



**HAL**  
open science

## **Topp-Leone Weibull-Lomax distribution: Properties, Regression Model and Applications**

Farrukh Jamal, Hesham Mohammed Reyad, M Arslan Nasir, Christophe Chesneau, M Akbar Ali Shah, Soha Othman Ahmed

► **To cite this version:**

Farrukh Jamal, Hesham Mohammed Reyad, M Arslan Nasir, Christophe Chesneau, M Akbar Ali Shah, et al.. Topp-Leone Weibull-Lomax distribution: Properties, Regression Model and Applications. 2019. hal-02270561

**HAL Id: hal-02270561**

**<https://hal.science/hal-02270561>**

Preprint submitted on 26 Aug 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Topp-Leone Weibull-Lomax distribution: Properties, Regression Model and Applications

FARRUKH JAMAL,\*<sup>1</sup>HESHAM MOHAMMED REYAD,<sup>2</sup>M. ARSLAN NASIR,<sup>3</sup>  
CHRISTOPHE CHESNEAU<sup>4</sup>, M. AKBAR ALI SHAH<sup>5</sup> and SOHA OTHMAN AHMED<sup>6</sup>

<sup>1</sup>Department of Statistics, Govt.S.A. Postgraduate College, DeraNawab Sahib, Pakistan

\*<sup>1</sup>Corresponding author E-mail: drfarrukh1982@gmail.com

<sup>2</sup> Department of Management Information Systems and Production Management, Qassim University, KSA

E-mail: hesham\_reyad@yahoo.com

<sup>3</sup>Department of Statistics, Govt.S.E. College, Bahawalpur, Pakistan

E-mail: arslannasir147@gmail.com

<sup>4</sup>Department of Mathematics, LMNO, University of Caen, France

E-mail: christophe.chesneau@gmail.com

<sup>5</sup>Department of Statistics, The Islamia University of Bahawalpur, Pakistan

E-mail: akbar.ali@iub.edu.pk

<sup>6</sup>Department of Applied Statistics, Cairo University, Egypt

E-mail: soha\_othman@yahoo.com

## ABSTRACT

In this paper, a new four parameter lifetime distribution called the Topp Leone Weibull-Lomax distribution is introduced. Some mathematical properties of the new distribution are studied including the quantile function, ordinary and incomplete moments, probability weighted moment, conditional moments, order statistics, stochastic ordering and stress strength reliability parameter. The regression model and the residual analysis for the new model are also investigated. The model parameters are estimated by using the maximum likelihood criterion and the behavior of these estimates are examined by conducting a simulation study. We prove empirically the importance and flexibility of the new distribution in modeling four data sets.

**Keywords:** Maximum likelihood, Regression model, Residual Analysis, Topp Leone-G Family.

**AMS Subject:** 9A60; 62E15; 62H10.

## 1. INTRODUCTION

The Lomax (or Pareto type II ) distribution was first introduced by Lomax (1954). It has many applications in different areas such as size of cities, income and wealth inequality, reliability modeling, actuarial science, medical and biological sciences. In the literature, different extensions of the Lomax distribution are available such as the exponentiated Lomax (ELx) (El-Bassiouny et al., 2015), beta-Lomax (BLx), Kumaraswamy-Lomax (KwL), McDonald-Lomax (McLx) (Lemonte and Cordeiro, 2013), gamma-Lomax (GLx) (Cordeiro et al., 2013), Weibull Lomax (WLx) (Tahir et al., 2015), Gompertz Lomax (GzLx) (Oguntunde et al., 2017) and half logistic Lomax (Anwar and Zahoor, 2018). The

probability density function (pdf) and cumulative distribution function (cdf) corresponding to the Weibull Lomax (WLx) distribution are given by, respectively,

$$g(x) = \alpha ab(1+bx)^{a\alpha-1} \left\{1 - (1+bx)^{-a}\right\}^{\alpha-1} \\ \times \exp\left\{-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}, \quad \alpha, a, b > 0, \quad x > 0, \quad (1)$$

$$G(x) = 1 - \exp\left\{-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}, \quad \alpha, a, b > 0, \quad x > 0, \quad (2)$$

where  $a$  and  $\alpha$  are shape parameters while  $b$  is a scale parameter.

On the other side, Al-Shomrani et al. (2016) introduced the Topp Leone-G (TL-G) family of distributions. The corresponding cdf and pdf are given by, respectively,

$$F_{TL-G}(x; \phi) = \left[1 - \bar{G}(x; \phi)^2\right]^\theta, \quad \theta > 0, \quad x \in R. \quad (3)$$

and

$$f_{TL-G}(x; \phi) = 2\theta g(x; \phi) \bar{G}(x; \phi) \left[1 - \bar{G}(x; \phi)^2\right]^{\theta-1}, \quad \theta > 0, \quad x \in R, \quad (4)$$

$G(x; \phi)$  is a cdf of a continuous distribution,  $g(x; \phi)$  is the corresponding pdf,  $\phi$  is the associated vector of parameters and  $\bar{G}(x; \phi) = 1 - G(x; \phi)$ .

The main thrust of this study is to introduce a new extension of the Lomax distribution called the Topp-Leone Weibull-Lomax (TLWLx for short) distribution by using the WLx as a baseline distribution in the TL-G family. Inserting (2) in (3), the cdf of the proposed model is given by

$$F(x) = \left[1 - \exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}\right]^\theta, \quad x > 0, \quad (5)$$

where  $\alpha > 0, \theta > 0, a > 0$  are the shape parameters and  $b > 0$  is the scale parameter.

The pdf and hazard function (hrf) corresponding to (5) are given by

$$f(x) = 2\theta \alpha ab(1+bx)^{a\alpha-1} \left\{1 - (1+bx)^{-a}\right\}^{\alpha-1} \exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\} \\ \times \left[1 - \exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}\right]^{\theta-1}, \quad x > 0, \alpha, \theta, a, b > 0 \quad (6)$$

and

$$\begin{aligned}
 h(x) &= 2\theta\alpha ab(1+bx)^{a\alpha-1} \left\{1-(1+bx)^{-a}\right\}^{\alpha-1} \exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\} \\
 &\quad \times \left[1-\exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}\right]^{\theta-1} \\
 &\quad \times \left[1-\left[1-\exp\left\{-2\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right)^\alpha\right\}\right]^\theta\right]^{-1}.
 \end{aligned}
 \tag{7}$$

The plots of the pdf and hrf of the TLWLx distribution are displayed in Fig. 1.

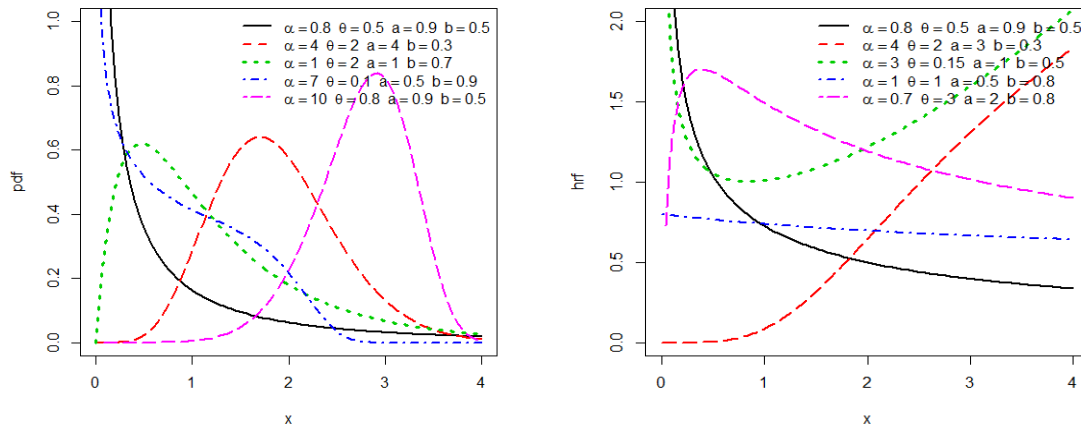


FIG. 1: Plots of the TLWLx pdf and hrf for selected parameters values.

Moreover, the basic motivations for TLWLx distribution in practice are the following:

- to produce a skewness for symmetrical distributions;
- to construct heavy-tailed distributions for modeling real data;
- to generate distributions with symmetric, left-skewed, right-skewed, reversed-J shape;
- to define new models with all types of the hrf monotonic as well as non monotonic;
- to provide consistently better fits and properties than other competitive models;
- to obtain a more flexible model by inducting just one extra shape parameter to the WLx model and to improve goodness-of-fit to real data.

In fact, we prove empirically that the proposed distribution provides better fits to four real data sets than other six extended Lomax distributions (see Section 8). These examples really show that the new distribution is a good alternative for modelling survival data. Moreover, the TLWLx density can be symmetric, left-skewed, right-skewed or reversed J-shape, whereas the hrf can be bathtub, unimodal, reversed J-shape, constant, monotonically increasing and decreasing shapes (see Fig 1).

The paper is outlined as follows. In Section 2, a useful expansion of TLWLx pdf is obtained. In Section 3, basic mathematical properties of the new model are investigated. Section 4 contains the maximum likelihood estimates for the parameters of TLWLx distribution. A simulation study is conducted in Section 5. In Section 6, the regression model for the TLWLx distribution is discussed. The residual analysis of the TLWLx distribution are carried out in Section 7. Section 8 and Section 9 consist of some applications and concluding remarks respectively.

## 2. Linear Representation of the Model Density

In this section, we give an infinite mixture representation of pdf corresponding to TLWLx distribution in terms of exp-Lomax distribution, which will be useful for further calculations.

Using binomial series expansion in (6), we have

$$f(x) = 2\alpha\theta ab (1 + bx)^{-a-1} \frac{[1 - (1 + bx)^{-a}]^{\alpha-1}}{(1 + bx)^{-a(\alpha+1)}} \\ \times \sum_{i=0}^{\infty} \binom{\theta-1}{i} (-1)^i \exp \left[ -2(i+1) \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^{\alpha} \right].$$

Using the power series expansion in the above equation, we get

$$f(x) = 2\alpha\theta ab (1 + bx)^{-a-1} \frac{[1 - (1 + bx)^{-a}]^{\alpha-1}}{(1 + bx)^{-a(\alpha+1)}} \\ \times \sum_{i=0}^{\infty} \binom{\theta-1}{i} (-1)^i \sum_{j=0}^{\infty} \frac{[-2(i+1)]^j}{j!} \frac{\{1 - (1 + bx)^{-a}\}^{\alpha j}}{\{(1 + bx)^{-a}\}^{\alpha j}}.$$

By applying the binomial formula, we obtain

$$f(x) = 2\alpha\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{k + \alpha(j+1)}{k} \binom{\theta-1}{i} (-1)^{i+j} \frac{[2(i+1)]^j}{j!} \\ \times ab(1 + bx)^{-a-1} \times \{1 - (1 + bx)^{-a}\}^{k+\alpha(j+1)-1}.$$

The above expression can be written as

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \phi(x) \Phi^{k+\alpha(j+1)-1}(x),$$

(8)

where  $a_{j,k} = 2\alpha\theta \sum_{i=0}^{\infty} \binom{\theta+1}{i} \binom{k+\alpha(j+1)}{k} (-1)^{i+j} \frac{[2(i+1)]^j}{j!}$ ,

where  $\phi(x)$  and  $\Phi(x)$  are the pdf and cdf of the Lomax distribution, respectively.

### 3. MATHEMATICAL PROPERTIES

In this section, we obtained some mathematical properties of the TLWLx distribution.

#### 3.1. QUANTILE FUNCTION

The quantile function of the TLWLx distribution say,  $Q(u) = F^{-1}(u)$  for  $u \in (0,1)$ ,  $\alpha, \theta, a > 0$  and  $b > 0$  is the solution of the non-linear equation:  $F(Q(u)) = Q(F(u)) = u$ . The quantile function (qf) of X is obtained by inverting (5) as

$$Q_X(u) = \frac{1}{b} \left[ \left\{ 1 + \left[ -\frac{1}{2} \log \left( 1 - u^{\frac{1}{\theta}} \right) \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{a}} - 1 \right]. \quad (9)$$

#### 3.2. MOMENTS AND RELATIVE MEASURES

The  $r$ th moment of the TLWLx distribution can be obtained by using the following relation

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f(x) dx.$$

Using the mixture representation of pdf given in (8), we have

$$\mu'_r = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \int_0^{\infty} x^r \phi(x) \Phi^{k+\alpha(j+1)-1}(x) dx.$$

Using the binomial series expansion and some algebra, we have

$$\begin{aligned} \mu'_r &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \sum_{l=0}^{\infty} \binom{k+\alpha(j+1)-1}{l} (-1)^l \times \frac{a}{b^r} \\ &\times B(r+1, a(l+1)-r), \end{aligned} \quad (10)$$

where  $B(c, d)$  is the complete beta function.

The incomplete moments we have

$$\begin{aligned} Q'_r &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \sum_{l=0}^{\infty} \binom{k+\alpha(j+1)-1}{l} (-1)^l \\ &\times \frac{a}{b^r} B_{bx}(r+1, a(l+1)-r), \end{aligned} \quad (11)$$

where  $B_x(c, d)$  is defined by

$$B_x(c, d) = \int_0^x \frac{t^{c-1}}{(1+t)^{c+d}} dt.$$

The measures of skewness and kurtosis of the TLWLx distribution can be obtained by using the following formulas

$$\text{Skewness}(X) = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{\frac{3}{2}}} \quad (12)$$

and

$$\text{Kurtosis}(X) = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}, \quad (13)$$

respectively. Fig. 2 shows the behavior of skewness and kurtosis corresponding to the TLWLx distribution.

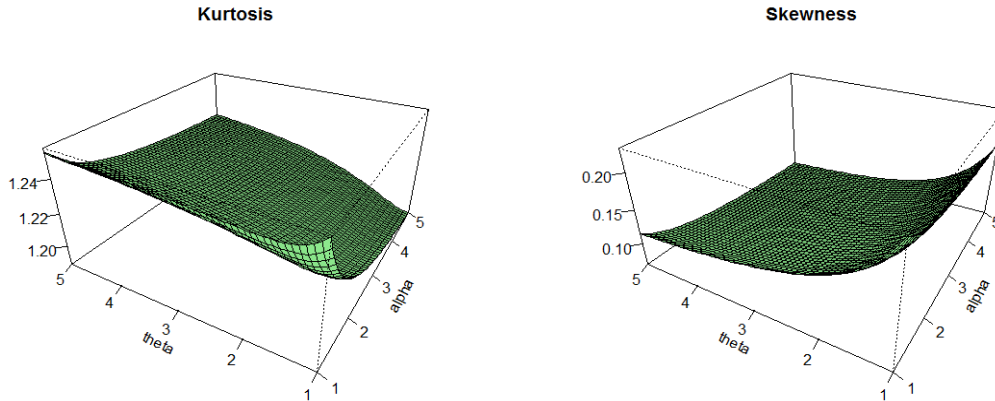


FIG. 2: The skewness and kurtosis of TLWLx ( $a=1.5$  and  $b=2.5$ ) distribution.

From Fig. 2, we can observe that  $\alpha$  has significant control on skewness and kurtosis.

### 3.3. PROBABILITY WEIGHTED MOMENTS

The  $(r + s)$ th probability weighted moments of a random variable  $X$  follows TLWLx distribution, say  $M_{r,s}$ , is given by

$$M_{r,s} = E(X^r F(X)^s) = \int_0^\infty x^r F(x)^s f(x) dx. \quad (14)$$

From (5) and (6), we have

$$f(x)F(x)^s = 2\theta\alpha ab(1+bx)^{a\alpha-1} \left[ 1 - (1+bx)^{-a} \right]^{\alpha-1}$$

$$\begin{aligned} & \times \exp \left\{ -2 \left( \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right)^\alpha \right\} \\ & \times \left( 1 - \exp \left\{ -2 \left( \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right)^\alpha \right\} \right)^{\theta(s+1)-1}. \end{aligned}$$

It follows from the binomial and exponential series formulas that

$$\begin{aligned} f(x)F(x)^s &= 2\theta\alpha ab \sum_{j=0}^{\infty} (-1)^j \binom{\theta(s+1)-1}{j} (1+bx)^{a\alpha-1} \left[ 1 - (1+bx)^{-a} \right]^{\alpha-1} \\ & \times \exp \left\{ -2(j+1) \left( \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right)^\alpha \right\}. \\ &= 2\theta\alpha ab \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{j+i} 2^i (j+1)^i}{i!} \binom{\theta(s+1)-1}{j} (1+bx)^{a\alpha(i+1)-1} \\ & \quad \times \left[ 1 - (1+bx)^{-a} \right]^{\alpha(i+1)-1}. \\ &= 2\theta\alpha ab \sum_j \sum_i \sum_{\ell=0}^{\infty} \frac{(-1)^{j+i+\ell} 2^i (j+1)^i}{i!} \binom{\theta(s+1)-1}{j} \binom{\alpha(i+1)-1}{\ell} \\ & \quad \times (1+bx)^{-a(\ell-\alpha(i+1))-1}, \end{aligned}$$

and finally, we have

$$f(x)F(x)^s = \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \delta_{i,\ell} h_{a(\ell-\alpha(i+1)),b}(x), \quad (15)$$

where

$$\delta_{i,\ell} = 2\theta\alpha \sum_{j=0}^{\infty} \frac{(-1)^{j+i+\ell} 2^i (j+1)^i}{i! (\ell - \alpha(i+1))} \binom{\theta(s+1)-1}{j} \binom{\alpha(i+1)-1}{\ell}.$$

Here  $h_{a(\ell-\alpha(i+1)),b}(x)$  is the Lomax distribution with shape and scale parameters  $a(\ell - \alpha(i+1))$  and  $b$  respectively.



Inserting (15) in (14), we obtained

$$M_{r,s} = \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \delta_{i,\ell} \frac{a(\ell - \alpha(i+1))}{b^r} B(r+1, a(\ell - \alpha(i+1)) - r). \quad (16)$$

### 3.4 CONDITIONAL MOMENTS AND MEAN DEVIATIONS

In connection with lifetime distributions, it is important to determine the conditional moments  $E(X^r|X > t)$ ,  $r = 1, 2, \dots$ , which are useful in predictive inference. The  $r$ th conditional moment of the TLWLx distribution can be obtained as

$$E(X|X > t) = \frac{1}{S(t)} \left[ E(X^r) - \int_0^t x^r f(x) dx \right],$$

where  $S(t) = 1 - F(t)$ .

From equations (5), (10) and (11), we have the following result

$$\begin{aligned} E(X^r|X > t) = & \left[ 1 - \left\{ 1 - \exp \left[ -2 \left( \frac{1 - (1 + bt)^{-a}}{(1 + bt)^{-a}} \right)^\alpha \right] \right\}^\theta \right]^{-1} \\ & \times \left[ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \sum_{l=0}^{\infty} \binom{k + \alpha(j+1) - 1}{l} (-1)^l \times \frac{a}{b^r} \times B(r+1, a(l+1) - r) \right. \\ & - \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} \sum_{l=0}^{\infty} \binom{k + \alpha(j+1) - 1}{l} (-1)^l \\ & \left. \times \frac{a}{b^r} B_{bt}(r+1, a(l+1) - r) \right]. \end{aligned}$$

Mean deviations from the mean and the median provide useful information about the characteristics of a population and it can be calculated from the first incomplete moment. The amount of dispersion in a population may be measured to some extent by all the deviations from the mean and median. The mean deviations of  $X$  about the mean  $\mu = E(X)$  and about the median  $M$  can be expressed as  $\delta = 2\mu F(\mu) - 2Q'_1(\mu)$  and  $\eta = \mu - 2Q'_1(M)$ , where  $F(\mu)$ ,  $Q'_1(\mu)$  and  $Q'_1(M)$  can be calculated from (5) and (11).

### 3.5. ORDER STATISTICS

Let  $X_{1:n} \leq X_{2:n}, \dots \leq X_{n:n}$  be the order statistics corresponding to a sample of size  $n$  from the TLWLx distribution. The pdf of  $X_{k:n}$ , the  $k$ th order statistic, is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) F^{j+i-1}(x). \quad (17)$$

From (5) and (6), we have

$$\begin{aligned}
& f(x) F^{j+i-1}(x) \\
&= 2\alpha\theta ab (1+bx)^{-a-1} \frac{[1-(1+bx)^{-a}]^{\alpha-1}}{(1+bx)^{-a(\alpha+1)}} \\
&\times \exp\left[-2\left\{\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right\}^{\alpha}\right] \\
&\times \left[1 - \exp\left[-2\left\{\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right\}^{\alpha}\right]\right]^{\theta(j+i)-1}.
\end{aligned}$$

Using binomial series expansion, we have

$$\begin{aligned}
f(x) F^{j+i-1}(x) &= 2\alpha\theta ab (1+bx)^{-a-1} \frac{[1-(1+bx)^{-a}]^{\alpha-1}}{(1+bx)^{-a(\alpha+1)}} \\
&\times \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \binom{\theta(j+i)-1}{k} (-1)^k \frac{[-2(k+1)]^q}{q!} \frac{\{1-(1+bx)^{-a}\}^{\alpha q}}{\{(1+bx)^{-a}\}^{\alpha q}}.
\end{aligned}$$

After some algebra, we have

$$\begin{aligned}
f(x) F^{j+i-1} &= 2\alpha\theta ab \\
&\times \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \binom{\theta(j+i)-1}{k} (-1)^k \frac{[-2(k+1)]^q}{q!} \binom{n+\alpha(q+1)}{n} \\
&\times (1+bx)^{-a-1} [1-(1+bx)^{-a}]^{n+\alpha(q+1)-1}. \tag{18}
\end{aligned}$$

Inserting (18) in (17), we obtain

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j 2\alpha\theta ab \\
&\times \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} \binom{\theta(j+1)-u}{k} (-1)^k \frac{[-2(k+1)]^q}{q!} \binom{n+\alpha(q+1)}{n} (1+bx)^{-a-1} [1-(1+bx)^{-a}]^{n+\alpha(q+1)-1}.
\end{aligned}$$

The above expression can be written as

$$f_{i:n}(x) = \sum_{q=0}^{\infty} \sum_{n=0}^{\infty} w_{n,q} h(x; n + \alpha(q+1)), \tag{19}$$

where,  $h(x; n + \alpha(q+1)) = (n + \alpha(q+1))ab(1+bx)^{-a-1}[1-(1+bx)^{-a}]^{n+\alpha(q+1)-1}$ .

and

$$w_{n,q} = \frac{n! 2\alpha\theta}{(i-1)!(n-i)!(n+\alpha(q+1))} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} \binom{n-i}{j} (-1)^j \binom{\theta(j+i)-1}{k} (-1)^k \frac{[-2(k+1)]^q}{q!} \binom{n+\alpha(q+1)}{n}.$$

### 3.6. STOCHASTIC ORDERING

Stochastic ordering has been recognized as an important tool in reliability theory and other fields to assess comparative behavior. Here we present a stochastic ordering result related to the TLWLx distribution.

Let  $X_1 \sim TLW - Lx(\alpha, \theta_1, a, b)$  and  $X_2 \sim TLW - Lx(\alpha, \theta_2, a, b)$ , then the pdf of  $X_1$  is given by

$$f(x) = 2\alpha\theta_1 ab (1 + bx)^{-a-1} \frac{[1 - (1 + bx)^{-a}]^{\alpha-1}}{(1 + bx)^{-a(\alpha+1)}} \times \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \\ \times \left[ 1 - \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \right]^{\theta_1-1},$$

and the pdf of  $X_2$  is given by

$$g(x) = 2\alpha\theta_2 ab (1 + bx)^{-a-1} \frac{[1 - (1 + bx)^{-a}]^{\alpha-1}}{(1 + bx)^{-a(\alpha+1)}} \times \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \\ \times \left[ 1 - \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \right]^{\theta_2-1}.$$

Then, we have

$$\frac{f(x)}{g(x)} = \frac{\theta_1}{\theta_2} \left[ 1 - \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \right]^{\theta_1-\theta_2}.$$

By differentiating with respect to  $x$ , we have

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{\theta_1}{\theta_2} (\theta_1 - \theta_2) [1 - \exp[-2w_i^\alpha]]^{\theta_1-\theta_2-1} \exp[-2w_i^\alpha] 2\alpha w_i^{\alpha-1} w_i',$$

where  $w_i = \frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}$  and  $w_i' = \frac{d}{dx} w_i$ .

If  $\theta_1 < \theta_2$  then  $\frac{d}{dx} \frac{f(x)}{g(x)} < 0$ , so this implies that likelihood ratio exists among  $X_1$  and  $X_2$  i.e.  $X_2 \leq_{lr} X_1$ .

### 3.7. STRESS STRENGTH MODEL

Let  $X_1 \sim TLWLx(\alpha, \theta_1, a, b)$  and  $X_2 \sim TLWLx(\alpha, \theta_2, a, b)$  then from (6) and (5), the pdf of  $X_1$  is given by

$$f_1(x) = 2\alpha\theta_1 ab (1 + bx)^{-a-1} \frac{[1 - (1 + bx)^{-a}]^{\alpha-1}}{(1 + bx)^{-a(\alpha+1)}} \times \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \\ \times \left[ 1 - \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \right]^{\theta_1-1},$$

and the cdf of  $X_2$  is given by

$$F_2(x) = \left[ 1 - \exp \left[ -2 \left\{ \frac{1 - (1 + bx)^{-a}}{(1 + bx)^{-a}} \right\}^\alpha \right] \right]^{\theta_2}.$$

Then,

$$\begin{aligned}
R = P(X_2 < X_1) &= \int_0^{\infty} f_1(x)F_2(x)dx \\
&= 2\alpha\theta_1ab \int_0^{\infty} (1+bx)^{-a-1} \frac{[1-(1+bx)^{-a}]^{\alpha-1}}{(1+bx)^{-a(\alpha+1)}} \times \exp\left[-2\left\{\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right\}^{\alpha}\right] \\
&\times \left[1 - \exp\left[-2\left\{\frac{1-(1+bx)^{-a}}{(1+bx)^{-a}}\right\}^{\alpha}\right]\right]^{\theta_1+\theta_2-1} dx. \\
&= \frac{\theta_1}{\theta_1+\theta_2}. \tag{22}
\end{aligned}$$

Note that, in the identically distributed case where  $\theta_1 = \theta_2$ , we find the well-known value:  $R = \frac{1}{2}$ .

#### 4. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the estimation of the maximum likelihood estimates (MLEs) for the model parameters of the TLWLx distribution.

Let  $x_1, x_2, \dots, x_n$  be a random sample from TLWLx distribution with parameter vector

$\Theta = (\theta, \alpha, a, b)^T$  and  $w_i = \frac{1-(1+bx_i)^{-a}}{(1+bx_i)^{-a}}$  then the corresponding log-likelihood function is given by

$$\begin{aligned}
l(x; \Theta) &= n \log(2\alpha\theta ab) + (a-1) \sum_{i=1}^n \log(1+bx_i) + (\alpha-1) \sum_{i=1}^n \log w_i - 2 \sum_{i=1}^n w_i^{\alpha} \\
&\quad + (\theta-1) \sum_{i=1}^n \log\{1 - \exp[-2w_i^{\alpha}]\}.
\end{aligned}$$

Taking derivative with respect to  $\theta, \alpha, a$  and  $b$ , we have

$$U_{\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log\{1 - \exp[-2w_i^{\alpha}]\} \tag{23}$$

$$U_{\alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log w_i - 2 \sum_{i=1}^n w_i^{\alpha} \log w_i + (\theta-1) \sum_{i=1}^n \frac{\exp[-2w_i^{\alpha}]2w_i^{\alpha} \log w_i}{1 - \exp[-2w_i^{\alpha}]}, \tag{24}$$

$$\begin{aligned}
U_a &= \frac{n}{a} + \sum_{i=1}^n \log(1+bx_i) + (\alpha-1) \sum_{i=1}^n \frac{w_i'(a)}{w_i} - 2\alpha \sum_{i=1}^n w_i^{\alpha-1} w_i'(a) \\
&\quad + \alpha(\theta-1) \sum_{i=1}^n \frac{\exp[-2w_i^{\alpha}]2w_i^{\alpha-1} w_i'(a)}{1 - \exp[-2w_i^{\alpha}]}, \tag{25}
\end{aligned}$$

$$U_b = \frac{n}{b} + (a-1) \sum_{i=1}^n \frac{x_i}{1+bx_i} + (\alpha-1) \sum_{i=1}^n \frac{w_i'(b)}{w_i} - 2\alpha \sum_{i=1}^n w_i^{\alpha-1} w_i'(b)$$

$$+\alpha(\theta - 1) \sum_{i=1}^n \frac{\exp[-2w_i^\alpha] 2w_i^{\alpha-1} w_i'(b)}{1 - \exp[-2w_i^\alpha]}. \quad (26)$$

The MLEs, say  $\hat{\Theta} = (\hat{\theta}, \hat{\alpha}, \hat{a}, \hat{b})$  of  $\Theta = (\theta, \alpha, a, b)^T$  can be obtained by equating the system of nonlinear equations (23) - (26) to zero and solving them simultaneously. This can be performed numerically by using any mathematical softwares (R, Matlab, Mathematica...).

## 5. SIMULATION STUDY

In this section, the performance of the MLEs of the parameters of TLWLx distribution is evaluated by means of Monte-Carlo simulation study. The precision of the MLEs is discussed by means of biases, mean square errors (MSEs) and coverage probabilities (CPs).  $N = 3000$  samples of sizes  $n = 50, 100, 200, 300$  and  $500$  from TLWLx distribution with  $\theta, \alpha, a$  and  $b$  are generated. The parametric values are  $\theta = 0.60, \alpha = 2.80, a = 1.20$  and  $b = 1.90$ . the estimated bias, MSEs and CPs can be obtained using following equations:

$$Bias_\varepsilon(n) = \frac{1}{N} \sum_{i=1}^N (\hat{\varepsilon}_i - \varepsilon)$$

$$MSE_\varepsilon(n) = \frac{1}{N} \sum_{i=1}^N (\hat{\varepsilon}_i - \varepsilon)^2$$

$$CP_\varepsilon(n) = \sum_{i=1}^N I(\hat{\varepsilon}_i - 1.95996s_{\hat{\varepsilon}_i}, \hat{\varepsilon}_i + 1.95996s_{\hat{\varepsilon}_i}),$$

where  $\varepsilon = (\theta, \alpha, a, b)$ . Fig. 3, 4 and 5 display the numerical results for above measures. We conclude the following results from Figures (3-5): (i) the estimated biases and MSEs decrease when the sample  $n$  increases (ii) the coverage probabilities of all parameters are approximately 0.95 and approaches to nominal value when the sample sizes increase.

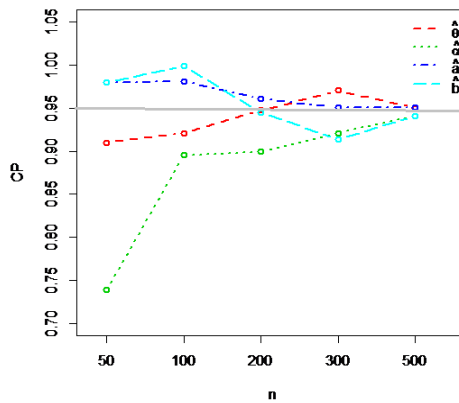


FIG. 3: The CPs for selected parameter values.

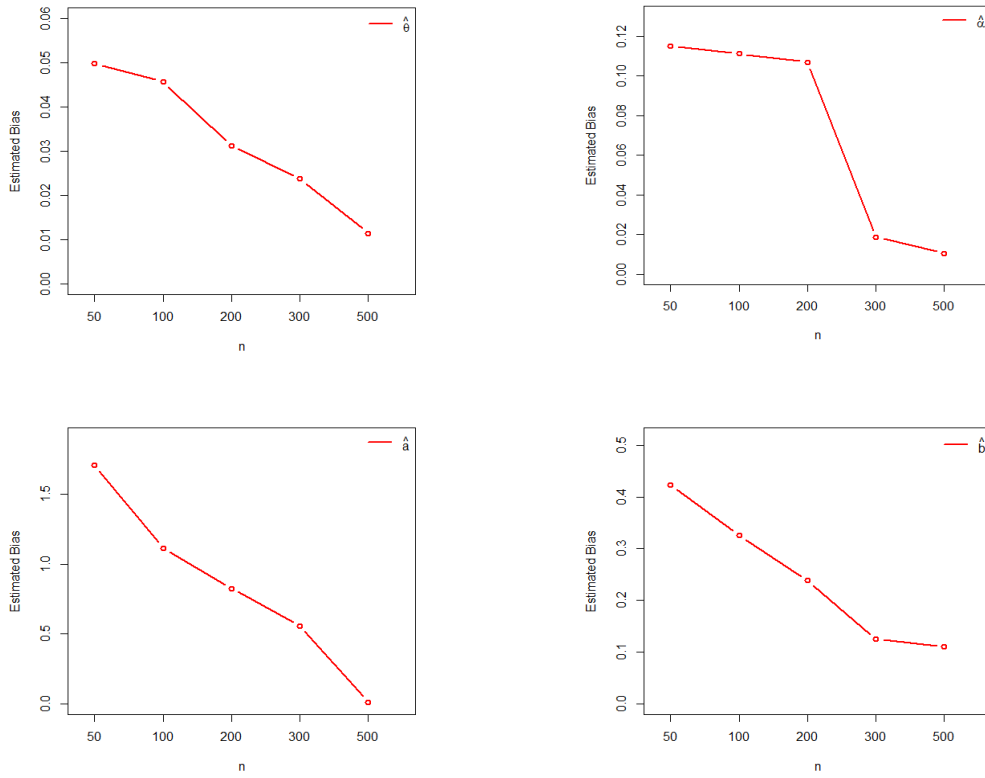


FIG. 4: The Biases for selected parameter values.

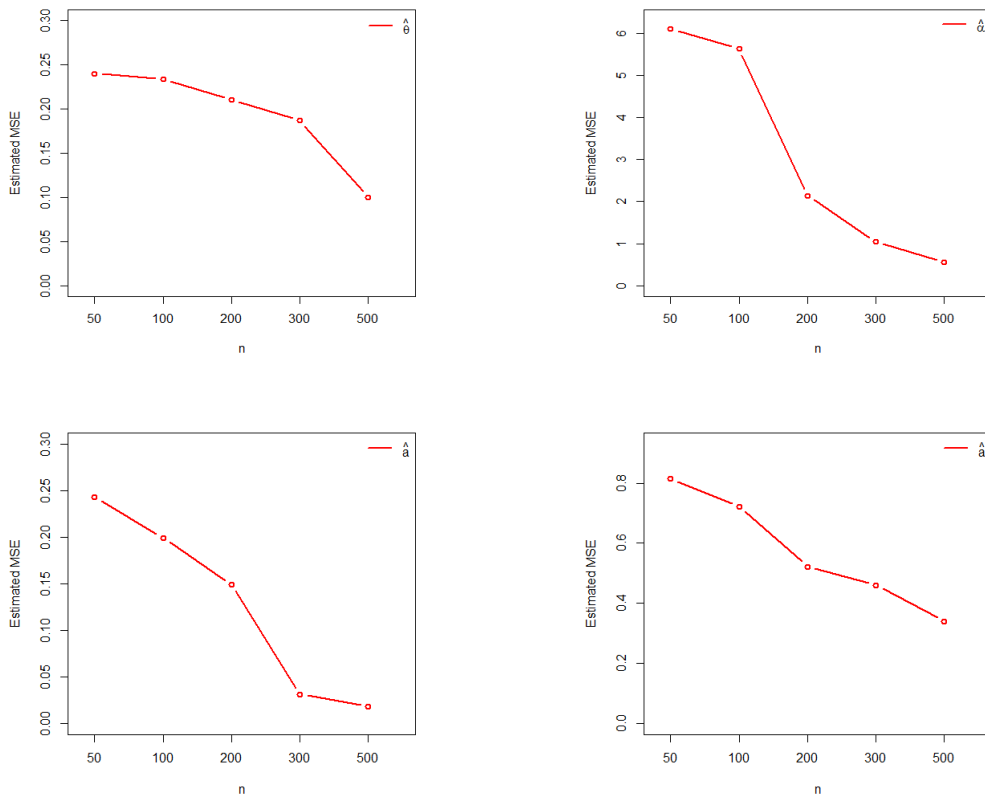


FIG 5: The MSEs for selected parameter values.

## 6. THE TLWLX REGRESSION MODEL FOR CENSORED DATA

In this section, we propose a parametric regression model using the TLWLx distribution. In survival analysis, there are different ways of proposing regression model. For example, the regression models in the form of location-scale frequently used in clinical trials.

Regression analysis of lifetimes involves specifications for the distribution of a lifetime,  $X$  given a vector of covariates denoted by  $w = (w_1, \dots, w_p)^T$ . The most important of the parametric regression models is to be able to allow the parameters to depend on  $w$ . We relate the parameter  $a$  to covariates by the log-linear link function  $a_i = \exp(w_i^T \beta)$ ,  $i = 1, \dots, n$ , where,  $\beta = (\beta_1, \dots, \beta_p)^T$  denotes the vector of regression coefficients. The TLWLx regression model, where the parameter  $a$  depends on  $w$ , is very useful in many practical situations. The survival function of  $X|w$  is given by

$$S(X|w) = 1 - \left\{ 1 - e^{\left[ -2 \left( \frac{1 - (1 + bx) - \exp(w^T \beta)}{(1 + bx) - \exp(w^T \beta)} \right)^\alpha \right]} \right\}^\theta \quad (27)$$

Equation (27) is referred to as the TLWLx parametric regression model with two regression structures. This regression model may be considered a new possibility to fit many different types of data.

Considering a sample  $(x_1, w_1), \dots, (x_n, w_n)$  of  $n$  be a sample of  $n$  independent observations, where the response variable  $x_i$  corresponds to the observed life time or censoring time for the  $i$ th individual. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let  $F$  and  $C$  be the sets of individuals for which  $x_i$  is the lifetime or censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters  $\Theta = (\beta^T, b, \alpha, \theta)^T$  from model (27) has the form  $l(\Theta) = \sum_{i \in F} l_i(\Theta) + \sum_{i \in C} l_i(\Theta)^{(c)}$ , where,  $l_i(\Theta) = \log(f(x_i|w_i))$ ,  $l_i^{(c)}(\Theta) = \log(S(x_i|w_i))$  and  $f(x_i|w_i), S(x_i|w_i)$  are the density and survival functions of  $X$ , respectively. The log-likelihood function for  $\Theta$  reduces to

$$\begin{aligned} l(\Theta) = & r \log(2\theta\alpha b) + \sum_{i \in F} (\alpha \exp(w_i^T \beta) - 1) \log(1 + bx_i) \\ & + (\alpha - 1) \sum_{i \in F} \log(1 - (1 + bx_i)^{-\exp(w_i^T \beta)}) - 2 \sum_{i \in F} \left\{ \frac{1 - (1 + bx_i)^{-\exp(w_i^T \beta)}}{(1 + bx_i)^{-\exp(w_i^T \beta)}} \right\}^\alpha \\ & + (\theta - 1) \sum_{i \in F} \log \left( 1 - e^{\left[ -2 \left( \frac{1 - (1 + bx_i) - \exp(w_i^T \beta)}{1 + bx_i - \exp(w_i^T \beta)} \right)^\alpha \right]} \right) \end{aligned}$$

$$+ \sum_{i \in C} \log \left\{ 1 - \left( 1 - e \left[ -2 \left( \frac{1 - (1 + bx_i)^{-\exp(w_i^T \beta)}}{(1 + bx_i)^{-\exp(w_i^T \beta)}} \right)^\alpha \right] \right)^\theta \right\}, \quad (28)$$

where  $r$  is the number of uncensored observations (failures). The MLEs for the parameter vector  $\Theta = (\alpha, \theta, a, b)^T$  can be obtained by maximizing the likelihood function. A mathematical software can be used in this regards.

## 7. RESIDUAL ANALYSIS

In the literature, various residuals were investigated, see for example, Ortega et al. (2008) and more recently, Pescim et al. (2017). In the context of survival analysis, the deviance residuals have been more widely used because they take into account the information of censored times (Silva et al., 2011). These residuals can also be adopted for the TLWLx regression model. The plot of the deviance residuals versus the observed times provides a consistent way to detect a typical observation and to verify the adequacy of the fitted model. The deviance residual can be expressed as

$$rD_i = \text{sign}(rM_i) \left\{ -2 [rM_i + \delta_i \log(\delta_i - rM_i)] \right\}^{0.5},$$

where

$$rM_i = \begin{cases} 1 + \log \left\{ 1 - \left( 1 - \exp \left[ -2 \left( \frac{1 - (1 + bx_i)^{-\exp(w_i^T \beta)}}{(1 + bx_i)^{-\exp(w_i^T \beta)}} \right)^\alpha \right] \right)^\theta \right\} & \text{if } \delta_i = 1, \\ \log \left\{ 1 - \left( 1 - \exp \left[ -2 \left( \frac{1 - (1 + bx_i)^{-\exp(w_i^T \beta)}}{(1 + bx_i)^{-\exp(w_i^T \beta)}} \right)^\alpha \right] \right)^\theta \right\} & \text{if } \delta_i = 0, \end{cases}$$

is the martingale residual,  $\text{sign}(\cdot)$  is a function that leads to the value  $s + 1$  if the argument positive and  $-1$  if the argument is negative, where  $\delta_i = 1$  indicates that the observation is uncensored and  $\delta_i = 0$  indicates that the observation is censored.

Atkinson (1985) suggested the construction of envelopes to enable better interpretation of the normal probability plot of the residuals. These envelopes are simulated confidence bands that contain the residuals, such that if the model is well-fitted, the majority of points will be within these bands and randomly distributed.

## 8. APPLICATIONS

In this section, we introduce four applications to real data to show the applicability of the TLWLx distribution. The first data set from Murthy et al. (2004). This data set consists of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that have not failed at the time of observation. The unit for measurement is 1000 h. The data are: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467,



0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663. This data is previously analyzed by Ramos et al. (2013) and Tahir et al. (2015).

The second data set are service times of 63 aircraft windshield from Tahir et al. (2015). The data are: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

The third data set represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Firstly, we derive the MLEs for the unknown parameters of all competitive models and then comparing the results via goodness-of-fit statistics; Anderson-Darling (A\*), Cramér–von Mises (W\*), AIC (Akaike information criterion) and BIC (Bayesian information criterion). The better model corresponds to smaller of these criteria. The values for the Kolmogorov Smirnov (KS) statistic and its p-value are also presented.

We compare the TLWLx distribution with those of the Topp-Leone Lomax (TLLx) (Al-Shomrani et al., 2016), Weibull Lomax (WLx), Kumaraswamy Lomax (KwLx), beta Lomax (BLx), exponentiated Lomax (ELx) and Gompertz Lomax (GzLx). The MLEs and some statistics of the models for the three data sets are introduced in Tables (1), (2), (3) (4), (5) and (6) respectively. The estimated pdfs and cdfs plots of all competitive distributions for the three data sets are displayed in Figures 6, 7 and 8 respectively.

TABLE 1: The MLEs for the first data set.

Distribution	Estimates with standard error in parenthesis				
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TLWLx	0.0661 (0.0335)	35.2065 (1.5988)	---	0.0622 (0.0405)	71.4986 (2.1653)
TLLx	---	101.1922 (7.6315)	---	38.6128 (4.1406)	3.5742 (0.6270)
WLx	---	0.0181 (0.0230)	1.0663 (0.6024)	3.9104 (2.6595)	1.8195 (1.9380)
KwLx	---	5.0482 (16.5557)	92.1146 (4.4556)	2.4990 (0.2963)	131.0185 (5.9635)
BLx	---	176.1935 (4.0512)	5.2683 (4.6575)	3.5588 (0.5245)	45.5857 (4.0564)
ELx	70.0101 (6.1184)	---	---	89.6983 (8.8184)	3.6758 (0.6300)
GzLx	0.0094 (0.0066)	0.3837 (0.2635)	---	2.4425 (2.3061)	12.2054 (11.7306)

TABLE 2: Some statistics for the models fitted to the first data set.

Distribution	Estimates with standard error in parenthesis						
	-LL	A*	W*	KS	P-value	AIC	BIC
TLWL <sub>x</sub>	<b>128.5736</b>	<b>0.4021</b>	<b>0.0440</b>	<b>0.05771</b>	<b>0.9397</b>	<b>263.1471</b>	<b>272.9177</b>
TLL <sub>x</sub>	141.8656	1.7895	0.2265	0.1211	0.1651	289.7312	297.0592
WL <sub>x</sub>	128.9776	0.56085	0.0789	0.08534	0.5656	265.1552	274.9258
KwL <sub>x</sub>	132.2679	0.6661	0.0660	0.0679	0.8281	272.5357	282.3063
BL <sub>x</sub>	138.6101	1.3943	0.1659	0.1046	0.3099	285.2202	294.9908
EL <sub>x</sub>	141.9122	1.8101	0.2298	0.1236	0.1484	289.8244	297.1524
GzL <sub>x</sub>	128.8574	0.6065	0.08848	0.0844	0.5799	264.1148	273.8854

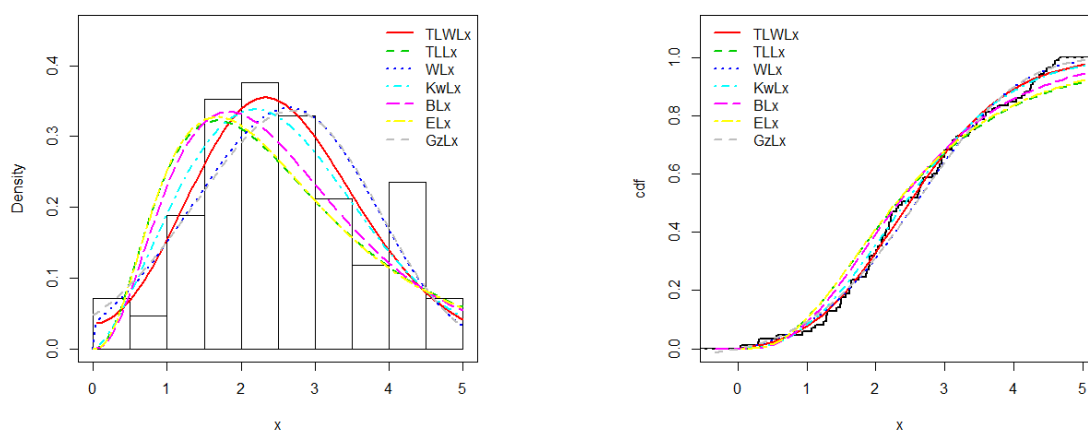


FIG. 6: Estimated pdfs and cdfs plots of the TLWL<sub>x</sub> distribution for data set 1.

TABLE 3: The MLEs for the second data set.

Distribution	Estimates with standard error in parenthesis				
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TLWL <sub>x</sub>	0.1099 (0.0047)	19.2450 (3.5809)	---	0.0842 (0.0662)	280.0032 (2.0273)
TLL <sub>x</sub>	---	51.7111 (5.0939)	---	18.4659 (18.9930)	1.9302 (0.3517)
WL <sub>x</sub>	---	0.1280 (0.3005)	0.9185 (0.4296)	3.9448 (3.9672)	3.0387 (8.5136)
KwL <sub>x</sub>	---	2.5300 (0.6350)	63.8330 (2.2054)	1.6703 (0.2572)	60.2547 (4.9082)
BL <sub>x</sub>	---	136.7057 (1.4524)	4.7006 (1.9155)	1.9250 (0.3192)	26.6660 (8.4992)
EL <sub>x</sub>	42.3281 (6.6741)	---	---	59.3865 (6.3460)	1.9282 (0.3496)
GzL <sub>x</sub>	0.0631 (0.4692)	0.3467 (2.7457)	---	4.3083 (11.7533)	11.9564 (2.2192)

TABLE 4: Some statistics for the models fitted to the second data set.

Distribution	Estimates with standard error in parenthesis						
	-LL	A*	W*	KS	P-value	AIC	BIC
TLWLx	<b>97.9650</b>	<b>0.2021</b>	<b>0.0289</b>	<b>0.0612</b>	<b>0.9607</b>	<b>203.93</b>	<b>212.5026</b>
TLLx	104.0120	1.3092	0.2163	0.1449	0.1281	214.0241	220.4535
WLx	98.1172	0.2417	0.0355	0.0660	0.9303	204.2345	212.8070
KwLx	100.8724	0.7396	0.1219	0.1145	0.3531	209.7447	218.3173
BLx	102.9920	1.1388	0.1880	0.1397	0.1553	213.984	222.5565
ELx	103.9526	1.2994	0.2146	0.14545	0.1258	213.905	220.3345
GzLx	98.1363	0.2555	0.03841	0.0675	0.9170	204.2726	212.8451

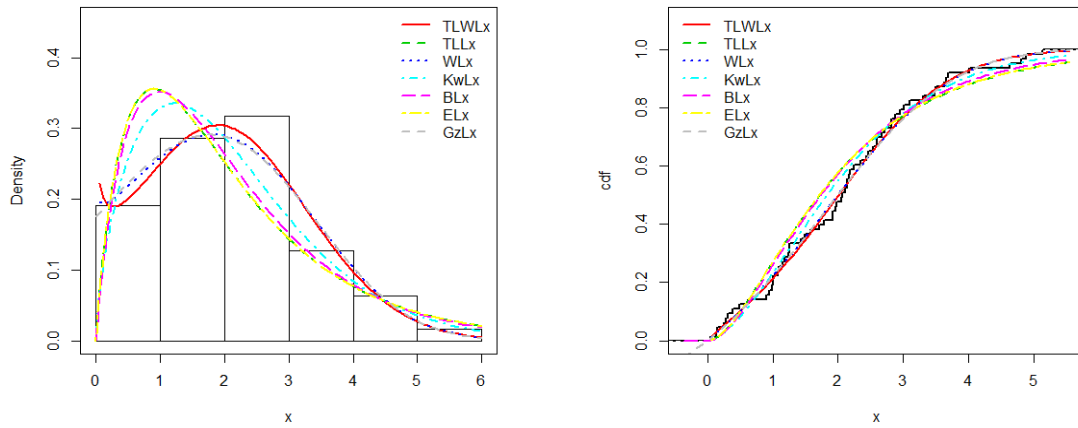


FIG. 7: Estimated pdfs and cdfs plots of the TLWLx distribution for data set 2.

TABLE 5: The MLEs for the third data set.

Distribution	Estimates with standard error in parenthesis				
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
TLWLx	0.0618 (0.0184)	39.7632 (9.5340)	---	0.9711 (0.5131)	12.2198 (2.5988)
TLLx	---	36.6009 (3.8435)	---	18.2082 (1.1266)	2.7983 (0.5622)
WLx	---	20.7373 (2.2730)	2.1978 (0.5650)	0.2554 (0.2483)	1.4122 (1.7884)
KwLx	---	0.8648 (3.5923)	4.1583 (8.5076)	2.3171 (0.5825)	17.9444 (2.7900)
BLx	---	22.8693 (5.1740)	2.5895 (0.3373)	2.61489 (0.5778)	12.4953 (8.5096)
ELx	26.8885 (1.6895)	---	---	26.3708 (3.4618)	2.8679 (0.5822267)
GzLx	0.0026 (0.0004)	0.1665 (0.0083)	---	0.2337 (0.0099)	10.8690 (0.01713)

TABLE 6: Some statistics for the models fitted to the third data set.

Distribution	Estimates with standard error in parenthesis						
	-LL	A*	W*	KS	P-value	AIC	BIC
TLWLx	<b>100.8678</b>	<b>0.4246</b>	<b>0.0666</b>	<b>0.079385</b>	<b>0.7546</b>	<b>209.7357</b>	<b>218.8423</b>
TLLx	103.0818	0.7157	0.10255	0.1089	0.3595	212.1635	218.9935
WLx	102.7830	0.7327	0.1124	0.1051	0.4031	213.5660	222.6726
KwLx	102.4251	0.6501	0.0969	0.0993	0.4763	212.8503	221.957
BLx	102.9073	0.7153	0.1054	0.1045	0.4108	213.8146	222.9213
ELx	103.1329	0.7172	0.1021	0.1086	0.3634	212.2659	219.0959
GzLx	104.8148	1.0984	0.18508	0.11452	0.3015	217.6295	226.7362

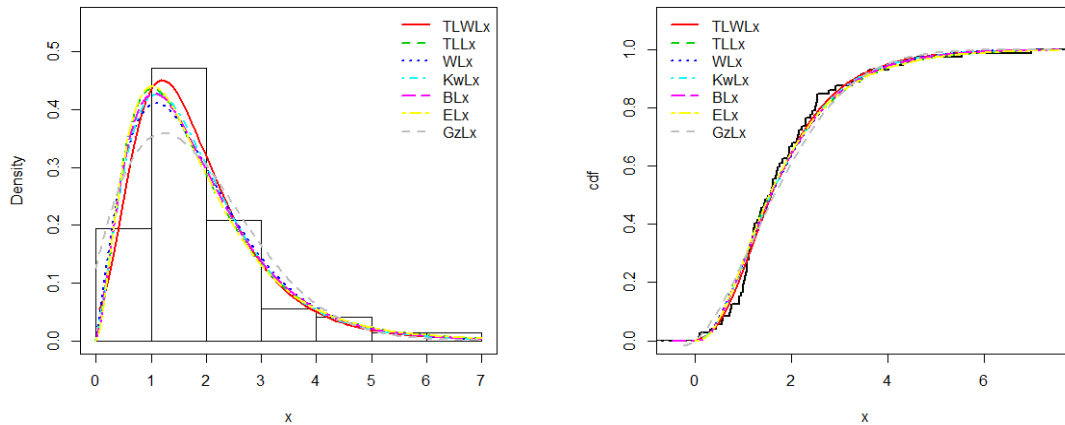


FIG. 8: Estimated pdfs and cdfs plots of the TLWLx distribution for data set 3.

The values in Tables 2, 4 and 6 showed that the TLWLx distribution has the smallest values for A\*, W\*, AIC, BIC, KS and largest P-values among all fitted models (TLLx, WLx, KwLx, BLx, ELx and GzLx) for the three real data sets. It is also clear from Figures 6, 7 and 8, that the TLWLx distribution provides the best fit as compare to TLLx, WLx, KwLx, BLx, ELx and GzLx for given three data sets. So, the TLWLx model could be chosen as the best model.

### 8.1. REGRESSION MODEL FOR TRANSFORMER TURN DATA

In this application, we use a real data set available in the book of (Nelson and Wayne; 2004) to study the TLWLx regression model in the presence of censoring. The data are given in Table (7) where the symbol (+) represents the censored data. The model parameters are estimated by maximum likelihood using the optim subroutine in R. We compared the proposed model with the Weibull distribution.

TABLE 7:Transformer turn data.

Voltage	Hours									
35.4kV	40.1	59.4	71.2	166.5	204.7	229.7	308.3	537.9	1002.3+	1002.3+
42.4kV	0.6	13.4	15.2	19.9	25.0	30.2	32.8	44.4	50.2+	56.2
46.7kV	3.1	8.3	8.9	9.0	13.6	14.9	16.1	16.9	21.3	48.1+

Firstly, we describe the data set. Next, we presented the MLEs and their standard errors (SEs) in parentheses, global deviance (GD) and the Akaike information criterion (AIC) and Bayesian information criterion (BIC) statistics. This data set presents a transformer life testing at high voltage resulted in the multiplied censored data below. All failures were turn-to-turn failures of the primary insulation considering three voltage levels: 35.4kV, 42.4kV and 46.7kV in which 10 samples were tested for each of the three levels. The variables considered in this study are  $x_i$  = time of failure in hours of the transformer,  $i = 1, \dots, 30$  and the three voltage levels (35.4kV, 42.4kV and 46.7kV) are defined by dummy variables, 35.4kV: ( $w_{i1} = 1, w_{i2} = 0$ ), 42.4kV: ( $w_{i1} = 0, w_{i2} = 1$ ), and 46.7kV: ( $w_{i1} = 0, w_{i2} = 0$ ).

The percentage of censored observations for survival times is 13%. We consider in a preliminary analysis only the times of survival and censoring. The TTT-plot is displayed in Fig.9, which indicates that the hrf associated with the data set has a decreasing shape, since the plot shows a first convex curvature.

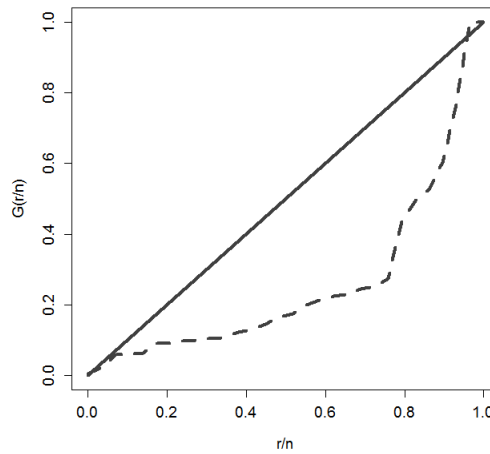


FIG.9: TTT-plot curve for transformer turn data.

The MLEs of the parameters  $a$ ,  $b$ ,  $\alpha$  and  $\theta$  (the corresponding SEs in parentheses) and the values of AIC, BIC and GD statistics are given in Table (8). We consider the TLWLx and Weibull distributions.

TABLE 8:MLEs of the model parameters for transformer turn data, the corresponding SEs (given in parentheses)and the GD, AIC and BIC statistics.

Model	Estimates with standard error in parenthesis				Statistics		
	$\hat{\alpha}$	$\hat{\theta}$	$\hat{a}$	$\hat{b}$	GD	AIC	BIC
TLWLx	0.5945 (0.4885)	2.9340 (3.7060)	0.3032 (0.2720)	0.1059 (0.1462)	285.5	293.5	299.1
Weibull	113.7722 (42.9772)	0.5671 (0.0836)	---	---	293.8	297.8	300.7

In order to evaluate, if the model is appropriate, Fig. 10a displays the estimated survival functions of the TLWLx and Weibull distributions and the empirical survival function. As an alternative to verify the quality of fit, Fig.10b displays the estimated hrfs of the TLWLx and Weibull distributions and the empirical hrf. We conclude that the TLWLx distribution provides a good fit to the current data.

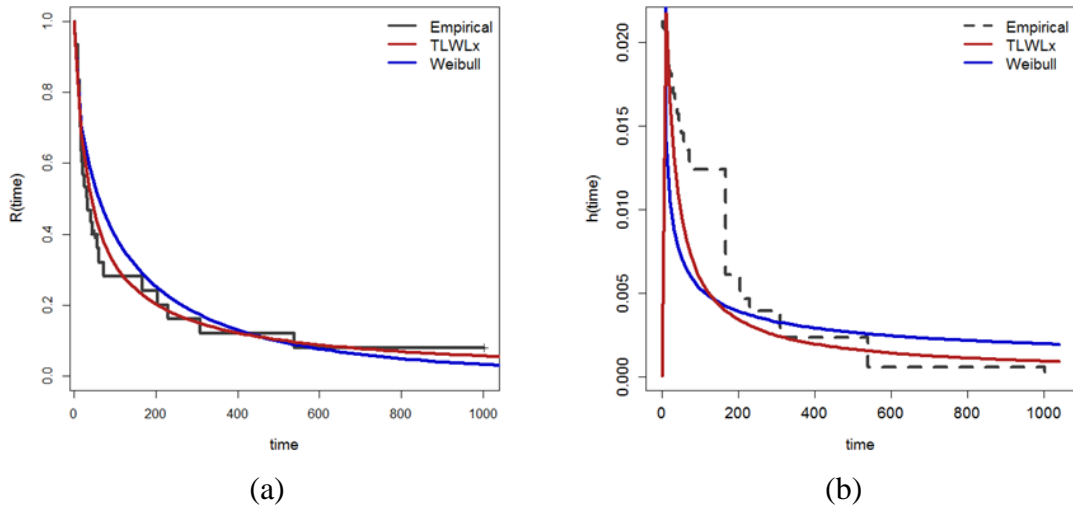


FIG 10: (a) Estimated survival function with adjustment of the TLWLx and weibull distributions and empirical survival for the transformer turn data and (b) Estimated hrf with adjustment of the TLWLx and weibull distributions and empirical hazard for the transformer turn data.

We consider the following structures regression

$$a = \exp(\beta_{10} + \beta_{11}w_{i1} + \beta_{12}w_{i2}), i = 1, \dots, 30,$$

to maximize the log-likelihood function (28) and thus obtain the MLEs of the parameters of the proposed model. Thus, in Table (9) we provide the parameter estimates, standard errors and significance of the MLEs. The figures in this table indicate that there is evidence that the presence of the only covariate  $w_1$  is significant at 5% level of significance, that is, there is a significant difference of the level 135.4Kv with respect to the level 46.7Kv.

TABLE 9: MLEs, SEs and p-values for regression model fitted to the transformer turn data.

Parameters	Estimate	SE	p-value
$\alpha$	0.9232	0.6419	---
$\theta$	1.7458	1.6861	---
$b$	0.0160	0.0198	---
$\beta_{10}$	0.6985	1.2013	0.5654
$\beta_{11}$	-2.0103	0.6276	0.0032
$\beta_{12}$	-0.4385	0.2974	0.1511

### Goodness of fit

We plot the residual analysis in Fig.11, the deviance component residual  $rD_i$  (see Section 9). Fig.11a shows some large residuals (observations #9, #10, #11, #19 and #30), although Fig.11b gives the normal probability plot with generated envelope, supports the

hypothesis that the TLWLx regression model is suitable for these data, since only one observation is outside the envelope.

In order to assess if the model is appropriate, the plot comparing the empirical survival function and estimated survival function for the TLWLx regression model is displayed in Fig. 12a. The curves displayed in this figure are the empirical survival and estimated survival functions determined. Based on this figure, we note that TLWLx regression model show satisfactory fit. We observed there is a significant difference of the level 35.4Kv with respect to the 46.7Kv. We also present in Fig.12b the fitted hazard functions. From these plots, we can observe a significant different between the survival curves. However, the TLWLx regression model presents a better fit to the current data.

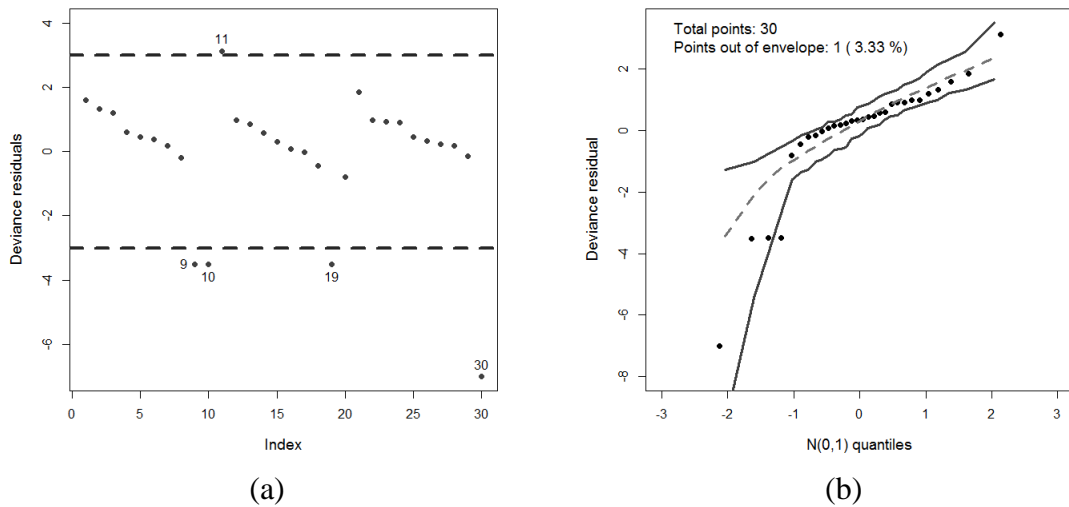


FIG. 11: (a) Index plot of the deviance residual for the transformer turn data and (b) Normal probability plot for the deviance component residual with envelopes from the fitted of TLWLx regression model to transformer turn data.

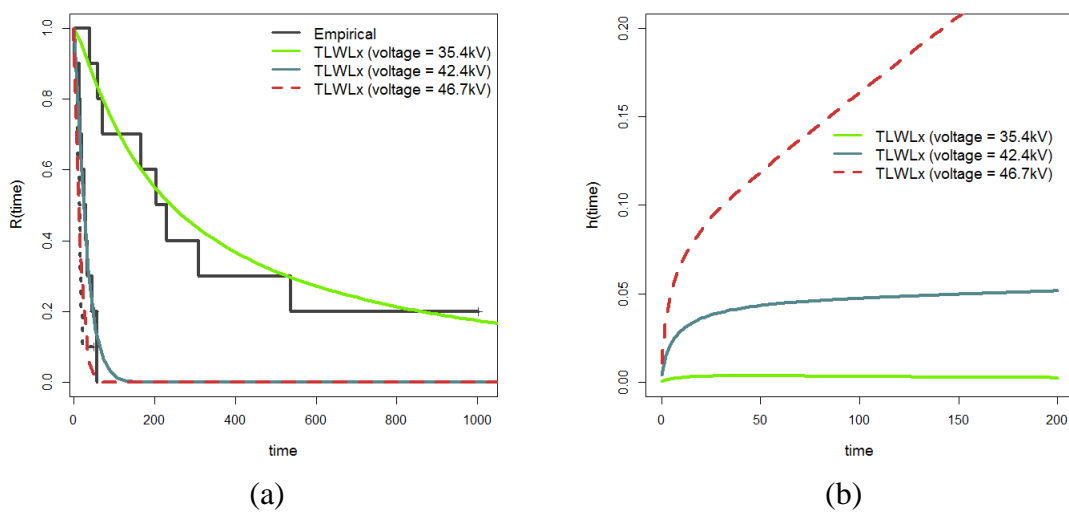


FIG. 12: (a) Estimated survival function for the TLWLx regression model and the empirical survival for all levels of 35.4, 42.4 and 46.7Kv to transformer turn data and (b) For all levels of 35.4, 42.4 and 46.7Kv, the fitted hazard functions to transformer turn data.

## 9. CONCLUSIONS

We have proposed a new four parameter distribution named as the Topp Leone Weibull Lomax distribution that extends the Lomax distribution. We have investigated main mathematical properties of the new model such as quantile function probability weighted moment, ordinary and incomplete moments, conditional moments, order statistics and stress strength model. The model parameters of the proposed model is estimated via the method of maximum likelihood. The regression model and the residual analysis are studied for the new distribution. The usefulness of the new model is illustrated via four real life applications. Numerical computations showed that the proposed distribution can be considered as good alternative to the Topp Leone Lomax, Weibull Lomax, Kumaraswamy Lomax, beta Lomax, exponentiated Lomax and Gompertz Lomax distributions.

## REFERENCES

- [1] Al-Shomrani, A., Arif, O., Shawky, K., Hanif, S. and Shahbaz, M. Q. (2016). Topp Leone family of distributions: some properties and application. *Pak.j.stat.oper.res.* 3: 443-451.
- [2] Anwar, M. and Zahoor, J. (2018). The half logistic Lomax distribution for lifetime modeling. *Journal of Probability and Statistics*; Article ID 3152807, 12 pages. <https://doi.org/10.1155/2018/3152807>
- [3] Atkinson, A.C. (1985). *Plots, transformations and regression: an introduction to graphical methods of diagnostic regression analysis*. Clarendon Press Oxford, Oxford.
- [4] Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *Am J Hyg*, 72: 130-148.
- [5] Cordeiro, G., Ortega E. and Popović, B. (2013). The gamma Lomax distribution. *J Stat Comput Simul*; 85(2):305–319
- [6] El-Bassiouny, A. H., Abdo, N. F. and Shahan, H. S. (2015). Exponential Lomax distribution. *International Journal of Computer applications*, 121(13): 24-29.
- [7] Lemonte, A. and Cordeiro, G. (2013). An extended Lomax distribution. *Statistics*, 47: 800-816.
- [8] Lomax, K. S. (1954). Business failures: another example of the analysis of failure data. *Journal of the American Statistical Association*; 49(268): 847-852.
- [9] Nelson, Wayne B. *Accelerated testing: statistical models, test plans, and data analysis*. John Wiley & Sons, 2004.
- [10] Murthy, D, N, P., Xie, M. and Jiang, R. *Weibull models*. Wiley; 2004.
- [11] Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O. and Odetunmbi, O. A. (2017). A New generalization of the Lomax distribution with increasing, decreasing and constant failure rate. *Modelling and Simulation in Engineering*, Article ID 6043169, 6 pages.
- [12] Pescim, R.R., Ortega, E.M.M., Cordeiro, G.M. and alizadeh, M. (2017). A new log location regression model: estimation, influence diagnostics and residual analysis. *Journal of Applied Statistics*, 44: 233-252.
- [13] Silva, G.O., Ortega, E.M.M. and Paula, G.A. (2011). Residual for log-Burr XII regression models in survival analysis. *Journal of Applied Statistics*, 38: 1435-1445.



- [14] Tahir, M. H., Cordeiro, G. M., Mansoor, M. and Zubair, M. (2015). The Weibull Lomax distribution: properties and applications. Hacettepe Journal of Mathematics and Statistics, 44(2): 461–480.