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Matthieu Dussaule, Ilya Gekhtman

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STABILITY PHENOMENA FOR MARTIN BOUNDARIES OF RELATIVELY HYPERBOLIC GROUPS

MATTHIEU DUSSAULE AND ILYA GEKHTMAN

ABSTRACT. Let Γ be a relatively hyperbolic group and let μ be an admissible symmetric finitely supported probability measure on Γ . We extend Floyd-Ancona type inequalities from [11] up to the spectral radius of μ . We then show that when the parabolic subgroups are virtually abelian, the Martin boundary of the induced random walk on Γ is stable in the sense of Picardello and Woess [28]. We also define a notion of spectral degenerescence along parabolic subgroups and give a criterion for strong stability of the Martin boundary in terms of spectral degenerescence. We prove that this criterion is always satisfied in small rank, so that in particular, the Martin boundary of an admissible symmetric finitely supported probability measure on a geometrically finite Kleinian group of dimension at most 5 is always strongly stable.

1. INTRODUCTION

1.1. Martin boundaries and stability. We consider a finitely generated group Γ together with a probability measure μ . The μ -random walk starting at e on Γ is defined as $X_n = g_1 \dots g_n$, where the g_k are independent, identically distributed according to μ random variables on Γ .

We always assume that the random walk is admissible: the support of μ generates the group Γ as a semi-group.

In many situations, one can understand the asymptotic behaviour of X_n in terms of geometric properties at large scale of Γ .

One way to do so is to try to compare geometric boundaries of the group, encoding how geodesics behave at infinity, with probabilistic boundaries, encoding how the random walk behaves asymptotically. In the groups we will study here, the random walk will always be transient, meaning that it almost surely goes to infinity. One can then define the *Martin boundary* as follows.

Define the Green function $G(\cdot, \cdot)$ by

$$G(x, y) = \sum_{n \geq 0} \mu^{*n}(x^{-1}y),$$

where μ^{*n} is the n th convolution power of μ . Precisely,

$$\mu^{*n} = \sum_{z_0=x, z_1, \dots, z_n=y} \mu(z_0^{-1}z_1)\mu(z_1^{-1}z_2)\dots\mu(z_{n-1}^{-1}z_n).$$

One can also define the Green function restricted to paths that stay in some fixed subset $A \subset \Gamma$. Precisely, we define $G(\cdot, \cdot; A)$ by

$$G(x, y; A) = \sum_{n \geq 0} \sum_{\substack{z_0=x, z_1, \dots, z_n=y \\ z_1, \dots, z_{n-1} \in A}} \mu(z_0^{-1}z_1)\mu(z_1^{-1}z_2)\dots\mu(z_{n-1}^{-1}z_n).$$

Notice that we do not require x and y to be in A .

Define then the Martin kernel $K(\cdot, \cdot)$ by

$$K(x, y) = \frac{G(x, y)}{G(e, y)}.$$

We endow Γ with the discrete topology. The Martin compactification of Γ and μ is the smallest compact metrizable set X such that Γ embeds as a dense and open set in X and such that the function $K(\cdot, \cdot)$ extends as a continuous function on $\Gamma \times X$. In other words, a sequence g_n in Γ converges to a point in the Martin compactification if and only if for every $g \in \Gamma$, $K(g, g_n)$ converges to some limit. The Martin boundary is the complement of Γ in X . The Martin compactification and the Martin boundary always abstractly exist and they are unique up to homeomorphism, see [21] or [30]. We will denote the Martin boundary by $\partial_\mu \Gamma$.

The random walk almost surely converges to a point in the Martin boundary (see [30]). Letting X_∞ be the corresponding random variable in $\partial_\mu \Gamma$, we can consider the law of X_∞ . This yields a probability measure ν on $\partial_\mu \Gamma$ which is called the *harmonic measure*. The Martin boundary, together with ν , is a model for the so-called *Poisson boundary*. We will not be interested in the Poisson boundary in the following, so we do not give more details and instead refer to [34], [23] or [22].

More generally, one can define the weighted Green function at r by

$$G_r(x, y) = \sum_{n \geq 0} r^n \mu^{*n}(x^{-1}y).$$

Let R be the radius of convergence of this power series, which we call the *spectral radius* of μ . For $r \leq R$, we then define similarly the r -Martin kernel by

$$K_r(x, y) = \frac{G_r(x, y)}{G_r(e, y)}$$

and the r -Martin compactification and boundary. As before, a sequence g_n in Γ converges to a point in the r -Martin compactification if and only if for every $g \in \Gamma$, $K_r(g, g_n)$ converges to some limit. The r -Martin compactification also always abstractly exist, see [36] and is unique up to homeomorphism. We will denote the r -Martin boundary by $\partial_{r\mu} \Gamma$.

One important aspect of the r -Martin boundary is that it gives a description of r -harmonic positive functions in the following sense. Recall that a function $f : \Gamma \rightarrow \mathbb{R}$ is called r -harmonic if for all $x \in \Gamma$

$$\sum_{y \in \Gamma} \mu(x^{-1}y) f(y) = r f(x).$$

This can be written as $Pf = rf$, where P is the Markov operator associated with μ . Every r -harmonic positive function can be represented as an integral over the Martin boundary. Precisely, for every such function f , there exists a probability measure ν_f on $\partial_{r\mu} \Gamma$ such that for all $x \in \Gamma$,

$$f(x) = \int_{\partial_{r\mu} \Gamma} K(x, \xi) \nu_f(d\xi).$$

In general, the measure ν_f is not unique. To obtain uniqueness, we restrict our attention to the minimal boundary that we now define.

A r -harmonic positive function f is called minimal if for every other r -harmonic positive function g satisfying $g \leq Cf$ for some constant C , we have $g = C'f$ for some constant C' . The r -minimal Martin boundary is then the set

$$\partial_{r\mu}^m \Gamma = \{\xi \in \partial_{r\mu} \Gamma, K(\cdot, \xi) \text{ is minimal harmonic}\}.$$

Let f be an r -harmonic positive function. Then, one can choose ν_f giving full measure to $\partial_{r\mu}^m \Gamma$ and in this case, ν_f is unique. This is the so-called Martin representation Theorem.

In [28], Picardello and Woess define the stability of the Martin boundary of a random walk (or more generally of a Markov chain) as follows (see also [36, IV.28.A]).

Definition 1.1. *Let Γ be a finitely generated group and μ be a probability measure on Γ . Let R be the spectral radius of μ . Say that the Martin boundary is stable if the following conditions hold.*

- (1) *For every $0 < r_1, r_2 < R$, the r_1 and r_2 -Martin compactifications are homeomorphic, that is, $K_{r_1}(\cdot, y_n)$ converges pointwise if and only if $K_{r_2}(\cdot, y_n)$ converges pointwise.*
- (2) *The identity on Γ extends to a continuous and equivariant surjective map ϕ_μ from $\Gamma \cup \partial_{r\mu} \Gamma$, $r < R$, to $\Gamma \cup \partial_{R\mu} \Gamma$. We then write $K_R(x, \xi) = K_R(x, \phi_\mu(\xi))$ for $\xi \in \partial_\mu \Gamma$.*
- (3) *The map $(r, x, \xi) \in (0, R] \times \Gamma \times \partial_\mu \Gamma \mapsto K_r(x, \xi)$ is continuous in the three variables (r, x, ξ) .*

Also say that the Martin boundary is strongly stable if the first condition holds for every $0 < r_1, r_2 \leq R$ so that in particular, the map ϕ_μ induces a homeomorphism from the r -Martin boundary to the R -Martin boundary.

In general, identifying the homeomorphism type of the Martin boundary is difficult and there are few examples for which we know whether the Martin boundary is stable. Often, all the r -Martin compactifications are homeomorphic for $r < R$ but not at $r = R$.

For example, when Γ is an abelian group of rank d with a finitely supported probability measure μ , the r -Martin boundary of a finitely supported probability measure μ on an abelian group of rank d is homeomorphic to a $d - 1$ -dimensional sphere for $r < R$, as proved by Ney and Spitzer in a particular case [27] whereas the R -Martin boundary is reduced to a point, see [36, §25.B]. In particular, the Martin boundary is stable but not strongly stable. Moreover, the r -Martin boundary is minimal for every $r \leq R$.

For nilpotent groups, we do not know in general the homeomorphism type of the r -Martin boundary. However, according to results of Margulis [26], we know that for finitely supported random walks, it is not reduced to a point for $r < R$, whereas the R -Martin boundary is reduced to a point. We do not know if the r -Martin boundary is minimal when $r < R$.

The situation in hyperbolic groups is different. Gouëzel [16] proved that the Martin boundary of a finitely supported symmetric probability measure on a nonelementary Gromov-hyperbolic group is strongly stable. For every $r \leq R$, the r -Martin boundary is always minimal and coincides with the Gromov boundary. This was already proved by Gouëzel and Lalley [17] for co-compact Fuchsian groups.

1.2. Random walks on relatively hyperbolic groups. In this paper, we are interested in symmetric finitely supported random walks on relatively hyperbolic groups. We will give a precise definition of these groups below. We are in particular interested in the case where the parabolic subgroups are virtually abelian, although a lot of our results still hold for any relatively hyperbolic group.

The 1-Martin boundary of a finitely supported probability measure on such a group is described in [10]. It is obtained from the Bowditch boundary by blowing up each parabolic point into a sphere of dimension one less than the rank of its stabilizer. Roughly speaking, those spheres appear as the Martin boundary of the induced random walk on the virtually abelian parabolic subgroups. Actually, the term induced random walk refers to the sub-Markov chain corresponding to first return transition kernel to the parabolic subgroup, see Section 2. So in other words, the 1-Martin boundary roughly consists of the Martin boundary of the parabolic subgroups, glued to the conical limit points.

We will prove that the Martin boundary on such groups is stable, giving a precise description of the homeomorphism type of the r -Martin boundary for every $r \leq R$. We will also prove that strong stability of the Martin boundary depends on the behaviour of the measure μ along parabolic subgroups.

Before giving a precise statement, let us give a brief explanation of what happens in the particular case of adapted random walks on a free product. Let $\Gamma = \Gamma_1 * \Gamma_2$ be a free product of two finitely generated groups Γ_1 and Γ_2 . Then, Γ is hyperbolic relative to Γ_1 and Γ_2 . Recall that a probability measure μ on Γ is called adapted if it can be written as

$$\mu = \alpha\mu_1 + (1 - \alpha)\mu_2,$$

where μ_i is a probability measure on Γ_i . Let G_r be the Green function for μ on Γ and let $G_r^{(i)}$ be the Green function for μ_i on Γ_i . Then, in [35], Woess shows that for every r , there exists $\rho_i(r)$ such that for all $x, y \in \Gamma_i$, $G_r(x, y) = G_{\rho_i(r)}^{(i)}(x, y)$. Moreover, letting R_i be the spectral radius of μ_i we have $\rho_i(r) < R_i$ whenever $r < R$. Call μ spectrally degenerate along Γ_i if $\rho_i(R) = R_i$. In other words, μ is spectrally degenerate along Γ_i if the spectral radius of the induced random walk on Γ_i equals the spectral radius of μ .

It is also proved in [35] that the r -Martin boundary of Γ consists of a copy of the $\rho_i(r)$ -Martin boundary of (Γ_i, μ_i) for each coset $\gamma\Gamma_i$ glued to the set of infinite words, see [35] for a precise statement. In particular, if the groups Γ_i are virtually abelian, then the Martin boundary of (Γ, μ) is stable if and only if the random walk is not spectrally degenerate along Γ_1 and Γ_2 .

We will generalize Woess's results to any relatively hyperbolic group with virtually abelian parabolic subgroups. We thus need to define a notion of spectral degenerescence along parabolic subgroups which mimics the one above. We will give a precise formulation in Section 2. If μ is spectrally degenerate along a parabolic subgroup H , we equivalently say that the parabolic limit point fixed by H is spectrally degenerate.

We will identify the R -Martin boundary with a geometric boundary that we now informally describe, see Section 2 for a precise construction. We start with the Bowditch boundary. If ξ is a parabolic limit point of Γ whose stabilizer is a virtually abelian group of rank d and if ξ is not spectrally degenerate, we replace ξ

by a sphere of dimension $d - 1$. We do not replace spectrally degenerate parabolic limit points.

We will also identify the r -Martin boundary, $r < R$, with a geometric boundary. In this case, we need to replace every parabolic limit point, spectrally degenerate or not, with a sphere of the appropriate dimension. We will use the following terminology.

- (1) Whenever $r < R$ we call a boundary obtained by replacing every parabolic fixed point in the Bowditch boundary with a sphere of dimension one less than the rank of its stabilizer a *r -geometric boundary*.
- (2) We call a boundary obtained by replacing every spectrally non-degenerate parabolic limit point in the Bowditch boundary with a sphere of the appropriate dimension a *R -geometric boundary*.

Note that the R -geometric boundary depends on the measure μ , whereas for $r < R$ the r -geometric boundary does not. Also note that the identity map on Γ extends to a continuous equivariant surjection from the r -geometric compactification to the R -geometric compactification, which consists of collapsing the added spheres into one point. Our main result is the following identification of the homeomorphism type of the Martin boundary.

Theorem 1.2. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a symmetric probability measure on Γ whose finite support generates Γ and let R be its spectral radius. Then for every $r \leq R$, the identity of Γ extends to an equivariant homeomorphism from the r -geometric boundary to the r -Martin boundary.*

In particular, the identity of Γ extends to a continuous equivariant surjection ϕ_μ from the r -Martin compactification to the R -Martin compactification.

We also prove that the Martin boundary is always stable.

Theorem 1.3. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a symmetric probability measure on Γ whose finite support generates Γ and let R be its spectral radius. Let $\partial_\mu \Gamma$ denote the r -Martin boundary for $0 < r < R$. The map $(r, x, \xi) \mapsto K_r(x, \xi)$ is continuous in $(x, r, \xi) \in (0, R] \times \Gamma \times \Gamma \cup \partial_\mu \Gamma$.*

In particular the Martin boundary is stable and it is strongly stable if and only if μ is spectrally non-degenerate.

Note that the third variable ξ is allowed to vary in the whole compactification $\Gamma \cup \partial_\mu \Gamma$ and not only in the boundary $\partial_\mu \Gamma$. We can also prove that the r -Martin boundary is always minimal.

Theorem 1.4. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a symmetric probability measure on Γ whose finite support generates Γ and let R be its spectral radius. Then, for every $0 < r \leq R$, the r -Martin boundary is minimal.*

We will also show that the random walk cannot be spectrally degenerate at parabolic subgroups of small rank. More precisely, let H be a parabolic subgroup which is virtually abelian of rank d . If $d \leq 4$, then the random walk is not spectrally degenerate at H . In particular, the Martin boundary is stable in small dimension. This implies the following.

Theorem 1.5. *Let Γ be the fundamental group of a geometrically finite hyperbolic manifold of dimension $d \leq 5$. Let μ be a symmetric probability measure whose finite support generates Γ . Let R be its spectral radius. Then, for every $r \leq R$, the r -Martin boundary coincides with the $CAT(0)$ boundary of Γ . In particular, the Martin boundary is strongly stable.*

We emphasize that our strong stability result cannot be extended to arbitrary relatively hyperbolic groups. Indeed, even for adapted measures on free products of abelian groups where one factor has rank at least 5, Candellero-Gilch [5] construct examples of both spectrally degenerate and spectrally non-degenerate measures, so the Martin boundary may or may not be strongly stable. In particular, Theorem 1.2 and Theorem 1.3 are optimal. Moreover, the Candellero-Gilch construction might be adapted to construct spectrally degenerate measures on any relatively hyperbolic group with respect to virtually abelian groups of large enough rank.

1.3. Relative form of Ancona inequalities up to the spectral radius. Let us give a few explanations on how we prove these results. Whenever f and g are two functions satisfying that there exists a constant C such that $\frac{1}{C}f \leq g \leq Cg$, we will write $f \asymp g$. If the constant depends on some parameter, we will not use this notation to avoid confusion, except if the parameter is clear from the context. Also, whenever $f \leq Cg$ for some constant C , we will write $f \lesssim g$.

Ancona [1] proved that the Martin boundary of a finitely supported admissible random walk on a hyperbolic group coincides with the Gromov boundary. To prove this, he used the following deviation inequalities. They state that if x, y, z are three points in Γ satisfying that y is on a geodesic $[x, z]$ from x to z , then we have

$$(1) \quad G(x, z) \asymp G(x, y)G(y, z).$$

The implicit constant only depends on the hyperbolicity parameters of the group. We can restate these inequalities saying that the random walk must pass within a bounded distance of y with high probability.

To prove that the Martin boundary is stable, Gouëzel [16] showed that these deviation inequalities still hold at the spectral radius. For every $r \leq R$, we have

$$(2) \quad G_r(x, z) \asymp G_r(x, y)G_r(y, z).$$

where the implicit constant does not depend on r . He actually proved a stronger version of them, which allowed him to prove Hölder regularity of the Martin kernels. More precisely, he proved that whenever two geodesics $[x, y]$ and $[x', y']$ fellow travel for a time at least n , then for all $r \leq R$, we have

$$(3) \quad \left| 1 - \frac{G_r(x, y)G_r(x', y')}{G_r(x', y)G_r(x, y')} \right| \leq C\rho^n$$

for some constants $C \geq 0$ and $0 < \rho < 1$ independent of r . We will call (2) weak Ancona inequalities and (3) strong Ancona inequalities up to the spectral radius in what follows.

For relatively hyperbolic groups, such inequalities are not expected to hold in general, for if y is deep in a parabolic subgroup, there is no chance to guarantee that the random walk passes within a bounded distance of y .

However, we can prove analogous inequalities if we restrict ourselves to so called transition points. Say a point y on a geodesic $\alpha = [x, z]$ is an (ϵ, η) -transition point of α if the length- η interval of α around y is not contained in the ϵ -neighborhood

of a single parabolic subgroup. We show that the inequalities (2) hold whenever y is an (ϵ, η) -transition point on a geodesic connecting x, z . Similarly, we prove (3) holds whenever there exist n distinct (ϵ, η) -transition points $z_i \in [x, y]$, $z'_i \in [x', y']$ with $d(z_i, z'_i) < D$. In both cases the implied constants are independent of $r \leq R$.

In fact, we prove a more general version of these inequalities, using Floyd functions. Fix a function $f(n) = \lambda^n$, $0 < \lambda < 1$. The Floyd distance viewed from $y \in \Gamma$, which we denote by $\delta_y^f(\cdot, \cdot)$ is the path metric on Γ obtained from rescaling the Cayley graph by declaring the length of an edge τ to be $f(d(\tau, y))$. The connection with transition points on word geodesics in relatively hyperbolic groups is as follows: if Γ is relatively hyperbolic, and f a suitable exponentially decaying function, then for each $\epsilon, \eta, D > 0$ there is a $\delta > 0$ such that whenever $x, y, z \in \Gamma$ and y is D close to an (ϵ, η) -transition point of $[x, z]$ we have $\delta_y^f(x, z) > \delta$. In hyperbolic groups, all points on a geodesic are transition points so this condition on the Floyd distance is equivalent to saying that y is within a bounded distance of $[x, z]$.

We say that a probability measure μ on a finitely generated group Γ satisfies the weak Floyd-Ancona inequalities up to the spectral radius if for every $\delta > 0$, and every $x, y, z \in \Gamma$ with $\delta_y^f(x, z) > \delta$ the inequality (2) holds. We say that μ satisfies the strong Floyd-Ancona inequalities up to the spectral radius if the inequality (3) holds whenever there is a sequence of n distinct points z_1, \dots, z_n such that for every i , $\delta_{z_i}^f(x, y) > \delta$ and $\delta_{z_i}^f(x', y') > \delta$. In both cases, the implied constants are required to depend only on $\delta > 0$ but not on $r \leq R$. Adapting the techniques of Gou ezel in [16], we prove the following.

Theorem 1.6. *[Theorem 3.6, Theorem 3.14] Let μ be a symmetric measure on a finitely generated group Γ , whose finite support generates Γ . Then, μ satisfies both weak and strong Floyd-Ancona inequalities up to the spectral radius with respect to any exponential Floyd function.*

We prove that these inequalities hold for admissible symmetric finitely supported measures on any finitely generated groups. However, the statement will only be non-vacuous if the Floyd boundary, which is the complement of the group Γ inside its completion for the Floyd distance, is infinite. The only known examples of finitely generated groups with infinite Floyd boundary are nonelementary relatively hyperbolic groups.

For $r = 1$, the weak Floyd-Ancona inequalities were proved in [11]. The strong Floyd-Ancona inequalities are new even for $r = 1$.

The results concerning the identification of the Martin boundary in this paper use only the weak Floyd-Ancona inequalities at the spectral radius. However, the strong inequalities are important for other reasons. In the setting of hyperbolic groups, these strong versions of Ancona inequalities for $r = 1$ were used by many authors including [20], [18], where they are the key ingredient to prove H older continuity of the Martin kernel on the Gromov boundary. The uniformity of these inequalities up to the spectral radius was established in [16], [17] and used to prove a local limit theorem for random walks on hyperbolic groups. In the context of relatively hyperbolic groups, the Floyd-Ancona inequalities proved here are used by the first author in in [8] to establish a weaker H older-type property and use it to prove a local limit theorem.

1.4. Some background on relatively hyperbolic groups and the Floyd distance. Let Γ be a finitely generated group. The action of Γ on a compact Hausdorff space T is called a convergence action if the induced action on triples of distinct points of T is properly discontinuous. Suppose $\Gamma \curvearrowright T$ is a convergence action. The set of accumulation points $\Lambda\Gamma$ of any orbit $\Gamma \cdot x$ ($x \in T$) is called the *limit set* of the action. As long as $\Lambda\Gamma$ has more than two points, it is uncountable and is the unique minimal closed Γ -invariant subset of T . The action is then said to be non-elementary. In this case, the orbit of every point in $\Lambda\Gamma$ is infinite. The action is *minimal* if $\Lambda\Gamma = T$.

A point $\zeta \in \Lambda\Gamma$ is called *conical* if there exists a sequence g_n of Γ and distinct points $\alpha, \beta \in \Lambda\Gamma$ such that $g_n\zeta \rightarrow \alpha$ and $g_n\eta \rightarrow \beta$ for all $\eta \in T \setminus \{\zeta\}$. The point $\zeta \in \Lambda\Gamma$ is called bounded parabolic if it is the unique fixed point of its stabilizer in Γ , which is infinite and acts co-compactly on $\Lambda\Gamma \setminus \{\zeta\}$. The stabilizers of bounded parabolic points are called maximal parabolic subgroups. The convergence action $\Gamma \curvearrowright T$ is called geometrically finite if every point of $\Lambda\Gamma \subset T$ is either conical or bounded parabolic. Since Γ is assumed to be finitely generated, every maximal parabolic subgroup is finitely generated too (see [12, Main Theorem (d)]). Then, by Yaman's results [37], it follows that if $\Gamma \curvearrowright T$ is a minimal geometrically finite action, then there exists a proper geodesic Gromov hyperbolic space X on which Γ acts properly discontinuously by isometries and a Γ -equivariant homeomorphism $T \rightarrow \partial X$.

Suppose now Ω is a collection of subgroups of Γ . We say that Γ is *hyperbolic relative to Ω* if there exists some compactum T on which Γ acts minimally and geometrically finitely and such that the maximal parabolic subgroups are the elements of Ω . Such a compactum is then unique up to Γ -equivariant homeomorphism [4] and is called the Bowditch boundary of (Γ, Ω) . The group Γ is said to be *non-elementary relatively hyperbolic* if it admits a non-elementary geometrically finite convergence action on some infinite compactum.

Let Γ be a finitely generated group and let S be a finite generating set. Consider then a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the two following conditions: $\sum_{n \geq 0} f(n)$ is finite and there exists a $\lambda \in (0, 1)$ such that $1 \geq f(n+1)/f(n) \geq \lambda$ for all $n \in \mathbb{N}$. The function f is called a *rescaling function* or a *Floyd-function*. When f is of the form $f(n) = \lambda^n$, we say that f is an exponential Floyd function.

Pick a basepoint $o \in \Gamma$ and rescale the Cayley graph $\text{Cay}(\Gamma, S)$ by declaring the length of an edge τ to be $f(d(o, \tau))$. The induced short-path metric on $\text{Cay}(\Gamma, S)$ is called the *Floyd metric* with respect to the basepoint o and Floyd function f and denoted by $\delta_o^f(\cdot, \cdot)$. Its Cauchy completion, whose topology does not depend on the basepoint, is called the Floyd compactification $\bar{\Gamma}_f$ and $\partial_f\Gamma = \bar{\Gamma}_f \setminus \Gamma$ is called the Floyd boundary. Karlsson [24] proved that the Floyd boundary is either uncountably infinite or reduced to 0, 1 or 2 points.

If Γ is non-elementary relatively hyperbolic and if the Floyd function f is not decreasing exponentially too fast, Gerasimov proved that the Floyd boundary is infinite. One can take f to be of the form $f(n) = \lambda^n$ for suitable λ . Moreover, there is a continuous Γ -equivariant surjection (*Floyd map*) from the Floyd boundary to the Bowditch boundary [13, Map theorem]. Furthermore, Gerasimov and Potyagailo [14, Theorem A] proved that the pre-image of any conical point by this map is a singleton and the pre-image of a parabolic fixed point p is the limit set for the action of its stabilizer Γ_p on $\partial_f\Gamma$. In particular if Γ_p is an amenable non-virtually cyclic

group then its limit set on the Floyd boundary is a point. Consequently, when Γ is hyperbolic relative to a collection of infinite amenable subgroups which are not virtually cyclic, the Floyd boundary is homeomorphic to the Bowditch boundary. This is in particular the case if the parabolic subgroups are virtually abelian.

We will mainly be interested in relatively hyperbolic groups in this paper. However, the deviation inequalities that we will prove in Section 3 will be formulated with the Floyd distance and will hold in any finitely generated group.

To conclude this section, we recall the definition of transition points.

Definition 1.7. *If α is a (finite or infinite) geodesic in $\text{Cay}(\Gamma, S)$ for the word metric, a point $p \in \alpha$ is said to be (ϵ, η) -deep if there is a $g \in \Gamma$, $P \in \Omega$ such that the part of α containing the points at distance at most η from p is contained in the ϵ -neighborhood of gP . Otherwise, $p \in \alpha$ is called an (ϵ, η) -transition point of α .*

The following result relates transition points to the Floyd metric.

Proposition 1.8. [15, Corollary 5.10] *For every $\epsilon > 0$, $\eta > 0$ and $D > 0$ there exists $\delta > 0$ such that if y is within word distance D of an (ϵ, η) -transition point of a word geodesic from x to z then $\delta_y^f(x, z) > \delta$.*

1.5. Organization of the paper. In Section 2, we properly define the notion of spectral degenerescence for a probability measure on a relatively hyperbolic group. We also define the geometric boundaries we will identify the Martin boundaries with and propose a construction using arguments of Dahmani [6].

In Section 3, we prove deviation inequalities. In particular, we prove Theorem 1.6, generalizing Gouëzel approach in [16]. We first prove that the probability of going from x to z , missing a ball centered on a point y satisfying $\delta_y^f(x, z) \geq \delta$, decays super-exponentially fast in the radius of the ball. We can then conclude that weak Floyd-Ancona inequalities hold, as in [11]. To prove that strong Floyd-Ancona inequalities also hold, we have to be more precise and we first prove a refined version of weak inequalities, see Proposition 3.12.

Section 4 is devoted to the proof of our main theorem. We first prove that if a sequence converges to a point in the geometric boundary, then it converges to a point in the Martin boundary. If the limit point in the boundary is a conical limit point, the result is given by [11], once weak Floyd-Ancona inequalities are established. We thus only have to prove the result when the limit is a parabolic limit point. We deal separately with the degenerate and non-degenerate case.

Essentially, the strategy goes as follows. When the parabolic limit point is spectrally degenerate, we prove that the Martin boundary of the induced chain on the corresponding parabolic subgroup is reduced to a point. We then have to prove that the Martin boundary of this induced chain embeds into the Martin boundary of the whole group. This is a difficult task in general, but this follows here from the fact that the Martin boundary of the induced chain on any fixed neighborhood of the parabolic subgroup is also reduced to a point. We can actually prove this when the parabolic subgroups are virtually nilpotent and we do not need to assume there that they are virtually abelian.

When the parabolic limit point is not spectrally degenerate, we use the results of [7] to prove that the Martin boundary of the induced chain on the corresponding parabolic subgroup is a Euclidean sphere at infinity. We again have to prove that this Martin boundary embeds in the whole Martin boundary. This was proved in

[10] for the 1-Martin boundary and the same strategy works here. However, some crucial technical details have to be changed so we rewrite the entire proof.

In Section 5, we prove Theorem 1.3 and Theorem 1.4. The proof of minimality is similar to the proof the the 1-Martin boundary is minimal in [11].

To prove continuity of $(r, x, \xi) \mapsto K_r(x, \xi)$, we again deal separately with conical limit points and parabolic limit points ξ . In the first case, continuity is a direct consequence of strong Ancona inequalities. Our proof also shows that the map $(r, x, \xi) \mapsto K_r(x, \xi)$ is continuous for every point in the Martin boundary, when Γ is hyperbolic. To the authors' knowledge, this is not stated anywhere, although this is implicit in [16], where there is a description of the r -Martin boundary for every $r \leq R$.

For parabolic limit points, we need to prove a form of continuity in r of the first return kernel to a parabolic limit group associated with $r\mu$. This uses in turn results of Section 4. Since we have an explicit formula for the Martin kernel of this induced first return kernel, given by the results of [7], we can prove continuity.

Finally, in Section 6, we prove that the Martin boundary is strongly stable in small dimension. Precisely, we prove that if H is a parabolic subgroup which is virtually abelian of rank d and if μ is spectrally degenerate along H , then $d \geq 5$. Roughly speaking, we prove this in two steps. We first show that the derivative at 1 of the Green function along a parabolic subgroup is always finite. We then show that if μ is spectrally degenerate along H , then the induced transition kernel p on H satisfies a local limit theorem of the form $p^{(n)}(e) \sim Cn^{-d/2}$, where $p^{(n)}$ is the n th convolution power of p . This is a version of the classical local limit theorem in \mathbb{Z}^d that we prove using results in [7], see also [36]. Since the Green function along a parabolic subgroup is given by $t \mapsto \sum_{n \geq 0} t^n p^{(n)}(e)$, the first derivative is finite if and only if $n \geq 5$.

2. SPECTRAL DEGENERESCENCE AND GEOMETRIC BOUNDARIES

Let Γ be a relatively hyperbolic group with respect to Ω and let μ be a probability measure on Γ . We now define properly the notion of spectral degenerescence along parabolic subgroups. We then define the geometric boundaries associated with μ .

We choose a finite set Ω_0 of representatives of conjugacy classes of elements of Ω . Let $H \in \Omega_0$. We denote by p_H the first return kernel to H . If $h, h' \in H$, then $p_H(h, h')$ is the probability that the μ -random walk, starting at h , eventually comes back to H and that its first return to H is at h' . In other words,

$$p_H(h, h') = \mathbb{P}_h(\exists n \geq 1, X_n = h', X_1, \dots, X_{n-1} \notin H).$$

More generally, for $r \in [0, R]$, we denote by $p_{H,r}$ the first return transition kernel to H for $r\mu$. Precisely, if $h, h' \in H$, then

$$p_{H,r}(h, h') = \sum_{n \geq 0} \sum_{\substack{g_1, \dots, g_{n-1} \\ \notin H}} r^n \mu(h^{-1}g_1)\mu(g_1^{-1}g_2)\dots\mu(g_{n-2}^{-1}g_{n-1})\mu(g_{n-1}^{-1}h').$$

We then denote by $p_{H,r}^{(n)}$ the convolution powers of this transition kernel, by $G_{H,r}(h, h'|t)$ the associated Green function, evaluated at t and by $R_H(r)$ the associated spectral radius, that is, the radius of convergence of $t \mapsto G_{H,r}(h, h'|t)$. For simplicity, write $R_H = R_H(R)$.

Definition 2.1. We say that the random walk is spectrally degenerate along H if $R_H(R) = 1$. We also say that μ is spectrally degenerate along H . We say that the random walk is spectrally non-degenerate if for every $H \in \Omega_0$, it is not spectrally degenerate along H . We also say that μ is spectrally non-degenerate.

Note that this definition does not depend on the choice of Ω_0 .

Lemma 2.2. Let Ω_0 and Ω_1 be two finite set of representatives of conjugacy classes of elements of Ω . Let $H_0 \in \Omega_0$ and $H_1 \in \Omega_1$ be such that $H_0 = gH_1g^{-1}$ for some g . Then, μ is spectrally degenerate along H_0 if and only if it is spectrally degenerate along H_1 .

To avoid a lengthy argument, we do not prove this lemma a priori and we will not use it in the following. It will be a consequence of our main result anyway, since the R -Martin boundary does not depend on the choice of Ω_0 .

Let ξ be a parabolic limit point. We say that ξ is spectrally degenerate (with respect to μ) if μ is spectrally degenerate along H , where $H \in \Omega_0$ is a conjugate of the stabilizer of ξ .

We now associate with μ two geometric boundaries. Actually, the first one is purely geometric and does not depend on μ . It consists of the Bowditch boundary blown-up at every parabolic point. The second one only depends on the spectral degenerescence of μ . Only the degenerate parabolic limit points are blown-up.

We do assume here that the parabolic subgroups are virtually abelian. Let H be such a parabolic subgroup. Then, there exists a finite index subgroup of H which is isomorphic to \mathbb{Z}^d . Any section $H/\mathbb{Z}^d \rightarrow H$ gives an identification between H and $\mathbb{Z}^d \times E$, where E is a finite set. An element $h \in H$ will be denoted by (x, k) , where $x \in \mathbb{Z}^d$ and $k \in E$. We define the geometric boundary ∂H of H as follows. A sequence $h_n = (x_n, k_n)$ of elements of H converges to a point in ∂H if and only if x_n tends to infinity and $\frac{x_n}{\|x_n\|}$ converges to some $\theta \in \mathbb{S}^{d-1}$, where $\|x\|$ is the Euclidean norm of $x \in \mathbb{Z}^d$. In other words, we glue a sphere at infinity on $\mathbb{Z}^d \times E$. Notice that this definition is independent of the coordinate k_n . Also notice that this boundary coincides with the CAT(0) boundary of H .

Definition 2.3 (First geometric boundary). Let Γ be a relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a probability measure on Γ whose support generates Γ as a semi-group. Let R be its spectral radius and let $r < R$. A r -geometric compactification is a metrizable compact space X such that Γ is a dense and open subset of X and such that the following holds. A sequence g_n of elements of Γ converges to some point $\xi \in X \setminus \Gamma$ if and only if

- (1) either g_n converges to a conical limit point,
- (2) or there exists a parabolic subgroup $H \in \Omega_0$ and there exists $g \in \Gamma$ such that the projection $\pi_{gH}(g_n)$ of g_n on gH satisfies that $g^{-1}\pi_{gH}(g_n)$ converges to a point in the geometric boundary of H .

We call the complement $X \setminus \Gamma$ of Γ in X a r -geometric boundary.

Definition 2.4 (Second geometric boundary). Let Γ be a relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a probability measure on Γ whose support generates Γ as a semi-group. Let R be its spectral radius. A R -geometric compactification is a metrizable compact space X such that Γ is a dense and open subset of X and such that the following holds. A sequence g_n of elements of Γ converges to some point $\xi \in X \setminus \Gamma$ if and only if

- (1) either g_n converges to a conical limit point,
- (2) or g_n converges to a degenerate parabolic limit point,
- (3) or there exists a non-degenerate parabolic subgroup $H \in \Omega_0$ and there exists $g \in \Gamma$ such that the projection $\pi_{gH}(g_n)$ of g_n on gH satisfies that $g^{-1}\pi_{gH}(g_n)$ converges to a point in the geometric boundary of H .

We call the complement $X \setminus \Gamma$ of Γ in X a R -geometric boundary.

It follows from the definitions that the identity of Γ extends to a continuous equivariant surjective map ϕ_μ from the r -geometric compactification to the R -geometric boundary. The induced map on the boundaries is obtained collapsing the geometric boundaries of the parabolic subgroups to one point.

For $r < R$, the r -geometric boundary coincides with the PBU-boundary defined in [10]. As noted in there, for $r < R$, any two r -geometric boundaries are equivariantly homeomorphic, see precisely [10, Lemma 3.4]. Actually, the same proof shows that any two R -geometric boundaries are also equivariantly homeomorphic. In particular, these definitions do not depend on the choice of Ω_0 . We will denote by $\partial_r\Gamma$ the r -geometric boundary, for $r \leq R$.

One can give a geometric construction of $\partial_r\Gamma$, working with the Cayley graph, as follows. In [6], Dahmani defines a boundary for a relatively hyperbolic group. Assume that the parabolic subgroups H are themselves endowed with a boundary ∂H . If those boundaries ∂H satisfy some technical condition, namely that finite sets fade at infinity (see [10, Definition A.4]), then one can define a boundary for Γ that consists of gluing the boundaries ∂H to the set of conical limit points, see [6, Theorem 3.1] for more details.

It is clear from the definitions that the trivial boundary consisting of one point satisfy the condition that finite sets fading at infinity. Also, it is proved in the appendix of [10] that the geometric boundary ∂H of a virtually abelian parabolic subgroup H , which consists of a sphere at infinity and which is defined above satisfies this condition as well, see precisely [10, Lemma A.6]. In particular, one can recover the R -geometric boundary as the Dahmani boundary choosing the trivial boundary for degenerate parabolic subgroups H and the sphere at infinity for non-degenerate ones. One can also recover the r -geometric boundary, $r < R$ as the Dahmani boundary choosing the sphere at infinity for every parabolic subgroup. We refer to [10, Appendix] for more details.

3. DEVIATION INEQUALITIES

The goal of this section is to prove weak and strong Floyd-Ancona inequalities.

3.1. Bounds for sums of Green functions over spheres. We first prove the following.

Proposition 3.1. *Let Γ be a group with nontrivial Floyd boundary and μ a symmetric probability measure on Γ whose finite support generates Γ . Let R be its spectral radius. The quantity*

$$\sum_{x \in \Gamma, d(e, x) = k} G_r^2(e, x)$$

is bounded independently of k and of $r \leq R$.

We need some preliminary results. We refer to [40] for definitions and background for statistically convex co-compact group actions and contracting elements. For our purposes, we will only use that such actions $\Gamma \curvearrowright (X, d)$ include the actions of groups with infinite Floyd boundary on their Cayley graphs [39, Lemma 7.2]. The following result is due to Yang [40, Corollary 1.12], who calls it the "Extension Lemma".

Lemma 3.2. *Let $\Gamma \curvearrowright X$ be a proper co-compact action with a contracting element with respect to a metric d on X . Let $o \in X$ be a basepoint. Then there is a $C > 0$ with the following property. For any $g, h \in \Gamma$ there is a contracting element $w \in G$ with $d(o, w \cdot o) < C$ such that*

$$|d(o, gwh \cdot o) - d(o, g \cdot o) - d(o, h \cdot o)| < C$$

and every geodesic in X from o to $gwh \cdot o$ passes within C of $g \cdot o$ and $gw \cdot o$.

As explained, Yang proved that if Γ is a group with nontrivial Floyd boundary, then its action on any Cayley graph has a contracting element, so we can reformulate the Extension Lemma as follows in our situation.

Lemma 3.3. *Let Γ be a group with nontrivial Floyd boundary. Then, there exists $C \geq 0$ and $\delta > 0$ such that for any $g, h \in \Gamma$, there exists $w \in \Gamma$ with $d(e, w) \leq C$ such that $\delta_g(e, gwh) \geq \delta$ and $\delta_g(e, gw) \geq \delta$ and $|d(e, gwh) - d(e, g) - d(e, h)| \leq C$. Moreover, every geodesic from e to gwh in the Cayley graph of Γ passes within C of g and gw .*

We also have the following lemma.

Lemma 3.4. *For every $K_1, K_2 > 0$, there is a constant $N > 0$ such that for all $k, l \in \mathbb{N}$, and $z \in \Gamma$ with $k + l - K_1 \leq |z| \leq k + l + K_1$ there are at most N triples (x, a, y) with $d(e, x) = k$, $d(e, y) = l$, $d(e, a) \leq K_1$ such that $xay = z$ and x is at distance at most K_2 from a geodesic from e to z .*

Proof. Consider such $x, y, z, a \in \Gamma$ with $xay = z$. Let α be a geodesic from e to z such that x is within K_2 of α . Then, x is within $2K_2$ of $\alpha(d(e, x))$, so there is a uniformly finite number of possibilities for such x . Since $d(e, a) \leq K_1$, there also is a finite number of possibilities for such a . This proves that there also is a finite number of possibilities for such y , which concludes the proof. \square

Proof of Proposition 3.1. We follow the proof of [16, Lemma 2.5]. First fix $r < R$. Write

$$u_k(r) = \sum_{x \in \Gamma, d(e, x) = k} G_r^2(e, x).$$

Using Lemma 3.3, to any x, y with $d(e, x) = k$, $d(e, y) = l$ we can associate a $\Psi(x, y) = xay$ with $l + K - C \leq d(e, \Psi(x, y)) < l + k + C$. We have

$$\begin{aligned} G_r^2(e, x)G_r^2(e, y) &\lesssim G_r^2(e, x)G_r^2(e, a)G_r^2(e, y) \\ &= G_r^2(e, x)G_r^2(x, xa)G_r^2(xa, xay) \\ &\lesssim G_r^2(e, xay). \end{aligned}$$

According to Lemma 3.4, any $g \in \Gamma$ has a finite number of preimage by Ψ , so that

$$\begin{aligned} u_k(r)u_l(r) &= \sum_{x \in S_k, y \in S_l} G^2(e, x)G_r^2(e, y) \lesssim \sum_{x \in S_k, y \in S_l} G_r^2(e, \Phi(x, y)) \\ &\lesssim \sum_{i=k+l-C}^{k+l+C} \sum_{z \in S_i} G_r^2(e, z) \lesssim \sum_{i=k+l-C}^{k+l+C} u_i(r). \end{aligned}$$

Since $r < R$, [17, Proposition 1.9] shows that $\sum_{g \in \Gamma} G_r^2(e, g)$ is finite. In particular, the sequence $u_k(r)$ is summable, and reaches its maximum $M(r)$ at some index $k_0(r)$. Using the previous equation with $k = l = k_0(r)$ we get $M(r)^2 \lesssim M(r)$ and thus $M(r) \lesssim 1$. We have thus proved that for $r < R$, $u_k(r)$ is bounded independently of k and r . Since $r \mapsto G_r(e, e)$ is non-decreasing, we can take the limit as $r \rightarrow R$, which completes the proof. \square

3.2. Super-exponential decay of the probability of missing balls. We consider a finitely generated group Γ and a symmetric admissible finitely supported probability measure μ on Γ . Let $f(n) = a^{-n}$ be an exponentially decaying Floyd function for Γ . Let $G = G_R$ denote the Green function at the spectral radius R . To simplify notations, also write $d(e, x) = |x|$ for $x \in \Gamma$. We prove here the following.

Proposition 3.5. *With these notations, there exists $c > 1$ and $\eta_0 \geq 0$ such that the following holds. For every $\delta > 0$, for every $x, y, z \in \Gamma$ with $\delta_z^f(x, y) > \delta$ we have for every $\eta \geq \eta_0$*

$$G(x, y; B_\eta(z)^c) \leq e^{-\delta c^\eta}.$$

Proof. By equivariance of the Green function we can assume without loss of generality that $z = e$. If the Floyd metric is a point the result trivially holds because for each $\delta > 0$ there are only finitely many $x \in \Gamma$ with $\delta_z^f(x, y) > \delta$ (see proof of [11, Theorem 4.1]). Therefore we assume the Floyd boundary is infinite.

We fix $\eta > 0$. We consider some $\epsilon > 0$ which will be chosen small enough later, independently of η . Let N be the smallest integer less than $\delta a^{\epsilon\eta}/2$. We will find subsets $A_0, \dots, A_{N+1} \subset \Gamma$ with $A_0 = \{x\}$ and $A_{N+1} = \{y\}$ satisfying the following.

- a) Any admissible path in Γ from x to y which does not intersect $B_\eta(y)$ passes successively through A_1, \dots, A_{N+1} .
- b) For each $i = 0, \dots, N$, we have

$$(4) \quad \sum_{a \in A_i, b \in A_{i+1}} G(a, b)^2 < 1/4.$$

The result will then follow as in [16, Lemma 2.6], as we now explain. Following [32] and [25], we introduce the operator

$$L_i : \ell^2(A_{i+1}) \rightarrow \ell^2(A_i)$$

defined by

$$L_i f(a) = \sum_{b \in A_{i+1}} G(a, b) f(b).$$

First, Condition a) above shows that

$$G(x, y; B_\eta(z)^c) \leq \sum_{a_1 \in A_1} \dots \sum_{a_N \in A_N} G(x, a_1) G(a_1, a_2) \dots G(a_N, y).$$

This can be written as

$$G(x, y; B_\eta(z)^c) \leq (L_0 \dots L_N \delta_y)(x),$$

so that $G(x, y; B_\eta(z)^c)$ is bounded by $\prod \|L_i\|$. Then, Cauchy-Schwarz shows that

$$\|L_i\| \leq \left(\sum_{a \in A_i, b \in A_{i+1}} G(a, b)^2 \right)^{1/2}.$$

Thus, Condition b) shows that $\|L_i\| \leq 1/2$, so that

$$G(x, y; B_\eta(z)^c) \leq 2^{-N} \leq 2^{-\delta \frac{1}{2} a^{\epsilon \eta} + 1}.$$

This will yield the desired inequality, so that we only have to construct the sets A_1, \dots, A_N to conclude the proof.

Let $K = \max_{g \in \text{supp } \mu} |g|$ be the maximal jump size of the random walk and let L be such that for all $g \in \Gamma$,

$$Kf(|g| - K) \leq Lf(|g|).$$

For $\theta \in \mathbb{R}$ let

$$A(\theta) = \left\{ g \in \Gamma : |g| > \eta, \delta_e^f(g, x) \in \left[\theta - La^{-|g|}, \theta + La^{-|g|} \right] \right\}.$$

For $i = 1, \dots, N$, we consider the interval

$$I_i = [(2i - 1)a^{-\epsilon \eta}, 2ia^{-\epsilon \eta}].$$

We will find $\theta_i \in I_i$, $i = 1, \dots, N$ so that $A_i = A(\theta_i)$ satisfy the conditions above.

We first shows that for any choice of $\theta_1, \dots, \theta_N$, Condition a) is satisfied. Consider a path for the random walk from x to y . Since $\theta_N \in I_N$, in particular we have $0 \leq \theta_N \leq \delta_e^f(x, y)$. Let g_{j_N} be the last point on the path with $\delta_e^f(x, g_{j_N}) \leq \theta_N$. By definition, $\delta_e^f(x, g_{j_N+1}) > \theta_N$. On the other hand,

$$\delta_e^f(x, g_{j_N+1}) \leq \delta_e^f(x, g_{j_N}) + \delta_e^f(g_{j_N}, g_{j_N+1}) \leq \theta_N + Ka^{-|g_{j_N+1}|+K} \leq \theta_N + La^{-|g_{j_N+1}|}$$

so that $g_{j_N+1} \in A(\theta_N)$. This proves that the path passes through A_N .

Now we claim that if η is large enough and ϵ is small enough, independently of η , then if the path passes through A_{i+1} , it first passes through A_i . This will settle the first condition. We thus consider $x_i \in A_{i+1}$ such that the path passes through x_i . Then, by definition

$$\delta_e^f(x, x_i) \geq \theta_{i+1} - La^{-|x_i|} \geq (2i + 1)a^{-\epsilon \eta} - La^{-|x_i|}.$$

Since $x_i \in A_{i+1}$, in particular, $|x_i| > \eta$. If η is large enough, then $a^{\eta/2} \geq L$, so if $\epsilon < 1/2$, then

$$a^{-\epsilon \eta} \geq La^{-\eta} > La^{-|x_i|}.$$

In particular,

$$\delta_e^f(x, x_i) > 2ia^{-\epsilon \eta} \geq \theta_i.$$

Consider the sub-path of the random walk from x to x_i . Let g_{j_i} be the last point on this sub-path with $\delta_e^f(x, g_{j_i}) \leq \theta_i$. The same proof as above shows that $g_{j_i+1} \in A_i$. This proves the claim.

We now find $\theta_1, \dots, \theta_N$, $\theta_i \in I_1$ satisfying Condition b). Let Leb denote the Lebesgue measure on \mathbb{R} and let m_j be the restriction of the measure $N \text{Leb}$ to

I_j . Consider the product space $\Omega = \prod_{j=1}^N I_j$, endowed with the product measure $m = \otimes_{j=1}^N m_j$. Define

$$f_i(\theta_1, \dots, \theta_N) = \sum_{g \in A(\theta_i), h \in A(\theta_{i+1})} G^2(g, h).$$

We will show that for each i ,

$$\int_{\Omega} f_i dm < a^{-\eta/4}.$$

It will follow that if η is large enough, then

$$\int \sum f_i < 1/4.$$

In particular, it will follow that there exists $(\theta_1, \dots, \theta_N)$ such that for every i ,

$$(5) \quad f_i(\theta_1, \dots, \theta_N) \leq 1/4.$$

This will prove (4). We give the argument for $i = 1, \dots, N-1$. The cases for f_0 and f_N are similar but simpler. Let $X_j(g) = \{\theta \in I_j : g \in A(\theta)\}$. By definition,

$$\int f_i dm = \sum_{g, h \in \Gamma} G^2(g, h) N \text{Leb}(X_i(g)) N \text{Leb}(X_{i+1}(h))$$

Note, $\text{Leb}(X_i(g)) \leq 2La^{-|g|}$ for each $i = 1, \dots, N$ and $g \in \Gamma$. Furthermore $X_i(g)$ is empty outside the set

$$A'_i = \{g \in \Gamma : |g| > \eta \text{ and } \delta_e^f(x, g) \in [(2i-1)a^{-\epsilon\eta} - La^{-\eta}, 2ia^{-\epsilon\eta} + La^{-\eta}]\}.$$

It follows that

$$\begin{aligned} \int f_i dm &\leq 4L^2 N^2 \sum_{g \in A'_i, h \in A'_{i+1}} G^2(g, h) a^{-|g|-|h|} \\ &\lesssim e^{2\epsilon\eta} \sum_{g \in A'_i, h \in A'_{i+1}} G^2(g, h) a^{-|h|-|g|}. \end{aligned}$$

The last expression is bounded above by

$$e^{2\epsilon\eta} \sum_{u \in \Gamma} N(u) G^2(e, u) a^{-|u|}$$

where $N(u)$ is the number of ways to decompose $u = g^{-1}h$ where $g \in A'_i, h \in A'_{i+1}$. Karlsson's visibility lemma [24, Lemma 1] implies that letting d be the maximum distance of a geodesic segment $[g, h]$ to e we have

$$\delta_e^f(g, h) \leq 4df(d) + 2 \sum_{j=d}^{\infty} f(j) = 4da^{-d} + 2a^{-d}/(1-a^{-1}) \lesssim a^{-d/2}.$$

The implied constant depends only on a . Note, if $g \in A'_i, h \in A'_{i+1}$, then

$$\delta_e^f(g, h) \geq a^{-\epsilon\eta} - 2La^{-\eta} \geq \frac{1}{2}a^{-\epsilon\eta}$$

if ϵ is small enough. It follows that $d \leq 2\epsilon\eta + C_a$ for a constant C_a depending only on a .

Let α be a geodesic segment from e to u in Γ , then $g\alpha$ is a geodesic segment from g to h . There exists a time j such that $g\alpha(j) \in B(e, 2\epsilon\eta + C_a)$. Thus,

$$g \in \bigcup_{j=0}^{|u|} \alpha(j)^{-1} B(e, 2\epsilon\eta + C_a)$$

which gives at most $(|u|+1)D_a e^{2\epsilon\eta v}$ possibilities for g for some v , since balls Γ grow at most exponentially. The constant D_a only depends on a . Arguing similarly for h we obtain

$$N(u) \lesssim (|u|+1)^2 e^{4\epsilon\eta v}.$$

Furthermore, note $\delta_e^f(g, h) \leq f(\eta - |u|/2)|u|$ whenever $|g|, |h| > \eta$. It follows that $a^{-\epsilon\eta} \lesssim a^{|u|/2-\eta}|u|$ so $|u| > \eta$ assuming ϵ is small enough. Thus,

$$(6) \quad \int f_i dm \lesssim e^{2\epsilon\eta} \sum_{|u|>\eta} (|u|+1)^2 e^{4\epsilon\eta v} G^2(e, u) a^{-|u|}$$

Proposition 3.1, together with (6) show that

$$\int f_i dm \lesssim e^{2\epsilon(1+2v)\eta} \sum_{n=\eta}^{\infty} (n+1)^2 a^{-n} \lesssim e^{2\epsilon(1+2v)\eta} a^{-\eta/2}.$$

Thus, letting ϵ be small enough (not depending on η) we obtain

$$\int f_i dm \leq a^{-\eta/4},$$

showing (5) and thus completing the proof. \square

Following the arguments of [11, Sections 4,5], we obtain as a corollary weak Floyd-Ancona inequalities at the spectral radius.

Theorem 3.6. *Let Γ be a finitely generated group and let μ be a symmetric probability measure whose finite support generates Γ . For any $\delta > 0$, there exists $C \geq 0$ such that for any $x, y, z \in \Gamma$ with $\delta_y^f(x, z) > \delta$, we have for every $r \leq R$*

$$\frac{1}{C} G_r(x, y) G_r(y, z) \leq G_r(x, z) \leq C G_r(x, y) G_r(y, z).$$

We do not give the proof here and we refer to [11, Theorem 5.1]. We will prove anyway a stronger result in the following subsection.

Following [11], we also obtain the following corollary, which is a consequence of Proposition 3.5, see [11, Theorem 5.2] for more details.

Corollary 3.7. *Let Γ be a finitely generated group and let μ be a symmetric probability measure whose finite support generates Γ . For every $\delta > 0$ and $\epsilon > 0$ there exists η such that for every x, y, z with $\delta_y^f(x, z) \geq \delta$,*

$$G(x, z; B_\eta(y)^c) \leq \epsilon G(x, z).$$

3.3. Floyd-Ancona inequalities. We prove a strong version of Floyd-Ancona inequalities. We start proving a stronger version of Proposition 3.5 and we consider a finitely generated group Γ . We define the function $h_t(\eta)$ by

$$h_t(\eta) = -\log \sup \{G(x, y; B_\eta(e)^c), \delta_e^f(x, y) \geq t\}.$$

Proposition 3.5 shows that $h_t(\eta) \geq tc^\eta$, where $c > 1$. Let $h(\eta) = h_{1/2\eta}(\eta)$. Then, h is super-linear, that is

$$(7) \quad \frac{h(\eta)}{\eta} \rightarrow \infty, \eta \rightarrow \infty.$$

Moreover, for every x, y such that $\delta_e^f(x, y) \geq \frac{1}{2\eta}$, we have

$$(8) \quad G(x, y; B_\eta(e)^c) \leq e^{-h(\eta)}.$$

We now define the notion of regularly entering a ball centered at e for a path as follows. What we call a path here is a finite sequence of elements of Γ , not necessarily adjacent in the Cayley graph. We fix $\delta > 0$ and $x, y \in \Gamma$ such that $\delta_e^f(x, y) \geq \delta$. Let $E_\eta(x)$, respectively $E_\eta(y)$, be the set of $z \in B_\eta(e)$ such that $\delta_e^f(x, z) \leq \frac{1}{\eta}$, respectively $\delta_e^f(z, y) \leq \frac{1}{\eta}$.

Definition 3.8. *Let $x, y \in \Gamma$ be such that $\delta_e^f(x, y) \geq \delta$. Call a path from x to y η -regular if*

- (1) *it enters $B_\eta(e)$,*
- (2) *its entrance point in $B_\eta(e)$ is in $E_\eta(x)$,*
- (3) *its exit point in $B_\eta(e)$ is in $E_\eta(y)$.*

Beware that the definition depends on δ . Denote by $\text{reg}(\eta, \delta)$ the set of η -regular paths from x to y , where $\delta_e(x, y) \geq \delta$. For technical reasons, we will need a slight modification of $\text{reg}(\eta, \delta)$ in the following.

Definition 3.9. *Let x, y be such that $\delta_e^f(x, y) \geq \delta$. We say that a path from x to y is in $\text{reg}'(\eta, \delta)$ if it satisfies the assumption of Definition 3.8, replacing $E_\eta(x)$ and $E_\eta(y)$ by $E_{2\eta}(x)$ and $E_{2\eta}(y)$ respectively.*

Note that we keep $B_\eta(e)$ unchanged and only replace $E_\eta(x)$ and $E_\eta(y)$ with $E_{2\eta}(x)$ and $E_{2\eta}(y)$ in Definition 3.9. We will actually both need to deal with paths in $\text{reg}(\eta, \delta)$ and with paths in $\text{reg}'(\eta, \delta)$ in our proofs below.

Let $G_r(x, y; \text{reg}'(\eta, \delta)^c)$ be the Green function at r restricted to paths that are not η -regular and $G_r(x, y; \text{reg}'(\eta, \delta))$ be the Green function at r restricted to η -regular paths. Precisely,

$$G_r(x, y; \text{reg}'(\eta, \delta)^c) = \sum_{n \geq 0} \sum_{\substack{(z_1, \dots, z_{n-1}) \\ \notin \text{reg}'(\eta, \delta)}} r^n \mu(x^{-1}z_1) \mu(z_1^{-1}z_2) \dots \mu(z_{n-1}^{-1}y)$$

and

$$G_r(x, y; \text{reg}'(\eta, \delta)) = \sum_{n \geq 0} \sum_{\substack{(z_1, \dots, z_{n-1}) \\ \in \text{reg}'(\eta, \delta)}} r^n \mu(x^{-1}z_1) \mu(z_1^{-1}z_2) \dots \mu(z_{n-1}^{-1}y).$$

We have the following enhanced version of Proposition 3.5.

Proposition 3.10. *For each $\delta > 0$ there exists a super-linear function h_δ such that for all $x, y, z \in \Gamma$ with $\delta_z^f(x, y) > \delta$, we have*

$$G(x, y; \text{reg}'(\eta, \delta)^c) \leq e^{-h_\delta(\eta)}.$$

Proof. Consider a path from x to y which is not in $\text{reg}'(\eta, \delta)$. Then, either it does not enter $B_\eta(e)$ or denoting by w_1 and w_2 its entrance and exit points in $B_\eta(e)$, either $\delta_e^f(x, w_1) > 1/2\eta$ or $\delta_e^f(w_2, y) > 1/2\eta$. Denote by $G(x, y; \text{reg}'_1(\eta, \delta)^c)$ the Green function restricted to paths satisfying the condition on w_1 and by $G(x, y; \text{reg}'_2(\eta, \delta)^c)$ the Green function restricted to paths satisfying the condition on w_2 . We thus have

$$G(x, y; \text{reg}'(\eta, \delta)^c) \leq G(x, y; B_\eta(e)^c) + G(x, y; \text{reg}'_1(\eta, \delta)^c) + G(x, y; \text{reg}'_2(\eta, \delta)^c).$$

Proposition 3.5 shows that

$$G(x, y; B_\eta(e)^c) \leq e^{-h_0(\eta)},$$

for some super-linear function h_0 .

We now deal with $G(x, y; \text{reg}'_1(\eta, \delta)^c)$. Conditioning on the first visit to $B_\eta(e)$, we have

$$G(x, y; \text{reg}'_1(\eta, \delta)^c) \leq \sum_{\substack{w_1 \in B_\eta(e) \\ \delta_e^f(x, w_1) > 1/2\eta}} G(x, w_1; B_\eta(e)^c) G(w_1, y).$$

Then, (7) shows that

$$G(x, w_1; B_\eta(e)^c) \leq e^{-h_1(\eta)}$$

for some super-linear function h_1 . Since balls grow at most exponentially and since $G(w_1, y)$ is uniformly bounded, $\sum_{w_1 \in B_\eta(e), \delta_e^f(x, w_1) > 1/2\eta} G(w_1, y)$ grows at most exponentially in η . Since h_1 is super-linear, up to changing h_1 , we get

$$G(x, y; \text{reg}'_1(\eta, \delta)^c) \leq e^{-h_1(\eta)}.$$

Similarly, we have

$$G(x, y; \text{reg}'_2(\eta, \delta)^c) \leq e^{-h_2(\eta)}.$$

Letting h_3 be the infimum of h_0, h_1, h_2 and $h = h_3 - \log 3$, we get

$$G(x, y; \text{reg}'(\eta, \delta)^c) \leq e^{-h(\eta)}.$$

Then, h is super-linear and only depends on δ , which concludes the proof. \square

Note that the same proof shows that

$$G(x, y; \text{reg}(\eta, \delta)^c) \leq e^{-h_\delta(\eta)}$$

for some super-linear function h_δ , although we will not need this in the following.

Definition 3.11. Let $\Omega \subset \Gamma$ and let $x \in \Gamma$. For $\lambda \geq 1$ and $c \geq 0$, we say that Ω is (λ, c) -starlike around x if for every $y \in \Omega$, there exists a path α from y to x that stay inside Ω and whose length is at most $\lambda d(y, x) + c$.

We deduce from Proposition 3.10 a strengthened version of weak Floyd-Ancona inequalities.

Proposition 3.12. Let Γ be a finitely generated group and let μ be a symmetric admissible finitely supported probability measure on Γ . For every $\lambda \geq 1, c \geq 0$ and $\delta \geq 0$, there exist $C \geq 0$ such that the following holds. Let x, y, z be three points in Γ with $\delta_z^f(x, y) \geq \delta$. Let $\Omega \subset \Gamma$ be a (λ, c) -starlike set around z . Then, for every $r \leq R$, we have

$$\frac{1}{C} G_r(x, z; \Omega) G_r(z, y; \Omega) \leq G_r(x, y; \Omega) \leq C G_r(x, z; \Omega) G_r(z, y; \Omega).$$

Note that taking $\Omega = \Gamma$, we recover Theorem 3.6.

Proof. We fix $r \leq R$. To simplify the notations, we do not refer to r in the following. In particular, we will write $G = G_r$ for the Green function. Notice that we can assume for simplicity that $z = e$. Also, the inequality

$$G(x, e; \Omega)G(e, y; \Omega) \leq CG(x, y; \Omega)$$

is always satisfied (see for example [16, (2.4)]) so we just need to prove the other inequality.

We fix η_0 that will be chosen large enough later. Consider a path for the random walk from x to y . Then, it is either η_0 -regular or not. Hence,

$$G(x, y; \Omega) \leq G(x, y; \text{reg}(\eta_0, \delta) \cap \Omega) + G(x, y; \text{reg}(\eta_0, \delta)^c \cap \Omega),$$

where $G(x, y; \text{reg}(\eta_0, \delta) \cap \Omega)$ denotes the Green function restricted to η_0 -regular paths that stay inside Ω and $G(x, y; \text{reg}(\eta_0, \delta)^c \cap \Omega)$ denotes the Green function restricted to paths that stay inside Ω and that are not η_0 -regular.

If a path is η_0 -regular, in particular, it enters $B_{\eta_0}(e)$, so that conditioning by the first visit to $B_{\eta_0}(e)$, we get

$$G(x, y; \text{reg}(\eta_0, \delta) \cap \Omega) \leq \sum_{w \in B_{\eta_0}(e)} G(x, w; B_{\eta_0}(e)^c \cap \Omega)G(w, y; \Omega).$$

Since Ω is (λ, c) -starlike around e , if $w \in B_{\eta_0}(e) \cap \Omega$, there is a path from w to e that stays inside Ω and whose length is bounded by $\lambda\eta_0 + c$. In particular, we have $G(x, w; \Omega) \leq C_{\eta_0}G(x, e; \Omega)$ and $G(w, y; \Omega) \leq C_{\eta_0}G(e, y; \Omega)$. Summing over all possible $w \in B_{\eta_0}$, we obtain

$$(9) \quad G(x, y; \text{reg}(\eta_0, \delta) \cap \Omega) \leq C'_{\eta_0} G(x, e; \Omega)G(e, y; \Omega).$$

We now find an upper bound of $G(x, y; \text{reg}(\eta_0, \delta)^c \cap \Omega)$. Among paths from x to y that are not η_0 -regular, some of them are $2\eta_0$ regular and the other ones are not. We thus have

$$(10) \quad \begin{aligned} G(x, y; \text{reg}(\eta_0, \delta)^c \cap \Omega) &\leq G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta) \cap \Omega) \\ &\quad + G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta)^c \cap \Omega). \end{aligned}$$

Consider a path α from x to y which is not η_0 -regular, but which is $2\eta_0$ -regular. Let u , respectively v be the entrance, respectively exit points in $B_{2\eta_0}$. If η_0 is large enough, then $\delta_e^f(u, v) \geq \delta/2$, since $\delta_e^f(x, u) \leq 1/2\eta_0$, $\delta_e^f(v, y) \leq 1/2\eta_0$ and $\delta_e^f(x, y) \geq \delta$. Let β be the sub-path of α from u to v and assume first that β enters $B_{\eta_0}(e)$. Let w_1 and w_2 be the entrance and exit point of β in $B_{\eta_0}(e)$. Since $B_{\eta_0}(e) \subset B_{2\eta_0}(e)$, w_1 and w_2 also are the entrance and exit point of α in $B_{\eta_0}(e)$. By definition, α is not η_0 -regular, so that either $\delta_e^f(x, w_1) > \frac{1}{\eta_0}$ or $\delta_e^f(w_2, y) > \frac{1}{\eta_0}$. In the first case, we get $\delta_e^f(u, w_1) > \frac{1}{2\eta_0}$, in the second case, we get $\delta_e^f(w_2, v) > \frac{1}{2\eta_0}$. In both cases, β is in $\text{reg}'(\eta_0, \delta/2)$. Note that this is also the case if β does not enter $B_{\eta_0}(e)$ at all. Conditioning on the first and last visit to $B_{2\eta_0}(e)$, we have

$$\begin{aligned} G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta) \cap \Omega) &\leq \sum_{u, v \in B_{2\eta_0}(e)} G(x, u; \Omega) \\ &\quad G(u, v; \text{reg}'(\eta_0)^c)G(v, y; \Omega). \end{aligned}$$

Then, Proposition 3.10 yields

$$G(u, v; \text{reg}'(\eta_0)^c) \leq e^{-h_{\delta/2}(\eta_0)}$$

for some super-linear function $h_{\delta/2}$. Since Ω is (λ, c) -starlike around e , if u and v are in Ω , then $G(u, e; \Omega) \geq p^{d(e, u)} \geq p^{2\eta_0}$ for some $p < 1$ and similarly, $G(e, v; \Omega) \geq p^{2\eta_0}$. Up to changing $h_{\delta/2}(\eta_0)$ into some other super-linear function, we thus have

$$G(u, v; \text{reg}'(\eta_0)^c) \leq e^{-h_{\delta/2}(\eta_0)} G(u, e; \Omega) G(e, v; \Omega).$$

Also,

$$G(x, u; \Omega) G(u, e; \Omega) \leq CG(x, e; \Omega)$$

and

$$G(e, v; \Omega) G(v, y; \Omega) \leq CG(e, y; \Omega).$$

This shows that

$$G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta) \cap \Omega) \leq C^2 e^{-h_{\delta/2}(\eta_0)} \sum_{u, v \in B_{2\eta_0}(e)} G(x, e; \Omega) G(e, y; \Omega).$$

Since balls grow at most exponentially, changing $h_{\delta/2}$ again, we get

$$G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta) \cap \Omega) \leq e^{-h_{\delta/2}(\eta_0)} G(x, e; \Omega) G(e, y; \Omega).$$

We deal with the second term in (10). We have by definition

$$G(x, y; \text{reg}(\eta_0, \delta)^c \cap \text{reg}(2\eta_0, \delta)^c \cap \Omega) \leq G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \Omega).$$

A path which is not $2\eta_0$ -regular can be $4\eta_0$ -regular or not. We thus have

$$\begin{aligned} G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \Omega) &\leq G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \text{reg}(4\eta_0, \delta) \cap \Omega) \\ &\quad + G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \text{reg}(4\eta_0, \delta)^c \cap \Omega). \end{aligned}$$

We again first deal with $G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \text{reg}(4\eta_0, \delta) \cap \Omega)$. Decomposing a path in $\text{reg}(2\eta_0, \delta)^c \cap \text{reg}(4\eta_0, \delta)$ according to its entrance and exit point in $B_{4\eta_0}(e)$, we similarly get

$$G(x, y; \text{reg}(2\eta_0, \delta)^c \cap \text{reg}(4\eta_0, \delta) \cap \Omega) \leq e^{-h_{\delta/2}(2\eta_0)} G(x, e; \Omega) G(e, y; \Omega).$$

Doing a similar decomposition for path which are not $2^i\eta_0$ -regular but which are $2^{i+1}\eta_0$ -regular, we finally get

$$(11) \quad G(x, y; \text{reg}(\eta_0, \delta)^c \cap \Omega) \leq \sum_{i \geq 0} e^{-h(2^i\eta_0)} G(x, e; \Omega) G(e, y; \Omega)$$

for some super-linear function h . The sum $\sum_{i \geq 0} e^{-h(2^i\eta_0)}$ is thus finite, so that combining (9) with (11), we have

$$G(x, y; \Omega) \leq CG(x, e; \Omega) G(e, y; \Omega).$$

This concludes the proof. \square

We can now prove that strong Floyd-Ancona inequalities hold. Recall the following definition from the introduction.

Definition 3.13. *Let x, y and x', y' be four points in Γ . Let $\delta \geq 0$. We say that the pairs (x, y) and (x', y') δ -fellow-travel for a time at least n if there exist points $z_1, \dots, z_n \in \Gamma$ such that for all i , $\delta_{z_i}^f(x, y) \geq \delta$ and $\delta_{z_i}^f(x', y') \geq \delta$.*

Theorem 3.14. *Let Γ be a finitely generated group and let μ be a symmetric admissible finitely supported probability measure on Γ . For every $\delta \geq 0$, there exist $K \geq 0$ and $0 < \rho < 1$ such that the following holds. For every x, y, x', y' such that the pairs (x, y) and (x', y') δ -fellow travel for a time at least n , we have for every $r \leq R$*

$$\left| \frac{G_r(x, y)G_r(x', y')}{G_r(x, y')G_r(x', y)} - 1 \right| \leq K\rho^n.$$

Before proving this theorem, we introduce the following terminology.

Definition 3.15. *Let α be a geodesic in $\text{Cay}(\Gamma, S)$ and let $\delta > 0$. A δ -transition point on α is a point y on α such that for every x, z on α such that x, y, z are aligned in this order, we have $\delta_y^f(x, z) \geq \delta$.*

Beware that the terminology transition point is also commonly used for relatively hyperbolic groups, as noted in the introduction. However, if Γ is relatively hyperbolic, then a transition in sens of Definition 1.7 is also a transition point in the sense of Definition 3.15, according to Proposition 1.8. We will denote by $\text{Tr}_\delta(\alpha)$ the set of δ -transition points on a geodesic α .

Lemma 3.16. *For every δ there exists $D(\delta) \geq 0$ and $\delta' \leq \delta$ such that the following holds. Let α be a geodesic with endpoints α_- and α_+ and let w be a point in Γ satisfying $\delta_w^f(\alpha_-, \alpha_+) \geq \delta$. Then there exists a δ' -transition point w' on α such that $d(w, w') \leq D(\delta)$.*

Proof. According to Karlsson's lemma [24, Lemma 1], w is within a bounded distance of a point w' on α . By definition of the Floyd distance, there exists δ' such that $\delta_{w'}(\alpha_-, \alpha_+) \geq \delta'$. Thus, we only need to prove that whenever w is on a geodesic α and satisfies $\delta_w^f(\alpha_-, \alpha_+) \geq \delta$, then w is a δ' -transition point on α for some δ' .

Assume this is not the case. Then, there exist a sequence w_n on geodesics $\alpha^{(n)}$ and a sequence of points x_n, y_n on $\alpha^{(n)}$ such that $\delta_{w_n}(\alpha_-^{(n)}, \alpha_+^{(n)}) \geq \delta$ and $\delta_{w_n}(x_n, y_n)$ tends to 0. By left invariance, we can assume that $w_n = w$ is fixed. Necessarily, x_n and y_n tend to infinity. Hence, up to changing x_n and y_n , the subgeodesics from $\alpha_-^{(n)}$ to x_n and from $\alpha_+^{(n)}$ to y_n are arbitrarily far away from w . In particular, the Floyd length of these subgeodesic, seen from w , goes to 0. Moreover, there is a sequence of paths from x_n to y_n whose Floyd length seen from w goes to 0. Concatenating these paths yields a path from $\alpha_-^{(n)}$ to $\alpha_+^{(n)}$ whose Floyd length goes to 0, contradicting the fact that $\delta_w(\alpha_-^{(n)}, \alpha_+^{(n)}) \geq \delta$. \square

Lemma 3.17. *For every $\delta > 0$ there exists $\Delta(\delta) \geq 0$ and $\delta_0 \leq \delta$ such that the following holds. Let $x, y, z \in \Gamma$ and consider a geodesic triangle with vertices x, y, z . Then any point w on $[x, y]$ satisfying $\delta_w^f(x, y) \geq \delta$ is within a distance at most $\Delta(\delta)$ from $\text{Tr}_{\delta_0}([x, z]) \cup \text{Tr}_{\delta_0}([y, z])$.*

Proof. Since $\delta_w^f(x, y) \geq \delta$, either $\delta_w^f(x, z) \geq \delta/2$ or $\delta_w^f(y, z) \geq \delta/2$. Thus, the result follows from Lemma 3.16. \square

We now prove Theorem 3.14.

Proof. Let $D = D(\delta)$ and δ' be the numbers given by Lemma 3.16. Then, for every i , there exist δ' -transition points \hat{z}_i on a geodesic $[x, y]$ such that $d(z_i, \hat{z}_i) \leq D$. Up to reindexing the z_i , we may assume that $d(\hat{z}_i, x) \leq d(\hat{z}_{i+1}, x)$. We fix some integer k

that we will choose later and we consider the points $z_k, z_{2k}, \dots, z_{mk}$, with $m = \lfloor n/k \rfloor$. By definition, for every i , $\hat{z}_{(i+1)k}$ is a δ' -transition point on the subgeodesic of $[x, y]$ from \hat{z}_{ik} to y . Let $\Delta = \Delta(\delta')$ and $\delta_0 \leq \delta'$ be the numbers given by Lemma 3.17 for δ' . We define sets Ω_i by

$$\Omega_i = \{w \in \Gamma, d(z_{(m-j+1)k}, \text{Tr}_{\delta_0}([w, y])) \leq \Delta + D\},$$

for some geodesic $[w, y]$ from w to y . We first prove the following.

Lemma 3.18. *If k is large enough, then $\Omega_{i+1} \subset \Omega_i$. Moreover, $x, x' \in \Omega_m$ and $y, y' \notin \Omega_1$.*

Proof. Let $w \in \Omega_{i+1}$ and consider a geodesic $[w, y]$ from w to y and a point \tilde{w} on $\text{Tr}_{\delta'}([w, y])$ with

$$(12) \quad d(\tilde{w}, z_{(m-i)k}) \leq \Delta + D.$$

Let $[w, \hat{z}_{(m-i)k}]$ be a geodesic from w to $\hat{z}_{(m-i)k}$ and $[\hat{z}_{(m-i)k}, y]$ the subgeodesic of $[x, y]$ from $\hat{z}_{(m-i)k}$ to y . Consider the triangle composed by these three geodesics. Since $\hat{z}_{(m-i+1)k}$ is a δ' -transition point on $[\hat{z}_{(m-i)k}, y]$, Lemma 3.17 shows that there is a δ_0 -transition point \tilde{z} which is either on $[w, y]$ or on $[w, \hat{z}_{(m-i)k}]$ satisfying

$$(13) \quad d(\tilde{z}, \hat{z}_{(m-i+1)k}) \leq \Delta.$$

If we prove that \tilde{z} is on $[w, y]$, then $w \in \Omega_i$ since $d(z_{(m-i+1)k}, \hat{z}_{(m-i+1)k}) \leq D$. To show that $\Omega_{i+1} \subset \Omega_i$, we just need to prove that if k is large enough, then \tilde{z} cannot be on $[w, \hat{z}_{(m-i)k}]$.

Since \tilde{w} is on $[w, y]$ and $d(\tilde{w}, \hat{z}_{(m-i)k})$ is uniformly bounded by (12), there exists c such that

$$d(w, y) \geq d(w, \hat{z}_{(m-i)k}) + d(\hat{z}_{(m-i)k}, y) - c.$$

Also, $\hat{z}_{(m-i+1)k}$ is by definition on a geodesic from $\hat{z}_{(m-i)k}$ to y , so we get

$$d(w, y) \geq d(w, \hat{z}_{(m-i)k}) + d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k}) + d(\hat{z}_{(m-i+1)k}, y) - c.$$

Finally, by the triangle inequality, $d(w, y) - d(\hat{z}_{(m-i+1)k}, y) \leq d(w, \hat{z}_{(m-i+1)k})$ so we finally get

$$(14) \quad d(w, \hat{z}_{(m-i+1)k}) \geq d(w, \hat{z}_{(m-i)k}) + d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k}) - c.$$

Now assume by contradiction that \tilde{z} is on $[w, \hat{z}_{(m-i)k}]$. Then, $\hat{z}_{(m-i+1)k}$ is within a bounded distance of $[w, \hat{z}_{(m-i)k}]$ by (13) and so there exists c' such that

$$(15) \quad d(w, \hat{z}_{(m-i)k}) \geq d(w, \hat{z}_{(m-i+1)k}) + d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k}) - c'.$$

Summing (14) and (15), we get

$$2d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k}) \leq c + c',$$

which cannot happen if $d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k})$ is large enough, which is true if k is large enough.

This concludes the first part of the lemma. We now prove that $x, x' \in \Omega_m$ and $y, y' \notin \Omega_1$. By symmetry in the definition of Ω_i , we just need to prove that $x \in \Omega_m$ and $y \notin \Omega_1$. By definition \hat{z}_k is in $\text{Tr}_{\delta_0}([x, y])$ and since $d(z_k, \hat{z}_k) \leq D$, $x \in \Omega_m$. Also, $\text{Tr}_{\delta_0}([y, y])$ is reduced to $\{y\}$ and if k is large enough, then $d(\hat{z}_m, y) > \Delta + D$, so $y \notin \Omega_1$. This concludes the proof of the lemma. \square

Let $h_1(w) = \frac{G_r(w,y)}{G_r(x',y)}$ and $h_2(w) = \frac{G_r(w,y')}{G_r(x',y')}$. Then, h_1 is r -harmonic everywhere except at y and h_2 is r -harmonic everywhere except at y' . In particular, both are r -harmonic on Ω_1 . We will construct functions h_1^j and h_2^j that are r -harmonic on Ω_{j+1} , with $h_1^0 = h_1$ and $h_2^0 = h_2$. We will also construct functions φ^j satisfying the following conditions

- a) $h_i^j = h_i^{j+1} + \varphi^{j+1}$ on Ω_{j+2} ,
- b) $h_i^j \geq \varphi^{j+1} \geq \beta_0 h_i^j$ for some $\beta_0 > 0$.

Before constructing them, let us show how we can conclude. By construction, we have $h_i^{j+1} \leq (1 - \beta_0)h_i^j$ and so $h_i^{m-1} \leq (1 - \beta_0)^{m-2}h_i$. By definition, on Ω_m , we have $h_i = \sum_j \varphi^j + h_i^{m-1}$. Thus, since $x \in \Omega_m$,

$$|h_1(x) - h_2(x)| = |h_1^{m-1}(x) - h_2^{m-1}(x)| \leq (1 - \beta_0)^{m-2}(h_1(x) + h_2(x)).$$

Proposition 3.12 shows that $\frac{G_r(x,y)}{G_r(x,z_k)G_r(z_k,y)}$ and $\frac{G_r(x',y)}{G_r(x',z_k)G_r(z_k,y)}$ are bounded away from 0 and infinity. The ratio of these two quantities is $\frac{h_1(x)G_r(x',z_k)}{G_r(x,z_k)}$ so that this quantity also is bounded away from 0 and infinity. Similarly, $\frac{h_2(x)G_r(x',z_k)}{G_r(x,z_k)}$ is bounded away from 0 and infinity. Finally, we see that $\frac{h_1(x)}{h_2(x)}$ is bounded away from 0 and infinity. We thus get

$$\left| \frac{h_1(x)}{h_2(x)} - 1 \right| \leq C(1 - \beta_0)^{m-2} \leq \frac{C}{(1 - \beta_0)^3} \left((1 - \beta_0)^{1/k} \right)^n.$$

This proves the theorem.

Let us construct h_i^j and φ^j by induction to conclude. Assume that the functions h_i^j and φ^j were constructed. Since h_1^j is r -harmonic on Ω_{j+1} , for $w \in \Omega_{j+2} \subset \Omega_{j+1}$, we have

$$(16) \quad h_1^j(w) = \sum_{w' \in \Omega_{j+1}^c} G(w, w'; \Omega_{j+1}) h_1^j(w').$$

We want to apply Proposition 3.12 to $G(w, w'; \Omega_{j+1})$. We thus have to prove that Ω_{j+1} is starlike in the sense of Definition 3.11.

Lemma 3.19. *There exist λ and c such that for every j , the set Ω_{j+1} is (λ, c) -starlike around $z_{(m-j)k}$.*

Proof. Let $w \in \Omega_{j+1}$. Then there exists a δ_0 -transition point w' on a geodesic $[w, y]$ such that $d(z_{(m-j)k}, w') \leq \Delta + D$. Consider the path α obtained by concatenating the subgeodesic α_1 of $[w, y]$ from w to w' and a geodesic α_2 from w' to $\hat{z}_{(m-j)k}$. Then, α_1 stays in Ω_{j+1} . Moreover, for any $w'' \in \Gamma$, w'' is a δ_0 -transition point on any geodesic $[w'', y]$ from w'' to y by definition. Hence, if w'' is on α_2 , then $d(z_{(m-j)k}, w'') \leq \Delta + D$ and so $w'' \in \Omega_{j+1}$. Thus, α itself is contained in Ω_{j+1} . Moreover, since α_1 is a geodesic, the length of α is at most $d(w, w') + \Delta \leq d(w, z_{(m-j)k}) + 2\Delta + 2D$. This concludes the proof of the lemma. \square

To apply Proposition 3.12, we also prove the following.

Lemma 3.20. *If k is large enough, then there exists δ_1 , only depending on k and δ , such that for $w \in \Omega_{j+2}$, $w' \notin \Omega_{j+1}$, we have $\delta_{z_{(m-j)k}}^f(w, w') \geq \delta_1$.*

Proof. We will use the thin-triangle property twice. Let $w' \notin \Omega_{j+1}$ and $w \in \Omega_{j+2}$. Consider the triangle consisting of geodesics $[\hat{z}_{(m-j-1)k}, w']$ from $\hat{z}_{(m-j-1)k}$ to w' , $[w', y]$ from w' to y and the subgeodesic $[\hat{z}_{(m-j-1)k}, y]$ of $[x, y]$ from $\hat{z}_{(m-j-1)k}$ to y .

Then $\hat{z}_{(m-j)k}$ is a δ' -transition point on $[\hat{z}_{(m-j-1)k}, y]$. Recall that we denote by $\Delta = \Delta(\delta')$ and $\delta_0 \leq \delta'$ the numbers given by Lemma 3.17 for δ' . Then, according to this lemma, there is a δ_0 -transition point \tilde{z} which lies either on $[w', y]$ or on $[\hat{z}_{(m-j-1)k}, w']$ and satisfying

$$(17) \quad d(\tilde{z}, \hat{z}_{(m-j)k}) \leq \Delta.$$

Since $w' \notin \Omega_{j+1}$, \tilde{z} has to lie on $[\hat{z}_{(m-j-1)k}, w']$.

Now, consider the triangle given by the geodesic $[\hat{z}_{(m-j-1)k}, w']$ and geodesics $[w, w']$ from w to w' and $[w, \hat{z}_{(m-j-1)k}]$ from w to $\hat{z}_{(m-j-1)k}$. Then, using again Lemma 3.17, there exist numbers $\Delta(\delta_0) \geq 0$ and $\delta'_0 \leq \delta_0$ and there exists a δ'_0 -transition point \tilde{z}' which lies either on $[w, w']$ or on $[w, \hat{z}_{(m-j-1)k}]$ such that $d(\tilde{z}, \tilde{z}') \leq \Delta(\delta_0)$. We prove by contradiction that if k is large enough, then \tilde{z}' cannot lie on $[w, \hat{z}_{(m-j-1)k}]$. Indeed, if it were the case, then $d(\tilde{z}', \hat{z}_{(m-j)k})$ would be uniformly bounded by (17) and so we would have

$$(18) \quad d(w, \hat{z}_{(m-j-1)k}) \geq d(w, \hat{z}_{(m-j)k}) + d(\hat{z}_{(m-j)k}, \hat{z}_{(m-j-1)k}) - c.$$

However, since $w \in \Omega_{j+2}$, $\hat{z}_{(m-j-1)k}$ is within a bounded distance of a point on a geodesic from w to y . Thus,

$$d(w, y) \geq d(w, \hat{z}_{(m-j-1)k}) + d(\hat{z}_{(m-j-1)k}, y) - c',$$

so that

$$(19) \quad d(w, \hat{z}_{(m-j)k}) \geq d(w, \hat{z}_{(m-j-1)k}) + d(\hat{z}_{(m-j-1)k}, \hat{z}_{(m-j)k}) - c'.$$

Summing (18) and (19), we would get

$$2d(\hat{z}_{(m-j-1)k}, \hat{z}_{(m-j)k}) \leq c + c'.$$

This cannot happen if $d(\hat{z}_{(m-i)k}, \hat{z}_{(m-i+1)k})$ is large enough, which is true if k is large enough. This proves that \tilde{z}' lies on $[w, w']$. In particular, this proves that $\hat{z}_{(m-j)k}$ is within a bounded distance of a transition point on $[w, w']$. Since $z_{(m-j)k}$ is within a bounded distance of $\hat{z}_{(m-j)k}$, this concludes the proof of the lemma. \square

We now fix k large enough so that all conditions above are satisfied. Proposition 3.12 shows that for $w \in \Omega_{j+2}$ and $w' \in \Omega_{j+1}$, we have

$$G(w, w'; \Omega_{j+1}) \asymp G(w, z_{(m-j-1)k}; \Omega_{j+1})G(z_{(m-j-1)k}, w'; \Omega_{j+1}).$$

Then, (16) shows that for $w \in \Omega_{j+2}$,

$$h_1(w) \asymp \sum_{w' \in \Omega_{j+1}} G(w, z_{(m-j-1)k}; \Omega_{j+1})G(z_{(m-j-1)k}, w'; \Omega_{j+1}).$$

In other words,

$$h_1(w) \asymp G(w, z_{(m-j-1)k}; \Omega_{j+1})h_1(z_{(m-j-1)k}).$$

Since $x' \in \Omega_{j+2}$, we also have

$$\frac{h_1^j(w)}{h_1^j(x')} \asymp \frac{G_r(w, z_{(m-j-1)k}; \Omega_{j+1})}{G_r(x', z_{(m-j-1)k}; \Omega_{j+1})}.$$

For small enough β , we set

$$\varphi^{j+1} = \beta h_1^j(x') \frac{G_r(w, z_{(m-j-1)k}; \Omega_{j+1})}{G_r(x', z_{(m-j-1)k}; \Omega_{j+1})}.$$

If β is small enough, then $h_1^j \geq \varphi^{j+1} \geq \beta_0 h_1^j$, for some fixed $\beta_0 > 0$. We then set $h_1^{j+1} = h_1^j - \varphi^{j+1}$. Notice that φ^{j+1} does not depend on h_1^j , but only on its value at x' . By induction, it only depends on $h_1^0(x') = 1$, so it does not depend on h_1 . We also have $h_2^j \geq \varphi^{j+1} \geq \beta_0 h_2^j$, so that the functions h_i^j and φ^{j+1} thus constructed satisfy both Conditions a) and b) above. This concludes the proof of the theorem. \square

Theorem 1.6 follows from Proposition 3.12 and Theorem 3.14.

4. DESCRIPTION OF THE MARTIN BOUNDARY

This section is devoted to the proofs of Theorem 1.2 and Theorem 1.4. We consider a non-elementary relatively hyperbolic group Γ and a symmetric probability measure μ whose finite support generates Γ . Recall the definition of the r -geometric boundary given in Section 2. Our goal is to prove that the r -geometric boundary coincides with the r -Martin boundary. We start proving the following.

Proposition 4.1. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection of virtually abelian subgroups Ω . Let μ be a symmetric probability measure whose finite support generates Γ and let R be its spectral radius. Let $r \leq R$. Let g_n be a sequence of elements of Γ . Assume that g_n converges to a point in the r -geometric boundary. Then, g_n converges to a point in the r -Martin boundary.*

We will deal separately with conical limit points and parabolic limit points. First, it is proved in [11] that as soon as weak Ancona inequalities (Theorem 3.6) are satisfied, then if g_n converges to a conical limit point, it also converges to the Martin boundary. It is also proved that the corresponding point in the Martin boundary is minimal and that two distinct conical limit points yield two distinct points in the Martin boundary. In particular, we have the following proposition.

Proposition 4.2. *Let Γ be a non-elementary relatively hyperbolic group. Let μ be a symmetric probability measure whose finite support generates Γ and let R be its spectral radius. Let $r \leq R$. Let g_n be a sequence of elements of Γ which converges to a conical limit point. Then, g_n converges to a minimal point in the r -Martin boundary.*

We thus are left with parabolic limit points.

4.1. Parabolic limit points: the degenerate case. We focus here on degenerate parabolic points and we prove the following. If g_n converges to a degenerate-parabolic limit point in the Bowditch boundary, then g_n converges in the R -Martin boundary. We assume that the parabolic subgroups are virtually nilpotent.

Proposition 4.3. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection of virtually nilpotent subgroups. Let μ be a symmetric probability measure whose finite support generates Γ and let R be its spectral radius. Let ξ be a degenerate parabolic limit point. Let g_n be a sequence of elements of Γ converging to ξ . Then, g_n converges to a point in the R -Martin boundary.*

Let H be a parabolic subgroup, so that H is virtually nilpotent. There is a finite index subgroup of H which is isomorphic to a nilpotent group \mathcal{N} . Any section $H/\mathcal{N} \rightarrow H$ identifies H as a set with $\mathcal{N} \times E$, where E is finite. Let $\eta \geq 0$. Then, the η -neighborhood of H , which we denote by $N_\eta(H)$, is identified with $H \times E_\eta$, where E_η is finite. Thus, $N_\eta(H)$ can also be identified with $\mathcal{N} \times E'_\eta$, where E'_η is finite. We will use the notation $E'_\eta = \{1, \dots, N_\eta\}$.

Actually, as a set, Γ can be H -equivariantly identified with $H \times \mathbb{N}$. Indeed, H acts by left multiplication on Γ and the quotient is countable. We order elements in the quotient according to their distance to H . It follows that

- (1) $N_\eta(H)$ can be \mathcal{N} -equivariantly identified with $\mathcal{N} \times \{1, \dots, N_\eta\}$ as above,
- (2) if $\eta \leq \eta'$, then $N_\eta \leq N_{\eta'}$. In other words, the set $\mathcal{N} \times \{1, \dots, N_\eta\}$, identified with $N_\eta(H)$ is a subset of $\mathcal{N} \times \{1, \dots, N_{\eta'}\}$ identified with $N_{\eta'}(H)$.

Recall from Section 2 that we denote by $p_{H,r}$ the first return kernel to H for $r\mu$. More generally, we define the first return kernel to $N_\eta(H)$ for $r\mu$ as

$$p_{H,\eta,r}(h, h') = \sum_{n \geq 0} \sum_{\substack{g_1, \dots, g_{n-1} \\ \notin N_\eta(H)}} r^n \mu(h^{-1}g_1)\mu(g_1^{-1}g_2)\dots\mu(g_{n-2}^{-1}g_{n-1})\mu(g_{n-1}^{-1}h').$$

To simplify the notation, whenever H , r and η are fixed, identifying h, h' with $(x, k), (x', k')$, $x, x' \in \mathcal{N}$ and $k, k' \in \{1, \dots, N_\eta\}$, we write

$$p_{H,\eta,r}((x, k), (x', k')) = p_{k,k'}(x, x').$$

Let $\mathcal{A} = \mathcal{N}/\mathcal{N}'$ be the abelianization of \mathcal{N} . Then, \mathcal{A} is abelian of rank d , so it is isomorphic to $\mathbb{Z}^d \times F$, where F is finite. Let $x \in \mathcal{N}$. Let $\pi(x)$ be the projection of x on \mathbb{Z}^d . That is, π is the composition of the first projection of $A = \mathbb{Z}^d \times F$ on \mathbb{Z}^d and the projection of \mathcal{N} on its abelianization \mathcal{A} . For $u \in \mathbb{R}^d$, we define for $j, k \in \{1, \dots, N_\eta\}$,

$$F_{j,k}(u) = \sum_{x \in \mathcal{N}} p_{j,k}(e, x) e^{\pi(x) \cdot u}.$$

We then let $F(u)$ be the matrix with entries $F_{j,k}(u)$. This matrix is analogous to the matrix $F(u)$ in [7] and [10], which is only defined for abelian groups (that is, we do not need the projection π in there).

By definition, the entries of the n th power $F(u)^n$ of $F(u)$ are given by

$$F(u)^n_{j,k} = \sum_{x \in \mathcal{N}} p_{j,k}^{(n)}(e, x) e^{\pi(x) \cdot u},$$

where $p_{j,k}^{(n)}(e, x)$ is the n th power convolution of $p_{j,k}(e, x)$. Recall that we assume that $\mu(e) > 0$. In particular, the matrix $F(u)$ is strongly irreducible, meaning that there exists n such that all the entries of $F(u)^n$ are positive. The Perron-Frobenius Theorem (see [31, Theorem 1.1]) shows that $F(u)$ has a dominant eigenvalue, which we denote by $\lambda(u)$.

According to [7, Proposition 3.5], the function $u \mapsto \lambda(u)$ is strongly convex where it is defined. The proof is only given for the abelian case (that is \mathcal{N} is abelian and we do not need to use the projection π). However, the exact same proof applies in our situation.

In [26], Margulis shows that positive harmonic functions on \mathcal{N} are constant on left cosets of \mathcal{N}' and thus define positive harmonic functions on the abelianization \mathcal{A} for the induced transition kernel. This result is still true for graphs with polynomial growth, according to [36, Section 25]. Moreover, the minimal harmonic function on

\mathcal{N} are in one-to-one correspondence with the minimal harmonic function on \mathcal{A} . We can actually be very precise about the minimal harmonic functions on \mathcal{N} .

Lemma 4.4. *The t -minimal harmonic functions for the transition kernel $p_{H,\eta,r}$ on $N_\eta(H) = \mathcal{N} \times \{1, \dots, N_\eta\}$ are exactly the functions of the form*

$$(x, k) \in \mathcal{N} \times \{1, \dots, N_\eta\} \mapsto C_k e^{\pi(x) \cdot u},$$

where $u \in \mathbb{R}^d$ satisfies $\lambda(u) = 1/t$ and where the vector with entries C_k is an eigenvector associated with $\lambda(u)$.

Proof. To simplify notations, if h is a function on $\mathcal{N} \times \{1, \dots, N_\eta\}$, we set

$$h_j(x) = h((x, j)).$$

If (x_0, j_0) is fixed and if h is t -harmonic, then [36, Theorem 25.8] states that the function $x \in \mathcal{N} \mapsto \frac{h_{j_0}(xx_0)}{h_{j_0}(x_0)}$ is an exponential, that is, there exists $u \in \mathbb{R}^d$ such that for all $x \in \mathcal{N}$

$$\frac{h_{j_0}(xx_0)}{h_{j_0}(x_0)} = e^{\pi(x) \cdot u}.$$

In particular, for every j , for every x , we have

$$h_j(x) = h_j(e) e^{\pi(x) \cdot u}$$

for some $u \in \mathbb{R}^d$. To find all possible u , let us write the t -harmonicity condition. We have

$$\begin{aligned} (20) \quad h_j(e) &= t \sum_{x \in \mathcal{N}} \sum_{1 \leq k \leq N} p_{j,k}(e, x) h_k(e) e^{\pi(x) \cdot u} \\ &= t \sum_{1 \leq k \leq N} \left(\sum_{x \in \mathcal{N}} p_{j,k}(e, x) e^{\pi(x) \cdot u} \right) h_k(e). \end{aligned}$$

Thus, the vector of \mathbb{R}^{N_η} with entries $h_j(e)$ is an eigenvector for $F(u)$ associated with the eigenvalue $1/t$. Since this vector has non-negative entries, the corresponding eigenvalue is necessarily the dominant eigenvalue $\lambda(u)$, according to the Perron-Frobenius Theorem [31, Theorem 1.1]. Thus, $\lambda(u) = 1/t$. We proved that as a set, the t -minimal Martin boundary is included in this set of functions. Moreover, the topology of pointwise convergence coincide with the Euclidean topology on $\{u, \lambda(u) = 1/t\}$. Thus, the t -minimal Martin boundary can be seen as a Borelian subset of $\{u, \lambda(u) = 1/t\}$.

Conversely, let h be a positive function as in the statement of the lemma. Then, we necessarily have $h_j(e) = C_j$. Moreover, (20) is satisfied, so

$$h_j(x) = th_j(e) e^{\pi(x) \cdot u} = t \sum_{x' \in \mathcal{N}} \sum_{1 \leq k \leq N} p_{j,k}(e, x') h_k(e) e^{\pi(xx') \cdot u}.$$

Let us sum over $x'' = xx'$ to obtain

$$h_j(x) = t \sum_{x'' \in \mathcal{N}} \sum_{1 \leq k \leq N} p_{j,k}(e, x^{-1}x'') h_k(e) e^{\pi(x'') \cdot u} = t \sum_{x,k} p_{j,k}(x, x'') h_k(x'').$$

Thus, h is t -harmonic. Let us prove that it is minimal. To do so, we use the Martin representation theorem and we write

$$h_j(x) = \int_{\{u, \lambda(u)=1/t\}} C_j(v) e^{\pi(x) \cdot v} d\nu(v)$$

for some finite measure ν on $\{u, \lambda(u) = 1/t\}$. For every $x \in \mathcal{N}$,

$$1 = \int_{\{u, \lambda(u)=1/t\}} \frac{C_j(v)}{h_j(e)} e^{\pi(x) \cdot (v-u)} d\nu(v).$$

If the support of ν were not reduced to u , then letting x vary, we would have that this integral is not bounded, which would be a contradiction. Thus, the support of ν is reduced to u and so h is minimal. This concludes the proof. \square

We arbitrarily choose a word distance $d_{\mathcal{N}}$ on \mathcal{N} . Following [10], for $M \geq 0$, the transition kernel $p_{H,\eta,r}$ is said to have exponential moments up to M if for every $k, k' \in \{1, \dots, N_\eta\}$,

$$\sum_{x \in \mathcal{N}} p_{k,k'}(e, x) e^{M d_{\mathcal{N}}(e, x)} < +\infty.$$

It is proved in [10] that up to enlarging η , $p_{H,\eta,1}$ has exponential moments up to M , where η depends on M . The proof crucially uses the fact that $1 < R$. It actually applies for $r < R$, but we will need the same result at the spectral radius, so we have to produce a new proof. This new proof will use the new deviation inequalities from Section 3.

Lemma 4.5. *Let $M \geq 0$. There exists $\eta_M \geq 0$ such that for every $\eta \geq \eta_M$, for every $r \leq R$, $p_{H,\eta,r}$ has exponential moments up to M .*

Proof. Let $x \in \mathcal{N}$ and $j, j' \in \{1, \dots, N_\eta\}$. If the first return to $N_\eta(H)$, starting at $g = (e, j)$ is at $g' = (x, j')$, then there is a trajectory of the random walk X_0, \dots, X_{n+1} such that $X_0 = (e, j)$, $X_{n+1} = (x, j')$ and $X_l \notin N_\eta(H)$ for $1 \leq l \leq n$. Recall that μ is finitely supported. Hence, $d(X_l, X_{l+1}) \leq d(\mu)$, where $d(\mu)$ only depends on μ . Hence, if $\eta \geq 3d(\mu)$, then $X_0, X_{n+1} \notin N_{2\eta/3}(H)$ as soon as $n \geq 1$, which will hold if $d_{\mathcal{N}}(e, x)$ is large enough. Moreover, any geodesic from X_l to X_{l+1} stays outside of $N_{\eta/3}(H)$, for $0 \leq l \leq n$.

Define a path α from X_0 to X_{n+1} , gluing together geodesics from X_l to X_{l+1} . Then, the length of α is at most $nd(\mu)$. The parabolic subgroup H together with the word distance is quasi-isometric to its subgroup \mathcal{N} together with the distance $d_{\mathcal{N}}$. In particular, the word distance between e and x is larger than $\Lambda d_{\mathcal{N}}(e, x)$, where Λ only depends on the quasi-isometry parameters.

We will use the following claim, given by [10, Lemma 5.12]. For $g_1, g_2 \in \Gamma$, we let g'_1 and g'_2 be their projection on H . There exists an $a_0 > 0$ such that the function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$\rho(\eta) = \inf\{d(g_1, g_2) : d(g'_1, g'_2) \geq a_0, d(g_i, H) > \eta\}$$

tends to infinity as $\eta \rightarrow \infty$.

Moreover, according to [10, Lemma 5.11], there exists c_0 such that for every $g_1, g_2 \in \Gamma$, we have

$$d(g'_1, g'_2) \leq d(g_1, g_2) + c_0.$$

Thus, we can choose consecutive points y_1, \dots, y_l on the path α among the points X_1, \dots, X_n which project on H on points $\tilde{y}_1, \dots, \tilde{y}_l$ such that the distance between \tilde{y}_i and \tilde{y}_{i+1} is between a_1 and $2a_1$, for $1 \leq i \leq l-1$ and such that $l \geq \frac{\Lambda d_{\mathcal{N}}(e, x)}{2a_0}$ where $a_1 \geq a_0$ will be chosen later (where a_0 is the constant above).

We thus proved that in order to go from g to g' , staying outside $N_\eta(H)$, the random walk has to pass through points y_1, \dots, y_l , with $l \geq a(\eta)d_{\mathcal{N}}(e, x)$, where $a(\eta)$ tends to infinity when η tends to infinity. Moreover, the random walk goes

from y_i to y_{i+1} staying outside $B(\tilde{y}_i, \eta/3)$. Finally, if a_1 is chosen large enough, the distance formula from [33] shows that $d(y_i, y_{i+1}) \leq \lambda\eta + c$ for some λ and c , see [33, Theorem 3.1]. We thus have

$$p_{H,\eta,r}(g, g') \leq \sum_{n \geq a(\eta)d_{\mathcal{N}}(e,x)} \sum_{y_1, \dots, y_n} G_r(g, y_1; B(\tilde{y}_1, \eta/3)^c) G_r(y_1, y_2; B(\tilde{y}_2, \eta/3)^c) \cdots G_r(y_n, g'; B(x, \eta/3)^c).$$

If a_1 is large enough, it follows from [33, Lemma 1.13 (2), Lemma 1.15 (2)] and from [19, Proposition 8.13] that \tilde{y}_i is within a bounded distance of a transition point in the sense of Definition 1.7 on a geodesic from y_{i-1} to y_i . According to Proposition 1.8, it thus satisfies that $\delta_{\tilde{y}_i}(y_{i-1}, y_i) \geq \delta$ for some uniform $\delta > 0$. Proposition 3.5 shows that there is a super-linear function h such that

$$p_{H,\eta,r}(g, g') \leq \sum_{n \geq a(\eta)d_{\mathcal{N}}(e,x)} \left(e^{-h(\eta/3)} \text{Card}B(e, \lambda\eta + c) \right)^n.$$

The cardinality of the ball $B(e, \lambda\eta + c)$ grows at most exponentially with η , so there exists $\rho_0 < 1$ and there exists η large enough such that

$$e^{-h(\eta/3)} \text{Card}B(e, \lambda\eta + c) \leq \rho_0 < 1.$$

Thus,

$$p_{H,\eta,r}(g, g') \leq \sum_{n \geq a(\eta)d_{\mathcal{N}}(e,x)} \rho_0^n \leq C \rho_0^{a(\eta)d_{\mathcal{N}}(e,x)}.$$

Choosing $a(\eta)$ large enough, that is, choosing η large enough, we have

$$\sum_{x \in \mathcal{N}} p_{j,k}(e, x) e^{M d_{\mathcal{N}}(e,x)} < +\infty.$$

This proves that for large enough η , $p_{H,\eta,r}$ has exponential moments up to M . \square

Lemma 4.6. *There exists η_0 such that for every $\eta \geq \eta_0$ and for every $r \leq R$, every point in the 1-Martin boundary of the transition kernel $p_{H,\eta,r}$ is a positive harmonic function for $p_{H,\eta,r}$.*

Proof. In [11], the authors prove the following. Whenever a transition kernel has super-exponential moments, every Martin kernel is a positive harmonic function. In our situation, up to choosing η large enough, the transition kernel we study has exponential moments up to M , for arbitrarily large M . We show that this is enough to conclude, adapting the proof of [11].

First, Harnack inequalities show that there exists some uniform $C > 1$ such that for every $g, g' \in N_\eta(H)$,

$$K_{H,\eta,R}(g, g') = K_R(g, g') \leq C^{d(e,\gamma)}.$$

Let us fix $M > 2 \log C$ and let us fix η_0 such that $p_{\eta,H,R}$ has exponential moments up to M for every $\eta \geq \eta_0$. For every $g \in N_\eta(H)$,

$$d(e, g) \leq \lambda d_{\mathcal{N}}(e, g) + \alpha\eta,$$

for some constants λ and α . Up to changing M , we see that for every $g \in N_\eta(H)$,

$$(21) \quad K_R(g, g') \leq C_\eta C^{d_{\mathcal{N}}(e,g)}$$

and we still have $M > 2 \log C$.

We introduce the linear operator P associated with the transition kernel $p_{H,\eta,R}$. That is, for every function ω on $N_\eta(H)$,

$$P\omega(g) = \sum_{g' \in N_\eta(H)} p_{H,\eta,R}(g, g')\omega(g').$$

We want to prove that for every ξ in the 1-Martin boundary of $p_{H,\eta,R}$, we have $PK(\cdot, \xi) = K(\cdot, \xi)$. Let g_n be a sequence of $N_\eta(H)$ converging to ξ . We need to prove that $PK(\cdot, g_n)$ both converges to $K(\cdot, \xi)$ and to $PK(\cdot, \xi)$.

By definition, the Green function $G_{H,\eta,R}(\cdot, g')$ is 1-harmonic for $p_{H,\eta,R}$ everywhere except at g' . Fixing g , for n large enough, $PG_{H,\eta,R}(g, g_n) = G_{H,\eta,R}(g, g_n)$, since g_n tends to infinity. Thus, for large enough n , $PK_{H,\eta,R}(g, g_n) = K_{H,\eta,R}(g, g_n)$ and so $PK(g, g_n)$ converges to $K(g, \xi)$.

We now prove that $PK(\cdot, g_n)$ also converges to $PK(\cdot, \xi)$. Let

$$\tilde{K}_n = |K(\cdot, g_n) - K(\cdot, \xi)|.$$

We just need to show that $P\tilde{K}_n$ converges to 0. Let us fix $\epsilon > 0$. According to (21),

$$\tilde{K}_n(g) \leq 2C_\eta C^{d_{\mathcal{N}}(e,x)},$$

where x is the projection of g on \mathcal{N} . Let us fix g and let us fix $L > d_{\mathcal{N}}(e, x)$. Decompose the transition kernel $p_{H,\eta,R}$ as $p_{H,\eta,R} = p_L + q_L$, where p_L is its restriction to $B_{\mathcal{N}}(e, L) = \{g \in N_\eta(H), d_{\mathcal{N}}(e, x) \leq L\}$. Denote by P_L and Q_L the associated linear operators, so that $P = P_L + Q_L$. Then,

$$Q_L \tilde{K}_n(g) \leq \sum_{g', d_{\mathcal{N}}(x, x') > L} p_{H,\eta,R}(g, g') 2C_\eta C^{d_{\mathcal{N}}(e, x')}.$$

Changing the sum over g' with a sum over $g^{-1}g'$, we get

$$Q_L \tilde{K}_n(g) \leq \sum_{g', d_{\mathcal{N}}(e, x') > L} p_{H,\eta,R}(e, g') 2C_\eta C^{d_{\mathcal{N}}(e, x) + d_{\mathcal{N}}(e, x')}$$

Recall that $d_{\mathcal{N}}(e, x) < L$. Consequently,

$$Q_L \tilde{K}_n(g) \leq 2C_\eta \sum_{\gamma', d_{\mathcal{N}}(e, \gamma') > L} p_{H,\eta,R}(e, \gamma') C^{2d_{\mathcal{N}}(e, \gamma')}.$$

Since $M > 2 \log C$ and since $p_{H,\eta,R}$ has exponential moments up to M , this last sum goes to 0 when L tends to infinity. We can choose L large enough so that $Q_L \tilde{K}_n(g) \leq \epsilon$. Now that L is fixed, $P_L \tilde{K}_n(g)$ tends to 0 when n tends to infinity, since the sum in the definition of P_L is finite. For n large enough, $P_L \tilde{K}_n(g) \leq \epsilon$, so that $P\tilde{K}_n(g) \leq 2\epsilon$. This proves that $P\tilde{K}_n(g)$ converges to 0 when n tends to infinity and this is true for every g . This concludes the proof. \square

Let $R_{\eta,r}(H)$ be the spectral radius of the transition kernel $p_{H,\eta,r}$. In particular, $R_{0,R}(H) = R_H$.

Lemma 4.7. *There exists η_0 such that for every $\eta \geq \eta_0$ and for every $r \leq R$, if $R_{\eta,r}(H) = 1$, then $\min_u \lambda(u) = 1$.*

Proof. According to [36, Lemma 7.2], if there exists a t -harmonic positive function, then $t \leq R_{\eta,r}(H)$. Assume that $R_{\eta,r}(H) = 1$. Lemma 4.6 shows that there exists a 1-harmonic function. This proves that 1 is the maximum of the t such that there exists a t -harmonic positive function.

According to the Martin representation theorem, there exists a t -harmonic function if and only if the t -minimal Martin boundary is non-empty. Lemma 4.4 shows that there exists a t -minimal harmonic function if and only if there exists $u \in \mathbb{R}^d$ such that $\lambda(u) = 1/t$. Thus, $1 = \max_u \frac{1}{\lambda(u)}$. This proves the lemma. \square

Assuming that μ is spectrally degenerate along H , we have $R_{0,R}(H) = 1$. We show that we also have $R_{\eta,R}(H) = 1$ for every $\eta \geq 0$.

Lemma 4.8. *Let $0 \leq \eta \leq \eta'$ and let $r \leq R$. If $R_{\eta,r}(H) = 1$, then $R_{\eta',r}(H) = 1$.*

Proof. Let $G_{H,\eta,r}(\cdot, \cdot|t)$ be the Green function at t , associated with the transition kernel $p_{H,\eta,r}$. We show that

$$(22) \quad G_{H,\eta,r}(e, e|t) \leq G_{H,\eta',r}(e, e|t).$$

By definition,

$$G_{H,\eta,r}(e, e|t) = \sum_{n \geq 0} t^n p_{H,\eta,r}^{(n)}(e, e).$$

In other words, $G_{H,\eta,r}(e, e|t)$ is the sum of the weight of every path from e to e that stays inside $N_\eta(H)$, multiplied with t at the power the length of this path. Here, the weight is the power of convolution of $p_{H,\eta,r}$.

By definition of this transition kernel, it is equivalent to sum the weight of every path from e to e (not necessarily staying inside $N_\eta(H)$, multiplied by t at the power the number of points on this path which are in $N_\eta(H)$. Here, the weight is the power of convolution of μ . Formally, we write this as

$$G_{H,\eta,r}(e, e|t) = \sum_{n \geq 0} \sum_{g_1, \dots, g_{n-1}} t^{\text{Card}\{j, g_j \in N_\eta(H)\}} \mu(g_1) \mu(g_1^{-1} g_2) \cdots \mu(g_{n-1}^{-1}).$$

Since $N_\eta(H) \subset N_{\eta'}(H)$, we have (22) and the lemma follows. \square

We can now use all the lemmas of this section to prove the following proposition.

Proposition 4.9. *Assume that μ is spectrally degenerate along H . Then, there exists η_0 such that for every $\eta \geq \eta_0$, the 1-Martin boundary of the transition kernel $p_{H,\eta,R}$ is reduced to a point.*

Proof. We first prove that the minimal Martin boundary of $p_{H,\eta,R}$ is reduced to a point. Indeed, according to Lemma 4.8, for every η , $R_{\eta,R}(H) = 1$. Thus, Lemma 4.7 shows that $\min \lambda(u) = 1$. Since λ is strictly convex, this proves that $\lambda(u) = 1$ has only one solution u_0 . Finally, Lemma 4.4 shows that there is only one positive minimal harmonic function and so the minimal Martin boundary is reduced to a point.

According to Lemma 4.6, if $\eta \geq \eta_0$, then the Martin kernels are positive harmonic functions for $p_{H,\eta,R}$. Since every positive harmonic function can be represented as an integral of the Martin kernel over the minimal boundary for some finite measure, this proves that all Martin kernels are proportional. Since they are all equal to 1 at e , this proves there is only one Martin kernel, which concludes the proof. \square

We can now prove Proposition 4.3.

Proof. Let us fix $\eta \geq \eta_0$, so that we can apply Proposition 4.9. Let g_n converge to a degenerate parabolic limit point ξ . Let H be the stabilizer of ξ and let $H_0 \in \Omega_0$ be such that $H = gH_0g^{-1}$. Then, by definition, μ is spectrally degenerate along H_0 .

Let $\pi_n = \pi_{gH_0}(g_n)$ be the projection of g_n on gH_0 . Since g_n converges to ξ , the sequence $g^{-1}\pi_n$ goes to infinity in H_0 . In particular, according to Proposition 4.9, $g^{-1}\pi_n$ converges in the 1-Martin boundary of $p_{H,\eta,R}$. That is, for every $h \in H$,

$$K_{H,\eta,R}(h, g^{-1}\pi_n) = \frac{G_{H,\eta,R}(h, g^{-1}\pi_n)}{G_{H,\eta,R}(e, g^{-1}\pi_n)}$$

converges to some limit.

More generally, if $h \in \Gamma$ is fixed, then $h \in N_\eta(H)$ for some η , so that Proposition 4.9 also shows that $K_{H,\eta,R}(h, g^{-1}\pi_n)$ converges to some limit. Recall that the Green function is invariant by left multiplication, so this also proves that for every $h \in \Gamma$, $K_{H,\eta,R}(h, \pi_n)$ converges to some limit. Moreover, according to [8, Lemma 3.1], the Martin kernel associated with $p_{H,\eta,R}$ at 1 and the Martin kernel associated with μ at R coincide. This proves that the limit of $K_{H,\eta,R}(h, \pi_n)$ does not depend on H and η . We denote this limit by $K_R(h, \xi)$. This also proves that $K_R(h, \pi_n)$ converges to $K_R(h, \xi)$.

To conclude, we want to prove that $K_R(h, g_n)$ also converges to the same limit. We fix $h \in \Gamma$ and we fix $\epsilon > 0$. Since π_n goes to infinity, the distance between the projection of h on H_0 and π_n also goes to infinity. As above, [33, Lemma 1.13 (2), Lemma 1.15 (2)] and [19, Proposition 8.13] show that π_n is within a bounded distance of a transition point, in the sense of Definition 1.7, on a geodesic from h to g_n . Proposition 1.8 then shows that $\delta_{\pi_n}^f(h, g_n) > \delta$ for some uniform δ that does not depend on n . Corollary 3.7 shows that there exists S such that

$$G_R(h, g_n; B_S(\pi_n)^c) \leq \epsilon G_R(h, g_n).$$

Similarly, for large enough n , we have

$$G_R(e, g_n; B_S(\pi_n)^c) \leq \epsilon G_R(e, g_n).$$

We decompose a path from h to g_n according to its last visit to $B_S(\pi_n)$. We have

$$G_R(h, g_n) = G_R(h, g_n; B_S(\pi_n)^c) + \sum_{u \in B_S(e)} G_R(h, \pi_n u) G_R(\pi_n u, g_n),$$

so that

$$(23) \quad (1 - \epsilon) G_R(h, g_n) \leq \sum_{u \in B_S(e)} G_R(h, \pi_n u) G_R(\pi_n u, g_n)$$

and similarly

$$(24) \quad (1 - \epsilon) G_R(e, g_n) \leq \sum_{u \in B_S(e)} G_R(e, \pi_n u) G_R(\pi_n u, g_n).$$

We also have

$$(25) \quad G_R(h, g_n) \geq \sum_{u \in B_S(e)} G_R(h, \pi_n u) G_R(\pi_n u, g_n)$$

and

$$(26) \quad G_R(e, g_n) \geq \sum_{u \in B_S(e)} G_R(e, \pi_n u) G_R(\pi_n u, g_n).$$

Moreover, for every $u \in B_S(e)$, the sequence $\pi_n u$ lies in $N_\eta(H)$ and converges to ξ . The above discussion shows that $K_R(h, \pi_n u)$ converges to $K_R(h, \xi)$. Thus for large

enough n , for every $u \in B_S(e)$,

$$(K_R(h, \xi) - \epsilon)G_R(e, \pi_n u) \leq G_R(h, \pi_n u) \leq (K_R(h, \xi) + \epsilon)G_R(e, \pi_n u).$$

In particular, (23) shows that

$$(27) \quad (1 - \epsilon)G_R(h, g_n) \leq (K_R(h, \xi) + \epsilon) \sum_{u \in B_S(e)} G_R(e, \pi_n u)G_R(\pi_n u, g_n).$$

Combining (26) and (27), we get

$$(1 - \epsilon)G_R(h, g_n) \leq (K_R(h, \xi) + \epsilon)G_R(e, g_n)$$

and so

$$K_R(h, g_n) \leq \frac{1}{1 - \epsilon}(K_R(h, \xi) + \epsilon).$$

Similarly, (24) and (25) yield

$$K_R(h, g_n) \geq (1 - \epsilon)(K_R(h, \xi) - \epsilon).$$

To sum-up, for large enough n , we have

$$(1 - \epsilon)(K_R(h, \xi) - \epsilon) \leq K_R(h, g_n) \leq \frac{1}{1 - \epsilon}(K_R(h, \xi) + \epsilon).$$

Since ϵ is arbitrary, this concludes the proof. \square

4.2. Parabolic limit points: the non-degenerate case. We now focus on non-degenerate parabolic points. We do assume that the parabolic subgroups are virtually abelian here. We prove the following. If g_n converges to a point in the geometric boundary of a non-degenerate parabolic subgroups, then g_n converges to a point in the R -Martin boundary. We also prove that for $r < R$, g_n converges to a point in the r -Martin boundary, whether the parabolic subgroup is non-degenerate or not. The formal statement is a bit technical.

Proposition 4.10. *Let Γ be a non-elementary relatively hyperbolic group with respect to a collection Ω of virtually abelian subgroups. Choose a finite set Ω_0 of representatives of conjugacy classes of elements of Ω . Let μ be a symmetric probability measure whose finite support generates Γ and let R be its spectral radius and $r \leq R$. Let ξ be a parabolic limit point and let $g \in \Gamma$ and $H \in \Omega_0$ be such that gHg^{-1} is the stabilizer of ξ . Let g_n be a sequence of elements of Γ such that the projection $\pi_{gH}(g_n)$ of g_n on gH satisfies that $g^{-1}\pi_{gH}(g_n)$ converges to the geometric boundary of H . Finally, assume either that $r < R$ or that μ is spectrally non-degenerate along H . Then, g_n converges to a point in the r -Martin boundary.*

Our goal is to prove this proposition. Let H be a parabolic subgroup, so that H is virtually abelian. There is a finite index subgroup of H which is isomorphic to \mathbb{Z}^d . As in Section 4.1, any section $H/\mathbb{Z}^d \rightarrow H$ identifies H as a set with $\mathbb{Z}^d \times E$, where E is finite. Moreover, for any $\eta \geq 0$,

- (1) $N_\eta(H)$ can be \mathbb{Z}^d -equivariantly identified with $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$,
- (2) if $\eta \leq \eta'$, then $N_\eta \leq N_{\eta'}$. In other words, the set $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$, identified with $N_\eta(H)$ is a subset of $\mathbb{Z}^d \times \{1, \dots, N_{\eta'}\}$ identified with $N_{\eta'}(H)$.

We still denote by $p_{H,r}$ the first return kernel to H for $r\mu$ and by $p_{H,\eta,r}$ the first return kernel to $N_\eta(H)$ for $r\mu$. Recall that

$$p_{H,\eta,r}(h, h') = \sum_{n \geq 0} \sum_{\substack{g_1, \dots, g_{n-1} \\ \notin N_\eta(H)}} r^n \mu(h^{-1}g_1) \mu(g_1^{-1}g_2) \dots \mu(g_{n-2}^{-1}g_{n-1}) \mu(g_{n-1}^{-1}h').$$

Again, to simplify notations, whenever H , r and η are fixed, identifying $h, h'' \in H$ with $(x, k), (x', k')$, $x, x' \in \mathbb{Z}^d$ and $k, k' \in \{1, \dots, N_\eta\}$, we write

$$p_{H, \eta, r}((x, k), (x', k')) = p_{k, k'}(x, x').$$

For $u \in \mathbb{R}^d$, we define for $j, k \in \{1, \dots, N_\eta\}$,

$$F_{j, k}(u) = \sum_{x \in \mathbb{Z}^d} p_{j, k}(e, x) e^{x \cdot u}.$$

We then let $F(u)$ be the matrix with entries $F_{j, k}(u)$. This is the same matrix as in Section 4.1, except that we do not need to project x on \mathbb{Z}^d , since we are not assuming that the parabolic subgroups are virtually nilpotent, but virtually abelian.

The entries of the n th power $F(u)^n$ of $F(u)$ are still given by

$$F(u)_{j, k}^n = \sum_{x \in \mathbb{Z}^d} p_{j, k}^{(n)}(e, x) e^{x \cdot u},$$

where $p_{j, k}^{(n)}(e, x)$ is the n th power convolution of $p_{j, k}(e, x)$. Moreover, the matrix $F(u)$ is still strongly irreducible and thus has a dominant eigenvalue, which we still denote by $\lambda(u)$. According to [7, Proposition 3.5], the function $u \mapsto \lambda(u)$ is strongly convex where it is defined. More precisely, let \mathcal{F} be the interior of the set where all the entries of $F(u)$ are finite. Then, $u \mapsto \lambda(u)$ is strictly convex on u .

In [7] and [10], there are two technical assumptions made on λ , in order to identify the 1-Martin boundary. Let

$$\mathcal{D} = \{u \in \mathcal{F}, \lambda(u) \leq 1\}.$$

Notice that F , λ , \mathcal{F} and \mathcal{D} all depend on η , but we omit this dependence in the notations. The two assumptions are as follows.

Assumption 1. The set \mathcal{D} is compact.

Assumption 2. The minimum of λ is strictly smaller than 1.

We will prove that for large enough η , these two assumptions are satisfied. This will allow us to prove Proposition 4.10. The first assumption is satisfied as soon as $p_{H, \eta, r}$ has sufficiently large exponential moments. According to Lemma 4.5, this is true as soon as η is large enough. Precisely, we have the following.

Lemma 4.11. [10, Proposition 4.8] *Let $r \leq R$. If η is large enough, then the transition kernel $p_{H, \eta, r}$ on $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$ satisfies Assumption 1.*

This settles Assumption 1. Note that we did not have to assume that μ is not spectrally degenerate along H . The fact that Assumption 2 is also satisfied for spectrally non-degenerate parabolic subgroups will be derived from the next lemmas. Recall that we denote by $R_{\eta, r}(H)$ the spectral radius of $p_{H, \eta, r}$.

Lemma 4.12. *There exists η_0 such that for every $\eta \geq \eta_0$ and for every $r \leq R$, we have $R_{\eta, r}(H) = 1$ if and only if $\min_u \lambda(u) = 1$.*

Proof. The "only if" part is given by Lemma 4.7. Let us prove the "if" part. Assume that $\min_u \lambda(u) = 1$ and that λ reaches its minimum at u_0 . Since λ is strictly convex, we have $\nabla \lambda(u_0) = 0$. Thus, [7, Proposition 3.14] shows that for large enough η , $p_{H, \eta, r}^{(n)}(e, e) \sim Cn^{-d/2}$. In particular, $R_{\eta, r}(H) = 1$. \square

Lemma 4.13. *Let $\eta \geq \eta_0$. Assume that $R_{\eta,r}(H) > 1$. Then, the transition kernel $p_{H,\eta,r}$ satisfies Assumption 2.*

Proof. This is a direct consequence of Lemma 4.12. \square

Lemma 4.14. *Let $\eta \geq 0$ and let $r < R$. Then $R_{\eta,r}(H) > 1$.*

Proof. Let $\epsilon > 0$ be such that $r(1 + \epsilon) < R$. By definition, if $g \neq g'$,

$$p_{H,\eta,r}(g, g') = \sum_{n \geq 1} \sum_{\substack{g_1, \dots, g_{n-1} \\ \notin N_\eta(H)}} r^n \mu(g^{-1}g_1) \mu(g_1^{-1}g_2) \dots \mu(g_{n-1}^{-1}g').$$

We deduce that

$$(1 + \epsilon)p_{H,\eta,r}(g, g') \leq p_{H,\eta,r(1+\epsilon)}(g, g'),$$

so that

$$(1 + \epsilon)^n p_{H,\eta,r}^{(n)}(g, g') \leq p_{H,\eta,r(1+\epsilon)}^{(n)}(g, g').$$

Denoting by $G_{H,\eta,r}(g, g'|t)$ the Green function associated with $p_{H,\eta,r}$ evaluated at t , we thus have

$$G_{H,\eta,r}(g, g'|1 + \epsilon) \leq G_{H,\eta,r(1+\epsilon)}(g, g').$$

Since $G_{H,\eta,r(1+\epsilon)}(g, g') = G_{r+r\epsilon}(g, g') < +\infty$ according to [8, Lemma 3.1], we finally get that $G_{H,\eta,r}(g, g'|1 + \epsilon)$ is finite. \square

Lemma 4.15. *Let $\eta \geq 0$. Then, $R_{\eta,R}(H) = 1$ if and only if $R_{0,R}(H) = 1$.*

Proof. The "if" part is given by Lemma 4.8. Let us prove the converse and assume that $R_{\eta,R}(H) = 1$. Then, according to Lemma 4.8, for every $\eta' \geq \eta$, we also have $R_{\eta',R}(H) = 1$. Proposition 4.9 shows that for large enough η' , the 1-Martin boundary of $p_{H,\eta',R}$ is reduced to a point.

We want to prove that the 1-Martin boundary of $p_{H,0,R}$ is also reduced to a point. Fix $g \in H$ and let g_n be a sequence of H going to infinity. Then, in particular, $g, g_n \in N_{\eta'}(H)$, so that $K_{H,\eta',R}(g, g_n)$ converges to some limit which does not depend on g_n , where $K_{H,\eta',R}$ is the Martin kernel associated with $p_{H,\eta',R}$. According to [8, Lemma 3.1], $K_{H,\eta',R} = K_{H,0,R}$. Consequently, $K_{H,0,R}(g, g_n)$ also converges to some limit, independently of g_n . We deduce that the 1-Martin boundary of $p_{H,0,R}$ is indeed reduced to a point.

According to Lemma 4.4, there is only one solution to $\lambda(u) = 1$ and since λ is strictly convex, 1 is the minimum of $\lambda(u)$. According to Lemma 4.7, $R_{0,R}(H) = 1$, which concludes the proof. \square

Corollary 4.16. *Let $H \in \Omega_0$ be a parabolic subgroup and assume either that $r < R$ or that μ is spectrally non-degenerate along H . Let $\eta \geq \eta_0$. Then, the transition kernel $p_{H,\eta,r}$ on $N_\eta(H)$ satisfies Assumption 2.*

We can finally prove Proposition 4.10.

Proof. This is a consequence of [10, Corollary 4.10]. However, the formulation is a bit different (the statement refers to sub-Markov chains rather than transition kernels satisfying Assumption 2). We give the full proof for completeness.

If η is large enough, the transition kernel $p_{H,\eta,r}$, which can be seen as a \mathbb{Z}^d -invariant kernel on $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$, satisfies both Assumption 1 and 2. According

to [7, Lemma 3.13], the set $\{u, \lambda(u) = 1\}$ is homeomorphic to the sphere \mathbb{S}^{d-1} . An explicit homeomorphism is given by

$$u \in \{u \in \mathbb{R}^d, \lambda(u) = 1\} \mapsto \frac{\nabla \lambda(u)}{\|\nabla \lambda(u)\|}.$$

This homeomorphism provides a homeomorphism φ between the geometric boundary of $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$ and \mathbb{S}^{d-1} constructed as follows. Let (x_n, j_n) be a sequence in $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$ converging to a point \tilde{x} in the geometric boundary $\partial(\mathbb{Z}^d \times \{1, \dots, N_\eta\})$. That is, x_n tends to infinity and $\frac{x_n}{\|x_n\|}$ converges to a point θ in the unit sphere \mathbb{S}^{d-1} . There exists a unique $u \in \{u \in \mathbb{R}^d, \lambda(u) = 1\}$ such that $\theta = \frac{\nabla \lambda(u)}{\|\nabla \lambda(u)\|}$. Then, define $\varphi(\tilde{x}) = u$.

Let $g \in N_\eta(H)$ and let g_n be a sequence of $N_\eta(H)$ converging to a point in the geometric boundary of H . Write $g = (x, j)$ and $g_n = (x_n, j_n)$. Then, by definition, x_n converges to some \tilde{x} in the geometric boundary of $\mathbb{Z}^d \times \{1, \dots, N_\eta\}$. Let $u = \varphi(\tilde{x})$. Then, [7, Proposition 3.27] shows that $K_r(g, g_n)$ converges to $C_j e^{u \cdot x}$. Here, C_j only depends on the coordinate j of $g = (x, j)$. In particular, C_j does not depend on j_n . Note that since $K_r(g, g_n)$ does not depend on η , its limit $C_j e^{u \cdot x}$ also is independent. Moreover, letting x vary, we see that u is independent of η , so that C_j also is independent of η .

To prove Proposition 4.10, we now consider a sequence g_n such that its projection h_n on H converges to ξ in the geometric boundary of H . The above discussion shows that for every fixed $g \in \Gamma$ and for every $L > 0$, for every u in the ball $B(e, L)$, we have

$$K_r(g, h_n u) \rightarrow C_g e^{u \cdot x(g)},$$

where u only depends on ξ and where C_g only depends on $d(g, H)$ and $x(g) \in \mathbb{Z}^d$ is the projection of g on \mathbb{Z}^d . Exactly like in the proof of Proposition 4.3, we can use the deviation inequalities from Section 3 to show that

$$(28) \quad K_r(g, g_n) \rightarrow C_g e^{u \cdot x(g)}.$$

This concludes the proof. \square

4.3. Topology of the Martin boundary. We end the proof of our main theorem. We thus consider a non-elementary relatively hyperbolic group Γ with respect to virtually abelian subgroups. We also consider a symmetric probability measure μ whose finite support generates Γ . We let R be its spectral radius.

We first note that Proposition 4.1 is a consequence of Proposition 4.2, Proposition 4.3 and Proposition 4.10.

This proposition gives a map $\xi \in \Gamma \cup \partial_r \Gamma \mapsto \tilde{\xi} \in \Gamma \cup \partial_{r\mu} \Gamma$. We will denote by $K_r(\cdot, \tilde{\xi})$ the corresponding limit Martin kernel. Our goal is to prove Theorem 1.2. We first prove the following lemma.

Lemma 4.17. *Fix $r \leq R$. Let ξ and ξ' be two distinct points in the r -geometric boundary. Then, there exists a sequence g_n such that*

- (1) *either $K_r(g_n, \tilde{\xi})$ stays bounded away from 0 whereas $K_r(g_n, \tilde{\xi}')$ converges to 0,*
- (2) *or $K_r(g_n, \tilde{\xi})$ tends to infinity whereas $K_r(g_n, \tilde{\xi}')$ stays bounded away from infinity.*

Proof. By definition of the geometric boundary, if $g_n \in \Gamma$ converges to ξ , then it converges to a point in the Bowditch boundary. In other words, ξ and ξ' define two points ζ, ζ' in the Bowditch boundary.

We first prove that if g_n converges to ζ and if $\zeta' \neq \zeta$, then $K(g_n, \tilde{\xi}')$ converges to 0. Indeed, consider a sequence g'_m converging to ζ' in the Bowditch boundary. Since the Floyd boundary covers the Bowditch boundary, according to the results of Gerasimov [13], there exists some uniform $\delta > 0$ such that $\delta_e^f(g_n, g'_m) \geq \delta$. According to Theorem 3.6, there exists $C \geq 0$ such that for every n and m , we have

$$G_r(g_n, g'_m) \leq CG_r(g_n, e)G_r(e, g'_m),$$

so that

$$K_r(g_n, g'_m) \leq CG_r(g_n, e).$$

Letting m tend to infinity, we get

$$K_r(g_n, \tilde{\xi}') \leq CG_r(g_n, e).$$

Since g_n goes to infinity, $G_r(g_n, e)$ converges to 0 and we get the desired result.

Next, we prove that we can find a sequence g_n converging to ζ such that $K(g_n, \tilde{\xi})$ stays bounded away from 0. Assume first that ζ is conical and fix a geodesic $[e, \zeta)$ from e to ζ . Since ζ is conical, there exists an infinite sequence g_n of transition points on $[e, \zeta)$ converging to ζ , see [38, Lemma 2.20]. Proposition 1.8 shows that for $n \leq m$, $\delta_{g_n}^f(e, g_m) \geq \delta$ for some uniform $\delta > 0$ and so Theorem 3.6 shows that there exists $C \geq 0$ such that

$$G_r(e, g_m) \leq CG_r(e, g_n)G_r(g_n, g_m).$$

Letting m tend to infinity, we see that $K_r(g_n, \tilde{\xi}) \geq \frac{1}{C} \frac{1}{G_r(e, g_n)}$, so that $K_r(g_n, \tilde{\xi})$ tends to infinity when n tends to infinity. Assume now that ζ is parabolic. For simplicity assume first that its stabilizer H is in Ω_0 . Then, there exists $u \in \mathbb{R}^d$ such that

$$K_r(g, \tilde{\xi}) = C_g e^{\pi(g) \cdot u},$$

where $\pi(g)$ is the projection of g on H and where C_g only depends on the distance from g to H . Indeed, either the 1-Martin boundary of $p_{H, \eta, r}$ is reduced to a point and then we can use Lemma 4.4, or it is homeomorphic to a sphere and we use the explicit form of the Martin kernel given by [7, Proposition 3.27]. If $u = 0$, then for any sequence g_n going to infinity in H , $K_r(g_n, \tilde{\xi})$ stays bounded. If not, then choose $\theta \in \mathbb{S}^{d-1}$ such that $\theta \cdot u > 0$ and let $g_n = (x_n, j_n)$ in $H = \mathbb{Z}^d \times \{1, \dots, N_\eta\}$ such that g_n goes to infinity and $\frac{g_n}{\|g_n\|}$ converges to θ . Then $K_r(g_n, \tilde{\xi})$ tends to infinity. In any case, we found a sequence g_n such that $K_r(g_n, \tilde{\xi})$ is bounded away from 0. If the stabilizer of ζ is not in Ω_0 , then we get the same result, multiplying everything on the left by some $g \in \Gamma$ such that gHg^{-1} is in Ω_0 .

We now end the proof. We begin with the case where $\zeta \neq \zeta'$. Then, we choose a sequence g_n converging to ζ such that $K(g_n, \tilde{\xi})$ stays bounded away from 0. Since $K(g_n, \tilde{\xi}')$ converges to 0, we are done. We thus assume that $\zeta = \zeta'$. The only possibility is that ζ is parabolic. As above, we first assume that its stabilizer H is in Ω_0 . Thus, there exists $u \neq u' \in \mathbb{R}^d$ such that

$$K_r(g, \tilde{\xi}) = C_g e^{\pi(g) \cdot u},$$

$$K_r(g, \tilde{\xi}') = C_g e^{\pi(g) \cdot u'}.$$

We have to be a little more precise than above. If $u = 0$, we choose θ such that $\theta \cdot u' < 0$. Otherwise, we choose θ such that $\theta \cdot u > 0$ and $\theta \cdot u' \leq 0$. This is always possible, whether $u' = 0$ or not. We then choose again a sequence $g_n = (x_n, j_n)$ in $H = \mathbb{Z}^d \times \{1, \dots, N_\eta\}$ such that g_n goes to infinity and $\frac{g_n}{\|g_n\|}$ converges to θ . Then, in the first case, $K_r(g_n, \tilde{\xi})$ stays bounded away from 0 whereas $K_r(g_n, \tilde{\xi}')$ converges to 0. In the second case, $K_r(g_n, \tilde{\xi})$ tends to infinity, whereas $K_r(g_n, \tilde{\xi}')$ stays bounded away from infinity. If H is not in Ω_0 , then we get the same result, multiplying everything on the left by some $g \in \Gamma$ such that gHg^{-1} is in Ω_0 . We dealt with every case, so this concludes the proof. \square

We can finish the proof of Theorem 1.2.

Proof. First, the map $\xi \in \Gamma \cup \partial_r \Gamma \mapsto \tilde{\xi} \in \Gamma \cup \partial_{r\mu} \Gamma$ is continuous. Indeed since the geometric compactifications are metrizable and since Γ is dense in these compactifications, it is enough to prove that whenever $g_n \in \Gamma$ converges to ξ , then \tilde{g}_n converges to $\tilde{\xi}$, which is given by Propositions 4.2, 4.3 and 4.10.

We now prove that it is also one-to-one. We just need to prove that whenever $\xi \neq \xi' \in \partial_r \Gamma$, then $\tilde{\xi} \neq \tilde{\xi}' \in \partial_{r\mu} \Gamma$. This is given by Lemma 4.17.

The Martin compactification also is metrizable and Γ is dense in this compactification. Let $\tilde{\xi} \in \partial_{r\mu} \Gamma$. Then, there exists $g_n \in \Gamma$ converging to $\tilde{\xi}$. Up to taking a subsequence, g_n converges to ξ in the r -geometric compactification, so $\tilde{\xi}$ is the image of ξ by the above map.

Thus, the map we constructed is one-to-one, onto and continuous. Since both spaces are metrizable and compact, this map is a homeomorphism. \square

5. MINIMALITY AND STABILITY OF THE MARTIN BOUNDARY

We prove here Theorem 1.3 and Theorem 1.4 and so we consider a relatively hyperbolic group with respect to virtually abelian subgroups. We also consider a symmetric probability measure μ whose finite support generates Γ .

5.1. Minimality. To prove minimality of the Martin boundary, we will use the following lemmas. The first one is a general result in potential theory.

Lemma 5.1. [2, Proposition II.1.6]. *Let p be a finitely supported transition kernel on a countable space E which is transient and irreducible. Let ϕ be a non-negative harmonic function on E and ν_ϕ the corresponding measure on the minimal Martin boundary $\partial^{min} E$. For μ_ϕ -almost every point \tilde{y} in $\partial^{min} E$, $\frac{G(y_n, x_0)}{\phi(y_n)}$ converges to 0 when y_n converges to \tilde{y} .*

We will also need the following result, which is a refinement of Lemma 4.17.

Lemma 5.2. *Consider a non-degenerate parabolic subgroup H and let $\xi_1 \neq \xi_2$ be two points in the geometric boundary of H and $\tilde{\xi}_1$ and $\tilde{\xi}_2$ the corresponding points in the Martin boundary. There exists a compact neighborhood \mathcal{U} of ξ_1 in the geometric boundary of H and a sequence g_n in H such that*

- (1) *either $K_r(g_n, \tilde{\xi})$ tends to infinity, uniformly over $\xi \in \mathcal{U}$ and $K_r(g_n, \tilde{\xi}_2)$ stays bounded away from infinity,*
- (2) *or $K_r(g_n, \tilde{\xi})$ stays uniformly bounded away from 0 over \mathcal{U} and $K_r(g_n, \tilde{\xi}_2)$ converges to 0.*

Proof. This is exactly [10, Corollary 6.2], but the authors only deal with the 1-Martin boundary there. We rewrite the proof for convenience. For every point in the geometric boundary of H , there exists $u \in \mathbb{R}^d$, with $\lambda(u) = 1$ and such that

$$K_r(g, \tilde{\xi}) = Ce^{\pi(g) \cdot u}.$$

The induced map $\partial H \rightarrow \{u, \lambda(u) = 1\}$ is a homeomorphism. Let u_1, u_2 be the points corresponding to ξ_1, ξ_2 . Since $\xi_1 \neq \xi_2$, there exists a compact neighborhood \mathcal{U} of u_1 such that $u_2 \notin \mathcal{U}$. We can see \mathcal{U} as a compact neighborhood of ξ_1 in ∂H .

First, assume that $u_1 \neq 0$. Then, we can choose \mathcal{U} such that $0 \notin \mathcal{U}$. We then choose $\theta \in \mathbb{S}^{d-1}$ such that $\theta \cdot u > 0$ for every $u \in \mathcal{U}$ and such that $\theta \cdot u_2 \leq 0$. Let $g_n = (x_n, j_n)$ be a sequence in $H = \mathbb{Z}^d \times \{1, \dots, N_\eta\}$ such that x_n goes to infinity and $\frac{x_n}{\|x_n\|}$ converges to θ . Then, $K_r(g_n, \tilde{\xi})$ tends to infinity whereas $K_r(g_n, \tilde{\xi}_2)$ stays bounded away from infinity.

Assume then that $u_1 = 0$. Then $u_2 \neq 0$. Choose θ such that $\theta \cdot u \geq 0$ for every $u \in \mathcal{U}$ and $\theta \cdot u_2 < 0$. Let $g_n = (x_n, j_n)$ be a sequence in $H = \mathbb{Z}^d \times \{1, \dots, N_\eta\}$ such that x_n goes to infinity and $\frac{x_n}{\|x_n\|}$ converges to θ . Then, $K_r(g_n, \tilde{\xi})$ stays bounded away from 0 whereas $K_r(g_n, \tilde{\xi}_2)$ converges to 0. This concludes the proof. \square

We can now prove Theorem 1.4.

Proof. Let $\tilde{\xi}_0 \in \partial_{r\mu}\Gamma$. Then, $K_r(\cdot, \tilde{\xi}_0)$ is an r -harmonic function. Let ν be the corresponding measure in the r -minimal Martin boundary, that is

$$K_r(\cdot, \tilde{\xi}_0) = \int_{\partial_{r\mu}^{min}\Gamma} K_r(\cdot, \tilde{\xi}) d\nu(\tilde{\xi}).$$

To prove that $\tilde{\xi}_0$ is minimal, we just need to prove that the support of ν is reduced to $\{\tilde{\xi}_0\}$. We will actually prove that any other point cannot be in the support of ν .

Let $\tilde{\xi}'_0$ be another point. Assume first that $\tilde{\xi}_0$ and $\tilde{\xi}'_0$ correspond to different points ζ_0 and ζ'_0 in the Bowditch boundary. Then, there exists a compact neighborhood \mathcal{U} of $\tilde{\xi}'_0$ such that for every $\tilde{\xi}$ in \mathcal{U} , $\delta_e^f(\zeta_0, \zeta'_0) \geq \delta$, so that Theorem 3.6 shows that if g_n converges to $\tilde{\xi} \in \mathcal{U}$, then

$$K_r(g_n, \tilde{\xi}_0) \leq CG_r(g_n, e).$$

This proves that $\frac{G_r(g_n, e)}{K_r(g_n, \tilde{\xi}_0)}$ cannot converge to 0 and so Lemma 5.1 shows that $\tilde{\xi}'_0$ cannot be in the support of ν .

Assume now that $\tilde{\xi}_0$ and $\tilde{\xi}'_0$ correspond to the same point in the Bowditch boundary. This point necessarily is a non-degenerate parabolic limit point. Lemma 5.2 shows that there is a neighborhood \mathcal{U} of $\tilde{\xi}'_0$ such that

- (1) either $K_r(g_n, \tilde{\xi})$ tends to infinity, uniformly over $\xi \in \mathcal{U}$ and $K_r(g_n, \tilde{\xi}_0)$ stays bounded away from infinity,
- (2) or $K_r(g_n, \tilde{\xi})$ stays uniformly bounded away from 0 over \mathcal{U} and $K_r(g_n, \tilde{\xi}_0)$ converges to 0.

By definition,

$$K_r(g_n, \tilde{\xi}_0) = \int_{\partial_{r\mu}^{min}\Gamma} K_r(g_n, \tilde{\zeta}) d\nu(\tilde{\zeta}) \geq \int_{\mathcal{U}} K_r(g_n, \tilde{\zeta}) d\nu(\tilde{\zeta}).$$

Thus, in the first case, $K_r(g_n, \tilde{\xi}_0) \geq \alpha_n \nu(U)$ for n large enough, where α_n tends to infinity and so $\mu(U) = 0$. In the second case, $K_r(g_n, \tilde{\xi}_0) \geq C\mu(U)$ for some constant C , for n large enough and so again, $\mu(U) = 0$. Hence, $\tilde{\xi}_0$ is not in the support of ν .

We proved that in every case, $\tilde{\xi}_0$ is not in the support of ν , so that the support of ν is reduced to $\{\tilde{\xi}_0\}$, hence $\tilde{\xi}_0$ is minimal. \square

5.2. Stability. We still consider a relatively hyperbolic group with respect to virtually abelian subgroups and a symmetric probability measure μ whose finite support generates Γ . According to Theorem 1.2, the r -Martin compactifications are all homeomorphic to each other for $r < R$ and there exists a continuous equivariant and surjective map ϕ_μ from the r -Martin compactification to the R -Martin compactification. Our goal is to prove that the map $(r, x, \xi) \mapsto K_r(x, \xi)$ is continuous in $(r, x, \xi) \in (0, R] \times \Gamma \times \partial_\mu \Gamma$.

Theorem 1.2 also gives a homeomorphism from the r -geometric compactification to the r -Martin compactification. We first deal with conical limit points, that is we assume that ξ is the image of a conical point by this homeomorphism.

Proposition 5.3. *Let $0 < r_0 \leq R$ and $x_0 \in \Gamma$. Let ξ_0 be a conical limit point and identify ξ_0 with a point in $\partial_{r_0\mu} \Gamma$. Then, $(r, x, \xi) \in (0, R] \times \Gamma \times \Gamma \cup \partial_\mu \Gamma \mapsto K_r(x, \xi)$ is continuous at (x_0, r_0, ξ_0) .*

Proof. Let $(s_n, y_n, \zeta_n) \in (0, R] \times \Gamma \times \Gamma \cup \partial_\mu \Gamma$ converge to r_0, x_0, ξ_0 . Since Γ is discrete, up to taking large enough n we can assume that $y_n = x$ for all n . Let z_m be a sequence in Γ converging to ξ_0 and $z'_{n,m}$ be a sequence in Γ converging to ζ_n , when m tends to infinity. Since ξ_0 is conical, [9, Lemma 2.2, Proposition 2.3] shows that there exists $\delta > 0$ such that the following holds. There exists a sequence of numbers k_n going to infinity such that for large enough m there are at least k_n points w_1, \dots, w_{k_n} such that for every i , $\delta_{w_i}^f(x, z_m) \geq \delta$ and $\delta_{w_i}^f(e, z_{n,m}) \geq \delta$. Strong Floyd-Ancona inequalities (Theorem 3.14) thus show that there exist constants $C \geq 0$ and $\rho < 1$, not depending on the points in Γ and not depending on s_n such that

$$\left| \frac{G_{s_n}(e, z_m)G_{s_n}(x, z'_{n,m})}{G_{s_n}(x, z_m)G_{s_n}(e, z'_{n,m})} - 1 \right| \leq C\rho^{k_n}.$$

Letting m tend to infinity, we get that

$$\left| \frac{K_{s_n}(x, \xi_n)}{K_{s_n}(x, \xi)} - 1 \right| \leq C\rho^{k_n}.$$

The same proof shows that there exists a sequence k'_m going to infinity when m go to infinity such that for large enough l , independently of s_n ,

$$\left| \frac{G_{s_n}(e, z_m)G_{s_n}(x, z_l)}{G_{s_n}(x, z_m)G_{s_n}(e, z_l)} - 1 \right| \leq C\rho^{k'_m}$$

so that letting l tend to infinity, we get

$$\left| \frac{K_{s_n}(x, \xi)}{K_{s_n}(x, z_m)} - 1 \right| \leq C\rho^{k'_m}.$$

This is also true replacing s_n with r , so that

$$\left| \frac{K_r(x, \xi)}{K_r(x, z_m)} - 1 \right| \leq C\rho^{k'_m}.$$

Now, fix $\epsilon > 0$ and let n and m be large enough so that $C\rho^{k_n} \leq \epsilon$ and $C\rho^{k'_m} \leq \epsilon$. Then,

$$\left| \frac{K_{s_n}(x, \xi_n)}{K_{s_n}(x, z_m)} - 1 \right| \leq \epsilon(1 + \epsilon) + \epsilon.$$

Note that

$$K_{s_n}(x, z_m) = \frac{G_{s_n}(x, z_m)}{G_{s_n}(e, z_m)} \leq C \frac{1}{G_{s_n}(x, e)}$$

and since s_n converges to $r \neq 0$, there exists C_x not depending on n such that

$$K_{s_n}(x, z_m) \leq C_x.$$

In particular, we get

$$|K_{s_n}(x, \xi_n) - K_{s_n}(x, z_m)| \leq C_x \epsilon (2 + \epsilon).$$

Similarly, we have

$$|K_r(x, \xi) - K_r(x, z_m)| \leq C'_x \epsilon.$$

Since x and z_m are fixed in Γ , $K_{s_n}(x, z_m)$ converges to $K_r(x, z_m)$. In particular, if n is large enough, then

$$|K_{s_n}(x, z_m) - K_r(x, z_m)| \leq \epsilon.$$

Finally, we get that for large enough n ,

$$|K_{s_n}(x, \xi_n) - K_r(x, \xi)| \leq C_x \epsilon (2 + \epsilon) + \epsilon + C'_x \epsilon.$$

Since ϵ is arbitrary, this concludes the proof. \square

As explained in the introduction, this proposition also shows that the Martin boundary is strongly stable in hyperbolic groups, although this is implicit in [16].

We can now deal with parabolic limit points and we fix a parabolic subgroup H . Whether μ is spectrally degenerate along H or not, Theorem 1.2 shows that if ξ is in the geometric boundary ∂H of H and if g_n in Γ converges to ξ , then $K_r(\cdot, g_n)$ converges to a point $K_r(\cdot, \xi)$ for every $r \leq R$. If μ is spectrally degenerate along H , then $\xi \in \partial H \mapsto K_R(\cdot, \xi)$ is not one-to-one, but we will still prove that $(r, x, \xi) \mapsto K_r(x, \xi)$ is continuous for $\xi \in \partial H$.

We will use the notations of Section 4.1 and Section 4.2. We denote by $p_{H, \eta, r}$ the first return kernel induced by $r\mu$ to the η -neighborhood $N_\eta(H)$ of H . Note that if $r \leq r'$, then $p_{H, \eta, r}(g, g') \leq p_{H, \eta, r'}(g, g')$ for every $g, g' \in N_\eta(H)$. We also consider the matrix $F^{(r)}(u)$ defined by

$$F_{j,k}^{(r)}(u) = \sum_{x \in \mathbb{Z}^d} p_{j,k}(e, x) e^{x \cdot u}$$

as in Section 4.2. We added the exponent r to insist on the dependency on this parameter. Also note that this matrix depends on η .

According to Lemma 4.5, for every M , there exists η such that $p_{H, \eta, r}$ has exponential moments up to M , independently of r . In particular, if η is large enough, the set of $u \in \mathbb{R}^d$ such that the coefficients of $F^{(r)}(u)$ are finite contains a set $\{u \in \mathbb{R}^d, \|u\| \leq M'\}$ which can be taken independently of r . We fix such a large η .

Lemma 5.4. *As a function of u , the matrix $F^{(r)}$ depends continuously on r .*

Proof. This follows from the fact that $r \mapsto p_{H, \eta, r}$ is non-decreasing and the fact that $p_{H, \eta, r}$ has exponential moments up to some M , which is independent of r . \square

Let ξ be a point in the geometric boundary ∂H of H . Identify ξ both with a point in the 1-Martin boundary of $(N_\eta(H), p_{H,\eta,r})$ and with a point in the r -Martin boundary of (Γ, μ) . If $h \in N_\eta(H)$, then we have an explicit formula for $K_r(h, \xi)$. Precisely, denote by $\lambda^{(r)}(u)$ the dominant eigenvalue of $F^{(r)}(u)$ and by $C^{(r)}(u)$, respectively $\nu^{(r)}(u)$, right, respectively left, eigenvectors associated with $\lambda^{(r)}(u)$. Then, $\lambda^{(r)}$, $C^{(r)}$ and $\nu^{(r)}$ depend continuously on $F^{(r)}$, so they depend continuously on r . Moreover, as explained in Section 4.2, there exists an homeomorphism ϕ_r between ∂H and $\{u, \lambda_r(u) = 1\}$. This homeomorphism is given by $u \mapsto \frac{\nabla \lambda_r(u)}{\|\nabla \lambda_r(u)\|}$. Finally, recall that we identify $N_\eta(H)$ with $\mathbb{Z}^d \times \{1, \dots, N\}$ and that $h \in N_\eta(H)$ is identified with (z, k) . Then, denoting by $u_r = \phi_r(\xi)$, [7, Proposition 3.27] shows that for every $h = (z, k) \in N_\eta(H)$, we have

$$K_r(h, \xi) = \frac{C_r(u_r)_k}{C_r(u_r)_0} e^{u_r \cdot z}.$$

This shows that $K_r(h, \xi)$ is continuous in r . Moreover, [7, Lemma 3.26] shows the following.

Lemma 5.5. *Let $h \in N_\eta(H)$ and h_n be a sequence in $N_\eta(H)$ converging to $\xi \in \partial H$. Then, $K_r(h, h_n)$ converges to $K_r(h, \xi)$. If h is fixed, then the convergence is uniform in r lying in a compact subset of $(0, R]$.*

We can now prove the statement analogous to Proposition 5.3 for parabolic limit points.

Proposition 5.6. *Let $0 < r_0 \leq R$ and $x_0 \in \Gamma$. Let ξ_0 be a point in ∂H for some parabolic subgroup H and identify ξ_0 with a point in $\partial_{r_0\mu}\Gamma$. Then, the map $(r, x, \xi) \in (0, R] \times \Gamma \times \Gamma \cup \partial_\mu\Gamma \mapsto K_r(x, \xi)$ is continuous at (x_0, r_0, ξ_0) .*

Proof. Let (s_n, y_n, ζ_n) converge to (r_0, x_0, ξ_0) . As in the proof of Proposition 5.3, we can assume that $y_n = x_0$ for every n . We can thus fix large enough η so that $y_n \in N_\eta(H)$ for every n . We can also, up to extracting subsequences, assume that either $\zeta_n \in \partial H$ for every n , or that $\zeta_n \notin \partial H$ for every n . In the first case, continuity is given by the above formulae.

Let us assume then that for every n , $\zeta_n \notin \partial H$. Then, the projection π_n of ζ_n on H is well defined up to a bounded distance in the Cayley graph. Moreover, π_n converges to ξ_0 in ∂H . Fix $\epsilon > 0$. Corollary 3.7 shows that there exists S , independent of n , such that

$$G_{s_n}(x_0, \zeta_n) \leq \frac{1}{1-\epsilon} \sum_{u \in B_S(\epsilon)} G_{s_n}(x_0, \pi_n u) G_{s_n}(\pi_n u, \zeta_n).$$

Also, we have

$$G_{s_n}(e, \zeta_n) \geq \sum_{u \in B_S(\epsilon)} G_{s_n}(e, \pi_n u) G_{s_n}(\pi_n u, \zeta_n).$$

Since $\pi_n u$ converges to ξ_0 in ∂H , Lemma 5.5 shows that for large enough n ,

$$|K_{s_n}(x_0, \pi_n u) - K_{s_n}(x_0, \xi_0)| \leq \epsilon.$$

Then, as in the proof of Proposition 4.3, we finally get that for large enough n ,

$$(1-\epsilon)(K_{s_n}(x_0, \xi_0) - \epsilon) \leq K_{s_n}(x_0, \zeta_n) \leq \frac{1}{1-\epsilon}(K_{s_n}(x_0, \xi_0) + \epsilon).$$

Finally, $K_{s_n}(x_0, \xi_0)$ converges to $K_{r_0}(x_0, \xi_0)$, so that $K_{s_n}(x_0, \zeta_n)$ also converges to $K_{r_0}(x_0, \xi_0)$, which concludes the proof. \square

6. STRONG STABILITY IN LOW DIMENSION

We first prove here the following proposition. We will then use it to prove strong stability of the Martin boundary when the parabolic subgroups have small rank.

Proposition 6.1. *Let Γ be a relatively hyperbolic group and let H a virtually abelian parabolic subgroup of rank d . Let μ be a probability measure on Γ whose symmetric finite support generates Γ . Assume that μ is spectrally degenerate along H . Then, $d \geq 5$.*

We begin with the following result proved in [17].

Lemma 6.2. [17, Proposition 1.9] *Let Γ be a finitely generated group and let μ be a probability measure on Γ , with spectral radius R . Then, for every $g, g' \in \Gamma$ and for every $r \leq R$, we have*

$$\frac{d}{dr}(rG_r(g, g')) = \sum_{g'' \in \Gamma} G_r(g, g'')G_r(g'', g').$$

We now prove the following key result.

Proposition 6.3. *Let Γ be a relatively hyperbolic group and let μ be a symmetric probability measure whose finite support generates Γ , with spectral radius R . Let H be a parabolic subgroup. Then, for every $r \leq R$ and for every $\eta \geq 0$, we have*

$$\sum_{g \in N_\eta(H)} G_r(e, g)G_r(g, e) < +\infty.$$

In other words, the derivative of the Green function along parabolic subgroups is finite at the spectral radius. Notice that we do not need to assume that the measure is spectrally non-degenerate along H .

Proof. We just need to prove the result for $r = R$, since the Green function is non-decreasing in r . Also, we just need to prove the result for $\eta = 0$. We first fix $r < R$, so that $\frac{d}{dr}G_r(e, e)$ is finite. According to Lemma 6.2, the sum $\sum_{g \in \Gamma} G_r(e, g)G_r(g, e)$ is thus finite.

Let $h_1, \dots, h_m \in H$. According to Lemma 3.3, there exists a_1 in $B_C(e)$ such that $\delta_{h_1}(e, h_1 a_1 h_2) \geq \delta$ for some uniform $C \geq 0$ and $\delta > 0$. We let $g_1 = h_1 a_1 h_2$. Then, there exists $a_2 \in B(e, C)$ such that $\delta_{g_1}(e, g_2) \geq \delta$, where $g_2 = g_1 a_2 h_3$. By induction, we see there exist a_1, \dots, a_{m-1} in $B(e, C)$ such that letting $g_i = h_1 a_1 h_2 \dots a_i h_{i+1}$, we have $\delta_{g_i}(e, g_{i+1}) \geq \delta$.

Also, whenever g is fixed, Lemma 3.4 shows there is a finite number of ways of writing g as $h_1 a_1 h_2$ as above. In particular, there exists C_1 such that there are at most C_1^m ways of writing g as $h_1 a_1 h_2 \dots a_{m-1} h_m$. We can thus find C_2 such that

$$C_2^m \sum_{g \in \Gamma} G_r(e, g)G_r(g, e) \geq \sum_{h_1, \dots, h_m \in H} G_r(e, g_m)G_r(g_m, e).$$

According to Theorem 3.6 (applied m times), we have

$$C_2^m \sum_{g \in \Gamma} G_r(e, g|r)G_r(g, e) \geq \frac{1}{C_3^{2m}} \sum_{h_1, \dots, h_m \in H} G_r(e, h_1)G_r(h_1, e)G_r(e, a_1)G_r(a_1, e) \dots G_r(e, a_{m-1})G_r(a_{m-1}, e)G_r(e, h_m)G_r(h_m, e).$$

Since $a_j \in B(e, C)$, $G_r(e, a_j)G_r(a_j, e) \geq C_4$ for some $C_4 \geq 0$.

We thus proved that for every $m \geq 1$,

$$\left(\sum_{h \in H} G_r(e, h) G_r(h, e) \right)^m \leq C_5^m \sum_{g \in \Gamma} G_r(e, g) G_r(g, e)$$

so that

$$\sum_{h \in H} G_r(e, h) G_r(h, e) \leq C_5 \left(\sum_{g \in \Gamma} G_r(e, g) G_r(g, e) \right)^{1/m}.$$

Letting m tend to infinity, we thus have

$$\sum_{h \in H} G_r(e, h) G_r(h, e) \leq C_5.$$

Since $r \mapsto G_r(e, h) G_r(h, e)$ is non-decreasing, we can now let r tend to R to obtain

$$\sum_{h \in H} G_R(e, h) G_R(h, e | R) \leq C_5,$$

which concludes the proof. \square

Let us recall the following classical result, which is a generalization of the local limit theorem in \mathbb{Z}^d .

Lemma 6.4. [36, Theorem 13.12] *Let H be a virtually abelian group of rank d . Let p be an H -invariant transition kernel on $H \times \{1, \dots, N\}$, with spectral radius R . Assume that p is strongly irreducible and has finite exponential moments. Then, if $R = 1$, there exists $C > 0$ such that*

$$p^{(n)}(e, e) \sim \frac{C}{n^{d/2}},$$

where $p^{(n)}$ denote the n th convolution power of p .

We now prove Proposition 6.1.

Proof. First, Lemme 4.5 shows that if η is chosen large enough, then $p_{\eta, H, R, \mu}$ has finite exponential moments. According to Lemma 6.2 and Proposition 6.3, for every $r \leq R$, the first derivative of the Green function $G_{\eta, H, r}$ associated with $p_{\eta, H, r}$ is finite at 1. According to Lemma 4.8, if we assume that μ is spectrally degenerate along H , then the spectral radius of $p_{\eta, H, R}$ is 1. We thus have an H -invariant transition kernel p on $H \times \{1, \dots, N\}$ with finite exponential moments, whose spectral radius is 1 and whose Green function has a finite derivative at 1. We just need to prove that this can only happen if the rank d of H is at least 5.

Lemma 6.4 shows that if p were strongly irreducible, then we would have

$$p^{(n)} \sim \frac{C}{n^{d/2}}.$$

Since the derivative of the Green function at 1 is given by $\sum_{n \geq 0} n p^{(n)}$. This sum is finite if and only if $\frac{d}{2} - 1 > 1$ or equivalently $d > 4$.

To conclude, let us show that we can assume that p is strongly irreducible. Since the initial random walk on Γ is irreducible, we already now that p is irreducible. We define a new transition kernel \tilde{p} on $N_\eta(H)$ by

$$\tilde{p}(h, h') = \alpha \delta_{h, h'} + (1 - \alpha) p(h, h'),$$

where $0 < \alpha < 1$. Then, \tilde{p} is strongly irreducible. Moreover, letting \tilde{G} be its Green function, [36, Lemma 9.2] shows that

$$\tilde{G}_t(e, e) = \frac{1}{1 - \alpha t} G_{\frac{(1-\alpha)t}{1-\alpha t}}(e, e).$$

The radius of convergence of $\frac{1}{1-\alpha t}$ is $\frac{1}{\alpha} > 1$. Also, if $t < \frac{1}{\alpha}$, then $\frac{(1-\alpha)t}{1-\alpha t} \geq 1$ if and only if $t \geq 1$. This proves that the radius of convergence of \tilde{G} also is 1 and the formula above shows that the first derivative of \tilde{G} is finite at 1.

We thus constructed a strongly irreducible H -invariant transition kernel p on $H \times \{1, \dots, N\}$ with finite exponential moments, whose spectral radius is 1 and whose Green function has a finite derivative at 1. This concludes the proof. \square

We get Theorem 1.5 as a corollary. Indeed, if Γ is the fundamental group of a geometrically finite hyperbolic manifold X , then it is relatively hyperbolic with respect to the cusp stabilizers, which are virtually abelian groups. If the dimension of X is at most 5, then the rank of the cusp stabilizers is at most 4, see for example [29, §5.4, §5.5] and [3] for more details on this. Hence, Proposition 6.1 shows that for any symmetric probability measure μ whose finite support generates Γ , μ cannot be spectrally degenerate along the parabolic subgroups. Thus, Theorem 1.2 shows that for every $r \leq R$, the r -Martin boundary coincides with the 1-geometric boundary. As noted in [11, Corollary 1.5], in this situation, this boundary also coincides with the CAT(0) boundary of Γ . \square

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