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# Continuous-Time Gain-Scheduled $H_\infty$ Controllers with Causality for Scheduling Parameters via Parameter-Dependent Lyapunov Functions\*

Masayuki Sato<sup>1</sup> and Dimitri Peaucelle<sup>2</sup>

## Abstract

When continuous-time Gain-Scheduled Output Feedback (GSOF) controllers are designed via Parameter-Dependent Lyapunov Functions (PDLFs), the GSOF controllers, in general, depend on not only the current scheduling parameters but also the derivatives of scheduling parameters. However, it is obvious that the derivatives of scheduling parameters are not available in real world; that is, the designed GSOF controllers cannot be implemented to practical systems. We therefore address the design problem of GSOF controllers via PDLFs while causality of scheduling parameters in GSOF controllers is kept. For this problem, we propose a new formulation in which the causality of scheduling parameters is kept by over-bounding the term causing the causality problem. It is also shown that our method is no more conservative than an existing method in the literature. A numerical example is included to demonstrate the usefulness and effectiveness of our method.

## I. INTRODUCTION

Gain-Scheduled (GS) controllers have been applied to various systems and their usefulness has consequently been well recognized. The classical GS controllers are designed by a rule of thumb; that is, several design points are first selected and Linear Time-Invariant (LTI) plant systems are obtained at those selected design points, then corresponding LTI controllers are designed for the LTI plant systems, and the satisfaction of design requirements is examined by a lot of simulations, such as, Monte-Carlo simulations, with a GS controller composed of the interpolated LTI controllers. This method surely works well, e.g. flight controller design for a missile [1], quad tilt wing unmanned aerial vehicle [2], F-16 [3], etc. (Hidden coupling terms should be sometimes addressed properly.) However, this methodology is not so practical if the satisfaction of design requirements fails at the final examination step. This is because it is not clear which step we should go back for the redesign of controllers.

After the proposition of GS Output Feedback (GSOF) controller design with Linear Matrix Inequality (LMI) in [4], many researchers have addressed GSOF controller design problem, e.g. [5]–[7], etc. The recent developments with respect to theoretical improvement and application examples are well summarized in a survey paper [8] and books [9]–[11]. Most of these recent methods address GS controller design for Linear Parameter-Varying (LPV) systems via Parameter-Dependent Lyapunov Functions (PDLFs) to reduce conservatism when the scheduling parameter variations are bounded.

Regarding the design method of GSOF controllers for LPV systems via PDLFs, in general, continuous-time GSOF controllers depend on the current scheduling parameters as well as the derivatives of scheduling parameters [12], [13]; that is, causality of scheduling parameters in GSOF controllers is broken. This property prevents the implementation of the designed GSOF controllers, because the ideal derivatives are not available in real world. Several methods tackling this issue have already been proposed, i.e. the use of structurally constrained PDLFs [12], [13], incorporation of filters for scheduling parameters [14], and over-bounding the term containing the derivatives of scheduling parameters [15]–[17]. The design method using structurally constrained PDLFs in [12], [13] is not so recommended due to the increase of conservatism, which has been illustrated with a numerical example in [16], [17]. The method in [14] is attractive, because the designed GSOF controllers do not depend on the derivatives of scheduling parameters and the guaranteed  $L_2$  performance asymptotically tends to the best achievable (but not causal for scheduling parameters) performance via PDLFs in [12], [13]. However, the implemented GSOF controllers become slightly complicated due to the filters for scheduling parameters. On the other hand, GSOF controllers designed by the method in [15]–[17] have no requirement of additional systems like filters, while they depend only on the current scheduling parameters. However, the adopted over-bounding method is merely a sufficient condition for the original problem, which leads to a large gap between the guaranteed  $L_2$  performance and the best achievable performance via PDLFs in [12], [13].

From the research background above, we propose a design method of continuous-time GSOF controllers via PDLFs with a new formulation for the over-bounding of the problematic term. This is an extension of the method in [16], [17]. The new formulation is obtained by use of Elimination lemma [18]–[20]. As a consequence, our new method is no more conservative than the method in [16], [17], which is also proved in this paper.

The remainder of this paper is as follows: Section II gives the preliminaries for our proposition; Section III shows our proposed method and the relation between our proposition and the method in [16], [17]; and conclusions are given in Section IV.

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We use the following notations:  $\mathbf{I}$  and  $\mathbf{0}$  respectively denote an identity matrix and a zero matrix of appropriate dimensions,  $\mathbb{R}_+$ ,  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times m}$  and  $\mathbb{S}_+^n$  respectively denote the sets of positive real numbers,  $n$ -dimensional real vectors,  $n \times m$ -dimensional real matrices and  $n \times n$ -dimensional positive definite real matrices,  $X^T$  for matrix  $X$  denotes the transpose of matrix  $X$ ,  $*$  denotes an abbreviated off-diagonal block in a symmetric matrix,  $\text{He}\{X\}$  denotes  $X + X^T$ , and  $\text{diag}(X_1, \dots, X_k)$  denotes a block-diagonal matrix composed of  $X_1, \dots, X_k$ . For a parameter-dependent matrix  $X(\theta)$ ,  $\dot{X}(\theta)$  denotes  $\sum_i \dot{\theta}_i \frac{\partial X(\theta)}{\partial \theta_i}$ .

## II. PRELIMINARIES

### A. Plant Definition

Let us consider the following continuous-time LPV plant.

$$G(\theta) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\theta) & B_1(\theta) & B_2(\theta) \\ C_1(\theta) & D_{11}(\theta) & D_{12}(\theta) \\ C_2(\theta) & D_{21}(\theta) & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^{n_w}$ ,  $u \in \mathbb{R}^{n_u}$ ,  $z \in \mathbb{R}^{n_z}$  and  $y \in \mathbb{R}^{n_y}$  respectively denote the state with its initial value being zero, the external input, the control input, the performance output and the measurement output, and matrices  $A(\theta)$ , etc. are supposed to have compatible dimensions. The vector  $\theta = [\theta_1 \ \theta_2 \ \dots \ \theta_q]$  represents the scheduling parameters whose values are available in real time. The scheduling parameters  $\theta_i$  and their derivatives  $\dot{\theta}_i$  are both supposed to lie in bounded intervals. Thus, the following holds with *a priori* defined convex polytope  $\Lambda_\theta$ .

$$(\theta, \dot{\theta}) \in \Lambda_\theta \quad (2)$$

### B. Controller Definition

For  $G(\theta)$ , we would like to design a full-order GSOF controller defined below.

$$K(\theta) : \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} A_K(\theta) & B_K(\theta) \\ C_K(\theta) & D_K(\theta) \end{bmatrix}}_{\bar{K}(\theta)} \begin{bmatrix} x_K \\ y \end{bmatrix}, \quad (3)$$

where  $x_K \in \mathbb{R}^n$  denotes the state with its initial value being zero, and matrices  $A_K(\theta)$ , etc. are supposed to have compatible dimensions.

Note that the controller  $K(\theta)$  is required to depend only on  $\theta$ ; that is, the controller should keep causality with respect to scheduling parameters.

*Remark 1:* If the state-space matrices depend on not only the scheduling parameters but also the derivatives of the scheduling parameters as in [12], [13], then the controller and its state-space matrices are denoted by  $K(\theta, \dot{\theta})$  and  $\bar{K}(\theta, \dot{\theta})$ , respectively, to clearly indicate the dependence on  $\dot{\theta}$ .  $\square$

### C. Problem Definition

The closed-loop system comprising  $G(\theta)$  and  $K(\theta)$  is given below.

$$G_{cl}(\theta) : \begin{bmatrix} \dot{x}_{cl} \\ z \end{bmatrix} = \begin{bmatrix} A_{cl}(\theta) & B_{cl}(\theta) \\ C_{cl}(\theta) & D_{cl}(\theta) \end{bmatrix} \begin{bmatrix} x_{cl} \\ w \end{bmatrix}, \quad (4)$$

where  $x_{cl} = [x^T \ x_K^T]^T$  denotes the state, and matrices  $A_{cl}(\theta)$ , etc. are straightforwardly calculated as follows:

$$\begin{aligned} \begin{bmatrix} A_{cl}(\theta) & B_{cl}(\theta) \\ C_{cl}(\theta) & D_{cl}(\theta) \end{bmatrix} &= \begin{bmatrix} A(\theta) & \mathbf{0} & B_1(\theta) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ C_1(\theta) & \mathbf{0} & D_{11}(\theta) \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} & B_2(\theta) \\ -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -D_{12}(\theta) \end{bmatrix} \bar{K}(\theta) \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ C_2(\theta) & \mathbf{0} & D_{21}(\theta) \end{bmatrix}. \end{aligned}$$

*Remark 2:* The block diagram of  $G_{cl}(\theta)$  is shown in Fig. 1. On the contrary, if the state-space matrices of the closed-loop system depend on not only the scheduling parameters but also the derivatives of the scheduling parameters, then the closed-loop system is denoted by  $G_{cl}(\theta, \dot{\theta})$  to clearly indicate the dependence on  $\dot{\theta}$ , as shown in Fig. 2.  $\square$

We now define our problem.

*Problem 1:* For given scalar  $\gamma_\infty \in \mathbb{R}_+$ , design a continuous-time GSOF controller  $K(\theta)$  defined in (3) such that the closed-loop system  $G_{cl}(\theta)$  is stabilized and (5) is satisfied for all admissible trajectories  $(\theta, \dot{\theta}) \in \Lambda_\theta$ .

$$\|z\|_2^2 < \gamma_\infty^2 \|w\|_2^2 \quad (5)$$

#### D. Existing Method

By following the same parametrization for Lyapunov matrix in [21], the following design method is well known via PDLFs.

*Lemma 1:* C.f. [21] For given scalar  $\gamma_\infty \in \mathbb{R}_+$ , if there exist continuously differentiable parameter-dependent matrices  $\mathcal{X}(\theta), \mathcal{Z}(\theta) \in \mathbb{S}_+^n$  and a parameter-dependent matrix  $\mathcal{K}(\theta) \in \mathbb{R}^{(n+n_u) \times (n+n_y)}$  such that (6) and (7) hold for all  $(\theta, \dot{\theta}) \in \Lambda_\theta$ , then, the controller  $K(\theta, \dot{\theta})$ , whose state-space matrices are given as  $\bar{K}(\theta, \dot{\theta})$  in (10) at the top of the next page with  $\mathcal{Y}(\theta) = \mathcal{X}(\theta) - \mathcal{Z}(\theta)^{-1}$ , stabilizes the closed-loop system  $G_{cl}(\theta, \dot{\theta})$  and satisfies (5) for all admissible trajectories  $(\theta, \dot{\theta}) \in \Lambda_\theta$ .

$$\begin{bmatrix} \mathcal{X}(\theta) & \mathbf{I} \\ \mathbf{I} & \mathcal{Z}(\theta) \end{bmatrix} \succ 0, \quad (6)$$

$$\Upsilon(\theta, \dot{\theta}) \prec 0, \quad (7)$$

where  $\Upsilon(\theta, \dot{\theta})$  is defined as

$$\Upsilon(\theta, \dot{\theta}) = \begin{bmatrix} \text{He}\{\Upsilon_A(\theta)\} + \begin{bmatrix} -\dot{\mathcal{X}}(\theta) & \mathbf{0} \\ \mathbf{0} & \dot{\mathcal{Z}}(\theta) \end{bmatrix} & * & \Upsilon_B(\theta) \\ \Upsilon_C(\theta) & -\gamma_\infty \mathbf{I} & \Upsilon_D(\theta) \\ * & * & -\gamma_\infty \mathbf{I} \end{bmatrix}, \quad (8)$$

with matrices  $\Upsilon_A(\theta)$ , etc. being defined in (9) at the top of the next page.  $\square$

*Remark 3:* The formulation above uses  $x_{cl}^T X_{cl}(\theta)^{-1} x_{cl}$  as a candidate of PDLFs. Here, the parameter-dependent matrix  $X_{cl}$  is defined as follows:

$$X_{cl}(\theta) = \begin{bmatrix} \mathcal{X}(\theta) & \mathcal{Y}(\theta) \\ \mathcal{Y}(\theta) & \mathcal{Y}(\theta) \end{bmatrix}. \quad (11)$$

The formulation in Lemma 1 can be also derived from the result in [12], [13] with  $N(\theta) = -\mathcal{X}(\theta)$  and  $M(\theta) = \mathcal{Y}(\theta) - \mathcal{X}(\theta)^{-1}$ .  $\square$

As illustrated in (10), the transition matrix of the GSOF controller depends on  $\theta$  as well as  $\dot{\theta}$ . To circumvent this impracticality, the following method has been proposed in [16].

*Lemma 2:* For given scalar  $\gamma_\infty \in \mathbb{R}_+$ , if there exist a scalar  $\varepsilon \in \mathbb{R}_+$ , continuously differentiable parameter-dependent matrices  $\mathcal{X}(\theta), \mathcal{Z}(\theta) \in \mathbb{S}_+^n$  and a parameter-dependent matrix  $\mathcal{K}(\theta) \in \mathbb{R}^{(n+n_u) \times (n+n_y)}$  such that (6), and (12) or (13) hold for all  $(\theta, \dot{\theta}) \in \Lambda_\theta$ , then, the controller  $K(\theta)$ , whose state-space matrices are given as  $\bar{K}(\theta)$  in (14) at the top of the next page with  $\mathcal{Y}(\theta) = \mathcal{X}(\theta) - \mathcal{Z}(\theta)^{-1}$ , stabilizes the closed-loop system  $G_{cl}(\theta)$  and satisfies (5) for all admissible trajectories  $(\theta, \dot{\theta}) \in \Lambda_\theta$ .

$$\begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \varepsilon \mathbf{I} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ * & -\varepsilon \mathcal{Z}(\theta) \end{bmatrix} \prec 0, \quad (12)$$

$$\begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \varepsilon \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ * & -\begin{bmatrix} \varepsilon \mathcal{Z}(\theta) & \mathbf{0} \\ \mathbf{0} & \varepsilon \mathcal{Z}(\theta) \end{bmatrix} \end{bmatrix} \prec 0, \quad (13)$$

where  $\Upsilon(\theta, \dot{\theta})$  in both inequalities has the same definition in Lemma 1, i.e. (8).  $\square$

Although the formulations (12) and (13) contain the multiplications of  $\varepsilon$  and  $\mathcal{Z}(\theta)$ , viz., they are not LMIs but LMIs with a single line search parameter  $\varepsilon$ , they produce a practical GSOF controller  $K(\theta)$  which do not depend on the derivatives of scheduling parameters.

A brief explanation for removing  $\dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1}$  from  $\bar{K}(\theta, \dot{\theta})$  in Lemma 2 is given below. Inequality (12) is equivalent to

the following two inequalities.

$$\begin{cases} \Upsilon(\theta, \dot{\theta}) + \text{diag} \left( \begin{bmatrix} \mathbf{0} & * \\ \dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1} & \mathbf{0} \end{bmatrix}, \mathbf{0}, \mathbf{0} \right) \\ \prec -\text{diag} \left( \begin{bmatrix} \varepsilon\mathcal{Z}(\theta)^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{\dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1}\dot{\mathcal{Z}}(\theta)}{\varepsilon} \end{bmatrix}, \mathbf{0}, \mathbf{0} \right) \preceq 0 \\ \varepsilon\mathcal{Z}(\theta) \succ 0 \end{cases}$$

Similarly, inequality (13) is equivalent to the following two inequalities.

$$\begin{cases} \Upsilon(\theta, \dot{\theta}) + \text{diag} \left( \begin{bmatrix} \mathbf{0} & * \\ \dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1} & \mathbf{0} \end{bmatrix}, \mathbf{0}, \mathbf{0} \right) \\ \prec -\text{diag} \left( \begin{bmatrix} \varepsilon\mathbf{I} \\ -\dot{\mathcal{Z}}(\theta) \end{bmatrix} \frac{\mathcal{Z}(\theta)^{-1}}{\varepsilon} \begin{bmatrix} \varepsilon\mathbf{I} & -\dot{\mathcal{Z}}(\theta) \end{bmatrix}, \mathbf{0}, \mathbf{0} \right) \preceq 0 \\ \varepsilon\mathcal{Z}(\theta) \succ 0 \end{cases}$$

Thus, the additional columns and rows in (12) and (13) produce  $\dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1}$ . This production removes  $\dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1}$  from the change-of-variables in (10) and consequently leads to  $\bar{K}(\theta)$  in (14).

Two formulations using (12) and (13) are both sufficient conditions for our problem. Although a numerical example in [16] demonstrates that the formulation (13) is slightly less conservative than the formulation (12), it has not been clarified which method is theoretically less conservative between the two formulations.

In the next section, we propose another formulation which is no more conservative than the two formulations in Lemma 2.

### III. MAIN RESULTS

We first show our formulation via Elimination lemma [18]–[20], and then show that our formulation is no more conservative than the two formulations in Lemma 2.

#### A. Proposed Method

Let us consider the following inequality:

$$\begin{aligned} & \begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \mathbf{0} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ * & \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \end{bmatrix} \\ & + \text{He} \left\{ \begin{bmatrix} \mathcal{Z}(\theta)^{-1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \end{bmatrix} R(\theta, \dot{\theta}) \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \right\}^T \prec 0, \end{aligned} \quad (15)$$

where  $\Upsilon(\theta, \dot{\theta})$  has the same definition in Lemmas 1 and 2, and  $R(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$  is a matrix to be designed.

By applying Elimination lemma, the existence of  $R(\theta, \dot{\theta})$  satisfying inequality (15) is equivalent to the satisfaction of the following two inequalities:

$$\begin{cases} \Upsilon(\theta, \dot{\theta}) \prec 0 \\ \Upsilon(\theta, \dot{\theta}) + \text{diag} \left( \begin{bmatrix} \mathbf{0} & * \\ \dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1} & \mathbf{0} \end{bmatrix}, \mathbf{0}, \mathbf{0} \right) \prec 0 \end{cases} \quad (16)$$

The former is a necessary condition for GSOF controller design, as given in Lemma 1; that is, if the former condition is not satisfied then GSOF controllers cannot be designed even if the causality of scheduling parameters is broken. The latter inequality is the inequality to be satisfied for our problem when  $\bar{K}(\theta)$  in (14) is considered. Thus, looking for  $\mathcal{X}(\theta)$ ,  $\mathcal{Z}(\theta)$ ,  $\mathcal{K}(\theta)$  and  $R(\theta, \dot{\theta})$  satisfying (15) is the problem which we should tackle.

However, inequality (15) contains  $\mathcal{Z}(\theta)$  and its inverse, which is not convenient to solve the condition. Now, matrix  $\mathcal{Z}(\theta)$  is supposed to be positive definite due to the requirement in (6). Thus, without loss of generality, matrix  $R(\theta, \dot{\theta})$  can be set as  $\mathcal{Z}(\theta)\mathcal{R}(\theta, \dot{\theta})$  with  $\mathcal{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ .

We then propose the following method for Problem 1.

*Theorem 1:* For given scalar  $\gamma_\infty \in \mathbb{R}_+$ , if there exist a matrix  $\mathcal{R}(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ , continuously differentiable parameter-dependent matrices  $\mathcal{X}(\theta), \mathcal{Z}(\theta) \in \mathbb{S}_+^n$  and a parameter-dependent matrix  $\mathcal{K}(\theta) \in \mathbb{R}^{(n+n_u) \times (n+n_u)}$  such that (6) and (17) hold for all  $(\theta, \dot{\theta}) \in \Lambda_\theta$ , then, the controller  $K(\theta)$ , whose state-space matrices are given as  $\bar{K}(\theta)$  in (14) at the top of this

page with  $\mathcal{Y}(\theta) = \mathcal{X}(\theta) - \mathcal{Z}(\theta)^{-1}$ , stabilizes the closed-loop system  $G_{cl}(\theta)$  and satisfies (5) for all admissible trajectories  $(\theta, \dot{\theta}) \in \Lambda_\theta$ .

$$\begin{aligned} & \begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \mathbf{0} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ * & \end{bmatrix} \\ & + \text{He} \left\{ \begin{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ -\mathcal{Z}(\theta) \end{bmatrix} \end{bmatrix} \mathcal{R}(\theta, \dot{\theta}) \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \end{bmatrix}^T \right\} \prec 0 \end{aligned} \quad (17)$$

*Proof:* By using Elimination lemma, the existence of  $\mathcal{R}(\theta, \dot{\theta})$  satisfying inequality (17) is equivalent to the feasibility of (16). The remaining of the proof using the latter inequality in (16) is the same as in [16], [17], thus it is omitted here. ■

Note that  $\Upsilon(\theta, \dot{\theta}) \prec 0$  is necessary for our GSOF controller design, which has already been mentioned above. Thus, the feasibility supposition of  $\Upsilon(\theta, \dot{\theta}) \prec 0$  does not introduce any additional implicit requirement to our method.

*Remark 4:* Inequality (17) is equivalently transformed to the following inequality after some algebraic manipulations.

$$\begin{aligned} & \begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\ * & \end{bmatrix} \\ & + \text{He} \left\{ \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ -\mathcal{Z}(\theta) \end{bmatrix} \end{bmatrix} \mathcal{R}(\theta, \dot{\theta})^{-1} \begin{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \end{bmatrix}^T \right\} \prec 0 \end{aligned} \quad (18)$$

□

### B. Conservatism Comparison

Regarding the formulations in Lemma 2 and Theorem 1, the following is claimed.

*Theorem 2:* For given scalar  $\gamma_\infty \in \mathbb{R}_+$ , if a GSOF controller is designed by Lemma 2, then it is always possible to design a GSOF controller satisfying the closed-loop stability and (5) by Theorem 1 with the same  $\gamma_\infty$ .

*Proof:* We first consider the case in which a GSOF controller is designed by using (12). Note that inequality (17) with  $\mathcal{R}(\theta, \dot{\theta}) = \varepsilon \mathbf{I}$  is given as follows:

$$\begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \varepsilon \mathbf{I} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ -2\varepsilon \mathcal{Z}(\theta) \end{bmatrix} \\ * & \end{bmatrix} \prec 0.$$

Since  $-2\varepsilon \mathcal{Z}(\theta) \prec -\varepsilon \mathcal{Z}(\theta) \prec 0$  holds due to the positivity of scalar  $\varepsilon$  and positive definiteness of matrix  $\mathcal{Z}(\theta)$ , it is obvious that inequality (17) is always feasible if inequality (12) holds.

We next consider the case in which a GSOF controller is designed by using (13). Multiplications of  $\text{diag} \{ \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}, [\mathbf{I} \quad \mathbf{I}] \}$  and its transpose to (13) from the left and the right respectively lead to

$$\begin{bmatrix} \Upsilon(\theta, \dot{\theta}) & \begin{bmatrix} \varepsilon \mathbf{I} \\ \dot{\mathcal{Z}}(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ -2\varepsilon \mathcal{Z}(\theta) \end{bmatrix} \\ * & \end{bmatrix} \prec 0,$$

which is the inequality (17) with  $\mathcal{R}(\theta, \dot{\theta}) = \varepsilon \mathbf{I}$ . Thus, inequality (17) is always feasible if inequality (13) holds. This completes the proof. ■

As described in the previous section, it has not been clarified which formulations in Lemma 2 is less conservative. This question is not yet cleared; however, from Theorem 2, we conclude that both formulations in Lemma 2 are conservative

than or equal to the formulation in Theorem 1 with  $\mathcal{R}(\theta, \dot{\theta})$  being set as  $r\mathbf{I}$  using a scalar  $r \in \mathbb{R}_+$ . Thus, if we use matrix  $\mathcal{R}(\theta, \dot{\theta})$  instead of  $r\mathbf{I}$  in Theorem 1, it can be expected that conservatism will be further reduced. Though, in exchange for conservatism reduction, iterative algorithm for solving Bilinear Matrix Inequality (BMI) must be applied due to the multiplication of  $\mathcal{Z}(\theta)$  and  $\mathcal{R}(\theta, \dot{\theta})$ . As a compromise, in a numerical example shown below, we set  $\mathcal{R}(\theta, \dot{\theta})$  as  $r\mathbf{I}$  with a scalar  $r \in \mathbb{R}_+$ , and conduct a line search for  $r$ .

In general, the formulations in Lemmas 1 and 2, and Theorem 1 are at least cubically parameter-dependent. Thus, some methods for solving such high-order parameter-dependent inequalities are required to solve them, i.e. Sum-Of-Square (SOS) relaxation [22]–[25], slack variable approach [26], etc. Slack variable approach in [26] is used in the numerical example shown below, similarly to the examples in [16], [17].

### C. Numerical Example

To illustrate conservatism reduction, we show the design results for the numerical example used in [16], [17], [27]. The state-space matrices of LPV system  $G(\theta)$  in (1) with a single scheduling parameter are defined as follows.

$$\begin{aligned}
 & \left[ \begin{array}{c|c|c} A(\theta) & B_1(\theta) & B_2(\theta) \\ \hline C_1(\theta) & D_{11}(\theta) & D_{12}(\theta) \\ \hline C_2(\theta) & D_{21}(\theta) & \end{array} \right] \\
 = & \left[ \begin{array}{ccc|cc} -4 & 3 & 5 & 1 & 0 \\ 0 & 7 & -5 & -2 & 16 \\ 0.1 & -2 & -3 & 1 & -10 \\ \hline 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 2 & \end{array} \right] \\
 & + \theta \left[ \begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & -5 & 0 & -5 \\ 2 & 5 & 1.5 & 0 & 3.5 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \end{array} \right], \tag{19}
 \end{aligned}$$

where  $\Lambda_\theta$  is  $[-3, 3] \times [-1, 1]$ .

The design results for this example using Theorem 1 are shown in Table I, where all parameter-dependent decision matrices are set as parametrically affine. The line search for  $r$  is conducted with 400 points linearly gridded over a logarithmic scale in  $[10^{-10}, 10^{10}]$ . For reference, the numbers of decision variables and LMI rows are also given.

The corresponding results using the method in [16], i.e. Lemma 2 in this paper, are shown in Table II, where all parameter-dependent decision matrices are set as parametrically affine. The line search for  $\varepsilon$  is the same as when using Theorem 1.

For reference, the design result using Theorem 1, i.e. line search parameter  $r$  against  $\gamma_\infty$ , is shown in Fig. 3. In the same figure, the design result using formulation (13) in Lemma 2, i.e. line search parameter  $\varepsilon$  against  $\gamma_\infty$ , is also shown.

Comparison of Tables I and II clearly indicates that Theorem 2 holds; that is, our proposition is no more conservative than the method in [16]. This property is also confirmed in Fig. 3.

Next, the design results using the method in [12] are also shown in Table III, where all parameter-dependent decision matrices are again set as parametrically affine. Please note that the case, in which  $X(\theta) = X_0 + \theta X_1$  and  $Y(\theta) = Y_0 + \theta Y_1$  are used, does not produce an implementable GSOF controller  $K(\theta)$  but an *impractical* GSOF controller  $K(\theta, \dot{\theta})$  due to the fact that the state-space matrices depend on not only the current scheduling parameters but also the derivatives of scheduling parameters. Thus, the value of 21.38 is the lower bound of achievable performance. Theorem 1 cannot produce GSOF controllers with  $\gamma_\infty$  being very close to the achievable performance bound; however, guaranteed performance, i.e. 21.73, is close to the bound. This illustrates the effectiveness of our method.

The numerical complexity of Theorem 1 is the same as the method of Lemma 2 using formulation (12), and its increase compared to that of the method in [12] is not so large. Thus, our proposed method has comparable numerical complexity as exiting methods in the literature.

## IV. CONCLUSIONS

We address continuous-time Gain-Scheduled Output Feedback (GSOF) controller design for Linear Parameter-Varying (LPV) systems via Parameter-Dependent Lyapunov Functions (PDLFs). This is one of the classical problems and there has been several methods to produce practical GSOF controllers which do not depend on the derivatives of scheduling parameters. However, existing methods have some drawbacks, i.e. conservatism due to structurally constrained Lyapunov matrix, slightly increase of system complexity due to the use of additional filters, and a certain gap between guaranteed performance and achievable performance due to the only sufficiency of derived formulations.

We propose a new formulation, in which Lyapunov matrix has no structural constraints, by using Elimination lemma for over-bounding the term causing the dependency of the derivatives of scheduling parameters. Thanks to the use of Elimination

TABLE I  
DESIGN RESULT USING THEOREM 1

$\gamma_\infty$	21.73
# of decision variables	185
# of LMI rows	88

TABLE II  
DESIGN RESULT USING LEMMA 2

	(12)	(13)
$\gamma_\infty$	22.07	22.06
# of decision variables	185	185
# of LMI rows	88	100

TABLE III  
DESIGN RESULT USING METHOD IN [12] WITH VARIOUS  $X(\theta)$  AND  $Y(\theta)$

$X(\theta)$	$X_0 + \theta X_1$	$X_0 + \theta X_1$	$X_0$	$X_0$
$Y(\theta)$	$Y_0 + \theta Y_1$	$Y_0$	$Y_0 + \theta Y_1$	$Y_0$
$\gamma_\infty$	21.38	$\infty$	27.50	$\infty$
# of decision variables	185	179	179	173
# of LMI rows	76	76	76	76

lemma, the derived formulation theoretically introduces no conservatism, and it is also proved that the derived formulation is no more conservative than an existing method in the literature. A numerical example illustrates the effectiveness of our proposed method.

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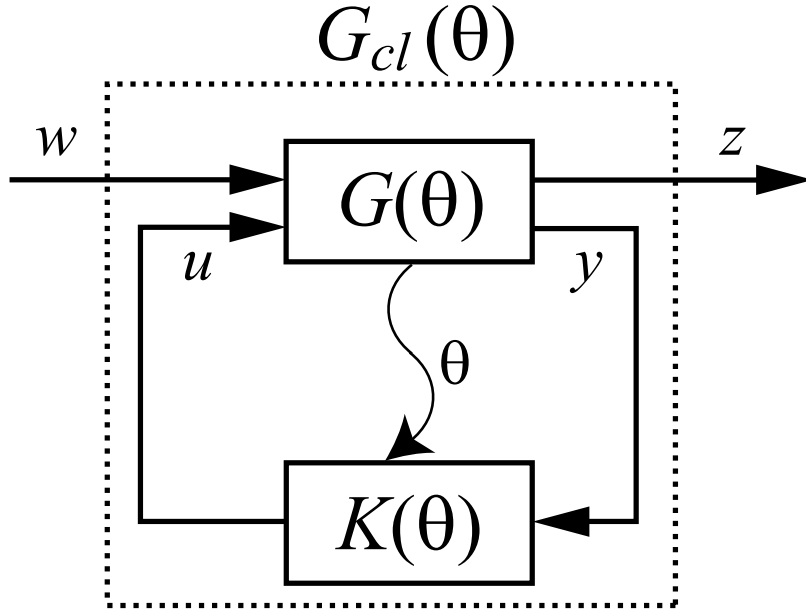


Fig. 1. Block diagram using causal GSOF controller  $K(\theta)$

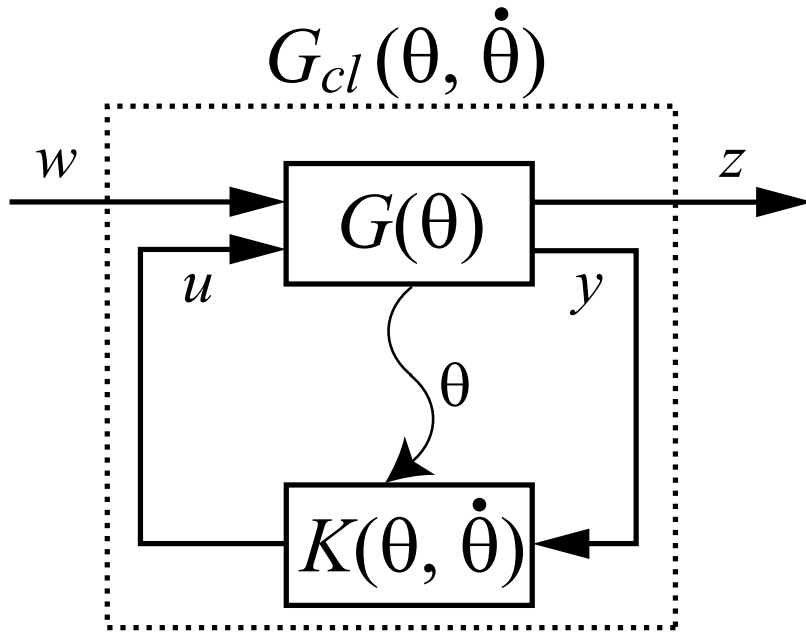


Fig. 2. Block diagram using *non-causal* GSOF controller  $K(\theta, \dot{\theta})$

$$\begin{bmatrix} \Upsilon_A(\theta) & \Upsilon_B(\theta) \\ \Upsilon_C(\theta) & \Upsilon_D(\theta) \end{bmatrix} = \begin{bmatrix} A(\theta)\mathcal{X}(\theta) & A(\theta) & B_1(\theta) \\ \mathbf{0} & \mathcal{Z}(\theta)A(\theta) & \mathcal{Z}(\theta)B_1(\theta) \\ \bar{C}_1(\theta)\mathcal{X}(\theta) & \bar{C}_1(\theta) & \bar{D}_{11}(\theta) \end{bmatrix} + \begin{bmatrix} \mathbf{0} & B_2(\theta) \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{D}_{12}(\theta) \end{bmatrix} \mathcal{K}(\theta) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & C_2(\theta) \\ \mathbf{0} & D_{21}(\theta) \end{bmatrix} \quad (9)$$

$$\bar{K}(\theta, \dot{\theta}) = \begin{bmatrix} -\mathcal{Z}(\theta)^{-1} & B_2(\theta) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left( \mathcal{K}(\theta) - \begin{bmatrix} \mathcal{Z}(\theta)A(\theta)\mathcal{X}(\theta) + \dot{\mathcal{Z}}(\theta)\mathcal{Z}(\theta)^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathcal{Y}(\theta)^{-1} & \mathbf{0} \\ -C_2(\theta)\mathcal{X}(\theta)\mathcal{Y}(\theta)^{-1} & \mathbf{I} \end{bmatrix} \quad (10)$$

$$\bar{K}(\theta) = \begin{bmatrix} -Z(\theta)^{-1} & B_2(\theta) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left( \mathcal{K}(\theta) - \begin{bmatrix} Z(\theta)A(\theta)\mathcal{X}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} \mathcal{Y}(\theta)^{-1} & \mathbf{0} \\ -C_2(\theta)\mathcal{X}(\theta)\mathcal{Y}(\theta)^{-1} & \mathbf{I} \end{bmatrix} \quad (14)$$

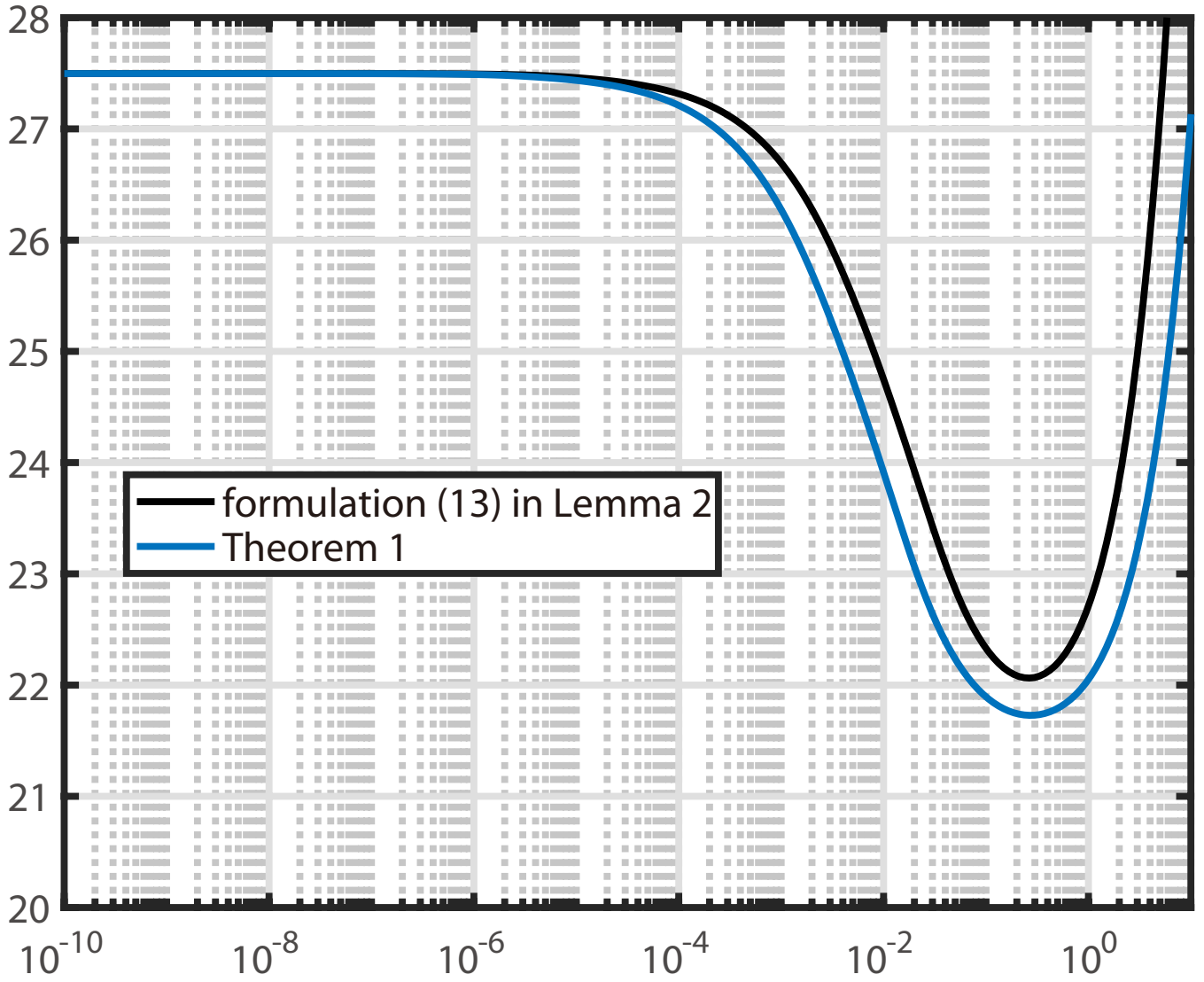


Fig. 3. Line search parameter ( $\varepsilon$  in Lemma 2 and  $r$  in Theorem 1) v.s.  $\gamma_\infty$