

# On the top-dimensional $\ell^2$ -Betti numbers

Damien Gaboriau\*, Camille Noûs†

June 26, 2020

## Abstract

The purpose of this note is to introduce a trick which relates the (non)-vanishing of the top-dimensional  $\ell^2$ -Betti numbers of actions with that of sub-actions. We provide three different types of applications: we prove that the  $\ell^2$ -Betti numbers of  $\text{Aut}(\mathbf{F}_n)$  and  $\text{Out}(\mathbf{F}_n)$  (and of their Torelli subgroups) do not vanish in degree equal to their virtual cohomological dimension, we prove that the subgroups of the 3-manifold groups have vanishing  $\ell^2$ -Betti numbers in degree 3 and 2 and we figure out the ergodic dimension of certain direct product of the form  $H \times A$  where  $A$  is infinite amenable.

## Résumé en Français

Le but de cette note est d'introduire une astuce qui relie l'annulation (ou la non-annulation) du nombre de Betti  $\ell^2$  en dimension maximale des actions d'un groupe avec l'annulation pour ses sous-actions. On fournit trois différents types d'applications : on montre que les nombres de Betti  $\ell^2$  de  $\text{Aut}(\mathbf{F}_n)$  et  $\text{Out}(\mathbf{F}_n)$  (et de leurs sous-groupe de Torelli) ne s'annulent pas en degré égal à leur dimension cohomologique virtuelle ; on prouve qu'un sous-groupe quelconque du groupe fondamental d'une variété compacte de dimension 3 a ses nombres de Betti  $\ell^2$  nuls en degré 3 et 2 et enfin, on parvient à déterminer la dimension ergodique de certains produits directs de la forme  $H \times A$  où  $A$  est moyennable infini.

**keywords:**  $\ell^2$ -Betti numbers, measured group theory, cohomological dimension, ergodic dimension,  $\text{Out}(\mathbf{F}_n)$ ,  $\text{Aut}(\mathbf{F}_n)$ , 3-dimensional manifolds.

**MSC:** 37A20, 19K56, 20F28, 20E15, 57Mxx.

## 1 Presentation of the results

The  $\ell^2$ -Betti numbers were introduced by Atiyah [Ati76], in terms of heat kernel, for free cocompact group actions on manifolds and were extended to the

---

\*CNRS & U.M.P.A., Ecole Normale Supérieure de Lyon, UMR 5669, 46 allée d'Italie, 69364 Lyon cedex 07, France.

†Laboratoire Cogitamus; <https://www.cogitamus.fr/>

framework of measured foliations by Connes [Con79]. They acquired the status of group invariants thanks to Cheeger and Gromov [CG86] who provided us with the definition of the  $\ell^2$ -Betti numbers of an arbitrary countable group  $\Gamma$ :

$$\beta_0^{(2)}(\Gamma), \beta_1^{(2)}(\Gamma), \beta_2^{(2)}(\Gamma), \dots, \beta_k^{(2)}(\Gamma), \dots$$

Their extension to standard probability measure preserving actions and equivalence relations by the first author [Gab02] opened the connection with the domain of orbit equivalence, offering in return some general by-products, for instance the  **$\ell^2$ -proportionality principle** [Gab02, Corollaire 0.2]: *If  $\Gamma$  and  $\Lambda$  are lattices in a locally compact second countable (lsc) group  $G$  with Haar measure  $\text{Vol}$ , then their  $\ell^2$ -Betti numbers are related as their covolumes:  $\frac{\beta_k^{(2)}(\Gamma)}{\text{Vol}(\Gamma \backslash G)} = \frac{\beta_k^{(2)}(\Lambda)}{\text{Vol}(\Lambda \backslash G)}$ .*

Over the years, the  $\ell^2$ -Betti numbers have been proved to provide very useful invariants in geometry, in 3-dimensional manifolds, in ergodic theory, in operator algebras and in many aspects of discrete group theory such as geometric, resp. measured, resp. asymptotic group theory. We refer to [Eck00] for an introduction to the subject and to the monographies [Lüc02, Kam19].

The term top-dimension used in the title may have different meanings. At first glance, we mean the dimension of some contractible simplicial complex on which our group  $\Gamma$  acts simplicially and properly (i.e., with finite stabilizers). For the purpose of computing  $\ell^2$ -Betti numbers, one can consider the action of some finite index subgroup of  $\Gamma$ . In many interesting cases, the group  $\Gamma$  is indeed virtually torsion-free. Then, the **virtual geometric dimension** (the minimal dimension of a contractible simplicial complex on which a finite index subgroup acts simplicially and freely) can be used as a better (i.e., lower) top-dimension for  $\Gamma$ . Observe that the  $\ell^2$ -Betti numbers must vanish in all degrees above this dimension. In view of the Eilenberg-Ganea Theorem [EG57] (see also [Bro82, Chapter VIII.7]), if the **virtual cohomological dimension (vcd)** of  $\Gamma$  is finite and greater than three then it coincides with the virtual geometric dimension. The vanishing or non-vanishing of  $\ell^2$ -Betti numbers in some degree is an invariant for lattices in the same lsc group (as the  $\ell^2$ -proportionality principle above indicates), and it is more generally an invariant of measure-equivalence [Gab02, Théorème 6.3]. In contrast, the virtual cohomological dimension is not: for instance cocompact versus non-cocompact lattices in  $\text{SL}(d, \mathbb{R})$  have different vcd. This nominates the **ergodic dimension** as a better notion of top-dimension. This is intrinsically an invariant of measured group theory introduced in [Gab02, Définition 6.4] (see Section 6 and also [Gab20]) which mixes geometry and ergodic theory. It is bounded above by the virtual geometric dimension and is often much less. Our trick (Theorems 1.9 and 5.1) also applies to it.

## 1.1 $\text{Aut}(\mathbf{F}_n)$ and $\text{Out}(\mathbf{F}_n)$

While the  $\ell^2$ -Betti numbers of many classic groups are quite well understood, this is far from true for the groups  $\text{Aut}(\mathbf{F}_n)$  and  $\text{Out}(\mathbf{F}_n)$  of automorphisms

(resp. outer automorphisms) of the free group  $\mathbf{F}_n$  on  $n \geq 3$  generators. These groups share many algebraic features with both the group  $\mathrm{GL}(n, \mathbb{Z})$  and with the mapping class group  $\mathrm{MCG}(S_g)$  of the surface  $S_g$  of genus  $g$ . One reason is that all these groups are (outer) automorphism groups of the most primitive discrete groups ( $\mathbf{F}_n$ ,  $\mathbb{Z}^n$  and  $\pi_1(S_g)$  respectively) and the three families begin with the same group  $\mathrm{Out}(\mathbf{F}_2) \simeq \mathrm{GL}(2, \mathbb{Z}) \simeq \mathrm{MCG}(S_1)$ . These empirical similarities have served as guiding lines for their study, see for instance [CV86, BV06, Vog06].

By the work of Borel [Bor85], the  $\ell^2$ -Betti numbers of the cocompact lattices of  $\mathrm{GL}(n, \mathbb{R})$  are known to all vanish when  $n \geq 3$ . The same holds for the non-cocompact ones like  $\mathrm{GL}(n, \mathbb{Z})$  by the  $\ell^2$ -proportionality principle. The mapping class group  $\mathrm{MCG}(S_g)$  is virtually torsion-free, and when  $g > 1$ , all its  $\ell^2$ -Betti numbers vanish except in degree equal to the middle dimension  $3g - 3$  of its Teichmüller space (see for instance [Kid08, Appendix D]). These behaviors are very common for  $\ell^2$ -Betti numbers of the classic groups: most of them vanish, and when a non-vanishing happens it is only in the middle dimension of “the associated symmetric space”.

Culler-Vogtmann [CV86] invented the Outer space  $CV_n$  as an analogue of the Teichmüller space in order to transfer (rarely straightforwardly) the geometric techniques of Thurston for the mapping class groups to  $\mathrm{Out}(\mathbf{F}_n)$ . It is also often thought of as an analogue of the symmetric space of lattices in Lie groups. It has dimension  $3n - 4$  and admits an  $\mathrm{Out}(\mathbf{F}_n)$ -equivariant deformation retraction onto a proper contractible simplicial complex, the spine of the outer-space, of dimension  $2n - 3$  which is thus exactly the virtual cohomological dimension of  $\mathrm{Out}(\mathbf{F}_n)$  [CV86, Corollary 6.1.3] (a lower bound being easy to obtain). An avatar of  $CV_n$  can be used to show that the virtual cohomological dimension of  $\mathrm{Aut}(\mathbf{F}_n)$  is  $2n - 2$  [Hat95, pp. 59-61].

### 1.1 Theorem

*The  $\ell^2$ -Betti numbers of the groups  $\mathrm{Out}(\mathbf{F}_n)$  and  $\mathrm{Aut}(\mathbf{F}_n)$  ( $n \geq 2$ ) do not vanish in degree equal to their virtual cohomological dimensions  $2n - 3$  (resp.  $2n - 2$ ):*

$$\beta_{2n-3}^{(2)}(\mathrm{Out}(\mathbf{F}_n)) > 0 \text{ and } \beta_{2n-2}^{(2)}(\mathrm{Aut}(\mathbf{F}_n)) > 0.$$

The rational homology of  $\mathrm{Out}(\mathbf{F}_n)$  is very intriguing. It was computed explicitly using computers by Ohashi [Oha08] up to  $n = 6$ . Then Bartholdi [Bar16] proved for  $n = 7$  that  $H_k(\mathrm{Out}(\mathbf{F}_7); \mathbb{Q})$  is trivial except for  $k = 0, 8, 11$ , when it is 1-dimensional. The non-zero classes for  $k = 8, 11$  were a total surprise, since they are not generated by Morita classes. Moreover, the rational homology of both  $\mathrm{GL}(n, \mathbb{Z})$  and  $\mathrm{MCG}(S_g)$  vanishes in the virtual cohomological dimension, and everyone expected the same would be true for  $\mathrm{Out}(\mathbf{F}_n)$ . In view of the Lück approximation [Lüc94], Theorem 1.1 implies that in degree equal to their vcd, the rational homology grows indeed linearly along towers. More precisely, these groups being residually finite [Bau63, Gro75], for every sequence of finite index normal subgroups  $(\Gamma_i)_i$  which is decreasing with trivial intersection in  $\mathrm{Out}(\mathbf{F}_n)$  (resp.  $\mathrm{Aut}(\mathbf{F}_n)$ ), then

$$\lim_{i \rightarrow \infty} \frac{\dim H_{2n-3}(\Gamma_i; \mathbb{Q})}{[\mathrm{Out}(\mathbf{F}_n) : \Gamma_i]} > 0, \quad \text{resp.} \quad \lim_{i \rightarrow \infty} \frac{\dim H_{2n-2}(\Gamma_i; \mathbb{Q})}{[\mathrm{Aut}(\mathbf{F}_n) : \Gamma_i]} > 0.$$

The mystery top-dimensional classes implicitly exhibited here for large finite index subgroups “come” from a poly-free subgroup  $\mathbf{F}_2 \times \mathbf{F}_2^{2n-4}$  of  $\text{Out}(\mathbf{F}_n)$ . In a work in progress with Laurent Bartholdi, we build on this remark to produce more explicit classes [BG20]. We also work on discovering other  $\ell^2$ -Betti numbers for  $\text{Out}(\mathbf{F}_n)$ . Results of Smillie and Vogtmann suggest that the (rational) Euler characteristic (equivalently the standard Euler characteristic of any torsion-free finite index subgroup) of  $\text{Out}(\mathbf{F}_n)$  should always be negative and this has been indeed proved very recently by Borinsky and Vogtmann [BV19]. A positive answer to the following question would deliver another demonstration.

### 1.2 Question

Do all  $\beta_k^{(2)}(\text{Out}(\mathbf{F}_n))$  for  $k \neq 2n - 3$  vanish?

Theorem 1.1 will be proved in section 3.

The canonical homomorphisms of  $\text{Aut}(\mathbf{F}_n)$  and  $\text{Out}(\mathbf{F}_n)$  to  $\text{GL}(n, \mathbb{Z})$  lead to the short exact sequences

$$\begin{aligned} 1 \rightarrow \mathcal{T}_n \rightarrow \text{Out}(\mathbf{F}_n) \xrightarrow{\phi_n} \text{GL}(n, \mathbb{Z}) \rightarrow 1. & \quad (1) \\ 1 \rightarrow \mathcal{K}_n \rightarrow \text{Aut}(\mathbf{F}_n) \rightarrow \text{GL}(n, \mathbb{Z}) \rightarrow 1. & \quad (2) \end{aligned}$$

The left hand side groups  $\mathcal{T}_n$  and  $\mathcal{K}_n$ , called the *Torelli groups*, have cohomological dimension  $2n - 4$  and  $2n - 3$  [BBM07].

### 1.3 Theorem

The  $\ell^2$ -Betti numbers of the Torelli groups  $\mathcal{T}_n$  and  $\mathcal{K}_n$  ( $n \geq 2$ ) do not vanish in degree equal to their virtual cohomological dimensions  $2n - 4$  (resp.  $2n - 3$ ):

$$\beta_{2n-4}^{(2)}(\mathcal{T}_n) \neq 0 \text{ and } \beta_{2n-3}^{(2)}(\mathcal{K}_n) \neq 0.$$

This is proved in Section 4.

## 1.2 Fundamental groups of compact manifolds

We now switch to another type of application. This one necessitates the full strength of the measured framework of Theorem 1.9 below. The (virtual) cohomological dimension of the fundamental group  $\pi_1(M)$  of a compact aspherical  $d$ -dimensional manifold  $M$  is clearly  $\leq d$ , with equality when  $M$  is closed. However, with Conley, Marks and Tucker-Drob we sharpened this in [CGMT] by showing that  $\Gamma = \pi_1(M)$  has ergodic dimension  $\leq d - 1$ . This means that with the help of an auxiliary probability measure preserving free  $\Gamma$ -action, one gains one on the top-dimension (see Section 7). And of course the smaller the ergodic dimension, the better the top-dimension. Thus the importance of Questions 7.1. So far, we obtain:

### 1.4 Theorem

Let  $\Gamma$  be the fundamental group of a compact connected **aspherical** manifold  $M$  of dimension  $d \geq 3$ . Let  $\Lambda \leq \Gamma$  be any subgroup. Then  $\beta_d^{(2)}(\Lambda) = 0$ . If moreover  $\beta_{d-1}^{(2)}(\Gamma) = 0$  then  $\beta_{d-1}^{(2)}(\Lambda) = 0$ .

Of course all the  $\ell^2$ -Betti numbers of  $\Lambda$  vanish in degree  $> d$ . Observe that the asphericity is a necessary condition in this statement since for instance  $\mathbf{F}_2^4$  is the fundamental group of some compact 4-manifold while its 4-th  $\ell^2$ -Betti number equals 1. Recall that the Singer Conjecture predicts that *the  $\ell^2$ -Betti numbers of a closed aspherical manifold  $M$  are concentrated in the middle dimension, i.e., if  $\beta_k^{(2)}(\pi_1(M)) > 0$  then  $2k = \text{the dimension of } M$* . The "moreover" assumption of Theorem 1.4 would then be satisfied automatically. The Singer Conjecture holds in particular for closed hyperbolic manifolds [Dod79]. Given the recent progress on 3-dimensional manifolds ([Per02, Per03], see also [KL08, BBB<sup>+</sup>10]), we obtain a more general statement:

### 1.5 Theorem

*Let  $\Gamma$  be the fundamental group of a connected compact 3-dimensional manifold. Every subgroup  $\Lambda \leq \Gamma$ , has vanishing  $\ell^2$ -Betti numbers in degree 2 and 3*

$$\beta_2^{(2)}(\Lambda) = \beta_3^{(2)}(\Lambda) = 0.$$

*In particular, if  $\Lambda$  is infinite then  $\chi^{(2)}(\Lambda) \in [-\infty, 0]$ .*

Here  $\chi^{(2)}(\Lambda)$  is the  $\ell^2$ -Euler characteristic of  $\Lambda$ . It coincides with the virtual Euler characteristic when the latter is defined. Observe that the 3-manifold in this theorem can have boundary, can be non-orientable and is not necessarily aspherical. While the vanishing in degree 3 for subgroups could have been expected, it is more surprising in degree 2. These results are proved in Section 7.

### 1.3 Ergodic dimension

Let's now switch to the third type of applications. The non-vanishing of the  $\ell^2$ -Betti number in some degree  $d$  for some subgroup  $\Lambda$  of a countable group  $\Gamma$  promotes clearly  $d$  to a lower bound of the virtual geometric dimension of  $\Gamma$ . Although the ergodic dimension is bounded above by the virtual geometric dimension,  $d$  is even a lower bound of the ergodic dimension of  $\Gamma$  [Gab02, Corollaire 3.17, Corollaire 5.9]. In case  $\beta_d^{(2)}(\Gamma) = 0$ , then  $d + 1$  is upgraded a lower bound:

### 1.6 Theorem

*If  $\Gamma$  is a countable discrete group of ergodic dimension (resp. virtual geometric dimension)  $\leq d$  and if  $\Lambda \leq \Gamma$  is any subgroup such that  $\beta_d^{(2)}(\Lambda) \neq 0$ , then  $\beta_d^{(2)}(\Gamma) \neq 0$  and the ergodic dimension of  $\Gamma$  is  $d$ .*

This statement is an immediate application of Theorem 5.1. It is worth recalling a result in this spirit: *If  $\Gamma$  is non-amenable and satisfies  $\beta_1^{(2)}(\Gamma) = 0$  then its ergodic dimension is  $\geq 2$  [Gab02, Prop. 6.10]*. The non-amenable assumption plays here the role of a subgroup with non-zero  $\beta_1^{(2)}$ . And this is not just an analogy since non-amenable groups contain, in a measurable sense, a free subgroup  $\mathbf{F}_2$  [GL09].

As a corollary, one computes the ergodic dimension of such groups as  $\mathbf{F}_2^d \times \mathbb{Z}$ : it is  $d + 1$ . As another example  $\text{Out}(\mathbf{F}_n) \times \mathbb{Z}^k$  (resp.  $\text{Aut}(\mathbf{F}_n) \times \mathbb{Z}^k$ ) has ergodic dimension  $2n - 2$  (resp.  $2n - 1$ ). More generally,

### 1.7 Corollary

*If  $\Lambda$  has ergodic dimension  $d$  and  $\beta_d^{(2)}(\Lambda) \neq 0$ , then for any infinite amenable group  $B$ , the direct sum  $\Lambda \times B$  has ergodic dimension  $d + 1$ .*

All the  $\ell^2$ -Betti numbers of  $\Lambda \times B$  equal 0. Observe that the condition  $\beta_d^{(2)}(\Lambda) \neq 0$  is necessary since for instance  $(\Lambda \times B) \times B = \Lambda \times (B \times B)$  has also ergodic dimension  $d + 1$ .

## 1.4 A top-dimensional $L^2$ -Betti number result

The different statements announced above use at some point variants of the general trick (Theorem 5.1) involving a probability measure preserving standard equivalence relation  $\mathcal{R}$  with countable classes (**pmp equivalence relation** for short), a standard sub-relation  $\mathcal{S}$  and a simplicial discrete  $\mathcal{R}$ -complex together with their  $L^2$ -Betti numbers<sup>1</sup>; see sections 5 and 6 where the notions are recalled. The specialization of Theorem 5.1 to proper actions (simplicial actions with finite stabilizers) which is appropriate for geometric dimension will be given its own proof in section 2 for the reader's convenience and as a warm-up to section 5. Let's denote by  $\beta_d^{(2)}(\Gamma \curvearrowright L)$  the  $d$ -th  $\ell^2$ -Betti number of the action of  $\Gamma$  on  $L$ , also denoted by countless different manners in the literature such as  $\beta_d(L, \Gamma)$ ,  $\beta_d^{(2)}(L, \Gamma)$ ,  $\beta_d^{(2)}(L : \Gamma)$  or  $b_{(2)}^d(L : \Gamma)$ .

### 1.8 Theorem (Proper actions version)

*Let  $\Gamma$  be a countable discrete group and  $\Lambda \leq \Gamma$  be a subgroup. If  $\Gamma \curvearrowright L$  is a proper action on a  $d$ -dimensional simplicial complex such that the restriction to  $\Lambda$  satisfies  $\beta_d^{(2)}(\Lambda \curvearrowright L) \neq 0$ , then  $\beta_d^{(2)}(\Gamma \curvearrowright L) \neq 0$ .*

Specializing Theorem 5.1 to a **contractible**  $\mathcal{R}$ -complex, one obtains a statement involving the  $L^2$ -Betti numbers of the pmp equivalence relation [Gab02, Théorème 3.13, Définition 3.14] and of its sub-relations. The minimal dimension of such a contractible complex defines the **geometric dimension** of  $\mathcal{R}$  (see the proof of Theorem 1.9).

### 1.9 Theorem (Geometric dimension of pmp equivalence relation)

*If  $\mathcal{R}$  is a pmp equivalence relation on the standard space  $(X, \mu)$  of geometric dimension  $\leq d$  for which the  $L^2$ -Betti number in degree  $d$  vanishes ( $\beta_d^{(2)}(\mathcal{R}, \mu) = 0$ ) then every standard sub-equivalence relation  $\mathcal{S} \leq \mathcal{R}$  satisfies  $\beta_d^{(2)}(\mathcal{S}, \mu) = 0$ .*

<sup>1</sup>Observe the debatable use introduced in [Gab02] of capital letter  $L^2$  for equivalence relations versus the cursive lowercase  $\ell^2$  for groups.

## Acknowledgments

The authors would like to thank Michel Boileau and Jean-Pierre Otal for enlightening discussions on 3-dimensional manifolds. They also thank Clara Löh and Karen Vogtmann for their valuable comments on a preliminary version. Camille Noûs embodies the collective's constitutive role in the creation of scientific knowledge, which is always part of a collegial organization and the legacy of previous work.

This work is supported by the ANR project GAMME (ANR-14-CE25-0004), by the CNRS and by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

## 2 Proof of Theorem 1.8 on proper simplicial actions

Recall [CG86, (2.8) p. 198] (see also [Gab02, Section 1.2]) that for a proper non-cocompact action  $\Gamma \curvearrowright L$ , the  $\ell^2$ -Betti numbers are defined as follows: Consider any increasing exhausting sequence  $(L_i)_{i \in \mathbb{N}}$  of cocompact  $\Gamma$ -invariant subcomplexes of  $L$ . For each dimension  $k$ , for each  $i \leq j$ , the inclusion  $L_i \subset L_j$  induces a  $\Gamma$ -equivariant map  $\bar{H}_k^{(2)}(L_i) \rightarrow \bar{H}_k^{(2)}(L_j)$  between the reduced  $\ell^2$ -homology spaces. The von Neumann  $\Gamma$ -dimension of the closure of the image  $\text{im} \left( \bar{H}_k^{(2)}(L_i) \rightarrow \bar{H}_k^{(2)}(L_j) \right)$  is decreasing in  $j$  and increasing in  $i$ . The  $k$ -th  $\ell^2$ -Betti number of the action is defined as:

$$\beta_k^{(2)}(\Gamma \curvearrowright L) := \lim_{i \rightarrow \infty} \nearrow \lim_{\substack{j \rightarrow \infty \\ j \geq i}} \searrow \dim_{\Gamma} \left( \overline{\text{im} \left( \bar{H}_k^{(2)}(L_i) \rightarrow \bar{H}_k^{(2)}(L_j) \right)} \right).$$

This is easily seen to be independent of the choice of the exhausting sequence. The  $k$ -th  $\ell^2$ -Betti number of the group  $\Gamma$  is defined as the  $k$ -th  $\ell^2$ -Betti number  $\beta_k^{(2)}(\Gamma \curvearrowright L)$  for any proper contractible (or even only  $k$ -contractible)  $\Gamma$ -complex  $L$  and this is independent of the choice of  $L$ .

The key observation is that for any  $d$ -dimensional complex  $M$  the reduced  $\ell^2$ -homology, defined from the  $\ell^2$ -chain complex

$$0 \longleftarrow C_0^{(2)}(M) \xleftarrow{\partial_1} C_1^{(2)}(M) \cdots \longleftarrow C_{d-1}^{(2)}(M) \xleftarrow{\partial_d} C_d^{(2)}(M) \longleftarrow 0$$

boils down in dimension  $d$  to the kernel of the boundary map

$$\bar{H}_d^{(2)}(M) = H_d^{(2)}(M) = \ker \partial_d^M := \ker \left( C_d^{(2)}(M) \xrightarrow{\partial_d} C_{d-1}^{(2)}(M) \right).$$

Of course, for the boundary operators to be bounded,  $M$  needs here to have **bounded geometry**, i.e., it admits a uniform bound on the **valencies** (the number of simplices a vertex belongs to).

Since the injective maps induced on  $\ell^2$ -chains by the inclusions  $L_i \subset L_j$  commute with boundaries, it follows that

$$\beta_d^{(2)}(\Gamma \curvearrowright L) := \lim_{i \rightarrow \infty} \lim_{\substack{j \rightarrow \infty \\ j \geq i}} \dim_{\Gamma} \operatorname{im} \left( \ker \partial_d^{L_i} \hookrightarrow \ker \partial_d^{L_j} \right) \quad (3)$$

$$= \lim_{i \rightarrow \infty} \nearrow \dim_{\Gamma} \ker \partial_d^{L_i}. \quad (4)$$

Consider, for the restricted action  $\Lambda \curvearrowright L$ , an increasing exhausting sequence  $(K_i)_{i \in \mathbb{N}}$  of cocompact  $\Lambda$ -invariant subcomplexes of  $L$ . By assumption, for  $i$  large enough,  $\dim_{\Lambda} \ker \partial_d^{K_i} \neq 0$ , so that  $\ker \partial_d^{K_i} \neq \{0\}$ . Let  $L_i := \cup_{\gamma \in \Gamma} \gamma K_i$  be the  $\Gamma$ -saturation of the  $K_i$ . It is  $\Gamma$ -invariant and  $\Gamma$ -cocompact. Again by commutation with boundaries of the injective maps induced on  $\ell^2$ -chains by the inclusion  $K_i \subset L_i$ , we also have  $\ker \partial_d^{L_i} \neq \{0\}$ . The  $\Gamma$ -saturation  $L_j$  of the  $K_j$  give an increasing exhausting sequence  $(L_j)_{j \in \mathbb{N}}$  of cocompact  $\Gamma$ -invariant subcomplexes of  $L$ . In view of formula (4) and since the von Neumann dimension is faithful, we have  $\beta_d^{(2)}(\Gamma \curvearrowright L) \neq 0$ .  $\blacksquare$

### 3 Proof of Theorem 1.1 that $\beta_{2n-3}^{(2)}(\operatorname{Out}(\mathbf{F}_n)) > 0$

We begin by recalling what is known about the  $\ell^2$ -Betti numbers of  $\operatorname{Aut}(\mathbf{F}_n)$  and  $\operatorname{Out}(\mathbf{F}_n)$ . The groups fit into a canonical short exact sequence

$$1 \rightarrow \mathbf{F}_n \rightarrow \operatorname{Aut}(\mathbf{F}_n) \xrightarrow{\theta_n} \operatorname{Out}(\mathbf{F}_n) \rightarrow 1. \quad (5)$$

When  $n = 2$ , the group  $\operatorname{Out}(\mathbf{F}_2) \simeq \operatorname{GL}(2, \mathbb{Z})$  admits a single non-vanishing  $\ell^2$ -Betti number, namely  $\beta_1^{(2)}(\operatorname{GL}(2, \mathbb{Z})) = 1/24$  in degree 1, exactly the middle dimension of its associated symmetric space and also the virtual geometric dimension of  $\operatorname{GL}(2, \mathbb{Z})$ . It follows that  $\operatorname{Aut}(\mathbf{F}_2)$  has an index 24 subgroup isomorphic with  $\mathbf{F}_2 \times \mathbf{F}_2$  so that its  $\ell^2$ -Betti numbers vanish except  $\beta_2^{(2)}(\operatorname{Aut}(\mathbf{F}_2)) = 1/24$  (see for instance Proposition 3.1). When  $n \geq 3$ , the kernel  $\mathcal{T}_n$  of  $\phi_n$  (sequence (1)) is a finitely generated infinite normal subgroup of infinite index by [Nie24, Mag35] (and clearly the same holds for the kernel  $\mathbf{F}_n$  of  $\theta_n$  (sequence (5)). It follows that  $\beta_1^{(2)}(\operatorname{Out}(\mathbf{F}_n)) = \beta_1^{(2)}(\operatorname{Aut}(\mathbf{F}_n)) = 0$  by using for instance [Lüc98, Theorem 3.3 (5)] or [Gab02, Théorème 6.8]: *The middle group  $H$  of a short exact sequence  $1 \rightarrow N \rightarrow H \rightarrow Q \rightarrow 1$  of infinite groups has  $\beta_1^{(2)}(H) = 0$  as soon as  $\beta_1^{(2)}(N) < \infty$ .* Remark that this is another instance where the strength of the  $L^2$  orbit equivalence theory allows one to obtain a more general result [Gab02, Théorème 6.8] in comparison with [Lüc98, Theorem 3.3 (5)] where a parasitic assumption remains on  $Q$  (containing an infinite order element or arbitrarily large finite subgroups) which always holds in a measurable sense. In higher degrees the same paradigm is used in [ST10, Corollary 1.8] (see the proof of Proposition 3.1). With Abért we proved that  $\beta_2^{(2)}(\operatorname{Out}(\mathbf{F}_n)) = 0$  for  $n \geq 5$  [AG20].



The reason for the non-vanishing of  $\beta_{2n-3}^{(2)}(\text{Out}(\mathbf{F}_n))$  and  $\beta_{2n-2}^{(2)}(\text{Aut}(\mathbf{F}_n))$  boils now down to the existence of subgroups of the form  $\mathbf{F}_2 \times \mathbf{F}_2^{2n-4}$  (resp.  $(\mathbf{F}_2 \times \mathbf{F}_2^{2n-4}) \times \mathbf{F}_n$ ), to the use of Proposition 3.1 and to an application of Theorem 1.8 applied to  $L$  = the spine of the Culler-Vogtmann space  $CV_n$  which is contractible, has dimension  $2n - 3$ , and is equipped with a proper action of  $\text{Out}(\mathbf{F}_n)$  [CV86] (and its avatar for  $\text{Aut}(\mathbf{F}_n)$ ).

Let  $(x_1, x_2, \dots, x_n)$  be a free base of the free group  $\mathbf{F}_n$ . Choose a rank 2 free subgroup  $V \leq \text{Out}(\mathbf{F}(x_1, x_2)) \simeq \text{GL}(2, \mathbb{Z})$  and pick a section  $U \leq \text{Aut}(\mathbf{F}(x_1, x_2))$  of it under  $\theta_2$  in the short exact sequence (5). The family of automorphisms  $\Phi(x_1) = \alpha(x_1), \Phi(x_2) = \alpha(x_2), \Phi(x_j) = l_j x_j r_j^{-1}$  ( $j \neq 1, 2$ ) for all choices of  $(\alpha, l_3, r_3, l_4, r_4, \dots, l_n, r_n) \in U \times \mathbf{F}(x_1, x_2)^{2n-4}$  defines a subgroup of  $\text{Aut}(\mathbf{F}_n)$  which is isomorphic to  $U \times \mathbf{F}_2^{2(n-2)} = \mathbf{F}_2 \times \mathbf{F}_2^{2(n-2)}$  and descends injectively to  $\Lambda_n \leq \text{Out}(\mathbf{F}_n)$  under  $\theta_n$ . This reproduces an argument from [BKK02]. Its pull-back  $\widetilde{\Lambda}_n := \theta_n^{-1}(\Lambda_n)$  is thus isomorphic to  $(U \times \mathbf{F}_2^{2(n-2)}) \times \mathbf{F}_n \simeq \mathbf{F}_2 \times (\mathbf{F}_2^{2(n-2)} \times \mathbf{F}_n)$  (the restriction of  $\theta_n$  to  $\widetilde{\Lambda}_n$  admits a section, thus the splitting). By Proposition 3.1, these poly-free groups satisfy  $\beta_{2n-3}^{(2)}(\Lambda_n) = 1$  and  $\beta_{2n-2}^{(2)}(\widetilde{\Lambda}_n) = n - 1$ . Then apply Theorem 1.6.  $\blacksquare$

### 3.1 Proposition (Poly-free groups)

Consider a group  $G = G_n$  obtained by a finite sequence  $(G_i)_{i=1}^n$  of extensions

$$1 \rightarrow G_i \rightarrow G_{i+1} \rightarrow Q_{i+1} \rightarrow 1, \quad (6)$$

where  $G_1$  and all the  $Q_i$  are finitely generated, non-cyclic free groups.

Then for all  $j$  the  $\beta_j^{(2)}(G_n)$  vanish except “in top-dimension”

$$\beta_n^{(2)}(G_n) = \beta_1^{(2)}(G_1) \prod_{i=2}^n \beta_1^{(2)}(Q_i) = (-1)^{n+1} \chi(G_n).$$

Proof: The statement is obtained by induction from the following:

1. the general results on cohomological/geometric dimension for extensions imply that the geometric dimension of  $G_n$  is  $\leq n$  [Bro82, Chapter VIII.2];
2. a result [Lüc98, Theorem 3.3 (5)], [ST10, Corollary 1.8] alluded to above: *Let  $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$  be a short exact sequence of infinite groups. If  $\beta_k^{(2)}(N) = 0$  for  $k = 0, 1, \dots, d - 1$  and  $\beta_d^{(2)}(N) < \infty$ , then  $\beta_k^{(2)}(\Gamma) = 0$  for  $k = 0, 1, \dots, d$ ;*
3. the multiplicativity of the Euler characteristic under extensions [Bro82, Chapter IX.7] and the coincidence of Euler and  $\ell^2$ -Euler characteristics [CG86, Proposition 0.4]:

$$\begin{aligned} \chi^{(2)}(G_{i+1}) &= \chi(G_i) \cdot \chi(Q_{i+1}) = (-1)^i \beta_i^{(2)}(G_i) (-1) \beta_1^{(2)}(Q_{i+1}) \\ &= \sum_{j=0}^{\infty} (-1)^j \beta_j^{(2)}(G_{i+1}) = (-1)^{i+1} \beta_{i+1}^{(2)}(G_{i+1}). \end{aligned}$$

$\blacksquare$

## 4 Proof of Theorem 1.3 for the Torelli subgroups

We continue with the notation of the previous section. Pick two elements that generate a free subgroup of rank 2 in the intersection of the commutator subgroup  $[\mathbf{F}_n, \mathbf{F}_n]$  with  $\mathbf{F}(x_1, x_2)$ , for instance  $u := [x_1, x_2]$  and  $v := [x_1^{-1}, x_2^{-1}]$ . The family of automorphisms  $\Phi(x_1) = x_1, \Phi(x_2) = x_2, \Phi(x_j) = l_j x_j r_j^{-1}$  ( $j \neq 1, 2$ ) for all choices of  $(l_3, r_3, l_4, r_4, \dots, l_n, r_n) \in \mathbf{F}(u, v)^{2n-4}$  defines a subgroup of  $\text{Aut}(\mathbf{F}_n)$  which is isomorphic to  $\mathbf{F}_2^{2n-4}$  and descends injectively under  $\theta_n$  (of the exact sequence (5)) to  $\Delta_n \leq \mathcal{T}_n \leq \text{Out}(\mathbf{F}_n)$ . Its pullback  $\widetilde{\Delta}_n := \theta_n^{-1}(\Delta_n)$  is thus a subgroup of  $\mathcal{K}_n$  isomorphic to  $\mathbf{F}_2^{2n-4} \times \mathbf{F}_n$ . Proposition 3.1 gives  $\beta_{2n-4}^{(2)}(\Delta_n) = 1$  and  $\beta_{2n-3}^{(2)}(\widetilde{\Delta}_n) = n - 1$ . The group  $\mathcal{T}_n$  has cohomological dimension  $2n - 4$  [BBM07]. By its general behavior under exact sequences and  $1 \rightarrow \mathbf{F}_n \rightarrow \mathcal{K}_n \rightarrow \mathcal{T}_n \rightarrow 1$ , the cohomological dimension of  $\mathcal{K}_n$  is  $2n - 3$ . Then apply Theorem 1.6.  $\blacksquare$

## 5 Proof of Theorem 1.9, measured theoretic version

Let's consider now the measured theoretic version below of Theorem 1.8. Theorem 1.9 will follow directly. We assume some familiarity with the foundations [Gab02] and refer to this for some background.

### 5.1 Theorem (Top-dimension $\beta_d^{(2)}$ , discrete $\mathcal{R}$ -complex version)

Let  $(X, \mu)$  be a standard probability measure space and let  $\mathcal{R}$  be a pmp equivalence relation. Assume  $\Sigma$  is  $d$ -dimensional simplicial discrete  $\mathcal{R}$ -complex with vanishing top-dimensional  $L^2$ -Betti number,  $\beta_d^{(2)}(\Sigma, \mathcal{R}, \mu) = 0$ . For any sub-equivalence relation  $\mathcal{S} \leq \mathcal{R}$  the  $L^2$ -Betti number of  $\Sigma$  seen as a simplicial discrete  $\mathcal{S}$ -complex also vanishes in degree  $d$ ,  $\beta_d^{(2)}(\Sigma, \mathcal{S}, \mu) = 0$ .

Proof of Th. 5.1: Recall [Dix69, Gab02] that a measurable bundle  $x \mapsto \Sigma_x$  over  $(X, \mu)$  of simplicial complexes with uniform bounded geometry delivers an integrated field of  $\ell^2$ -chain complexes  $C_k^{(2)}(\Sigma) = \int_X^\oplus C_k^{(2)}(\Sigma_x) d\mu(x)$ , and that the field of boundary operators can be integrated into a continuous operator  $\partial_k = \int_X^\oplus (\partial_{k,x} : C_k^{(2)}(\Sigma_x) \rightarrow C_{k-1}^{(2)}(\Sigma_x)) d\mu(x)$ .

By commutation of the diagram involving the boundary operators and the injective operators induced by inclusion, one gets:

### 5.2 Claim

Let  $\Theta$  and  $\Omega$  be measurable bundles  $x \mapsto \Theta_x$  and  $x \mapsto \Omega_x$  over  $(X, \mu)$  of simplicial complexes both with a bounded geometry. If  $\Theta \subset \Omega$  then

$$\ker \left( \partial_k : C_d^{(2)}(\Theta) \rightarrow C_{k-1}^{(2)}(\Theta) \right) \hookrightarrow \ker \left( \partial_k : C_d^{(2)}(\Omega) \rightarrow C_{k-1}^{(2)}(\Omega) \right). \quad (7)$$

Recall from [Gab02, Définition 2.6 and Définition 2.7]) that a simplicial discrete (or smooth)  $d$ -dimensional  $\mathcal{R}$ -complex  $\Sigma$  is an  $\mathcal{R}$ -equivariant measurable bundle  $x \mapsto \Sigma_x$  of simplicial complexes over  $(X, \mu)$

- that is discrete (the  $\mathcal{R}$ -equivariant field of 0-dimensional cells  $\Sigma^{(0)} : x \mapsto \Sigma_x^{(0)}$  admits a Borel fundamental domain); and
- such that ( $\mu$ -almost) every fiber  $\Sigma_x$  is  $\leq d$ -dimensional and  $\Sigma_x$  is  $d$ -dimensional for a non-null set of  $x \in X$ .

Recall that such an  $\mathcal{R}$ -complex is called **uniformly locally bounded (ULB)** if  $\Sigma^{(0)}$  admits a finite measure fundamental domain (for its natural fibered measure) and if it admits a uniform bound on the valency of ( $\mu$ -almost) every vertex  $v \in \Sigma^{(0)}$  (uniform bounded geometry). Recall the definition of the  $L^2$ -Betti numbers of the  $\mathcal{R}$ -complex  $\Sigma$  [Gab02, Définition 3.7 and Proposition 3.9]: Choose any sequence  $(\Sigma_i)_i$  of ULB  $\mathcal{R}$ -invariant subcomplexes of  $\Sigma$  (given by the sequence of bundles  $x \mapsto \Sigma_{i,x}$ ) which is increasing ( $\Sigma_i \subset \Sigma_{i+1}$ ) and exhausting ( $\cup_i \Sigma_i = \Sigma$ ). Let's call such a sequence a **good  $\mathcal{R}$ -exhaustion** of  $\Sigma$ . Let  $\mathcal{M}(\mathcal{R})$  be the von Neumann algebra of  $\mathcal{R}$ . The continuous boundary operators  $\partial_k$

$$0 \xleftarrow{\partial_0} C_0^{(2)}(\Sigma_i) \xleftarrow{\partial_1} C_1^{(2)}(\Sigma_i) \xleftarrow{\partial_2} \dots \xleftarrow{\partial_k} C_k^{(2)}(\Sigma_i) \xleftarrow{\partial_{k+1}} \dots,$$

are  $\mathcal{M}(\mathcal{R})$ -equivariant between the Hilbert  $\mathcal{M}(\mathcal{R})$ -modules  $C_k^{(2)}(\Sigma_i)$ .

The reduced  $L^2$ -homology of  $\Sigma_i$  is defined as expected as the Hilbert  $\mathcal{M}(\mathcal{R})$ -module quotient of the kernel by the closure of the image:

$$\bar{H}_k^{(2)}(\Sigma_i) = \frac{\ker \left( \partial_k : C_k^{(2)}(\Sigma_i) \rightarrow C_{k-1}^{(2)}(\Sigma_i) \right)}{\overline{\text{im}} \partial_{k+1} \left( \partial_{k+1} : C_{k+1}^{(2)}(\Sigma_i) \rightarrow C_k^{(2)}(\Sigma_i) \right)}.$$

The inclusions  $\Sigma_i \subset \Sigma_j$  (for  $i \leq j$ ) induce Hilbert  $\mathcal{M}(\mathcal{R})$ -module operators  $C_k^{(2)}(\Sigma_i) \rightarrow C_k^{(2)}(\Sigma_j)$  that descend to Hilbert  $\mathcal{M}(\mathcal{R})$ -module operators  $\bar{H}_k^{(2)}(\Sigma_i) \xrightarrow{J_{i,j}} \bar{H}_k^{(2)}(\Sigma_j)$ . The  $k$ -th  $L^2$ -Betti number is the double limit of the von Neumann dimension of the closure of the image of these maps:

$$\beta_k^{(2)}(\Sigma, \mathcal{R}, \mu) = \lim_{i \rightarrow \infty} \nearrow \lim_{\substack{j \rightarrow \infty \\ j \geq i}} \searrow \dim_{\mathcal{M}(\mathcal{R})} \overline{\text{im}} \left( \bar{H}_k^{(2)}(\Sigma_i) \xrightarrow{J_{i,j}} \bar{H}_k^{(2)}(\Sigma_j) \right).$$

### 5.3 Claim

*In the particular case when  $k = d$  is the top-dimension of  $\Sigma$  and  $(\Sigma_i)_i$  is a good  $\mathcal{R}$ -exhaustion of  $\Sigma$ , then we have the equivalence:  $\beta_d^{(2)}(\Sigma, \mathcal{R}, \mu) > 0$  if and only if  $\ker \left( \partial_d : C_d^{(2)}(\Sigma_i) \rightarrow C_{d-1}^{(2)}(\Sigma_i) \right) \neq \{0\}$  for a large enough  $i$ .*

Proof: Since  $C_{d+1}^{(2)}(\Sigma_i) = \{0\}$  then  $\bar{H}_d^{(2)}(\Sigma_i) = \ker \left( \partial_d : C_d^{(2)}(\Sigma_i) \rightarrow C_{d-1}^{(2)}(\Sigma_i) \right)$  for every  $i$ . Thus by Claim 5.2

$$\overline{\text{im}} \bar{H}_d^{(2)}(\Sigma_i) \xrightarrow{J_{i,j}} \bar{H}_d^{(2)}(\Sigma_j) = \ker \left( \partial_d : C_d^{(2)}(\Sigma_i) \rightarrow C_{d-1}^{(2)}(\Sigma_i) \right).$$

Then  $\beta_d^{(2)}(\Sigma, \mathcal{R}, \mu) = \lim_{i \rightarrow \infty} \nearrow \dim_{\mathcal{M}(\mathcal{R})} \ker \left( \partial_d : C_d^{(2)}(\Sigma_i) \rightarrow C_{d-1}^{(2)}(\Sigma_i) \right)$ . The claim 5.3 follows by faithfulness: the property that the von Neumann dimension is non zero if and only if the Hilbert module is non zero. ■

The  $d$ -dimensional simplicial discrete  $\mathcal{R}$ -complex  $\Sigma$  is also an  $\mathcal{S}$ -complex with the same properties. Let  $(\Omega_i)_i$  be a good  $\mathcal{R}$ -exhaustion of  $\Sigma$  and let  $(\Theta_i)_i$  be a similar good  $\mathcal{S}$ -exhaustion of  $\Sigma$  such that  $\Theta_i \subset \Omega_i$  (one can for instance consider the intersection of a good  $\mathcal{S}$ -exhaustion of  $\Sigma$  with the good  $\mathcal{R}$ -exhaustion  $(\Omega_i)_i$ ). Assume by contraposition the  $\beta_d^{(2)}(\Sigma, \mathcal{S}, \mu) > 0$ . It follows from Claim 5.3 that  $\ker \left( \partial_d : C_d^{(2)}(\Theta_i) \rightarrow C_{d-1}^{(2)}(\Theta_i) \right) \neq \{0\}$  for  $\Theta_i$  and a large enough  $i$ . Then the same holds,  $\ker \left( \partial_d : C_d^{(2)}(\Omega_i) \rightarrow C_{d-1}^{(2)}(\Omega_i) \right) \neq \{0\}$ , for  $\Omega_i$  by Claim 5.2. It follows that  $\beta_d^{(2)}(\Sigma, \mathcal{R}, \mu) > 0$  by Claim 5.3. This completes the proof of Th. 5.1. ■

As for the proof of Theorem 1.9, recall from [Gab02, Définition 3.18] that  $\mathcal{R}$  has *geometric dimension*  $\leq d$  if it admits a contractible  $d$ -dimensional simplicial discrete  $\mathcal{R}$ -complex  $\Sigma$  (see [Gab02, Définition 2.6 and Définition 2.7]). Recall also the definition of the  $L^2$ -Betti numbers of  $\mathcal{R}$  [Gab02, Définition 3.14, Théorème 3.13]:  $\beta_k^{(2)}(\mathcal{R}, \mu) := \beta_k^{(2)}(\Sigma, \mathcal{R}, \mu)$  where  $\Sigma$  is any contractible simplicial discrete  $\mathcal{R}$ -complex. A contractible  $d$ -dimensional simplicial discrete  $\mathcal{R}$ -complex  $\Sigma$  is also an  $\mathcal{S}$ -complex with the same properties, so that it can be used to compute the  $L^2$ -Betti numbers of  $\mathcal{S}$ . Thus Theorem 1.9 is a specialisation of Theorem 5.1 when  $\Sigma$  is contractible. ■

## 6 Proof of Theorem 1.6 and Corollary 1.7

Recall from [Gab02, Définition 6.4] that a group  $\Gamma$  has *ergodic dimension*  $\leq d$  if it admits a probability measure preserving **free** action  $\Gamma \curvearrowright^\alpha (X, \mu)$  on some standard space such that the orbit equivalence relation  $\mathcal{R}_\alpha$  has geometric dimension  $\leq d$ . Equivalently, it admits a  $\Gamma$ -equivariant bundle  $\Sigma : x \mapsto \Sigma_x$  over  $(X, \mu)$  of contractible simplicial complexes of dimension  $\leq d$  which is measurable and discrete. See [Gab02, Gab20] for more information on ergodic dimension.

**Proof of Theorem 1.6:** Assume  $\Gamma$  has ergodic dimension  $\leq d$  and that this is witnessed by  $\Gamma \curvearrowright^\alpha (X, \mu)$  and  $\Sigma$ , a free pmp  $\Gamma$ -action and a contractible  $d$ -dimensional simplicial discrete  $\mathcal{R}_\alpha$ -complex. The restriction  $\omega$  of the action  $\alpha$  to  $\Lambda$  being also free, the complex  $\Sigma$  computes both the  $\ell^2$ -Betti numbers of  $\Gamma$  and of  $\Lambda$ ; more precisely,  $\beta_k^{(2)}(\Sigma, \mathcal{R}_\alpha, \mu) = \beta_k^{(2)}(\Gamma)$  and (considering  $\Sigma$  as an  $\mathcal{R}_\omega$ -complex)  $\beta_k^{(2)}(\Sigma, \mathcal{R}_\omega, \mu) = \beta_k^{(2)}(\Lambda)$  [Gab02, Corollaire 3.16]. The  $L^2$ -Betti numbers of  $\Sigma$  vanish strictly above its dimension  $d$ . If moreover  $\beta_d^{(2)}(\Sigma, \mathcal{R}_\alpha, \mu) = \beta_d^{(2)}(\Gamma) = 0$ , then applying Theorem 5.1 gives  $\beta_d^{(2)}(\Sigma, \mathcal{R}_\omega, \mu) = \beta_d^{(2)}(\Lambda) = 0$ . ■

**Proof of Corollary 1.7:** By [CG86, Theorem 0.2 and Proposition 2.7], all the  $\ell^2$ -Betti numbers of  $\Gamma = \Lambda \times B$  equal 0, in particular  $\beta_{d+1}^{(2)}(\Gamma) = 0$ . By

Theorem 1.6, the ergodic dimension of  $\Gamma$  is  $\geq d + 1$ . On the other hand, the ergodic dimension of  $B$  is 1 by Ornstein-Weiss [OW80] and the ergodic dimension of a direct sum is bounded above by the sum of the ergodic dimensions of the factors. ■

## 7 Proof of Theorems 1.4 and 1.5 on manifolds

Proof of Theorem 1.4: By [CGMT] the fundamental group  $\Gamma = \pi_1(M)$  of a compact connected **aspherical** manifold  $M$  of dimension  $d \geq 3$  has ergodic dimension  $\leq d - 1$ . Then apply Theorem 1.6. ■

Any improvement on the ergodic dimension of  $\pi_1(M)$  would produce in return a corresponding improvement in Theorem 1.4.

### 7.1 Question

*What is the ergodic dimension of the fundamental group of a closed connected **hyperbolic**  $d$ -manifold  $M$  of dimension? Is it  $d/2$  when  $d$  is even and  $(d+1)/2$  when  $d$  is odd? More generally, is the ergodic dimension of the fundamental group of a closed connected aspherical manifold of dimension  $d$  bounded above by  $(d+1)/2$ ?*

Proof of Theorem 1.5: Let  $\Gamma$  be the fundamental group of a connected compact 3-dimensional manifold  $M$ . If  $M$  is non-orientable, then the fundamental group of its orientation covering  $\tilde{M} \rightarrow M$  has index 2 in  $\pi_1(M)$  so that  $\tilde{\Lambda} := \Lambda \cap \pi_1(\tilde{M})$  has index  $i = 1$  or  $i = 2$  in  $\Lambda$  and  $\beta_k^{(2)}(\tilde{\Lambda}) = [\Lambda : \tilde{\Lambda}] \beta_k^{(2)}(\Lambda)$  for every  $k$ . Thus, without loss of generality, one can assume that  $M$  is orientable.

Recall that a compact 3-manifold  $M$  is **prime** when every connected sum decomposition  $M = N_1 \# N_2$  is trivial in the sense that either  $N_1$  or  $N_2 \simeq \mathbb{S}^2$ . Except for  $\mathbb{S}^1 \times \mathbb{S}^2$ , the orientable prime manifolds  $M$  are **irreducible**: once the potential boundary spheres have been filled in with 3-balls (which produces  $M'$  and does not change the fundamental group), every embedded 2-sphere bounds a 3-ball.

### 7.2 Theorem (Kneser-Milnor [Kne29, Mil62])

*Let  $M^3$  be a connected compact orientable manifold. It can be decomposed as a connected sum (along separating spheres)  $M = M_1 \# M_2 \# \dots \# M_k$  whose pieces  $M_j$  are prime; i.e., either are*

$$\left\{ \begin{array}{l} - \text{copies of } \mathbb{S}^1 \times \mathbb{S}^2 \text{ (thus } \pi_1(M_j) \simeq \mathbb{Z} \text{), or} \\ - \text{irreducible manifolds } \left\{ \begin{array}{l} - \text{that either have finite } \pi_1, \text{ or} \\ - \pi_1(M_j) \text{ is the fundamental group of an} \\ \text{aspherical orientable 3-manifold } M'_j. \end{array} \right. \end{array} \right.$$

It follows that the fundamental group of  $M$  decomposes as a free product  $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \dots * \pi_1(M_k)$  of copies of  $\mathbb{Z}$ , of finite groups and of  $\pi_1$  of aspherical 3-manifolds; for these,  $\beta_k^{(2)}(\pi_1(M_j) \curvearrowright \tilde{M}_j) = \beta_k^{(2)}(\pi_1(M_j))$ .

The second  $\ell^2$ -Betti number of the fundamental group of a compact connected orientable irreducible non-exceptional aspherical 3-manifold vanishes [LL95, Theorem 0.1]:  $\beta_2^{(2)}(\pi_1(M_j) \curvearrowright \widetilde{M}_j) = 0$ . By the work of Perelman and his proof of Thurston’s geometrization conjecture [Per02, Per03] (see also [KL08, BBB+10]) exceptional manifolds do not exist.

It follows (by the  $\ell^2$ -version of Mayer-Vietoris [CG86]) that  $\beta_2^{(2)}(\pi_1(M)) = 0$  for every connected compact 3-manifold  $M$ .

The above free product decomposition implies that  $\pi_1(M)$  has virtual geometric dimension  $\leq 3$ . Moreover by [CGMT],  $\pi_1(M)$  has ergodic dimension  $\leq 2$ . Theorem 1.5 then follows from Theorem 1.6. When  $\Lambda$  is infinite,  $\beta_0^{(2)}(\Lambda) = 0$  and  $\chi^{(2)}(\Lambda) = -\beta_1^{(2)}(\Lambda) \in [-\infty, 0]$ .

We now give an alternative argument avoiding the use of the unpublished article [CGMT]. If  $M$  is an aspherical orientable 3-manifold with boundary, then its fundamental group has geometric dimension  $\leq 2$ . Otherwise, by Thurston’s geometrization conjecture (now established), an aspherical orientable 3-manifold can be decomposed along a disjoint union of embedded tori into pieces which carry a geometric structure. This delivers a further decomposition of its fundamental group as a graph of groups with edge groups isomorphic to  $\mathbb{Z}^2$ . The fundamental group  $\pi_1(M)$  eventually follows decomposed as a graph of groups with edge groups isomorphic to either  $\{1\}$  or  $\mathbb{Z}^2$ . The vertex groups  $\Gamma_i$  have ergodic dimension  $\leq 2$ . More precisely, the  $\Gamma_i$  are either

- amenable: they have ergodic dimension  $\leq 1$  by [OW80]; or
- a cocompact lattice in the isometry group of one of the Thurston’s geometries: when it is non-amenable,  $\Gamma_i$  is measure equivalent with some non-cocompact lattice  $\Gamma'_i$  in the isometry group of  $\mathbb{H}^3, \mathbb{H}^2 \times \mathbb{R}$  or  $\widetilde{\text{PSL}(2, \mathbb{R})}$  ( $\Gamma'_i$  has geometric dimension  $\leq 2$ ). Then  $\Gamma_i$  has ergodic dimension  $\leq 2$  [Gab02, Proposition 6.5]; or
- the fundamental group of an aspherical complex of dimension  $\leq 2$  (by a deformation retraction of a 3-dimensional manifold with boundary).

By Mayer-Vietoris [CG86] and by triviality of  $\beta_p^{(2)}$  ( $p = 1, 2$ ) for amenable groups,  $\beta_2^{(2)}(\pi_1(M))$  equals to the sum of the  $\beta_2^{(2)}(\Gamma_i)$  of the vertex groups. Since  $\beta_2^{(2)}(\pi_1(M)) = 0$ , all the vertex groups  $\Gamma_i$  satisfy  $\beta_2^{(2)}(\Gamma_i) = 0$ . The same holds for their subgroups by Theorem 1.6.

A subgroup  $\Lambda$  of  $\pi_1(M)$  decomposes, by Bass-Serre theory [Ser77], as a graph of groups whose edge groups are subgroups of  $\mathbb{Z}^2$  and vertex groups are subgroups of the  $\Gamma_i$ . Again by Mayer-Vietoris,  $\beta_2^{(2)}(\Lambda) = 0$ . ■

## References

- [AG20] M. Abért and D. Gaboriau. Higher dimensional cost and profinite actions. in preparation, 2020. 8
- [Ati76] M. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. In *Colloque “Analyse et Topologie” en l’Honneur de Henri*

- Cartan (Orsay, 1974)*, pages 43–72. Astérisque, SMF, No. 32–33. Soc. Math. France, Paris, 1976. [1](#)
- [Bar16] L. Bartholdi. The rational homology of the outer automorphism group of  $\mathbf{F}_7$ . *New York J. Math.*, 22:191–197, 2016. [3](#)
- [Bau63] G. Baumslag. Automorphism groups of residually finite groups. *J. London Math. Soc.*, 38:117–118, 1963. [3](#)
- [BBB<sup>+</sup>10] L. Bessières, G. Besson, M. Boileau, S. Maillot, and J. Porti. *Geometrisation of 3-manifolds*, volume 13 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010. [5](#), [14](#)
- [BBM07] M. Bestvina, K.-U. Bux, and D. Margalit. Dimension of the Torelli group for  $\text{Out}(F_n)$ . *Invent. Math.*, 170(1):1–32, 2007. [4](#), [10](#)
- [BG20] L. Bartholdi and D. Gaboriau. Around the homology of  $\text{Out}(F_n)$ . in preparation, 2020. [4](#)
- [BKK02] M. Bestvina, M. Kapovich, and B. Kleiner. Van Kampen’s embedding obstruction for discrete groups. *Invent. Math.*, 150(2):219–235, 2002. [9](#)
- [Bor85] A. Borel. The  $L^2$ -cohomology of negatively curved Riemannian symmetric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10:95–105, 1985. [3](#)
- [Bro82] K. Brown. *Cohomology of groups*. Springer-Verlag, New York, 1982. [2](#), [9](#)
- [BV06] M. R. Bridson and K. Vogtmann. Automorphism groups of free groups, surface groups and free abelian groups. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 301–316. Amer. Math. Soc., Providence, RI, 2006. [3](#)
- [BV19] M. Borinsky and K. Vogtmann. The Euler characteristic of  $\text{Out}(F_n)$ . *arXiv e-prints*, page arXiv:1907.03543, Jul 2019. [4](#)
- [CG86] J. Cheeger and M. Gromov.  $L_2$ -cohomology and group cohomology. *Topology*, 25(2):189–215, 1986. [2](#), [7](#), [9](#), [12](#), [14](#)
- [CGMT] C. T. Conley, D. Gaboriau, A. S. Marks, and R. D. Tucker-Drob. One-ended spanning subforests and treeability of groups. *preprint*. [4](#), [13](#), [14](#)
- [Con79] A. Connes. Sur la théorie non commutative de l’intégration. In *Algèbres d’opérateurs (Sém., Les Plans-sur-Bex, 1978)*, pages 19–143. Springer, Berlin, 1979. [2](#)

- [CV86] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986. [3](#), [9](#)
- [Dix69] J. Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*. Gauthier-Villars Éditeur, Paris, 1969. Deuxième édition, revue et augmentée, Cahiers Scientifiques, Fasc. XXV. [10](#)
- [Dod79] J. Dodziuk.  $L^2$  harmonic forms on rotationally symmetric Riemannian manifolds. *Proc. Amer. Math. Soc.*, 77(3):395–400, 1979. [5](#)
- [Eck00] B. Eckmann. Introduction to  $l_2$ -methods in topology: reduced  $l_2$ -homology, harmonic chains,  $l_2$ -Betti numbers. *Israel J. Math.*, 117:183–219, 2000. Notes prepared by Guido Mislin. [2](#)
- [EG57] S. Eilenberg and T. Ganea. On the Lusternik-Schnirelmann category of abstract groups. *Ann. of Math. (2)*, 65:517–518, 1957. [2](#)
- [Gab02] D. Gaboriau. Invariants  $L^2$  de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, 95:93–150, 2002. [2](#), [5](#), [6](#), [7](#), [8](#), [10](#), [11](#), [12](#), [14](#)
- [Gab20] D. Gaboriau. On the ergodic dimension. in preparation, 2020. [2](#), [12](#)
- [GL09] D. Gaboriau and R. Lyons. A measurable-group-theoretic solution to von Neumann's problem. *Invent. Math.*, 177(3):533–540, 2009. [5](#)
- [Gro75] E. K. Grossman. On the residual finiteness of certain mapping class groups. *J. London Math. Soc. (2)*, 9:160–164, 1974/75. [3](#)
- [Hat95] A. Hatcher. Homological stability for automorphism groups of free groups. *Comment. Math. Helv.*, 70(1):39–62, 1995. [3](#)
- [Kam19] H. Kammeyer. *Introduction to  $\ell^2$ -invariants*. Springer-Verlag, 2019. [2](#)
- [Kid08] Y. Kida. The mapping class group from the viewpoint of measure equivalence theory. *Mem. Amer. Math. Soc.*, 196(916):viii+190, 2008. [3](#)
- [KL08] B. Kleiner and J. Lott. Notes on Perelman's papers. *Geom. Topol.*, 12(5):2587–2855, 2008. [5](#), [14](#)
- [Kne29] H. Kneser. Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 38:248–259, 1929. [13](#)
- [LL95] J. Lott and W. Lück.  $L^2$ -topological invariants of 3-manifolds. *Invent. Math.*, 120(1):15–60, 1995. [14](#)



- [Lüc94] W. Lück. Approximating  $L^2$ -invariants by their finite-dimensional analogues. *Geom. Funct. Anal.*, 4(4):455–481, 1994. [3](#)
- [Lüc98] W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and  $L^2$ -Betti numbers. II. Applications to Grothendieck groups,  $L^2$ -Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998. [8](#), [9](#)
- [Lüc02] W. Lück.  *$L^2$ -invariants: theory and applications to geometry and  $K$ -theory*, volume 44. Springer-Verlag, Berlin, 2002. [2](#)
- [Mag35] W. Magnus. Über  $n$ -dimensionale Gittertransformationen. *Acta Math.*, 64(1):353–367, 1935. [8](#)
- [Mil62] J. Milnor. A unique decomposition theorem for 3-manifolds. *Amer. J. Math.*, 84:1–7, 1962. [13](#)
- [Nie24] J. Nielsen. Die Isomorphismengruppe der freien Gruppen. *Math. Ann.*, 91(3-4):169–209, 1924. [8](#)
- [Oha08] R. Ohashi. The rational homology group of  $\text{Out}(F_n)$  for  $n \leq 6$ . *Experiment. Math.*, 17(2):167–179, 2008. [3](#)
- [OW80] D. Ornstein and B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980. [13](#), [14](#)
- [Per02] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv Mathematics e-prints*, page math/0211159, Nov 2002. [5](#), [14](#)
- [Per03] G. Perelman. Ricci flow with surgery on three-manifolds. *arXiv Mathematics e-prints*, page math/0303109, Mar 2003. [5](#), [14](#)
- [Ser77] J.-P. Serre. *Arbres, amalgames,  $SL_2$* . Astérisque, No. 46. Société Mathématique de France, Paris, 1977. [14](#)
- [ST10] R. Sauer and A. Thom. A spectral sequence to compute  $L^2$ -Betti numbers of groups and groupoids. *J. Lond. Math. Soc. (2)*, 81(3):747–773, 2010. [8](#), [9](#)
- [Vog06] K. Vogtmann. The cohomology of automorphism groups of free groups. In *International Congress of Mathematicians. Vol. II*, pages 1101–1117. Eur. Math. Soc., Zürich, 2006. [3](#)