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LARGENESS AND EQUATIONAL PROBABILITY IN GROUPS

KHALED JABER AND FRANK O. WAGNER

ABSTRACT. We define k -genericity and k -largeness for a subset of a group, and determine the value of k for which a k -large subset of G^n is already the whole of G^n , for various equationally defined subsets. We link this with the inner measure of the set of solutions of an equation in a group, leading to new results and/or proofs in equational probabilistic group theory.

1. INTRODUCTION

In probabilistic group theory we are interested in what proportion of (tuples of) elements of a group have a particular property; if this property is given by an equation, we talk about *equational probability*. In [9] a notion of *largeness* was introduced for a subset of a group, and it was shown that certain equational properties of a group hold everywhere as soon as they hold largely. In this paper, we shall introduce a quantitative version of largeness, and deduce some results in equational probabilistic group theory.

Throughout this paper, G will be a group and μ a left-invariant probability measure on some algebra of subsets of G .

- Example 1.1.** (1) G finite, μ the counting measure.
 (2) G_1 a group, μ_1 a left-invariant measure on G_1 , and $G = G_1^n$ with the product measure $\mu = \mu_1^n$.
 (3) More generally, G_1 a group, $G \leq G_1^n$ and μ a left-invariant measure on G .
 (4) G arbitrary and the measure algebra reduced to $\{\emptyset, G\}$. While this set-up trivialises the probability statements, the largeness results remain meaningful.

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If X is a measurable subset of G we can interpret $\mu(X)$ as the probability that a random element of G lies in X . If H is another group, $f : G \rightarrow H$ is a function and $c \in H$ some constant, we put $\mu(f(x) = c) = \mu(\{g \in G : f(g) = c\})$.

Example 1.2. Let G_1 be a group, $G \leq G_1^n$ a subgroup, $\bar{g} \in G_1^m$ constants, and $w(\bar{x}, \bar{y})$ a word in $\bar{x}\bar{y}$ and their inverses, with $|\bar{x}| = n$ and $|\bar{y}| = m$. Then $w(\bar{x}, \bar{g})$ induces a function from G to G_1 .

We shall now list some known results, starting with Frobenius in 1895.

Fact 1.3. *Let G be a finite group.*

- **Frobenius 1895** [5] *If n divides $|G|$ then the number of solutions of $x^n = 1$ is a multiple of n . In particular, $\mu(x^n = 1) \geq \frac{n}{|G|}$.*
- **Miller 1907** [14] *If G is non-abelian, then $\mu(x^2 = 1) \leq \frac{3}{4}$.*
- **Laffey 1976** [11] *If G is a 3-group not of exponent 3 then $\mu(x^3 = 1) \leq \frac{7}{9}$.*
- **Laffey 1976** [12] *If p is prime and divides $|G|$, but G is not a p -group, then $\mu(x^p = 1) \leq \frac{p}{p+1}$.*
- **Laffey 1979** [13] *If G is not a 2-group, then $\mu(x^4 = 1) \leq \frac{8}{9}$.*
- **Iiyori, Yamaki 1991** [8] *If n divides $|G|$ and $X = \{g \in G : g^n = 1\}$ has cardinality n , then X forms a subgroup of G .*
- **Erdős, Turan, 1968** [3] *If $k(G)$ is the number of conjugacy classes in G , then $\mu([x, y] = 1) = \frac{k(G)}{|G|}$.*
- **Joseph 1977** [10], **Gustafson 1973** [6] *If G is non-abelian, then $\mu([x, y] = 1) \leq \frac{5}{8}$.*
- **Neumann, 1989** [16] *For any real $r > 0$ there are $n_1(r)$ and $n_2(r)$ such that if $\mu([x, y] = 1) \geq r$ then G contains normal subgroups $H \leq K$ such that K/H is abelian, $|G : K| \leq n_1(r)$ and $|H| \leq n_2(r)$.*
- **Barry, MacHale, Ní Shé, 2006** [1] *If $\mu([x, y] = 1) > \frac{1}{3}$ then G is supersoluble.*
- **Heffernan, MacHale, Ní Shé, 2014** [7] *If $\mu([x, y] = 1) > \frac{7}{24}$ then G is metabelian. If $\mu([x, y] = 1) > \frac{83}{675}$ then G is abelian-by-nilpotent.*

In Section 2 we shall introduce largeness and prove the main connection between largeness and measure, Lemma 2.5, which will be used throughout the rest of the paper. Section 3 deals with central elements, or more generally FC and BFC groups. We shall treat equations of the form $x^n = c$ for arbitrary c in Section 4, recovering Miller's result for

$n = 2$, and a weaker bound than Laffey for $n = 3$ (namely $\frac{6}{7}$). In Section 5 we shall consider commutator equations; while our methods allow us to deal with more complicated commutators, they are too general to obtain the bounds from Fact 1.3. Section 6 deals with nilpotent groups via linearisation, and the short Section 7 places Sherman's autocommutativity degree in our context.

Notation. We shall write $x^y = y^{-1}xy$, $x^{-y} = (x^{-1})^y = y^{-1}x^{-1}y$ and $[x, y] = x^{-1}y^{-1}xy = y^{-x}y = x^{-1}x^y$.

2. LARGENESS AND PROBABILITY

The following notion of largeness was introduced in [9].

Definition 2.1. If $X \subseteq G$, we say that X is *k-large* in G if the intersection of any k left translates of X is non-empty, and X is *k-generic* in G if k left translates of X cover G . A subset X is *large* if it is k -large for all k ; it is *generic* if it is k -generic for some k .

Of course, analogous notions exist for right and two-sided genericity/largeness. Both genericity and largeness are notions of prominence, increasing with k for largeness and decreasing with k for genericity. Clearly, if $X \subseteq G$ and X is $(k-)$ large/generic, so is any left or right translate or superset of X . Largeness and genericity are complementary:

Lemma 2.2. *Let $X \subseteq G$. Then X is 1-large if and only if $X \neq \emptyset$, and X is 1-generic if and only if $X = G$. More generally, X is k -large if and only if $G \setminus X$ is not k -generic. Finally, X is k -generic/large if and only if $X \cap Y \neq \emptyset$ for all k -large/generic $Y \subseteq G$.*

Proof. We only show the last assertion. If X is not k -generic/large, then $Y := G \setminus X$ is k -large/generic, and $X \cap Y = \emptyset$. Conversely, if X is k -generic, say $G = \bigcup_{i < k} g_i X$, and Y is k -large, then

$$\begin{aligned} \emptyset \neq \bigcap_{i < k} g_i Y &= G \cap \bigcap_{i < k} g_i Y = \bigcup_{i < k} g_i X \cap \bigcap_{i < k} g_i Y \\ &= \bigcup_{i < k} (g_i X \cap \bigcap_{i < k} g_i Y) \subseteq \bigcup_{i < k} (g_i X \cap g_i Y) = \bigcup_{i < k} g_i (X \cap Y). \end{aligned}$$

Thus $X \cap Y \neq \emptyset$. □

Remark 2.3. If $\phi : G \rightarrow H$ is an epimorphism and $X \subseteq G$ is $(k-)$ large/generic, so is $\phi(X) \subseteq H$. Conversely, if $Y \subseteq H$ is $(k-)$ large/generic in H , so is $\phi^{-1}[Y]$ in G .

In particular, if $X \subseteq G \times H$ is $(k-)$ large/generic, so are the projections to each coordinate. Conversely, if $X \subseteq G$ and $Y \subseteq H$ are $(k-)$ large,

so is $X \times Y \subseteq G \times H$; if X is k -generic and Y is ℓ -generic, $X \times Y$ is $k\ell$ -generic.

Lemma 2.4. *Suppose X is $k\ell$ -large in G and $H \leq G$ is a subgroup of index k . Then $X \cap H$ is ℓ -large in H .*

Proof. Let $(g_i : i < k)$ be coset representatives of H in G , and consider $(h_j : j < \ell)$ in H . By $k\ell$ -largeness of X in G there is $x \in \bigcap_{i < k, j < \ell} g_i h_j X$. As $\bigcup_{i < k} g_i H = G$, there is $i_0 < k$ with $x \in g_{i_0} H$. But then

$$g_{i_0}^{-1}x \in H \cap \bigcap_{i < k, j < \ell} g_{i_0}^{-1}g_i h_j X \subseteq H \cap \bigcap_{j < \ell} h_j X = \bigcap_{j < \ell} h_j (X \cap H),$$

so $X \cap H$ is ℓ -large. \square

The link between largeness and probability is given by the following lemma, which will be used throughout the paper. Recall that the *inner measure* of an arbitrary subset X of a measurable group G is

$$\mu_*(X) = \sup\{\mu(Y) : Y \subseteq X \text{ measurable}\},$$

and the *outer measure* is given by

$$\mu^*(X) = \inf\{\mu(Y) : Y \supseteq X \text{ measurable}\}.$$

Clearly the inner measure is superadditive, the outer measure is subadditive, and $\mu_*(X) + \mu^*(G \setminus X) = 1$.

Lemma 2.5. *If X is k -generic in G , then $\mu^*(X) \geq \frac{1}{k}$. If $\mu_*(X) > 1 - \frac{1}{k}$ then X is k -large in G .*

Proof. If X is k -generic there are g_1, \dots, g_k in G with $G = \bigcup_{i \leq k} g_i X$. Hence

$$1 = \mu^*(G) = \mu^*\left(\bigcup_{i \leq k} g_i X\right) \leq \sum_{i \leq k} \mu^*(g_i X) = k \mu^*(X)$$

by left invariance, whence $\mu^*(X) \geq \frac{1}{k}$.

Now if X is not k -large, its complement is k -generic, so $\mu^*(G \setminus X) \geq \frac{1}{k}$. But then $\mu_*(X) \leq 1 - \frac{1}{k}$. \square

These bounds are strict, as we can take X a subgroup of index k (resp. its complement).

Remark 2.6. For any group G the set $(G \times \{1\}) \cup (\{1\} \times G)$ is 2-large in G^2 ; if G is infinite it is of measure 0.

We shall now prove some results about finite groups, which owing to their non-linearity do not generalise easily to the measurable context.

Remark 2.7. Let G be a finite group of order n , and $X \subseteq G$ a non-empty proper subset of size m . Then X is $(n - m + 1)$ -generic and at most m -large, since we can form the union of X with $n - m$ translates of X to cover all the $n - m$ points of $G \setminus X$, and we can intersect X with m translates of X to remove all m points of X .

Theorem 2.8. *Let G be a finite group of order n , and $X \subseteq G$ a non-empty proper subset of size m . If $m > n - \frac{1}{2} - \sqrt{n - \frac{3}{4}}$, then X is 2-generic. Hence if $m < \frac{1}{2} + \sqrt{n - \frac{3}{4}}$ then X is not 2-large.*

Proof. If $m > n - \frac{1}{2} - \sqrt{n - \frac{3}{4}}$, then

$$n - \frac{3}{4} > (n - m - \frac{1}{2})^2 = (n - m)(n - m - 1) + \frac{1}{4}.$$

Put $Z = \{xy^{-1} : x, y \in G \setminus X\}$. Then

$$|Z| \leq (n - m)(n - m - 1) + 1 < n,$$

so there is $g \in G \setminus Z$. But if $h \in G \setminus (X \cup gX)$, then $h, g^{-1}h \in G \setminus X$, and $g = h(g^{-1}h)^{-1} \in Z$, a contradiction. Thus $G = X \cup gX$ and X is 2-generic.

The second assertion follows by taking complements. \square

Theorem 2.9. *Let G be a finite group of order n . If the exponent of G does not divide ℓ then $\mu(x^\ell = 1) \leq 1 - \frac{1}{\sqrt{2n}}$, where μ is the counting measure.*

Proof. Put $X = \{g \in G : g^\ell = 1\}$, of size $m < n$, and take any $g \in G \setminus X$. Note that $X \cap gX \cap C_G(g)$ is empty, as otherwise there would be $y \in C_G(g)$ with $y^\ell = 1 = (gy)^\ell$, whence $g^\ell = 1$ and $g \in X$.

Thus $|C_G(g)| \leq 2|G \setminus X|$. Moreover $g^G \cap X = \emptyset$, and

$$|G|/|C_G(g)| = |g^G| \leq |G \setminus X|.$$

Thus $n = |G| \leq 2|G \setminus X|^2$ and $\sqrt{\frac{n}{2}} \leq n - m$, whence

$$\mu(x^\ell = 1) = \frac{m}{n} \leq \frac{n - \sqrt{\frac{n}{2}}}{n} = 1 - \frac{1}{\sqrt{2n}}. \quad \square$$

Definition 2.10. Let $f : G \rightarrow H$ be a function, and $c \in H$. The equation $f(x) = c$ is *k-largely satisfied* in G if $\{g \in G : f(g) = c\}$ is k -large in G . By abuse of notation, if $G = G_1^n$ and $x = (x_1, \dots, x_n)$, we shall also say that $f(x_1, \dots, x_n) = c$ is k -largely satisfied in G_1 .

3. FC-GROUPS

In this section we shall work in the set-up of Example 1.2: G_1 will be a group, $G \leq G_1^n$, $w(\bar{x}, \bar{y})$ a word in $\bar{x}\bar{y}$ and their inverses with $n = |\bar{x}|$ and $m = |\bar{y}|$, $\bar{g} \in G_1^m$ and $c \in G_1$ constants, and $f(\bar{x}) = w(\bar{x}, \bar{g})$.

Recall that a group is *FC* if the centraliser of any element has finite index; it is *BFC* if the index is bounded independently of the element.

We shall first need a preparatory lemma. For two tuples $\bar{g} = (g_i : i < k)$ and $\bar{g}' = (g'_i : i < k)$ in G_1^k we shall put $\bar{g}^{-1} = (g_i^{-1} : i < k)$ and $\bar{g} \cdot \bar{g}' = (g_i g'_i : i < k)$.

Lemma 3.1. *Suppose $\bar{g}, \bar{g}' \in G_1^m$ and $\bar{h}, \bar{h}' \in G_1^n$ are such that all elements from $\bar{g}\bar{h}$ commute with all elements from $\bar{g}'\bar{h}'$. Then*

$$w(\bar{h} \cdot \bar{h}', \bar{g} \cdot \bar{g}') = w(\bar{h}, \bar{g}) w(\bar{h}', \bar{g}').$$

Proof. Obvious. □

Theorem 3.2. *Let G_1 be an FC-group. If the equation $w(\bar{x}, \bar{g}) = c$ is largely satisfied in G then it is identically satisfied in G .*

Proof. Consider $\bar{h} \in G$, and $C = C_{G_1}(\bar{g}, \bar{h})$, a subgroup of finite index in G_1 . Put $H = C^n \cap G$, a subgroup of finite index in G , and $X = \{\bar{h}' \in G : w(\bar{h}', \bar{g}) = c\}$. Then $X \cap \bar{h}^{-1}X \cap H$ is large in H , whence non-empty. So there is $\bar{x} \in H$ with

$$w(\bar{1}, \bar{g}) w(\bar{x}, \bar{1}) = w(\bar{x}, \bar{g}) = c = w(\bar{h} \cdot \bar{x}, \bar{g}) = w(\bar{h}, \bar{g}) w(\bar{x}, \bar{1}).$$

Hence $w(\bar{h}, \bar{g}) = w(\bar{1}, \bar{g})$ for all $\bar{h} \in G$, and $w(\bar{1}, \bar{g}) = w(\bar{x}, \bar{g}) = c$. □

For a *BFC*-group, we can bound the degree of largeness needed:

Theorem 3.3. *Suppose every centraliser of a single element has index at most k in G_1 . If the equation $w(\bar{x}, \bar{g}) = c$ is $2k^{n^2+mn}$ -largely satisfied in G then it is identically satisfied in G .*

Proof. In the notation of the previous proof, $C = C_{G_1}(\bar{g}, \bar{h})$ has index at most k^{n+m} in G_1 , so

$$|G : H| = |G : G \cap C^n| \leq |G_1^n : C^n| = |G_1 : C|^n \leq (k^{n+m})^n = k^{n^2+mn}.$$

Now $2k^{n^2+mn}$ -largeness of X in G implies k^{n^2+mn} -largeness of $X \cap \bar{h}^{-1}X$ in G , whence 1-largeness of $X \cap \bar{h}^{-1}X \cap H$ in H . So we can find the \bar{x} required to finish the proof. □

Corollary 3.4. *Suppose every centraliser of a single element has index at most k in G_1 . If $w(\bar{x}, \bar{g}) = c$ is not an identity on G , then*

$$\mu_*(w(\bar{x}, \bar{g}) = c) \leq 1 - \frac{1}{2k^{n^2+mn}}.$$

Proof. If $\mu_*(w(\bar{x}, \bar{g}) = c) > 1 - \frac{1}{2k^{n^2+mn}}$, then $\{\bar{x} \in G : w(\bar{x}, \bar{g}) = c\}$ is $2k^{n^2+mn}$ -large in G by Lemma 2.5, and $w(\bar{x}, \bar{g}) = c$ is identically satisfied in G by Theorem 3.3. \square

Remark 3.5. This holds in particular for the equation $x^\ell = c$, with $n = 1$ and $m = 0$.

If the group is central-by-finite, the largeness needed does not depend on the number of parameters.

Corollary 3.6. *Suppose $Z(G_1)$ has index k in G_1 . If the equation $w(\bar{x}, \bar{g}) = c$ is $2k^n$ -largely satisfied in G then it is identically satisfied in G .*

Proof. $H = G \cap Z(G_1)^n$ has index at most k^n in G . We finish as in Theorem 3.3. \square

Corollary 3.7. *If $|G_1 : Z(G_1)| \leq k$ and $w(\bar{x}, \bar{g}) = c$ is not an identity in G , then $\mu_*(w(\bar{x}, \bar{g}) = 1) \leq 1 - \frac{1}{2k^n}$.* \square

Of course, for an abelian group G_1 we have $k = 1$ in the above results.

Remark 3.8. If $w(\bar{x}, \bar{g}) = c$ is 2-largely satisfied in G^n , then it is identically satisfied in the abelian quotient G/G' . If moreover G is a BFC-group, then G' is finite by B.H. Neumann's Lemma [15], and G^n satisfies a finite disjunction $\bigvee_{c' \in G'} w(\bar{x}, \bar{g}) = cc'$.

We can also deduce results for central elements just from 2-largeness (although for infinite index $|G_1 : Z(G_1)|$ there is no reason that if X is large in G the intersection $X \cap Z(G_1)^n$ is still large in $G \cap Z(G_1)^n$).

Theorem 3.9. *If $w(\bar{x}, \bar{g}) = c$ is 2-largely satisfied in G , then $w(\bar{x}, \bar{1}) = 1$ identically on $G \cap Z(G_1)^n$.*

Proof. Consider $\bar{h} \in G \cap Z(G_1)^n$. Put $X = \{\bar{h}' \in G : w(\bar{h}', \bar{g}) = 1\}$. Then $X \cap \bar{h}^{-1}X$ is non-empty, so there is $\bar{x} \in G$ with

$$w(\bar{x}, \bar{g}) = c = w(\bar{h} \cdot \bar{x}, \bar{g}) = w(\bar{h}, \bar{1}) w(\bar{x}, \bar{g}).$$

Hence $w(\bar{h}, \bar{1}) = 1$. \square

Corollary 3.10. *If $x_1^{k_1} \cdots x_n^{k_n} = c$ is 2-largely satisfied in G^n and $k = \gcd(k_1, \dots, k_n)$, then $x^k = 1$ identically on $Z(G)$.*

Proof. We have $x_1^{k_1} \cdots x_n^{k_n} = 1$ on $Z(G)$. Putting $x_i = g \in Z(G)$ and $x_j = 1$ for $j \neq i$ we have $g^{k_i} = 1$ for all $1 \leq i \leq n$. The result follows. \square

Corollary 3.11. *If the exponent of $Z(G)$ does not divide $\gcd(k_1, \dots, k_n)$, then $\mu_*(x_1^{k_1} \cdots x_n^{k_n} = c) \leq \frac{1}{2}$.* \square

4. BURNSIDE AND ENGEL EQUATIONS

In Remark 3.5 we have already seen that if every centraliser of a single element has index at most k in G , then $\mu_*(x^m = c) \leq 1 - \frac{1}{2k}$ unless the exponent of G divides m . In this case necessarily $c = x^m = 1$.

We shall first prove Miller's Theorem mentioned in the introduction.

Theorem 4.1. *Let $c \in G$. If $x^2 = c$ is 4-largely satisfied in G , then G is abelian of exponent 2, and $c = 1$.*

Proof. Fix $g, h \in G$. Then there is x with $c = x^2 = (gx)^2 = (hx)^2 = (ghx)^2$. But this implies $x^{-1}gx = g^{-1}$, $x^{-1}hx = h^{-1}$ and $x^{-1}ghx = (gh)^{-1}$. On the other hand,

$$x^{-1}ghx = x^{-1}gx x^{-1}hx = g^{-1}h^{-1} = (hg)^{-1}.$$

Hence $gh = hg$ and G is abelian. But now $c = x^2 = (gx)^2 = g^2x^2 = g^2c$, whence $g^2 = 1$. \square

If G satisfies 4-largely $xax = b$ for some $a, b \in G$, then it satisfies 4-largely $(ax)^2 = ab$, whence $x^2 = ab$. Hence G is abelian of exponent 2, and $a = b$.

Corollary 4.2. *If G is not of exponent 2 or $a \neq b$, then $\mu_*(xax = b) \leq \frac{3}{4}$.* \square

Recall that the n^{th} Engel condition is the condition $[x, {}_n y] = 1$, where $[x, {}_1 y] = [x, y]$ and $[x, {}_{n+1} y] = [[x, {}_n y], y]$. Note that

$$[x, y, y] = [y^{-x}y, y] = y^{-1}y^xy^{-1}y^{-x}yy = [y^{-x}, y]^y.$$

Thus the 2-Engel condition $[x, y, y] = 1$ is equivalent to $[y^{-x}, y] = 1$, that is all conjugacy classes being commutative.

Theorem 4.3. *If G satisfies 7-largely $x^3 = 1$ then G is 2-Engel.*

Proof. Put $X = \{g \in G : g^3 = 1\}$. For $g, h \in G$ consider

$$x \in X \cap g^{-1}X \cap h^{-1}X \cap gX \cap (gh)^{-1}X \cap gh^{-1}X \cap gh^{-1}g^{-1}X.$$

Then $(yx)^3 = 1$ for $y \in \{1, g, h, g^{-1}, gh, hg^{-1}, ghg^{-1}\}$, which means that $xyx = y^{-1}x^{-1}y^{-1}$. We calculate the product xhx^2gx in two ways:

$$\begin{aligned} xhx^2gx &= (xhx)(xgx) = h^{-1}(x^{-1}h^{-1}g^{-1}x^{-1})g^{-1} \\ &= h^{-1}ghxghg^{-1} \quad \text{and} \\ xhx^2gx &= xh(g^{-1}x)^{-1}x = xh(g^{-1}x)^2x = (xhg^{-1}x)g^{-1}x^2 \\ &= gh^{-1}(x^{-1}gh^{-1}g^{-1}x^{-1}) = gh^{-1}ghg^{-1}xghg^{-1}. \end{aligned}$$

Thus $h^{-1}gh = gh^{-1}ghg^{-1}$ and $g^hg = gg^h$. As $h \in G$ was arbitrary, the conjugacy class of g is commutative; as g was arbitrary, all conjugacy classes are commutative. \square

Theorem 4.4. *Let G be 2-Engel. If G satisfies 2-largely $x^3 = 1$ then G has exponent 3.*

Proof. For any $g \in G$ there is $x \in G$ with $x^3 = (gx)^3 = 1$. As x^G is commutative,

$$g^x g^{-1} g^{x^{-1}} = x^{-1} g x g^{-1} x g x^{-1} = g x^{-g} x x^g x^{-1} = g x^{-g} x^g x x^{-1} = g.$$

Since g^G is commutative, we have

$$g^3 = g^2 g^x g^{-1} g^{x^{-1}} = g^2 g^{-1} g^{x^{-1}} g^x = (gx)^3 = 1. \quad \square$$

Corollary 4.5. *If G satisfies 7-largely $x^3 = 1$, then G has exponent 3. If G is not of exponent 3 then $\mu_*(x^3 = 1) \leq \frac{6}{7}$. If moreover G is 2-Engel, then $\mu_*(x^3 = 1) \leq \frac{1}{2}$.* \square

Note that the bound $\frac{6}{7}$ is not as good as Laffey's bound $\frac{7}{9}$ cited in the introduction.

Problem 4.6. A group which satisfies 5-largely $x^3 = 1$, is it 2-Engel? This would improve our bound to $\frac{4}{5}$.

Corollary 4.7. *If $|G : Z(G)| \leq 7$ and G satisfies 7-largely $x^3 = c$ for some $c \in G$, then $c = 1$ and G has exponent 3.*

Proof. $\{x \in G : x^3 = c\} \cap Z(G)$ is 1-large, whence non-empty, and contains an element z . But now there is $x \in G$ with $x^3 = 1 = (zx)^3 = z^3 x^3 = c x^3$, whence $c = 1$. We finish by Corollary 4.5. \square

If $|G : Z(G)|$ is prime, then G is abelian, and 2-largeness is sufficient by Corollary 3.10.

5. COMMUTATOR EQUATIONS

Consider the equation $[x, g] = c$ for some $c, g \in G$. Since $\{x \in G : [x, g] = c\}$ is a coset of $C_G(g)$ or empty, and a coset of a proper subgroup cannot be 2-large, it follows that if G satisfies 2-largely $[x, g] = c$ then $g \in Z(G)$ and $c = 1$. The following argument generalises this result.

Theorem 5.1. *Suppose $f : G \rightarrow H$ satisfies $f(xx') = f(x)^h f(x')$ for some $h \in H$ which depends on $x, x' \in G$. If G_0 and G_1 are groups, $f_0 : G_0 \rightarrow H$ and $f_1 : G_1 \rightarrow H$ are functions such that $G_0 \times G \times G_1$ satisfies k -largely $f_0(x_0) f(x) f_1(x_1) = c$ for some $k \geq 2$, then $f(G) = 1$ and $G_0 \times G_1$ satisfies k -largely $f_0(x_0) f_1(x_1) = c$.*

Proof. Fix $g \in G$. By 2-largeness there is $(x_0, x, x_1) \in G_0 \times G \times G_1$ such that

$$f_0(x_0) f(x) f(x_1) = c = f_0(x_0) f(gx) f(x_1).$$

Thus $f(x) = f(gx) = f(g)^h f(x)$ and $f(g) = 1$. It follows that $f_0(x_0) f(x) f_1(x_1) = f_0(x_0) f_1(x_1)$ on $G_0 \times G \times G_1$. The result follows. \square

Corollary 5.2. *If G satisfies 2-largely $\prod_{i < n} [x_i, g_i] = c$ for some $g_i \in G$, then $g_i \in Z(G)$ for all $i < n$ and $c = 1$. If not all g_i are central or $c \neq 1$ then $\mu_*(\prod_{i < n} [x_i, g_i] = c) \leq \frac{1}{2}$.*

Proof. We have $[xx', y] = [x, y]^{x'} [x', y]$. Now use Theorem 5.1. \square

Remark 5.3. Theorem 5.1 also holds if $f(xx') = f(x')f(x)^h$, with almost the same proof. Hence Corollary 5.2 also holds if some factors are of the form $[g_i, x_i]$.

Gustafson [6] has shown that $\mu_2([x, y] = 1) \leq \frac{1}{2}(1 + \mu(Z(G))) \leq \frac{5}{8}$ for a non-abelian compact topological group G , where μ is the Haar measure on G and μ_2 the product measure on G^2 . Pournaki and Sobhani [17] have generalised this to calculate that $\mu([x, y] = g) < \frac{1}{2}$ for any $g \neq 1$ in a finite group, using Rusin's classification [18] of all finite groups with $\mu([x, y] = 1) > \frac{11}{32}$ (see also [4]). We have only been able to establish results using 4-largeness, giving the bound of $\frac{3}{4}$ in Corollary 5.7, so the following two problems remain open:

- Problem 5.4.** (1) If G satisfies 2-largely $[x, y] = 1$, is $G' = C_2$ and $G/Z(G)$ of exponent 2, or $G' = C_3$ and $G/Z(G) = S_3$?
 (2) If G satisfies 2-largely $[x, y] = c$ for some $c \in G$, is $c = 1$?

Theorem 5.5. *If $w(\bar{x}, \bar{g})[x, y] = c$ is satisfied 4-largely in G^{n+1} , where $x \in \bar{x}$ and $y \notin \bar{x}$, then G is abelian and $w(\bar{x}, \bar{g}) = c$.*

Proof. For any $h \in G$ the set

$$\{(\bar{x}, x, y) : w(\bar{x}, \bar{g})[x, y] = c = w(\bar{x}, \bar{g})[x, hy]\}$$

is 2-large in G^{n+1} . Hence $\{(x, y) \in G^2 : [x, y] = [x, hy]\}$ is 2-large in G^2 . Now $[x, hy] = [x, y][x, h]^y$, so $[x, h] = 1$ is satisfied 2-largely in G , whence $h \in Z(G)$. It follows that G is abelian. But then $w(\bar{x}, \bar{g}) = c$ is satisfied 4-largely in G^n , and must be an identity in G by commutativity and Corollary 3.6. \square

Corollary 5.6. *If G is a group with $\mu_*(w(\bar{x}, \bar{g})[x, y] = c) > \frac{3}{4}$, then G is abelian satisfying $w(\bar{x}, \bar{g}) = c$.* \square

Corollary 5.7. *If G satisfies 4-largely $[x, y] = c$, then G is abelian and $c = 1$. If G is not abelian or $c \neq 1$, then $\mu_*([x, y] = c) \leq \frac{3}{4}$.* \square

Remark 5.8. The same holds for the equation $xcy = yc'x$ with $c \neq c'$: putting $x' = xc$ and $y' = yc'$, this is equivalent to $[x', y'] = c^{-1}c'$.

Theorem 5.9. *Let $g, h \in G$ and $k = \min\{|G : C_G(g)|, |G : C_G(h)|\}$. If G satisfies k -largely $[g, h^x] = 1$, then g^G and h^G commute.*

Proof. If $k = |G : C_G(h)|$, then $\{x \in G : [g, h^x] = 1\} \cap C_G(h)$ is 1-large, whence non-empty, and $[g, h] = 1$. Now note that for any $a \in G$ also $|G : C_G(h^a)| = k$ and $[g, h^{ax}] = 1$ is satisfied k -largely, whence $[g, h^a] = 1$ and $[g, h^G] = 1$.

If $k = |G : C_G(g)|$, then $\{x \in G : [g^{x^{-1}}, h] = 1\} \cap C_G(g)$ is 1-large (still on the left) and non-empty, whence $[g, h] = 1$ and we finish as above. \square

Corollary 5.10. *If $[g^G, h^G]$ is non-trivial for some $g, h \in G$, then $\mu_*([g, h^x] = 1) \leq 1 - \frac{1}{k}$, where $k = \min\{|G : C_G(g)|, |G : C_G(h)|\}$. \square*

Theorem 5.11. *If $g, h, c \in G$ and $[x, g, h] = c$ is $2k$ -largely satisfied, where $k = |G : C_G(h)|$, then $[G, g, h] = 1$. Similarly, if $[g, x, h] = c$ is $2k$ -largely satisfied for some $c \in Z(G)$, then $[g, G, h] = 1$.*

Proof. Choose $a \in G$. Then the set $X = \{x \in G : [x, g, h] = c = [ax, g, h]\}$ is k -large, and for $x \in X$ we have

$$[x, g, h] = c = [ax, g, h] = [[a, g]^x [x, g], h] = [[a, g]^x, h]^{[x, g]} [x, g, h],$$

whence $[[a, g]^x, h] = 1$. By Theorem 5.9 we have $[a, g, h] = 1$.

If $[g, x, h] = c$ is $2k$ -largely satisfied with $c \in Z(G)$, then for $a \in G$ we obtain a k -large $X \subseteq G$ such that for $x \in X$ we have

$$[g, x, h] = c = [g, ax, h] = [[g, x][g, a]^x, h] = [g, x, h]^{[g, a]^x} [[g, a]^x, h],$$

whence $[[g, a]^x, h] = 1$, and $[g, a, h] = 1$ by Theorem 5.9. \square

Corollary 5.12. *If $g, h \in G$ and $k = |G : C_G(h)|$, then $[G, g, h] \neq 1$ implies $\mu_*([x, g, h] = c) \leq 1 - \frac{1}{2k}$ for any $c \in G$, and $[g, G, h] \neq 1$ implies $\mu_*([g, x, h] = c) \leq 1 - \frac{1}{2k}$ for any $c \in Z(G)$. \square*

We shall now generalise Corollary 5.7 to higher nilpotency classes. However, the proof requires an additional assumption.

Theorem 5.13. *Suppose $s < \omega$ is such that for all $i < k$ there is a set A_i of size at most s such that $Z(G/Z_i(G)) = C_{G/Z_i(G)}(A_i)$. If G satisfies $2(s+1)^k$ -largely $[x_0, x_1, \dots, x_k] = c$, then $c = 1$ and G is nilpotent of class at most k .*

Proof. We use induction on k . For $k = 1$ note that $s \geq 1$ (otherwise G is abelian and we are done), so the result follows from Corollary 5.7.

Now suppose the assertion is true for k , and

$$X = \{\bar{x} \in G^{k+2} : [x_0, x_1, \dots, x_{k+1}] = c\}$$

is $2(s+1)^{k+1}$ -large in G^{k+2} . If $A_0 = \{a_i : i < s\}$ consider the projection Y of $X \cap \bigcap_{i < s} (1, \dots, 1, a_i^{-1})X$ to the first $k+1$ coordinates, and note that it is $2(s+1)^k$ -large. Then for all $(x_0, \dots, x_k) \in Y$ there is $y \in G$ such that

$$[x_0, \dots, x_k, y] = c = [x_0, \dots, x_k, a_i y] = [x_0, \dots, x_k, y] [x_0, \dots, x_k, a_i]^y$$

for all $i < s$, whence $[x_0, \dots, x_k] \in Z(G)$. By inductive assumption $G/Z(G)$ is nilpotent of class at most k , and we are done. \square

Corollary 5.14. *Let s be as above. If G is not nilpotent of class at most k or $c \neq 1$, then $\mu_*([x_0, x_1, \dots, x_k] = c) \leq 1 - \frac{1}{2}(s+1)^{-k}$. \square*

Remark 5.15. Recall that an \mathfrak{Mc} -group is a group G such that for every subset A there is a finite subset $A_0 \subseteq A$ such that $C_G(A) = C_G(A_0)$. Equivalently, G satisfies the ascending (or the descending) chain condition on centralisers. Roger Bryant [2] has shown that in an \mathfrak{Mc} -group, for every iterated centre $Z_i(G)$ there is a finite set A_i such that $Z(G/Z_i(G)) = C_{G/Z_i(G)}(A_i)$. So in an \mathfrak{Mc} -group we can find some s as needed for Theorem 5.13 and Corollary 5.14.

Problem 5.16. To what extent do we need the \mathfrak{Mc} -condition (or similar) in Theorem 5.13 and Corollary 5.13? It is not needed for nilpotency class 1 (Corollary 5.7). In general, assuming just 2^{k+1} -largeness of $[x_0, \dots, x_k] = c$, we obtain that $\{\bar{x} \in G^k : [x_0, \dots, x_{k-1}] \in C_G(g)\}$ is 2^k -large in G^k for any $g \in G$. Does this imply $\gamma_k(G) \leq C_G(g)$, or even $\gamma_k(G) \leq Z(G)$?

6. NILPOTENT GROUPS

We shall first introduce the notion of a supercommutator from [9].

Definition 6.1. Any variable and any constant from G is a *supercommutator*; if v and w are supercommutators, then v^{-1} and $[v, w]$ are supercommutators.

Alternatively, we could have said that x, x^{-1} and g are supercommutators for any variable x and any $g \in G$, and that if v and w are supercommutators, so is $[v, w]$.

Definition 6.2. The set $\text{Var}(v)$ of variables of a supercommutator v is defined by $\text{Var}(x) = \{x\}$, $\text{Var}(g) = \emptyset$, $\text{Var}(v^{-1}) = \text{Var}(v)$, and $\text{Var}([v, w]) = \text{Var}(v) \cup \text{Var}(w)$. We put $\text{var}(v) = |\text{Var}(v)|$, the *variable number* of v . If \bar{x} is a tuple of variables, we put $\text{Var}_{\bar{x}} = \text{Var}(v) \cap \bar{x}$, $\text{Var}'_{\bar{x}}(v) = \text{Var}(v) \setminus \bar{x}$, $\text{var}_{\bar{x}}(v) = |\text{Var}_{\bar{x}}(v)|$ and $\text{var}'_{\bar{x}}(v) = |\text{Var}'_{\bar{x}}(v)|$.

Clearly $\text{var}([v, v']) \geq \max\{\text{var}(v), \text{var}(v')\}$, and similarly for $\text{var}_{\bar{x}}$ and $\text{var}'_{\bar{x}}$.

Lemma 6.3. *Let $H \trianglelefteq G$ and $v(\bar{x}, \bar{z})$ a supercommutator.*

- (1) *v defines a function from $H^{|\bar{x}\bar{z}|}$ to $\gamma_{\text{var}(v)}(H)$.*
- (2) *If $\text{var}_{\bar{x}}(v) > 0$ and \bar{x}, \bar{y} and \bar{z} are pairwise disjoint, then*

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}),$$

where Φ is a product of supercommutators whose factors w satisfy

- (†) *$\text{Var}_{\bar{z}}(w) = \text{Var}_{\bar{z}}(v)$, and if $x_i \in \text{Var}_{\bar{x}}(v)$ then $x_i \in \text{Var}(w)$ or $y_i \in \text{Var}(w)$, and both possibilities occur for at least one i .*

- (3) *If $v(\bar{x}, \bar{z})$ is a product of supercommutators whose factors w satisfy $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}}(w) \geq n$, then*

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}),$$

where Φ is a product of supercommutators whose factors w satisfy $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}}(w) > n$.

Proof. (1) is proved as in [9, Lemme 6(1)] by induction, using that $\gamma_n(H)$ is characteristic in H , whence normal in G , and $[\gamma_n(H), \gamma_m(H)] \leq \gamma_{n+m}(H)$. We shall show (2) by induction on the construction of v .

If $v = x \in \bar{x}$ we have $v(yx) = yx = xy[y, x] = v(x)v(y)[y, x]$; if $v = x^{-1}$ we have $v(yx) = x^{-1}y^{-1} = v(x)v(y)$. This leaves the case $v = [v_1, v_2]$ for two supercommutators v_1 and v_2 . We shall assume $\text{var}_{\bar{x}}(v_1) > 0$ and $\text{var}_{\bar{x}}(v_2) > 0$ (the case $\text{var}_{\bar{x}}(v_1)\text{var}_{\bar{x}}(v_2) = 0$ is analogous, but simpler).

By inductive hypothesis, there are Φ_i for $i = 1, 2$, products of supercommutators satisfying (†) relative to v_i , such that

$$v_i(\bar{y} \cdot \bar{x}, \bar{z}) = v_i(\bar{x}, \bar{z}) v_i(\bar{y}, \bar{z}) \Phi_i.$$

Then

$$\begin{aligned} v(\bar{y} \cdot \bar{x}, \bar{z}) &= [v_1(\bar{y} \cdot \bar{x}, \bar{z}), v_2(\bar{y} \cdot \bar{x}, \bar{z})] \\ &= [v_1(\bar{x}, \bar{z}) v_1(\bar{y}, \bar{z}) \Phi_1, v_2(\bar{x}, \bar{z}) v_2(\bar{y}, \bar{z}) \Phi_2] \\ &= [v_1(\bar{x}, \bar{z}), v_2(\bar{x}, \bar{z})] [v_1(\bar{y}, \bar{z}), v_2(\bar{y}, \bar{z})] \Phi = v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi, \end{aligned}$$

where Φ is a product of supercommutators $[w, w']$

- (i) where $w \in \Phi_1 \cup \{v_1(\bar{x}, \bar{z}), v_1(\bar{y}, \bar{z})\}$ and $w' \in \Phi_2 \cup \{v_2(\bar{x}, \bar{z}), v_2(\bar{y}, \bar{z})\}$, except for $[v_1(\bar{x}, \bar{z}), v_2(\bar{x}, \bar{z})]$ and $[v_1(\bar{y}, \bar{z}), v_2(\bar{y}, \bar{z})]$; it is clear that these must satisfy (†).
- (ii) where one of w, w' is from (i), so $[w, w']$ satisfies (†).
- (iii) where one of w, w' is equal to $v(\bar{x}, \bar{z})$ and the other contains at least one y_i , or one is equal to $v(\bar{y}, \bar{z})$ and the other contains at least one x_i ; again $[w, w']$ satisfies (†).

- (iv) which are obtained iteratively from supercommutators from (ii) and (iii) by commutation with other supercommutators, thus satisfying (\dagger) .

Here (i) takes care of the commutators of various factors of the two products, while (ii)–(iv) takes care of the correct order. Note that the only factor without a variable y_i is $v(\bar{x}, \bar{z})$ and the only factor without a variable x_j is $v(\bar{y}, \bar{z})$.

To show (3) note first that for a single supercommutator v the factorisation given in (2) satisfies the requirement. So for a product of supercommutators, we apply (2) to every factor, and then use commutators to get them into the right order. Note that we never have to commute a $w(\bar{x}, \bar{z})$ with a $w'(\bar{x}, \bar{z})$, or a $w(\bar{y}, \bar{z})$ with a $w'(\bar{y}, \bar{z})$, as they already appear in the correct order with respect to one another. It follows that all new commutators satisfy (\dagger) , whence $\text{var}'_{\bar{x}} > n$. \square

Theorem 6.4. *If G is nilpotent of class k and v is a product of supercommutators w with $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}}(w) \geq n$ such that G satisfies $\max\{2^{k-n}, 1\}$ -largely $v(\bar{x}, \bar{g}) = c$, then $c = 1$.*

Proof. This is true for $n \geq k$, as then $\text{var}(w) = \text{var}_{\bar{x}}(w) + \text{var}'_{\bar{x}}(w) \geq 1 + n$, and

$$c = w(\bar{x}, \bar{g}) \in \gamma_{\text{var}(w)} G \leq \gamma_{n+1} G = \{1\}$$

for some $\bar{x} \in G$.

Now suppose it is true for $n + 1 \leq k$, and let $v(\bar{x}, \bar{z})$ be a product of supercommutators w with $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}} \geq n$, such that H satisfies 2^{k-n} -largely $v(\bar{x}, \bar{g}) = c$. By Lemma 6.3 there is Φ , a product of supercommutators whose factors w satisfy $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}}(w) > n$, such that

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}).$$

Choose $\bar{h} \in G$ with $v(\bar{h}, \bar{g}) = c$. If $X = \{\bar{x} \in G : v(\bar{x}, \bar{g}) = c\}$, then X is 2^{k-n} -large, and $Y = X \cap \bar{h}^{-1}X$ is 2^{k-n-1} -large. Moreover, for $\bar{x} \in Y$ we have

$$\Phi(\bar{x}, \bar{h}, \bar{g}) = v(\bar{h}, \bar{g})^{-1} v(\bar{x}, \bar{g})^{-1} v(\bar{h} \cdot \bar{x}, \bar{g}) = c^{-1} c^{-1} c = c^{-1}.$$

By hypothesis $c^{-1} = 1$ and we are done. \square

Theorem 6.5. *If G is nilpotent of class k and satisfies 2^k -largely an equation $v(\bar{x}, \bar{g}) = c$, then it satisfies $v(\bar{x}, \bar{g}) = c$.*

Proof. Bringing all the constants to the right-hand side, we may assume that $v(\bar{x}, \bar{z})$ is a product of supercommutators w with $\text{var}_{\bar{x}}(w) > 0$. By Lemma 6.3 there is Φ , a product of supercommutators whose factors w satisfy $\text{var}_{\bar{x}}(w) > 0$ and $\text{var}'_{\bar{x}}(w) > 0$, such that

$$v(\bar{y} \cdot \bar{x}, \bar{z}) = v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}).$$

Fix $\bar{h} \in G$. Then

$$\Phi(\bar{x}, \bar{h}, \bar{g}) = v(\bar{h}, \bar{g})^{-1} c^{-1} c = v(\bar{h}, \bar{g})^{-1}$$

2^{k-1} -largely on G . By Theorem 6.4 we have $v(\bar{h}, \bar{g}) = 1$. So $v(\bar{x}, \bar{g})$ is constant. \square

Corollary 6.6. *If G is nilpotent of class k and $x^n = c$ is true 2^k -largely, then $c = 1$ and the exponent of G divides n .*

Proof. Immediate from Theorem 6.5. \square

Corollary 6.7. *If G is nilpotent of class k and $\mu_*(x^n = c) > 1 - 2^{-k}$, then $c = 1$ and the exponent of G divides n .* \square

7. AUTOCOMMUTATIVITY

The notion of autocommutativity has been introduced by Sherman in 1975 [19].

Definition 7.1. Let G be a finite group, Σ a group of automorphisms of G , and H a subgroup of G . The *degree of autocommutativity relative to $(H; \Sigma)$* is given by

$$\text{ac}(H; \Sigma) = \frac{|\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}|}{|\Sigma| \cdot |H|}.$$

It gives the probability that a random element of H is fixed by a random automorphism in Σ .

Note that $\text{ac}(H; \Sigma) = \mu(\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\})$, where μ is the counting measure on $\Sigma \times H$.

Theorem 7.2. *Let $H \leq G$ be finite groups, Σ a group of automorphisms of G , and suppose that $\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}$ is 4-large in $\Sigma \times H$. Then $H \leq \text{Fix}(\Sigma)$.*

Proof. Given $\sigma \in \Sigma$ and $g \in H$, by 4-largeness there are $x \in H$ and $\tau \in \Sigma$ with

$$\tau(x) = x, \quad (\sigma \circ \tau)(x) = x, \quad \tau(gx) = gx \quad \text{and} \quad (\sigma \circ \tau)(gx) = gx.$$

Then

$$gx = \sigma(\tau(gx)) = \sigma(gx) = \sigma(g)\sigma(x) = \sigma(g)\sigma(\tau(x)) = \sigma(g)x,$$

whence $g = \sigma(g)$. \square

Corollary 7.3. *If $H \leq G$ are finite groups and Σ is a group of automorphisms of G with $H \not\leq \text{Fix}(\Sigma)$, then $\text{ac}(H; \Sigma) \leq \frac{3}{4}$.*

Proof. If $\text{ac}(H; \Sigma) > \frac{3}{4}$ then $\{(\sigma, g) \in \Sigma \times H : \sigma(g) = g\}$ is 4-large in $\Sigma \times H$ by Lemma 2.5. Hence $H \leq \text{Fix}(\Sigma)$ by Theorem 7.2. \square

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