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Abstract: In this paper, we introduce two new concepts of conditional random set in a Banach space: The conditional interior and the conditional closure. The conditional interior is an open version of the conditional core, as recently introduced by Lépinette and Molchanov, and may be seen as a measurable version of the topological interior. The conditional closure generalizes the concept of conditional support. These concepts are useful for applications in mathematical finance and conditional optimization.

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1. Introduction

The conditional essential supremum and infimum of a real-valued random variable X have been introduced in [2]. A generalization is then proposed in [5] for vector-valued random variables with respect to random preference relations. Actually, these two concepts are related to the notion of conditional core as introduced in [6] for random sets in Banach spaces. A conditional core of a set-valued mapping X is defined as the largest $\mathcal{H}$-measurable random set X' such that $X' \subset X$. This concept provides a natural conditional risk measure that generalizes the concept of essential infimum for multi-asset portfolios in mathematical finance. Applications are deduced in geometrical market models with transaction costs, see [7], [6] and the theory with transaction costs developed in [4].

In this paper, we first introduce the open version of the conditional core proposed in [6]. Precisely, if $\mathcal{H}$ is a sub σ-algebra on a complete probability space, the conditional interior of a set-valued mapping X is defined as the largest $\mathcal{H}$-measurable random open set X' such that $X' \subset X$. It may be seen as a measurable version of the classical interior in topology. One of our main contribution is to show the existence and uniqueness of such a conditional interior for an arbitrary random set in a separable Banach space. Then, the dual concept, i.e. the conditional closure, is introduced as a generalization of the conditional support of a real-valued random variable to a family of vector-valued random variables. A numerical application of the latter is deduced in conditional random optimization: We show that an essential supremum is a pointwise supremum on a conditional closure.

The paper is organized as follows. In section 2, we recall the definition and usual properties of measurable random sets in Banach spaces. Then, we introduce the notion of conditional interior and show the existence of such sets in Banach spaces. In Section 3, the notion of conditional closure is introduced and the main application is formulated.

2. Conditional interior

In the following, we first recall the concept of measurable random set. We then introduce the concept of conditional interior of any random set. Existence is deduced from the existence of the conditional core of any closed random set, see [6].

Definition 2.1. *Let T be a separable Banach space with the Borel σ-algebra $\mathcal{B}(T)$. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space and $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. A random set Γ is a set-valued mapping that assigns to each $\omega \in \Omega$ a subset $\Gamma(\omega)$ of T. We write $\Gamma : \Omega \twoheadrightarrow T$. We say that Γ is $\mathcal{H}$-measurable if for any open set $O \in T$*

$$\{\omega \in \Omega,\ \Gamma(\omega) \cap O \neq \emptyset\} \in \mathcal{H}.$$

We say that Γ is $\mathcal{H}$-graph-measurable if

$$\operatorname{graph}(\Gamma) := \{(\omega, x) \in \Omega \times T :\ x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{B}(T).$$

When Γ is closed (resp. open) almost surely, we say that Γ is closed (resp. open).

Remark 2.2. *By [8, Theorem 14.8], if Γ is closed-valued and $\mathcal{H}$-measurable, then Γ is $\mathcal{H}$-graph-measurable.*

Definition 2.3. *Let $\Gamma : \Omega \twoheadrightarrow T$ be a random set. We say that the $\mathcal{F}$-measurable random variable $\xi : \Omega \to T$ is an $\mathcal{F}$-measurable selection of Γ if $\xi(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$. The set of such selections is denoted by $\mathcal{L}(\Gamma, \mathcal{F})$.*

The following result may be found in [3, Th. 4.4] and is usually mentioned as the *measurable selection argument*.

Theorem 2.4 (Measurable selection argument)**.** *If X is a $\mathcal{F}$-graph-measurable random set which is non empty a.s., then $\mathcal{L}^0(X, \mathcal{F}) \neq \emptyset$*

The following result is proven in [6].

Proposition 2.5. *Suppose that X is a $\mathcal{F}$-graph-measurable random set which is non empty a.s. Then, $\mathrm{cl}\,(X)$ is $\mathcal{F}$-graph-measurable and admits a Castaing representation, i.e.*

$$\mathrm{cl}\,(X) = \mathrm{cl}\,(\xi_n : \ n \geq 1),$$

where $(\xi_n)_{n\geq 1}$ is a countable family of measurable selections of X. Moreover, we have $\mathrm{cl}\,(\mathcal{L}^0(X, \mathcal{F})) = \mathcal{L}^0(\mathrm{cl}\,(X), \mathcal{F})$.

If $x \in T$ and $r \geq 0$, $B(x, r)$ denotes the open ball in T of center x and radius r while $\bar{B}(x, r)$ is its closure. For any $A \subseteq T$ and $\lambda \in \mathbb{R}$, we use the convention that $\lambda \times A = \{\lambda a : \ a \in A\}$. In particular, $0 \times A = \{0\}$. Recall the following definition, see [6]:

Definition 2.6. *The $\mathcal{H}$-*core *(also called conditional core), $\mathbf{m}(\Gamma|\mathcal{H})$, of a set-valued mapping Γ is the largest $\mathcal{H}$-graph-measurable random set Γ' such that $\Gamma' \subseteq \Gamma$ a.s.*

The natural question is whether such a set exits. In the case where Γ is closed, then the conditional core $\mathbf{m}(\Gamma|\mathcal{H})$ exists by [6] and we have $\mathcal{L}(\mathbf{m}(\Gamma|\mathcal{H}), \mathcal{H}) = \mathcal{L}(\Gamma, \mathcal{H})$ in the case where $\mathcal{L}(\Gamma, \mathcal{H}) \neq \emptyset$. If $\gamma \in \mathcal{L}(T, \mathcal{H})$ and $r \in \mathcal{L}(\mathbb{R}_+, \mathcal{F})$, it is easy to see that $\mathbf{m}(\bar{B}(\gamma, r)|\mathcal{H}) = \bar{B}(\gamma, \operatorname{ess\,inf}_{\mathcal{H}} r)$. Notice that the conditional core does not depend on the σ-algebra $\mathcal{F}$. Actually, we may choose $\mathcal{F}$ as the largest one composed of all the subparts of Ω. The conditional core plays a role in mathematical finance as it naturally appears when considering the dynamics of a self-financing discret-time portfolio process of the form $V_{t-1} \in V_t + G_t$, $t \geq 1$, where G_t is the solvency set, see [4], i.e. $V_{t-1} \in \mathbf{m}(V_t + G_t | \mathcal{F}_{t-1})$, see [7].

We now introduce an open version of the conditional core:

Definition 2.7. *The $\mathcal{H}$-graph-measurable interior $\mathbf{o}(\Gamma|\mathcal{H})$ of a set-valued mapping Γ is the largest $\mathcal{H}$-graph-measurable random open set Γ' such that $\Gamma' \subseteq \Gamma$ a.s. if existence holds.*

Notice that the uniqueness of such conditional interior automatically holds by definition. We first show the existence for a random open set.

Theorem 2.8. *Let us consider a $\mathcal{F}$-graph-measurable open random set $\mathcal{O}$. Then, the $\mathcal{H}$-graph-measurable interior of $\mathcal{O}$ exists.*

Proof. By [6, Proposition 2.6], $\mathrm{cl}\,\mathcal{O}$ and $\partial\mathcal{O} := \mathrm{cl}\,\mathcal{O} \setminus \mathcal{O}$ are closed $\mathcal{F}$-graph-measurable random sets. When $\partial\mathcal{O} = \emptyset$, we set $d(x, \partial\mathcal{O}) = \infty$ for all $x \in T$. Otherwise, $\partial\mathcal{O}$ admits a Castaing representation on $\{\partial\mathcal{O} \neq \emptyset\}$ by Proposition 2.5. We deduce that the random continuous mapping $x \mapsto d(x, \partial\mathcal{O})$ is $\mathcal{F}$-measurable. We deduce that the sets $F^n := \{x : \ d(x, \partial\mathcal{O}) \geq 1/n\} \cap \mathrm{cl}\,\mathcal{O}$, $n \geq 1$, are closed $\mathcal{F}$-graph-measurable random subsets of $\mathcal{O}$ and we have $\mathcal{O} = \cup_n F^n$. Let us define the $\mathcal{H}$-graph-measurable open random set

$$\mathbf{o}(\mathcal{O}|\mathcal{H}) = \mathrm{int}\ (\cup_n \mathbf{m}(F^n|\mathcal{H})) \subseteq \mathcal{O},$$

and let us show that this is the largest $\mathcal{H}$-graph-measurable open set of $\mathcal{O}$. To do so, let $\mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ be a $\mathcal{H}$-graph-measurable open subset of $\mathcal{O}$. As previously, we may write $\mathcal{O}_{\mathcal{H}} = \cup_n H^n$ where $H^n := \{x : \ d(x, \partial\mathcal{O}_{\mathcal{H}}) \geq 1/n\} \cap \mathrm{cl}\,\mathcal{O}_{\mathcal{H}}$ are $\mathcal{H}$-graph-measurable closed subsets of $\mathcal{O}_{\mathcal{H}}$.

We claim that $H^n \subseteq F^n$. Indeed, if $x \in H^n$, it suffices to show that $d(x, \partial\mathcal{O}_{\mathcal{H}}) \leq d(x, \partial\mathcal{O})$ a.s. In the contrary case, on a non null set, there exists $o \in \partial\mathcal{O}$ such that $o \in B(x, d(x, \partial\mathcal{O}_{\mathcal{H}}))$. By Lemma 4.1, this implies that $o \in \mathcal{O}_{\mathcal{H}} \subseteq \mathcal{O}$ hence a contradiction.

As H^n is $\mathcal{H}$-graph-measurable and closed, $H^n \subseteq \mathbf{m}(F^n|\mathcal{H})$ for all n. We deduce that $\mathcal{O}_{\mathcal{H}} \subseteq \cup_n \mathbf{m}(F^n|\mathcal{H})$ hence $\mathcal{O}_{\mathcal{H}} \subseteq \mathbf{o}(\mathcal{O}|\mathcal{H})$. □

We then deduce the general case:

Corollary 2.9. *For any $\mathcal{F}$-graph-measurable random set Γ, the $\mathcal{H}$-graph-measurable interior of Γ exists and $\mathbf{o}(\Gamma|\mathcal{H}) = \mathbf{o}(\mathrm{int}\,\Gamma|\mathcal{H})$.*

3. Conditional closure and application

We are now in position to introduce the concept of conditional closure. The existence is proved in the following theorem which is the second main contribution of this paper.

Theorem 3.1. *For any $\mathcal{F}$-graph-measurable random set Γ, there exists a smallest $\mathcal{H}$-graph-measurable closed random set containing Γ a.s. denoted by $\mathrm{cl}\,(\Gamma|\mathcal{H})$ and called the $\mathcal{H}$-graph-measurable conditional closure of Γ. We have*

$$\mathrm{cl}\,(\Gamma|\mathcal{H}) = T \setminus \mathbf{o}(T \setminus \Gamma|\mathcal{H}).$$

Moreover, for all $\gamma \in \mathrm{cl}\,(\Gamma|\mathcal{H})$ and $\epsilon \in \mathcal{L}((0,\infty),\mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathrm{P}(H) > 0$, we have $\mathrm{P}(\{\Gamma \cap B(\gamma,\epsilon) \neq \emptyset\} \cap H) > 0$.

Proof. The first part is a direct consequence of Corollary 2.9. Suppose that $\mathrm{P}(\{\Gamma \cap B(\gamma,\epsilon) \neq \emptyset\} \cap H) = 0$. Therefore, by definition of the conditional closure as a smallest set, we have

$$\mathrm{cl}\,(\Gamma|\mathcal{H}) = \mathrm{cl}\,(\Gamma|\mathcal{H})1_{\Omega\setminus H} + \mathrm{cl}\,(\Gamma|\mathcal{H}) \cap (T \setminus B(\gamma,\epsilon))\,1_H.$$

Indeed, the $\mathcal{H}$-graph-measurable set in the r.h.s. above contains Γ by assumption hence it contains $\mathrm{cl}\,(\Gamma|\mathcal{H})$. We get a contradiction since $\gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}),\mathcal{H})$ a.s. by assumption. □

Corollary 3.2. *Let $X \in \mathcal{L}(T,\mathcal{F})$. There exists a smallest $\mathcal{H}$-graph-measurable random closed set denoted by $\mathrm{supp}_{\mathcal{H}}(X)$ s.t. $\mathrm{P}(X \in \mathrm{supp}_{\mathcal{H}}(X)) = 1$. It is given by $\mathrm{supp}_{\mathcal{H}}(X) = \mathrm{cl}\,(\{X\}|\mathcal{H})$ and is called the $\mathcal{H}$-conditional support of X. Moreover, for all $\gamma \in \mathcal{L}(\mathrm{supp}_{\mathcal{H}}(X),\mathcal{H})$ and $\epsilon \in \mathcal{L}((0,\infty),\mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathrm{P}(H) > 0$, we have $\mathrm{P}(\{X \in B(\gamma,\epsilon)\} \cap H) > 0$.*

Notice that the conditional support is necessarily non empty a.s. hence it admits a measurable selection. In the following, we adopt the notation $kA = \{ka :\ a \in A\}$ for any subset $A \subseteq T$ and $k \in A$.

Lemma 3.3. *Let $H \in \mathcal{H}$. Then, $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) = 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})$, for every $\mathcal{F}$-graph-measurable set Γ.*

Proof. First, observe that $\mathrm{P}(\Gamma 1_H \subseteq 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})) = 1$ as $1_H \mathrm{cl}\,(\Gamma|\mathcal{H}) = \{0\}$ on $\Omega\setminus H$. We deduce that $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) \subseteq 1_H \mathrm{cl}\,(\Gamma|\mathcal{H})$ a.s. and $\mathrm{cl}\,(\Gamma 1_H|\mathcal{H}) = \{0\}$ on $\Omega\setminus H$. Let us define $Z = \mathrm{cl}\,(\Gamma 1_H|\mathcal{H})1_H + \mathrm{cl}\,(\Gamma|\mathcal{H})1_{\Omega\setminus H}$. As $\Gamma \subseteq \Gamma 1_H$ on H, we deduce that $\Gamma \subseteq Z$ a.s. hence $\mathrm{cl}\,(\Gamma|\mathcal{H}) \subseteq Z$ a.s. Therefore, $\mathrm{cl}\,(\Gamma|\mathcal{H})1_H \subseteq Z1_H$ and finally $\mathrm{cl}\,(\Gamma|\mathcal{H})1_H \subseteq \mathrm{cl}\,(\Gamma 1_H|\mathcal{H})$ as the latter set is $\{0\}$ on $\Omega \setminus H$. □

The second main contribution is the following. It allows to compute numerically an essential supremum as a pointwise supremum on the conditional closure. This result is a generalization of [1, Proposition 2.9] and should be useful in robust finance.

Theorem 3.4. *Let $h(\omega, x)$, $x \in T$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(T)$-measurable and is l.s.c in x. Let Γ be a closed $\mathcal{F}$-graph-measurable set of T. Then,*

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} = \sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n), \text{ a.s.},$$

where $(\gamma_n)_{n \in \mathbb{N}}$ is a Castaing representation of $\mathrm{cl}\,(\Gamma|\mathcal{H})$.

Proof. As $\mathrm{cl}\,(\Gamma|\mathcal{H})$ is $\mathcal{H}$-graph-measurable and closed, it admits a Castaing representation (see Theorem 14.5 in [8]) $\mathrm{cl}\,(\Gamma|\mathcal{H}) = \mathrm{cl}\{\gamma_n : \ n \in \mathbf{N}\}$, where $\gamma_n \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})$ for all n. Notice that we may assume that the equality holds everywhere on Ω, see the proof of [4, Proposition 5.4.4]. We first prove that

$$\sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n).$$

Let $(\gamma_n)_n \subset \mathrm{cl}\{\gamma_n : \ n \in \mathbf{N}\} = \mathrm{cl}\,(\Gamma|\mathcal{H})$, then $h(\gamma_n) \leq \sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$. Therefore, $\sup_n h(\gamma_n) \leq \sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$. Moreover, if $x \in \mathrm{cl}\,(\Gamma|\mathcal{H})$, we may write almost surely $x = \lim_n \gamma_n$ where $(\gamma_n)_n \subset \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})$ is a subsequence. By lower semicontinuity, we deduce that $h(x) \leq \liminf_n h(\gamma_n)$. Thus, $h(x) \leq \sup_n h(\gamma_n)$ and $\sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) \leq \sup_n h(\gamma_n)$ on Ω. The equality is then deduced.

Since h is $\mathcal{H} \otimes \mathcal{B}(T)$-measurable, $\sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)$ is $\mathcal{H}$-measurable. Moreover, as $\Gamma \subseteq \mathrm{cl}\,(\Gamma|\mathcal{H})$ a.s., we have $h(\gamma) \leq \sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x)$, for every $\gamma \in \mathcal{L}(\Gamma, \mathcal{F})$. We deduce that

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) : \ \gamma \in \mathcal{L}(\Gamma, \mathcal{F})\} \leq \sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) \text{ a.s.}$$

To show the reverse inequality, we first show the equality

$$\sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma), \ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

To see it, notice that

$$\sup_{x \in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma), \ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}), \mathcal{H})\}.$$

If $\gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}),\mathcal{H})$, then $\gamma \in \mathrm{cl}\{\gamma_n :\ n \in \mathbf{N}\}$ i.e. $\gamma = \lim_n \gamma_n$ for a subsequence and, by semi-continuity,

$$h(\gamma) = \liminf_n h(\gamma_n) \leq \sup_n h(\gamma_n) = \sup_{x\in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x).$$

Since the family $\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H})\}$ is directed upward, we also deduce that $\sup_{x\in \mathrm{cl}\,(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma^n)$ where $\gamma^n \in \mathcal{L}(\mathrm{cl}\,(\Gamma|\mathcal{H}),\mathcal{H})$.

At last, consider any $\mathcal{H}$-measurable selection γ of $\mathrm{cl}\,(\Gamma|\mathcal{H})$ and a deterministic sequence $\epsilon_n > 0$ with $\lim_n \epsilon_n = 0$. By Corollary 3.1, $\Lambda_n = \{\Gamma\cap \bar{B}(\gamma,\epsilon_n) \neq \emptyset\} \in \mathcal{F}$ satisfies $\mathrm{P}(\Lambda_n|\mathcal{H}) > 0$ a.s. Indeed, in the contrary case, on a non null $\mathcal{H}$-measurable set $\tilde{\Lambda}_n$, we have $\mathrm{P}(\tilde{\Lambda}_n \cap \Lambda_n|\mathcal{H}) = 0$ hence $\mathrm{P}(\tilde{\Lambda}_n \cap \Lambda_n) = 0$, i.e. a contradiction with Corollary 3.1. By a measurable selection argument, it is possible to construct $\hat{\gamma}_n \in L^0(\Gamma,\mathcal{F})$ such that $\hat{\gamma}_n \in \bar{B}(\gamma,\epsilon_n)$ on Λ_n. We define $\hat{\Lambda}_n = \{\hat{\gamma}_n \in \bar{B}(\gamma,\epsilon_n)\}$. Since $\Lambda_n \subseteq \hat{\Lambda}_n$, we have $\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}) > 0$ a.s. Moreover,

$$\begin{aligned}
\mathrm{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\}1_{\hat{\Lambda}_n} &\geq h(\hat{\gamma}_n)1_{\hat{\Lambda}_n},\\
\mathrm{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F})\}\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}) &\geq \mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}). \qquad (3.1)
\end{aligned}$$

The last inequality holds almost surely, i.e. on a set we denoted by $A^n \in \mathcal{H}$ such that $\mathrm{P}(A^n) = 1$. Moreover, there exists $B^n \in \mathcal{H}$ such that $\mathrm{P}(B_n) = 1$ and

$$\mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) = \int_{\mathbb{R}^d} h(x)1_{x\in\bar{B}(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}).$$

To obtain this equality, it suffices to use a regular version $\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H})$ of the conditional law of $\hat{\gamma}_n$. Moreover, we obtain the following pointwise inequality satisfied for each $\omega \in C_n = A_n \cap B_n$:

$$\begin{aligned}
\int_{\mathbb{R}^d} h(x)1_{x\in\bar{B}(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}) &\geq \inf_{z\in\bar{B}(\gamma,\epsilon_n)} h(z)\int_{\mathbb{R}^d} 1_{x\in\bar{B}(\gamma,\epsilon_n)}\mathrm{P}(\hat{\gamma}_n \in dx|\mathcal{H}),\\
&\geq \inf_{z\in\bar{B}(\gamma,\epsilon_n)} h(z)\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}). \qquad (3.2)
\end{aligned}$$

Note that we do not need for the infinimum $\inf_{z\in\bar{B}(\gamma,\epsilon_n)} h(z)$ to be $\mathcal{H}$-measurable as the integral above is computed with respect to x for each fixed $\omega \in C_n$. Precisely, $\inf_{z\in\bar{B}(\gamma,\epsilon_n)} h(z)$ does not depend on x. Since $\mathrm{P}(\hat{\Lambda}_n|\mathcal{H}) > 0$, we then deduce by (3.1) that

$$\mathrm{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma,\mathcal{F}\} \geq \inf_{z\in\bar{B}(\gamma,\epsilon_n)} h(z), \text{ a.s.}$$

for every $n \geq 1$. Therefore,

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma, \mathcal{F}\} \geq \lim_n \inf_{z \in \bar{B}(\gamma, \epsilon_n)} h(z), \text{ a.s.}$$

Since $\bar{B}(\gamma, \epsilon_n)$ is a.s. compact and h is a.s. lower semi-continuous, we deduce that $\inf_{z \in \bar{B}(\gamma,\epsilon_n)} h(z) = h(z_n)$ where $z_n \in \bar{B}(\gamma, \epsilon_n)$ converges pointwise to γ as $n \to \infty$. We finally deduce by lower semi-continuity that

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma, \mathcal{F}\} \geq \liminf_n h(z_n) \geq h(\gamma).$$

This inequality holds for any $\mathcal{H}$-measurable selection of $\mathrm{cl}(\Gamma|\mathcal{H})$. Therefore, $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \mathcal{L}(\Gamma, \mathcal{F}\} \geq \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma),\ \gamma \in \mathcal{L}(\mathrm{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}$. The conclusion of the lemma follows. □

Recall that a set Λ of measurable random variables is said $\mathcal{F}$-decomposable if for any finite partition $(F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}$ of Ω, and for every family $(\gamma_i)_{i=1,\cdots,n}$ of Λ, we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. In the following, we denote by $\Sigma(\Lambda)$ the $\mathcal{F}$-decomposable envelop of Λ, i.e. the smallest $\mathcal{F}$-decomposable family containing Λ. Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} :\ n \geq 1,\ (\gamma_i)_{i=1,\cdots,n} \subseteq \Lambda,\ (F_i)_{i=1,\cdots,n} \subseteq \mathcal{F} \text{ s.t.} \sum_{i=1}^n F_i = \Omega \right\}.$$

The closure $\overline{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if Λ is not decomposable. By [4, Proposition 5.4.3], there exists a $\mathcal{F}$-graph-measurable closed random set $\sigma(\Lambda)$ such that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all measurable selectors of $\sigma(\Lambda)$.

Corollary 3.5. *Let $h(\omega, x)$, $x \in T$, be a real-valued mapping such that h is $\mathcal{H} \otimes \mathcal{B}(T)$-measurable and is l.s.c in x. Let us consider a family Λ of measurable random variables so that $\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$ is the set of all measurable selectors of some $\mathcal{F}$-graph-measurable random closed set $\sigma(\Lambda)$. Then,*

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \Lambda\} = \operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma) :\ \gamma \in \overline{\Sigma}(\Lambda)\} = \sup_{x \in \mathrm{cl}\,(\sigma(\Lambda)|\mathcal{H})} h(x).$$

Proof. Notice that for any finite partition $(F_i)_{i=1,\cdots,n} \subseteq \mathcal{F}$ of Ω, $n \geq 1$, and for every family $(\gamma_i)_{i=1,\cdots,n}$ of Λ, we have

$$h\left(\sum_{i=1}^n \gamma_i 1_{F_i}\right) = \sum_{i=1}^n h(\gamma_i) 1_{F_i}.$$

Therefore, as $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma):\ \gamma\in\Lambda\}\geq h(\gamma)$ a.s. for any $\gamma\in\Lambda$, we deduce that $\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma):\ \gamma\in\Lambda\}\geq h(\gamma)$ a.s. for any $\gamma\in\Sigma(\Lambda)$, and finally for all $\gamma\in\overline{\Sigma}(\Lambda)$ since h is l.s.c. We deduce that

$$\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma):\ \gamma\in\Lambda\}\geq\operatorname{ess\,sup}_{\mathcal{H}}\{h(\gamma):\ \gamma\in\overline{\Sigma}(\Lambda)\}$$

and the equality holds since $\Lambda\subseteq\overline{\Sigma}(\Lambda)$. The last equality is deduced from Theorem 3.4. □

Lemma 3.6. *Consider $X\in\mathcal{L}(T,\mathcal{F})$. Then, we have*

$$\begin{aligned}
\operatorname{ess\,inf}_{\mathcal{H}}X &= \inf\operatorname{supp}_{\mathcal{H}}(X), \quad \operatorname{ess\,sup}_{\mathcal{H}}X=\sup\operatorname{supp}_{\mathcal{H}}(X),\\
\operatorname{ess\,inf}_{\mathcal{H}}X &\in \operatorname{supp}_{\mathcal{H}}(X), \quad \text{on the set } \{\operatorname{ess\,inf}_{\mathcal{H}}X>-\infty\},\\
\operatorname{ess\,sup}_{\mathcal{H}}X &\in \operatorname{supp}_{\mathcal{H}}(X), \quad \text{on the set } \{\operatorname{ess\,sup}_{\mathcal{H}}X<\infty\}.
\end{aligned}$$

Proof. The two first statements are deduced from Theorem 3.4 with $h(x)=x$ and $\Gamma=\{X\}$. Suppose that $\operatorname{ess\,inf}_{\mathcal{H}}X\notin\operatorname{supp}_{\mathcal{H}}(X)$ on some non null subset $\Lambda\in\mathcal{H}$ of $\{\operatorname{ess\,inf}_{\mathcal{H}}X>-\infty\}$. By a selection measurable argument, we deduce the existence of $r\in\mathcal{L}(\mathbb{R}_+,\mathcal{H})$ such that $r>0$ and $[\operatorname{ess\,inf}_{\mathcal{H}}X-r;\operatorname{ess\,inf}_{\mathcal{H}}X+r]\subseteq T\setminus\operatorname{supp}_{\mathcal{H}}(X)$ on Λ. As $X\in\operatorname{supp}_{\mathcal{H}}(X)$ a.s., we deduce that $X\geq\operatorname{ess\,inf}_{\mathcal{H}}X+r$ on Λ, which contradicts the definition of $\operatorname{ess\,inf}_{\mathcal{H}}X$. The fourth statement is similarly shown. □

4. Appendix

Lemma 4.1. *Let $\mathcal{O}$ be an open set in a normed space. For every $x\in\mathcal{O}$, $B(x,d(x,\partial\mathcal{O}))\subseteq\mathcal{O}$.*

Proof. Let us consider $r^*=\sup R$ where R is the non empty set of all $r>0$ such that $B(x,r)\subseteq\mathcal{O}$. It is trivial that $B(x,r^*)\subseteq\mathcal{O}$. So, for all $o\in\partial\mathcal{O}$, $d(x,o)\geq r^*$ hence $r^*\leq d(x,\partial\mathcal{O})$. Moreover, by definition of r^*, for all n there exists $z^n\in B(x,r^*+n^{-1})$ such that $z^n\notin\mathcal{O}$ hence $r^*\leq\|z^n-x\|\leq r^*+n^{-1}$. As the sequence $(z_n)_n$ is bounded, we deduce by a compactness argument that for a subsequence $z_n\to z$ as $n\to\infty$. Then, $\|z-x\|=r^*$ hence $z\in\operatorname{cl}\mathcal{O}$. In the case where $z\in\mathcal{O}$, $z^n\in\mathcal{O}$ for n large enough since $\mathcal{O}$ is open, which yields a contradiction. Therefore, $z\in\partial\mathcal{O}$. This implies that $r^*=d(x,z)\geq d(x,\partial\mathcal{O})$ and finally $r^*=d(x,\partial\mathcal{O})$. □

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