numerical purposes. In a further work about the RDCH model, we will propose a numerical scheme that preserves the physical properties of the solution.

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