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LAGRANGIAN CONFIGURATIONS AND SYMPLECTIC CROSS-RATIOS

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Abstract. We consider moduli spaces of cyclic configurations of N lines in a $2n$-dimensional symplectic vector space, such that every set of n consecutive lines generates a Lagrangian subspace. We study geometric and combinatorial problems related to these moduli spaces, and prove that they are isomorphic to quotients of spaces of symmetric linear difference operators with monodromy -1.

The symplectic cross-ratio is an invariant of two pairs of 1-dimensional subspaces of a symplectic vector space. For $N = 2n + 2$, the moduli space of Lagrangian configurations is parametrized by $n + 1$ symplectic cross-ratios. These cross-ratios satisfy a single remarkable relation, related to tridiagonal determinants and continuants, given by the Pfaffian of a Gram matrix.

Contents

1. Introduction

Throughout this article, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$. We consider configurations of one-dimensional subspaces in the standard symplectic space $\mathbb{K}^{2n}$, or equivalently, configurations of points in the contact projective space $\mathbb{KP}^{2n-1}$, modulo the action of the symplectic group $\mathrm{Sp}(2n,\mathbb{K})$.

Definition. For $N \geqslant 2n$, define an (n,N)-*Lagrangian configuration* over $\mathbb{K}$ to be a cyclically ordered N-tuple $(X_1,\dots,X_N)$ of lines through the origin in the symplectic space $\mathbb{K}^{2n}$ with the following properties (by cyclically ordered, we mean that the indices are read modulo N):

(i) Every n consecutive lines span a Lagrangian subspace: $\langle X_{i+1},\dots,X_{i+n}\rangle$ is Lagrangian for all i.

(ii) Every $2n$ consecutive lines span the entire symplectic space: $\langle X_{i+1},\dots,X_{i+2n}\rangle = \mathbb{K}^{2n}$ for all i.

Equivalently, we may speak of *Legendrian configurations* of points in the contact projective space $\mathbb{KP}^{2n-1}$. We have formulated our results in the symplectic setting, but let us mention that Legendrian configurations in $\mathbb{RP}^3$ may be viewed as discrete analogs of *Legendrian knots.* In another direction, one may consider cyclically ordered N-tuples of Lagrangian subspaces $(L_1,L_2,\dots,L_N)$ in $\mathbb{R}^{2n}$ such that every two consecutive subspaces L_i and L_{i+1} are "maximally non-transversal". These configurations are in some sense dual to Lagrangian configurations, and the *Maslov index* may be applied to study them; see [2]. Continuous versions were treated in [16].

Suppose that an arbitrary cyclically ordered N-tuple $(X_1,\dots,X_N)$ of lines through the origin in $\mathbb{K}^{2n}$ has Property (i) above. We will see in Lemma 2.2 that then it has Property (ii) if and only if the subspace $\langle X_i, X_{i+n}\rangle$ is not isotropic for any i. It turns out that $\mathrm{Sp}(2n,\mathbb{K})$ does not act freely on all Lagrangian configurations, but for $N > 2n$, it does act freely on configurations in which $\langle X_i, X_j\rangle$ is not isotropic except when forced to be so by Property (i); see Proposition 2.7. We refer to such configurations as generic:

Definition. An (n,N)-Lagrangian configuration $(X_1,\dots,X_N)$ is *generic* if $\langle X_i, X_j\rangle$ is non-isotropic whenever the N-cyclic distance between i and j is at least n, that is, $|i-j|$ is not congruent to any of $0,\dots,n-1$ modulo N. We denote the $\mathrm{Sp}(2n,\mathbb{K})$-moduli space of generic (n,N)-configurations over $\mathbb{K}$ by $\mathcal{L}_{n,N}(\mathbb{K})$.

The space $\mathcal{L}_{n,N}(\mathbb{K})$ is the main object of our study. We will see that $\mathrm{Sp}(2n,\mathbb{K})$ acts freely on generic configurations, implying that $\mathcal{L}_{n,N}(\mathbb{K})$ is a variety of dimension $n(N-2n-1)$, and moreover, $\mathcal{L}_{n,N}(\mathbb{R})$ and $\mathcal{L}_{n,N}(\mathbb{C})$ are smooth real and complex manifolds of this dimension, respectively. We construct a collection of *symplectic cross-ratios* which are $\mathrm{Sp}(2n,\mathbb{K})$-invariants of Lagrangian configurations over $\mathbb{K}$, and in some cases show that these cross-ratios form a coordinate ring on $\mathcal{L}_{n,N}(\mathbb{K})$. They satisfy certain relations, which we calculate explicitly as Pfaffians for $N = 2n+2$, the simplest non-trivial case. These Pfaffians are closely related to the classical determinants of continued fractions known as *continuants;* see [3].

Observe that for $n = 1$ the Lagrangian condition is vacuous, so $\mathcal{L}_{1,N}(\mathbb{K})$ is essentially the classical moduli space $\mathcal{M}_{0,N}$ of configurations of N points on the projective line. We regard $\mathcal{L}_{n,N}(\mathbb{K})$ as a multi-dimensional symplectic variant of $\mathcal{M}_{0,N}$. The only previously studied configurations in symplectic space we know of are triangles and skew lines; see [22] and Section 2.8 of [15]. To the best of our knowledge, Lagrangian configurations have not been considered before. We believe that they deserve further study; in particular, it would be interesting to investigate the topology of $\mathcal{L}_{n,N}(\mathbb{K})$. It seems plausible that the topological invariants of Legendrian knots, for instance, the Maslov class and the Bennequin invariant, as well as more general invariants, can be expressed in terms of cross-ratios of a Lagrangian configuration.

Relations to dynamical systems also seem promising. Moduli spaces of cyclic configurations of points in $\mathbb{RP}^n$ (without any Legendrian condition) carry a family of discrete integrable systems, including the pentagram map and its generalizations; see [6, 8, 9, 18, 19, 21]. We believe that $\mathcal{L}_{n,N}(\mathbb{K})$ also supports interesting discrete dynamical systems.

1.1. **Example: hexagons in $\mathbb{KP}^3$.** Our main geometric result is a description of $\mathcal{L}_{n,2n+2}$. The simplest case, $\mathcal{L}_{1,4}(\mathbb{K})$, is the moduli space of quadrilaterals in $\mathbb{KP}^1$. It has been known since ancient times to be 1-dimensional and parametrized by the classical cross-ratio. Therefore the first new case is $\mathcal{L}_{2,6}(\mathbb{K})$, the $\mathrm{Sp}(4,\mathbb{K})$-moduli space of generic $(2,6)$-Lagrangian configurations, i.e., Legendrian hexagons in $\mathbb{KP}^3$.

Given any $(2,6)$-configuration, choosing a non-zero point on each of the six lines gives a hexagon $(x_0,\dots,x_5)$ in $\mathbb{K}^4$. It turns out to be natural to regard this hexagon as the 6-antiperiodic sequence $(x_i)_{i\in\mathbb{Z}}$ defined by $x_{i\pm 6} := -x_i$. Then

$$\omega(x_i, x_{i+3}) = \omega(x_{i+3}, x_{i+6}), \qquad \omega(x_i, x_{i+2}) = \omega(x_{i+2}, x_{i+6}),$$

where here and throughout the paper ω is the standard symplectic form on $\mathbb{K}^{2n}$ (see Section 2.5). The sequences $\big(\omega(x_i, x_{i+2})\big)_{i\in\mathbb{Z}}$ and $\big(\omega(x_i, x_{i+3})\big)_{i\in\mathbb{Z}}$ are 6-periodic and 3-periodic, respectively.

The Lagrangian conditions are $\omega(x_i, x_{i+1}) = 0$ and $\omega(x_i, x_{i+2}) \neq 0$. Thus we may say that we are considering hexagons whose sides are of "symplectic length zero", but whose "symplectic subdiameters" are non-zero. The generic configurations are those with non-zero "symplectic diameters": $\omega(x_i, x_{i+3}) \neq 0$.

The three "diametric symplectic cross-ratios"

$$c_i := \frac{\omega(x_i, x_{i+3})\,\omega(x_{i+1}, x_{i+4})}{\omega(x_i, x_{i+4})\,\omega(x_{i+1}, x_{i+3})}, \quad i = 0,\, 1,\, 2,$$

depend only on the original configuration of lines, not the choice of points x_i, and are symplectic invariants. We will see that they form an essentially complete set of invariants parametrizing $\mathcal{L}_{2,6}(\mathbb{K})$. As an example, we illustrate c_0 by the following diagram.

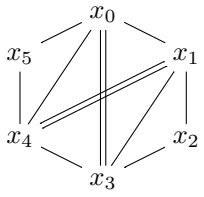

Figure 1. The cross-ratio $c_0 = \frac{\omega(x_0,x_3)\,\omega(x_1,x_4)}{\omega(x_0,x_4)\,\omega(x_1,x_3)}$ on $\mathcal{L}_{2,6}(\mathbb{K})$.

Observe that the space of all $(2,6)$-Lagrangian configurations is 12-dimensional: there are three degrees of freedom for each of the six points, and six Lagrangian conditions. As mentioned earlier, the 10-dimensional group $\mathrm{Sp}(4,\mathbb{K})$ acts freely on the generic configurations, so $\mathcal{L}_{2,6}(\mathbb{K})$ is 2-dimensional. Therefore the three cross-ratios cannot be independent. In fact, they satisfy the relation

$$\frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} = 1. \tag{1.1}$$

For $\mathbb{K} = \mathbb{C}$ this analysis can be reformulated as follows. By rescaling, the x_i can be normalized so that the symplectic subdiameters $\omega(x_i, x_{i+2})$ are all 1. Then the symplectic diameters $a_i := \omega(x_i, x_{i+3})$ become, up to an overall choice of sign, symplectic invariants. Indeed, here $c_i = a_i a_{i+1}$, so (1.1) becomes

$$a_0 a_1 a_2 = a_0 + a_1 + a_2. \tag{1.2}$$

For $\mathbb{K} = \mathbb{R}$, it may happen that only complex rescalings can bring all subdiameters to 1. However, the required scale factors are always either real or pure imaginary. There are four possibilities: the normalized x_i with i even are either all real or all pure imaginary, and similarly for i odd. If the normalized x_i are all real or all imaginary, then the a_i are all real, while if the normalized x_i are half real and half imaginary, then the a_i are all imaginary.

Let us remark that up to permutation, $(c_0, c_1, c_2) = (2, 3, 6)$ is the only *Egyptian fraction* solution of (1.1). It arises from $(a_0, a_1, a_2) = (1, 2, 3)$, the only positive integer solution of (1.2). Integer solutions of the multi-dimensional analogs of these relations are discussed in [3, 17].

1.2. **Example: the Gauss relations.** In the final section of this article we make some initial remarks on the relations between the symplectic cross-ratios of $\mathcal{L}_{n,2n+3}$. Historically, the earliest examples of relations between cross-ratios arose in Gauss' *pentagramma mirificum* [7], which is $\mathcal{M}_{0,5}(\mathbb{K})$, or equivalently, $\mathcal{L}_{1,5}(\mathbb{K})$, the moduli space of pentagons in $\mathbb{KP}^1$.

As we did for Legendrian hexagons, given five points in $\mathbb{KP}^1$, lift them to non-zero points $x_0, \cdots, x_4$ in $\mathbb{K}^2$ and extend to a 5-antiperiodic sequence $(x_i)_{i\in\mathbb{Z}}$ via $x_{i\pm 5} := -x_i$. Gauss discovered that the 5-periodic sequence of cross-ratios $d_i := \omega(x_{i-3}, x_i)\omega(x_{i-2}, x_{i-1})/\omega(x_{i-3}, x_{i-2})\omega(x_{i-1}, x_i)$ satisfy the relations

$$d_i d_{i+1} = d_{i+3} + 1. \tag{1.3}$$

These five *Gauss relations* completely determine the varietal structure of $\mathcal{M}_{0,5}$. They can be rewritten in the remarkable form

$$\begin{pmatrix} d_0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d_1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d_2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d_3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d_4 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This relates the topic to two classical subjects: the theory of continued fractions and the theory of linear difference equations. The Gauss relations were the main motivation for Coxeter [4] to develop the notion of frieze patterns, relating projective geometry to combinatorics. Friezes provide a special parametrization of $\mathcal{M}_{0,N}$; see [13] and the appendix of [14].

We regard the relations between the symplectic cross-ratios of $\mathcal{L}_{n,N}$ as multi-dimensional analogs of the Gauss relations. Building on preliminary versions of this article, Morier-Genoud [12] has studied the combinatorial aspects of $\mathcal{L}_{2,N}(\mathbb{C})$, the moduli space of Legendrian N-gons in $\mathbb{CP}^3$. Her work indicates that in general, $\mathcal{L}_{n,N}(\mathbb{K})$ has a rich combinatorial structure related to friezes.

1.3. **Outline of results.** It is natural to ask for a coordinate system on the moduli space $\mathcal{L}_{n,N}(\mathbb{K})$ of (n, N)-Lagrangian configurations. In this article we show that this question is vacuous when N is $2n$ or $2n+1$, and answer it when N is $2n+2$. Our coordinates are given by the symplectic cross-ratio, a direct analog of the classical cross-ratio: we show that $\mathcal{L}_{n,2n+2}(\mathbb{K})$ is parametrized by symplectic cross-ratios and determine its structure as an algebraic variety. We expect that symplectic cross-ratios parametrize $\mathcal{L}_{n,N}(\mathbb{K})$ for all N. The exposition is organized as follows.

In Section 2 we define symplectic cross-ratios and show that they provide continuous invariants on (n, N)-Lagrangian configurations for $N \geqslant 2n+2$. We also deduce the dimension of $\mathcal{L}_{n,N}(\mathbb{K})$ and define *opposite* configurations and equivalence classes, which over $\mathbb{R}$ are distinguished by *sign invariants.*

Section 3 contains our main geometric results, which we summarize here:

- $(n, 2n)$-configurations are all generic and equivalent over both $\mathbb{C}$ and $\mathbb{R}$.
- $(n, 2n+1)$-configurations are all generic. Over $\mathbb{C}$ they are all equivalent, and over $\mathbb{R}$ there are two equivalence classes, which are opposite.
- $(n, 2n+2)$-configurations admit $n+1$ diametric symplectic cross-ratios $c_0, \ldots, c_n$:

$$c_i := \frac{\omega(x_i, x_{i+n+1})\,\omega(x_{i+1}, x_{i+n+2})}{\omega(x_i, x_{i+n+2})\,\omega(x_{i+1}, x_{i+n+1})}, \tag{1.4}$$

  the x_i being arbitrary non-zero points on the lines of the configuration.

  Over $\mathbb{C}$, generic $(n, 2n+2)$-configurations are equivalent if and only if they have the same diametric cross-ratios. Over $\mathbb{R}$, generic $(n, 2n+2)$-configurations with the same cross-ratios are either equivalent or in opposite equivalence classes.

  The moduli space $\mathcal{L}_{n,2n+2}(\mathbb{K})$ is n-dimensional. The $n+1$ cross-ratios satisfy the relation (3.2), and any collection of non-zero $\mathbb{K}$-scalars $(c_0, \ldots, c_n)$ satisfying (3.2) is the set of cross-ratios of an $(n, 2n+2)$-configuration over $\mathbb{K}$. Thus the cross-ratios are coordinates describing $\mathcal{L}_{n,2n+2}(\mathbb{K})$ as an algebraic hypersurface in $\mathbb{K}^{n+1}$. These results are presented in Theorem 1.

For $(n, 2n+2)$-configurations the proof has several components and is given in Section 4.

In Section 5 we present certain normalized choices of the x_i generalizing Section 1.1. For n even there is an essentially unique choice such that the symplectic subdiameters $\omega(x_i, x_{i+n})$ are all 1. This normalization provides an alternate coordinate system on $\mathcal{L}_{n,2n+2}(\mathbb{K})$: the symplectic diameters $a_0, \dots, a_n$, where $a_i := \omega(x_i, x_{i+n+1})$. These diameters are determined up to an overall choice of sign, and for $\mathbb{K} = \mathbb{R}$ they are either all real or all pure imaginary. In this coordinate system the relation (3.2) can be written in terms of the celebrated classical determinants called *continuants*. This connection is emphasized in Section 3.3; see Theorem 2.

For n odd, one cannot in general choose the points x_i so that the symplectic subdiameters are all 1, but one can choose them so that the subdiameters alternate between a scalar μ and its reciprocal, and the two alternating products of diameters are equal: $a_0 a_2 \cdots a_{n-1} = a_1 a_3 \cdots a_n$. Here μ is determined up to a $(2n+2)^{\text{nd}}$ root of unity, and the a_i are determined up to an overall $(n+1)^{\text{st}}$ root of unity.

An old idea of projective differential geometry consists in representing geometric objects such as curves or configurations of points via differential or difference operators. Following this approach, in Section 6 we realize the moduli space of Lagrangian configurations as the quotient by rescaling of the space of symmetric linear difference equations with periodic coefficients and antiperiodic solutions; see Theorem 3.

We conclude in Section 7 with a preliminary discussion of $\mathcal{L}_{n,2n+3}$, including a general result on normalizations in the case that $N/\operatorname{GCD}(n,N)$ is odd, and relations on cross-ratios for $\mathcal{L}_{2,7}$ and $\mathcal{L}_{3,9}$, the moduli spaces of generic Legendrian heptagons in $\mathbb{KP}^3$ and Legendrian nonagons in $\mathbb{KP}^5$.

2. Lagrangian configurations and their moduli spaces

In this section we collect some basic properties of Lagrangian configurations and the action of $\mathrm{Sp}(2n,\mathbb{K})$ on them. We prove that the action is free on generic configurations and introduce two types of invariants: continuous invariants known as symplectic cross-ratios, and certain discrete sign invariants.

2.1. **Symplectic cross-ratios.** Consider two pairs of points in $(\mathbb{K}^{2n}, \omega)$, (x_1, x_2) and (y_1, y_2), such that $\omega(x_1, y_2)$ and $\omega(x_2, y_1)$ are non-zero. We define their *symplectic cross-ratio* to be

$$[x_1, x_2; y_1, y_2] := \frac{\omega(x_1, y_1)\,\omega(x_2, y_2)}{\omega(x_1, y_2)\,\omega(x_2, y_1)}. \tag{2.1}$$

The symplectic cross-ratio is obviously invariant with respect to both the action of the symplectic group $\mathrm{Sp}(2n,\mathbb{K})$ and rescalings $x_i \mapsto \lambda_i x_i$ and $y_i \mapsto \mu_i y_i$. Therefore it is in fact a symplectic invariant of two pairs of one-dimensional subspaces in $\mathbb{K}^{2n}$, or equivalently, of two pairs of points in $\mathbb{KP}^{2n-1}$. Observe the symmetries

$$[x_2, x_1; y_1, y_2] = [x_1, x_2; y_1, y_2]^{-1}, \qquad [y_1, y_2; x_1, x_2] = [x_1, x_2; y_1, y_2]. \tag{2.2}$$

Remark. For $n > 1$, (2.1) is not the only symplectic invariant of a quadruple $(\mathbb{K}x_1, \mathbb{K}x_2, \mathbb{K}y_1, \mathbb{K}y_2)$ of lines in $\mathbb{K}^{2n}$. However, it *is* the only such invariant if

- $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ are *isotropic,* i.e., $\omega(x_1, x_2)$ and $\omega(y_1, y_2)$ are 0, and
- (x_1, x_2, y_1, y_2) is *generic* under this condition.

In the one-dimensional case, (2.1) is nothing but the classical cross-ratio of 4 points on the projective line. In affine coordinates, it is given by the usual formula:

$$[x_1, x_2; y_1, y_2] = \frac{(x_1 - y_1)\,(x_2 - y_2)}{(x_1 - y_2)\,(x_2 - y_1)}.$$

It is the unique $\mathrm{PSL}(2,\mathbb{K})$-invariant of (x_1, x_2, y_1, y_2). Different partitions of the points into two pairs give six different cross-ratios, but any one of them determines the others.

Remark. The cross-ratio plays a fundamental role in many areas, from projective geometry to mathematical physics; for an overview, see [11]. It is the discrete version of the Schwarzian derivative; see, e.g., [20]. Different versions of multi-dimensional symplectic cross-ratios have been considered. One is an invariant of a quadruples of Lagrangian planes related to the Maslov index; again, see [20] for a survey. Another is a unitary group invariant defined in the complex setting; see [5]. The symplectic cross-ratio (2.1) is the most straightforward generalization of the one-dimensional cross-ratio. It has been used to construct symplectic projective invariants in other settings; see, e.g., [15] (p. 367) and [22].

2.2. **Gram matrices.** Given a collection $x_1, \dots, x_m$ of vectors in $\mathbb{K}^{2n}$, we define their *ω-Gram matrix* $\Omega(x_1, \dots, x_m)$ to be the $m \times m$ matrix whose entries are their inner products:

$$\Omega(x_1, \dots, x_m)_{ij} := \omega(x_i, x_j).$$

We will use the following standard lemma throughout the paper. Its proof is elementary and is omitted.

Lemma 2.1. *Suppose that $x_1, \dots, x_m$ and $x'_1, \dots, x'_m$ are two collections of m vectors in $\mathbb{K}^{2n}$.*

(i) *$\Omega(x_1, \dots, x_m)$ is of rank at most $2n$.*

(ii) *$\Omega(x_1, \dots, x_m)$ is of rank $2n$ if and only if $\langle x_1, \dots, x_m \rangle = \mathbb{K}^{2n}$.*

(iii) *If $\Omega(x_1, \dots, x_m)$ and $\Omega(x'_1, \dots, x'_m)$ are equal and of rank $2n$, then there is a unique symplectic transformation T such that $T(x_i) = x'_i$ for all i.*

Lemma 2.2. *Fix $N \geqslant 2n$ and let $(x_1, \dots, x_N)$ be an N-tuple of points in $\mathbb{K}^{2n}$. Define $(x_i)_{i \in \mathbb{Z}}$ via $x_{i \pm N} := -x_i$, and assume that $\langle x_{i+1}, \dots, x_{i+n} \rangle$ is Lagrangian for all i. Then $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$ is an (n, N)-Lagrangian configuration if and only if $\omega(x_i, x_{i+n}) \neq 0$ for all i.*

Proof. We must show that $\langle x_{i+1}, \dots, x_{i+2n} \rangle = \mathbb{K}^{2n}$ for all i if and only if $\omega(x_i, x_{i+n}) \neq 0$ for all i. Consider the ω-Gram matrix $\Omega^i := \Omega(x_{i+1}, \dots, x_{i+2n})$. By Lemma 2.1(ii), $\det(\Omega^i) \neq 0$ if and only if $x_{i+1}, \dots, x_{i+2n}$ form a basis of $\mathbb{K}^{2n}$. The Lagrangian condition implies that the block 2×2 form of Ω^i is $\left(\begin{smallmatrix} 0 & A \\ -A^T & 0 \end{smallmatrix}\right)$, where A is upper triangular with diagonal entries $\omega(x_{i+r}, x_{i+r+n})$, $1 \leqslant r \leqslant n$. □

2.3. **Continuous invariants.** Given an (n, N)-Lagrangian configuration $(X_1, \dots, X_N)$, fix *representatives:* non-zero points x_i on the lines X_i. As in Section 1.1, extend the N-tuple $(x_1, \dots, x_N)$ to an N-antiperiodic sequence

$$(x_i)_{i \in \mathbb{Z}}, \qquad x_{i \pm N} := -x_i. \tag{2.3}$$

Write ω_{ij} for $\omega(x_i, x_j)$, and observe that

$$\omega_{i+N,j} = \omega_{ji} = -\omega_{ij}, \qquad \omega_{j,i+N} = \omega_{ij}. \tag{2.4}$$

Of course the ω_{ij} depend on the choice of the x_i, but the symplectic cross-ratios of the configuration do not. Define projective symplectic invariants

$$c_{i_1 i_2 j_1 j_2} := [x_{i_1}, x_{i_2}; x_{j_1}, x_{j_2}] = \frac{\omega_{i_1 j_1}\, \omega_{i_2 j_2}}{\omega_{i_1 j_2}\, \omega_{i_2 j_1}}. \tag{2.5}$$

Note that $c_{i_1 i_2 j_1 j_2}$ has the symmetries (2.2) and in addition is invariant under the addition of N to any of the four indices. Due to the Lagrangian condition many of the $c_{i_1 i_2 j_1 j_2}$ vanish or are not well defined. To be precise, define the *N-cyclic distance* $|i - j|_N$ between any two integers i and j to be their "separation modulo N":

$$|i - j|_N := \min\bigl\{|i - j + qN| : q \in \mathbb{Z}\bigr\}.$$

The Lagrangian condition is $\omega_{ij} = 0$ for $|i - j|_N < n$. Therefore $c_{i_1 i_2 j_1 j_2}$ is either zero or undefined unless $|i_{\varepsilon_i} - j_{\varepsilon_j}|_N \geqslant n$ for ε_i and ε_j either 1 or 2. Moreover, if either $i_1 = i_2$ or $j_1 = j_2$, then $c_{i_1 i_2 j_1 j_2}$ is 1 if defined. It follows that there are no non-trivial cross-ratios when N is $2n$ or $2n + 1$, and the only cross-ratios of configurations with $N = 2n + 2$ taking values other than 0 and 1 are the c_i of (1.4):

$$c_i = c_{i,i+1,i+n+1,i+n+2} = \frac{\omega_{i,i+n+1}\, \omega_{i+1,i+n+2}}{\omega_{i-n,i}\, \omega_{i+1,i+n+1}}. \tag{2.6}$$

Observe that for $N = 2n + 2$ these cross-ratios are $(n + 1)$-periodic. Lemma 2.2 shows that in this case they are also always well defined.

We conjecture that in general, (2.5) is a coordinate ring on $\mathcal{L}_{n,N}(\mathbb{K})$ with polynomial relations. This is confirmed only for $N \leqslant 2n + 2$. It would be interesting to find minimal cyclically invariant subsets of (2.5) providing such coordinate rings.

2.4. **Opposite configurations and sign invariants.** Note that the negation of a symplectic form is also a symplectic form; in particular, $-\omega$ is a symplectic form on $\mathbb{K}^{2n}$. This leads to the notion of opposite Lagrangian configurations. To be concrete, observe that the $2n \times 2n$ matrix $Q = \left(\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{smallmatrix}\right)$ has the property $\omega(Qx, Qy) = -\omega(x, y)$ for all x and y in $\mathbb{K}^{2n}$.

In light of the fact that conjugation by Q preserves $\mathrm{Sp}(2n, \mathbb{K})$, the following lemma is clear.

Lemma 2.3. (i) *If $(X_1, \dots, X_N)$ is a Lagrangian configuration, then so is $(QX_1, \dots, QX_N)$. We refer to them as* opposites.

(ii) *Opposite configurations have the same cross-ratios, and if one is generic, then so is the other.*

(iii) *If two Lagrangian configurations are equivalent, then their opposites are also equivalent.*

Therefore we may speak of *opposite equivalence classes* in $\mathcal{L}_{n,N}(\mathbb{K})$. Because Q is not symplectic, a configuration is not *a priori* equivalent to its opposite. In order to resolve the situation we define the *sign invariants.* Let us write sgn for the sign function:

$$\mathrm{sgn} : \mathbb{R}\backslash\{0\} \to \{\pm 1\}.$$

Let $(\mathbb{R}x_1, \dots, \mathbb{R}x_N)$ be a real (n, N)-Lagrangian configuration, and suppose that $j_0, j_1, \dots, j_r$ are any integers such that $j_0 \equiv j_r$ modulo N. If $\omega_{j_{s-1}, j_s} \neq 0$ for all s, consider the product $\prod_1^r \omega_{j_{s-1}, j_s}$. Suppose we rescale each x_j by some λ_j. Because the λ_j are by definition N-periodic, $\lambda_{j_0} = \lambda_{j_r}$, so the product rescales by the positive quantity $\prod_1^r \lambda_{j_s}^2$. This gives the following lemma.

Lemma 2.4. *Let $(\mathbb{R}x_1, \dots, \mathbb{R}x_N)$ be a real (n, N)-Lagrangian configuration. If $j_0, \dots, j_r$ are integers such that $j_0 \equiv j_r$ modulo N and $\omega_{j_{s-1}, j_s} \neq 0$ for all s, then $\mathrm{sgn}\big(\prod_1^r \omega_{j_{s-1}, j_s}\big)$ is an invariant of the configuration.*

The next proposition elucidates equivalence and inequivalence of opposite configurations. We write GCD for the greatest common divisor function.

Proposition 2.5. (i) *Opposite complex configurations are equivalent.*

(ii) *Opposite real (n, N)-configurations are inequivalent if $N/\mathrm{GCD}(n, N)$ is odd.*

(iii) *Opposite real generic (n, N)-configurations are inequivalent for $N > 2n$.*

Proof. Let $(\mathbb{K}x_1, \dots \mathbb{K}x_N)$ be a configuration. For (i), note that iQ is symplectic and $iQ\mathbb{C}x_j = Q\mathbb{C}x_j$.

For (ii), note that the sign invariant $\mathrm{sgn}\big(\prod_{s=1}^{N/\mathrm{GCD}(n,N)} \omega_{(s-1)n, sn}\big)$ negates under passage to the representatives Qx_j of the opposite configuration.

In the setting of (iii), it is always possible to find a sequence $j_0, \dots, j_r$ with r odd, $j_0 \equiv j_r$ modulo N, and $|j_s - j_{s-1}|_N \geqslant n$, whence the invariant $\mathrm{sgn}\big(\prod_1^r \omega_{j_{s-1}, j_s}\big)$ distinguishes between the configuration and its opposite. □

2.5. **Dimensions and standard configurations.** We now discuss the dimension of $\mathcal{L}_{n,N}$, which determines the number of independent relations which must be satisfied by any coordinate ring of symplectic invariants of configurations. It turns out that for $N \geqslant 2n + 2$, this dimension is always strictly less than the number of non-trivial symplectic cross-ratios. As stated in Section 1.3, at $N = 2n + 2$ the dimension is n and there are $n + 1$ invariants c_i, so there must be one relation. It is given in Theorem 1.

Let us begin by defining a convenient normal form for sets of representatives of configurations.

Definition. Throughout this paper we write $\{e_1, \dots, e_n,\ f_1, \dots, f_n\}$ for the standard basis of $\mathbb{K}^{2n}$, and ω for the standard symplectic form on it. Thus for $1 \leqslant i, j \leqslant n$,

$$\omega(e_i, e_j) = 0, \qquad \omega(e_i, f_j) = \delta_{ij}, \qquad \omega(f_i, f_j) = 0.$$

We define the group $\mathrm{Sp}(2n, \mathbb{K})$ with respect to ω.

Let $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$ be an (n, N)-Lagrangian configuration. The representatives $x_1, \dots x_N$ are said to be *standard* if $x_1, \dots, x_{2n}$ may be written in terms of the standard basis as follows: for some vectors $g_i \in \langle f_1, \dots, f_{i-1}\rangle$, we have

$$x_1 = e_1, \quad x_2 = e_2, \quad \dots, \quad x_n = e_n, \qquad x_{n+1} = f_1, \quad x_{n+2} = f_2 + g_2, \quad \dots, \quad x_{2n} = f_n + g_n.$$

Lemma 2.6. (i) *Every (n, N)-Lagrangian configuration is symplectically equivalent to a configuration with standard representatives.*

(ii) *In a configuration with standard representatives, $\omega(e_j, g_i) = 0$ for $j < i + 2n - N$.*

(iii) *In a generic configuration with standard representatives, $\omega(e_j, g_i) \neq 0$ for $i + 2n - N \leqslant j < i$.*

Proof. Given an (n, N)-configuration $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$, consider the Gram matrix $\Omega(x_1, \dots, x_{2n})$. Because it has the form noted in the proof of Lemma 2.2, Lemma 2.1(iii) shows that there is a symplectic transformation mapping x_i to e_i and x_{n+i} to a multiple of $f_i + g_i$ for $1 \leqslant i \leqslant n$, g_1 being zero and $g_2, \dots, g_n$ having the desired form. To complete the proof, rescale the representatives. For (ii) and (iii), refer to the definitions of Lagrangian and generic. □

Proposition 2.7. *For $N > 2n$, the variety of (n, N)-Lagrangian configurations is nN-dimensional.* $\mathrm{Sp}(2n, \mathbb{K})$ *acts freely on generic configurations, and so $\mathcal{L}_{n,N}(\mathbb{K})$ is $n(N - 2n - 1)$-dimensional.*

Proof. Consider the process of constructing an (n, N)-configuration by choosing first X_1, then X_2, and so on to X_N. Count the number of degrees of freedom available in choosing each X_i as follows. There are $2n - 1$ degrees of freedom for X_1, as it is simply an arbitrary line. There are $2n - 2$ degrees of freedom for X_2, as $\langle X_1, X_2\rangle$ must be isotropic, and $2n - 3$ for X_3, as $\langle X_1, X_2, X_3\rangle$ must be isotropic. Continuing, for $i \leqslant n$ we find that there are $2n - i$ degrees for X_i. For $n \leqslant i \leqslant N - n + 1$ there are n degrees for X_i, as the only constraints arise from the requirement that $\langle X_{i-n+1}, \dots, X_i\rangle$ be isotropic.

There are $n - 1$ degrees for X_{N-n+2}, as in addition to the above isotropy requirement, $\langle X_{N-n+2}, X_1\rangle$ must be isotropic. Continuing, for $N - n + 1 \leqslant i \leqslant N$ there are $N + 1 - i$ degrees for X_i. In particular, all of the X_i can be chosen, and there are a total of nN degrees of freedom in choosing the configuration.

To complete the proof, it suffices to prove that any symplectic transformation T stabilizing a generic standard configuration $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$ is the identitity Id. Keeping in mind that T must stabilize $\mathbb{K}x_i$ but not necessarily x_i, we find that it must be of the form $\left(\begin{smallmatrix} D & 0 \\ 0 & D^{-1}\end{smallmatrix}\right)$ for some diagonal matrix D. Because $N > 2n$, $\omega(e_{i-1}, g_i) \neq 0$, forcing D to be scalar. Finally, genericity implies that $\omega(e_n, x_{2n+1})$ and $\omega(f_1, x_{2n+1})$ are both non-zero. Hence the condition that T stabilize $\mathbb{K}x_{2n+1}$ forces $D = \mathrm{Id}$. □

Corollary 2.8. $\mathcal{L}_{n,N}(\mathbb{R})$ *and* $\mathcal{L}_{n,N}(\mathbb{C})$ *are smooth manifolds.*

2.6. **The case $N > 2n + 2$.** We conclude this section with a few remarks on configurations with $N > 2n + 2$. We begin with a corollary of Lemma 2.2. Let us single out the symplectic cross-ratios

$$\gamma_{ij} := c_{i,j+n,j,i+n} = \frac{\omega_{ij}\,\omega_{i+n,j+n}}{\omega_{i,i+n}\,\omega_{j,j+n}}. \tag{2.7}$$

Corollary 2.9. *Let $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$ be an (n, N)-Lagrangian configuration. Then γ_{ij} is defined for all i and j, and it is 0 for $|i - j|_N < n$ and 1 for $|i - j|_N = n$. The configuration is generic if and only if γ_{ij} is non-zero whenever $|i - j|_N \geqslant n$.*

Observe that $\gamma_{i+N,j} = \gamma_{ji} = \gamma_{ij}$. For $N = 2n + 2$, all non-trivial symplectic cross-ratios are γ_{ij}'s, as $c_i = \gamma_{i+1,i-n}$. This is not true for $N > 2n + 2$: for example, for $(2, 7)$-configurations, $c_{1,2,4,5}$ is not a γ_{ij}. However, we do have the relation

$$c_{i_1 i_2 j_1 j_2} c_{i_1+n,i_2+n,j_1+n,j_2+n} = \frac{\gamma_{i_1 j_1}\, \gamma_{i_2 j_2}}{\gamma_{i_1 j_2}\, \gamma_{i_2 j_1}}.$$

3. The main results

In this section we describe the moduli space $\mathcal{L}_{n,N}(\mathbb{K})$ of generic (n, N)-configurations over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ for N equal to $2n$, $2n+1$, and $2n+2$. In the first two cases it is trivial and is described in Proposition 3.1. Let us mention that a related result is proven in [15] for three lines in $\mathbb{K}^{2n}$.

The first non-trivial case, $\mathcal{L}_{n,2n+2}(\mathbb{K})$, is described in Theorem 1: there are $n + 1$ cross-ratios, which satisfy a single relation. In the case that n is even and $\mathbb{K} = \mathbb{C}$, Theorem 2 provides a combinatorial interpretation of this relation.

3.1. The cases $N = 2n$ and $N = 2n + 1$.

Proposition 3.1. (i) *For $N = 2n$ or $2n + 1$, all (n, N)-Lagrangian configurations are generic.*

(ii) *$\mathcal{L}_{n,2n}(\mathbb{K})$ is a single point for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

(iii) *$\mathcal{L}_{n,2n+1}(\mathbb{R})$ consists of two opposite points, and $\mathcal{L}_{n,2n+1}(\mathbb{C})$ is a single point.*

Proof. Part (i) is clear from Lemma 2.2 and the fact that $|i-j|_{2n+1}$ never exceeds n. For (ii), Lemma 2.6 shows that all configurations are equivalent to $(\mathbb{K}e_1, \dots, \mathbb{K}e_n, \mathbb{K}f_1, \dots, \mathbb{K}f_n)$.

For (iii), Lemma 2.6 shows that any configuration is equivalent to some $(\mathbb{K}x_1, \dots, \mathbb{K}x_{2n+1})$ with $x_i = e_i$ and $x_{n+i} = f_i + b_i f_{i-1}$ for $1 \leqslant i \leqslant n$, where b_i is a non-zero scalar for $i > 1$. Define $b_1 := 1$, apply the diagonal symplectic transformation $e_i \mapsto e_i \prod_1^i b_j^{-1}$ and $f_i \mapsto f_i \prod_1^i b_j$, and rescale to deduce that we may assume $b_i = 1$ for all i.

Combine the Lagrangian condition with genericity to see that x_{2n+1} can be rescaled to take the form $f_n + b_0 \sum_1^n (-1)^i e_i$ for some scalar $b_0 \neq 0$. For $\mathbb{K} = \mathbb{R}$ and $b_0 > 0$ or $\mathbb{K} = \mathbb{C}$, apply the symplectic transformation $e_i \mapsto b_0^{1/2} e_i$ and $f_i \mapsto b_0^{-1/2} f_i$ and rescale to arrive at $x_{2n+1} = f_n + \sum_1^n (-1)^i e_i$. For $\mathbb{K} = \mathbb{R}$ and $b_0 < 0$, use $(-b_0)^{1/2}$ in place of $b_0^{1/2}$ to arrive at $x_{2n+1} = f_n - \sum_1^n (-1)^i e_i$. By Proposition 2.5, the two signs of b_0 give opposite real equivalence classes. □

Combining this argument with Section 2.4 yields the following corollary. For $\varepsilon = \pm 1$, consider the $(n, 2n + 1)$-configurations with representatives

$$\Big(e_1, e_2, \dots, e_n, \quad \varepsilon f_1,\, \varepsilon(f_1 + f_2),\, \varepsilon(f_2 + f_3), \dots, \varepsilon(f_{n-1} + f_n), \quad \varepsilon f_n + \sum_i^n (-1)^i e_i\Big). \tag{3.1}$$

Observe that these representatives have $\omega_{i,n+i} = \varepsilon$ for all i.

Corollary 3.2. (i) *The unique element of $\mathcal{L}_{n,2n+1}(\mathbb{C})$ is the class of (3.1) for $\varepsilon = 1$.*

(ii) *The two opposite elements of $\mathcal{L}_{n,2n+1}(\mathbb{R})$ are the classes of (3.1) for $\varepsilon = \pm 1$.*

(iii) *An arbitrary $(n, 2n+1)$-Lagrangian configuration $(\mathbb{R}x_1, \dots, \mathbb{R}x_{2n+1})$ over $\mathbb{R}$ is equivalent to (3.1) with $\varepsilon = \operatorname{sgn}\big(\prod_{i=0}^{2n} \omega_{i,n+i}\big)$.*

3.2. The first non-trivial case: $N = 2n + 2$.

We now give the description of $\mathcal{L}_{n,2n+2}(\mathbb{K})$, one of our main results. Recall that the symplectic cross-ratios c_i given in (1.4) and (2.6) are $(n + 1)$-periodic, are the only non-trivial cross-ratios on $(n, 2n + 2)$-configurations, and are non-zero on generic configurations. We use the standard notation $\lfloor x \rfloor$ for the integer part of a real number x.

Theorem 1. (i) *On $\mathcal{L}_{n,2n+2}(\mathbb{K})$, the $n+1$ symplectic cross-ratios $c_0, \ldots, c_n$ satisfy the relation*

$$0 = \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} (-1)^r \sum_{\substack{0 \leqslant i_1 < \cdots < i_r \leqslant n, \\ |i_s - i_{s'}|_{n+1} \geqslant 2 \ \forall \ s \neq s'}} \prod_{i=1}^{r} \frac{1}{c_{i_s}}. \tag{3.2}$$

(ii) *$(c_0, \ldots, c_n)$ is a coordinate ring on $\mathcal{L}_{n,2n+2}(\mathbb{K})$: any two generic $(n, 2n+2)$-equivalence classes with the same cross-ratios are equal for $\mathbb{K} = \mathbb{C}$, and either equal or opposite for $\mathbb{K} = \mathbb{R}$.*

(iii) *(3.2) is the only relation on the cross-ratios: if $c_0, \ldots, c_n$ are arbitrary non-zero scalars in $\mathbb{K}$ satisfying (3.2), then they are the cross-ratios of some generic $(n, 2n+2)$-configuration over $\mathbb{K}$.*

Thus $\mathcal{L}_{n,2n+2}(\mathbb{K})$ may be viewed as a dense open subset of the algebraic hypersurface in $\mathbb{K}^{n+1}$ defined by multiplying (3.2) by $c_0 \cdots c_n$. We prove Theorem 1 in Section 4.

Examples. (a) For $\mathcal{L}_{1,4}(\mathbb{K})$, the relation on the two cross-ratios of quadrilaterals in $\mathbb{KP}^1$ simply relates two forms of the classical cross-ratio:

$$\frac{1}{c_0} + \frac{1}{c_1} = 1.$$

(b) For $\mathcal{L}_{2,6}(\mathbb{K})$, the relation on the three cross-ratios c_0, c_1, and c_2 of Legendrian hexagons in $\mathbb{KP}^3$ was given in (1.1).

(c) The moduli space $\mathcal{L}_{3,8}(\mathbb{K})$ of Legendrian octagons in $\mathbb{KP}^5$ is parametrized by the four cross-ratios c_0, c_1, c_2, and c_3, subject to the relation

$$\frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} - \frac{1}{c_0 c_2} - \frac{1}{c_1 c_3} = 1.$$

(d) For $\mathcal{L}_{4,10}(\mathbb{K})$, the Legendrian decagons in $\mathbb{KP}^7$, the five cross-ratios $c_0, \ldots, c_4$ satisfy

$$\frac{1}{c_0} + \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4} - \frac{1}{c_0 c_2} - \frac{1}{c_1 c_3} - \frac{1}{c_2 c_4} - \frac{1}{c_3 c_0} - \frac{1}{c_4 c_1} = 1.$$

The following analog of Figure 1 depicts c_0 in this setting.

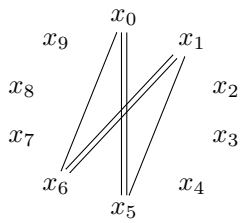

Figure 2. The cross-ratio $c_0 = \frac{\omega_{0,5}\,\omega_{1,6}}{\omega_{0,6}\,\omega_{1,5}}$ on $\mathcal{L}_{4,10}$.

3.3. **The cyclic continuant.** In Section 5 we will describe certain normalized choices of representatives of $(n, 2n+2)$-Lagrangian configurations. The nature of the normalizations depends on the field $\mathbb{K}$ and the parity of n. The situation is simplest for n even and $\mathbb{K} = \mathbb{C}$, the case generalizing the hexagonal example in Section 1.1. At this point we state the relevant normalization, Proposition 3.3, along with the corresponding specialization of Theorem 1, Theorem 2. A notable feature of this specialization is that the relation (3.2) is expressed in terms of continuants. The proofs will be given in Section 5.1.

The classical *continuant* is the tridiagonal determinant

$$K_n(a_1,\dots,a_n) := \begin{vmatrix} a_1 & 1 & & & \\ 1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & a_{n-1} & 1 \\ & & & 1 & a_n \end{vmatrix}. \tag{3.3}$$

This remarkable polynomial has a long history. It arises in the theory of continued fractions and many other areas; see for example [3] and references therein.

The main topic of [3] is the *cyclic continuant,* defined as

$$R_{n+1}(a_0,\dots,a_n) := K_{n+1}(a_0,\dots,a_n) - K_{n-1}(a_1,\dots,a_{n-1}). \tag{3.4}$$

It too is related to continued fractions, via the identity

$$R_{n+1}(a_0,,\dots,a_n) = \operatorname{tr}\begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{n-1} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As an illustration, let us display $R_5(a_0,a_1,a_2,a_3,a_4)$:

$$a_0a_1a_2a_3a_4 - \big(a_0a_1a_2 + a_1a_2a_3 + a_2a_3a_4 + a_3a_4a_0 + a_4a_0a_1\big) + \big(a_0+a_1+a_2+a_3+a_4\big).$$

In the next two statements, let $(X_0,\dots,X_{2n+1})$ be a generic $(n,2n+2)$-Lagrangian configuration over $\mathbb{K}$ with representatives $x_0,\dots,x_{2n+1}$, inner products $\omega_{ij} := \omega(x_i,x_j)$, and cross-ratios $c_0,\dots,c_n$. Recall from Section 1.3 that we refer to the $(2n+2)$-periodic sequence $(\omega_{i,i+n})_i$ as the set of *symplectic subdiameters* of $(x_i)_i$, and the $(n+1)$-periodic sequence $(\omega_{i,i+n+1})_i$ as the set of *symplectic diameters.*

Proposition 3.3. *For n even and $\mathbb{K}=\mathbb{C}$, $(X_0,\dots,X_{2n+1})$ admits exactly four choices of representatives whose symplectic subdiameters are all 1. Fix such a choice, $x_0,\dots,x_{2n+1}$, and set $a_i := \omega_{i,i+n+1}$.*

(i) *The four choices are $(x_i)_i$, $(-x_i)_i$, $((-1)^i x_i)_i$, and $(-(-1)^i x_i)_i$.*

(ii) *The first two choices in (i) have symplectic diameters $(a_i)_i$, and the second two have symplectic diameters $(-a_i)_i$.*

(iii) *The cross-ratios of the configuration are $c_i = a_ia_{i+1}$. The symplectic diameters satisfy*

$$a_i^2 = \frac{c_ic_{i+2}\cdots c_{i+n}}{c_{i+1}c_{i+3}\cdots c_{i+n-1}}.$$

This result shows that the symplectic diameters of the choices of representatives whose symplectic subdiameters are all 1 are invariants of the configuration, defined up to an overall sign. We refer to them as the *normalized symplectic diameters* and denote them by $\pm(a_0,\dots,a_n)$.

In reading the next result, keep in mind that K_n and hence also R_n are of parity $(-1)^n$ under negation of all their arguments, so $R_{n+1}(a_0,\dots,a_n)$ vanishes if and only if $R_{n+1}(-a_0,\dots,-a_n)$ vanishes.

Theorem 2. *Let n be even.*

(i) *The cyclic continuant of the normalized symplectic diameters on $\mathcal{L}_{n,2n+2}(\mathbb{C})$ vanishes:*

$$R_{n+1}(a_0,\dots,a_n) = 0.$$

(ii) *Equivalence classes in $\mathcal{L}_{n,2n+2}(\mathbb{C})$ with the same normalized symplectic diameters are equal.*

(iii) *If $\pm(a_0,\dots,a_n)$ are arbitrary non-zero complex scalars with cyclic continuant zero, then they are the normalized symplectic diameters of some equivalence class in $\mathcal{L}_{n,2n+2}(\mathbb{C})$.*

As noted above, Proposition 3.3 and Theorem 2 are proven in Section 5.1.

4. Pfaffians and the proof of Theorem 1

This section contains the proof of our main result, Theorem 1.

4.1. **Tridiagonal determinants and the proof of Theorem 1**(i)**.** The following formula is well-known and easily proven by induction.

Proposition 4.1. *The tridiagonal matrix*

$$A_n := \begin{pmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & a_{23} & & & \\ & \ddots & \ddots & \ddots & & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} & \\ & & & a_{n,n-1} & a_{nn} \end{pmatrix} \tag{4.1}$$

has determinant

$$\det(A_n) = \Big(\prod_{k=1}^{n} a_{kk}\Big) \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \sum_{\substack{1 \leqslant i_1 < \cdots < i_r < n, \\ i_s + 1 < i_{s+1} \ \forall \ 1 \leqslant s \leqslant r}} \prod_{s=1}^{r} \frac{a_{i_s,i_s+1}\, a_{i_s+1,i_s}}{a_{i_s i_s}\, a_{i_s+1,i_s+1}}. \tag{4.2}$$

Remark. When the sub- and superdiagonal entries of A_n are all 1, its determinant is in fact the continuant K_n. In this case Euler discovered a pleasing interpretation of Proposition 4.1, which generalizes as follows: to write out all summands of (4.2), start with $a_{11} \cdots a_{nn}$, and for each set of disjoint adjacent pairs $(i, i+1)$, $1 \leqslant i < n$, replace $a_{ii}a_{i+1,i+1}$ by $-a_{i,i+1}a_{i+1,i}$. We refer to this process as *Euler's replacement algorithm.*

Examples. Indicating pairs by parentheses, $\det(A_3)$, $\det(A_4)$, and $\det(A_5)$ are, respectively,

- $a_{11}a_{22}a_{33} - (a_{12}a_{21})a_{33} - a_{11}(a_{23}a_{32})$,
- $a_{11}a_{22}a_{33}a_{44} - (a_{12}a_{21})a_{33}a_{44} - a_{11}(a_{23}a_{32})a_{44} - a_{11}a_{22}(a_{34}a_{43}) + (a_{12}a_{21})(a_{34}a_{43})$,
- $a_{11}a_{22}a_{33}a_{44}a_{55} - (a_{12}a_{21})a_{33}a_{44}a_{55} - a_{11}(a_{23}a_{32})a_{44}a_{55} - a_{11}a_{22}(a_{34}a_{43})a_{55} - a_{11}a_{22}a_{33}(a_{45}a_{54})$
  $+ (a_{12}a_{21})(a_{34}a_{43})a_{55} + (a_{12}a_{21})a_{33}(a_{45}a_{54}) + a_{11}(a_{23}a_{32})(a_{45}a_{54})$.

Recall now that the *Pfaffian* $\mathrm{pf}(S)$ is a polynomial in the entries of a skew-symmetric matrix S whose square is $\det(S)$. Consider the $(2n+2) \times (2n+2)$ matrix

$$\Omega_n := \begin{pmatrix} \omega_{0,n}E & A \\ -A^T & \omega_{n+1,2n+1}E \end{pmatrix}, \tag{4.3}$$

with ingredients defined as follows: E is the skew-symmetric $(n+1) \times (n+1)$ matrix $e_{1,n+1} - e_{n+1,1}$ (e_{ij} being the elementary matrix whose ij^{th} entry is 1 and whose other entries are 0), A is given by

$$A = \begin{pmatrix} \omega_{0,n+1} & \omega_{0,n+2} & & & & \\ \omega_{1,n+1} & \omega_{1,n+2} & \omega_{1,n+3} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \omega_{n-1,2n-1} & \omega_{n-1,2n} & \omega_{n-1,2n+1} & \\ & & & \omega_{n,2n} & \omega_{n,2n+1} \end{pmatrix}, \tag{4.4}$$

and the ω_{ij} are arbitrary scalars. As a visual aid, we illustrate Ω_n in long form:

$$\Omega_n = \begin{pmatrix} & & & & & & & & & & & & \\ & & & & & & \omega_{0,n} & \omega_{0,n+1} & \omega_{0,n+2} & & & & \\ & & & & & & & \omega_{1,n+1} & \omega_{1,n+2} & \omega_{1,n+3} & & & \\ & & & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & & \ddots & \ddots & \omega_{n-1,2n+1} & \\ -\omega_{0,n} & & & & & & & & & & \omega_{n,2n} & \omega_{n,2n+1} & \\ -\omega_{0,n+1} & -\omega_{1,n+1} & & & & & & & & & & \omega_{n+1,2n+1} & \\ -\omega_{0,n+2} & \ddots & \ddots & & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & & & \\ & & & & & -\omega_{n,2n} & & & & & & & \\ & & & & -\omega_{n-1,2n+1} & -\omega_{n,2n+1} & -\omega_{n+1,2n+1} & & & & & & \end{pmatrix}. \tag{4.5}$$

Proposition 4.2. *The Pfaffian* $\operatorname{pf}(\Omega_n)$ *is given by the expression*

$$(-1)^{n(n+1)/2}\Big(\prod_{k=0}^{n}\omega_{k,k+n+1}\Big)\sum_{r=0}^{\lfloor (n+1)/2\rfloor}(-1)^r \sum_{\substack{0\leqslant i_1<\cdots<i_r\leqslant n,\\ |i_s-i_{s'}|_{n+1}\geqslant 2\ \forall\ s\neq s'}}\prod_{i=1}^{r}\frac{\omega_{i_s,i_s+n+2}\,\omega_{i_s+1,i_s+n+1}}{\omega_{i_s,i_s+n+1}\,\omega_{i_s+1,i_s+n+2}}, \tag{4.6}$$

where whenever $\omega_{i,2n+2}$ *appears, it should be replaced by* $\omega_{0,i}$.

Proof. Given any $m\times m$ matrix A, let A_{mid} be the $(m-2)\times(m-2)$ matrix obtained by deleting the top and bottom rows and left and right columns of A. The following lemma is proven in [3].

Lemma 4.3. *For any* $m\times m$ *matrix* A *and any scalars* x *and* y*, one has*

$$\operatorname{pf}\begin{pmatrix} xE & A \\ -A^T & yE\end{pmatrix} = (-1)^{m(m-1)/2}\big(\det(A)-xy\det(A_{\mathrm{mid}})\big). \tag{4.7}$$

Applying both (4.2) and (4.7) to (4.3) yields (4.6). □

Remark. The Pfaffian (4.6) may be interpreted via a cyclic version of Euler's replacement algorithm, the "cyclic replacement algorithm": to write out all summands of $(-1)^{n(n+1)/2}\operatorname{pf}(\Omega_n)$, start with

$$\omega_{0,n+1}\omega_{1,n+2}\cdots\omega_{n,2n+1},$$

and for each set of disjoint cyclically adjacent pairs $(i,i+1)$, $0\leqslant i\leqslant n$, replace $\omega_{i,i+n+1}\omega_{i+1,i+n+2}$ by $-\omega_{i,i+n+2}\omega_{i+1,i+n+1}$. *Cyclically adjacent* indicates that the pair $(n,n+1)$ is read as $(n,0)$. When it is in the set of pairs, replace $\omega_{0,n+1}\omega_{n,2n+1}$ by $-\omega_{0,n}\omega_{n+1,2n+1}$. To explain, note that by the replacement rule $\omega_{i,2n+2}=\omega_{0,i}$,

$$\omega_{0,n+1}\omega_{n,2n+1}=\omega_{n,2n+1}\omega_{n+1,2n+2},\qquad \omega_{n,2n+2}\omega_{n+1,2n+1}=\omega_{0,n}\omega_{n+1,2n+1}.$$

Examples. Let us give (4.6) explicitly for small n. As in the examples below Proposition 4.1, we indicate pairs with parentheses. By (4.7) and (4.4), the summands from sets not containing the special pair $(n,0)$ add up to an $(n+1)\times(n+1)$ tridiagonal determinant, and the summands from sets containing it add up to $\omega_{0,n}\omega_{n+1,2n+1}$ times an $(n-1)\times(n-1)$ tridiagonal determinant. To emphasize this, we have separated the two types of terms with square brackets and factored $\omega_{0,n}\omega_{n+1,2n+1}$ out of the second type.

Note that at $n = 1$ there are two ways to delete the lone pair $(0, 1)$: as $(0, 1)$, or as $(1, 0)$. For $n = 1$, 2, and 3, $(-1)^{n(n+1)/2}\operatorname{pf}(\Omega_n)$ is, respectively,

- $\Big[\omega_{02}\omega_{13} - (\omega_{03}\omega_{12})\Big] - (\omega_{01}\omega_{23})\Big[1\Big]$,
- $\Big[\omega_{03}\omega_{14}\omega_{25} - (\omega_{04}\omega_{13})\omega_{25} - \omega_{03}(\omega_{15}\omega_{24})\Big] - (\omega_{02}\omega_{35})\Big[\omega_{14}\Big]$,
- $\Big[\omega_{04}\omega_{15}\omega_{26}\omega_{37} - (\omega_{05}\omega_{14})\omega_{26}\omega_{37} - \omega_{04}(\omega_{16}\omega_{25})\omega_{37} - \omega_{04}\omega_{15}(\omega_{27}\omega_{36}) + (\omega_{05}\omega_{14})(\omega_{27}\omega_{36})\Big]$
$$- (\omega_{03}\omega_{47})\Big[\omega_{15}\omega_{26} - (\omega_{16}\omega_{25})\Big].$$

Observe that dividing by $\omega_{0,n+1}\omega_{1,n+2}\cdots\omega_{n,2n+1}$ leads to the examples in Section 3.2, and compare the terms in square brackets to the examples below Proposition 4.1.

Proof of (3.2). Suppose now that $(\mathbb{K}x_0, \ldots, \mathbb{K}x_{2n+1})$ is an $(n, 2n + 2)$-Lagrangian configuration, and revert to our customary notation $\omega_{ij} = \omega(x_i, x_j)$. By the Lagrangian condition, the ω-Gram matrix $\Omega(x_0, \ldots, x_{2n+1})$ is precisely the matrix Ω_n in (4.3), and by Lemma 2.1(ii),

$$\det\big(\Omega(x_0, \ldots, x_{2n+1})\big) = 0.$$

Hence Proposition 4.2 yields the following corollary.

Corollary 4.4. *Let $(\mathbb{K}x_0, \ldots, \mathbb{K}x_{2n+1})$ be an $(n, 2n + 2)$-Lagrangian configuration. Then*

$$0 = \Big(\prod_{k=0}^{n}\omega_{k,k+n+1}\Big)\sum_{r=0}^{\lfloor(n+1)/2\rfloor}(-1)^r\sum_{\substack{0\leqslant i_1<\cdots<i_r\leqslant n,\\ |i_s-i_{s'}|_{n+1}\geqslant 2\ \forall\ s\neq s'}}\prod_{i=1}^{r}\frac{1}{c_{i_s}}. \tag{4.8}$$

This in turn yields (3.2) of Theorem 1(i), because generic $(n, 2n+2)$-configurations have $\omega_{k,k+n+1} \neq 0$. Note that (4.6) is polynomial in the ω_{ij}, so after cancellation, Corollary 4.4 gives a non-trivial relation even on non-generic configurations.

4.2. **Proof of Theorem 1**(ii)**.** Suppose that $(\mathbb{K}x_1, \ldots, \mathbb{K}x_{2n+2})$ and $(\mathbb{K}\tilde{x}_1, \ldots, \mathbb{K}\tilde{x}_{2n+2})$ are two generic $(n, 2n + 2)$-Lagrangian configurations with the same cross-ratios $c_1, \ldots, c_{n+1}$. Following our convention $\omega_{ij} := \omega(x_i, x_j)$, we set $\tilde{\omega}_{ij} := \tilde{\omega}(x_i, x_j)$.

By Lemma 2.1(iii), in order to prove the two configurations equivalent it suffices to find a renormalization $x_i \mapsto \lambda_i x_i$ such that $\lambda_i\lambda_j\omega_{ij} = \tilde{\omega}_{ij}$ for all i and j. By the Lagrangian condition, we need only do this for $j = i + n$ and $i + n + 1$. The argument depends on the parity of n.

The case of n even. It is important to keep in mind that here $\mathrm{GCD}(n, 2n + 2) = 2$. Hence the subset of lines in an $(n, 2n + 2)$-configuration whose indices have a given parity may be written in either of the following ways:

$$\{\mathbb{K}x_i, \mathbb{K}x_{i+2}, \mathbb{K}x_{i+4}, \ldots, \mathbb{K}x_{i+2n}\} = \{\mathbb{K}x_i, \mathbb{K}x_{i+n}, \mathbb{K}x_{i+2n}, \ldots, \mathbb{K}x_{i+n^2}\}.$$

We begin with the case $\mathbb{K} = \mathbb{C}$. For $i = 1, \ldots, 2n + 2$, fix a square root χ_i of $\tilde{\omega}_{i,i+n}/\omega_{i,i+n}$. Extend $(2n + 2)$-periodically to define χ_i for $i \in \mathbb{Z}$. Set

$$\lambda_i := \frac{\chi_i\chi_{i+2n}\chi_{i+4n}\cdots\chi_{i+n^2}}{\chi_{i+n}\chi_{i+3n}\cdots\chi_{i+(n-1)n}} = \prod_{r=0}^{n}\chi_{i+rn}^{(-1)^r} \tag{4.9}$$

and check that $\lambda_i\lambda_{i+n} = \tilde{\omega}_{i,i+n}/\omega_{i,i+n}$ for all i.

Replacing x_i by $\lambda_i x_i$, we may assume $\omega_{i,i+n} = \tilde{\omega}_{i,i+n}$ for all i. Write ρ_i for the ratio $\tilde{\omega}_{i,i+n+1}/\omega_{i,i+n+1}$, which is $(n + 1)$-periodic. Use the fact that the configurations have the same cross-ratios to obtain

$\rho_i\rho_{i+1} = 1$ for all i. Hence

$$1 = \frac{(\rho_i\rho_{i+1})(\rho_{i+2}\rho_{i+3})\cdots(\rho_{i+n}\rho_{i+n+1})}{(\rho_{i+1}\rho_{i+2})(\rho_{i+3}\rho_{i+4})\cdots(\rho_{i+n-1}\rho_{i+n})} = \rho_i^2.$$

Deduce that the ρ_i are either all 1 or all -1. In the former case we are done. In the latter case, rescale again, replacing x_i by $(-1)^i x_i$. This leaves the $\omega_{i,i+n}$ unchanged and negates the $\omega_{i,i+n+1}$, so again we are done.

Example. For $n = 2$ and $n = 4$, the following diagrams depict the equations giving the scale factors (4.9) sending $\omega_{i,i+n}$ to $\tilde{\omega}_{i,i+n}$ for i even. Those for i odd are constructed independently.

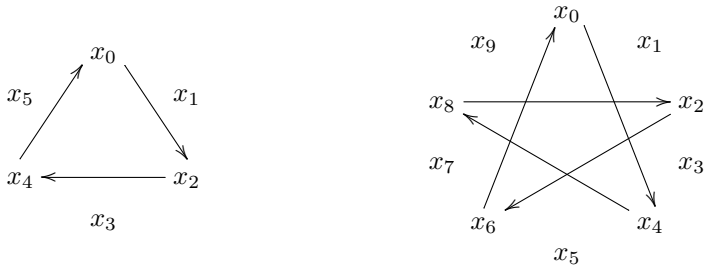

Figure 3. The rescaling scheme for $\mathcal{L}_{2,6}$ and $\mathcal{L}_{4,10}$.

Now take $\mathbb{K} = \mathbb{R}$. Because $(2n+2)/\operatorname{GCD}(n, 2n+2) = n+1$ is odd, the sign invariant

$$\operatorname{sgn}\Big(\prod_{r=0}^{n} \omega_{rn,(r+1)n}\Big) = \operatorname{sgn}\Big(\prod_{s=0}^{n} \omega_{2s,2s+n}\Big) \tag{4.10}$$

reverses under passage to the opposite configuration. Therefore, replacing $(\mathbb{R}x_1, \ldots, \mathbb{R}x_{2n+2})$ by its opposite if necessary, we may assume that

$$\operatorname{sgn}\Big(\prod_{s=0}^{n} \omega_{2s,2s+n}\Big) = \operatorname{sgn}\Big(\prod_{s=0}^{n} \tilde{\omega}_{2s,2s+n}\Big). \tag{4.11}$$

Under this assumption we will show that the scale factors (4.9) are all real, so the two configurations are equivalent over $\mathbb{R}$. This will complete the proof of Theorem 1(ii) for even n.

To prove λ_i real, we must prove λ_i^2 positive. From (4.9),

$$\lambda_i^2 = \prod_{r=0}^{n} \Big(\frac{\tilde{\omega}_{i+rn,i+(r+1)n}}{\omega_{i+rn,i+(r+1)n}}\Big)^{(-1)^r}, \qquad \operatorname{sgn}(\lambda_i^2) = \operatorname{sgn}\Big(\prod_{j \equiv i \bmod 2} \tilde{\omega}_{j,j+n}/\omega_{j,j+n}\Big).$$

For i even, this is positive by (4.11). To prove it positive for i odd, we must prove that

$$\operatorname{sgn}\Big(\prod_{s=0}^{n} \omega_{2s+1,2s+1+n}\Big) = \operatorname{sgn}\Big(\prod_{s=0}^{n} \tilde{\omega}_{2s+1,2s+1+n}\Big). \tag{4.12}$$

Check that

$$\prod_{i=0}^{n} c_i = \Big(\prod_{i=0}^{n} \omega_{i,i+n+1}\Big)^2 \Big(\prod_{i=0}^{2n+1} \omega_{i,i+n}\Big)^{-1}. \tag{4.13}$$

Because the two configurations have the same cross-ratios, we must have

$$\operatorname{sgn}\Big(\prod_{i=0}^{2n+1} \omega_{i,i+n}\Big) = \operatorname{sgn}\Big(\prod_{i=0}^{2n+1} \tilde{\omega}_{i,i+n}\Big).$$

Therefore (4.11) implies (4.12).

The case of n odd. Here $\mathrm{GCD}(n, 2n+2) = 1$, so in contrast with the case that n is even, the entire set of lines in an $(n, 2n+2)$-configuration may be listed with increments of n:

$$\{\mathbb{K}x_0, \mathbb{K}x_1, \mathbb{K}x_2, \dots, \mathbb{K}x_{2n+1}\} = \{\mathbb{K}x_0, \mathbb{K}x_n, \mathbb{K}x_{2n}, \dots, \mathbb{K}x_{(2n+1)n}\}.$$

Set $\lambda_0 := 1$ and define $\lambda_n, \dots, \lambda_{(2n+1)n}$ recursively by $\lambda_{rn} := \lambda_{(r-1)n}\tilde{\omega}_{(r-1)n,rn}/\omega_{(r-1)n,rn}$ for $1 \leqslant r \leqslant 2n+1$. This leads to

$$\lambda_{rn} = \prod_{s=1}^{r} \Big(\frac{\tilde{\omega}_{(s-1)n,sn}}{\omega_{(s-1)n,sn}}\Big)^{(-1)^{r-s}}.$$

At this point we have $\lambda_i\lambda_{i+n} = \tilde{\omega}_{i,i+n}/\omega_{i,i+n}$ except possibly at $i \equiv -n$ modulo $2n+2$, where

$$\lambda_{-n}\lambda_0 = \prod_{s=1}^{2n+1} \Big(\frac{\tilde{\omega}_{(s-1)n,sn}}{\omega_{(s-1)n,sn}}\Big)^{(-1)^{s-1}}.$$

We claim that this is in fact $\tilde{\omega}_{-n,0}/\omega_{-n,0}$. The proof reduces to proving that the expression

$$\frac{\omega_{0,n}\omega_{2n,3n}\cdots\omega_{2n^2,(2n+1)n}}{\omega_{n,2n}\omega_{3n,4n}\cdots\omega_{(2n+1)n,(2n+2)n}} = \frac{\omega_{0,n}\omega_{2,n+2}\cdots\omega_{2n,3n}}{\omega_{1,n+1}\omega_{3,n+3}\cdots\omega_{2n+1,3n+1}}$$

does not change if all the ω's are replaced with $\tilde{\omega}$'s. This is true, because as the reader may check, it is equal to

$$\frac{c_0c_2\cdots c_{n-1}}{c_1c_3\cdots c_n},$$

and the ω's and $\tilde{\omega}$'s have the same cross-ratios.

Thus we may replace x_i by $\lambda_i x_i$, giving $\omega_{i,i+n} = \tilde{\omega}_{i,i+n}$ for all i. Then by equality of cross-ratios, $\omega_{i,i+n+1}\omega_{i+1,i+n+2}$ is equal to $\tilde{\omega}_{i,i+n+1}\tilde{\omega}_{i+1,i+n+2}$, or, equivalently,

$$\big(\tilde{\omega}_{i,i+n+1}/\omega_{i,i+n+1}\big)^{(-1)^i} \tag{4.14}$$

is independent of i.

If $\mathbb{K} = \mathbb{C}$, let χ be a square root of (4.14). If $\mathbb{K} = \mathbb{R}$, let χ be a square root of its magnitude. Observe that for any δ, the rescaling $x_i \mapsto \delta^{(-1)^i}x_i$ leaves $\omega_{i,i+n}$ unchanged and multiplies $\omega_{i,i+n+1}$ by $\delta^{2(-1)^i}$. Therefore in the case $\mathbb{K} = \mathbb{C}$, replacing the x_i by $\chi^{(-1)^i}x_i$ gives $\omega_{ij} = \tilde{\omega}_{ij}$, proving the two configurations equivalent. In the case $\mathbb{K} = \mathbb{R}$, the same argument proves them equivalent when (4.14) is positive.

In the case that $\mathbb{K} = \mathbb{R}$ and (4.14) is negative, this argument leaves us with $\omega_{i,i+n} = \tilde{\omega}_{i,i+n}$ and $\omega_{i,i+n+1} = -\tilde{\omega}_{i,i+n+1}$. Replacing the x_i by the additional rescaling $(-1)^i x_i$ then gives $\omega_{ij} = -\tilde{\omega}_{ij}$, proving the configurations opposite.

4.3. **Proof of Theorem 1**(iii)**.** Suppose that $c_0, \dots, c_n$ are non-zero scalars in $\mathbb{K}$ satisfying (3.2), and extend them $(n+1)$-periodically to $(c_i)_{i\in\mathbb{Z}}$. We will construct an $(n, 2n+2)$-Lagrangian configuration $(\mathbb{K}x_0, \mathbb{K}x_1, \dots, \mathbb{K}x_{2n+1})$ having the given scalars c_i as cross-ratios.

As an intermediate step, we construct from the c_i scalars ω_{ij} that will be equal to $\omega(x_i, x_j)$. By (2.4) and the Lagrangian condition, it is only necessary to construct a $(2n+2)$-periodic sequence $\omega_{i,i+n}$ and an $(n+1)$-periodic sequence $\omega_{i,i+n+1}$ such that the c_i are given by (2.6): then $\omega_{i,i+n+2} = \omega_{i-n,i}$, and the remaining ω_{ij} are 0.

The ω_{ij} for n even. Over $\mathbb{C}$, set $\omega_{i,i+n} := 1$ for all i and fix an $(n+1)$-periodic sequence σ_i of square roots of the c_i: $\sigma_i^2 = c_i$. It is then simple to check that (2.6) is satisfied if we set

$$\omega_{i,i+n+1} := \frac{\sigma_i\sigma_{i+2}\cdots\sigma_{i+n}}{\sigma_{i+1}\sigma_{i+3}\cdots\sigma_{i+n-1}}. \tag{4.15}$$

Over $\mathbb{R}$, the same process works if $c_0c_1\cdots c_n$ is positive: the individual σ_i may not be real, but the $\omega_{i,i+n+1}$ are because their squares are positive.

If $c_0c_1\cdots c_n$ is negative, it suffices to modify the construction as follows: set $\omega_{i,i+n} := (-1)^i$, let σ_i be an $(n+1)$-periodic sequence of square roots of the $-c_i$, and again define the $\omega_{i,i+n+1}$ by (4.15).

The ω_{ij} for n odd. Observe that in this case, (2.6) implies $\prod_{i=0}^{2n+1} \omega_{i,i+n}^{(-1)^i} = \prod_0^n c_i^{(-1)^i}$.

We begin with an asymmetric choice of the ω_{ij} that works over any field. Define $\omega_{0,n} := \prod_0^n c_i^{(-1)^i}$, and for $1 \leqslant i \leqslant 2n+1$, set $\omega_{i,i+n} := 1$. Check that it suffices to set $\omega_{0,n+1} := 1$, $\omega_{1,n+2} := c_0$, and in general, for $0 \leqslant k \leqslant \frac{1}{2}(n-1)$,

$$\omega_{2k,2k+n+1} := \frac{c_1 c_3 \cdots c_{2k-1}}{c_0 c_2 \cdots c_{2k-2}}, \qquad \omega_{2k+1,2k+n+2} := \frac{c_0 c_2 \cdots c_{2k}}{c_1 c_3 \cdots c_{2k-1}}.$$

Over $\mathbb{C}$ it is possible to choose the ω_{ij} more symmetrically, as described in Section 1.3. Fix an $(n+1)$-periodic sequence η_i such that $\eta_i^{2n+2} = c_i$. Define the $\omega_{i,i+n}$ by

$$\mu := \frac{\eta_0 \eta_2 \cdots \eta_{n-1}}{\eta_1 \eta_3 \cdots \eta_n}, \qquad \omega_{i,i+n} := \mu^{(-1)^i}.$$

The reader may check that then (2.6) is satisfied by fixing $\omega_{0,n+1}$ arbitrarily and setting

$$\omega_{2k,2k+n+1} := \mu^{4k} \omega_{0,n+1} \frac{c_1 c_3 \cdots c_{2k-1}}{c_0 c_2 \cdots c_{2k-2}}, \qquad \omega_{2k+1,2k+n+2} := \mu^{-4k-2} \omega_{0,n+1}^{-1} \frac{c_0 c_2 \cdots c_{2k}}{c_1 c_3 \cdots c_{2k-1}}$$

for $0 \leqslant k \leqslant \frac{1}{2}(n-1)$. Requiring $a_0 a_2 \cdots a_{n-1} = a_1 a_3 \cdots a_n$ leads to the most symmetric choice:

$$\omega_{i,i+n+1} := \frac{\eta_i^n \eta_{i+2}^{n-4} \cdots \eta_{i+n-1}^{-n+2}}{\eta_{i+1}^{n-2} \eta_{i+3}^{n-6} \cdots \eta_{i+n}^{-n}} = \prod_{j=0}^{n} \eta_{i+j}^{(-1)^j (n-2j)}.$$

The x_i for n arbitrary. Suppose now that scalars ω_{ij} in $\mathbb{K}$ have been chosen so as to satisfy (2.6). From these scalars we will construct an $(n, 2n+2)$-Lagrangian configuration $(\mathbb{K}x_0, \ldots, \mathbb{K}x_{2n+1})$ over $\mathbb{K}$ which satisfies $\omega(x_i, x_j) = \omega_{ij}$, and therefore has the given cross-ratios $c_0, \ldots, c_n$.

The representatives x_i are almost standard: we set $x_i := e_i$ for $1 \leqslant i \leqslant n$, and

$$\begin{aligned} x_{1+n} &:= \omega_{1,1+n} f_1, \qquad x_{2+n} := \omega_{1,2+n} f_1 + \omega_{2,2+n} f_2, \\ x_{i+n} &:= \omega_{i-2,i+n} f_{i-2} + \omega_{i-1,i+n} f_{i-1} + \omega_{i,i+n} f_i, \quad 3 \leqslant i \leqslant n. \end{aligned}$$

It is immediate that with these definitions, $\omega(x_i, x_j) = \omega_{ij}$ for $1 \leqslant i, j \leqslant 2n$.

The requirement that $\omega(x_i, x_j) = \omega_{ij}$ for $1 \leqslant i \leqslant 2n$ and $j = 2n+1$ or $2n+2$ now determines x_{2n+1} and $x_0 = -x_{2n+2}$. We find that

$$x_{2n+1} = \omega_{n-1,2n+1} f_{n-1} + \omega_{n,2n+1} f_n + \omega_{1+n,2n+1} \sum_{i=1}^{n} (-1)^i \Big(\prod_{j=1}^{i} \omega_{j,j+n}^{-1} \Big) d_i e_i,$$

where $d_1 = 1$, $d_2 = \omega_{1,2+n}$, and the d_i with $3 \leqslant i \leqslant n$ satisfy the recursion relation

$$d_i = \omega_{i-1,i+n} d_{i-1} - \omega_{i-1,i+n-1} \omega_{i-2,i+n} d_{i-2}. \tag{4.16}$$

In the same way we obtain

$$x_0 = -\omega_{0,n} f_n - \sum_{i=1}^{n} (-1)^i \Big(\prod_{j=1}^{i} \omega_{j,j+n}^{-1} \Big) d'_i e_i,$$

where $d'_1 = \omega_{0,1+n}$, $d'_2 = \omega_{0,1+n}\omega_{1,2+n} - \omega_{0,2+n}\omega_{1,1+n}$, and the d'_i with $3 \leqslant i \leqslant n$ also satisfy (4.16).

In order to clarify (4.16), consider for any integers $0 \leqslant j \leqslant i \leqslant n$ the following truncation of the tridiagonal matrix A in (4.4):

$$A_{j,i} = \begin{pmatrix} \omega_{j,j+n+1} & \omega_{j,j+n+2} & & & & \\ \omega_{j+1,j+n+1} & \omega_{j+1,j+n+2} & \omega_{j+1,j+n+3} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \omega_{i-1,i+n-1} & \omega_{i-1,i+n} & \omega_{i-1,i+n+1} & \\ & & & \omega_{i,i+n} & \omega_{i,i+n+1} & \end{pmatrix}.$$

Let us write $\Delta_{j,i}$ for $\det(A_{j,i})$, and adopt the convention $\Delta_{j,j-1} := 1$ and $\Delta_{j,j-2} := 0$. It is clear that the $\Delta_{j,i}$ satisfy a shifted version of (4.16):

$$\Delta_{j,i} = \omega_{i,i+n+1}\Delta_{j,i-1} - \omega_{i,i+n}\omega_{i-1,i+n+1}\Delta_{j,i-2}.$$

Therefore $d_i = \Delta_{1,i-1}$ and $d'_i = \Delta_{0,i-1}$ for $1 \leqslant i \leqslant n$.

The only remaining condition is $\omega(x_0, x_{2n+1}) = 0$. After some simplification it reduces to

$$\Delta_{0,n} - \omega_{0,n}\omega_{n+1,2n+1}\Delta_{1,n-1} = 0. \tag{4.17}$$

Recall the matrices Ω_n and A from (4.3). Because A is $A_{0,n}$, Proposition 4.2 and (4.7) show that (4.17) is equivalent to (4.8). Because we assumed that the given c_i satisfy (3.2), these conditions hold. This completes the proof of Theorem 1.

5. Normalized configurations

As noted in Section 3.3, in this section we describe certain normalized choices of representatives of $(n, 2n+2)$-configurations. As in that section, let $(X_0, \dots, X_{2n+1})$ be a generic $(n, 2n+2)$-Lagrangian configuration over $\mathbb{K}$ with representatives $x_0, \dots, x_{2n+1}$, inner products $\omega_{ij} := \omega(x_i, x_j)$, and cross-ratios $c_0, \dots, c_n$.

5.1. **The case of n even and $\mathbb{K} = \mathbb{C}$.** The results in this case were stated in Section 3.3. Here we give their proofs.

Proof of Proposition 3.3. By the constructions in Section 4.3, there exists a generic configuration $(\mathbb{C}\tilde{x}_0, \dots, \mathbb{C}\tilde{x}_{2n+1})$ with $\tilde{\omega}_{i,i+n} = 1$ and cross-ratios $c_0, \dots, c_n$. By Theorem 1(ii), it is equivalent to $(X_0, \dots, X_{2n+1})$. The images $x_0, \dots, x_{2n+1}$ of $\tilde{x}_0, \dots, \tilde{x}_{2n+1}$ under the equivalence have $\omega_{i,i+n} = 1$ for all i.

To see that there are exactly four such choices of $(x_i)_i$, suppose that $(\lambda_i x_i)_i$ is another. Observe that then $\lambda_{i+n} = \lambda_i^{-1}$, so $\lambda_{i+rn} = \lambda_i^{(-1)^r}$. But $\lambda_{i+n(n+1)} = \lambda_i$ by periodicity, so $\lambda_i = \pm 1$, whence $\lambda_{i+n} = \lambda_i$, and $\lambda_i = \lambda_j$ for $i \equiv j$ modulo 2. This proves (i). The remaining statements are immediate. □

Proof of Theorem 2. Part (i) follows from Section 4.1: by Lemma 4.3, for normalized representatives $(x_0, \dots, x_{2n+1})$ the Pfaffian of the ω-Gram matrix is, up to a sign, $R_{n+1}(a_0, \dots, a_n)$. Parts (ii) and (iii) follow from Theorem 1(ii) and (iii) and Proposition 3.3(iii): the cross-ratios determine $\pm(a_0, \dots, a_n)$ and vice versa. □

5.2. **The case of n even and $\mathbb{K} = \mathbb{R}$.** Recall from Section 4.2 the sign invariants

$$\varepsilon_0 := \operatorname{sgn}\Big(\prod_{s=0}^{n} \omega_{2s,2s+n}\Big), \qquad \varepsilon_1 := \operatorname{sgn}\Big(\prod_{s=0}^{n} \omega_{2s+1,2s+1+n}\Big), \qquad \varepsilon_c := \operatorname{sgn}\Big(\prod_{i=0}^{n} c_i\Big)$$

of the configuration $(X_0, \dots, X_{2n+1})$: ε_0 is (4.10), ε_1 is the left side of (4.12), and by (4.13), $\varepsilon_c = \varepsilon_0\varepsilon_1$.

Proposition 5.1. *For n even and $\mathbb{K} = \mathbb{R}$, $(X_0, \dots, X_{2n+1})$ admits exactly four choices of representatives whose symplectic subdiameters are $\omega_{i,i+n} = \varepsilon_{i \bmod 2}$. Fix such a choice, $x_0, \dots, x_{2n+1}$, and denote its symplectic diameters $\omega_{i,i+n+1}$ by $a_i^{\mathbb{R}}$.*

(i) *In terms of* $(x_i)_i$*, the four choices are as in Proposition 3.3(i). The first two have symplectic diameters* $(a_i^{\mathbb{R}})_i$*, and the second two have symplectic diameters* $(-a_i^{\mathbb{R}})_i$*. In particular,* $\pm(a_i^{\mathbb{R}})_i$ *is an invariant of the configuration, the collection of its* normalized real symplectic diameters.

(ii) *If* $\varepsilon_c = 1$*, then the normalized symplectic diameters coincide with the normalized real symplectic diameters:* $\pm(a_i)_i = \pm(a_i^{\mathbb{R}})_i$.

(iii) *If* $\varepsilon_c = -1$*, then* $\pm(a_i)_i = \pm\sqrt{-1}\,(a_i^{\mathbb{R}})_i$.

Proof. By the constructions in Section 4.3, there exists a generic configuration $(\mathbb{R}\tilde{x}_0, \dots, \mathbb{R}\tilde{x}_{2n+1})$ with cross-ratios c_i, such that if $\varepsilon_c = 1$, then $\tilde{\omega}_{i,i+n} = 1$ for all i, and if $\varepsilon_c = -1$, then $\tilde{\omega}_{i,i+n} = (-1)^i$ for all i. By Theorem 1(ii), this configuration is equivalent either to $(X_0, \dots, X_{2n+1})$ or its opposite, and so, replacing $(\mathbb{R}\tilde{x}_0, \dots, \mathbb{R}\tilde{x}_{2n+1})$ by its opposite if necessary, we may assume that it is equivalent to $(X_0, \dots, X_{2n+1})$. Then, recalling that passage to opposites negates inner products, we find that the images $x_0, \dots, x_{2n+1}$ of $\tilde{x}_0, \dots, \tilde{x}_{2n+1}$ under the equivalence have $\omega_{i,i+n} = \varepsilon_{i \bmod 2}$.

The proof of (i) goes exactly as in Proposition 3.3. For (ii) and (iii), fix a choice of $\sqrt{-1}$. In (ii), if ε_0 and ε_1 are both -1, then the representatives of the complex normalization of Section 3.3 may be taken to be $\sqrt{-1}\,(x_i)_i$, while if they are both 1, then the real and complex normalizations coincide.

In (iii), if $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$, the $\mathbb{C}$-normalized representatives may be taken to be x_i for i even and $\sqrt{-1}\,x_i$ for i odd, while if $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$, they may be taken to be $\sqrt{-1}\,x_i$ for i even and x_i for i odd. To summarize, in all cases the $\mathbb{C}$-normalized representatives are $\left(\sqrt{-1}^{(1-\varepsilon_{i \bmod 2})/2} x_i\right)_i$. The relation between $\pm(a_i)_i$ and $\pm(a_i^{\mathbb{R}})_i$ now follows easily. □

5.3. **The case of n odd and $\mathbb{K} = \mathbb{C}$.** Recall from Section 4.3 that here $\prod_{i=0}^{2n+1} \omega_{i,i+n}^{(-1)^i} = \prod_0^n c_i^{(-1)^i}$.

Proposition 5.2. *For n odd, $\mathbb{K} = \mathbb{C}$, and μ any $(2n+2)^{\text{nd}}$ root of $\prod_0^n c_i^{(-1)^i}$, $(X_0, \dots, X_{2n+1})$ admits exactly $(2n+2)$ choices of representatives such that $\omega_{i,i+n} = \mu^{(-1)^i}$ for all i and $\prod_0^n \omega_{i,i+n+1}^{(-1)^i} = 1$. Fix such a choice, $x_0, \dots, x_{2n+1}$, and set $a_i := \omega_{i,i+n+1}$.*

(i) *The $2n+2$ choices are $(\delta^{(-1)^i} x_i)_i$, where δ runs over the $(2n+2)^{\text{nd}}$ roots of unity.*

(ii) *The symplectic diameters corresponding to any given choice of δ are $(\delta^{2(-1)^i} a_i)_i$.*

(iii) *The cross-ratios of the configuration are $c_i = \mu^{2(-1)^i} a_i a_{i+1}$.*

Proof. The discussion in Section 4.3 shows that for any $(2n+2)^{\text{nd}}$ root μ of $\prod_0^n c_i^{(-1)^i}$, there exists a configuration $(\mathbb{C}\tilde{x}_0, \dots, \mathbb{C}\tilde{x}_{2n+1})$ with $\tilde{\omega}_{i,i+n} = \mu^{(-1)^i}$ and cross-ratios $c_0, \dots, c_n$. By Theorem 1(ii), it is equivalent to $(X_0, \dots, X_{2n+1})$. The images $x_0, \dots, x_{2n+1}$ of $\tilde{x}_0, \dots, \tilde{x}_{2n+1}$ under the equivalence have $\omega_{i,i+n} = \mu^{(-1)^i}$ for all i.

For (i), first check that if a renormalization $x_i \mapsto \lambda_i x_i$ preserves $\omega_{i,i+n}$ for all i, then it is of the form $\lambda_i = \delta^{(-1)^i}$ for some δ. Then check that $\prod_0^n \omega_{i,i+n+1}^{(-1)^i} = 1$ if and only if $\delta^{2n+2} = 1$. The remaining statements are clear. □

We remark that the general rescaling $x_i \mapsto \lambda_i x_i$ going between normalizations as above with different choices of μ is $\lambda_i := \xi^{(-1)^i(ip+q)}$, where ξ is any primitive $(2n+2)^{\text{nd}}$ root of unity and p and q are arbitrary elements of $\mathbb{Z}_{2n+2}$. It transforms $\omega_{i,i+n}$ from $\mu^{(-1)^i}$ to $(\xi^{-pn}\mu)^{(-1)^i}$ and $\omega_{i,i+n+1}$ from a_i to $(-1)^p \xi^{2(-1)^i(ip+q)} a_i$, i.e.,

$$\left(\mu, a_i\right)_i \mapsto \left(\xi^{-pn}\mu, (-1)^p\, \xi^{2(-1)^i(ip+q)} a_i\right)_i.$$

We will not formally state the specialization of Theorem 1 corresponding to the normalization in Proposition 5.2, but let us describe the specialization of the relation (3.2): it becomes the vanishing of the quantity obtained from the product $a_0 \cdots a_n$ by applying the "μ-cyclic replacement rule": replace

cyclically adjacent pairs $a_i a_{i+1}$ by $-\mu^{-2(-1)^i}$. For example, at $n = 1$ and 3,

$$0 = a_0 a_1 - \mu^{-2} - \mu^2,$$

$$0 = a_0 a_1 a_2 a_3 - \mu^{-2}(a_0 a_1 + a_2 a_3) + \mu^{-4} + \mu^4.$$

5.4. **The case of n odd and $\mathbb{K} = \mathbb{R}$.** Here we have only found natural normalizations under certain positivity conditions.

Proposition 5.3. *Suppose that n is odd, $\mathbb{K} = \mathbb{R}$, and $\prod_0^n c_i^{(-1)^i}$ is positive, and let μ be its positive $(2n+2)^{\text{nd}}$ root. Then $(X_0, \dots, X_{2n+1})$ admits choices of representatives such that $\omega_{i,i+n} = \mu^{(-1)^i}$ for all i.*

If both $c_0 c_2 \cdots c_{n+1}$ and $c_1 c_3 \cdots c_n$ are positive, then exactly two such choices satisfy in addition $\prod_0^n \omega_{i,i+n+1}^{(-1)^i} = 1$. Otherwise there is no such choice.

Proof. Given any representatives $(x_i)_i$, rescale to $\lambda_i x_i$, where $\lambda_0 := 1$ and

$$\lambda_{rn}\lambda_{(r+1)n}\omega_{rn,(r+1)n} = \mu^{(-1)^i}$$

for $0 < r < 2n+2$. Check that this proves the first paragraph.

In the second paragraph we can only use further rescalings preserving the subdiameters: $x_i \mapsto \delta^{(-1)^i} x_i$. Such rescalings multiply $\prod_0^n \omega_{i,i+n+1}^{(-1)^i}$ by δ^{2n+2}, so we can choose δ to make the product 1 if and only if $\prod_0^n \omega_{i,i+n+1}$ is positive. To complete the proof, observe that $c_0 c_2 \cdots c_{n-1}$ is $\mu^{n+1} \prod_0^n \omega_{i,i+n+1}$. There are two choices because the sign of δ is irrelevant. □

6. Symmetric linear difference equations and the closure of $\mathcal{L}_{n,N}(\mathbb{K})$

In this section we present general results relating Lagrangian configurations to non-degenerate symmetric linear difference equations of degree $2n$. The solution space of such an equation has a natural symplectic form, generalizing the Wronski determinant. When the equation has N-periodic coefficients and monodromy $-\mathrm{Id}$, there is a simple way to construct a particular Lagrangian configuration in its solution space. This yields a projection from the space of all such equations to equivalence classes of Lagrangian configurations.

6.1. **Linear difference operators.** Let T be the *shift operator*, acting on infinite sequences $(V_i)_{i\in\mathbb{Z}}$ by $(TV)_i := V_{i-1}$. A *linear difference operator over* $\mathbb{K}$ is a polynomial expression in T and its inverse,

$$A = a^n T^n + a^{n-1} T^{n-1} + \cdots + a^m T^m, \tag{6.1}$$

where $m \leqslant n$ are arbitrary integers and the coefficients a^ℓ are sequences $(a_i^\ell)_{i\in\mathbb{Z}}$ of $\mathbb{K}$-scalars. Such operators act on sequences (V_i) of $\mathbb{K}$-scalars, the coefficients acting by multiplication: $(aV)_i := a_i V_i$.

- A is said to be of *order* $n - m$ if both a^m and a^n are non-zero.
- A is said to be *non-degenerate* if both a_i^m and a_i^n are non-zero for all i.
- A is said to be N-*periodic* if $a_i^\ell = a_{i+N}^\ell$ for all ℓ and i.

Definition. The *adjoint* A^* of a linear difference operator A is defined by

$$V \cdot (A^* W) = (AV) \cdot W,$$

where $V \cdot W := \sum_{i\in\mathbb{Z}} V_i W_i$, an inner product on scalar sequences with only finitely many non-zero terms.

It is simple to check that $T^* = T^{-1}$. This is the discrete analog of the fact that translation is the exponential of the derivation $\frac{d}{dx}$, and $\frac{d}{dx}^* = -\frac{d}{dx}$. It is also clear that $(AB)^* = B^* A^*$ for any operators A and B. In particular, writing $(T^{\ell'} a^\ell)$ for the multiplication operator $(T^{\ell'} a^\ell)_i = a_{i-\ell'}^\ell$, one obtains the following lemma.

Lemma 6.1. $\left(\sum_{\ell=m}^n a^\ell T^\ell\right)^* = \sum_{\ell=m}^n (T^{-\ell} a^\ell)\, T^{-\ell}$.

Definition. If an operator A satisfies $A = A^*$, it is *self-adjoint,* or *symmetric.* In this case, for some $n \geqslant 0$ there exist sequences $a^0, \dots, a^n$ such that

$$A = a^0 + \sum_{\ell=1}^{n} \bigl(a^\ell\, T^\ell + (T^{-\ell} a^\ell)\, T^{-\ell}\bigr). \tag{6.2}$$

Remark. The spectral theory of linear difference operators is quite similar to that of linear differential operators; see [10] and references therein. Operators with periodic or antiperiodic solutions play a special role in [10], where they are called "superperiodic".

6.2. **Linear difference equations.** The *linear difference equation* corresponding to a linear difference operator A is $AV = 0$. We denote the space of solutions of this equation, the kernel of A, by $\mathcal{K}(A)$:

$$\mathcal{K}(A) := \{V : AV = 0\}.$$

Lemma 6.2. *Let A be a non-degenerate linear difference operator over $\mathbb{K}$ of order p. For any $i_0 \in \mathbb{Z}$ and any $\mathbb{K}$-scalars $c_{i_0+1}, \cdots, c_{i_0+p}$, there is a unique solution (V_i) of the equation $AV = 0$ satisfying the initial conditions $V_i = c_i$ for $i_0 < i \leqslant i_0 + p$. In particular, $\mathcal{K}(A)$ is a p-dimensional vector space over $\mathbb{K}$.*

The proof of this lemma is immediate. Note that the symmetric operator (6.2) is non-degenerate if and only if $a_i^n \neq 0$ for all i. Let us write the equation $AV = 0$ explicitly in this case:

$$a_i^n\, V_{i-n} + \cdots + a_i^1\, V_{i-1} + a_i^0\, V_i + a_{i+1}^1\, V_{i+1} + \cdots + a_{i+n}^n\, V_{i+n} = 0 \ \text{ for all } i. \tag{6.3}$$

Corollary 6.3. *Given a non-degenerate symmetric difference operator A over $\mathbb{K}$ of degree $2n$ as in (6.3), for all $i \in \mathbb{Z}$ there is a unique element $V^i(A)$ of the kernel $\mathcal{K}(A)$ such that*

$$\bigl(V_{i-n}^i(A),\, V_{i-n+1}^i(A),\, \dots,\, V_{i+n-1}^i(A),\, V_{i+n}^i(A)\bigr) := \Bigl(-\frac{1}{a_i^n},\, 0,\, 0,\, \dots,\, 0,\, \frac{1}{a_{i+n}^n}\Bigr). \tag{6.4}$$

For any i, $\bigl\{V^{i+1}(A),\, V^{i+2}(A),\, \dots,\, V^{i+2n}(A)\bigr\}$ is a basis of $\mathcal{K}(A)$.

An important property of non-degenerate symmetric operators is the existence of a natural symplectic form on their kernels. Before giving the general result, we describe the simplest case.

Example. The operator $L := T - a + T^{-1}$ is known as the discrete Sturm-Liouville (or Hill, or Schrödinger) operator. It is non-degenerate and symmetric, and the classical *Wronski determinant*

$$\mathcal{W}(V, V') := \begin{vmatrix} V_{i-1} & V'_{i-1} \\ V_i & V'_i \end{vmatrix}$$

is a well-defined symplectic form on its kernel $\mathcal{K}(L)$. To understand this, check that when $L(V)$ and $L(V')$ are zero, $\mathcal{W}(V, V')$ is independent of the choice of i:

$$\begin{vmatrix} V_i & V'_i \\ V_{i+1} & V'_{i+1} \end{vmatrix} = -\begin{vmatrix} V_i & V'_i \\ V_{i-1} & V'_{i-1} \end{vmatrix} - a_i \begin{vmatrix} V_i & V'_i \\ V_i & V'_i \end{vmatrix} = \begin{vmatrix} V_{i-1} & V'_{i-1} \\ V_i & V'_i \end{vmatrix}.$$

Remark. The continuant (3.3) may be viewed as an element of $\mathcal{K}(L)$: the Sturm-Liouville difference equation is

$$V_{i-1} - a_i V_i + V_{i+1} = 0, \tag{6.5}$$

and the initial conditions $(V_{-1}, V_0) = (0, 1)$ give $V_n = K_n(a_0, \dots, a_{n-1})$. In fact, continuants are the simplest members of the series of *André determinants,* which satisfy linear difference equations of higher order; see [1] and also [13].

We now define a multidimensional version of the Wronski determinant. It is a discrete analog of the symplectic form on the solution space of the symmetric linear differential equation studied in [16].

Definition. Fix a non-degenerate symmetric linear difference operator A over $\mathbb{K}$ of order $2n$, as in (6.2). Given two elements V and V' of the kernel $\mathcal{K}(A)$ and any $i \in \mathbb{Z}$, set

$$\mathcal{W}_A^i(V,V') := \sum_{\ell=1}^{n} \sum_{m=i+1}^{i+\ell} a_m^\ell \begin{vmatrix} V_{m-\ell} & V'_{m-\ell} \\ V_m & V'_m \end{vmatrix}. \tag{6.6}$$

Lemma 6.4. (i) *$\mathcal{W}_A^i$ is independent of i and is a symplectic form $\mathcal{W}_A$ on $\mathcal{K}(A)$.*

(ii) *Writing V^i for the solution $V^i(A)$ of Corollary 6.3, $\mathcal{W}_A(V^i,V^j) = V^i_j = -V^j_i$.*

(iii) *In particular, $\mathcal{W}_A(V^i,V^j) = 0$ for $|i-j| < n$, and $\mathcal{W}_A(V^{j-n},V^j) = 1/a_j^n$.*

Proof. Let us use the shorthand $\left|\begin{smallmatrix} j \\ k \end{smallmatrix}\right|$ for $\left|\begin{smallmatrix} V_j & V'_j \\ V_k & V'_k \end{smallmatrix}\right|$. It is helpful to expand $\mathcal{W}_A^i(V,V')$ as

$$\left(a^1_{i+1}\left|\begin{smallmatrix} i \\ i+1 \end{smallmatrix}\right|\right) + \left(a^2_{i+1}\left|\begin{smallmatrix} i-1 \\ i+1 \end{smallmatrix}\right| + a^2_{i+2}\left|\begin{smallmatrix} i \\ i+2 \end{smallmatrix}\right|\right) + \left(a^3_{i+1}\left|\begin{smallmatrix} i-2 \\ i+1 \end{smallmatrix}\right| + a^3_{i+2}\left|\begin{smallmatrix} i-1 \\ i+2 \end{smallmatrix}\right| + a^3_{i+3}\left|\begin{smallmatrix} i \\ i+3 \end{smallmatrix}\right|\right)$$

$$+ \cdots + \left(a^n_{i+1}\left|\begin{smallmatrix} i+1-n \\ i+1 \end{smallmatrix}\right| + a^n_{i+2}\left|\begin{smallmatrix} i+2-n \\ i+2 \end{smallmatrix}\right| + \cdots + a^n_{i+n}\left|\begin{smallmatrix} i \\ i+n \end{smallmatrix}\right|\right).$$

To prove that $\mathcal{W}_A^i$ is independent of i, verify that $\mathcal{W}_A^i(V,V') - \mathcal{W}_A^{i-1}(V,V')$ is

$$\left(a^1_{i+1}\left|\begin{smallmatrix} i \\ i+1 \end{smallmatrix}\right| - a^1_i\left|\begin{smallmatrix} i-1 \\ i \end{smallmatrix}\right|\right) + \left(a^2_{i+2}\left|\begin{smallmatrix} i \\ i+2 \end{smallmatrix}\right| - a^2_i\left|\begin{smallmatrix} i-2 \\ i \end{smallmatrix}\right|\right) + \cdots + \left(a^n_{i+n}\left|\begin{smallmatrix} i \\ i+n \end{smallmatrix}\right| - a^n_i\left|\begin{smallmatrix} i-n \\ i \end{smallmatrix}\right|\right).$$

Convert $\left|\begin{smallmatrix} i-\ell \\ i \end{smallmatrix}\right|$ to $-\left|\begin{smallmatrix} i \\ i-\ell \end{smallmatrix}\right|$ and use (6.3), $\left|\begin{smallmatrix} i \\ i \end{smallmatrix}\right| = 0$, and $AV = AV' = 0$ to check that this is

$$\begin{vmatrix} V_i & V'_i \\ (AV)_i & (AV')_i \end{vmatrix} = \begin{vmatrix} V_i & V'_i \\ 0 & 0 \end{vmatrix} = 0.$$

Thus we may write simply $\mathcal{W}_A$ for $\mathcal{W}_A^i$. Clearly it is a skew-symmetric bilinear form on $\mathcal{K}(A)$.

For (ii) and (iii), it suffices to check from the definitions that for any $V' \in \mathcal{K}(A)$,

$$\mathcal{W}_A^{i-1}(V^i,V') = a_i^n \begin{vmatrix} V^i_{i-n} & V'_{i-n} \\ V^i_i & V'_i \end{vmatrix} = -V'_i.$$

To prove that $\mathcal{W}_A$ is non-degenerate, recall Lemma 2.1 and consider the matrix $\Omega_{\mathcal{W}_A}$ of $\mathcal{W}_A$ in the basis $\{V^{j-n+1}, V^{j+1}, \dots, V^{j+n}\}$: for $1 \leqslant r,s \leqslant 2n$, $(\Omega_{\mathcal{W}_A})_{rs} := \mathcal{W}_A(V^{r+j-n}, V^{s+j-n})$. The result will follow if we prove $\det(\Omega_{\mathcal{W}_A}) \neq 0$. Applying (iii), we find

$$\Omega_{\mathcal{W}_A} = \begin{pmatrix} 0 & T \\ -T^t & 0 \end{pmatrix},$$

where T is an $n \times n$ upper triangular matrix with diagonal entries $(a^n_{j+1})^{-1}, (a^n_{j+2})^{-1}, \dots, (a^n_{j+n})^{-1}$. □

Rescaling. Suppose that λ is a non-vanishing sequence over $\mathbb{K}$: a sequence $(\lambda_i)_{i\in\mathbb{Z}}$ of non-zero $\mathbb{K}$-scalars. Given an operator A, we define its *rescaling by* λ to be the operator $\lambda^{-1} \circ A \circ \lambda^{-1}$.

Lemma 6.5. *Let A be a non-degenerate symmetric linear difference operator over $\mathbb{K}$ of order $2n$, as in (6.2), and let λ be a non-vanishing sequence over $\mathbb{K}$. Let $\tilde{A}$ be the rescaling $\lambda^{-1}A\lambda^{-1}$.*

(i) *$\tilde{A}$ is a non-degenerate symmetric operator over $\mathbb{K}$ of order $2n$. Its coefficients $\tilde{a}^\ell$ are*

$$\tilde{a}^\ell_i = \lambda_i^{-1}\lambda^{-1}_{i-\ell}a^\ell_i.$$

(ii) *If A and λ are N-periodic, then $\tilde{A}$ is too.*

(iii) *λ is a symplectic map from $\big(\mathcal{K}(A), \mathcal{W}_A\big)$ to $\big(\mathcal{K}(\tilde{A}), \mathcal{W}_{\tilde{A}}\big)$.*

(iv) *$\lambda\big(V^i(A)\big) = \lambda_i^{-1}V^i(\tilde{A})$.*

Proof. We leave (i), (ii), and $\lambda\big(\mathcal{K}(A)\big) = \mathcal{K}(\tilde{A})$ to the reader. To prove λ symplectic, verify

$$\mathcal{W}^i_{\tilde{A}}(\lambda V, \lambda V') := \sum_{\ell=1}^{n} \sum_{m=i+1}^{i+\ell} \tilde{a}^\ell_m \lambda_m \lambda_{m-\ell} \begin{vmatrix} V_{m-\ell} & V'_{m-\ell} \\ V_m & V'_m \end{vmatrix}.$$

Because $\tilde{a}^\ell_m \lambda_m \lambda_{m-\ell} = a^\ell_m$, this is simply $\mathcal{W}^i_A(V, V')$.

For (iv), use Corollary 6.3 to check that $\lambda\big(V^i(A)\big)_j = \lambda_i^{-1} V^i(\tilde{A})_j$ for $i - n \leqslant j \leqslant i + n$. By (iii), both $\lambda\big(V^i(A)\big)$ and $\lambda_i^{-1} V^i(\tilde{A})$ are in $\mathcal{K}(\tilde{A})$, so by Lemma 6.2 they are equal. □

6.3. **Periodic operators, monodromy, and Lagrangian configurations.** Difference equations corresponding to N-periodic operators do not necessarily have N-periodic solutions. However, we do have the following lemma. Its proof is immediate from the obvious fact that an operator is N-periodic if and only if it commutes with T^N.

Lemma 6.6. *Suppose that A is an N-periodic linear difference operator. Then T^N preserves the kernel $\mathcal{K}(A)$. It is called the* monodromy operator *M_A of A:*

$$M_A := T^N|_{\mathcal{K}(A)} : \mathcal{K}(A) \to \mathcal{K}(A).$$

In the case of non-degenerate symmetric operators, the monodromy is symplectic:

Lemma 6.7. *Suppose that A is a non-degenerate N-periodic symmetric linear difference operator of order $2n$. Then the monodromy operator M_A preserves the symplectic form $\mathcal{W}_A$ on $\mathcal{K}(A)$.*

Proof. We must prove that $\mathcal{W}_A(T^N V, T^N V') = \mathcal{W}_A(V, V')$ for all elements V and V' of $\mathcal{K}(A)$. Recall that $\mathcal{W}_A$ may be expressed as $\mathcal{W}^i_A$ for any i. Use the fact that $(T^N a^\ell) = a^\ell$ for all ℓ to check that $\mathcal{W}^i_A(T^N V, T^N V') = \mathcal{W}^{i-N}_A(V, V')$. □

Our main result in Section 6 is Theorem 3, the most general result of the paper. It states that a certain set of difference operators may be projected to symplectic equivalence classes of Lagrangian configurations. In order to define this projection we make two preliminary definitions.

Definition. For $N \geqslant 2n$, let $\overline{\mathcal{L}}_{n,N}(\mathbb{K})$ be the $\mathrm{Sp}(2n, \mathbb{K})$-moduli space of symplectic equivalence classes of *all* (n, N)-Lagrangian configurations over $\mathbb{K}$, both generic and non-generic.

Definition. For $N \geqslant 2n$, let $\mathcal{E}_{n,N}(\mathbb{K})$ be the set of non-degenerate N-periodic symmetric linear difference operators over $\mathbb{K}$ of order $2n$ with monodromy $-\mathrm{Id}$.

Remarks.

- $\overline{\mathcal{L}}_{n,N}(\mathbb{K})$ is a closure of $\mathcal{L}_{n,N}(\mathbb{K})$.
- For both geometric and analytic reasons, imposing the condition that the monodromy be Id in the definition of $\mathcal{E}_{n,N}(\mathbb{K})$ would be less natural; cf. [13, 10] for the $\mathrm{SL}(2n)$-analog.
- Suppose that A is a non-degenerate N-periodic symmetric linear difference operator of order $2n$. Fix initial conditions $V_{i_0+1}, \ldots, V_{i_0+2n}$, and let V be the corresponding solution of $AV = 0$. It is easy to see that each entry V_i of V depends polynomially on the quantities $(a^n_k)^{\pm 1}, a^{n-1}_k, \ldots, a^0_k$ for $i_0 < k \leqslant i_0 + N$. It follows that the same is true of M_A, and so $\mathcal{E}_{n,N}$ is an algebraic variety.

  We will see that $\mathcal{E}_{n,N}$ projects to $\overline{\mathcal{L}}_{n,N}$, with fibers given by rescaling. Recall from Proposition 2.7 that $\overline{\mathcal{L}}_{n,N}$ is $n(N-2n-1)$-dimensional. The set of periodic rescalings has N parameters, so the dimension of $\mathcal{E}_{n,N}$ is $(n+1)N - n(2n+1)$. Thus the number of independent constraints imposed on a periodic symmetric linear difference operator by specifying its monodromy to be $-\mathrm{Id}$ is the dimension of the symplectic group preserving $\mathcal{W}_A$, as one would predict from Lemma 6.7.

Proposition 6.8. *Suppose that A is in $\mathcal{E}_{n,N}(\mathbb{K})$. Fix arbitrarily an identification of the symplectic space $\big(\mathcal{K}(A), \mathcal{W}_A\big)$ with the standard symplectic space $\big(\mathbb{K}^{2n}, \omega\big)$, and let $v_i \in \mathbb{K}^{2n}$ be the image under this identification of the element $V^i(A)$ of $\mathcal{K}(A)$ defined in Corollary 6.3.*

(i) *The v_i are N-antiperiodic.*

(ii) $\big(\mathbb{K}v_1, \dots, \mathbb{K}v_N\big)$ *is an* (n, N)*-Lagrangian configuration.*

(iii) *There is a map* $P : \mathcal{E}_{n,N}(\mathbb{K}) \to \overline{\mathcal{L}}_{n,N}(\mathbb{K})$*, defined by*

$$P(A) := \textit{the symplectic equivalence class of } \big(\mathbb{K}v_1, \dots, \mathbb{K}v_N\big).$$

(iv) $P(A)$ *and* $P(-A)$ *are opposite configurations.*

Proof. The fact that the monodromy M_A is $-\mathrm{Id}$ translates to the statement that $V^{i+N}(A) = -V^i(A)$, giving (i). For (ii), apply Lemmas 2.2 and 6.4(iii) and use the fact that $\mathcal{W}_A\big(V^i(A), V^j(A)\big) = \omega(v_i, v_j)$ by construction. For (iii), note that the symplectic equivalence class of $(\mathbb{K}v_1, \dots, \mathbb{K}v_N)$ is independent of the choice of symplectic identification of $\mathcal{K}(A)$ with $\mathbb{K}^{2n}$.

For (iv), use the facts that $\mathcal{K}(-A) = \mathcal{K}(A)$, $V^i(-A) = -V^i(A)$, and $\mathcal{W}_{-A} = -\mathcal{W}_A$. □

Theorem 3. (i) $P : \mathcal{E}_{n,N}(\mathbb{K}) \to \overline{\mathcal{L}}_{n,N}(\mathbb{K})$ *is surjective.*

(ii) $P(A) = P(\tilde{A})$ *if and only if* $\tilde{A}$ *is a rescaling* $\lambda^{-1}A\lambda^{-1}$ *of* A *by an* N*-periodic* λ*.*

Proof. We proceed by a series of lemmas. For (i), fix an (n, N)-Lagrangian configuration $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$. As usual, extend the representatives N-antiperiodically to $(x_i)_{i\in\mathbb{Z}}$ and write ω_{ij} for $\omega(x_i, x_j)$. In order to construct an operator A in $\mathcal{E}_{n,N}(\mathbb{K})$ such that $P(A)$ is the class of $(\mathbb{K}x_1, \dots, \mathbb{K}x_N)$, for all i define

$$a_i^n := 1/\omega_{i-n,i}. \tag{6.7}$$

Keeping in mind that $\{x_{i-n+1}, \dots, x_{i+n}\}$ is a basis of $\mathbb{K}^{2n}$, define $a_i^{n-1}, \dots, a_i^{-n}$ by the equation

$$a_i^n x_{i-n} + a_i^{n-1} x_{i-n+1} + \dots + a_i^{-n+1} x_{i+n-1} + a_i^{-n} x_{i+n} = 0. \tag{6.8}$$

Define A by $(AV)_i := a_i^n V_{n-i} + \dots a_i^{-n} V_{n+i}$. The next two lemmas concern this difference operator.

Lemma 6.9. (i) A *is non-degenerate,* N*-periodic, and symmetric.*

(ii) *For* $1 \leqslant p \leqslant n$*, the coefficients* a_i^{n-p} *are given by*

$$a_i^{n-p} = \sum_{m=0}^{p-1} \sum_{0<p_1<\dots<p_m<p} -(-1)^m \frac{\omega_{i-n,i+p_1}\,\omega_{i-n+p_1,i+p_2}\cdots\omega_{i-n+p_{m-1},i+p_m}\,\omega_{i-n+p_m,i+p}}{\omega_{i-n,i}\,\omega_{i-n+p_1,i+p_1}\cdots\omega_{i-n+p_m,i+p_m}\,\omega_{i-n+p,i+p}}. \tag{6.9}$$

Examples. Observe that (6.9) has 2^{p-1} summands. Let us give the first three cases as examples. The summand at $m = 0$ is understood to be $-\omega_{i-n,i+p}/\omega_{i-n,i}\,\omega_{i-n+p,i+p}$, so we have

$$a_i^{n-1} = \frac{-1}{\omega_{i-n,i}}\Big(\frac{\omega_{i-n,i+1}}{\omega_{i-n+1,i+1}}\Big),$$

$$a_i^{n-2} = \frac{-1}{\omega_{i-n,i}}\Big(\frac{\omega_{i-n,i+2}}{\omega_{i-n+2,i+2}} - \frac{\omega_{i-n,i+1}\,\omega_{i-n+1,i+2}}{\omega_{i-n+1,i+1}\,\omega_{i-n+2,i+2}}\Big),$$

$$a_i^{n-3} = \frac{-1}{\omega_{i-n,i}}\Big(\frac{\omega_{i-n,i+3}}{\omega_{i-n+3,i+3}} - \frac{\omega_{i-n,i+1}\,\omega_{i-n+1,i+3}}{\omega_{i-n+1,i+1}\,\omega_{i-n+3,i+3}} - \frac{\omega_{i-n,i+2}\,\omega_{i-n+2,i+3}}{\omega_{i-n+2,i+2}\,\omega_{i-n+3,i+3}} + \frac{\omega_{i-n,i+1}\,\omega_{i-n+1,i+2}\,\omega_{i-n+2,i+3}}{\omega_{i-n+1,i+1}\,\omega_{i-n+2,i+2}\,\omega_{i-n+3,i+3}}\Big).$$

Proof. Apply $\omega(x_{i-r}, \cdot)$ to (6.8) to obtain

$$0 = a_i^n \omega_{i-r,i-n} + a_i^{n-1}\omega_{i-r,i-n+1} + a_i^{n-2}\omega_{i-r,i-n+2} + \dots + a_i^{-n}\omega_{i-r,i+n}. \tag{6.10}$$

Consider the case $r = 0$. By the Lagrangian condition, here only the leftmost and rightmost terms on the right hand side are non-zero. We obtain

$$a_i^{-n} = 1/\omega_{i,i+n} = a_{i+n}^n. \tag{6.11}$$

Thus A is non-degenerate and satisfies the symmetry condition $a_i^{-\ell} = a_{i+\ell}^{\ell}$ for $\ell = n$.

Now consider the cases $r = \pm p$ with $1 \leqslant p \leqslant n$. By the Lagrangian condition, for $r = -p$ only the leftmost $p+1$ terms on the right hand side are non-zero, while for $r = p$ only the rightmost $p+1$ terms are non-zero. We obtain

$$a_i^{n-p} = \frac{-1}{\omega_{i-n+p,i+p}}\Big(\frac{\omega_{i-n,i+p}}{\omega_{i-n,i}} + a_i^{n-1}\,\omega_{i-n+1,i+p} + \cdots + a_i^{n-p+1}\,\omega_{i-n+p-1,i+p}\Big), \tag{6.12}$$

$$a_i^{p-n} = \frac{-1}{\omega_{i-p,i+n-p}}\Big(\frac{\omega_{i-p,i+n}}{\omega_{i,i+n}} + a_i^{1-n}\,\omega_{i-p,i+n-1} + \cdots + a_i^{p-1-n}\,\omega_{i-p,i+n-p+1}\Big). \tag{6.13}$$

A straightforward induction argument from (6.12) gives (6.9): the first term of (6.12) is the $m = 0$ term of (6.9), and the term $-a_i^{n-q}\omega_{i-n+q,i+p}/\omega_{i-n+p,i+p}$ of (6.12) gives those terms of (6.9) with $p_m = q$. A parallel argument from (6.13) yields a closed formula for a_i^{p-n}:

$$a_i^{p-n} = \sum_{m=0}^{p-1} \sum_{0<p_1<\cdots<p_m<p} -(-1)^m\, \frac{\omega_{i-p,i+n-p_m}\,\omega_{i-p_m,i+n-p_{m-1}}\cdots\omega_{i-p_2,i+n-p_1}\,\omega_{i-p_1,i+n}}{\omega_{i-p,i+n-p}\,\omega_{i-p_m,i+n-p_m}\cdots\omega_{i-p_1,i+n-p_1}\,\omega_{i,i+n}}. \tag{6.14}$$

To finish proving that A is symmetric, we must prove $a_i^{-\ell} = a_{i+\ell}^{\ell}$ for $0 \leqslant \ell < n$. Note that

$$(p_1,\, p_2,\, \ldots,\, p_m) \mapsto (p - p_m,\, p - p_{m-1},\, \ldots,\, p - p_1)$$

is an involution of the index set of the inner summation in (6.9). Use this to verify that replacing i by $i + n - p$ in (6.9) gives (6.14). This completes the proof of the lemma: the fact that A is N-periodic is now immediate from $\omega_{i+N,j+N} = \omega_{ij}$. □

Lemma 6.10. (i) *A lies in $\mathcal{E}_{n,N}(\mathbb{K})$.*

(ii) *The solutions $V^i(A)$ defined in Corollary 6.3 are given by $V^i_j(A) = \omega_{ij}$.*

Proof. We begin with (ii). Abbreviate $V^i(A)$ by V^i. By (6.4), (6.7), and (6.11),

$$\begin{aligned}\big(V^i_{i-n},\, V^i_{i-n+1},\, \ldots,\, V^i_{i+n-1},\, V^i_{i+n}\big) &:= \Big(\omega_{i,i-n},\, 0,\, \ldots,\, 0,\, \omega_{i,i+n}\Big)\\ &:= \Big(\omega_{i,i-n},\, \omega_{i,i-n+1}\ldots,\, \omega_{i,i+n-1},\, \omega_{i,i+n}\Big).\end{aligned}$$

Consider (6.10): since $i - r$ is arbitrary, we see that $(\omega_{ij})_j$ lies in $\mathcal{K}(A)$. By the above equation, it has the same initial conditions as V^i, and so (ii) follows from Lemma 6.2.

In light of Lemma 6.9(i), to prove (i) it suffices to prove that (6.8) has monodromy $-\mathrm{Id}$. Because the V^i span $\mathcal{K}(A)$, this reduces to $V^i_{j+N} = -V^i_j$ for all i, j. By (ii), this follows from $x_{j+N} = -x_j$. □

At this point we have proven Theorem 3(i): by Lemmas 6.4 and 6.10, the element A of $\mathcal{E}_{n,N}(\mathbb{K})$ constructed in Lemma 6.9 has solutions $V^i(A)$ satisfying $\mathcal{W}_A(V^i, V^j) = V^i_j = \omega(x_i, x_j)$. Therefore by Lemma 2.1(iii) there is an element of $\mathrm{Sp}(2n, \mathbb{K})$ carrying the x_i to the v_i of Proposition 6.8, and so $P(A)$ is the class of the Lagrangian configuration originally given.

We now turn to Theorem 3(ii). The fact that $P(A) = P(\tilde{A})$ if $A = \lambda\tilde{A}\lambda$ is immediate from Lemma 6.5: λ is a symplectic map carrying $V^i(A)$ to a multiple of $V^i(\tilde{A})$. Conversely, suppose that $P(A) = P(\tilde{A})$. Reviewing Proposition 6.8, we find that this means there is a symplectic map Λ from $\big(\mathcal{K}(A), \mathcal{W}_A\big)$ to $\big(\mathcal{K}(\tilde{A}), \mathcal{W}_{\tilde{A}}\big)$ carrying $V^i(A)$ to a non-zero multiple of $V^i(\tilde{A})$, for all i. Define λ by setting λ_i^{-1} to be this multiple. Because the sequences $V^i(A)$ and $V^i(\tilde{A})$ are both N-antiperiodic, λ is N-periodic.

Let $\hat{A} := \lambda\tilde{A}\lambda$. By Lemma 6.5, λ is a symplectic map from $\big(\mathcal{K}(\hat{A}), \mathcal{W}_{\hat{A}}\big)$ to $\big(\mathcal{K}(\tilde{A}), \mathcal{W}_{\tilde{A}}\big)$ carrying $V^i(\hat{A})$ to $\lambda_i^{-1}V^i(\tilde{A})$. Therefore $\lambda^{-1}\circ\Lambda$ is a symplectic map from $\big(\mathcal{K}(A), \mathcal{W}_A\big)$ to $\big(\mathcal{K}(\hat{A}), \mathcal{W}_{\hat{A}}\big)$ carrying $V^i(A)$ to $V^i(\hat{A})$. The following lemma shows that $A = \hat{A}$, completing the proof of Theorem 3. □

Lemma 6.11. *Let A and $\hat{A}$ be elements of $\mathcal{E}_{n,N}(\mathbb{K})$. Write a^ℓ and $\hat{a}^\ell$ for the coefficients of A and $\hat{A}$, V^i and $\hat{V}^i$ for $V^i(A)$ and $V^i(\hat{A})$, and ν_{ij} and $\hat{\nu}_{ij}$ for the inner products $\mathcal{W}_A(V^i, V^j)$ and $\mathcal{W}_{\hat{A}}(\hat{V}^i, \hat{V}^j)$, respectively. The following statements are equivalent:*

(i) $A = \hat{A}$, *i.e.*, $a^\ell = \hat{a}^\ell$ *for* $0 \leqslant \ell \leqslant n$.
(ii) *There exists a symplectic map* $\sigma : \big(\mathcal{K}(A), \mathcal{W}_A\big) \to \big(\mathcal{K}(\hat{A}), \mathcal{W}_{\hat{A}}\big)$ *such that* $\sigma(V^i) = \hat{V}^i$ *for all* i.
(iii) $\nu_{ij} = \hat{\nu}_{ij}$ *for all* i *and* j.
(iv) $V^i = \hat{V}^i$ *for all* i.

Proof. It is immediate that (i) implies (ii), (iii), and (iv), and (ii) implies (iii). By Lemma 2.1(iii), (iii) implies (ii), and (iii) and (iv) are equivalent by Lemma 6.4(ii). In order to prove that (iii) and (iv) imply (i), we will prove that for $0 \leqslant p \leqslant n$ the a_i^{n-p} are given by (6.7) and (6.9) with ν_{ij} replacing ω_{ij}.

For $p = 0$, recall that by Lemma 6.4(iii), $a_i^n = 1/\nu_{i-n,i}$. For $p > 0$ it suffices to prove that (6.12) holds with ν replacing ω, and for this it suffices to prove that (6.10) holds with ν replacing ω. This equation in turn results from applying $\mathcal{W}_A(V^{i-r}, \cdot)$ to (6.8) with $V^\bullet$ replacing $x_\bullet$, so finally we come down to proving the vector equation

$$a_i^n V^{i-n} + a_i^{n-1} V^{i-n+1} + \cdots + a_{i+n-1}^{n-1} V^{i+n-1} + a_{i+n}^n V^{i+n} = 0$$

for all i. Because V^j is itself in $\mathcal{K}(A)$, we know that the scalar equation

$$a_i^n V_{i-n}^j + a_i^{n-1} V_{i-n+1}^j + \cdots + a_{i+n-1}^{n-1} V_{i+n-1}^j + a_{i+n}^n V_{i+n}^j = 0$$

holds for all i. To complete the proof, recall from Lemma 6.4(ii) that $V_k^j = -V_j^k$. □

7. The case $N = 2n + 3$

In this section, let $(X_1, \ldots, X_N)$ be a generic complex (n, N)-Lagrangian configuration with representatives $(x_1, \ldots, x_N)$. Extend them to an N-antiperiodic sequence and write ω_{ij} for $\omega(x_i, x_j)$. We conclude the article with a discussion of the case $N = 2n + 3$: we generalize the five Gauss relations (1.3) on $\mathcal{L}_{1,5}(\mathbb{C})$ to $2n + 3$ relations on the $2n + 3$ basic symplectic cross-ratios of $\mathcal{L}_{n,2n+3}(\mathbb{C})$.

These relations are obtained by means of the symmetric linear difference operators associated to Lagrangian configurations in Theorem 3. The computations actually consist in solving the system of equations given by the condition that the operators have monodromy $-\mathrm{Id}$. We remark that Theorem 2(i) may be obtained via the same method.

There are two sequences of non-trivial cross-ratios on $\mathcal{L}_{n,2n+3}$: the c_i of (1.4), and the $\gamma_{i,i+n+1}$ of (2.7), which we will abbreviate by γ_i. Both are $(2n + 3)$-periodic:

$$c_i := c_{i,i+1,i+n+1,i+n+2} = \frac{\omega_{i,i+n+1}\,\omega_{i+1,i+n+2}}{\omega_{i,i+n+2}\,\omega_{i+1,i+n+1}}, \quad \gamma_i := c_{i-2,i,i+n,i+n+1} = \frac{\omega_{i-2,i+n}\,\omega_{i,i+n+1}}{\omega_{i-2,i+n+1}\,\omega_{i,i+n}}.$$

We have $\gamma_i = c_{i-1}c_{i+n}$, so all cross-ratios on $\mathcal{L}_{n,2n+3}$ may be written in terms of the c_i. We will prove that over $\mathbb{C}$ these $2n+3$ cross-ratios determine the equivalence class of their Lagrangian configuration (we expect that over $\mathbb{R}$ they determine it up to opposites). By Proposition 2.7, $\mathcal{L}_{n,2n+3}(\mathbb{C})$ is $2n$-dimensional, so the space of relations on the c_i must have Krull dimension 3. We conjecture that the $2n + 3$ relations we present generate the full space of relations.

If $\omega_{i,i+n} = 1$ for all i, as in Section 3.3, then the representatives x_i are said to be *normalized.* We begin with a general lemma permitting us to restrict our consideration to such representatives.

Lemma 7.1. *For any complex* (n, N)*-Lagrangian configuration* $(X_1, \ldots, X_N)$ *with* $N/\operatorname{GCD}(n, N)$ *odd, there exist exactly* $2^{\operatorname{GCD}(n,N)}$ *normalized choices of representatives.*

Proof. Following Section 4.2, let $(\tilde{x}_1, \ldots, \tilde{x}_N)$ be any representatives of $(X_1, \ldots, X_N)$, with corresponding inner products $\tilde{\omega}_{ij}$. Fix an N-periodic sequence χ_i such that $\chi_i^2 = \tilde{\omega}_{i,i+n}$. Mimicking (4.9), set

$$\lambda_i := \frac{\chi_i \chi_{i+2n} \chi_{i+4n} \cdots \chi_{i+(q-1)n}}{\chi_{i+n} \chi_{i+3n} \cdots \chi_{i+(q-2)n}} = \prod_{r=0}^{q-1} \chi_{i+rn}^{(-1)^r},$$

where q denotes $N/\operatorname{GCD}(n, N)$. Check that $\lambda_i \lambda_{i+n} = \tilde{\omega}_{i,i+n}$, so $x_i := \tilde{x}_i/\lambda_i$ satisfies $\omega_{i,i+n} = 1$.

The fact that $\lambda_i^2 = \prod_{r=0}^{q-1} \tilde{\omega}_{i+rn,i+(r+1)n}^{(-1)^r}$ implies that each q-tuple $(\lambda_i, \lambda_{i+n}, \ldots, \lambda_{i+(q-1)n})$ is determined up to a single choice of overall sign. The lemma follows. □

Henceforth let $(X_1, \ldots, X_{2n+3})$ be a generic complex $(n, 2n+3)$-Lagrangian configuration, and fix a normalized choice of representatives x_i. Recall from Section 3.3 the $(n+1)$-periodic sequence $a_i := \omega_{i,i+n+1}$ of symplectic diameters of normalized $(n, 2n+2)$-configurations. The analog here is the $(2n+3)$-periodic sequence of *symplectic main diagonals,* defined by the same formula as the a_i:

$$d_i := \omega_{i,i+n+1}.$$

Note that this notation is consistent with (1.3), and

$$c_i = d_i d_{i+1}/d_{i-n-1}, \qquad \gamma_i = c_{i-1}c_{i+n} = d_i d_{i+n}.$$

Corollary 7.2. (i) *If $3 \nmid n$, then $(X_1, \ldots, X_{2n+3})$ has two normalized choices of representatives: $(x_i)_i$ and $(-x_i)_i$. Both have the same d_i: the configuration determines its main diagonals.*

(ii) *If $3 \mid n$, then $(X_1, \ldots, X_{2n+3})$ has eight normalized choices of representatives: $\epsilon_i x_i$, where $\epsilon_i = \pm 1$ and depends only on i modulo 3. The corresponding main diagonals are $\epsilon_i \epsilon_{i+1} d_i$.*

Proposition 7.3. *Generic equivalence classes in $\mathcal{L}_{n,2n+3}(\mathbb{C})$ with the same cross-ratios c_i are equal.*

Proof. Let x_i and $\tilde{x}_i$ be normalized representatives of two Lagrangian configurations having the same cross-ratios: $\tilde{c}_i = c_i$. It suffices to show that the $\tilde{x}_i$ may chosen so that the two sets of main diagonals are the same, i.e., $\tilde{d}_i = d_i$, as then $\tilde{\omega}_{ij} = \omega_{ij}$ for all i and j. Observe that

$$c_{i-1}\gamma_{i+1} = d_{i-1}d_i d_{i+1}, \text{ and for any } r, \ \frac{\gamma_i \gamma_{i+2n} \cdots \gamma_{i+2rn}}{\gamma_{i+n}\gamma_{i+3n}\cdots\gamma_{i+(2r-1)n}} = d_i d_{i+(2r+1)n}.$$

Write n in the form $3m+s$, where $s \in \{-1, 0, 1\}$. Take $r = m$ above and apply $(2n+3)$-periodicity to see that $d_i d_{i+s}$ is determined by the cross-ratios. If $s = \pm 1$, dividing $c_{i-1}\gamma_{i+1}$ by this gives d_{i-s} as a function of the cross-ratios. Thus for $3 \nmid m$, the cross-ratios determine the main diagonals.

If $3 \mid m$, i.e., $s = 0$, taking $r = m$ gives d_i^2, and so the cross-ratios determine the main diagonals up to sign: $\tilde{d}_i = \delta_i d_i$ for some $\delta_i = \pm 1$. Using $\gamma_i = d_i d_{i+n}$, $\tilde{\gamma}_i = \gamma_i$, and $\mathrm{GCD}(n, 2n+3) = 3$, we find that δ_i depends only on i modulo 3. Applying $c_{i-1}\gamma_{i+1} = d_{i-1}d_i d_{i+1}$, we obtain $\delta_{i-1}\delta_i\delta_{i+1} = 1$. Therefore by Corollary 7.2 it is possible to modify the $\tilde{x}_i$ so that $\tilde{d}_i = d_i$. □

Recall now the difference operator A constructed from the representatives x_i in (6.8). In the normalized case the formula (6.9) for its coefficients simplifies, as the denominators are all 1:

$$a_i^{n-p} = \sum_{m=0}^{p-1} \sum_{0<p_1<\cdots<p_m<p} -(-1)^m \, \omega_{i-n,i+p_1}\, \omega_{i-n+p_1,i+p_2} \cdots \omega_{i-n+p_{m-1},i+p_m}\, \omega_{i-n+p_m,i+p}.$$

Observe that the factor $\omega_{i-n+p_{\ell-1},i+p_\ell}$ is equal to $d_{i-n+p_\ell-1}$ if $p_\ell - p_{\ell-1}$ is 1, is equal to d_{i+p_ℓ} if $p_\ell - p_{\ell-1}$ is 2, is equal to 1 if $p_\ell - p_{\ell-1}$ is 3, and is equal to 0 otherwise. The initial cases are

$$a_i^n = 1, \qquad a_i^{n-1} = -d_{i-n}, \qquad a_i^{n-2} = d_{i-n}d_{i-n+1} - d_{i+2},$$

$$a_i^{n-3} = -d_{i-n}d_{i-n+1}d_{i-n+2} + d_{i-n}d_{i+3} + d_{i-n+2}d_{i+2} - 1.$$

Proposition 7.4. *The $2n+3$ symplectic main diagonals d_i of the normalized representatives x_i of $(X_1, \ldots, X_{2n+3})$ satisfy the following $2n+3$ polynomial relations on the d_i (for $n = 1$, take a_i^2 to be 0):*

$$0 = a_i^2 + d_{i+n+1}a_i^1 + d_i a_i^0 + a_{i+1}^1. \tag{7.1}$$

Proof. Simply take $r = -(n+1)$ in (6.10). □

Legendrian pentagons in $\mathbb{CP}^1$. This is $\mathcal{L}_{1,5}$, the Gaussian case discussed in Section 1.2. Here (7.1) reduces to (1.3). Using $d_i = c_{i-2}c_{i+1}/c_{i+2}$, the relations may be stated in terms of the c_i:

$$\frac{1}{c_i} + \frac{1}{c_{i-1}c_{i+1}} = 1.$$

Legendrian heptagons in $\mathbb{CP}^3$. For $\mathcal{L}_{2,7}$, (7.1) reads

$$d_{i-1}d_id_{i+1} - d_{i-3}d_{i-1} - d_{i+1}d_{i+3} - d_i + 1 = 0.$$

Using $d_i = c_{i-1}c_ic_{i+3}/c_{i-2}c_{i+1}$, this becomes

$$\frac{1}{c_{i-3}} + \frac{1}{c_{i+3}} + \frac{1}{c_{i-2}c_{i+2}} - \frac{1}{c_{i-3}c_ic_{i+3}} = 1.$$

Legendrian nonagons in $\mathbb{CP}^5$. For $\mathcal{L}_{3,9}$, (7.1) yields

$$d_{i-4}d_{i-3}d_{i+3}d_{i+4} - d_{i-4}d_{i-3}d_{i-1} - d_{i-3}d_id_{i+3} - d_{i+1}d_{i+3}d_{i+4}$$
$$-d_{i-4}d_{i+4} + d_{i-1}d_{i+1} + d_{i-3} + d_i + d_{i+3} = 0.$$

Using $c_i = d_id_{i+1}/d_{i-4}$, $c_{i-1}c_{i+3} = \gamma_i = d_id_{i+3}$, and $c_{i-1}\gamma_{i+1} = d_{i-1}d_id_{i+1}$, this may be rewritten as

$$\frac{1}{c_{i-1}} + \frac{1}{c_i} + \frac{1}{c_{i+1}} + \frac{1}{c_{i-1}c_{i+1}} + \frac{1}{c_{i-2}c_{i+2}} - \frac{1}{c_{i-4}c_ic_{i+1}} - \frac{1}{c_{i-2}c_ic_{i+2}} - \frac{1}{c_{i-1}c_ic_{i+4}} = 1.$$

We close with a few general remarks. Note that for $3 \nmid n$, (7.1) can always be written as a rational relation on the c_i, because the d_i are rational functions of the c_i. In light of the situation for $\mathcal{L}_{3,9}$, we expect that this is in fact true for all n. Also, although we have worked only over $\mathbb{C}$ in this section, it should be easy to show that the relations we have given on the c_i hold also over $\mathbb{R}$.

Finally, let us reiterate our conjecture regarding (7.1). Because $\mathcal{L}_{n,2n+3}$ is $2n$-dimensional and the cross-ratios $(c_1, \ldots, c_{2n+3})$ form a coordinate ring on it, the space of relations on the c_i must be of Krull dimension 3. We conjecture that the $2n+3$ relations (7.1) generate the full relation space.

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