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# LARGENESS AND EQUATIONAL PROBABILITY IN GROUPS 

KHALED JABER AND FRANK O. WAGNER


#### Abstract

We define $k$-genericity and $k$-largeness for a subset of a group, and determine the value of $k$ for which a $k$-large subset of $G^{n}$ is already the whole of $G^{n}$, for various equationally defined subsets. We link this with the inner measure of the set of solutions of an equation in a group, leading to new results and/or proofs in equational probabilistic group theory.


## 1. Introduction

In probabilistic group theory we are interested in what proportion of (tuples of) elements of a group have a particular property; if this property is given by an equation, we talk about equational probability. In [9] a notion of largeness was introduced for a subset of a group, and it was shown that certain equational properties of a group hold everywhere as soon as they hold largely. In this paper, we shall introduce a quantitative version of largeness, and deduce some results in equational probabilistic group theory.

Throughout this paper, $G$ will be a group and $\mu$ a left-invariant probability measure on some algebra of subsets of $G$.

Example 1.1. (1) $G$ finite, $\mu$ the counting measure.
(2) $G_{1}$ a group, $\mu_{1}$ a left-invariant measure on $G_{1}$, and $G=G_{1}^{n}$ with the product measure $\mu=\mu_{1}^{n}$.
(3) More generally, $G_{1}$ a group, $G \leq G_{1}^{n}$ and $\mu$ a left-invariant measure on $G$.
(4) $G$ arbitrary and the measure algebra reduced to $\{\emptyset, G\}$. While this set-up trivialises the probability statements, the largeness results remain meaningful.

[^0]If $X$ is a measurable subset of $G$ we can interpret $\mu(X)$ as the probability that a random element of $G$ lies in $X$. If $H$ is another group, $f: G \rightarrow H$ is a function and $c \in H$ some constant, we put $\mu(f(x)=c)=\mu(\{g \in G: f(g)=c\})$.
Example 1.2. Let $G_{1}$ be a group, $G \leq G_{1}^{n}$ a subgroup, $\bar{g} \in G_{1}^{m}$ constants, and $w(\bar{x}, \bar{y})$ a word in $\bar{x} \bar{y}$ and their inverses, with $|\bar{x}|=n$ and $|\bar{y}|=m$. Then $w(\bar{x}, \bar{g})$ induces a function from $G$ to $G_{1}$.

We shall now list some known results, starting with Frobenius in 1895.

Fact 1.3. Let $G$ be a finite group.

- Frobenius 1895 [5] If $n$ divides $|G|$ then the number of solutions of $x^{n}=1$ is a multiple of $n$. In particular, $\mu\left(x^{n}=1\right) \geq$ $\frac{n}{|G|}$.
- Miller 1907 [14] If $G$ is non-abelian, then $\mu\left(x^{2}=1\right) \leq \frac{3}{4}$.
- Laffey 1976 [11] If $G$ is a 3-group not of exponent 3 then $\mu\left(x^{3}=1\right) \leq \frac{7}{9}$.
- Laffey 1976 [12] If $p$ is prime and divides $|G|$, but $G$ is not a $p$-group, then $\mu\left(x^{p}=1\right) \leq \frac{p}{p+1}$.
- Laffey 1979 [13] If $G$ is not a 2-group, then $\mu\left(x^{4}=1\right) \leq \frac{8}{9}$.
- Iiyori, Yamaki 1991 [8] If $n$ divides $|G|$ and $X=\{g \in G$ : $\left.g^{n}=1\right\}$ has cardinality $n$, then $X$ forms a subgroup of $G$.
- Erdős, Turan, 1968 [3] If $k(G)$ is the number of conjugacy classes in $G$, then $\mu([x, y]=1)=\frac{k(G)}{|G|}$.
- Joseph 1977 [10], Gustafson 1973 [6] If $G$ is non-abelian, then $\mu([x, y]=1) \leq \frac{5}{8}$.
- Neumann, 1989 [16] For any real $r>0$ there are $n_{1}(r)$ and $n_{2}(r)$ such that if $\mu([x, y]=1) \geq r$ then $G$ contains normal subgroups $H \leq K$ such that $K / H$ is abelian, $|G: K| \leq n_{1}(r)$ and $|H| \leq n_{2}(r)$.
- Barry, MacHale, Ní Shé, 2006 [1] If $\mu([x, y]=1)>\frac{1}{3}$ then $G$ is supersoluble.
- Heffernan, MacHale, Ní Shé, 2014 [7] If $\mu([x, y]=1)>\frac{7}{24}$ then $G$ is metabelian. If $\mu([x, y]=1)>\frac{83}{675}$ then $G$ is abelian-by-nilpotent.

In Section 2 we shall introduce largeness and prove the main connection between largeness and measure, Lemma 2.5, which will be used throughout the rest of the paper. Section 3 deals with central elements, or more generally FC and BFC groups. We shall treat equations of the form $x^{n}=c$ for arbitrary $c$ in Section 4, recovering Miller's result for
$n=2$, and a weaker bound than Laffey for $n=3$ (namely $\frac{6}{7}$ ). In Section 5 we shall consider commutator equations; while our methods allow us to deal with more complicated commutators, they are too general to obtain the bounds from Fact 1.3. Section 6 deals with nilpotent groups via linearisation, and the short Section 7 places Sherman's autocommutativity degree in our context.
Notation. We shall write $x^{y}=y^{-1} x y, x^{-y}=\left(x^{-1}\right)^{y}=y^{-1} x^{-1} y$ and $[x, y]=x^{-1} y^{-1} x y=y^{-x} y=x^{-1} x^{y}$.

## 2. Largeness and Probability

The following notion of largeness was introduced in [9].
Definition 2.1. If $X \subseteq G$, we say that $X$ is $k$-large in $G$ if the intersection of any $k$ left translates of $X$ is non-empty, and $X$ is $k$-generic in $G$ if $k$ left translates of $X$ cover $G$. A subset $X$ is large if it is $k$-large for all $k$; it is generic if it is $k$-generic for some $k$.

Of course, analogous notions exist for right and two-sided genericity/largeness. Both genericity and largeness are notions of prominence, increasing with $k$ for largeness and decreasing with $k$ for genericity. Clearly, if $X \subseteq G$ and $X$ is ( $k$-)large/generic, so is any left or right translate or superset of $X$. Largeness and genericity are cocomplementary:
Lemma 2.2. Let $X \subseteq G$. Then $X$ is 1-large if and only if $X \neq \emptyset$, and $X$ is 1-generic if and only if $X=G$. More generally, $X$ is $k$-large if and only if $G \backslash X$ is not $k$-generic. Finally, $X$ is $k$-generic/large if and only if $X \cap Y \neq \emptyset$ for all $k$-large/generic $Y \subseteq G$.

Proof. We only show the last assertion. If $X$ is not $k$-generic/large, then $Y:=G \backslash X$ is $k$-large/generic, and $X \cap Y=\emptyset$. Conversely, if $X$ is $k$-generic, say $G=\bigcup_{i<k} g_{i} X$, and $Y$ is $k$-large, then

$$
\begin{aligned}
\emptyset & \neq \bigcap_{i<k} g_{i} Y=G \cap \bigcap_{i<k} g_{i} Y=\bigcup_{i<k} g_{i} X \cap \bigcap_{i<k} g_{i} Y \\
& =\bigcup_{i<k}\left(g_{i} X \cap \bigcap_{i<k} g_{i} Y\right) \subseteq \bigcup_{i<k}\left(g_{i} X \cap g_{i} Y\right)=\bigcup_{i<k} g_{i}(X \cap Y) .
\end{aligned}
$$

Thus $X \cap Y \neq \emptyset$.
Remark 2.3. If $\phi: G \rightarrow H$ is an epimorphism and $X \subseteq G$ is $(k$-)large/generic, so is $\phi(X) \subseteq H$. Conversely, if $Y \subseteq H$ is $(k$ )large/generic in $H$, so is $\phi^{-1}[X]$ in $G$.

In particular, if $X \subseteq G \times H$ is ( $k$-)large/generic, so are the projections to each coordinate. Conversely, if $X \subseteq G$ and $Y \subseteq H$ are ( $k$-)large,
so is $X \times Y \subseteq G \times H$; if $X$ is $k$-generic and $Y$ is $\ell$-generic, $X \times Y$ is $k \ell$-generic.

Lemma 2.4. Suppose $X$ is $k \ell$-large in $G$ and $H \leq G$ is a subgroup of index $k$. Then $X \cap H$ is $\ell$-large in $H$.

Proof. Let $\left(g_{i}: i<k\right)$ be coset representatives of $H$ in $G$, and consider $\left(h_{j}: j<\ell\right)$ in $H$. By $k \ell$-largeness of $X$ in $G$ there is $x \in$ $\bigcap_{i<k, j<\ell} g_{i} h_{j} X$. As $\bigcup_{i<k} g_{i} H=G$, there is $i_{0}<k$ with $x \in g_{i_{0}} H$. But then

$$
g_{i_{0}}^{-1} x \in H \cap \bigcap_{i<k, j<\ell} g_{i_{0}}^{-1} g_{i} h_{j} X \subseteq H \cap \bigcap \bigcap_{j<\ell} h_{j} X=\bigcap_{j<\ell} h_{j}(X \cap H)
$$

so $X \cap H$ is $\ell$-large.
The link between largeness and probability is given by the following lemma, which will be used throughout the paper. Recall that the inner measure of an arbitrary subset $X$ of a measurable group $G$ is

$$
\mu_{*}(X)=\sup \{\mu(Y): Y \subseteq X \text { measurable }\}
$$

and the outer measure is given by

$$
\mu^{*}(X)=\inf \{\mu(Y): Y \supseteq X \text { measurable }\} .
$$

Clearly the inner measure is superadditive, the outer measure is subadditive, and $\mu_{*}(X)+\mu^{*}(G \backslash X)=1$.
Lemma 2.5. If $X$ is $k$-generic in $G$, then $\mu^{*}(X) \geq \frac{1}{k}$. If $\mu_{*}(X)>1-\frac{1}{k}$ then $X$ is $k$-large in $G$.
Proof. If $X$ is $k$-generic there are $g_{1}, \ldots, g_{k}$ in $G$ with $G=\bigcup_{i \leq k} g_{i} X$. Hence

$$
1=\mu^{*}(G)=\mu^{*}\left(\bigcup_{i \leq k} g_{i} X\right) \leq \sum_{i \leq k} \mu^{*}\left(g_{i} X\right)=k \mu^{*}(X)
$$

by left invariance, whence $\mu^{*}(X) \geq \frac{1}{k}$.
Now if $X$ is not $k$-large, its complement is $k$-generic, so $\mu^{*}(G \backslash X) \geq$ $\frac{1}{k}$. But then $\mu_{*}(X) \leq 1-\frac{1}{k}$.

These bounds are strict, as we can take $X$ a subgroup of index $k$ (resp. its complement).

Remark 2.6. For any group $G$ the set $(G \times\{1\}) \cup(\{1\} \times G)$ is 2-large in $G^{2}$; if $G$ is infinite it is of measure 0 .

We shall now prove some results about finite groups, which owing to their non-linearity do not generalise easily to the measurable context.

Remark 2.7. Let $G$ be a finite group of order $n$, and $X \subseteq G$ a nonempty proper subset of size $m$. Then $X$ is $(n-m+1)$-generic and at most $m$-large, since we can form the union of $X$ with $n-m$ translates of $X$ to cover all the $n-m$ points of $G \backslash X$, and we can intersect $X$ with $m$ translates of $X$ to remove all $m$ points of $X$.

Theorem 2.8. Let $G$ be a finite group of order $n$, and $X \subseteq G$ a nonempty proper subset of size $m$. If $m>n-\frac{1}{2}-\sqrt{n-\frac{3}{4}}$, then $X$ is 2-generic. Hence if $m<\frac{1}{2}+\sqrt{n-\frac{3}{4}}$ then $X$ is not 2-large.

Proof. If $m>n-\frac{1}{2}-\sqrt{n-\frac{3}{4}}$, then

$$
n-\frac{3}{4}>\left(n-m-\frac{1}{2}\right)^{2}=(n-m)(n-m-1)+\frac{1}{4} .
$$

Put $Z=\left\{x y^{-1}: x, y \in G \backslash X\right\}$. Then

$$
|Z| \leq(n-m)(n-m-1)+1<n
$$

so there is $g \in G \backslash Z$. But if $h \in G \backslash(X \cup g X)$, then $h, g^{-1} h \in G \backslash X$, and $g=h\left(g^{-1} h\right)^{-1} \in Z$, a contradiction. Thus $G=X \cup g X$ and $X$ is 2-generic.

The second assertion follows by taking complements.
Theorem 2.9. Let $G$ be a finite group of order $n$. If the exponent of $G$ does not divide $\ell$ then $\mu\left(x^{\ell}=1\right) \leq 1-\frac{1}{\sqrt{2 n}}$, where $\mu$ is the counting measure.

Proof. Put $X=\left\{g \in G: g^{\ell}=1\right\}$, of size $m<n$, and take any $g \in G \backslash X$. Note that $X \cap g X \cap C_{G}(g)$ is empty, as otherwise there would be $y \in C_{G}(g)$ with $y^{\ell}=1=(g y)^{\ell}$, whence $g^{\ell}=1$ and $g \in X$.

Thus $\left|C_{G}(g)\right| \leq 2|G \backslash X|$. Moreover $g^{G} \cap X=\emptyset$, and

$$
|G| /\left|C_{G}(g)\right|=\left|g^{G}\right| \leq|G \backslash X|
$$

Thus $n=|G| \leq 2|G \backslash X|^{2}$ and $\sqrt{\frac{n}{2}} \leq n-m$, whence

$$
\mu\left(x^{\ell}=1\right)=\frac{m}{n} \leq \frac{n-\sqrt{\frac{n}{2}}}{n}=1-\frac{1}{\sqrt{2 n}}
$$

Definition 2.10. Let $f: G \rightarrow H$ be a function, and $c \in H$. The equation $f(x)=c$ is $k$-largely satisfied in $G$ if $\{g \in G: f(g)=c\}$ is $k$-large in $G$. By abuse of notation, if $G=G_{1}^{n}$ and $x=\left(x_{1}, \ldots, x_{n}\right)$, we shall also say that $f\left(x_{1}, \ldots, x_{n}\right)=c$ is $k$-largely satisfied in $G_{1}$.

## 3. FC-Groups

In this section we shall work in the set-up of Example 1.2: $G_{1}$ will be a group, $G \leq G_{1}^{n}, w(\bar{x}, \bar{y})$ a word in $\bar{x} \bar{y}$ and their inverses with $n=|\bar{x}|$ and $m=|\bar{y}|, \bar{g} \in G_{1}^{m}$ and $c \in G_{1}$ constants, and $f(\bar{x})=w(\bar{x}, \bar{g})$.

Recall that a group is $F C$ if the centraliser of any element has finite index; it is $B F C$ if the index is bounded independently of the element.
We shall first need a preparatory lemma. For two tuples $\bar{g}=\left(g_{i}\right.$ : $i<k)$ and $\bar{g}^{\prime}=\left(g_{i}^{\prime}: i<k\right)$ in $G_{1}^{k}$ we shall put $\bar{g}^{-1}=\left(g_{i}^{-1}: i<k\right)$ and $\bar{g} \cdot \bar{g}^{\prime}=\left(g_{i} g_{i}^{\prime}: i<k\right)$.
Lemma 3.1. Suppose $\bar{g}, \bar{g}^{\prime} \in G_{1}^{m}$ and $\bar{h}, \bar{h}^{\prime} \in G_{1}^{n}$ are such that all elements from $\bar{g} \bar{h}$ commute with all elements from $\bar{g}^{\prime} \bar{h}^{\prime}$. Then

$$
w\left(\bar{h} \cdot \bar{h}^{\prime}, \bar{g} \cdot \bar{g}^{\prime}\right)=w(\bar{h}, \bar{g}) w\left(\bar{h}^{\prime}, \bar{g}^{\prime}\right)
$$

Proof. Obvious.
Theorem 3.2. Let $G_{1}$ be an FC-group. If the equation $w(\bar{x}, \bar{g})=c$ is largely satisfied in $G$ then it is identically satisfied in $G$.
Proof. Consider $\bar{h} \in G$, and $C=C_{G_{1}}(\bar{g}, \bar{h})$, a subgroup of finite index in $G_{1}$. Put $H=C^{n} \cap G$, a subgroup of finite index in $G$, and $X=$ $\left\{\bar{h}^{\prime} \in G: w\left(\bar{h}^{\prime}, \bar{g}\right)=c\right\}$. Then $X \cap \bar{h}^{-1} X \cap H$ is large in $H$, whence non-empty. So there is $\bar{x} \in H$ with

$$
w(\overline{1}, \bar{g}) w(\bar{x}, \overline{1})=w(\bar{x}, \bar{g})=c=w(\bar{h} \cdot \bar{x}, \bar{g})=w(\bar{h}, \bar{g}) w(\bar{x}, \overline{1}) .
$$

Hence $w(\bar{h}, \bar{g})=w(\overline{1}, \bar{g})$ for all $\bar{h} \in G$, and $w(\overline{1}, \bar{g})=w(\bar{x}, \bar{g})=c$.
For a $B F C$-group, we can bound the degree of largeness needed:
Theorem 3.3. Suppose every centraliser of a single element has index at most $k$ in $G_{1}$. If the equation $w(\bar{x}, \bar{g})=c$ is $2 k^{n^{2}+m n}$-largely satisfied in $G$ then it is identically satisfied in $G$.
Proof. In the notation of the previous proof, $C=C_{G_{1}}(\bar{g}, \bar{h})$ has index at most $k^{n+m}$ in $G_{1}$, so

$$
|G: H|=\left|G: G \cap C^{n}\right| \leq\left|G_{1}^{n}: C^{n}\right|=\left|G_{1}: C\right|^{n} \leq\left(k^{n+m}\right)^{n}=k^{n^{2}+m n}
$$

Now $2 k^{n^{2}+m n}$-largeness of $X$ in $G$ implies $k^{n^{2}+m n}$-largeness of $X \cap \bar{h}^{-1} X$ in $G$, whence 1-largeness of $X \cap \bar{h}^{-1} X \cap H$ in $H$. So we can find the $\bar{x}$ required to finish the proof.

Corollary 3.4. Suppose every centraliser of a single element has index at most $k$ in $G_{1}$. If $w(\bar{x}, \bar{g})=c$ is not an identity on $G$, then

$$
\mu_{*}(w(\bar{x}, \bar{g})=c) \leq 1-\frac{1}{2 k^{n^{2}+m n}}
$$

Proof. If $\mu_{*}(w(\bar{x}, \bar{g})=c)>1-\frac{1}{2 k^{n^{2}+m n}}$, then $\{\bar{x} \in G: w(\bar{x}, \bar{g})=c\}$ is $2 k^{n^{2}+m n}$-large in $G$ by Lemma 2.5, and $w(\bar{x}, \bar{g})=c$ is identically satisfied in $G$ by Theorem 3.3.
Remark 3.5. This holds in particular for the equation $x^{\ell}=c$, with $n=1$ and $m=0$.

If the group is central-by-finite, the largeness needed does not depend on the number of parameters.
Corollary 3.6. Suppose $Z\left(G_{1}\right)$ has index $k$ in $G_{1}$. If the equation $w(\bar{x}, \bar{g})=c$ is $2 k^{n}$-largely satisfied in $G$ then it is identically satisfied in $G$.

Proof. $H=G \cap Z\left(G_{1}\right)^{n}$ has index at most $k^{n}$ in $G$. We finish as in Theorem 3.3.

Corollary 3.7. If $\left|G_{1}: Z\left(G_{1}\right)\right| \leq k$ and $w(\bar{x}, \bar{g})=c$ is not an identity in $G$, then $\mu_{*}(w(\bar{x}, \bar{g})=1) \leq 1-\frac{1}{2 k^{n}}$.

Of course, for an abelian group $G_{1}$ we have $k=1$ in the above results.
Remark 3.8. If $w(\bar{x}, \bar{g})=c$ is 2-largely satisfied in $G^{n}$, then it is identically satisfied in the abelian quotient $G / G^{\prime}$. If moreover $G$ is a $B F C$-group, then $G^{\prime}$ is finite by B.H. Neumann's Lemma [15], and $G^{n}$ satisfies a finite disjunction $\bigvee_{c^{\prime} \in G^{\prime}} w(\bar{x}, \bar{g})=c c^{\prime}$.

We can also deduce results for central elements just from 2-largeness (although for infinite index $\left|G_{1}: Z\left(G_{1}\right)\right|$ there is no reason that if $X$ is large in $G$ the intersection $X \cap Z\left(G_{1}\right)^{n}$ is still large in $\left.G \cap Z\left(G_{1}\right)^{n}\right)$.
Theorem 3.9. If $w(\bar{x}, \bar{g})=c$ is 2-largely satisfied in $G$, then $w(\bar{x}, \overline{1})=$ 1 identically on $G \cap Z\left(G_{1}\right)^{n}$.
Proof. Consider $\bar{h} \in G \cap Z\left(G_{1}\right)^{n}$. Put $X=\left\{\bar{h}^{\prime} \in G: w\left(\bar{h}^{\prime}, \bar{g}\right)=1\right\}$. Then $X \cap \bar{h}^{-1} X$ is non-empty, so there is $\bar{x} \in G$ with

$$
w(\bar{x}, \bar{g})=c=w(\bar{h} \cdot \bar{x}, \bar{g})=w(\bar{h}, \overline{1}) w(\bar{x}, \bar{g}) .
$$

Hence $w(\bar{h}, \overline{1})=1$.
Corollary 3.10. If $x_{1}^{k_{1}} \cdots x_{n}^{k_{n}}=c$ is 2-largely satisfied in $G^{n}$ and $k=$ $\operatorname{gcd}\left(k_{1}, \ldots, k_{n}\right)$, then $x^{k}=1$ identically on $Z(G)$.
Proof. We have $x_{1}^{k_{1}} \cdots x_{n}^{k_{n}}=1$ on $Z(G)$. Putting $x_{i}=g \in Z(G)$ and $x_{j}=1$ for $j \neq i$ we have $g^{k_{i}}=1$ for all $1 \leq i \leq n$. The result follows.

Corollary 3.11. If the exponent of $Z(G)$ does not divide $\operatorname{gcd}\left(k_{1}, \ldots, k_{n}\right)$, then $\mu_{*}\left(x_{1}^{k_{1}} \cdots x_{n}^{k_{n}}=c\right) \leq \frac{1}{2}$.

## 4. Burnside and Engel Equations

In Remark 3.5 we have already seen that if every centraliser of a single element has index at most $k$ in $G$, then $\mu_{*}\left(x^{m}=c\right) \leq 1-\frac{1}{2 k}$ unless the exponent of $G$ divides $m$. In this case necessarily $c=x^{m}=1$.

We shall first prove Miller's Theorem mentioned in the introduction.
Theorem 4.1. Let $c \in G$. If $x^{2}=c$ is 4-largely satisfied in $G$, then $G$ is abelian of exponent 2 , and $c=1$.
Proof. Fix $g, h \in G$. Then there is $x$ with $c=x^{2}=(g x)^{2}=(h x)^{2}=$ $(g h x)^{2}$. But this implies $x^{-1} g x=g^{-1}, x^{-1} h x=h^{-1}$ and $x^{-1} g h x=$ $(g h)^{-1}$. On the other hand,

$$
x^{-1} g h x=x^{-1} g x x^{-1} h x=g^{-1} h^{-1}=(h g)^{-1} .
$$

Hence $g h=h g$ and $G$ is abelian. But now $c=x^{2}=(g x)^{2}=g^{2} x^{2}=g^{2} c$, whence $g^{2}=1$.

If $G$ satisfies 4-largely $x a x=b$ for some $a, b \in G$, then it satisfies 4-largely $(a x)^{2}=a b$, whence $x^{2}=a b$. Hence $G$ is abelian of exponent 2 , and $a=b$.
Corollary 4.2. If $G$ is not of exponent 2 or $a \neq b$, then $\mu_{*}(x a x=b) \leq$ $\frac{3}{4}$.

Recall that the $n^{\text {th }}$ Engel condition is the condition $\left[x,{ }_{n} y\right]=1$, where $[x, 1 y]=[x, y]$ and $\left[x,{ }_{n+1} y\right]=\left[\left[x,{ }_{n} y\right], y\right]$. Note that

$$
[x, y, y]=\left[y^{-x} y, y\right]=y^{-1} y^{x} y^{-1} y^{-x} y y=\left[y^{-x}, y\right]^{y}
$$

Thus the 2-Engel condition $[x, y, y]=1$ is equivalent to $\left[y^{-x}, y\right]=1$, that is all conjugacy classes being commutative.
Theorem 4.3. If $G$ satisfies 7-largely $x^{3}=1$ then $G$ is 2 -Engel.
Proof. Put $X=\left\{g \in G: g^{3}=1\right\}$. For $g, h \in G$ consider

$$
x \in X \cap g^{-1} X \cap h^{-1} X \cap g X \cap(g h)^{-1} X \cap g h^{-1} X \cap g h^{-1} g^{-1} X .
$$

Then $(y x)^{3}=1$ for $y \in\left\{1, g, h, g^{-1}, g h, h g^{-1}, g h g^{-1}\right\}$, which means that $x y x=y^{-1} x^{-1} y^{-1}$. We calculate the product $x h x^{2} g x$ in two ways:

$$
\begin{aligned}
x h x^{2} g x & =(x h x)(x g x)=h^{-1}\left(x^{-1} h^{-1} g^{-1} x^{-1}\right) g^{-1} \\
& =h^{-1} g h x g h g^{-1} \quad \text { and } \\
x h x^{2} g x & =x h\left(g^{-1} x\right)^{-1} x=x h\left(g^{-1} x\right)^{2} x=\left(x h g^{-1} x\right) g^{-1} x^{2} \\
& =g h^{-1}\left(x^{-1} g h^{-1} g^{-1} x^{-1}\right)=g h^{-1} g h g^{-1} x g h g^{-1} .
\end{aligned}
$$

Thus $h^{-1} g h=g h^{-1} g h g^{-1}$ and $g^{h} g=g g^{h}$. As $h \in G$ was arbitrary, the conjugacy class of $g$ is commutative; as $g$ was arbitrary, all conjugacy classes are commutative.

Theorem 4.4. Let $G$ be 2-Engel. If $G$ satisfies 2-largely $x^{3}=1$ then $G$ has exponent 3.
Proof. For any $g \in G$ there is $x \in G$ with $x^{3}=(g x)^{3}=1$. As $x^{G}$ is commutative,

$$
g^{x} g^{-1} g^{x^{-1}}=x^{-1} g x g^{-1} x g x^{-1}=g x^{-g} x x^{g} x^{-1}=g x^{-g} x^{g} x x^{-1}=g .
$$

Since $g^{G}$ is commutative, we have

$$
g^{3}=g^{2} g^{x} g^{-1} g^{x^{-1}}=g^{2} g^{-1} g^{x^{-1}} g^{x}=(g x)^{3}=1
$$

Corollary 4.5. If $G$ satisfies 7 -largely $x^{3}=1$, then $G$ has exponent 3. If $G$ is not of exponent 3 then $\mu_{*}\left(x^{3}=1\right) \leq \frac{6}{7}$. If moreover $G$ is 2 -Engel, then $\mu_{*}\left(x^{3}=1\right) \leq \frac{1}{2}$.

Note that the bound $\frac{6}{7}$ is not as good as Laffey's bound $\frac{7}{9}$ cited in the introduction.
Problem 4.6. A group which satisfies 5 -largely $x^{3}=1$, is it 2 -Engel? This would improve our bound to $\frac{4}{5}$.
Corollary 4.7. If $|G: Z(G)| \leq 7$ and $G$ satisfies 7 -largely $x^{3}=c$ for some $c \in G$, then $c=1$ and $G$ has exponent 3 .
Proof. $\left\{x \in G: x^{3}=c\right\} \cap Z(G)$ is 1-large, whence non-empty, and contains an element $z$. But now there is $x \in G$ with $x^{3}=1=(z x)^{3}=$ $z^{3} x^{3}=c x^{3}$, whence $c=1$. We finish by Corollary 4.5.

If $|G: Z(G)|$ is prime, then $G$ is abelian, and 2-largeness is sufficient by Corollary 3.10.

## 5. Commutator Equations

Consider the equation $[x, g]=c$ for some $c, g \in G$. Since $\{x \in G$ : $[x, g]=c\}$ is a coset of $C_{G}(g)$ or empty, and a coset of a proper subgroup cannot be 2-large, it follows that if $G$ satisfies 2-largely $[x, g]=c$ then $g \in Z(G)$ and $c=1$. The following argument generalises this result.
Theorem 5.1. Suppose $f: G \rightarrow H$ satisfies $f\left(x x^{\prime}\right)=f(x)^{h} f\left(x^{\prime}\right)$ for some $h \in H$ which depends on $x, x^{\prime} \in G$. If $G_{0}$ and $G_{1}$ are groups, $f_{0}: G_{0} \rightarrow H$ and $f_{1}: G_{1} \rightarrow H$ are functions such that $G_{0} \times G \times G_{1}$ satisfies $k$-largely $f_{0}\left(x_{0}\right) f(x) f_{1}\left(x_{1}\right)=c$ for some $k \geq 2$, then $f(G)=1$ and $G_{0} \times G_{1}$ satisfies $k$-largely $f_{0}\left(x_{0}\right) f_{1}\left(x_{1}\right)=c$.
Proof. Fix $g \in G$. By 2-largeness there is $\left(x_{0}, x, x_{1}\right) \in G_{0} \times G \times G_{1}$ such that

$$
f_{0}\left(x_{0}\right) f(x) f\left(x_{1}\right)=c=f_{0}\left(x_{0}\right) f(g x) f\left(x_{1}\right) .
$$

Thus $f(x)=f(g x)=f(g)^{h} f(x)$ and $f(g)=1$. It follows that $f_{0}\left(x_{0}\right) f(x) f_{1}\left(x_{1}\right)=f_{0}\left(x_{0}\right) f_{1}\left(x_{1}\right)$ on $G_{0} \times G \times G_{1}$. The result follows.

Corollary 5.2. If $G$ satisfies 2-largely $\prod_{i<n}\left[x_{i}, g_{i}\right]=c$ for some $g_{i} \in$ $G$, then $g_{i} \in Z(G)$ for all $i<n$ and $c=1$. If not all $g_{i}$ are central or $c \neq 1$ then $\mu_{*}\left(\prod_{i<n}\left[x_{i}, g_{i}\right]=c\right) \leq \frac{1}{2}$.
Proof. We have $\left[x x^{\prime}, y\right]=[x, y]^{x^{\prime}}\left[x^{\prime}, y\right]$. Now use Theorem 5.1.
Remark 5.3. Theorem 5.1 also holds if $f\left(x x^{\prime}\right)=f\left(x^{\prime}\right) f(x)^{h}$, with almost the same proof. Hence Corollary 5.2 also holds if some factors are of the form $\left[g_{i}, x_{i}\right]$.

Gustafson [6] has shown that $\mu_{2}([x, y]=1) \leq \frac{1}{2}\left(1+\mu(Z(G)) \leq \frac{5}{8}\right.$ for a non-abelian compact topological group $G$, where $\mu$ is the Haar measure on $G$ and $\mu_{2}$ the product measure on $G^{2}$. Pournaki and Sobhani $[17]$ have generalised this to calculate that $\mu([x, y]=g)<\frac{1}{2}$ for any $g \neq 1$ in a finite group, using Rusin's classification [18] of all finite groups with $\mu([x, y]=1)>\frac{11}{32}$ (see also [4]). We have only been able to establish results using 4-largeness, giving the bound of $\frac{3}{4}$ in Corollary 5.7, so the following two problems remain open:

Problem 5.4. (1) If $G$ satisfies 2-largely $[x, y]=1$, is $G^{\prime}=C_{2}$ and $G / Z(G)$ of exponent 2 , or $G^{\prime}=C_{3}$ and $G / Z(G)=S_{3}$ ?
(2) If $G$ satisfies 2-largely $[x, y]=c$ for some $c \in G$, is $c=1$ ?

Theorem 5.5. If $w(\bar{x}, \bar{g})[x, y]=c$ is satisfied 4-largely in $G^{n+1}$, where $x \in \bar{x}$ and $y \notin \bar{x}$, then $G$ is abelian and $w(\bar{x}, \bar{g})=c$.

Proof. For any $h \in G$ the set

$$
\{(\bar{x}, x, y): w(\bar{x}, \bar{g})[x, y]=c=w(\bar{x}, \bar{g})[x, h y]\}
$$

is 2-large in $G^{n+1}$. Hence $\left\{(x, y) \in G^{2}:[x, y]=[x, h y]\right\}$ is 2-large in $G^{2}$. Now $[x, h y]=[x, y][x, h]^{y}$, so $[x, h]=1$ is satisfied 2-largely in $G$, whence $h \in Z(G)$. It follows that $G$ is abelian. But then $w(\bar{x}, \bar{g})=c$ is satisfied 4-largely in $G^{n}$, and must be an identity in $G$ by commutativity and Corollary 3.6.
Corollary 5.6. If $G$ is a group with $\mu_{*}(w(\bar{x}, \bar{g})[x, y]=c)>\frac{3}{4}$, then $G$ is abelian satisfying $w(\bar{x}, \bar{g})=c$.
Corollary 5.7. If $G$ satisfies 4-largely $[x, y]=c$, then $G$ is abelian and $c=1$. If $G$ is not abelian or $c \neq 1$, then $\mu_{*}([x, y]=c) \leq \frac{3}{4}$.
Remark 5.8. The same holds for the equation $x c y=y c^{\prime} x$ with $c \neq c^{\prime}$ : putting $x^{\prime}=x c$ and $y^{\prime}=y c^{\prime}$, this is equivalent to $\left[x^{\prime}, y^{\prime}\right]=c^{-1} c^{\prime}$.

Theorem 5.9. Let $g, h \in G$ and $k=\min \left\{\left|G: C_{G}(g)\right|,\left|G: C_{G}(h)\right|\right\}$. If $G$ satisfies $k$-largely $\left[g, h^{x}\right]=1$, then $g^{G}$ and $h^{G}$ commute.

Proof. If $k=\left|G: C_{G}(h)\right|$, then $\left\{x \in G:\left[g, h^{x}\right]=1\right\} \cap C_{G}(h)$ is 1large, whence non-empty, and $[g, h]=1$. Now note that for any $a \in G$ also $\left|G: C_{G}\left(h^{a}\right)\right|=k$ and $\left[g, h^{a x}\right]=1$ is satisfied $k$-largely, whence $\left[g, h^{a}\right]=1$ and $\left[g, h^{G}\right]=1$.

If $k=\left|G: C_{G}(g)\right|$, then $\left\{x \in G:\left[g^{x^{-1}}, h\right]=1\right\} \cap C_{G}(g)$ is 1-large (still on the left) and non-empty, whence $[g, h]=1$ and we finish as above.

Corollary 5.10. If $\left[g^{G}, h^{G}\right]$ is non-trivial for some $g, h \in G$, then $\mu_{*}\left(\left[g, h^{x}\right]=1\right) \leq 1-\frac{1}{k}$, where $k=\min \left\{\left|G: C_{G}(g)\right|,\left|G: C_{G}(h)\right|\right\}$.
Theorem 5.11. If $g, h, c \in G$ and $[x, g, h]=c$ is $2 k$-largely satisfied, where $k=\left|G: C_{G}(h)\right|$, then $[G, g, h]=1$. Similarly, if $[g, x, h]=c$ is $2 k$-largely satisfied for some $c \in Z(G)$, then $[g, G, h]=1$.
Proof. Choose $a \in G$. Then the set $X=\{x \in G:[x, g, h]=c=$ [ $a x, g, h]\}$ is $k$-large, and for $x \in X$ we have

$$
[x, g, h]=c=[a x, g, h]=\left[[a, g]^{x}[x, g], h\right]=\left[[a, g]^{x}, h\right]^{[x, g]}[x, g, h],
$$

whence $\left[[a, g]^{x}, h\right]=1$. By Theorem 5.9 we have $[a, g, h]=1$.
If $[g, x, h]=c$ is $2 k$-largely satisfied with $c \in Z(G)$, then for $a \in G$ we obtain a $k$-large $X \subseteq G$ such that for $x \in X$ we have

$$
[g, x, h]=c=[g, a x, h]=\left[[g, x][g, a]^{x}, h\right]=[g, x, h]^{[g, a]^{x}}\left[[g, a]^{x}, h\right],
$$

whence $\left[[g, a]^{x}, h\right]=1$, and $[g, a, h]=1$ by Theorem 5.9.
Corollary 5.12. If $g, h \in G$ and $k=\left|G: C_{G}(h)\right|$, then $[G, g, h] \neq 1$ implies $\mu_{*}([x, g, h]=c) \leq 1-\frac{1}{2 k}$ for any $c \in G$, and $[g, G, h] \neq c$ implies $\mu_{*}([g, x, h]=c) \leq 1-\frac{1}{2 k}$ for any $c \in Z(G)$.

We shall now generalise Corollary 5.7 to higher nilpotency classes. However, the proof requires an additional assumption.

Theorem 5.13. Suppose $s<\omega$ is such that for all $i<k$ there is $a$ set $A_{i}$ of size at most $s$ such that $Z\left(G / Z_{i}(G)\right)=C_{G / Z_{i}(G)}\left(A_{i}\right)$. If $G$ satisfies $2(s+1)^{k}$-largely $\left[x_{0}, x_{1}, \ldots, x_{k}\right]=c$, then $c=1$ and $G$ is nilpotent of class at most $k$.
Proof. We use induction on $k$. For $k=1$ note that $s \geq 1$ (otherwise $G$ is abelian and we are done), so the result follows from Corollary 5.7.

Now suppose the assertion is true for $k$, and

$$
X=\left\{\bar{x} \in G^{k+2}:\left[x_{0}, x_{1}, \ldots, x_{k+1}\right]=c\right\}
$$

is $2(s+1)^{k+1}$-large in $G^{k+2}$. If $A_{0}=\left\{a_{i}: i<s\right\}$ consider the projection $Y$ of $X \cap \bigcap_{i<s}\left(1, \ldots, 1, a_{i}^{-1}\right) X$ to the first $k+1$ coordinates, and note that it is $2(s+1)^{k}$-large. Then for all $\left(x_{0}, \ldots, x_{k}\right) \in Y$ there is $y \in G$ such that

$$
\left[x_{0}, \ldots, x_{k}, y\right]=c=\left[x_{0}, \ldots, x_{k}, a_{i} y\right]=\left[x_{0}, \ldots, x_{k}, y\right]\left[x_{0}, \ldots, x_{k}, a_{i}\right]^{y}
$$

for all $i<s$, whence $\left[x_{0}, \ldots, x_{k}\right] \in Z(G)$. By inductive assumption $G / Z(G)$ is nilpotent of class at most $k$, and we are done.

Corollary 5.14. Let $s$ be as above. If $G$ is not nilpotent of class at most $k$ or $c \neq 1$, then $\mu_{*}\left(\left[x_{0}, x_{1}, \ldots, x_{k}\right]=c\right) \leq 1-\frac{1}{2}(s+1)^{-k}$.
Remark 5.15. Recall that an $\mathfrak{M c}_{\mathrm{c}}$-group is a group $G$ such that for every subset $A$ there is a finite subset $A_{0} \subseteq A$ such that $C_{G}(A)=$ $C_{G}\left(A_{0}\right)$. Equivalently, $G$ satisfies the ascending (or the descending) chain condition on centralisers. Roger Bryant [2] has shown that in an $\mathfrak{M c}$ c-group, for every iterated centre $Z_{i}(G)$ there is a finite set $A_{i}$ such that $Z\left(G / Z_{i}(G)\right)=C_{G / Z_{i}(G)}\left(A_{i}\right)$. So in an $\mathfrak{M c}_{\text {c-group we can find some }}$ $s$ as needed for Theorem 5.13 and Corollary 5.14.

Problem 5.16. To what extent do we need the $\mathfrak{M c}$ c-condition (or similar) in Theorem 5.13 and Corollary 5.13? It is not needed for nilpotency class 1 (Corollary 5.7). In general, assuming just $2^{k+1}$-largeness of $\left[x_{0}, \ldots, x_{k}\right]=c$, we obtain that $\left\{\bar{x} \in G^{k}:\left[x_{0}, \ldots, x_{k-1}\right] \in C_{G}(g)\right\}$ is $2^{k}$-large in $G^{k}$ for any $g \in G$. Does this imply $\gamma_{k}(G) \leq C_{G}(g)$, or even $\gamma_{k}(G) \leq Z(G)$ ?

## 6. Nilpotent groups

We shall first introduce the notion of a supercommutator from [9].
Definition 6.1. Any variable and any constant from $G$ is a supercommutator; if $v$ and $w$ are supercommutators, then $v^{-1}$ and $[v, w]$ are supercommutators.

Alternatively, we could have said that $x, x^{-1}$ and $g$ are supercommutators for any variable $x$ and any $g \in G$, and that if $v$ and $w$ are supercommutators, so is $[v, w]$.
Definition 6.2. The set $\operatorname{Var}(v)$ of variables of a supercommutator $v$ is defined by $\operatorname{Var}(x)=\{x\}, \operatorname{Var}(g)=\emptyset, \operatorname{Var}\left(v^{-1}\right)=\operatorname{Var}(v)$, and $\operatorname{Var}([v, w]=\operatorname{Var}(v) \cup \operatorname{Var}(w)$. We put $\operatorname{var}(v)=|\operatorname{Var}(v)|$, the variable number of $v$. If $\bar{x}$ is a tuple of variables, we put $\operatorname{Var}_{\bar{x}}=\operatorname{Var}(v) \cap \bar{x}$, $\operatorname{Var}_{\bar{x}}^{\prime}(v)=\operatorname{Var}(v) \backslash \bar{x}, \operatorname{var}_{\bar{x}}(v)=\left|\operatorname{Var}_{\bar{x}}(v)\right|$ and $\operatorname{var}_{\bar{x}}^{\prime}(v)=\left|\operatorname{Var}_{\bar{x}}^{\prime}(v)\right|$.

Clearly $\operatorname{var}\left(\left[v, v^{\prime}\right]\right) \geq \max \left\{\operatorname{var}(v), \operatorname{var}\left(v^{\prime}\right)\right\}$, and similarly for $\operatorname{var}_{\bar{x}}$ and $\operatorname{var}_{\bar{x}}^{\prime}$.

Lemma 6.3. Let $H \unlhd G$ and $v(\bar{x}, \bar{z})$ a supercommutator.
(1) $v$ defines a function from $H^{|\bar{x} \bar{z}|}$ to $\gamma_{\operatorname{var}(v)}(H)$.
(2) If $\operatorname{var}_{\bar{x}}(v)>0$ and $\bar{x}, \bar{y}$ and $\bar{z}$ are pairwise disjoint, then

$$
v(\bar{y} \cdot \bar{x}, \bar{z})=v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z})
$$

where $\Phi$ is a product of supercommutators whose factors $w$ satisfy
$(\dagger) \operatorname{Var}_{\bar{z}}(w)=\operatorname{Var}_{\bar{z}}(v)$, and if $x_{i} \in \operatorname{Var}_{\bar{x}}(v)$ then $x_{i} \in \operatorname{Var}(w)$ or $y_{i} \in \operatorname{Var}(w)$, and both possibilities occur for at least one $i$.
(3) If $v(\bar{x}, \bar{z})$ is a product of supercommutators whose factors $w$ satisfy $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime}(w) \geq n$, then

$$
v(\bar{y} \cdot \bar{x}, \bar{z})=v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}),
$$

where $\Phi$ is a product of supercommutators whose factors $w$ satisfy $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime}(w)>n$.

Proof. (1) is proved as in [9, Lemme 6(1)] by induction, using that $\gamma_{n}(H)$ is characteristic in $H$, whence normal in $G$, and $\left[\gamma_{n}(H), \gamma_{m}(H)\right] \leq$ $\gamma_{n+m}(H)$. We shall show (2) by induction on the construction of $v$.

If $v=x \in \bar{x}$ we have $v(y x)=y x=x y[y, x]=v(x) v(y)[y, x]$; if $v=x^{-1}$ we have $v(y x)=x^{-1} y^{-1}=v(x) v(y)$. This leaves the case $v=\left[v_{1}, v_{2}\right]$ for two supercommutators $v_{1}$ and $v_{2}$. We shall assume $\operatorname{var}_{\bar{x}}\left(v_{1}\right)>0$ and $\operatorname{var}_{\bar{x}}\left(v_{2}\right)>0$ (the case $\operatorname{var}_{\bar{x}}\left(v_{1}\right) \operatorname{var}_{\bar{x}}\left(v_{2}\right)=0$ is analogous, but simpler).

By inductive hypothesis, there are $\Phi_{i}$ for $i=1,2$, products of supercommutators satisfying $(\dagger)$ relative to $v_{i}$, such that

$$
v_{i}(\bar{y} \cdot \bar{x}, \bar{z})=v_{i}(\bar{x}, \bar{z}) v_{i}(\bar{y}, \bar{z}) \Phi_{i}
$$

Then

$$
\begin{aligned}
v(\bar{y} \cdot \bar{x}, \bar{z}) & =\left[v_{1}(\bar{y} \cdot \bar{x}, \bar{z}), v_{2}(\bar{y} \cdot \bar{x}, \bar{z})\right] \\
& =\left[v_{1}(\bar{x}, \bar{z}) v_{1}(\bar{y}, \bar{z}) \Phi_{1}, v_{2}(\bar{x}, \bar{z}) v_{2}(\bar{y}, \bar{z}) \Phi_{2}\right] \\
& =\left[v_{1}(\bar{x}, \bar{z}), v_{2}(\bar{x}, \bar{z})\right]\left[v_{1}(\bar{y}, \bar{z}), v_{2}(\bar{y}, \bar{z})\right] \Phi=v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi,
\end{aligned}
$$

where $\Phi$ is a product of supercommutators $\left[w, w^{\prime}\right]$
(i) where $w \in \Phi_{1} \cup\left\{v_{1}(\bar{x}, \bar{z}), v_{1}(\bar{y}, \bar{z})\right\}$ and $w^{\prime} \in \Phi_{2} \cup\left\{v_{2}(\bar{x}, \bar{z}), v_{2}(\bar{y}, \bar{z})\right\}$, except for $\left[v_{1}(\bar{x}, \bar{z}), v_{2}(\bar{x}, \bar{z})\right]$ and $\left[v_{1}(\bar{y}, \bar{z}), v_{2}(\bar{y}, \bar{z})\right]$; it is clear that these must satisfy $(\dagger)$.
(ii) where one of $w, w^{\prime}$ is from (i), so $\left[w, w^{\prime}\right]$ satisfies ( $\dagger$ ).
(iii) where one of $w, w^{\prime}$ is equal to $v(\bar{x}, \bar{z})$ and the other contains at least one $y_{i}$, or one is equal to $v(\bar{y}, \bar{z})$ and the other contains at least one $x_{i}$; again $\left[w, w^{\prime}\right]$ satisfies $(\dagger)$.
(iv) which are obtained iteratively from supercommutators from (ii) and (iii) by commutation with other supercommutators, thus satisfying $(\dagger)$.
Here (i) takes care of the commutators of various factors of the two products, while (ii)-(iv) takes care of the correct order. Note that the only factor without a variable $y_{i}$ is $v(\bar{x}, \bar{z})$ and the only factor without a variable $x_{j}$ is $v(\bar{y}, \bar{z})$.

To show (3) note first that for a single supercommutator $v$ the factorisation given in (2) satisfies the requirement. So for a product of supercommutators, we apply (2) to every factor, and then use commutators to get them into the right order. Note that we never have to commute a $w(\bar{x}, \bar{z})$ with a $w^{\prime}(\bar{x}, \bar{z})$, or a $w(\bar{y}, \bar{z})$ with a $w^{\prime}(\bar{y}, \bar{z})$, as they already appear in the correct order with respect to one another. It follows that all new commutators satisfy $(\dagger)$, whence $\operatorname{var}_{\bar{x}}^{\prime}>n$.
Theorem 6.4. If $G$ is nilpotent of class $k$ and $v$ is a product of supercommutators $w$ with $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime}(w) \geq n$ such that $G$ satisfies $\max \left\{2^{k-n}, 1\right\}$-largely $v(\bar{x}, \bar{g})=c$, then $c=1$.
Proof. This is true for $n \geq k$, as then $\operatorname{var}(w)=\operatorname{var}_{\bar{x}}(w)+\operatorname{var}_{\bar{x}}^{\prime}(w) \geq$ $1+n$, and

$$
c=w(\bar{x}, \bar{g}) \in \gamma_{\operatorname{var}(w)} G \leq \gamma_{n+1} G=\{1\}
$$

for some $\bar{x} \in G$.
Now suppose it is true for $n+1 \leq k$, and let $v(\bar{x}, \bar{z})$ be a product of supercommutators $w$ with $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime} \geq n$, such that $H$ satisfies $2^{k-n}$-largely $v(\bar{x}, \bar{g})=c$. By Lemma 6.3 there is $\Phi$, a product of supercommutators whose factors $w$ satisfy $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime}(w)>$ $n$, such that

$$
v(\bar{y} \cdot \bar{x}, \bar{z})=v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z}) .
$$

Choose $\bar{h} \in G$ with $v(\bar{h}, \bar{g})=c$. If $X=\{\bar{x} \in G: v(\bar{x}, \bar{g})=c\}$, then $X$ is $2^{k-n}$-large, and $Y=X \cap \bar{h}^{-1} X$ is $2^{k-n-1}$-large. Moreover, for $\bar{x} \in Y$ we have

$$
\Phi(\bar{x}, \bar{h}, \bar{g})=v(\bar{h}, \bar{g})^{-1} v(\bar{x}, \bar{g})^{-1} v(\bar{h} \cdot \bar{x}, \bar{g})=c^{-1} c^{-1} c=c^{-1} .
$$

By hypothesis $c^{-1}=1$ and we are done.
Theorem 6.5. If $G$ is nilpotent of class $k$ and satisfies $2^{k}$-largely an equation $v(\bar{x}, \bar{g})=c$, then it satisfies $v(\bar{x}, \bar{g})=c$.
Proof. Bringing all the constants to the right-hand side, we may assume that $v(\bar{x}, \bar{z})$ is a product of supercommutators $w$ with $\operatorname{var}_{\bar{x}}(w)>0$. By Lemma 6.3 there is $\Phi$, a product of supercommutators whose factors $w$ satisfy $\operatorname{var}_{\bar{x}}(w)>0$ and $\operatorname{var}_{\bar{x}}^{\prime}(w)>0$, such that

$$
v(\bar{y} \cdot \bar{x}, \bar{z})=v(\bar{x}, \bar{z}) v(\bar{y}, \bar{z}) \Phi(\bar{x}, \bar{y}, \bar{z})
$$

Fix $\bar{h} \in G$. Then

$$
\Phi(\bar{x}, \bar{h}, \bar{g})=v(\bar{h}, \bar{g})^{-1} c^{-1} c=v(\bar{h}, \bar{g})^{-1}
$$

$2^{k-1}$-largely on $G$. By Theorem 6.4 we have $v(\bar{h}, \bar{g})=1$. So $v(\bar{x}, \bar{g})$ is constant.
Corollary 6.6. If $G$ is nilpotent of class $k$ and $x^{n}=c$ is true $2^{k}$-largely, then $c=1$ and the exponent of $G$ divides $n$.
Proof. Immediate from Theorem 6.5.
Corollary 6.7. If $G$ is nilpotent of class $k$ and $\mu_{*}\left(x^{n}=c\right)>1-2^{-k}$, then $c=1$ and the exponent of $G$ divides $n$.

## 7. Autocommutativity

The notion of autocommutativity has been introduced by Sherman in 1975 [19].
Definition 7.1. Let $G$ be a finite group, $\Sigma$ a group of automorphisms of $G$, and $H$ a subgroup of $G$. The degree of autocommutativity relative to $(H ; \Sigma)$ is given by

$$
\operatorname{ac}(H ; \Sigma)=\frac{|\{(\sigma, g) \in \Sigma \times H: \sigma(g)=g\}|}{|\Sigma| \cdot|H|} .
$$

It gives the probability that a random element of $H$ is fixed by a random automorphism in $\Sigma$.

Note that $\operatorname{ac}(H ; \Sigma)=\mu(\{(\sigma, g) \in \Sigma \times H: \sigma(g)=g\})$, where $\mu$ is the counting measure on $\Sigma \times H$.
Theorem 7.2. Let $H \leq G$ be finite groups, $\Sigma$ a group of automorphisms of $G$, and suppose that $\{(\sigma, g) \in \Sigma \times H: \sigma(g)=g\}$ is 4-large in $\Sigma \times H$. Then $H \leq \operatorname{Fix}(\Sigma)$.
Proof. Given $\sigma \in \Sigma$ and $g \in H$, by 4-largeness there are $x \in H$ and $\tau \in \Sigma$ with

$$
\tau(x)=x, \quad(\sigma \circ \tau)(x)=x, \quad \tau(g x)=g x \quad \text { and } \quad(\sigma \circ \tau)(g x)=g x .
$$

Then

$$
g x=\sigma(\tau(g x))=\sigma(g x)=\sigma(g) \sigma(x)=\sigma(g) \sigma(\tau(x))=\sigma(g) x,
$$

whence $g=\sigma(g)$.
Corollary 7.3. If $H \leq G$ are finite groups and $\Sigma$ is a group of automorphisms of $G$ with $H \not \leq$ Fix $(\Sigma)$, then $\operatorname{ac}(H ; \Sigma) \leq \frac{3}{4}$.
Proof. If $\operatorname{ac}(H ; \Sigma)>\frac{3}{4}$ then $\{(\sigma, g) \in \Sigma \times H: \sigma(g)=g\}$ is 4-large in $\Sigma \times H$ by Lemma 2.5. Hence $H \leq \operatorname{Fix}(\Sigma)$ by Theorem 7.2.

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