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Abstract

In the context of bivariate random variables $(Y^{(1)}, Y^{(2)})$, the marginal expected shortfall, defined as $\mathbb{E}(Y^{(1)}|Y^{(2)} \geqslant Q_2(1-p))$ for p small, where Q_2 denotes the quantile function of $Y^{(2)}$, is an important risk measure, which finds applications in areas like, e.g., finance and environmental science. We consider estimation of the marginal expected shortfall when the random variables of main interest $(Y^{(1)}, Y^{(2)})$ are observed together with a random covariate X, leading to the concept of the conditional marginal expected shortfall. The asymptotic behavior of an estimator for this conditional marginal expected shortfall is studied for a wide class of conditional bivariate distributions, with heavy-tailed marginal conditional distributions, and where p tends to zero at an intermediate rate.
Keywords: Marginal expected shortfall, empirical process, Pareto-type distribution, tail dependence.

1 Introduction

In the past years, many risk measures have been introduced in the literature, and these have been used to determine the amount of an asset to be kept in reserve in the financial framework. The most famous of these are the Value-At-Risk (VaR) defined for a random variable X as the $p-$quantile

$$Q(p) := \inf\{x \geqslant 0 : \mathbb{P}(X \leqslant x) \geqslant p\}, \text{ for } p \in (0,1),$$

and the Conditional Tail Expectation (CTE) defined as

$$CTE_p[X] = \mathbb{E}(X|X > Q(p)), \text{ for } p \in (0,1).$$

The latter risk measure is more conservative than the VaR for a same level of degree of confidence (see Landsman and Valdez, 2003) and it also satisfies the desirable property of being a coherent risk measure as defined by Artzner *et al.* (1999). For all these reasons, the CTE has been extensively studied and also it has been generalized to the multivariate framework, see, e.g., Cai and Li (2005), Cai *et al.* (2015), and Di Bernardino and Prieur (2018). More precisely, if $(Y^{(1)}, Y^{(2)})$ denote a pair of risk factors, the CTE can be extended into $\mathbb{E}(Y^{(1)}|Y^{(2)} > Q_2(p))$, where $Q_2(p)$ is the $p-$quantile of the risk $Y^{(2)}$. In such a multivariate context, this risk measure

is well-known as the Marginal Expected Shortfall (MES). It was introduced by Acharya *et al.* (2017), and used to measure the contribution of a financial institution to an overall systemic risk. The ongoing global credit crisis and other former financial crises have demonstrated the vital aspect of adequate risk measurement. For a financial firm, the MES is defined as its short-run expected equity loss conditional on the market taking a loss greater than its VaR. The MES is very simple to compute and therefore easy for regulators to consider. When estimating this risk measure, one often has the availability of additional information given by covariates, and these are important to take into account in order to obtain more precise estimates. This leads to the concept of conditional marginal expected shortfall.

In this paper, we will consider the estimation of the conditional marginal expected shortfall when the random variables of main interest $(Y^{(1)}, Y^{(2)})$ are recorded together with a random covariate $X \in \mathbb{R}^d$. We will denote by $F_j(.|x)$ the continuous conditional distribution function of $Y^{(j)}, j = 1, 2$, given $X = x$, and use the notation $\overline{F}_j(.|x)$ for the conditional survival function and $U_j(.|x)$ for the associated tail quantile function defined as $U_j(.|x) = \inf\{y : F_j(y|x) \geqslant 1 - 1/.\}$. Also, we will denote by f_X the density function of the covariate X and by x_0 a reference position such that $x_0 \in \mathrm{Int}(S_X)$, the interior of the support $S_X \subset \mathbb{R}^d$ of f_X, which is assumed to be non-empty. Our aim will be to estimate the conditional marginal expected shortfall, given $X = x_0$, and defined as

$$\theta_p = \mathbb{E}\left[Y^{(1)} \Big| Y^{(2)} \geqslant U_2\left(\frac{1}{p}\Big|x_0\right); x_0\right],$$

where p is small.

The remainder of the paper is organized as follows. In Section 2, we introduce our estimator for the conditional marginal expected shortfall and we establish its main asymptotic properties. All the proofs of the results are postponed to Section 3.

2 Estimator and asymptotic properties

We assume $Y^{(1)}$ and $Y^{(2)}$ follow a conditional Pareto-type model.

Assumption $(\mathcal{D})$ *For all $x \in S_X$, the conditional survival functions of $Y^{(j)}$, $j = 1, 2$, satisfy*

$$\overline{F}_j(y|x) \quad = \quad A_j(x) y^{-1/\gamma_j(x)} \left(1 + \frac{1}{\gamma_j(x)} \delta_j(y|x)\right),$$

where $A_j(x) > 0$, $\gamma_j(x) > 0$, and $|\delta_j(.|x)|$ is normalised regularly varying with index $-\beta_j(x)$, $\beta_j(x) > 0$, i.e.,

$$\delta_j(y|x) \quad = \quad B_j(x) \exp\left(\int_1^y \frac{\varepsilon_j(u|x)}{u} du\right),$$

with $B_j(x) \in \mathbb{R}$ and $\varepsilon_j(y|x) \to -\beta_j(x)$ as $y \to \infty$. Moreover, we assume $y \to \varepsilon_j(y|x)$ to be a continuous function.

Under Assumption $(\mathcal{D})$, $F_1(.|x)$ and $F_2(.|x)$ have density functions. Indeed, straightforward differentiation gives

$$f_j(y|x) = \frac{A_j(x)}{\gamma_j(x)} y^{-1/\gamma_j(x)-1} \left[1 + \left(\frac{1}{\gamma_j(x)} - \varepsilon_j(y|x)\right) \delta_j(y|x)\right], j = 1, 2. \tag{1}$$

Now, let $(Y_i^{(1)}, Y_i^{(2)}, X_i)$, $i = 1, \ldots, n$, be independent copies of $(Y^{(1)}, Y^{(2)}, X)$. We consider estimating the conditional marginal expected shortfall when $p \to 0$ at an intermediate rate, i.e. $p = k/n$, where $k, n \to \infty$ such that $k/n \to 0$. A natural idea is then to study

$$\widehat{\theta}_n := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) Y_i^{(1)} \mathbb{1}_{\{Y_i^{(2)} \geqslant \widehat{U}_2(n/k|x_0)\}},$$

where $\widehat{U}_2(.|x_0)$ is an estimator for $U_2(.|x_0)$, to be introduced later, and $K_{h_n}(.) := K(./h_n)/h_n^d$, with K a joint density function on $\mathbb{R}^d$, h_n is a positive non-random sequence of bandwidths with $h_n \to 0$ if $n \to \infty$, $\mathbb{1}_A$ is the indicator function on the event A.

To simplify the situation, let us assume for the moment that $U_2(.|x_0)$ is known and consider

$$\widetilde{\theta}_n := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) Y_i^{(1)} \mathbb{1}_{\{Y_i^{(2)} \geqslant U_2(n/k|x_0)\}}.$$

Clearly, assuming $F_1(y|x_0)$ is strictly increasing in y, we have

$$\begin{aligned}
\widetilde{\theta}_n &= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{Y_i^{(1)} \geqslant s\}} \mathbb{1}_{\{Y_i^{(2)} \geqslant U_2(n/k|x_0)\}} ds \\
&= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{Y_i^{(1)} \geqslant s,\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} ds \\
&= \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant (k/n)[(n/k)\overline{F}_1(s|x_0)],\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} ds \\
&= -U_1(n/k|x_0) \int_0^\infty \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant (k/n) s_n(u),\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\}} du^{-\gamma_1(x_0)},
\end{aligned}$$

where $s_n(u) := \frac{n}{k} \overline{F}_1(u^{-\gamma_1(x_0)} U_1(n/k|x_0)|x_0)$. Note that under $(\mathcal{D})$, we have $s_n(u) \to u$ as $n \to \infty$.

The key statistic to consider is thus, for $x_0 \in \mathrm{Int}(S_X)$,

$$T_n(y_1, y_2|x_0) := \frac{1}{k} \sum_{i=1}^{n} K_{h_n}(x_0 - X_i) \mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0) \leqslant k/n\, y_1,\, \overline{F}_2(Y_i^{(2)}|x_0) \leqslant k/n\, y_2\}},$$

where $y_1, y_2 > 0$.

As a first main result we study the weak convergence of the process

$$\left\{\sqrt{kh_n^d}\,\frac{T_n(y_1,y_2|x_0)-\mathbb{E}(T_n(y_1,y_2|x_0))}{y_1^\eta},\ \ y_1,y_2\in(0,T]\right\}, \tag{2}$$

for any $T>0$, finite, and $0\leqslant\eta<1/2-\chi$, where $\chi>0$ small. This will require some further assumptions.

In order to deal with the regression context, the functions appearing in $F_1(y|x)$ and $F_2(y|x)$ are assumed to satisfy the following Hölder conditions. Let $\|.\|$ denote some norm on $\mathbb{R}^d$.

Assumption $(\mathcal{H})$ *There exist positive constants* M_{f_X}, M_{A_j}, M_{γ_j}, M_{B_j}, M_{ε_j}, η_{f_X}, η_{A_j}, η_{γ_j}, η_{B_j} *and* η_{ε_j}, *where* $j=1,2$, *such that for all* $x,z\in S_X$*:*

$$\begin{aligned}
|f_X(x)-f_X(z)| &\leqslant M_{f_X}\|x-z\|^{\eta_{f_X}},\\
|A_j(x)-A_j(z)| &\leqslant M_{A_j}\|x-z\|^{\eta_{A_j}},\\
|\gamma_j(x)-\gamma_j(z)| &\leqslant M_{\gamma_j}\|x-z\|^{\eta_{\gamma_j}},\\
|B_j(x)-B_j(z)| &\leqslant M_{B_j}\|x-z\|^{\eta_{B_j}},\\
\sup_{y\geqslant 1}|\varepsilon_j(y|x)-\varepsilon_j(y|z)| &\leqslant M_{\varepsilon_j}\|x-z\|^{\eta_{\varepsilon_j}}.
\end{aligned}$$

We also impose a condition on the kernel function K, which is a standard condition in local estimation.

Assumption $(\mathcal{K})$ K *is a bounded density function on* $\mathbb{R}^d$, *with support* S_K *included in the unit ball in* $\mathbb{R}^d$.

Next, a uniform convergence result is needed for the joint conditional distribution of $(Y^{(1)},Y^{(2)})$. Let $R_t(y_1,y_2|x):=t\mathbb{P}(\overline{F}_1(Y^{(1)}|x)\leqslant y_1/t,\,\overline{F}_2(Y^{(2)}|x)\leqslant y_2/t|X=x)$.

Assumption $(\mathcal{R})$ *For all* $x\in S_X$ *we have*

$$\lim_{t\to\infty}R_t(y_1,y_2|x)=R(y_1,y_2|x),$$

uniformly in $y_1,y_2\in(0,T]$, *for any* $T>0$, *and* $x\in B(x_0,h_n)$.

The weak convergence of (2) is then established in the following theorem.

Theorem 2.1 *Assume* $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $(\mathcal{R})$ *with* $x\to R(y_1,y_2|x)$ *being a continuous function,* $x_0\in Int(S_X)$ *with* $f_X(x_0)>0$, *and* $y\to F_j(y|x_0)$, $j=1,2$, *are strictly increasing. Consider sequences* $k\to\infty$ *and* $h_n\to 0$ *as* $n\to\infty$, *in such a way that* $k/n\to 0$, $kh_n^d\to\infty$ *and* $h_n^{\eta_{\gamma_1}\wedge\eta_{\gamma_2}\wedge\eta_{\varepsilon_1}\wedge\eta_{\varepsilon_2}}\ln n/k\to 0$. *Then for* $\eta\in[0,1/2-\chi)$, *where* $\chi>0$, *small, we have,*

$$\sqrt{kh_n^d}\,\frac{T_n(y_1,y_2|x_0)-\mathbb{E}(T_n(y_1,y_2|x_0))}{y_1^\eta}\rightsquigarrow\frac{W(y_1,y_2)}{y_1^\eta}, \tag{3}$$

in $D((0,T]^2)$*, for any* $T>0$*, where* $W(y_1,y_2)$ *is a zero centered Gaussian process with covariance function*

$$\mathbb{E}(W(y_1,y_2)W(\bar{y}_1,\bar{y}_2)) = \|K\|_2^2 f_X(x_0) R(y_1\wedge\bar{y}_1, y_2\wedge\bar{y}_2|x_0).$$

We also introduce the following weak convergence result for a related process. This process will be useful in establishing the asymptotic properties of the quantile estimator $\widehat{U}_2(n/k|x_0)$. Let $\widehat{f}_n(x_0) := 1/n\sum_{i=1}^n K_{h_n}(x_0-X_i)$ be a classical kernel density estimator.

Theorem 2.2 *Assume* $(\mathcal{D})$*,* $(\mathcal{H})$*,* $(\mathcal{K})$*, and* $x_0\in Int(S_X)$ *with* $f_X(x_0)>0$*. Consider sequences* $k\to\infty$ *and* $h_n\to 0$ *as* $n\to\infty$*, in such a way that* $k/n\to 0$*,* $kh_n^d\to\infty$*,* $h_n^{\eta_{\varepsilon_2}}\ln n/k\to 0$*,* $\sqrt{kh_n^d}\,h_n^{\eta_{f_X}\wedge\eta_{A_2}}\to 0$*,* $\sqrt{kh_n^d}\,h_n^{\eta_{\gamma_2}}\ln n/k\to 0$*,* $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}}\to 0$*, and* $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}}\ln n/k\to 0$*. Then, we have*

$$\sqrt{kh_n^d}\left(\frac{T_n(\infty,y_2|x_0)}{\widehat{f}_n(x_0)} - y_2\right) \rightsquigarrow \frac{W(\infty,y_2)}{f_X(x_0)}$$

in $D((0,T])$*, for any* $T>0$*, where* $W(\infty,y_2)$ *is a zero centered Gaussian process with covariance function*

$$\mathbb{E}(W(\infty,y_2)W(\infty,\bar{y}_2)) = \|K\|_2^2 f_X(x_0)(y_2\wedge\bar{y}_2).$$

The joint weak convergence of the above two processes can be established by showing the joint finite dimensional weak convergence of them, combined with joint tightness. The joint finite dimensional convergence can be established by using the Cramér-Wold device (van der Vaart, 1998, p. 16). This is a standard, but tedious, calculation which is for brevity omitted from the paper. Note that the joint tightness follows from the individual tightness (similarly to Lemma 1 in Bai and Taqqu, 2013).

Now, generalise $\widetilde{\theta}_n$ to $\widetilde{\theta}_n(y_2)$, defined as

$$\widetilde{\theta}_n(y_2) = \frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0-X_i)Y_i^{(1)}\mathbb{1}_{\{Y_i^{(2)}\geqslant U_2(n/(ky_2)|x_0)\}}.$$

Assuming $F_1(y|x_0)$ strictly increasing in y, we have

$$\widetilde{\theta}_n(y_2) = -U_1\left(\frac{n}{k}\middle|x_0\right)\int_0^\infty T_n(s_n(u),y_2|x_0)du^{-\gamma_1(x_0)}.$$

Proposition 2.1 *Assume* $(\mathcal{D})$*,* $(\mathcal{H})$*,* $(\mathcal{K})$*,* $(\mathcal{R})$ *with* $x\to R(y_1,y_2|x)$ *being a continuous function,* $x_0\in Int(S_X)$ *with* $f_X(x_0)>0$*, and* $y\to F_j(y|x_0)$*,* $j=1,2$*, are strictly increasing. Consider sequences* $k\to\infty$ *and* $h_n\to 0$ *as* $n\to\infty$*, in such a way that* $k/n\to 0$*,* $kh_n^d\to\infty$ *and* $h_n^{\eta_{\gamma_1}\wedge\eta_{\gamma_2}\wedge\eta_{\varepsilon_1}\wedge\eta_{\varepsilon_2}}\ln n/k\to 0$*. Then, for* $\gamma_1(x_0)<1/2-\kappa$*, with* $\kappa>0$ *small, we have*

$$\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\frac{\sqrt{kh_n^d}}{U_1(n/k|x_0)}\left[\widetilde{\theta}_n(y_2)-\mathbb{E}(\widetilde{\theta}_n(y_2))\right]+\int_0^\infty W(u,y_2)du^{-\gamma_1(x_0)}\right|\xrightarrow{\mathbb{P}}0.$$

The main result of this paper is the asymptotic normality of $\overline{\theta}_n := \widehat{\theta}_n/\widehat{f}_n(x_0)$, which is an estimator for the conditional marginal expected shortfall $\theta_{k/n}$. Note that $\widehat{\theta}_n = \widetilde{\theta}_n(\widehat{e}_n)$, where $\widehat{e}_n := \frac{n}{k}\,\overline{F}_2(\widehat{u}_n U_2(\frac{n}{k}|x_0)|x_0)$ with $\widehat{u}_n := \widehat{U}_2(\frac{n}{k}|x_0)/U_2(\frac{n}{k}|x_0)$. To estimate $U_2(.|x_0)$ we use $\widehat{U}_2(.|x_0) := \inf\{y : \widehat{F}_{n,2}(y|x_0) \geqslant 1 - 1/.\}$ with

$$\widehat{F}_{n,2}(y|x_0) := \frac{\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{Y_i^{(2)} \leqslant y\}}}{\sum_{i=1}^n K_{h_n}(x_0 - X_i)},$$

the empirical kernel estimator of the unknown conditional distribution function of $Y^{(2)}$ given $X = x_0$. See for instance Daouia *et al.* (2011).

In order to obtain the weak convergence of $\overline{\theta}_n$, we need to introduce the following second order condition.

Assumption $(\mathcal{S})$**.** *There exist $\beta > \gamma_1(x_0)$ and $\tau < 0$ such that, as $t \to \infty$*

$$\sup_{x \in B(x_0,h_n)} \sup_{0<y_1<\infty, \frac{1}{2} \leqslant y_2 \leqslant 2} \frac{|R_t(y_1,y_2|x) - R(y_1,y_2|x_0)|}{y_1^{\beta} \wedge 1} = O(t^{\tau}).$$

Theorem 2.3 *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $(\mathcal{S})$ with $x \to R(y_1,y_2|x)$ being a continuous function, and $y \to F_j(y|x_0)$, $j = 1,2$, are strictly increasing. Let $x_0 \in Int(S_X)$ such that $f_X(x_0) > 0$. Consider sequences $k = \lfloor n^{\alpha}\ell_1(n) \rfloor$ and $h_n = n^{-\Delta}\ell_2(n)$, where ℓ_1 and ℓ_2 are slowly varying functions at infinity, with $\alpha \in (0,1)$ and*

$$\max\Bigg(\frac{\alpha}{d + 2\gamma_1(x_0)(\eta_{A_1} \wedge \eta_{\gamma_1})}, \frac{\alpha}{d + 2(1-\gamma_1(x_0))(\eta_{A_2} \wedge \eta_{\gamma_2} \wedge \eta_{B_2} \wedge \eta_{\varepsilon_2} \wedge \eta_{f_X})},$$
$$\frac{\alpha}{d} - \frac{2(1-\alpha)\gamma_1^2(x_0)\beta_1(x_0)}{d + d(\beta_1(x_0) + \varepsilon)\gamma_1(x_0)}, \frac{\alpha - 2(1-\alpha)(\gamma_1(x_0) \wedge (\beta_2(x_0)\gamma_2(x_0)) \wedge (-\tau))}{d}\Bigg) < \Delta < \frac{\alpha}{d}.$$

Then, for $\gamma_1(x_0) < 1/2 - \kappa$, with $\kappa > 0$ small, we have

$$\sqrt{kh_n^d}\left(\frac{\overline{\theta}_n}{\theta_{k/n}} - 1\right) \rightsquigarrow -(1-\gamma_1(x_0))\frac{W(\infty,1)}{f_X(x_0)} + \frac{1}{f_X(x_0)}\,\frac{\int_0^{\infty} W(s,1)ds^{-\gamma_1(x_0)}}{\int_0^{\infty} R(s,1|x_0)ds^{-\gamma_1(x_0)}}.$$

The conditions on k and h_n in Theorem 2.3 are due to the method of proof of the auxiliary result given in Lemma 3.4. Also in Cai *et al.* (2015) one needed a condition on the growth of k, but in the context without covariates.

3 Appendix

Lemma 3.1 *Assume $(\mathcal{D})$ and $(\mathcal{H})$ and $x_0 \in Int(S_X)$. Let $(t_n)_{n\geqslant 1}$ and $(h_n)_{n\geqslant 1}$ be arbitrary sequences satisfying $t_n \to \infty$ and $h_n \to 0$ such that $h_n^{\eta_{\gamma_j} \wedge \eta_{\varepsilon_j}} \ln t_n \to 0$, as $n \to \infty$, and $0 \leqslant \eta < 1$. Then*

$$\left|\frac{t_n \overline{F}_j(U_j(t_n/y|x_0)|x)}{y^{\eta}} - y^{1-\eta}\right| \to 0, \ \text{as } n \to \infty,$$

uniformly in $y \in (0,T]$ *and* $x \in B(x_0,h_n)$.

Proof

First note that, by continuity of $y \to F_j(y|x)$,

$$t_n\overline{F}_j(U_j(t_n/y|x_0)|x) = y\,\frac{\overline{F}_j(U_j(t_n/y|x_0)|x)}{\overline{F}_j(U_j(t_n/y|x_0)|x_0)}.$$

Then, from condition $(\mathcal{D})$, and a straightforward decomposition,

$$\begin{aligned}
&\left|\frac{t_n\overline{F}_j(U_j(t_n/y|x_0)|x)}{y^\eta} - y^{1-\eta}\right| \\
&\leqslant\ y^{1-\eta}\left\{\left|\frac{A_j(x)}{A_j(x_0)}-1\right|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)}\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)}\right. \\
&\quad+\left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)}-1\right|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)} \\
&\quad\left.+\left|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)}-1\right|\right\}.
\end{aligned}$$

Each of the absolute differences in the right-hand side of the above display can be handled by condition $(\mathcal{H})$. Obviously, for some constant C,

$$\left|\frac{A_j(x)}{A_j(x_0)}-1\right| \leqslant Ch_n^{\eta_{A_j}}, \quad \text{for} \quad x \in B(x_0,h_n).$$

Next, using the inequality $|e^z-1| \leqslant e^{|z|}|z|$, we have, for some constant C (not necessarily equal to the one introduced above),

$$\left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)}-1\right| \leqslant e^{Ch_n^{\eta\gamma_j}\ln U_j(t_n/y|x_0)}Ch_n^{\eta\gamma_j}\ln U_j(t_n/y|x_0).$$

For distributions satisfying $(\mathcal{D})$, one easily verifies that

$$U_j(t_n|x_0) = (A_j(x_0))^{\gamma_j(x_0)}t_n^{\gamma_j(x_0)}(1+a_j(t_n|x_0)) \tag{4}$$

where $|a_j(.|x_0)|$ is regularly varying with index equal to $-\gamma_j(x_0)\beta_j(x_0)$. Hence, for some constants C_1 and C_2, not depending on y, one gets for $x \in B(x_0,h_n)$ and n large,

$$\left|(U_j(t_n/y|x_0))^{1/\gamma_j(x_0)-1/\gamma_j(x)}-1\right| \leqslant C_1t_n^{C_2h_n^{\eta\gamma_j}}y^{-C_2h_n^{\eta\gamma_j}}\left(h_n^{\eta\gamma_j}\ln t_n - h_n^{\eta\gamma_j}\ln y\right).$$

Finally, for n large,

$$\begin{aligned}
&\left|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)}-1\right| \\
&\leqslant\ C|\delta_j(U_j(t_n/y|x_0)|x_0)|\left\{\left|\frac{\delta_j(U_j(t_n/y|x_0)|x)}{\delta_j(U_j(t_n/y|x_0)|x_0)}-1\right|+\left|\frac{1}{\gamma_j(x)}-\frac{1}{\gamma_j(x_0)}\right|\right\}.
\end{aligned}$$

By the assumptions on δ_j we obtain

$$\left|\frac{\delta_j(U_j(t_n/y|x_0)|x)}{\delta_j(U_j(t_n/y|x_0)|x_0)}-1\right| \leqslant \left|\frac{B_j(x)}{B_j(x_0)}-1\right| e^{\int_1^{U_j(t_n/y|x_0)}\frac{\varepsilon_j(u|x)-\varepsilon_j(u|x_0)}{u}du} + \left|e^{\int_1^{U_j(t_n/y|x_0)}\frac{\varepsilon_j(u|x)-\varepsilon_j(u|x_0)}{u}du}-1\right|,$$

and, hence, using $(\mathcal{H})$, for $x \in B(x_0,h_n)$ and n large,

$$\left|\frac{1+\frac{1}{\gamma_j(x)}\delta_j(U_j(t_n/y|x_0)|x)}{1+\frac{1}{\gamma_j(x_0)}\delta_j(U_j(t_n/y|x_0)|x_0)}-1\right| \leqslant C_1\left[h_n^{\eta_{\gamma_j}\wedge\eta_{B_j}} + t_n^{C_2h_n^{\eta_{\varepsilon_j}}}y^{-C_2h_n^{\eta_{\varepsilon_j}}}\left(h_n^{\eta_{\varepsilon_j}}\ln t_n - h_n^{\eta_{\varepsilon_j}}\ln y\right)\right].$$

Combining the above results establishes the lemma. ■

Lemma 3.2 *Assume $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$ and $(\mathcal{R})$ with $x \to R(y_1,y_2|x)$ being a continuous function, and $x_0 \in Int(S_X)$ such that $f_X(x_0)>0$. Consider sequences $k\to\infty$ and $h_n\to 0$ as $n\to\infty$ in such a way that $k/n\to 0$ and $h_n^{\eta_{\gamma_1}\wedge\eta_{\gamma_2}\wedge\eta_{\varepsilon_1}\wedge\eta_{\varepsilon_2}}\ln n/k\to 0$. Then, as $n\to\infty$*

$$\begin{aligned}\mathbb{E}(T_n(y_1,y_2|x_0)) &\to f_X(x_0)R(y_1,y_2|x_0),\\ kh_n^d\mathbb{V}ar(T_n(y_1,y_2|x_0)) &\to \|K\|_2^2 f_X(x_0)R(y_1,y_2|x_0).\end{aligned}$$

Proof

We have

$$\begin{aligned}\mathbb{E}(T_n(y_1,y_2|x_0)) &= \frac{n}{k}\mathbb{E}\left[K_{h_n}(x_0-X)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1,\,\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2\}}\right]\\ &= \frac{n}{k}\int_{S_K}K(v)\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1,\,\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2|X=x_0-h_nv)\\ &\qquad\qquad \times f_X(x_0-h_nv)dv\\ &= \int_{S_K}K(v)R(y_1,y_2|x_0-h_nv)f_X(x_0-h_nv)dv\\ &\quad+\int_{S_K}K(v)\left[\frac{n}{k}\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1,\,\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2|X=x_0-h_nv)\right.\\ &\qquad\qquad \left.-R(y_1,y_2|x_0-h_nv)\right]f_X(x_0-h_nv)dv\\ &=: T_{1,n}+T_{2,n}.\end{aligned}$$

Concerning $T_{1,n}$, by the continuity of $f_X(x)$ and $R(y_1,y_2|x)$ at x_0, we have that f_X and R are bounded in a neighborhood of x_0, and hence, by Lesbesgue's dominated convergence theorem

$$T_{1,n}\to f_X(x_0)R(y_1,y_2|x_0),\ \text{as } n\to\infty.$$

As for $T_{2,n}$,

$$\begin{aligned}|T_{2,n}| \leqslant \sup_{v\in S_K}&\left|\frac{n}{k}\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1,\,\overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2|X=x_0-h_nv)\right.\\ &\left.-R(y_1,y_2|x_0-h_nv)\right|\int_{S_K}K(v)f_X(x_0-h_nv)dv,\end{aligned}$$

and note that

$$\begin{aligned}
&\mathbb{P}(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ y_1,\ \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ y_2|X = x_0 - h_n v) \\
&= \mathbb{P}\left(\overline{F}_1(Y^{(1)}|x_0 - h_n v) \leqslant \frac{k}{n}\frac{n}{k}\overline{F}_1(U_1(n/(ky_1)|x_0)|x_0 - h_n v),\right. \\
&\quad \left.\overline{F}_2(Y^{(2)}|x_0 - h_n v) \leqslant \frac{k}{n}\frac{n}{k}\overline{F}_2(U_2(n/(ky_2)|x_0)|x_0 - h_n v)|X = x_0 - h_n v\right).
\end{aligned}$$

Then, by the result of Lemma 3.1 and the uniformity of the convergence in Assumption $(\mathcal{R})$, we have that $T_{2,n} \to 0$ as $n \to \infty$.

Now, consider the variance. We have

$$\begin{aligned}
kh_n^d \mathbb{V}ar(T_n(y_1, y_2|x_0)) &= \frac{nh_n^d \mathbb{V}ar(K_{h_n}(x_0 - X)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ y_2\}})}{k} \\
&= \|K\|_2^2 \frac{n}{k}\mathbb{E}\left[\frac{1}{h_n^d \|K\|_2^2} K^2\left(\frac{x_0 - X}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ y_2\}}\right] \\
&\quad - \frac{kh_n^d}{n}\left\{\frac{n}{k}\mathbb{E}\left[K_{h_n}(x_0 - X)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ y_1,\, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ y_2\}}\right]\right\}^2,
\end{aligned}$$

from which the result follows. ■

Proof of Theorem 2.1

To prove the result we will make use of empirical process theory with changing function classes, see for instance van der Vaart and Wellner (1996). To this aim we start by introducing some notation. Let P be the distribution measure of $(Y^{(1)}, Y^{(2)}, X)$, and denote the expected value under P, the empirical version and empirical process as follows

$$Pf := \int f dP, \quad \mathbb{P}_n f := \frac{1}{n}\sum_{i=1}^n f\left(Y_i^{(1)}, Y_i^{(2)}, X_i\right), \quad \mathbb{G}_n f := \sqrt{n}(\mathbb{P}_n - P)f,$$

for any real-valued measurable function $f : \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$. For a function class $\mathcal{F}$, let $N_{[]}(\varepsilon, \mathcal{F}, L_2(P))$, denote the minimal number of $\varepsilon-$brackets needed to cover $\mathcal{F}$. The bracketing integral is then defined as

$$J_{[]}(\delta, \mathcal{F}, L_2(P)) = \int_0^\delta \sqrt{\ln N_{[]}(\varepsilon, \mathcal{F}, L_2(P))}d\varepsilon.$$

We introduce our sequence of classes $\mathcal{F}_n$ on $\mathbb{R}^2 \times \mathbb{R}^d$ as

$$\mathcal{F}_n := \left\{(u, z) \to f_{n,y}(u, z),\ \ y \in (0, T]^2\right\}$$

where

$$f_{n,y}(u, z) := \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z)\frac{\mathbb{1}_{\left\{\overline{F}_1(u_1|x_0) \leqslant k/n\ y_1,\, \overline{F}_2(u_2|x_0) \leqslant k/n\ y_2\right\}}}{y_1^\eta}.$$

Denote also by F_n an envelope function of the class $\mathcal{F}_n$. Now, according to Theorem 19.28 in van der Vaart (1998), the weak convergence of the stochastic process (3) follows from the following four conditions. Let ρ_{x_0} be a semimetric, possibly depending on x_0, making $(0,T]^2$ totally bounded. We have to prove that

$$\begin{aligned}
\sup_{\rho_{x_0}(y,\bar{y})\leqslant\delta_n} P(f_{n,y}-f_{n,\bar{y}})^2 &\longrightarrow 0 \text{ for every } \delta_n \searrow 0, &(5)\\
PF_n^2 &= O(1), &(6)\\
PF_n^2 \mathbb{1}_{\{F_n>\varepsilon\sqrt{n}\}} &\longrightarrow 0 \text{ for every } \varepsilon>0, &(7)\\
J_{[]}(\delta_n,\mathcal{F}_n,L_2(P)) &\longrightarrow 0 \text{ for every } \delta_n \searrow 0, &(8)
\end{aligned}$$

along with the pointwise convergence of the covariance function.

We start with verifying condition (5), with $\rho_{x_0}(y,\bar{y}) := |y_1-\bar{y}_1|+|y_2-\bar{y}_2|$. Denote $A_{n,y} := \{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1, \overline{F}_2(Y^{(2)}|x_0)\leqslant k/n\ y_2\}$. We have then

$$\begin{aligned}
P(f_{n,y}-f_{n,\bar{y}})^2 &= \frac{nh_n^d}{k}\,\mathbb{E}\left[K_{h_n}^2(x_0-X)\left(\frac{\mathbb{1}_{A_{n,y}}}{y_1^\eta}-\frac{\mathbb{1}_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\right]\\
&= \frac{nh_n^d}{k}\,\mathbb{E}\left[K_{h_n}^2(x_0-X)\mathbb{E}\left[\left(\frac{\mathbb{1}_{A_{n,y}}}{y_1^\eta}-\frac{\mathbb{1}_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\middle| X\right]\right]. \qquad (9)
\end{aligned}$$

We consider now three cases.

Case 1: $y_1\wedge\bar{y}_1\leqslant\delta_n$. Assume without loss of generality that $y_1\leqslant\bar{y}_1$. By expanding the square in the above conditional expectation and using the fact that, e.g., $A_{n,y}\subset\{\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1\}$, we obtain the following inequality

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\mathbb{1}_{A_{n,y}}}{y_1^\eta}-\frac{\mathbb{1}_{A_{n,\bar{y}}}}{\bar{y}_1^\eta}\right)^2\middle| X=x\right] &\leqslant \frac{3P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1|X=x)}{y_1^{2\eta}}\\
&\quad+\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ \bar{y}_1|X=x)}{\bar{y}_1^{2\eta}},
\end{aligned}$$

which, after substituting in (9) leads to

$$\begin{aligned}
P(f_{n,y}-f_{n,\bar{y}})^2 &\leqslant 3\frac{n}{k}\int_{S_K}K^2(v)\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1|X=x_0-h_nv)}{y_1^{2\eta}}f_X(x_0-h_nv)dv\\
&\quad+\frac{n}{k}\int_{S_K}K^2(v)\frac{P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ \bar{y}_1|X=x_0-h_nv)}{\bar{y}_1^{2\eta}}f_X(x_0-h_nv)dv.
\end{aligned}$$

Now note that

$$P(\overline{F}_1(Y^{(1)}|x_0)\leqslant k/n\ y_1|X=x_0-h_nv)=\overline{F}_1\left(U_1(n/(ky_1)|x_0)|x_0-h_nv\right),$$

which, together with the result of Lemma 3.1, motivates the following decomposition

$$\begin{aligned}
P(f_{n,y} - f_{n,\bar{y}})^2 &\leqslant 3y_1^{1-2\eta} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
&\quad + 3 \int_{S_K} K^2(v) \left[\frac{1}{y_1^{2\eta}} \frac{n}{k} \overline{F}_1 \left(U_1(n/(ky_1)|x_0) | x_0 - h_n v \right) - y_1^{1-2\eta} \right] f_X(x_0 - h_n v) dv \\
&\quad + \bar{y}_1^{1-2\eta} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
&\quad + \int_{S_K} K^2(v) \left[\frac{1}{\bar{y}_1^{2\eta}} \frac{n}{k} \overline{F}_1 \left(U_1(n/(k\bar{y}_1)|x_0) | x_0 - h_n v \right) - \bar{y}_1^{1-2\eta} \right] f_X(x_0 - h_n v) dv.
\end{aligned}$$

Using Lemma 3.1 and the fact that $\rho_{x_0}(y, \bar{y}) \leqslant \delta_n$ which implies $\bar{y}_1 \leqslant 2\delta_n$, we get

$$P(f_{n,y} - f_{n,\bar{y}})^2 \leqslant 5\delta_n^{1-2\eta} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv + o(1),$$

where the $o(1)$ term does not depend on y_1 and $\bar{y}_1$.

Case 2: $y_1 \wedge \bar{y}_1 > \delta_n$ *and* $y_2 \wedge \bar{y}_2 \leqslant \delta_n$. Assume without loss of generality that $y_2 \leqslant \bar{y}_2$. Similarly to the approach followed in Case 1, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\mathbb{1}_{A_{n,y}}}{y_1^{\eta}} - \frac{\mathbb{1}_{A_{n,\bar{y}}}}{\bar{y}_1^{\eta}} \right)^2 \middle| X = x \right] &\leqslant \frac{3P(\overline{F}_2(Y^{(2)}|x_0) \leqslant k/n \; y_2 | X = x)}{(y_1 \wedge \bar{y}_1)^{2\eta}} \\
&\quad + \frac{P(\overline{F}_2(Y^{(2)}|x_0) \leqslant k/n \; \bar{y}_2 | X = x)}{(y_1 \wedge \bar{y}_1)^{2\eta}},
\end{aligned}$$

and thus

$$\begin{aligned}
P(f_{n,y} - f_{n,\bar{y}})^2 &\leqslant \frac{3y_2}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
&\quad + \frac{3y_2^{2\eta}}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) \left[\frac{1}{y_2^{2\eta}} \frac{n}{k} \overline{F}_2 \left(U_2(n/(ky_2)|x_0) | x_0 - h_n v \right) - y_2^{1-2\eta} \right] f_X(x_0 - h_n v) dv \\
&\quad + \frac{\bar{y}_2}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\
&\quad + \frac{\bar{y}_2^{2\eta}}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) \left[\frac{1}{\bar{y}_2^{2\eta}} \frac{n}{k} \overline{F}_2 \left(U_2(n/(k\bar{y}_2)|x_0) | x_0 - h_n v \right) - \bar{y}_2^{1-2\eta} \right] f_X(x_0 - h_n v) dv.
\end{aligned}$$

Again by Lemma 3.1 and using that $\bar{y}_2 \leqslant 2\delta_n$ we have that

$$P(f_{n,y} - f_{n,\bar{y}})^2 \leqslant 5\delta_n^{1-2\eta} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv + o(1),$$

where the $o(1)$ term does not depend on y_2 and $\bar{y}_2$.

Case 3: $y_1 \wedge \bar{y}_1 > \delta_n$ *and* $y_2 \wedge \bar{y}_2 > \delta_n$. Let $y \vee \bar{y}$ denote the vector with the component-wise maxima of y and $\bar{y}$, and similarly $y \wedge \bar{y}$ is the vector with the component-wise mimima of y and $\bar{y}$. Then

$$P(f_{n,y} - f_{n,\bar{y}})^2 \;\leqslant\; \frac{nh_n^d}{k}\,\mathbb{E}\left[K_{h_n}^2(x_0 - X)\mathbb{E}\left[\left(\frac{\mathbb{1}_{A_{n,y\vee\bar{y}}}}{(y_1 \wedge \bar{y}_1)^\eta} - \frac{\mathbb{1}_{A_{n,y\wedge\bar{y}}}}{(y_1 \vee \bar{y}_1)^\eta}\right)^2 \middle| X\right]\right].$$

Note that

$$\left(\frac{\mathbb{1}_{A_{n,y\vee\bar{y}}}}{(y_1 \wedge \bar{y}_1)^\eta} - \frac{\mathbb{1}_{A_{n,y\wedge\bar{y}}}}{(y_1 \vee \bar{y}_1)^\eta}\right)^2 = \left(\frac{1}{y_1^\eta} - \frac{1}{\bar{y}_1^\eta}\right)^2 \mathbb{1}_{A_{n,y\wedge\bar{y}}} + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}}(\mathbb{1}_{A_{n,y\vee\bar{y}}} - \mathbb{1}_{A_{n,y\wedge\bar{y}}}), \quad (10)$$

which leads to

$$\begin{aligned}
&P(f_{n,y} - f_{n,\bar{y}})^2 \\
&\leqslant \frac{(y_1^\eta - \bar{y}_1^\eta)^2}{(y_1\bar{y}_1)^{2\eta}}\frac{n}{k}\int_{S_K} K^2(v)P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\; y_1 \wedge \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\; y_2 \wedge \bar{y}_2 \middle| X = x_0 - h_n v\right) \\
&\qquad\qquad \times f_X(x_0 - h_n v)dv \\
&\quad + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}}\frac{n}{k}\int_{S_K} K^2(v)\left[P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\; y_1 \vee \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\; y_2 \vee \bar{y}_2 \middle| X = x_0 - h_n v\right)\right. \\
&\quad \left. - P\left(\overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\; y_1 \wedge \bar{y}_1, \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\; y_2 \wedge \bar{y}_2 \middle| X = x_0 - h_n v\right)\right] f_X(x_0 - h_n v)dv \\
&=: Q_{1,n} + Q_{2,n}.
\end{aligned}$$

As for $Q_{1,n}$, we easily obtain

$$Q_{1,n} \leqslant \frac{(y_1^\eta - \bar{y}_1^\eta)^2}{(y_1\bar{y}_1)^{2\eta}}\int_{S_K} K^2(v)\frac{n}{k}\overline{F}_1\left(U_1(n/(k\; y_1 \wedge \bar{y}_1)|x_0)|x_0 - h_n v\right) f_X(x_0 - h_n v)dv.$$

Now, by the mean value theorem, applied to $(y_1^\eta - \bar{y}_1^\eta)^2$, and a decomposition motivated by Lemma 3.1,

$$\begin{aligned}
&Q_{1,n} \\
&\leqslant (y_1 \wedge \bar{y}_1)^{-1-2\eta}(y_1 - \bar{y}_1)^2\int_{S_K} K^2(v)f_X(x_0 - h_n v)dv \\
&\quad + (y_1 \wedge \bar{y}_1)^{-2}(y_1 - \bar{y}_1)^2\int_{S_K} K^2(v)\left[\frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}}\frac{n}{k}\overline{F}_1\left(U_1(n/(k\; y_1 \wedge \bar{y}_1)|x_0)\middle|x_0 - h_n v\right) - (y_1 \wedge \bar{y}_1)^{1-2\eta}\right] \\
&\qquad\qquad \times f_X(x_0 - h_n v)dv.
\end{aligned}$$

This then gives

$$Q_{1,n} \;\leqslant\; \delta_n^{1-2\eta}\int_{S_K} K^2(v)f_X(x_0 - h_n v)dv + o(1),$$

where the $o(1)$ term does not depend on y_1 and $\bar{y}_1$.

Concerning $Q_{2,n}$, we have the following inequality

$$
\begin{aligned}
&Q_{2,n}\\
&\leqslant \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) P\left(k/n\ y_1 \wedge \bar{y}_1 \leqslant \overline{F}_1(Y^{(1)}|x_0) \leqslant k/n\ y_1 \vee \bar{y}_1 \Big| X = x_0 - h_n v\right) f_X(x_0 - h_n v) dv\\
&\quad + \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) P\left(k/n\ y_2 \wedge \bar{y}_2 \leqslant \overline{F}_2(Y^{(2)}|x_0) \leqslant k/n\ y_2 \vee \bar{y}_2 \Big| X = x_0 - h_n v\right) f_X(x_0 - h_n v) dv.\\
&=: Q_{2,1,n} + Q_{2,2,n}.
\end{aligned}
$$

We only give details about $Q_{2,1,n}$, the term $Q_{2,2,n}$ can be handled analogously. Direct computations give

$$
Q_{2,1,n} = \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) \int_{U_1(n/(k(y_1 \wedge \bar{y}_1))|x_0)}^{U_1(n/(k(y_1 \vee \bar{y}_1))|x_0)} f_1(y|x_0 - h_n v) dy f_X(x_0 - h_n v) dv,
$$

and, after substituting $u = (n/k)\overline{F}_1(y|x_0)$, we have

$$
Q_{2,1,n} = \frac{1}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{S_K} K^2(v) \int_{y_1 \wedge \bar{y}_1}^{y_1 \vee \bar{y}_1} \frac{f_1(U_1(n/(ku)|x_0)|x_0 - h_n v)}{f_1(U_1(n/(ku)|x_0)|x_0)} du f_X(x_0 - h_n v) dv.
$$

Using (1) and arguments similar to those used in the proof of Lemma 3.1 one obtains for n large and some small $\kappa > 0$,

$$
\frac{f_1(U_1(n/(ku)|x_0)|x_0 - h_n v)}{f_1(U_1(n/(ku)|x_0)|x_0)} \leqslant C u^{-\kappa},
$$

where C does not depend on y_1 and $\bar{y}_1$. Then, for n large enough,

$$
\begin{aligned}
Q_{2,1,n} &\leqslant \frac{C}{(y_1 \wedge \bar{y}_1)^{2\eta}} \int_{y_1 \wedge \bar{y}_1}^{y_1 \vee \bar{y}_1} u^{-\kappa} du \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv\\
&\leqslant \frac{C}{(y_1 \wedge \bar{y}_1)^{2\eta}} (y_1 \wedge \bar{y}_1)^{-\kappa} (y_1 \vee \bar{y}_1 - y_1 \wedge \bar{y}_1)\\
&\leqslant C \delta_n^{1-2\eta-\kappa}.
\end{aligned}
$$

Combining all the above we have verified (5).

Now, we move to the proof of (6). A natural envelope function of the class $\mathcal{F}_n$ is

$$
F_n(u, z) := \sqrt{\frac{n h_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant kT/n\}}}{[(n/k)\,\overline{F}_1(u_1|x_0)]^{\eta}}.
$$

This yields

$$
\begin{aligned}
PF_n^2 &= \left(\frac{n}{k}\right)^{1-2\eta} h_n^d\, \mathbb{E}\left(K_{h_n}^2(x_0-X)\mathbb{E}\left[\frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{(\overline{F}_1(Y^{(1)}|x_0))^{2\eta}}\middle| X\right]\right)\\
&= \left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)\mathbb{E}\left[\frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{(\overline{F}_1(Y^{(1)}|x_0))^{2\eta}}\middle| X=x_0-h_nv\right] f_X(x_0-h_nv)dv\\
&= \left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)\int_{U_1(n/(kT)|x_0)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\\
&\quad+\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)\int_{U_1(n/(kT)|x_0)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}\\
&\qquad\qquad\times\left\{\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{2\eta}-1\right\}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\\
&=: Q_{3,n}(T)+Q_{4,n}(T).
\end{aligned}
$$

Concerning $Q_{3,n}(T)$ we obtain by direct integration and a slight adjustment of Lemma 3.1, for large n

$$
\begin{aligned}
Q_{3,n}(T) &= \frac{1}{1-2\eta}\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)[\overline{F}_1(U_1(n/(kT)|x_0)|x_0-h_nv)]^{1-2\eta}f_X(x_0-h_nv)dv\\
&= \frac{T^{1-2\eta}}{1-2\eta}\int_{S_K} K^2(v)f_X(x_0-h_nv)dv\\
&\quad+\frac{1}{1-2\eta}\int_{S_K} K^2(v)\left[\left(\frac{n}{k}\overline{F}_1(U_1(n/(kT)|x_0)|x_0-h_nv)\right)^{1-2\eta}-T^{1-2\eta}\right]f_X(x_0-h_nv)dv\\
&\leqslant CT^{1-2\eta-\kappa},
\end{aligned}
\tag{11}
$$

for $\kappa<1-2\eta$.

Concerning $Q_{4,n}(T)$, combining $(\mathcal{D})$ with $(\mathcal{H})$ gives the following bound, for n large,

$$
\begin{aligned}
\left|\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{2\eta}-1\right| &\leqslant C_1\Big(h_n^{\eta_{A_1}}+y^{C_2h_n^{\eta_{\gamma_1}}}h_n^{\eta_{\gamma_1}}\ln y+|\delta_1(y|x_0)|h_n^{\eta_{B_1}}\\
&\qquad+|\delta_1(y|x_0)|y^{C_3h_n^{\eta_{\varepsilon_1}}}h_n^{\eta_{\varepsilon_1}}\ln y\Big).
\end{aligned}
\tag{12}
$$

Each of the terms in the right-hand side of the above inequality needs now be used in $Q_{4,n}(T)$, leading to the terms $Q_{4,j,n}(T)$, $j=1,\ldots,4$, studied below. First

$$
Q_{4,1,n}(T):=h_n^{\eta_{A_1}}\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)\int_{U_1(n/(kT)|x_0)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv.
$$

This term is clearly of smaller order than $Q_{3,n}(T)$ studied above and hence $Q_{4,1,n}(T)=O(1)$.
For the second term in the right-hand side of (12) we need to study

$$
Q_{4,2,n}(T):=h_n^{\eta_{\gamma_1}}\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K} K^2(v)\int_{t_n(T)}^{\infty}y^{\xi_{1,n}}\ln y\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv
$$

where $t_n(T) := U_1(n/(kT)|x_0)$ and $\xi_{1,n} := C_2 h_n^{\eta\gamma_1}$. Let $p_n(y) := \xi_{1,n} y^{\xi_{1,n}-1} \ln y + y^{\xi_{1,n}-1}$. Applying integration by parts on the inner integral gives, for n large enough,

$$\begin{aligned} Q_{4,2,n}(T) &= \left(\frac{n}{k}\right)^{1-2\eta} \frac{h_n^{\eta\gamma_1} \ln(t_n(T))[t_n(T)]^{\xi_{1,n}}}{1-2\eta} \int_{S_K} K^2(v) \left[\overline{F}_1(t_n(T)|x_0 - h_n v)\right]^{1-2\eta} f_X(x_0 - h_n v) dv \\ &\quad + \left(\frac{n}{k}\right)^{1-2\eta} \frac{h_n^{\eta\gamma_1}}{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} p_n(y) \left[\overline{F}_1(y|x_0 - h_n v)\right]^{1-2\eta} dy f_X(x_0 - h_n v) dv \\ &=: Q_{4,2,1,n}(T) + Q_{4,2,2,n}(T). \end{aligned}$$

We obtain, for n large enough

$$\begin{aligned} Q_{4,2,1,n}(T) &\leqslant C h_n^{\eta\gamma_1} \ln(t_n(T))[t_n(T)]^{\xi_{1,n}} T^{1-2\eta-\kappa} \\ &= O(1), \end{aligned}$$

by (4) and the fact that $h_n^{\eta\gamma_1} \ln(n/k) \to 0$ as $n \to \infty$.
Now consider $Q_{4,2,2,n}(T)$. We have

$$\begin{aligned} Q_{4,2,2,n}(T) &= \frac{h_n^{\eta\gamma_1} T^{1-2\eta}}{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(T)}^{\infty} p_n(y) \left(\frac{\overline{F}_1(y|x_0 - h_n v)}{\overline{F}_1(y|x_0)}\right)^{1-2\eta} \left(\frac{\overline{F}_1(y|x_0)}{\overline{F}_1(t_n(T)|x_0)}\right)^{1-2\eta} dy \\ &\qquad \times f_X(x_0 - h_n v) dv. \end{aligned}$$

For n large, with $\xi_{2,n} = C h_n^{\eta\varepsilon_1}$,

$$\left(\frac{\overline{F}_1(y|x_0 - h_n v)}{\overline{F}_1(y|x_0)}\right)^{1-2\eta} \leqslant C y^{\xi_{1,n}} \left(1 + y^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln y\right).$$

Substituting $u = y/t_n(T)$ we get

$$\begin{aligned} Q_{4,2,2,n}(T) &\leqslant C h_n^{\eta\gamma_1} T^{1-2\eta} [t_n(T)]^{1+\xi_{1,n}} \int_{S_K} K^2(v) \int_1^{\infty} p_n(t_n(T)u) u^{\xi_{1,n}} \left(1 + (t_n(T)u)^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln(t_n(T)u)\right) \\ &\qquad \times \left(\frac{\overline{F}_1(t_n(T)u|x_0)}{\overline{F}_1(t_n(T)|x_0)}\right)^{1-2\eta} du f_X(x_0 - h_n v) dv. \end{aligned}$$

Since $\overline{F}_1(.|x_0)$ is regularly varying, we can apply the Potter bound (see, e.g., de Haan and Ferreira, 2006, Proposition B.1.9), and obtain, for n large enough

$$\begin{aligned} Q_{4,2,2,n}(T) &\leqslant C h_n^{\eta\gamma_1} T^{1-2\eta} [t_n(T)]^{2\xi_{1,n}} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\ &\quad \times \int_1^{\infty} \left(\xi_{1,n} u^{\xi_{1,n}-1} \ln(t_n(T)) + \xi_{1,n} u^{\xi_{1,n}-1} \ln u + u^{\xi_{1,n}-1}\right) u^{\xi_{1,n} - (1/\gamma_1(x_0) - \delta)(1-2\eta)} \\ &\qquad \times \left(1 + (t_n(T)u)^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln(t_n(T)u)\right) du, \end{aligned}$$

where $0 < \delta < 1/\gamma_1(x_0)$. After tedious computations one gets

$$\begin{aligned} Q_{4,2,2,n}(T) &\leqslant C T^{1-2\eta} h_n^{\eta\gamma_1} [t_n(T)]^{2\xi_{1,n}} \left\{1 + h_n^{\eta\gamma_1} \ln(t_n(T)) + [t_n(T)]^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln(t_n(T))\right\} \\ &= O(1), \end{aligned}$$

by (4) and the fact that $h_n^{\eta_{\gamma_1}\wedge\eta_{\varepsilon_1}}\ln(n/k)\to 0$ as $n\to\infty$. Hence, $Q_{4,2,n}(T)=O(1)$.

Finally, the two last terms $Q_{4,3,n}(T)$ and $Q_{4,4,n}(T)$ can be dealt with similarly as the two previous ones since

$$\begin{aligned}
Q_{4,3,n}(T) &:= h_n^{\eta_{B_1}}\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K}K^2(v)\int_{t_n(T)}^{\infty}\frac{|\delta_1(y|x_0)|}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\\
&\leqslant \left(\sup_{y\geqslant t_n(T)}|\delta_1(y|x_0)|\right)h_n^{\eta_{B_1}}\left(\frac{n}{k}\right)^{1-2\eta}\\
&\quad\times\int_{S_K}K^2(v)\int_{t_n(T)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
Q_{4,4,n}(T) &:= h_n^{\eta_{\varepsilon_1}}\left(\frac{n}{k}\right)^{1-2\eta}\int_{S_K}K^2(v)\int_{t_n(T)}^{\infty}\frac{|\delta_1(y|x_0)|y^{\xi_{2,n}}\ln y}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\\
&\leqslant \left(\sup_{y\geqslant t_n(T)}|\delta_1(y|x_0)|\right)h_n^{\eta_{\varepsilon_1}}\left(\frac{n}{k}\right)^{1-2\eta}\\
&\quad\times\int_{S_K}K^2(v)\int_{t_n(T)}^{\infty}\frac{y^{\xi_{2,n}}\ln y}{(\overline{F}_1(y|x_0-h_nv))^{2\eta}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv. \qquad (14)
\end{aligned}$$

This yields $Q_{4,3,n}(T)=O(1)$ and $Q_{4,4,n}(T)=O(1)$. Combining all these results, we get (6).

Now, we want to show (7). To this aim, for any $\alpha\in(0,1/\eta-2)$, we have

$$\begin{aligned}
PF_n^2\mathbb{1}_{\{F_n>\varepsilon\sqrt{n}\}} &\leqslant \frac{1}{\varepsilon^{\alpha}n^{\alpha/2}}PF_n^{2+\alpha}\\
&= \frac{1}{\varepsilon^{\alpha}n^{\alpha/2}}\left(\frac{nh_n^d}{k}\right)^{1+\frac{\alpha}{2}}\mathbb{E}\left(K_{h_n}^{2+\alpha}(x_0-X)\frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{[(n/k)\,\overline{F}_1(Y^{(1)}|x_0)]^{\eta(2+\alpha)}}\right)\\
&= \frac{1}{\varepsilon^{\alpha}}\frac{1}{(kh_n^d)^{\alpha/2}}\left(\frac{n}{k}\right)^{1-\eta(2+\alpha)}\int_{S_K}K^{2+\alpha}(v)\mathbb{E}\left(\frac{\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant kT/n\}}}{[\overline{F}_1(Y^{(1)}|x_0)]^{\eta(2+\alpha)}}\middle|X=x_0-h_nv\right)\\
&\qquad\qquad\times f_X(x_0-h_nv)dv\\
&= \frac{1}{\varepsilon^{\alpha}}\frac{1}{(kh_n^d)^{\alpha/2}}\left(\frac{n}{k}\right)^{1-\eta(2+\alpha)}\\
&\quad\times\left\{\int_{S_K}K^{2+\alpha}(v)\int_{t_n(T)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{\eta(2+\alpha)}}dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\right.\\
&\quad+\int_{S_K}K^{2+\alpha}(v)\int_{t_n(T)}^{\infty}\frac{1}{(\overline{F}_1(y|x_0-h_nv))^{\eta(2+\alpha)}}\\
&\qquad\qquad\left.\times\left[\left(\frac{\overline{F}_1(y|x_0-h_nv)}{\overline{F}_1(y|x_0)}\right)^{\eta(2+\alpha)}-1\right]dF_1(y|x_0-h_nv)f_X(x_0-h_nv)dv\right\}.
\end{aligned}$$

The terms into brackets can be studied similarly as $Q_{j,n}(T), j = 3, 4$, and thus (7) is established since $kh_n^d \to \infty$.

Finally we verify (8). Without loss of generality assume $T = 1$ and consider, for $a, \theta, \tilde{\theta} < 1$, the classes

$$\begin{aligned}
\mathcal{F}_n^{(1)}(a) &:= \{f_{n,y} \in \mathcal{F}_n : y_1 \leqslant a\}, \\
\mathcal{F}_n^{(2)}(a) &:= \{f_{n,y} \in \mathcal{F}_n : y_1 > a, y_2 \leqslant a\}, \\
\mathcal{F}_n(\ell, m) &:= \{f_{n,y} \in \mathcal{F}_n : \theta^{\ell+1} \leqslant y_1 \leqslant \theta^{\ell}, \tilde{\theta}^{m+1} \leqslant y_2 \leqslant \tilde{\theta}^{m}\},
\end{aligned}$$

where $\ell = 0, \ldots, \lfloor \ln a / \ln \theta \rfloor$ and $m = 0, \ldots, \left\lfloor \ln a / \ln \tilde{\theta} \right\rfloor$. We start by showing that $\mathcal{F}_n^{(1)}(a)$ is an $\varepsilon-$bracket, for n sufficiently large. Clearly

$$\begin{aligned}
0 \leqslant f_{n,y}(u,z) &\leqslant \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ y_1\}}}{[(n/k)\,\overline{F}_1(u_1|x_0)]^{\eta}} \\
&\leqslant h_n^{d/2} (n/k)^{1/2-\eta} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ a\}}}{(\overline{F}_1(u_1|x_0))^{\eta}} := u_{1,n}(u,z).
\end{aligned}$$

Then

$$\begin{aligned}
Pu_{1,n}^2 &= \left(\frac{n}{k}\right)^{1-2\eta} \int_{S_K} K^2(v) \int_{t_n(a)}^{\infty} \frac{1}{(\overline{F}_1(y|x_0))^{2\eta}} dF_1(y|x_0 - h_n v) f_X(x_0 - h_n v) dv \\
&= Q_{3,n}(a) + Q_{4,n}(a).
\end{aligned}$$

Thus, one can obtain the result from the above analysis of $Q_{3,n}(T)$ and $Q_{4,n}(T)$, taking into account that the various constants involved in these will not depend on a.

Concerning $Q_{3,n}(a)$, according to (11), for n large

$$Q_{3,n}(a) \leqslant Ca^{1-2\eta-\kappa},$$

where C does not depend on a. Now, taking $a = \varepsilon^{3/(1-2\eta)}$, for n large enough and ε small we have $|Q_{3,n}(a)| \leqslant \varepsilon^2$.

Concerning $Q_{4,n}(a)$, we use the same decomposition as for $Q_{4,n}(T)$ based on (12), which entails that, for n large enough, ε small and some small $\zeta > 0$

$$\begin{aligned}
Q_{4,1,n}(a) &\leqslant \varepsilon^2, \\
Q_{4,2,1,n}(a) &\leqslant Ch_n^{\eta\gamma_1} \ln(t_n(a)) [t_n(a)]^{\xi_{1,n}} a^{1-2\eta-\kappa} \\
&\leqslant C(1 + |\ln a|) a^{-\zeta} a^{1-2\eta-\kappa} \\
&\leqslant Ca^{1-2\eta-2\kappa},
\end{aligned}$$

with C a constant not depending on a, since from (4) and for n large,

$$h_n^{\eta\gamma_1} \ln t_n(a) \leqslant C(1 + |\ln a|).$$

Also, for n large, and some small $\zeta > 0$

$$\begin{aligned} Q_{4,2,2,n}(a) &\leqslant Ca^{1-2\eta} h_n^{\eta\gamma_1} [t_n(a)]^{2\xi_{1,n}} \left\{ 1 + h_n^{\eta\gamma_1} \ln(t_n(a)) + [t_n(a)]^{\xi_{2,n}} h_n^{\eta\varepsilon_1} \ln(t_n(a)) \right\} \\ &\leqslant Ca^{1-2\eta} h_n^{\eta\gamma_1} a^{-\zeta} (1 + |\ln a| + a^{-\zeta}(1 + |\ln a|)) \\ &\leqslant Ca^{1-2\eta-\kappa}, \end{aligned}$$

where C does not depend on a. Hence, for n large and ε small we obtain $Q_{4,2,2,n}(a) \leqslant \varepsilon^2$. Using (13) and (14), we have also $Q_{4,3,n}(a) \leqslant \varepsilon^2$ and $Q_{4,4,n}(a) \leqslant \varepsilon^2$. Combining all the terms we get $Pu_{1,n}^2 \leqslant \varepsilon^2$ for n large.

Next consider $\mathcal{F}_n^{(2)}(a)$. Then

$$0 \leqslant f_{n,y}(u,z) \leqslant \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_2(u_2|x_0) \leqslant k/n\ a\}}}{a^\eta} =: u_{2,n}(u,z),$$

and

$$\begin{aligned} Pu_{2,n}^2 &= \frac{1}{a^{2\eta}} \frac{n}{k} \int_{S_K} K^2(v) \overline{F}_2(Q_2(1 - k/n\ a|x_0)|x_0 - h_n v) f_X(x_0 - h_n v) dv \\ &\leqslant \varepsilon^2, \end{aligned}$$

when n is large enough and for ε small.

Finally, we consider $\mathcal{F}_n(\ell, m)$. We obtain the following bounds

$$\begin{aligned} \underline{u}_n(u,z) := \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell+1},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m+1}\}}}{\theta^{\ell\eta}} \quad \leqslant \quad f_{n,y}(u,z) \leqslant \\ \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m}\}}}{\theta^{(\ell+1)\eta}} =: \overline{u}_n(u,z). \end{aligned}$$

Then

$$\begin{aligned} P(\overline{u}_n - \underline{u}_n)^2 &= \frac{nh_n^d}{k} \mathbb{E} \left[K_{h_n}^2(x_0 - X) \left(\frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m}\}}}{\theta^{(\ell+1)\eta}} \right.\right. \\ &\qquad \left.\left. - \frac{\mathbb{1}_{\{\overline{F}_1(u_1|x_0) \leqslant k/n\ \theta^{\ell+1},\, \overline{F}_2(u_2|x_0) \leqslant k/n\ \tilde{\theta}^{m+1}\}}}{\theta^{\ell\eta}} \right)^2 \right]. \end{aligned}$$

The difference of the indicator functions can be decomposed as in (10), and subsequent calculations follow arguments similar to those used in the verification of (5), Case 3. Taking $\theta = 1 - \varepsilon^3$ and $\tilde{\theta} = 1 - a$, gives for n large enough and ε small that $P(\overline{u}_n - \underline{u}_n)^2 \leqslant \varepsilon^2$.

Combining the above, for n large and ε small one obtains that the cover number by bracketing is of the order $\varepsilon^{-4-3/(1-2\eta)}$, and hence (8) is satisfied.

To conclude the proof, we comment on the pointwise convergence of the covariance function, which is given by $Pf_{n,y}f_{n,\bar{y}} - Pf_{n,y}Pf_{n,\bar{y}}$. We have

$$\begin{aligned} Pf_{n,y}f_{n,\bar{y}} &= \frac{\|K\|_2^2}{(y_1\bar{y}_1)^\eta}\frac{n}{k}\mathbb{E}\left[\frac{1}{\|K\|_2^2 h_n^d}K^2\left(\frac{x_0 - X}{h_n}\right)\mathbb{1}_{A_{n,y\wedge\bar{y}}}\right] \\ &\to \|K\|_2^2 f_X(x_0)\frac{R(y_1\wedge\bar{y}_1, y_2\wedge\bar{y}_2|x_0)}{(y_1\bar{y}_1)^\eta}, \end{aligned}$$

as $n \to \infty$, by the arguments used in the proof of Lemma 3.2. Also

$$\begin{aligned} Pf_{n,y} &= \sqrt{\frac{kh_n^d}{n}}\frac{1}{y_1^\eta}\frac{n}{k}\mathbb{E}\left[K_{h_n}(x_0 - X)\mathbb{1}_{A_{n,y}}\right] \\ &\to 0, \end{aligned}$$

as $n \to \infty$. ■

Proof of Theorem 2.2

Recall that

$$T_n(\infty, y_2|x_0) = \frac{1}{k}\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{\overline{F}_2(Y_i^{(2)}|x_0)\leqslant\frac{k}{n}y_2\}}.$$

We follow the lines of proof of Theorem 2.1. We introduce the sequence of classes $\widetilde{\mathcal{F}}_n$ on $\mathbb{R}\times\mathbb{R}^d$ as

$$\widetilde{\mathcal{F}}_n := \{(u,z)\to\widetilde{f}_{n,y}(u,z), y\in(0,T]\}$$

where

$$\widetilde{f}_{n,y}(u,z) := \sqrt{\frac{nh_n^d}{k}}K_{h_n}(x_0 - z)\mathbb{1}_{\{\overline{F}_2(u|x_0)\leqslant\frac{k}{n}y\}}.$$

We have to verify the conditions (5)-(8) in the proof of Theorem 2.1 for the new functions $\widetilde{f}_{n,y}$. Without loss of generality, we may assume that $y > \overline{y}$. Thus, we have

$$\begin{aligned} P\left(\widetilde{f}_{n,y} - \widetilde{f}_{n,\overline{y}}\right)^2 &= \frac{nh_n^d}{k}\mathbb{E}\left[K_{h_n}^2(x_0 - X)\left(\mathbb{1}_{\{\overline{F}_2(Y^{(2)}|x_0)\leqslant\frac{k}{n}y\}} - \mathbb{1}_{\{\overline{F}_2(Y^{(2)}|x_0)\leqslant\frac{k}{n}\overline{y}\}}\right)\right] \\ &= \frac{n}{k}\int_{S_K}K^2(v)\left[\overline{F}_2\left(U_2\left(\frac{n}{ky}\Big|x_0\right)\Big|x_0 - h_nv\right) - \overline{F}_2\left(U_2\left(\frac{n}{k\overline{y}}\Big|x_0\right)\Big|x_0 - h_nv\right)\right] \\ &\qquad\qquad \times f_X(x_0 - h_nv)dv \\ &= (y - \overline{y})\int_{S_K}K^2(v)f_X(x_0 - h_nv)dv \\ &\quad + \int_{S_K}K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky}\Big|x_0\right)\Big|x_0 - h_nv\right) - y\right]f_X(x_0 - h_nv)dv \\ &\quad - \int_{S_K}K^2(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{k\overline{y}}\Big|x_0\right)\Big|x_0 - h_nv\right) - \overline{y}\right]f_X(x_0 - h_nv)dv \\ &\leqslant \delta_n\int_{S_K}K^2(v)f_X(x_0 - h_nv)dv + o(1), \end{aligned}$$

with a $o(1)-$term which is uniform in y and $\overline{y}$ by Lemma 3.1. This yields (5).

Now, concerning (6) we can use the following envelope function of the class $\widetilde{\mathcal{F}}_n$

$$\widetilde{F}_n(u,z) := \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \mathbb{1}_{\{\overline{F}_2(u|x_0) \leqslant \frac{k}{n} T\}}$$

from which we deduce that

$$P\widetilde{F}_n^2 = \frac{n}{k} \int_{S_K} K^2(v) \overline{F}_2 \left(U_2 \left(\frac{n}{kT} \Big| x_0 \right) \Big| x_0 - h_n v \right) f_X(x_0 - h_n v) dv = O(1).$$

Next condition (7) is also a direct consequence of the definition of the envelope since

$$\begin{aligned} P\widetilde{F}_n^2 \mathbb{1}_{\{\widetilde{F}_n > \varepsilon\sqrt{n}\}} &\leqslant \frac{1}{\varepsilon^\alpha n^{\alpha/2}} P\widetilde{F}_n^{2+\alpha} \\ &\leqslant \frac{1}{\varepsilon^\alpha (kh_n^d)^{\alpha/2}} \frac{n}{k} \int_{S_K} K^{2+\alpha}(v) \overline{F}_2 \left(U_2 \left(\frac{n}{kT} \Big| x_0 \right) \Big| x_0 - h_n v \right) f_X(x_0 - h_n v) dv = o(1) \end{aligned}$$

as soon as $kh_n^d \to \infty$.

Finally, concerning (8), again without loss of generality we assume $T = 1$ and divide $[0,1]$ into m intervals of length $1/m$. Then, for $y \in [(i-1)/m, i/m]$ we have the bounds

$$\underline{u}_n(u,z) := \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \mathbb{1}_{\{\overline{F}_2(u|x_0) \leqslant \frac{k}{n} \frac{i-1}{m}\}} \leqslant \widetilde{f}_{n,y}(u,z) \leqslant \sqrt{\frac{nh_n^d}{k}} K_{h_n}(x_0 - z) \mathbb{1}_{\{\overline{F}_2(u|x_0) \leqslant \frac{k}{n} \frac{i}{m}\}} =: \overline{u}_n(u,z)$$

from which we deduce that

$$\begin{aligned} P(\underline{u}_n - \overline{u}_n)^2 &= \frac{1}{m} \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv \\ &\quad + \int_{S_K} K^2(v) \left[\frac{n}{k} \overline{F}_2 \left(U_2 \left(\frac{n}{k} \frac{m}{i} \Big| x_0 \right) \Big| x_0 - h_n v \right) - \frac{i}{m} \right] f_X(x_0 - h_n v) dv \\ &\quad - \int_{S_K} K^2(v) \left[\frac{n}{k} \overline{F}_2 \left(U_2 \left(\frac{n}{k} \frac{m}{i-1} \Big| x_0 \right) \Big| x_0 - h_n v \right) - \frac{i-1}{m} \right] f_X(x_0 - h_n v) dv \\ &\leqslant \varepsilon^3 \int_{S_K} K^2(v) f_X(x_0 - h_n v) dv + 2\varepsilon^3 \end{aligned}$$

when $m = \lceil \frac{1}{\varepsilon^3} \rceil$. If ε is small and n large, then $P(\underline{u}_n - \overline{u}_n)^2 \leqslant \varepsilon^2$.

Consequently

$$\sqrt{kh_n^d} \left[T_n(\infty, y_2 | x_0) - \mathbb{E}(T_n(\infty, y_2 | x_0)) \right] \rightsquigarrow W(\infty, y_2),$$

in $D((0,T])$.

Now, remark that

$$\begin{aligned}\mathbb{E}(T_n(\infty,y_2|x_0)) &= y_2 f_X(x_0)+O\left(h_n^{\eta_{f_X}}\right)\\ &\quad +f_X(x_0)\int_{S_K}K(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_nv\right)-y_2\right]dv\\ &\quad +\int_{S_K}K(v)\left[\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_nv\right)-y_2\right]\left[f_X(x_0-h_nv)-f_X(x_0)\right]dv.\end{aligned}$$

Following the lines of proof of Lemma 3.1, we deduce that

$$\left|\frac{n}{k}\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0-h_nv\right)-y_2\right|\leqslant C\left\{h_n^{\eta_{A_2}}+h_n^{\eta_{\gamma_2}}\ln\frac{n}{k}+|\delta_2(U_2(n/k|x_0)|x_0)|\left(h_n^{\eta_{B_2}}+h_n^{\eta_{\varepsilon_2}}\ln\frac{n}{k}\right)\right\}$$

from which we obtain

$$\begin{aligned}\mathbb{E}(T_n(\infty,y_2|x_0)) &= y_2 f_X(x_0)+O\left(h_n^{\eta_{f_X}\wedge\eta_{A_2}}\right)+O\left(h_n^{\eta_{\gamma_2}}\ln\frac{n}{k}\right)+O\left(|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}}\right)\\ &\quad +O\left(|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}}\ln\frac{n}{k}\right)\end{aligned}$$

with $O-$terms which are uniform in $y_2\in(0,T]$. This implies that, under the assumptions of Theorem 2.2, we have

$$\sqrt{kh_n^d}\left[T_n(\infty,y_2|x_0)-y_2f_X(x_0)\right]\rightsquigarrow W(\infty,y_2), \tag{15}$$

in $D((0,T])$.

Finally,

$$\sqrt{kh_n^d}\left(\frac{T_n(\infty,y_2|x_0)}{\widehat{f}_n(x_0)}-y_2\right)=\sqrt{kh_n^d}\left(\frac{T_n(\infty,y_2|x_0)}{f_X(x_0)}-y_2\right)-\frac{T_n(\infty,y_2|x_0)}{\widehat{f}_n(x_0)f_X(x_0)}\sqrt{\frac{k}{n}}\sqrt{nh_n^d}\left(\widehat{f}_n(x_0)-f_X(x_0)\right),$$

from which Theorem 2.2 follows. ■

3.1 Proof of Proposition 2.1

We use the decomposition

$$\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\frac{\sqrt{kh_n^d}}{U_1(n/k|x_0)}\left[\widetilde{\theta}_n(y_2)-\mathbb{E}(\widetilde{\theta}_n(y_2))\right]+\int_0^\infty W(u,y_2)du^{-\gamma_1(x_0)}\right|\leqslant I_1(T)+\sum_{i=2}^4 I_{i,n}(T),$$

where

$$
\begin{aligned}
I_1(T) &:= \sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\int_T^\infty W(u,y_2)du^{-\gamma_1(x_0)}\right|,\\
I_{2,n}(T) &:= \sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\int_T^\infty \sqrt{kh_n^d}\left[T_n(s_n(u),y_2|x_0)-\mathbb{E}\left(T_n(s_n(u),y_2|x_0)\right)\right]du^{-\gamma_1(x_0)}\right|,\\
I_{3,n}(T) &:= \sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\int_0^T \left\{\sqrt{kh_n^d}\left[T_n(s_n(u),y_2|x_0)-\mathbb{E}\left(T_n(s_n(u),y_2|x_0)\right)\right]-W(s_n(u),y_2)\right\}du^{-\gamma_1(x_0)}\right|,\\
I_{4,n}(T) &:= \sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\left|\int_0^T \left[W(s_n(u),y_2)-W(u,y_2)\right]du^{-\gamma_1(x_0)}\right|.
\end{aligned}
$$

Similarly to the proof of Proposition 2 in Cai *et al.* (2015), it is sufficient to show that for any $\varepsilon>0$, there exists $T_0=T_0(\varepsilon)$ such that

$$\mathbb{P}(I_1(T_0)>\varepsilon)<\varepsilon, \tag{16}$$

and $n_0=n_0(T_0)$ such that, for any $n>n_0$

$$\mathbb{P}(I_{j,n}(T_0)>\varepsilon)<\varepsilon, \text{ for } j=2,3 \text{ and } 4.$$

Clearly

$$I_1(T)\leqslant \sup_{u\geqslant T,\frac{1}{2}\leqslant y_2\leqslant 2}|W(u,y_2)|T^{-\gamma_1(x_0)}.$$

Since a rescaled version of our Gaussian process $W(.,.)$ gives the one in Cai *et al.* (2015), according to their Lemma 2, we have $\sup_{0<u<\infty,\frac{1}{2}\leqslant y_2\leqslant 2}|W(u,y_2)|<\infty$ with probability one. This implies that there exists $T_1=T_1(\varepsilon)$ such that

$$\mathbb{P}\left(\sup_{0<u<\infty,\frac{1}{2}\leqslant y_2\leqslant 2}|W(u,y_2)|>T_1^{\gamma_1(x_0)}\varepsilon\right)<\varepsilon,$$

from which we deduce that, for any $T>T_1$

$$\mathbb{P}\left(I_1(T)>\varepsilon\right)\leqslant \mathbb{P}\left(\sup_{0<u<\infty,\frac{1}{2}\leqslant y_2\leqslant 2}|W(u,y_2)|>T_1^{\gamma_1(x_0)}\varepsilon\right)<\varepsilon.$$

Consequently (16) holds for $T_0>T_1$.

We continue with the term $I_{2,n}(T)$. We have

$$
\begin{aligned}
&\mathbb{P}\left(I_{2,n}(T)>\varepsilon\right)\\
&\leqslant \mathbb{P}\left(\sup_{y_1\geqslant T,\frac{1}{2}\leqslant y_2\leqslant 2}\left|\sqrt{kh_n^d}\left[T_n(s_n(y_1),y_2|x_0)-\mathbb{E}(T_n(s_n(y_1),y_2|x_0))\right]\right|>\varepsilon\, T^{\gamma_1(x_0)}\right)\\
&\leqslant \mathbb{P}\left(\sup_{y_1\geqslant T,\frac{1}{2}\leqslant y_2\leqslant 2}\left|\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n\frac{K}{\|K\|_\infty}\left(\frac{x_0-X_i}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|x_0)\leqslant\frac{k}{n}s_n(y_1),\,\overline{F}_2(Y_i^{(2)}|x_0)\leqslant\frac{k}{n}y_2\}}\right.\right.\right.\\
&\qquad\left.\left.\left.-\mathbb{E}\left(\frac{K}{\|K\|_\infty}\left(\frac{x_0-X}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|x_0)\leqslant\frac{k}{n}s_n(y_1),\,\overline{F}_2(Y^{(2)}|x_0)\leqslant\frac{k}{n}y_2\}}\right)\right]\right|>\frac{\varepsilon\, T^{\gamma_1(x_0)}}{\|K\|_\infty}\sqrt{\frac{kh_n^d}{n}}\right)\\
&\leqslant \mathbb{P}\left(\sup_{(y_1,y_2)\in[0,1]^2}\left|\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^n\frac{K}{\|K\|_\infty}\left(\frac{x_0-X_i}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y_i^{(1)}|X_i)\leqslant y_1,\,\overline{F}_2(Y_i^{(2)}|X_i)\leqslant y_2\}}\right.\right.\right.\\
&\qquad\left.\left.\left.-\mathbb{E}\left(\frac{K}{\|K\|_\infty}\left(\frac{x_0-X}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(Y^{(1)}|X)\leqslant y_1,\,\overline{F}_2(Y^{(2)}|X)\leqslant y_2\}}\right)\right]\right|>\frac{\varepsilon\, T^{\gamma_1(x_0)}}{\|K\|_\infty}\sqrt{\frac{kh_n^d}{n}}\right).
\end{aligned}
$$

Define now the function class

$$
g_{n,y}(u,z):=\frac{K}{\|K\|_\infty}\left(\frac{x_0-z}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(u_1|z)\leqslant y_1,\,\overline{F}_2(u_2|z)\leqslant y_2\}}.
$$

Clearly, if we divide $[0,1]$ into m intervals of length $1/m$ on each axis, we have, for $y\in[(i-1)/m,i/m]\times[(j-1)/m,j/m]$,

$$
\begin{aligned}
\underline{g}_n(u,z) &:= \frac{K}{\|K\|_\infty}\left(\frac{x_0-z}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(u_1|z)\leqslant\frac{i-1}{m},\,\overline{F}_2(u_2|z)\leqslant\frac{j-1}{m}\}}\\
&\leqslant g_{n,y}(u,z)\leqslant\frac{K}{\|K\|_\infty}\left(\frac{x_0-z}{h_n}\right)\mathbb{1}_{\{\overline{F}_1(u_1|z)\leqslant\frac{i}{m},\,\overline{F}_2(u_2|z)\leqslant\frac{j}{m}\}}=:\overline{g}_n(u,z).
\end{aligned}
$$

Thus

$$
P\left(\overline{g}_n-\underline{g}_n\right)^2\leqslant\frac{2}{m}\mathbb{E}\left(\frac{K^2}{\|K\|_\infty^2}\left(\frac{x_0-X_i}{h_n}\right)\right)\leqslant\frac{2}{m}\leqslant\varepsilon^2,
$$

as soon as $m=\lceil\frac{2}{\varepsilon^2}\rceil$. Then the bracketing number is of the order ε^{-4} for n large, which implies that, according to Theorem 2.14.9 in van der Vaart and Wellner (1996)

$$
\mathbb{P}\left(I_{2,n}(T)>\varepsilon\right)\leqslant C\,\varepsilon^4\frac{T^{4\gamma_1(x_0)}}{\|K\|_\infty^4}\left(\frac{kh_n^d}{n}\right)^2\longrightarrow 0
$$

as $n\to\infty$.

Now, to study $I_{3,n}(T)$, remark that for any $T>0,\exists\, n_1=n_1(T):\forall n>n_1:s_n(T)<T+1$. Hence for $n>n_1$ and any $\eta_0\in(\gamma_1(x_0),1/2-\kappa)$:

$$
\begin{aligned}
\mathbb{P}\left(I_{3,n}(T)>\varepsilon\right) \leqslant \mathbb{P}\left(\sup_{0<y_1\leqslant T+1,\frac{1}{2}\leqslant y_2\leqslant 2}\left|\frac{\sqrt{kh_n^d}[T_n(y_1,y_2|x_0)-\mathbb{E}(T_n(y_1,y_2|x_0))]-W(y_1,y_2)}{y_1^{\eta_0}}\right|\right.\\
\left.\times\left|\int_0^T[s_n(u)]^{\eta_0}du^{-\gamma_1(x_0)}\right|>\varepsilon\right).
\end{aligned}
$$

According to Lemma 3 in Cai *et al.* (2015)

$$\left|\int_0^T [s_n(u)]^{\eta_0} du^{-\gamma_1(x_0)}\right| \longrightarrow \frac{\gamma_1(x_0)}{\eta_0-\gamma_1(x_0)} T^{\eta_0-\gamma_1(x_0)},$$

which, combining with our Theorem 2.1, entails that there exists $n_2(T) > n_1(T)$ such that $\forall n > n_2(T)$, $\mathbb{P}(I_{3,n}(T) > \varepsilon) < \varepsilon$.

Finally, concerning $I_{4,n}(T)$, we first remark that according to Lemma 2 in Cai *et al.* (2015), we have for $\eta_0 \in (\gamma_1(x_0), 1/2)$ and any $T > 0$, with probability one,

$$\sup_{0<y_1\leqslant T, \frac{1}{2}\leqslant y_2\leqslant 2} \frac{|W(y_1,y_2)|}{y_1^{\eta_0}} < \infty.$$

Then, applying Lemma 3 in Cai *et al.* (2015) with $S = T, S_0 = T+1$ and $g = W$, we deduce that there exists $n_3(T)$ such that for $n > n_3(T)$ we have $\mathbb{P}(I_{4,n}(T) > \varepsilon) < \varepsilon$.
This achieves the proof of Proposition 2.1. ■

In order to prove Theorem 2.3 we need some auxiliary results. Define for $u > 0$ and $v \in S_K$

$$\begin{aligned}
\widetilde{s}_n(u) &:= \frac{n}{k}\,\overline{F}_1\left(u^{-\gamma_1(x_0)} U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0 - h_n v\right), \\
t_n(y_2) &:= \frac{n}{k}\,\overline{F}_2\left(U_2\left(\frac{n}{ky_2}\Big|x_0\right)\Big|x_0 - h_n v\right).
\end{aligned}$$

Lemma 3.3 *Assume* $(\mathcal{D})$ *and* $(\mathcal{H})$ *and* $x_0 \in Int(S_X)$. *Consider sequences* $k \to \infty$ *and* $h_n \to 0$ *as* $n \to \infty$, *in such a way that* $k/n \to 0$ *and* $h_n^{\eta_{\varepsilon_1}\wedge\eta_{\gamma_1}} \ln\frac{n}{k} \to 0$. *Then, we have, for any* $u \leqslant T_n \to \infty$ *such that* $kT_n/n \to 0$ *and* $0 < \varepsilon < \beta_1(x_0)$, *that*

$$\begin{aligned}
\left|\widetilde{s}_n(u) - u\right| \;\leqslant\; & Cu\Big\{h_n^{\eta_{A_1}} + h_n^{\eta_{\gamma_1}} \ln\frac{n}{k} + h_n^{\eta_{\gamma_1}} |\ln u| u^{\pm C h_n^{\eta_{\gamma_1}}} \\
& + \left|\delta_1\left(U_1\left(\frac{n}{k}\Big|x_0\right)\Big|x_0\right)\right| \left[1 + u^{\pm C h_n^{\eta_{\gamma_1}}} h_n^{\eta_{\gamma_1}} |\ln u|\right] \\
& \times \Big[u^{\gamma_1(x_0)\beta_1(x_0)}\left(1 + u^{\pm\gamma_1(x_0)\varepsilon}\right)\left(h_n^{\eta_{B_1}} + u^{-C h_n^{\eta_{\varepsilon_1}}} h_n^{\eta_{\varepsilon_1}}\left(|\ln u| + \ln\frac{n}{k}\right)\right) \\
& + u^{\gamma_1(x_0)(\beta_1(x_0)\pm\varepsilon)} + \left|u^{\gamma_1(x_0)\beta_1(x_0)} - 1\right|\Big]\Big\},
\end{aligned}$$

where $u^{\pm\bullet}$ *means* $u^{\bullet}$ *if* u *is greater than 1, and* $u^{-\bullet}$ *if* u *is smaller than 1.*

Proof

Using Assumption $(\mathcal{D})$, we have

$$\begin{aligned}
\widetilde{s}_n(u) &= \frac{\overline{F}_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{\overline{F}_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)} \\
&= \frac{A_1(x_0-h_nv)}{A_1(x_0)}\left(U_1\left(\frac{n}{k}\middle|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}} \\
&\quad\times\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left|\widetilde{s}_n(u)-u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\right| &\leqslant u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\left\{\left|\frac{A_1(x_0-h_nv)}{A_1(x_0)}-1\right|\left(U_1\left(\frac{n}{k}\middle|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}\right. \\
&\qquad\times\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}\right| \\
&\quad+\left|\left(U_1\left(\frac{n}{k}\middle|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}-1\right| \\
&\qquad\times\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}\right| \\
&\quad\left.+\left|\frac{1+\frac{1}{\gamma_1(x_0-h_nv)}\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{1+\frac{1}{\gamma_1(x_0)}\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}-1\right|\right\} \\
&=: u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}}\{T_1+T_2+T_3\}.
\end{aligned}$$

Using Assumption $(\mathcal{H})$ and the inequality $|e^x-1|\leqslant|x|\,e^{|x|}$, we deduce that, for n large,

$$\left|\frac{A_1(x_0-h_nv)}{A_1(x_0)}-1\right| \leqslant C\,h_n^{\eta_{A_1}} \tag{17}$$

$$\left|\left(U_1\left(\frac{n}{k}\middle|x_0\right)\right)^{\frac{1}{\gamma_1(x_0)}-\frac{1}{\gamma_1(x_0-h_nv)}}-1\right| \leqslant C\,h_n^{\eta_{\gamma_1}}\ln\frac{n}{k}. \tag{18}$$

Now, direct computations yield, for n large,

$$\begin{aligned}
T_3 &\leqslant C\left|\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right|\left\{\left|\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_nv)}-1\right|\left|\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}\right|\right. \\
&\quad\left.+\left|\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0-h_nv\right)}{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}\frac{\delta_1\left(u^{-\gamma_1(x_0)}U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}{\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)}-1\right|\right\}.
\end{aligned}$$

Using the assumed form for $\delta_1(y|x)$, $(\mathcal{H})$, and the uniform bound from Proposition B.1.10 in de Haan and Ferreira (2006) with $0 < \varepsilon < \beta_1(x_0)$, we obtain, for n large, that

$$\begin{aligned} T_3 &\leqslant C\left|\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right|\left\{h_n^{\eta_{\gamma_1}} + u^{\gamma_1(x_0)\beta_1(x_0)}\left(1+u^{\pm\gamma_1(x_0)\varepsilon}\right)\right. \\ &\quad \left.\times\left[h_n^{\eta_{B_1}} + u^{-Ch_n^{\eta_{\varepsilon_1}}} h_n^{\eta_{\varepsilon_1}}\left(|\ln u| + \ln\frac{n}{k}\right)\right] + u^{\gamma_1(x_0)(\beta_1(x_0)\pm\varepsilon)} + \left|u^{\gamma_1(x_0)\beta_1(x_0)} - 1\right|\right\}. \quad (19) \end{aligned}$$

Since

$$\begin{aligned} \left|\widetilde{s}_n(u) - u\right| &\leqslant \left|\widetilde{s}_n(u) - u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_n v)}}\right| + u\left|u^{\frac{\gamma_1(x_0)-\gamma_1(x_0-h_n v)}{\gamma_1(x_0-h_n v)}} - 1\right| \\ &\leqslant \left|\widetilde{s}_n(u) - u^{\frac{\gamma_1(x_0)}{\gamma_1(x_0-h_n v)}}\right| + Cu^{1\pm Ch_n^{\eta_{\gamma_1}}} h_n^{\eta_{\gamma_1}} |\ln u|, \quad (20) \end{aligned}$$

combining (17), (18), (19) with (20), Lemma 3.3 is established. ■

Lemma 3.4 *Assume $(\mathcal{D})$, $(\mathcal{H})$, $\gamma_1(x_0) < 1$ and $x_0 \in Int(S_X)$. For sequences $k = \lfloor n^\alpha \ell_1(n) \rfloor$ and $h_n = n^{-\Delta}\ell_2(n)$, where ℓ_1 and ℓ_2 are slowly varying functions at infinity, with $\alpha \in (0,1)$ and*

$$\max\left(\frac{\alpha}{d+2\gamma_1(x_0)(\eta_{A_1}\wedge\eta_{\gamma_1})}, \frac{\alpha}{d+2(1-\gamma_1(x_0))(\eta_{A_2}\wedge\eta_{\gamma_2}\wedge\eta_{B_2}\wedge\eta_{\varepsilon_2})}, \frac{\alpha}{d} - \frac{2(1-\alpha)\gamma_1^2(x_0)\beta_1(x_0)}{d+d(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)}, \frac{\alpha-2(1-\alpha)\gamma_1(x_0)}{d}\right) < \Delta < \frac{\alpha}{d},$$

one has that

$$\sup_{v\in S_K}\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\sqrt{kh_n^d}\left|\int_0^\infty \left[R\left(\widetilde{s}_n(u), t_n(y_2)|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \longrightarrow 0$$

and

$$\sup_{\frac{1}{2}\leqslant y_2\leqslant 2}\sqrt{kh_n^d}\left|\int_0^\infty \left[R\left(s_n(u), y_2|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right| \longrightarrow 0.$$

Proof

We use the following decomposition along with the Lipschitz property of the function R:

$$
\begin{aligned}
&\sqrt{kh_n^d}\left|\int_0^\infty \left[R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right| \\
&\qquad\leqslant \sqrt{kh_n^d}\left|\int_0^{\delta_n} \left[R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_{\delta_n}^{T_n} \left[R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_{T_n}^{\infty} \left[R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)-R(u,y_2|x_0)\right]du^{-\gamma_1(x_0)}\right| \\
&\qquad\leqslant \sqrt{kh_n^d}\left|\int_0^{\delta_n} R\left(\widetilde{s}_n(u),t_n(y_2)|x_0\right)du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad +\sqrt{kh_n^d}\left|\int_0^{\delta_n} R(u,y_2|x_0)du^{-\gamma_1(x_0)}\right| \\
&\qquad\quad -\sqrt{kh_n^d}\int_{\delta_n}^{T_n}\left[|\widetilde{s}_n(u)-u|+|t_n(y_2)-y_2|\right]du^{-\gamma_1(x_0)} \\
&\qquad\quad +2\sup_{u\geqslant 0,\frac{1}{2}-\zeta\leqslant y_2\leqslant 2+\zeta} R(u,y_2|x_0)\sqrt{kh_n^d}\,T_n^{-\gamma_1(x_0)} \\
&\qquad=: T_1+T_2+T_3+T_4,
\end{aligned}
$$

for $\zeta>0$ small and where $\delta_n\to 0$ and $T_n\to\infty$, as $n\to\infty$.

Now, since $R(y_1,y_2|x_0)\leqslant y_1\wedge y_2$, using Lemma 3.3, and assuming $h_n^{\eta_{\varepsilon_1}\wedge\eta_{\gamma_1}}|\ln\delta_n|\to 0$, we obtain after tedious calculations, for n large,

$$
\begin{aligned}
T_1+T_2 &\leqslant -2\sqrt{kh_n^d}\int_0^{\delta_n} u\,du^{-\gamma_1(x_0)}-\sqrt{kh_n^d}\int_0^{\delta_n}|\widetilde{s}_n(u)-u|\,du^{-\gamma_1(x_0)} \\
&\leqslant C\sqrt{kh_n^d}\delta_n^{1-\gamma_1(x_0)}. \qquad (21)
\end{aligned}
$$

As for T_3, using again Lemma 3.3 and following the lines of proof of Lemma 3.1, we have, for n large,

$$
\begin{aligned}
T_3 &\leqslant -\sqrt{kh_n^d}\int_0^{T_n}|\widetilde{s}_n(u)-u|\,du^{-\gamma_1(x_0)}-\sqrt{kh_n^d}\int_{\delta_n}^{T_n}|t_n(y_2)-y_2|\,du^{-\gamma_1(x_0)} \\
&\leqslant C\sqrt{kh_n^d}\,T_n^{1-\gamma_1(x_0)}\left\{h_n^{\eta_{A_1}}+h_n^{\eta_{\gamma_1}}\ln\frac{n}{k}+h_n^{\eta_{\gamma_1}}\ln T_n+\left|\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right|T_n^{(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)}\right\} \\
&\quad +C\sqrt{kh_n^d}\,\delta_n^{-\gamma_1(x_0)}\left\{h_n^{\eta_{A_2}}+h_n^{\eta_{\gamma_2}}\ln\frac{n}{k}+\left|\delta_2\left(U_2\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right|\left[h_n^{\eta_{B_2}}+h_n^{\eta_{\varepsilon_2}}\ln\frac{n}{k}\right]\right\} \qquad (22)
\end{aligned}
$$

assuming $h_n^{\eta_{\varepsilon_1}\wedge\eta_{\gamma_1}}\ln T_n\to 0$.

Finally

$$
T_4 \leqslant C\sqrt{kh_n^d}\,T_n^{-\gamma_1(x_0)}. \qquad (23)
$$

Take $\delta_n = h_n^{\xi}$ and $T_n = n^{\kappa}$, with ξ and κ positive numbers, and $0 < \varepsilon < \beta_1(x_0)$. Combining (21), (22) and (23), the first part of Lemma 3.4 follows if the sequences δ_n and T_n are chosen such that

$$\begin{aligned}
\alpha - \Delta\left[d - 2\xi\gamma_1(x_0) + 2(\xi \wedge \eta_{A_2} \wedge \eta_{B_2} \wedge \eta_{\gamma_2} \wedge \eta_{\varepsilon_2})\right] &< 0,\\
\alpha - \Delta d - 2\kappa\gamma_1(x_0) &< 0,\\
\alpha - \Delta d + 2\kappa(1-\gamma_1(x_0)) - 2\Delta(\eta_{A_1} \wedge \eta_{\gamma_1}) &< 0,\\
\alpha - \Delta d - 2(1-\alpha)\gamma_1(x_0)\beta_1(x_0) + 2\kappa[1 + (\beta_1(x_0)+\varepsilon)\gamma_1(x_0) - \gamma_1(x_0)] &< 0.
\end{aligned}$$

Note that this is possible if we proceed as follows:
• α and Δ are chosen as stated in Lemma 3.4;
• κ is chosen such that

$$\frac{\alpha - \Delta d}{2\gamma_1(x_0)} < \kappa < \min\left(1-\alpha, \frac{2\Delta(\eta_{A_1} \wedge \eta_{\gamma_1}) - (\alpha - \Delta d)}{2(1-\gamma_1(x_0))}, \frac{2(1-\alpha)\gamma_1(x_0)\beta_1(x_0) - (\alpha - \Delta d)}{2[1 - \gamma_1(x_0) + (\beta_1(x_0)+\varepsilon)\gamma_1(x_0)]}\right);$$

• ξ is chosen such that

$$\frac{\alpha - \Delta d}{2\Delta(1-\gamma_1(x_0))} < \xi < \eta_{A_2} \wedge \eta_{\gamma_2} \wedge \eta_{B_2} \wedge \eta_{\varepsilon_2}.$$

Note that the choices of κ and ξ only depend on those of α and Δ.

The second part of Lemma 3.4 is similar, although simpler. Indeed, a decomposition of the quantity of interest this time into two parts yields

$$\begin{aligned}
&\sqrt{kh_n^d}\left|\int_0^{\infty}\left[R\left(s_n(u), y_2|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right|\\
&\leqslant \sqrt{kh_n^d}\left|\int_0^{T_n}\left[R\left(s_n(u), y_2|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right|\\
&\quad + \sqrt{kh_n^d}\left|\int_{T_n}^{\infty}\left[R\left(s_n(u), y_2|x_0\right) - R(u, y_2|x_0)\right] du^{-\gamma_1(x_0)}\right|\\
&\leqslant -\sqrt{kh_n^d}\int_0^{T_n}\left|s_n(u) - u\right| du^{-\gamma_1(x_0)} + 2\sup_{u\geqslant 0, \frac{1}{2}\leqslant y_2\leqslant 2} R(u, y_2|x_0)\sqrt{kh_n^d}\, T_n^{-\gamma_1(x_0)}\\
&\leqslant -\sqrt{kh_n^d}\frac{|\delta_1(U_1(\frac{n}{k}|x_0)|x_0)|}{|\gamma_1(x_0) + \delta_1(U_1(\frac{n}{k}|x_0)|x_0)|}\int_0^{T_n} u\left|\frac{\delta_1(u^{-\gamma_1(x_0)}U_1(\frac{n}{k}|x_0)|x_0)}{\delta_1(U_1(\frac{n}{k}|x_0)|x_0)} - 1\right| du^{-\gamma_1(x_0)} + C\sqrt{kh_n^d}\, T_n^{-\gamma_1(x_0)}\\
&\leqslant C\sqrt{kh_n^d}\left|\delta_1\left(U_1\left(\frac{n}{k}\middle|x_0\right)\middle|x_0\right)\right| T_n^{1-\gamma_1(x_0)+(\beta_1(x_0)+\varepsilon)\gamma_1(x_0)} + C\sqrt{kh_n^d}\, T_n^{-\gamma_1(x_0)}.
\end{aligned}$$

This achieves the proof of Lemma 3.4. ■

Lemma 3.5 *Assume* $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $x_0 \in Int(S_X)$ *with* $f_X(x_0) > 0$ *and* $y \to F_2(y|x_0)$ *is strictly increasing. Consider sequences* $k \to \infty$ *and* $h_n \to 0$ *as* $n \to \infty$, *in such a way that* $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{f_X} \wedge \eta_{A_2}} \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{\gamma_2}} \ln n/k \to 0$,

$\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{B_2}} \to 0$, *and* $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)|h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$. *Then, for any sequence* u_n *satisfying*

$$\sqrt{kh_n^d}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \to c \in \mathbb{R},$$

as $n \to \infty$, *we have*

$$\sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \rightsquigarrow \frac{W(\infty,1)}{f_X(x_0)}.$$

Proof

First remark that

$$\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{Y_i^{(2)}>u_n\}}}{\overline{F}_2(u_n|x_0)} = \frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)}\, T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\Big|x_0\right), \quad a.s. \ .$$

Since, with $r_n := \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}$,

$$\begin{aligned}
&\left|r_n\left[\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)}\, T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\Big|x_0\right) - f_X(x_0)\right] - W(\infty,1)\right| \\
\leqslant\ & \left|\sqrt{kh_n^d}\left[T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\Big|x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0)f_X(x_0)\right] - W\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\right)\right| \\
&+\sqrt{kh_n^d}\left|\sqrt{\frac{\overline{F}_2(u_n|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)}} - 1\right|\left|T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\Big|x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0)f_X(x_0)\right| \\
&+\left|W\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\right) - W(\infty,1)\right| \\
&+r_n\left|\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right|\left|T_n\left(\infty, \frac{n}{k}\overline{F}_2(u_n|x_0)\Big|x_0\right) - \frac{n}{k}\overline{F}_2(u_n|x_0)f_X(x_0)\right|,
\end{aligned}$$

we have by (15) combined with the Skorohod construction that

$$r_n\left(\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{Y_i^{(2)}>u_n\}}}{\overline{F}_2(u_n|x_0)} - f_X(x_0)\right) \rightsquigarrow W(\infty,1).$$

Then

$$\begin{aligned}
r_n\left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) &= r_n\left(\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{Y_i^{(2)}>u_n\}}}{\overline{F}_2(u_n|x_0)f_X(x_0)} - 1\right) \\
&\quad + r_n\frac{f_X(x_0) - \hat{f}_n(x_0)}{f_X(x_0)\hat{f}_n(x_0)}\,\frac{\frac{1}{n}\sum_{i=1}^n K_{h_n}(x_0 - X_i)\mathbb{1}_{\{Y_i^{(2)}>u_n\}}}{\overline{F}_2(u_n|x_0)} \\
&\rightsquigarrow \frac{W(\infty,1)}{f_X(x_0)}.
\end{aligned}$$

■

Lemma 3.6 *Assume* $(\mathcal{D})$, $(\mathcal{H})$, $(\mathcal{K})$, $x_0 \in Int(S_X)$ *with* $f_X(x_0) > 0$ *and* $y \to F_2(y|x_0)$ *is strictly increasing. Consider sequences* $k \to \infty$ *and* $h_n \to 0$ *as* $n \to \infty$, *in such a way that* $k/n \to 0$, $kh_n^d \to \infty$, $h_n^{\eta_{\varepsilon_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{f_X} \wedge \eta_{A_2}} \to 0$, $\sqrt{kh_n^d}\, h_n^{\eta_{\gamma_2}} \ln n/k \to 0$, $\sqrt{kh_n^d}|\delta_2(U_2(n/k|x_0)|x_0)| \to 0$. *Then, as* $n \to \infty$, *we have*

$$\sqrt{kh_n^d}\,(\widehat{u}_n - 1) \rightsquigarrow \frac{\gamma_2(x_0)W(\infty,1)}{f_X(x_0)}.$$

Proof

To prove the lemma we will use the idea of Wretman (1978), applied to our situation. We have, for $z \in \mathbb{R}$, and u_n from Lemma 3.5 taken as $U_2(n/k|x_0)(1 + z/\sqrt{kh_n^d})$, that

$$\begin{aligned}
&\mathbb{P}\left(\sqrt{kh_n^d}\,(\widehat{u}_n - 1) \leqslant z\right) \\
&\quad = \mathbb{P}\left(\sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\widehat{\overline{F}}_{n,2}(u_n|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \leqslant \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right)\right).
\end{aligned}$$

We have that in the present context

$$a_n := \sqrt{nh_n^d \overline{F}_2(u_n|x_0)}\left(\frac{\overline{F}_2(U_2(n/k|x_0)|x_0)}{\overline{F}_2(u_n|x_0)} - 1\right) \to \frac{z}{\gamma_2(x_0)}.$$

Let H_n denote the distribution function of $\sqrt{nh_n^d \overline{F}_2(u_n|x_0)}(\widehat{\overline{F}}_{n,2}(u_n|x_0)/\overline{F}_2(u_n|x_0) - 1)$, and H is the distribution function of $W(\infty,1)/f_X(x_0)$. Then by Lemma 3.5 and by continuity of H one has that $H_n(a_n) \to H(z/\gamma_2(x_0))$, as $n \to \infty$, hence the result of the lemma. ■

Proof of Theorem 2.3

We have the following decomposition:

$$\begin{aligned}
\sqrt{kh_n^d}\left(\frac{\widehat{\theta}_n}{f_X(x_0)\theta_{k/n}} - 1\right) &= \frac{U_1(n/k|x_0)}{\theta_{k/n}}\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widehat{\theta}_n}{U_1(n/k|x_0)} - \mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}}\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right) - \frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right) \\
&= \frac{U_1(n/k|x_0)}{\theta_{k/n}}\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)} - \mathbb{E}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)}\right)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}}\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)}\right) - \mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right)\right) \\
&\quad + \frac{U_1(n/k|x_0)}{\theta_{k/n}}\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right) - \frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right) \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

First, remark that the common factor of the three terms, $U_1(n/k|x_0)/\theta_{k/n}$ can be handled in a similar way as in Proposition 1 in Cai *et al.* (2015), i.e., as $n \to \infty$

$$\frac{U_1(n/k|x_0)}{\theta_{k/n}} \longrightarrow \frac{-1}{\int_0^\infty R(s,1|x_0)ds^{-\gamma_1(x_0)}}.$$

Thus the three terms without this factor need to be studied.

We start with T_1. Note that

$$\sqrt{kh_n^d}(\widehat{e}_n - 1) = -\frac{f_2(\tilde{u}_n U_2(n/k|x_0)|x_0)U_2(n/k|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)}\sqrt{kh_n^d}(\widehat{u}_n - 1),$$

where $\tilde{u}_n$ is a random value between $\widehat{u}_n$ and 1. By the continuous mapping theorem we have then

$$\frac{f_2(\tilde{u}_n U_2(n/k|x_0)|x_0)U_2(n/k|x_0)}{\overline{F}_2(U_2(n/k|x_0)|x_0)} \xrightarrow{\mathbb{P}} \frac{1}{\gamma_2(x_0)},$$

and hence by Lemma 3.6

$$\sqrt{kh_n^d}(\widehat{e}_n - 1) \rightsquigarrow -W(\infty,1)/f_X(x_0). \tag{24}$$

This implies that

$$\mathbb{P}\left(|\widehat{e}_n - 1| > (kh_n^d)^{-1/4}\right) \to 0.$$

Hence, with probability tending to one,

$$\begin{aligned}
&\left|\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)} - \mathbb{E}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)}\right)\right) + \frac{1}{f_X(x_0)}\int_0^\infty W(s,1)ds^{-\gamma_1(x_0)}\right| \\
\leqslant\ &\sup_{|y-1|\leqslant(kh_n^d)^{-1/4}}\left|\frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\frac{\widetilde{\theta}_n(y)}{U_1(n/k|x_0)} - \mathbb{E}\left(\frac{\widetilde{\theta}_n(y)}{U_1(n/k|x_0)}\right)\right) + \frac{1}{f_X(x_0)}\int_0^\infty W(s,y)ds^{-\gamma_1(x_0)}\right| \\
&+\frac{1}{f_X(x_0)}\sup_{|y-1|\leqslant(kh_n^d)^{-1/4}}\left|\int_0^\infty [W(s,y) - W(s,1)]ds^{-\gamma_1(x_0)}\right|.
\end{aligned}$$

The first term of the right-hand side tends to 0 in probability by our Proposition 2.1, whereas the second term can be handled similarly as in the proof of Proposition 3 in Cai *et al.* (2015). Consequently

$$T_1 \rightsquigarrow \frac{1}{\int_0^\infty R(s,1|x_0)ds^{-\gamma_1(x_0)}}\frac{1}{f_X(x_0)}\int_0^\infty W(s,1)ds^{-\gamma_1(x_0)}. \tag{25}$$

Next step consists to look at T_2. To this aim, remark that for y equal either to 1 or $\widehat{e}_n$, we have

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}\left(T_n(s_n(u), y|x_0)\right) du^{-\gamma_1(x_0)} \\
&= \int_0^\infty \int_{S_K} K(v) R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
&= \int_0^\infty \int_{S_K} K(v) R\left(\widetilde{s}_n(u), t_n(y)|x_0\right) f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
&\quad + \int_0^\infty \int_{S_K} K(v) \left[R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) - R\left(\widetilde{s}_n(u), t_n(y)|x_0\right)\right] f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
&= \int_0^\infty R\left(u, y|x_0\right) du^{-\gamma_1(x_0)} \int_{S_K} K(v) f_X(x_0 - h_n v) dv \\
&\quad + \int_{S_K} K(v) \int_0^\infty \left[R\left(\widetilde{s}_n(u), t_n(y)|x_0\right) - R(u, y|x_0)\right] du^{-\gamma_1(x_0)} f_X(x_0 - h_n v) dv \\
&\quad + \int_0^\infty \int_{S_K} K(v) \left[R_{\frac{n}{k}}\left(\widetilde{s}_n(u), t_n(y)|x_0 - h_n v\right) - R\left(\widetilde{s}_n(u), t_n(y)|x_0\right)\right] f_X(x_0 - h_n v) dv du^{-\gamma_1(x_0)} \\
&=: \widetilde{T}_{2,1} + \widetilde{T}_{2,2} + \widetilde{T}_{2,3}.
\end{aligned}
$$

By Lemma 3.4, Assumptions $(\mathcal{S})$ and $(\mathcal{H})$ we obtain

$$
\begin{aligned}
\widetilde{T}_{2,1} &= f_X(x_0) \int_0^\infty R\left(u, y|x_0\right) du^{-\gamma_1(x_0)} + O_{\mathbb{P}}\left(h_n^{\eta_{f_X}}\right), \\
\widetilde{T}_{2,2} &= o_{\mathbb{P}}\left(\frac{1}{\sqrt{k h_n^d}}\right), \\
|\widetilde{T}_{2,3}| &\leqslant - \sup_{x \in B(x_0, h_n)} \sup_{0 < y_1 < \infty, \frac{1}{2} \leqslant y_2 \leqslant 2} \frac{|R_{n/k}(y_1, y_2|x) - R(y_1, y_2|x_0)|}{y_1^\beta \wedge 1} \\
&\quad \times \int_{S_K} K(v) \int_0^\infty \left([\widetilde{s}_n(u)]^\beta \wedge 1\right) du^{-\gamma_1(x_0)} f_X(x_0 - h_n v) dv \\
&= O_{\mathbb{P}}\left(\left(\frac{n}{k}\right)^\tau\right).
\end{aligned}
$$

Note that the integral appearing in the bound for $|\widetilde{T}_{2,3}|$ is finite for n large, as $\widetilde{s}_n(u) \leqslant C u^{1-\xi}$ for $u \in (0, 1/2], \xi \in (0, (\beta - \gamma_1(x_0))/\beta)$ and n large. Consequently, under our assumptions and using the homogeneity of the $R-$function and the mean value theorem combining with (24), we

have

$$
\begin{aligned}
& \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}\left(\frac{\widetilde{\theta}_n(\widehat{e}_n)}{U_1(n/k|x_0)}\right)-\mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right)\right)\\
= & \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\int_0^\infty \mathbb{E}\left(T_n(s_n(u),1|x_0)\right)du^{-\gamma_1(x_0)}-\int_0^\infty \mathbb{E}\left(T_n(s_n(u),\widehat{e}_n|x_0)\right)du^{-\gamma_1(x_0)}\right)\\
= & \sqrt{kh_n^d}\left(\int_0^\infty R\left(u,1|x_0\right)du^{-\gamma_1(x_0)}-\int_0^\infty R\left(u,\widehat{e}_n|x_0\right)du^{-\gamma_1(x_0)}\right)+o_{\mathbb{P}}(1)\\
= & \sqrt{kh_n^d}\left(1-\widehat{e}_n^{1-\gamma_1(x_0)}\right)\int_0^\infty R\left(u,1|x_0\right)du^{-\gamma_1(x_0)}+o_{\mathbb{P}}(1)\\
\rightsquigarrow & (1-\gamma_1(x_0))\frac{W(\infty,1)}{f_X(x_0)}\int_0^\infty R\left(u,1|x_0\right)du^{-\gamma_1(x_0)}.
\end{aligned}
$$

This implies that

$$
T_2 \rightsquigarrow -(1-\gamma_1(x_0))\frac{W(\infty,1)}{f_X(x_0)}. \tag{26}
$$

Finally, for T_3 we have,

$$
\begin{aligned}
& \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(\mathbb{E}\left(\frac{\widetilde{\theta}_n(1)}{U_1(n/k|x_0)}\right)-\frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right)\\
= & \frac{\sqrt{kh_n^d}}{f_X(x_0)}\left(-\int_0^\infty \mathbb{E}(T_n(s_n(u),1|x_0))du^{-\gamma_1(x_0)}-\frac{f_X(x_0)\theta_{k/n}}{U_1(n/k|x_0)}\right)\\
= & \sqrt{kh_n^d}\int_0^\infty \left[R_{n/k}(s_n(u),1|x_0)-R(u,1|x_0)\right]du^{-\gamma_1(x_0)}+o(1)\\
= & \sqrt{kh_n^d}\int_0^\infty \left[R_{n/k}(s_n(u),1|x_0)-R(s_n(u),1|x_0)\right]du^{-\gamma_1(x_0)}\\
& +\sqrt{kh_n^d}\int_0^\infty \left[R(s_n(u),1|x_0)-R(u,1|x_0)\right]du^{-\gamma_1(x_0)}+o(1)\\
=: & \ \widetilde{T}_{3,1}+\widetilde{T}_{3,2}+o(1),
\end{aligned}
$$

where

$$
\begin{aligned}
|\widetilde{T}_{3,1}| &\leqslant \sqrt{kh_n^d}\sup_{x\in B(x_0,h_n)}\sup_{0<y_1<\infty,\frac{1}{2}\leqslant y_2\leqslant 2}\frac{|R_{n/k}(y_1,y_2|x)-R(y_1,y_2|x_0)|}{y_1^\beta\wedge 1}\left|\int_0^\infty\left([s_n(u)]^\beta\wedge 1\right)du^{-\gamma_1(x_0)}\right|\\
&= O\left(\sqrt{kh_n^d}\left(\frac{n}{k}\right)^\tau\right),\\
\widetilde{T}_{3,2} &= o(1).
\end{aligned}
$$

Overall, we have then

$$
T_3 = o(1). \tag{27}
$$

Combining (25), (26) and (27), and following the argument as at the end of the proof of Theorem 2.2, we can establish the result of Theorem 2.3. ■

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