1-2-3 Conjecture in Digraphs: More Results and Directions

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Abstract

Horňak, Przybyło and Woźniak recently proved that, a small class of obvious exceptions apart, every digraph can be 4-arc-weighted so that, for every arc $\overrightarrow{uv}$, the sum of weights incoming to u is different from the sum of weights outgoing from v. They conjectured a stronger result, namely that the same statement with 3 instead of 4 should also be true. We verify this conjecture in this work.

This work takes place in a recent "quest" towards a directed version of the 1-2-3 Conjecture, the variant above being one of the last introduced ones. We take the occasion of this work to establish a summary of all results known in this field, covering known upper bounds, complexity aspects, and choosability. On the way we prove additional results which were missing in the whole picture. We also mention the aspects that remain open.

Keywords: 1-2-3 Conjecture; Directed variants; Bounds; Complexity; Choosability.

1. Introduction

Let G be an undirected graph, and ω be an edge-weighting of G. From ω, we can associate to every vertex v the *sum* $\sigma(v)$ of the weights on its incident edges. We call ω *neighbour-sum-distinguishing* if $\sigma(u) \neq \sigma(v)$ for every edge uv of G. It turns out that G always admits such neighbour-sum-distinguishing edge-weightings, unless it includes K_2 as a connected component. Thus, we call G *nice* whenever it has no such connected component.

Whenever G is nice, it is legitimate to wonder what is the smallest k such that G admits neighbour-sum-distinguishing k-edge-weightings; the smallest such k for G is denoted $\chi_\Sigma(G)$. Karoński, Łuczak and Thomason conjectured in 2004 that for every nice graph G, the value of $\chi_\Sigma(G)$ should never exceed 3. This is known as the **1-2-3 Conjecture** nowadays [11].

1-2-3 Conjecture (Karoński, Łuczak, Thomason [11])**.** *For every nice graph G, we have* $\chi_\Sigma(G) \leq 3$.

So far, the best known result towards the 1-2-3 Conjecture is due to Kalkowski, Karoński and Pfender [10], who proved that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G. It has to be known also that there exist graphs G with $\chi_\Sigma(G) = 3$; however, it is NP-complete to decide whether $\chi_\Sigma(G) \leq 2$ is true for a given G, as first proved by Dudek and Wajc [9]. For more results and details, refer to [14] for a survey by Seamone.

In recent years, there have been quite a few efforts for bringing the 1-2-3 Conjecture to digraphs. Note that, by an arc-weighting of a digraph D, each vertex v gets associated

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two sums $\sigma^-(v), \sigma^+(v)$, being the sum of weights on the arcs incoming and outgoing, respectively, to and from v. Since there are two sum parameters to play with, there are several natural ways for defining a directed 1-2-3 Conjecture; and one can expect them to show different behaviours that might be or be not reminiscent of that behind the original 1-2-3 Conjecture. As will be described in Sections 4 and 5, things actually turned out to be rather deceiving from that perspective.

We are going to discuss most known directed variations of the 1-2-3 Conjecture. Most of the studied variants ask, for any arc $\overrightarrow{uv}$ of a digraph (arc-weighted by ω), that one of the parameters $\sigma^-(u), \sigma^+(u)$ is different from one of $\sigma^-(v), \sigma^+(v)$. To get a consistent terminology, we deal with these variants as follows. To each symbol $\alpha \in \{-,+\}$, we associate a parameter: $-$ is associated to σ^- while $+$ is associated to σ^+. Now, for two symbols $\alpha, \beta \in \{-,+\}$, we say that an arc-weighting of a digraph D is (α, β)*-distinguishing* if, for every arc $\overrightarrow{uv}$ of D, the parameter of u associated to α is different from the parameter of v associated to β. When writing $\chi_{\alpha,\beta}(D)$, we refer to the least k such that D admits (α, β)-distinguishing k-arc-weightings, if any. When referring to the (α, β) *variant* of the 1-2-3 Conjecture, we mean the variant involving (α, β)-distinguishing arc-weightings.

Note that this terminology already allows to encapsulate four natural directed variants of the 1-2-3 Conjecture (with the $(-,-)$ variant being actually identical to the $(+,+)$ variant, up to reversing arc directions [4]). Among these four variants, the $(-,+)$ variant is the most recent one, as it was only introduced in 2018 by Horňák, Przybyło and Woźniak [7]. They first noticed that $\chi_{-,+}(D)$ is not defined for digraphs having an arc $\overrightarrow{uv}$ such that u is a source (vertex without arcs incoming) and v is a sink (vertex without arcs outgoing). Such an arc is called an *ss-arc*. For digraphs with no ss-arcs, it turns out that $\chi_{-,+}$ is not bounded by an absolute constant, but this is only because of the presence of so-called *lonely arcs*, which are arcs $\overrightarrow{uv}$ such that $d^+(u) = d^-(v) = 1$ (i.e., $\overrightarrow{uv}$ is the only arc outgoing from u, and the only arc incoming to v). Regarding the $(-,+)$ variant, a digraph is called *nice* whenever it has no ss-arcs nor lonely arcs; this terms makes sense because $\chi_{-,+}$ becomes bounded by a constant for digraphs without such bad arcs.

Theorem 1.1 (Horňák, Przybyło, Woźniak [7])**.** *For every nice digraph D, we have $\chi_{-,+}(D) \leq 4$.*

Horňák, Przybyło and Woźniak did not come up with an example of digraph showing that Theorem 1.1 is tight; they thus left the following conjecture open, which would be reminiscent of the 1-2-3 Conjecture:

Conjecture 1.2 (Horňák, Przybyło, Woźniak [7])**.** *For every nice digraph D, we have $\chi_{-,+}(D) \leq 3$.*

As a support to Conjecture 1.2, its authors proved it for several families of digraphs, including tournaments and symmetric digraphs.

Our original intention in this paper is to pursue the work of Horňák, Przybyło and Woźniak, by providing two more results on the $(-,+)$ variant of the 1-2-3 Conjecture. Our first main result, given in Section 2, is a proof of Conjecture 1.2. Our second result, in Section 3, is a series of complexity results on the $(-,+)$ variant; in particular, we prove that there is no "good" characterization of nice digraphs D with $\chi_{-,+}(D) \leq 2$, unless P=NP.

We also take the occasion of this work to summarize, in Table 1 of Section 4, all results known so far on directed variants of the 1-2-3 Conjecture. The aspects we survey include known bounds, complexity aspects, and choosability. In the same section, we provide a few additional results for the purpose of filling in some missing results. Some aspects remain totally open though; such open questions are raised in concluding Section 5.

2. Proof of Conjecture 1.2

To get their upper bound of 4 on $\chi_{-,+}$, Horňák, Przybyło and Woźniak, in [7], made use of the following relationship between graph weighting and digraph weighting, first used in this area by Barme, Bensmail, Przybyło and Woźniak in [1] to deal with the $(+,-)$ variant of the 1-2-3 Conjecture. To a digraph D, we can *associate* the following bipartite graph $B(D)$:

- To each vertex v of D, we associate two vertices v^+ and v^- in $B(D)$.
- For each arc $\overrightarrow{uv}$ of D, we add the edge u^+v^- to $B(D)$.

Barme *et al.* noticed that, for the $(+,-)$ variant, finding a $(+,-)$-distinguishing k-arc-weighting of D is equivalent to finding a neighbour-sum-distinguishing k-edge-weighting of $B(D)$. This is because, given an arc-weighting ω of D and its corresponding edge-weighting ω' of $B(D)$ (obtained in the obvious way by translating the weights directly from an arc to the corresponding edge), the value of any $\sigma^+_\omega(v)$ is exactly $\sigma_{\omega'}(v^+)$, while $\sigma^-_\omega(v)$ is exactly $\sigma_{\omega'}(v^-)$. Furthermore, for every arc $\overrightarrow{uv}$ of D, in $B(D)$ we have the edge u^+v^-; hence, when ω' is neighbour-sum-distinguishing, we have $\sigma_{\omega'}(u^+) \neq \sigma_{\omega'}(v^-)$ which depicts exactly that we require $\sigma^+_\omega(u) \neq \sigma^-_\omega(v)$ in the $(+,-)$ variant.

The proof of Theorem 1.1 by Horňák *et al.* in [7] again relies on this notion of associated bipartite graph. For the $(-,+)$ variant, however, we note that this transformation is not sufficient any more. Indeed, consider a digraph D and its associated bipartite graph $B(D)$. For every arc $\overrightarrow{uv}$ of D, we have a corresponding edge u^+v^- in $B(D)$. In a neighbour-sum-distinguishing edge-weighting of $B(D)$, we do require $\sigma(u^+)$ to be different from $\sigma(v^-)$, which is not representative of what we require in a $(-,+)$-distinguishing arc-weighting of D, namely that $\sigma^-(u)$ gets different from $\sigma^+(v)$. In $B(D)$, it is possible that $\sigma(u^-)$ gets equal to $\sigma(v^+)$, as u^- and v^+ might not be adjacent. The crucial point is that edge-weighting $B(D)$ gives an arc-weighting of D that is equivalent in terms of obtained sums; however, it is not equivalent in terms of sum constraints, because, from the point of view of the constraints, the structure of $B(D)$ is not representative of that of D.

To overcome this point, Horňák *et al.* build neighbour-sum-distinguishing edge-weightings of $B(D)$ that, when derived to D, yield $(-,+)$-distinguishing arc-weightings no matter what the sum constraints actually are. To that aim, they weight $B(D)$ so that the $\sigma(v^+)$'s are different from all $\sigma(v^-)$'s; this way, back in D, this yields an arc-weighting where the $\sigma^+(v)$'s are different from the $\sigma^-(v)$'s. This is done by making sure the incident sums range in two disjoint sets.

We prove our main result in this section using the same idea, but with a more refined analysis. More precisely:

Theorem 2.1. *Every nice connected bipartite graph G with bipartition $U \cup V$ has a neighbour-sum-distinguishing 3-edge-weighting ω where:*

- *for every $u \in U$, we have $\sigma(u) \in \mathcal{U}$ and*
- *for every $v \in V$, we have $\sigma(v) \in \mathcal{V}$,*

for

- $\mathcal{U} := \{0,3\} \cup \{3k+1 : k \geq 1\}$ *and*
- $\mathcal{V} := \{0,1,2\} \cup \{3k-1, 3k : k \geq 2\}$.

To make it clearer, we have

$$\mathcal{U} := \{0, 3, 4, 7, 10, 13, 16, ...\}$$

and

$$\mathcal{V} := \{0, 1, 2, 5, 6, 8, 9, 11, 12, 14, 15, ...\}.$$

The value 0 in both $\mathcal{U}$ and $\mathcal{V}$ is to catch vertices with degree 0, which can occur in associated bipartite graphs. We note that the sets $\mathcal{U}$ and $\mathcal{V}$ are quite restrictive. Notably, for every vertex $u \in U$ with $d(u) = 1$, its unique incident edge must be weighted 3. However, the sums in $\mathcal{U}$ and $\mathcal{V}$ that must be reached for vertices become more "regular" as soon as the degree is large enough. For instance, every vertex with degree at least 4 cannot have sum in $\{1, 2, 3\}$ which are the peculiar values of the sets $\mathcal{U}$ and $\mathcal{V}$. Vertices with small degree will indeed be the most troublesome vertices in our upcoming proof of Theorem 2.1.

Before proceeding to proving Theorem 2.1, let us first prove it holds in particular cases.

Observation 2.2. *Theorem 2.1 holds when G is a nice star.*

Proof. Since G is nice, it has at least two leaves. If the leaves lie in U, then assigning weight 3 to all edges is correct, as, that way, we get $\sigma(u) = 3 \in \mathcal{U}$ for every $u \in U$, while we get $\sigma(v) = 3k$ for the unique $v \in V$, where $\sigma(v) \in \mathcal{V}$ since $k \geq 2$. If the leaves lie in V, then we consider two cases. Assume first there are $2k+1$ leaves, for $k \geq 1$. Then assigning weight 1 to $k+1$ edges and weight 2 to the remaining k edges is correct, as, that way, we get $\sigma(u) = k+1+2k = 3k+1$ for the unique $u \in U$, where $\sigma(u) \in \mathcal{U}$ since $k \geq 1$, while we get $\sigma(v) \in \{1, 2\} \subset \mathcal{V}$ for every $v \in V$. Assume now there are $2k$ leaves, for $k \geq 1$. Then assigning weight 1 to $k-1$ edges and weight 2 to the remaining $k+1$ edges is correct, as, here, we get $\sigma(u) = k-1+2(k+1) = 3k+1$ for the unique $u \in U$, where $\sigma(u) \in \mathcal{U}$ since $k \geq 1$, while we get $\sigma(v) \in \{1, 2\} \subset \mathcal{V}$ for every $v \in V$. □

Observation 2.3. *Theorem 2.1 holds when G is a nice path.*

Proof. Since G is nice, it is a path $x_1..., x_n$ with length at least 2. We prove the result in three steps, distinguishing whether the two end-vertices are in U or V.

- Assume $x_1, x_n \in U$. If G has length 2, then a desired weighting is obtained when assigning weight 3 to the two edges: this way, x_1 and x_3 both have sum $3 \in \mathcal{U}$, while x_2 has sum $6 \in \mathcal{V}$. If G has length 4, then a desired weighting is obtained when assigning weights $3, 2, 2, 3$ as traversing the edges from x_1 to x_5: this way, x_1 and x_5 both have sum $3 \in \mathcal{U}$, x_3 has sum $4 \in \mathcal{U}$, while x_2 and x_4 have sum $5 \in \mathcal{V}$.

  We now prove the general case by induction. Let $x_1, ..., x_{2k+1}$ be a path of length $2k$ ($k \geq 3$) where $x_1, x_{2k+1} \in U$. We here remove the last four edges $x_{2k-3}x_{2k-2}$, $x_{2k-2}x_{2k-1}$, $x_{2k-1}x_{2k}$ and $x_{2k}x_{2k+1}$ from G. By the induction hypothesis, the resulting smaller path G' admits a desired 3-edge-weighting, which we wish to extend to the four removed edges. Note that, in G', vertex x_{2k-3} has degree 1; for its sum to be in $\mathcal{U}$, its unique incident edge must thus be weighted 3. We extend the weighting to G by assigning weight 1 to $x_{2k-3}x_{2k-2}$ and $x_{2k-2}x_{2k-1}$, and weight 3 to $x_{2k-1}x_{2k}$ and $x_{2k}x_{2k+1}$. This way, the sum of x_{2k-3} and x_{2k-1} becomes $4 \in \mathcal{U}$, the sum of x_{2k+1} is $3 \in \mathcal{U}$, the sum of x_{2k-2} is $2 \in \mathcal{V}$, and the sum of x_{2k} is $6 \in \mathcal{V}$.

- Assume $x_1 \in U$ and $x_n \in V$. We here remove the edge $x_{n-1}x_n$ from G, resulting in a smaller path G' having its two end-vertices in U. In the previous case, we have seen that such a path can be correctly 3-edge-weighted. Since x_{n-1} has degree 1 in G', for its sum to belong to $\mathcal{U}$ its unique incident edge must be weighted 3. We extend this weighting to $x_{n-1}x_n$, thus to G, by assigning weight 1 to $x_{n-1}x_n$. This way, the sum of x_{n-1} becomes $4 \in \mathcal{U}$, while the sum of x_n is $1 \in \mathcal{V}$.

- Assume $x_1, x_n \in V$. Just as in the previous case, we remove the edge $x_{n-1}x_n$ from G, resulting in a smaller path G' with end-vertices $x_1 \in V$ and $x_{n-1} \in U$. By the previous item, it can be correctly weighted, and necessarily x_{n-1} has sum 3 by any 3-edge-weighting. Then we can again extend this weighting to G by assigning weight 1 to $x_{n-1}x_n$, for the same reasons. $\square$

Observation 2.4. *Theorem 2.1 holds when G is an even cycle.*

Proof. Let G be an even-length cycle $x_1, ..., x_{2k}x_1$. Assume that $x_i \in U$ for every odd $i \geq 1$, and $x_i \in V$ for every even $i \geq 2$. Let G' be the graph obtained from G by removing x_{2k}. This G' is a path with both end-vertices in U. According to Observation 2.3, G' admits a desired 3-edge-weighting. By that weighting, since x_1 and x_{2k-1} have degree 1, their unique incident edge must be weighted 3 so that their sum gets in $\mathcal{U}$. We extend this edge-weighting to G by assigning weight 1 to both x_1x_{2k} and $x_{2k-1}x_{2k}$. This way, the sum of x_1 and x_{2k-1} becomes $4 \in \mathcal{U}$, while x_{2k} gets sum $2 \in \mathcal{V}$. $\square$

We now proceed with the proof of our main result.

Proof of Theorem 2.1. We may assume that U has vertices of degree at least 2, as otherwise G would be a star, in which case the result holds by Observation 2.2. Let thus $u^* \in U$ be a vertex with degree at least 2. From u^*, we deduce a partition $L_0 \cup L_1 \cup ... \cup L_d$ of G into *layers* where each L_i contains the vertices at distance i from u^*. In particular, $L_0 = \{u^*\}$, and U contains the vertices from the even layers (i.e., with even index) while V contains the vertices from the odd layers (i.e., with odd index). Since G is bipartite, no edge joining a vertex in L_{2k} and a vertex in $L_{2k'}$ exists (and similarly for L_{2k+1} and $L_{2k'+1}$). More precisely, by the way the L_i's were obtained, every edge joins vertices in two consecutive layers. From the point of view of a vertex w of G in layer L_i, an incident edge ww' is called *upward* (resp. *downward*) if $w' \in L_{i-1}$ (resp. $w' \in L_{i+1}$). Note that every vertex but u^* is incident to an upward edge. Similarly, an *upward path* (resp. *downward path*) from w is a path starting from w that repeatedly traverses upward edges (resp. downward edges). We say a vertex w' is a *descendant* of a vertex w if there is a downward path from w to w'. A descendant w' of w is a *son* of w if ww' is a (downward) edge.

We try to deduce a desired 3-edge-weighting of G through the following iterative *layer-by-layer process*. We consider the vertices of L_d in arbitrary order first, then those of L_{d-1}, and so on until considering the vertices of L_1 (layer L_0 does not have to be considered). Each time a vertex w is considered that way, we want to weight its incident upward edges so that the sum of w lies in the corresponding $\mathcal{U}$ or $\mathcal{V}$. More precisely, when considering w, the process has already been applied to its sons, meaning that the downward edges incident to w have already been weighted. Vertex w thus has a *current sum*, which we need to alter in a satisfactory way by weighting its at least one incident upward edges.

We claim that this can always be done correctly; we distinguish two main cases, depending on whether $w \in U$ (i.e., $w \in L_{2k}$) or $w \in V$ (i.e., $w \in L_{2k+1}$):

- Assume $w \in U$. First, suppose w is incident to only one upward edge. If the current sum of w is 0, 1 or 2, then we assign weight 3, 2 or 1, respectively, to the upward edge so that $\sigma(w) = 3 \in \mathcal{U}$. If the current sum of w is $3k$, $3k+1$ or $3k+2$ for some $k \geq 1$, then we assign weight 1, 3 or 2, respectively, to the upward edge so that

$\sigma(w) = 3k' + 1$ with $k' \geq 1$, and thus $\sigma(w) \in \mathcal{U}$. Second, suppose w is incident to at least two upward edges. Here, we first assign weight 3 to all incident upward edges but one. Updating the current sum of w , we are left with one incident upward edge to weight, which can be done similarly as in the previous case. Note, in particular, that the current sum of w is at least 3 due to some incident upward edges being weighted 3.

- Assume $w \in V$. First, suppose w is incident to only one upward edge. If the current sum of w is 0, 1 or 2, then we assign weight 1, 1 or 3, respectively, to the upward edge so that $\sigma(w) \in \{1, 2, 5\} \subset \mathcal{V}$. If the current sum of w is $3k$, $3k + 1$ or $3k + 2$ for some $k \geq 1$, then we assign weight 3, 2 or 1, respectively, to the upward edge so that $\sigma(w) = 3k'$ for some $k' \geq 2$, and get $\sigma(w) \in \mathcal{V}$. Second, suppose w is incident to at least two upward edges. Here, as in the previous case, we first assign weight 3 to all upward edges incident to w but one. That way, the current sum of w is at least 3, and there is one remaining upward edge to be weighted. Then we can proceed just as in the previous case.

We process all vertices of G that way, layer by layer, starting from the vertices in the deepest layers and finishing with those in L_1. Note that it results in a weighting of all edges of G. Furthermore, every vertex in $V(G) \setminus \{u^*\}$ has its sum belonging to the corresponding of $\mathcal{U}$ and $\mathcal{V}$. If the sum of u^* inherited from the weighting of its incident downward edges belongs to $\mathcal{U}$, then we are done. So we can assume this is not the case.

Let us consider the moment where a vertex $w \in L_1$ is considered in the process. We note that if w has degree 1, then its unique (upward) incident edge can be weighted either 1 or 2 (so that the sum gets in $\mathcal{V}$). Similarly, if the current sum of w is at least 3, then there are actually two ways to weight its incident upward edge going to u^*. Namely:

- If the current sum of w is $3k$ for some $k \geq 1$, then wu^* can be weighted either 2 or 3 so that the sum of w gets $3k + 2 = 3(k + 1) - 1$ or $3k + 3 = 3(k + 1)$, thus in $\mathcal{V}$.
- If the current sum of w is $3k + 1$ for some $k \geq 1$, then wu^* can be weighted either 1 or 2 so that the sum of w gets $3k + 2 = 3(k + 1) - 1$ or $3k + 3 = 3(k + 1)$, thus in $\mathcal{V}$.
- If the current sum of w is $3k + 2$ for some $k \geq 1$, then wu^* can be weighted either 3 or 1 so that the sum of w gets $3k + 5 = 3(k + 2) - 1$ or $3k + 3 = 3(k + 1)$, thus in $\mathcal{V}$.

A vertex of L_1 is said to be a *choice vertex* if, at the moment we consider it in the layer-by-layer process, there are two possible ways for weighting its unique incident upward edge (going to u^*). If L_1 has at least two choice vertices, then we are done. Indeed, let w_1^* and w_2^* be two choice vertices of L_1. We repeat the whole layer-by-layer weighting process, making sure to finish with w_1^* and w_2^*. This means we are left with weighting $u^*w_1^*$ and $u^*w_2^*$. By definition, from the point of view of w_1^* and w_2^*, there are two possible correct weights x_1, x_2 for $u^*w_1^*$, and there are two possible correct weights y_1, y_2 for $u^*w_2^*$. No matter what x_1, x_2, y_1, y_2 are, there is always one combination that makes the sum of u^* lie in $\mathcal{U}$. Namely:

- If $\{x_1, x_2\} = \{1, 2\}$ and $\{y_1, y_2\} = \{1, 2\}$, then we can alter the sum of u^* by any value in $\{2, 3, 4\}$. If the current sum of u^* is 0, 1, 2, or 3, then we can weight the two remaining edges so that the sum of u^* is altered by 3, 2, 2 or 4, respectively, so that the sum becomes 3, 3, 4 or 7, respectively, thus in $\mathcal{U}$. If the current sum of u^* is $3k + 1$, $3k + 2$ or $3k + 3$ for some $k \geq 1$, then we can alter the sum by 3, 2 or 4, respectively, so that it becomes $3(k+1)+1$, $3(k+1)+1$ or $3(k+2)+1$, respectively, thus in $\mathcal{U}$.

- If $\{x_1, x_2\} = \{1, 2\}$ and $\{y_1, y_2\} = \{1, 3\}$, then we can alter the sum of u^* by any value in $\{2, 3, 4, 5\}$, and the previous case applies.

- If $\{x_1, x_2\} = \{1, 2\}$ and $\{y_1, y_2\} = \{2, 3\}$, then we can alter the sum of u^* by any value in $\{3, 4, 5\}$. If the current sum of u^* is 0, 1, 2, or 3, then we can weight the two remaining edges so that the sum of u^* is altered by 3, 3, 5 or 4, respectively, so that the sum becomes 3, 4, 7 or 7, respectively, thus in $\mathcal{U}$. If the current sum of u^* is $3k+1$, $3k+2$ or $3k+3$ for some $k \geq 1$, then we can alter the sum by 3, 5 or 4, respectively, so that it becomes $3(k+1)+1$, $3(k+2)+1$ or $3(k+2)+1$, respectively, thus in $\mathcal{U}$.

- If $\{x_1, x_2\} = \{1, 3\}$ and $\{y_1, y_2\} = \{1, 3\}$, then we can alter the sum of u^* by any value in $\{2, 4, 6\}$. If the current sum of u^* is 0, 1, 2, or 3, then we can weight the two remaining edges so that the sum of u^* is altered by 4, 6, 2 or 4, respectively, so that the sum becomes 4, 7, 4 or 7, respectively, thus in $\mathcal{U}$. If the current sum of u^* is $3k+1$, $3k+2$ or $3k+3$ for some $k \geq 1$, then we can alter the sum by 6, 2 or 4, respectively, so that it becomes $3(k+2)+1$, $3(k+1)+1$ or $3(k+2)+1$, respectively, thus in $\mathcal{U}$.

- If $\{x_1, x_2\} = \{1, 3\}$ and $\{y_1, y_2\} = \{2, 3\}$, then we can alter the sum of u^* by any value in $\{3, 4, 5, 6\}$. The same arguments as in the third case above apply.

- If $\{x_1, x_2\} = \{2, 3\}$ and $\{y_1, y_2\} = \{2, 3\}$, then we can alter the sum of u^* by any value in $\{4, 5, 6\}$. If the current sum of u^* is 0, 1, 2, or 3, then we can weight the two remaining edges so that the sum of u^* is altered by 4, 6, 5 or 4, respectively, so that the sum becomes 4, 7, 7 or 7, respectively, thus in $\mathcal{U}$. If the current sum of u^* is $3k+1$, $3k+2$ or $3k+3$ for some $k \geq 1$, then we can alter the sum by 6, 5 or 4, respectively, so that it becomes $3(k+2)+1$, $3(k+2)+1$ or $3(k+2)+1$, respectively, thus in $\mathcal{U}$.

So we may assume that L_1 contains at most one choice vertex. As said earlier, if a vertex w in L_1 is not a choice vertex, then its degree must be 2 or 3. In the latter case, this is because if the degree of w is at least 4, then it has at least three incident downward edges, meaning that, when considering w in the layer-by-layer process, its current sum is at least 3.

In what follows, we prove that there is actually a way to have weight choices for a limited number of non-choice vertices of L_1. Let us illustrate this with the simplest case. Assume L_1 has a choice vertex w_1^*. Recall that $d(u^*) \geq 2$, so let w_2^* be any other (non-choice) vertex of L_1. Let us consider a longest downward path P from w_2^* to a deepest vertex w in some L_{2k+1} (thus in V). Since w_2^* is incident to downward edges (because it has degree more than 1), P has length at least 1. Note also that it might be that $w_2^* = w$, in case where the descendants of w_2^* are actually its sons, i.e., its sons have no sons.

Now let us perform the layer-by-layer process again, with the exception that we do not actually assign a weight to the edges of P, but instead "memorize" the possible weights that can correctly be assigned to them. This is similar to applying the process from bottom to top on the edges of $G - u^*$ but those of P, and then repeating the process from bottom to top on the edges of P, taking all already weighted edges into account when computing the current sums. We say that P is a *choice path* if, whatever the other edge weights are, when considering every edge of P from bottom to top, there are always two possible weights that can correctly be assigned to it. Note that if P is indeed showed to be a choice path, then we are done, because w_2^* can then be viewed as a choice vertex itself:

- Assume the edge e of P incident to w_2^* can be assigned either of weights 1 and 2. If the current sum of w_2^* is 0, then we can consider an edge-weighting assigning weight 1 to e and 1 to $u^*w_2^*$ (so that w_2^* gets sum $2 \in \mathcal{V}$), or assigning weight 2 to e and 3 to $u^*w_2^*$ (so that w_2^* gets sum $5 \in \mathcal{V}$). If the current sum of w_2^* is 1, then we can consider an edge-weighting assigning weight 1 to e and 3 to $u^*w_2^*$ (so that w_2^* gets sum $5 \in \mathcal{V}$), or assigning weight 2 to e and 2 to $u^*w_2^*$ (so that w_2^* gets sum $5 \in \mathcal{V}$). If the current sum of w_2^* is 2, then we can consider an edge-weighting assigning weight 1 to e and 2 or 3 to $u^*w_2^*$ (so that w_2^* gets sum 5 or 6 in $\mathcal{V}$). As already proved when defining choice vertices, we are done as well as soon as the current sum is at least 3.
- Assume e can be assigned either of weights 1 and 3. Then by assigning weight 3 to e, the current sum of w_2^* gets at least 3. As mentioned earlier, this is a favourable case for having two possible ways of weighting $u^*w_2^*$.
- Assume e can be assigned either of weights 2 and 3. The same argument as in the previous case applies.

We now prove that P is indeed a choice path. Let us consider the end-vertex $w \in P$ of P. We prove that its incident upward edge e in P is subject to a choice.

- First, if w does not have any son, then, in the layer-by-layer process, the current sum is 0 when considering the vertex. If w has degree 1, then e can be assigned any of weights 1 and 2 since $1, 2 \in \mathcal{V}$. If w has degree more than 1, then we first assign weight 3 to all its incident edges different from e. This way, the current sum of w becomes $3k$ for some $k \geq 1$. Then e can be assigned any of weights 2 or 3, since $3k' - 1$ and $3k'$ are in $\mathcal{V}$ for all $k' \geq 2$.
- Second, assume that w has a son $w' \in U$. By the choice of P, w' does not have any son. Consider the moment where w' is treated in the layer-by-layer process. If w is the only neighbour of w', then we can correctly assign weight 3 to $w'w$ so that w' gets sum $3 \in \mathcal{U}$. That way, when later considering w in the process, its current sum will be at least 3, and the same arguments as in the previous case apply. So lastly assume that w' has more than one neighbour. When consider w', we assign weight 3 to all edges incident to w' but one different from $w'w$, which we assign weight 1. That way, w' gets sum $3k + 1$ for some $k \geq 1$, thus in $\mathcal{U}$. Again, this makes w having current sum at least 3 because of $w'w$. Then two choices are available at e.

It now remains to prove that all other edges of P also are subject to choices. Assume xx' and $x'x''$ are two consecutive edges of P, where $x \in L_i$, $x' \in L_{i+1}$ and $x'' \in L_{i+2}$, and that $x'x''$ can be weighted in two different ways. We prove that xx' also does.

- Assume the possible choices for $x'x''$ are 1 and 2, and consider the moment where x' is treated in the layer-by-layer process.
  - Assume $x' \in V$. If x' has incident upward edges different from xx', then we assign them weight 3. If, when virtually assigning to $x'x''$ any of weights 1 and 2, the current sum of x' becomes at least 3, then again there are two

choices as xx'. This means that the current sum of x' is 0, i.e., xx' and $x'x''$ are the only edges incident to x'. Then we could here assign weight 1 to $x'x''$ and weight 1 to xx' so that x' gets sum $2 \in \mathcal{V}$, or weight 2 to $x'x''$ and weight 3 to xx' so that x' gets sum $5 \in \mathcal{V}$. Thus 1 and 3 are two possible choices for xx'.

- Otherwise, $x' \in U$. Again, if x' is incident to upward edges different from $x'x$, then we assign them weight 3. If the current sum of x' is 0, then we can virtually assign weight 1 to $x'x''$ and 2 or 3 to xx' so that x' gets sum 3 or 4 in $\mathcal{U}$; 2 and 3 are then two choices for xx'. If the current sum of x' is 1, then we can virtually assign weight 1 to $x'x''$ and 1 or 2 to xx' so that x' gets sum 3 or 4 in $\mathcal{U}$; 1 and 2 are then two choices for xx'. If the current sum of x' is 2, then we can virtually assign weight 1 to $x'x''$ and 1 to xx' so that x' gets sum $4 \in \mathcal{U}$, and assign weight 2 to $x'x''$ and 3 to xx' so that x' gets sum $7 \in \mathcal{U}$; 1 and 3 are then two choices for xx'. So the current sum of x' is at least 3. Note that, by virtually assigning weight 1 to $x'x''$ and weights $1, 2, 3$ to xx', we can alter the sum of x' by any of $2, 3, 4$. By virtually assigning weight 2 to $x'x''$ and weights $1, 2, 3$ to xx', we can alter the sum of x' by any of $3, 4, 5$. Thus there are two ways to alter the sum by 3 and 4, and they involve assigning different weights to xx'. Note that the two ways for altering the sum by 2 and 5 involve assigning different weights to $x'x$ as well. We are now done: Consider the next smallest values $3k + 1$ and $3(k + 1) + 1$ that are strictly larger than the current sum of x'. By the previous arguments, they can be reached in two different ways which involve assigning different weights to xx'. We thus have two choices.

- Assume now the possible choices for $x'x''$ are 1 and 3.
  - Assume $x' \in V$. This time, when virtually assigning weight 3 to $x'x''$, the current sum of x' gets at least 3. As said earlier, there are two choices for $x'x$ here.
  - Otherwise, $x' \in U$. Again, we assign weight 3 to all upward edges different from $x'x$ incident to x', if any. If the current sum of x' is 0, then we can virtually assign weight 1 to $x'x''$ and 2 or 3 to xx' so that x' gets sum 3 or 4 in $\mathcal{U}$; 2 and 3 are then two choices for xx'. If the current sum of x' is 1, then we can virtually assign weight 1 to $x'x''$ and 1 or 2 to xx' so that x' gets sum 3 or 4 in $\mathcal{U}$; 1 and 2 are then two choices for xx'. If the current sum of x' is 2, then we can virtually assign weight 1 to $x'x''$ and 1 to xx' so that x' gets sum $4 \in \mathcal{U}$, and assign weight 3 to $x'x''$ and 2 to xx' so that x' gets sum $7 \in \mathcal{U}$; 1 and 2 are then two choices for xx'. So the current sum of x' is at least 3. Note that, by virtually assigning weight 1 to $x'x''$ and weights $1, 2, 3$ to xx', we can alter the sum of x' by any of $2, 3, 4$. By virtually assigning weight 3 to $x'x''$ and weights $1, 2, 3$ to xx', we can alter the sum of x' by any of $4, 5, 6$. Thus there are two ways to alter the sum by 4, and they involve assigning different weights to xx'. Note that the two ways for altering the sum by 2 and 5 involve assigning different weights to $x'x$ as well, and similarly for 3 and 6. We are then done by the same arguments as earlier.

- Assume finally that the possible choices for $x'x''$ are 2 and 3.
  - Assume $x' \in V$. Again, when virtually assigning weight 3 to $x'x''$, the current sum of x' gets at least 3 and we are done.

– Otherwise, $x' \in U$. We assign weight 3 to all upward edges different from $x'x$ incident to x', if any. If the current sum of x' is 0, then we can virtually assign weight 2 to $x'x''$ and 1 or 2 to xx' so that x' gets sum 3 or 4 in $\mathcal{U}$; 1 and 2 are then two choices for xx'. If the current sum of x' is 1, then we can virtually assign weight 2 to $x'x''$ and 1 to xx' so that x' gets sum $4 \in \mathcal{U}$, or virtually assign weight 3 to $x'x''$ and 3 to xx' so that x' gets sum $7 \in \mathcal{U}$; 1 and 3 are then two choices for xx'. If the current sum of x' is 2, then we can virtually assign weight 2 to $x'x''$ and 3 to xx' so that x' gets sum $7 \in \mathcal{U}$, and assign weight 3 to $x'x''$ and 2 to xx' so that x' gets sum $7 \in \mathcal{U}$; 2 and 3 are then two choices for xx'. So the current sum of x' is at least 3. Note that, by virtually assigning weight 2 to $x'x''$ and weights $1, 2, 3$ to xx', we can alter the sum of x' by any of $3, 4, 5$. By virtually assigning weight 3 to $x'x''$ and weights $4, 5, 6$ to xx', we can alter the sum of x' by any of $4, 5, 6$. Thus there are two ways to alter the sum by 5, and they involve assigning different weights to xx'. Note that the two ways for altering the sum by 4 and 6 involve assigning different weights to $x'x$ as well. Again, we are thus done.

Thus, P is indeed a choice path, and, by our assumption, we thus have two choices for weighting each of $u^*w_1^*$ and $u^*w_2^*$. Taking the current sum of u^* into account, as pointed out earlier there must be a combination of two weights that guarantees that the sum of u^* eventually lies in $\mathcal{U}$, as required. Thus, we are done if L_1 has one choice vertex.

We are thus left with the case when no vertex of L_1 is a choice vertex. This implies that in G all vertices of V must be of degree 2 or 3 as otherwise we could just choose another u^* that would guarantee the existence of a choice vertex in L_1. Also, no vertex of U can be of degree 1, as otherwise its neighbour in V would be a choice vertex when choosing as u^* another of its neighbours being of degree more than 1 (which exists since G is not a star).

Let us choose w_1^* and w_2^* two arbitrary vertices of L_1. For each $i = 1, 2$, let P_i be a longest downward path starting from w_i^* and ending at another vertex $x_i^* \in V$ in a lowest layer. Possibly these paths intersect, and also we possibly have $x_1^* = x_2^*$. If P_1 and P_2 do not intersect and x_1^* and x_2^* do not share a son, then it is easy to see that the arguments used earlier show that both P_1 and P_2 are choice paths (as P_1 and P_2 then get sufficiently independent). This occurs for instance when u^* is a cut-vertex and w_1^* and w_2^* belong to different connected components of $G - u^*$. From this, we can further suppose that no vertex of U is a cut-vertex. In what follows, we prove that by carefully studying how P_1 and P_2 interact, we can deduce choice paths ending in w_1^* and w_2^*, respectively, concluding the proof.

As a first case, let us assume that P_1 and P_2 intersect in a vertex y different from x_1^* and x_2^*. This implies that y has degree at least 3. We here distinguish two cases:

- $y \in U$. In that case, we consider P_2' the subpath of P_2 that goes from w_2^* to the vertex $y' \in V$ such that $yy' \in P_2$ and y is the son of y'. We claim that some edges of G can be weighted so that P_1 and P_2' are choice paths. To see this holds, let us assign weight 3 to yy'. This way, when later considering y' in the layer-by-layer process, its current sum will be at least 3, and as seen earlier this means that, no matter how the other edges incident to y' are weighted, there is always two ways to weight the edge of P_2' incident to y'. This is because $\mathcal{V}$ contains all values $3k - 1$ and $3k$ for every $k \geq 2$. The fact that yy' is weighted 3 has no influence on the fact that P_1 is a choice path, as long as this assigned weight is taken into account when considering y in the process.

- $y \in V$. In that case, we consider P'_2 the subpath of P_2 that goes from w^*_2 to the vertex $y'' \in V$ such that $y''y' \in P_2$ (where y' is defined as in the previous case) and y' is the son of y''. We here proceed as follows. When considering y in the layer-by-layer process, we do not assign a weight to yy' right away. When later considering y', we first assign weight 3 to $y'y''$, and then choose the weights of its other incident edges (including yy') so that the sum of y' gets in $\mathcal{U}$. Note that this is possible since the current sum of y' is at least 3 and $3k+1 \in \mathcal{U}$ for every $k \geq 1$. This makes y'' have current sum at least 3, and, as explained earlier, this makes P'_2 be a choice path. On the other hand, if z denotes the son of y on P_1, then we note that assigning the biggest choice (which is at least 2) as the weight of yz makes y get current sum at least 3 (due to the weighting of yy'). Since $y \in V$, as already seen this means that at least two choices are available when weighting the upward edge incident to y in P_1. Thus P_1 remains a choice path.

Thus we may assume that P_1 and P_2 do not intersect on inner vertices. Assume now they intersect on their end-vertex $x = x^*_1 = x^*_2$. Recall that x is a vertex of V. Let us denote by z_1 and z_2 the vertices in U such that $z_1x^*_1$ is the last edge of P_1, and similarly $z_2x^*_2$ is the last edge of P_2. We may assume that z_1 and z_2 have no other sons, as otherwise we could choose P_1 and P_2 so that they do not intersect (and then we fall in the last case below). Thus z_1 and z_2 have degree at least 2, and only one son each. Denote by z'_1 and z'_2 the other neighbours of z_1 and z_2, respectively, in P_1 and P_2, respectively. That is, z'_1z_1 is a downward edge of P_1, and similarly z'_2z_2 is a downward edge of P_2. By definition of P_1 and P_2, all sons of x (if any) have no descendants. If x has no sons, then, when considering x in the layer-by-layer process, we assign weight 1 to both xz_1 and xz_2, and weight 3 to all remaining incident upward edges. This makes x get sum 2 (if it has degree 2) or $3k+2$ (otherwise) for $k \geq 1$, thus in $\mathcal{V}$. When later considering z_1 (and similarly z_2) in the process, the current sum will be 1, and thus assigning weight 3 to all incident upward edges will make z_1 (resp. z_2) have sum $3k+1$ for some $k \geq 1$, thus in $\mathcal{U}$. When later considering z'_1 and z'_2, the current sum will thus be at least 3; as already mentioned, this implies that P'_1 and P'_2, the subpaths of P_1 and P_2 from w^*_1 to z'_1 and w^*_2 to z'_2 become choice paths.

Assume now x has sons. Actually, x has only one son y, as otherwise x would be a vertex of V with degree at least 4, and there would be a better choice as u^*. Also y cannot be of degree 1, as otherwise there would be a choice of u^* for which it would neighbour a vertex (y) neighbouring a vertex with degree 1, and that would thus be a choice vertex. However, since y is in a deepest layer, all its incident edges are upward edges. In this situation, we proceed as follows. When we consider y in the layer-by-layer process, we assign weight 3 to all but one incident upward edge different from xy. That way, y gets sums $3k+1$ for some $k \geq 1$, thus in $\mathcal{U}$. When later considering x, its current sum is thus 3; we here assign weight 1 to both xz_1 and xz_2 so that x (which recall has degree 3) gets sum $5 \in \mathcal{V}$. When later considering z_1 (and similarly z_2), its current sum is 1; we here assign weight 3 to all upward edges (thus including $z_1z'_1$ (resp. $z_2z'_2$)) so that z_1 (resp. z_2) gets sum $3k+1$ for some $k \geq 1$, thus in $\mathcal{U}$. We now get the situation where z'_1 and z'_2 have current sum at least 3. By previous arguments, this means that the subpath of P_1 from w^*_1 to z'_1 is a choice path, and similarly for the subpath of P_2 from w^*_2 to z'_2.

We may thus now assume that P_1 and P_2 do not intersect at all. Recall that x^*_1 and x^*_2 are in V, and their sons (if any) are in U and have no descendants. If at least one of x^*_1 and x^*_2 has no sons, then P_1 and P_2 are independent and are choice paths by arguments used earlier. So we may assume they both have sons. If the total number of sons of x^*_1

and x_2^* is at least 2, then we can proceed as follows. Let y_1 and y_2 be two of their sons, where y_1 neighbours x_1^* (and perhaps x_2^*) and y_2 neighbours x_2^* (and perhaps x_1^*). When considering y_1 in the layer-by-layer process, we can correctly (i.e., so that its sum gets in $\mathcal{U}$) weight all edges incident to y_1 so that $y_1x_1^*$ is weighted 3. This is because $3 \in \mathcal{U}$ and also $3k+1 \in \mathcal{U}$ for $k \geq 1$. Similarly, when considering y_2 in the process, we can correctly weight all edges incident to y_2 so that $y_2x_2^*$ is weighted 3. Then, by arguments used earlier, we deduce that P_1 and P_2 are choice paths, since x_1^* and x_2^* will have current sum at least 3 when considered by the process. So lastly assume that x_1^* and x_2^* have the same unique son y. If y has a third neighbour, then, when considering y in the process, we can assign weight 3 to all incident upward edges but one different from yx_1^* and yx_2^*, and weight 1 to the remaining edge. This way, the sum of y is some $3k+1$ for $k \geq 2$, and is thus in $\mathcal{U}$. Furthermore, the current sum of both x_1^* and x_2^* gets at least 3, and, as seen earlier, this means that P_1 and P_2 are choice paths.

So y can be assumed to be of degree 2. Now, if at least one of x_1^* and x_2^*, say x_1^*, is incident to a third (upward) edge, then we can proceed as follows. When considering y in the process, we can assign weight 1 to yx_1^* and weight 3 to yx_2^* so that y gets sum $4 \in \mathcal{U}$. When later considering x_2^* in the process, its current sum will be at least 3, implying that P_2 is a choice path. When later considering x_1^*, we can first assign weight 3 to its incident upward edge not in P_1; that way, x_1^* gets current sum at least 3, and P_1 becomes a choice path as well.

Thus, the last case we get to consider is that where y, x_1^* and x_2^* are vertices of degree 2. Free to consider y as u^*, we can assume that u^*, w_1^* and w_2^* are of degree 2 as well. Since G is not a cycle, as otherwise Observation 2.4 would apply, we get that at least one of P_1 and P_2 has an inner vertex being of degree at least 3 (thus distinct from u^*, w_1^*, w_2^*, x_1^*, x_2^*, y). Assume thus P_2 has inner vertices of degree at least 3, and consider one, z. Let z' be the neighbour of z in P_2 so that $z'z$ is a downward edge and z is a son of z'. We note that none of the edges incident to z not in P_2 can be upward. This is because otherwise:

- If $z \in U$, then, we considering z in the layer-by-layer process, we can always assign weight 3 to zz', since we can then adjust the sum of z, by weighting another upward edge, so that it gets in $\mathcal{U}$. This way, $z' \in V$ gets current sum at least 3, and the subpath of P_2 going from w_2^* to z' is a choice path. This has no effect on P_1, which is thus a choice path.

- If $z \in V$, then, when considering z in the layer-by-layer process, we can here assign weight 3 to an incident upward edge different from zz'. This way, the current sum of z gets at least 3, and, as seen earlier, this means that the subpath of P_2 going from w_2^* to z is a choice path. Again, P_1 is now a choice path.

Thus, the edges not in P_2 incident to z are downward edges. Now consider the path P_2' obtained by starting from w_2^*, following P_2 until reaching z, and then leaving P_2 through a downward edge not in P_2, and going downward as long as possible until reaching a deepest vertex of V. We note that P_2' cannot intersect P_2 on a lower vertex, as otherwise that vertex would be a vertex of P_2 with degree at least 3 incident to an upward edge not in P_2 (case we have handled earlier). Since we are going downward, we also cannot meet P_1, as otherwise we would find a vertex with degree at least 3 in P_1 with an incident upward edge not in P_1 (also a similar case as earlier). Furthermore, since x_1^*, x_2^* and y were shown to be of degree 2, this path P_2' cannot reach these vertices. This means that P_1 and P_2' are two disjoint paths having their end-vertices different from w_1^* and w_2^* sharing no sons. Then P_1 and P_2 are two independent paths, thus choice paths, and we are done. □

As a corollary of Theorem 2.1, we can now prove Conjecture 1.2:

Theorem 2.5. *For every nice digraph D, we have $\chi_{-,+}(D) \leq 3$.*

Proof. Consider $B(D)$ the bipartite graph associated to D, with bipartition $U \cup V$ (where, say, U contains the v^+'s and V contains the v^-'s). Since D is nice, so is $B(D)$. By Theorem 2.1, every connected component of $B(D)$ admits a 3-edge-weighting verifying the properties in the statement. Invoking this result with preserving the bipartition of these connected components as in $B(D)$, we get that $B(D)$ itself has a 3-edge-weighting where vertices in U have sum in $\mathcal{U}$ while vertices in V have sum in $\mathcal{V}$. As described earlier, we derive this 3-edge-weighting of $B(D)$ to a 3-arc-weighting of D, by assigning to any arc $\overrightarrow{uv}$ the weight of u^+v^-. By that weighting, all vertices v of D get $\sigma^+(v)$ in $\mathcal{U}$ and $\sigma^-(v)$ in $\mathcal{V}$. By the definition of $\mathcal{U}$ and $\mathcal{V}$, and because D is nice, this resulting arc-weighting is $(-,+)$-distinguishing. $\square$

3. Complexity aspects of the $(-,+)$ variant

We first prove that for every $k \geq 3$, deciding whether a digraph D verifies $\chi_{-,+}(D) \leq k$ is equivalent to the k-Colouring problem (Given a graph G, decide whether $\chi(G) \leq k$, i.e., whether it admits a *proper k-colouring*, meaning a partition of its vertex set into k stable sets), and is thus NP-complete.

Theorem 3.1. *For every $k \geq 3$, it is NP-complete to decide whether $\chi_{-,+}(D) \leq k$ holds for a given digraph D.*

Proof. Let $k \geq 3$ be fixed. It is obvious that the problem is in NP, so let us focus on proving its NP-hardness. This is done by reduction from the k-Colouring problem, which is well-known to be NP-complete. From a given undirected graph G, we construct, in polynomial time, a digraph D such that $\chi(G) \leq k$ if and only if $\chi_{-,+}(D) \leq k$.

The construction of D is achieved as follows. Arbitrarily denote by $v_1, ..., v_n$ the vertices of G, and let $\overrightarrow{G}$ be the (acyclic) orientation of G obtained by orienting every edge v_iv_j from v_i to v_j if $i < j$, or from v_j to v_i otherwise. For every vertex v_i in G, we add an arc $\overrightarrow{a_ib_i}$ to D, where a_i, b_i are two new vertices. Now, for every arc $\overrightarrow{v_iv_j}$ of $\overrightarrow{G}$, thus with $i < j$, we add the arc $\overrightarrow{b_ia_j}$ to D. Finally, for every $i = 1, ..., n$, we add $d^-_D(a_i) + 1$ arcs outgoing from b_i and going to new pendant vertices $c_{i,1}, ..., c_{i,d^-_D(a_i)+1}$ with indegree 1 and outdegree 0. Clearly, the construction of D is achieved in polynomial time.

Note that, in D, for each arc $\overrightarrow{a_ib_i}$, we have $d^+(a_i) = d^-(b_i) = 1$. Hence, for every arc-weighting ω of D, we have $\sigma^+(a_i) = \sigma^-(b_i) = \omega(\overrightarrow{a_ib_i})$. Furthermore, for every arc of the form $\overrightarrow{b_ic_{i,j}}$, we cannot get $\sigma^-(b_i) = \sigma^+(c_{i,j})$ as $d^-(b_i) = 1 > 0 = d^+(c_{i,j})$. Thus, from the point of view of that arc, $\omega(\overrightarrow{b_ic_{i,j}})$ can be assigned any value without creating any sum conflict involving its two ends; let us thus assign weight k to all such arcs from now on. Under that hypothesis, note that for every arc $\overrightarrow{a_ib_i}$ also we cannot get $\sigma^-(a_i) = \sigma^+(b_i)$, as

$$\sigma^-(a_i) \leq k \cdot d^-(a_i) < k \cdot d^-(a_i) + 1 \leq \sigma^+(b_i).$$

Thus, conflicts can only involve arcs of the form $\overrightarrow{b_ia_j}$ for some $i < j$. Since $d^-(b_i) = d^+(a_j) = 1$, we get that, for ω to be $(-,+)$-distinguishing, the weights $\omega(\overrightarrow{a_ib_i})$ and $\omega(\overrightarrow{a_jb_j})$ must be different. Thus, every two such arcs $\overrightarrow{a_ib_i}$ and $\overrightarrow{a_jb_j}$ being "adjacent" this way must receive different weights. This depicts the fact that the edge v_iv_j is present in G, and v_i, v_j must thus receive different colours by a proper k-colouring of G. The equivalence between edge-weighting D and vertex-colouring G should thus be easy to visualise now. $\square$

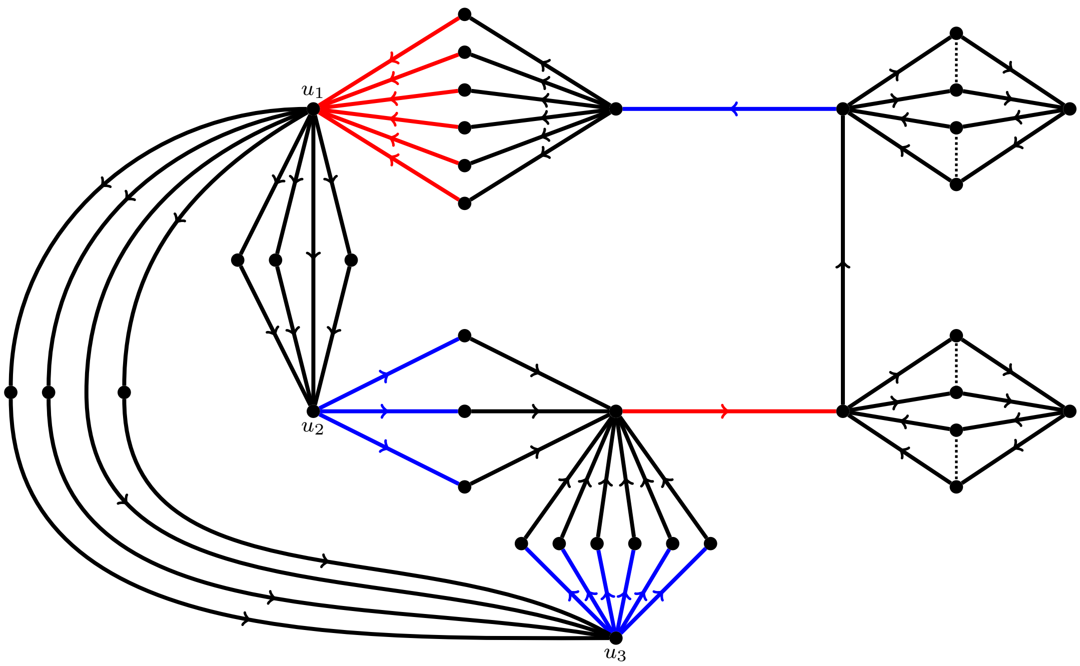

Figure 1: The digraph H used in the proof of Theorem 3.4. Dotted lines represent many vertices with the same neighbourhood.

The rest of this section is devoted to proving the counterpart of Theorem 3.1 for $k = 2$, in Theorem 3.4 below. We actually prove the result for nice digraphs, which is of interest as $\chi_{-,+}$ is bounded by 3 for these digraphs (recall Theorem 2.5). We first need need to introduce the digraph H depicted in Figure 1 and highlight some of its properties we will use.

Lemma 3.2. *H is nice. Furthermore, assuming dotted lines represent sufficiently many vertices, we have $\chi_{-,+}(H) \leq 2$, and, in every $(-,+)$-distinguishing 2-arc-weighting of H:*

- *all red arcs are assigned weight 2;*
- *all blue arcs are assigned weight 1.*

Therefore, $\sigma^+(u_2) = 3$ and $\sigma^+(u_3) = 6$.

Proof. Note first that H has no source nor sink, and thus no ss-arc. Furthermore, it can be checked that, for every arc $\overrightarrow{uv}$, either u has another outgoing arc or v has another incoming arc. Thus, H has no lonely arcs and it is nice.

In H, it can also be noted that there are some red arcs $\overrightarrow{uv}$ where 1) $\delta^+(u) = 1$, and 2) there is a blue arc $\overrightarrow{xy}$ such that $\delta^-(y) = 1$ and $\overrightarrow{yu}$ is an arc. Clearly, whenever a red arc and a blue arc verify these assumptions, they should be assigned different weights by a $(-,+)$-distinguishing 2-arc-weighting of H. Also, we note that all arcs incoming to u_1 are red arcs, all arcs outgoing from u_2 are blue arcs, and all arcs outgoing from u_3 are blue arcs, while $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_1u_3}$ are arcs. From these arguments and the connection between the blue arcs and red arcs, it can be deduced that either all red arcs must be weighted 1 and all blue arcs must be weighted 2, or conversely. The first of these two cases is not correct, as we would get $\sigma^-(u_1) = 6$ (since all arcs incoming to u_1 are red) and $\sigma^+(u_2) = 6$ (since all arcs outgoing from u_2 are blue), while $\overrightarrow{u_1u_2}$ is an arc of H. In the second case, we get $\sigma^-(u_1) = 12$ and $\sigma^+(u_2) = 3$, which yields no conflict between u_1 and u_2. Also, by the same reasoning, we get $\sigma^+(u_3) = 6$.

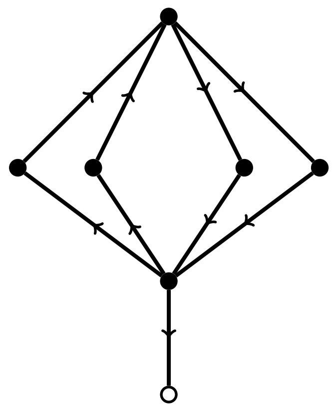

Figure 2: The variable gadget D_v, its white vertex being the root.

To complete the proof, we claim that assigning weight 1 to the remaining arcs (that are not red or blue) indeed yields a $(-,+)$-distinguishing 2-arc-weighting of D, assuming the number of vertices in the dotted parts are conveniently chosen. The main argument is that, under that hypothesis, most black arcs $\overrightarrow{uv}$ are "unbalanced" in the sense that they verify either $d^-(u) > 2d^+(v)$ or $2d^-(u) < d^+(v)$; when this is the case, it is not possible to have $\sigma^-(u) = \sigma^+(v)$ for such an arc. $\square$

We will also be using the *variable gadget* D_v depicted in Figure 2. Its white vertex is called the *root* of D_v, while the unique arc incident to the root is called the *root arc*. It can easily be checked that D_v fulfils the following.

Observation 3.3. *D_v is nice. Furthermore, we have $\chi_{-,+}(D_v) \leq 2$, and there exist $(-,+)$-distinguishing 2-arc-weightings where the root arc is weighted 1, and $(-,+)$-distinguishing 2-arc-weightings where the root arc is weighted 2. This remains true if an arbitrary number of pending arcs outgoing from the root is added to D_v.*

We are now ready to prove our main result in this section.

Theorem 3.4. *It is NP-complete to decide whether $\chi_{-,+}(D) \leq 2$ holds for a given nice digraph D.*

Proof. We again focus on proving the NP-hardness, which we establish through a reduction from Monotone Not-All-Equal 3-SAT, which is well-known to be NP-hard [12]. In this problem, we are given a 3CNF formula F whose each clause contains three distinct variables, and we aim at finding a *nae truth assignment* to the variables, i.e., a truth assignment such that every clause gets at least one true variable and at least one false variable. Note that we might assume that all clauses of F have three different variables. We construct, in polynomial time, a nice digraph D such that F has a nae truth assignment if and only if $\chi_{-,+}(D) \leq 2$.

The construction is as follows. First add a copy of the digraph H (Figure 1) to D. In the sequel, when referring to the vertices u_2, u_3 of D, we mean those of H. We then consider every variable x_i of F. Assume x_i appears in the x distinct clauses $C_{j_1}, ..., C_{j_x}$ of F. We add a copy D_{x_i} of the variable gadget D_v (Figure 2) to D (with root denoted r_i), and add x arcs $\overrightarrow{r_i v_{i,j_1}}, ...\overrightarrow{r_i v_{i,j_x}}$ where $v_{i,j_1}, ..., v_{i,j_x}$ are new pendant vertices. Now consider every clause C_j of F. We add to D a new *clause vertex* c_j, and, assuming C_j contains the distinct variables $x_{i_1}, x_{i_2}, x_{i_3}$, we add the arcs $\overrightarrow{v_{x_{i_1}} c_j}, \overrightarrow{v_{x_{i_2}} c_j}, \overrightarrow{v_{x_{i_3}} c_j}$. Finally, we add the arcs $\overrightarrow{c_j u_2}, \overrightarrow{c_j u_3}$ to D, where u_2, u_3 belong to H added earlier.

We claim that we have the desired equivalence, for the following reasons. Weighting the root arc of a variable gadget D_{x_i} either 1 or 2 by a 2-arc-weighting of D should be thought of as setting the variable x_i to true or false, respectively. Note that, by construction, the root arc $\overrightarrow{xy}$ of D_{x_i} verifies $\sigma^-(y) = 1$, and, for every clause C_j that contains x_i, we also have $\sigma^+(v_{i,j}) = 1$ while $\overrightarrow{v_{i,j}c_j}$ is an arc. Thus, $\overrightarrow{xy}$ and $\overrightarrow{v_{i,j}c_j}$ must get different weights by a $(-,+)$-distinguishing 2-arc-weighting of D. This is true for every clause C_j that contains x_i. Thus, as soon as $\overrightarrow{xy}$ is weighted, the different weight is forced on every arc of the form $\overrightarrow{v_{i,j}c_j}$; this models the fact that, by a nae truth assignment of F, a variable brings its truth value to every clause that contains it.

Now, we note that, for every clause vertex c_j, its value of σ^- by an arc-weighting of D is inherited from all arcs of the form $\overrightarrow{v_{i,j}c_j}$. There are only three such arcs, implying that $\sigma^-(c_j)$ ranges in the set $\{3, 4, 5, 6\}$. However, due to the presence of the arcs $\overrightarrow{c_ju_2}, \overrightarrow{c_ju_3}$, actually $\sigma^-(c_j)$ cannot take value 3 or 6 by a $(-,+)$-distinguishing 2-arc-weighting of D, by Lemma 3.2. Since these two values correspond to the cases where the three arcs incoming to c_j are all weighted 1 or all weighted 2, by the previous analogy this models the fact that, by a nae truth assignment of F, a clause is not regarded as satisfied when all its variables get assigned the same truth value.

Let us conclude by mentioning that D is indeed nice. This is because all gadgets we have used for its construction are nice, and it can easily be checked that combining them as we did did not introduce ss-arcs or lonely arcs. Also, because the gadgets admit $(-,+)$-distinguishing 2-arc-weightings and the connexion between them is rather sparse, the existence of a $(-,+)$-distinguishing 2-arc-weighting of D only relies on the equivalence with satisfying F in a nae way. These arguments conclude the proof. $\square$

4. Results to date and filling results

In Table 1, we have listed all progress we are aware of regarding directed variants of the 1-2-3 Conjecture defined over the parameters σ^-, σ^+. Each row is dedicated to a variant, the first column indicating the variant (it indicates, for every arc $\overrightarrow{uv}$, which parameters of u (left of $\neq$ symbol) and v (right of $\neq$ symbol) are required to differ). The column **Bound** indicates the best upper bound we know on the chromatic index of a variant. The presence of a star indicates that a bound is tight. The column **Complexity** gives information on whether determining the index is easy (in P) or not (NP-complete). For those variants where the upper bound of 2 was proved, such concerns barely make sense, as the value of the index is 1 only for digraphs whose adjacent vertices have particular degree properties (which can be checked in polynomial time). The column **Choosability** indicates whether the best known upper bound on an index also applies to the list context (see below for more details).

For every result appearing in a cell of Table 1, we mention where it was proved (either a previous reference of the literature, and/or results in the current paper). Some of the results actually follow from a combination of existing results; for every such combination that might be not obvious for someone not familiar to the field, we add an explaining result later in the section. For every cell for which we have no clue, more details are provided in concluding Section 5 (see the corresponding Questions).

We note that the variant in the first row is actually equivalent to the original 1-2-3 Conjecture, as the orientation could just be dropped. Finally, let us mention that the variant presented as the last row of Table 1 is a new directed variant of the 1-2-3 Conjecture, which, to the best of our knowledge, has not been considered before. It relies on the following ideas. By an arc-weighting of a digraph D, we have two parameters associated to

	Bound	**Complexity**	**Choosability**
$\sigma^- + \sigma^+ \neq \sigma^- + \sigma^+$	≤ 5 [10]	≤ 1: in P ≤ 2: NPC [9]	$\leq \Delta^- + \Delta^+ + 1$ [6]
$\lvert\sigma^- - \sigma^+\rvert \neq \lvert\sigma^- - \sigma^+\rvert$	≤ 2* [5]	≤ 1: in P	≤ 2* [8]
$\sigma^+ \neq \sigma^+$ (or $\sigma^- \neq \sigma^-$)	≤ 3* [4]	≤ 1: in P ≤ 2: NPC [4]	≤ 3* **Thm. 4.3**
$\sigma^+ \neq \sigma^-$	≤ 3* [1]	≤ 1: in P ≤ 2: in P [1, 15], **Thm. 4.2**	unknown **Qst. 5.1**
$\sigma^- \neq \sigma^+$ lonely allowed	$\leq \max\{3, \Delta^+ + \Delta^- + 1\}$ **Thm. 4.1**	≤ 1: in P $\leq k$: NPC ($k \geq 2$) **Thm. 3.1, 3.4**	unknown **Qst. 5.3**
$\sigma^- \neq \sigma^+$ nice digraphs	≤ 3* **Thm. 2.5**	≤ 1: in P ≤ 2: NPC **Thm. 3.4**	unknown **Qst. 5.2**
$(\sigma^-, \sigma^+) \neq (\sigma^-, \sigma^+)$	≤ 2* **Thm. 4.4**	≤ 1: in P	≤ 2* **Thm. 4.4**

Table 1: All results known so far on directed variants of the 1-2-3 Conjecture.

any vertex v, namely $\sigma^-(v)$ and $\sigma^+(v)$. In this new variant, we ask all adjacent vertices u, v to be distinguished by at least one of their parameters. That is, at least one of $\sigma^-(u), \sigma^+(u)$ should be different from at least one of $\sigma^-(v), \sigma^+(v)$. We prove that this variant is far easier than the 1-2-3 Conjecture, see Theorem 4.4.

We start off by proving the bound in Row "$\sigma^- \neq \sigma^+$ (lonely allowed)", Column **Bound** of Table 1.

Theorem 4.1. *Let D be a digraph with no ss-arc. Then*

$$\chi_{-,+}(D) \leq \max\{3, \Delta^+(D) + \Delta^-(D) + 1\}.$$

Proof. If D has no lonely arcs, then $\chi_{-,+}(D) \leq 3$ by Theorem 2.5. Now assume that D has lonely arcs. We first make an observation on the consequences of removing lonely arcs in D. We note that removing a lonely arc $\overrightarrow{uv}$ from D does not create new lonely arcs. Indeed, if an arc $\overrightarrow{xy}$ becomes lonely in $D - \overrightarrow{uv}$, then it would mean either that $\overrightarrow{uv}$ and $\overrightarrow{xy}$ share a common head or tail in D, a contradiction to $\overrightarrow{uv}$ being lonely, or that $\overrightarrow{xy}$ was already lonely in D. However, let us point out that repeatedly removing lonely arcs from D can create ss-arcs.

To get a $(-,+)$-distinguishing $\max\{3, \Delta^+(D) + \Delta^-(D) + 1\}$-arc-weighting of D, we proceed as follows. We first repeatedly remove all lonely arcs from D, which gives a digraph D' without lonely arcs. Forgetting about the sources and sinks of D' being joined by an ss-arc, using Theorem 2.5 we can deduce $(-,+)$-distinguishing 3-arc-weighting of D' where the only conflicts are along arcs $\overrightarrow{uv}$ such that u is a source (thus $\sigma^-(u) = 0$) and v is a sink (thus $\sigma^+(v) = 0$). Another way to see this is considering the nice digraph D'' obtained from D' by adding a new dummy pending arc incoming to every source, deducing a $(-,+)$-distinguishing 3-arc-weighting of D'', and then considering the edge-weighting back in D', by removing the dummy arcs.

Recall that D is nice; thus, the ss-arcs of D' result from the removal of lonely arcs. In a final step, we extend the 3-edge-weighting of D' to a $\max\{3, \Delta^+(D) + \Delta^-(D) + 1\}$-arc-weighting of D (by thus weighting the lonely arcs), thus removing the sum conflicts along ss-arcs. We proceed as follows. Let $\overrightarrow{uv}$ be such a lonely arc. There are $\delta^-(u) + \delta^+(v) \leq$

$\Delta^-(D)+\Delta^+(D)$ constraints around $\overrightarrow{uv}$, namely $\sigma^-(v)$ should not be equal to any $\sigma^+(w)$ where $\overrightarrow{vw}$ is an arc, and similarly for $\sigma^+(u)$ and $\sigma^-(w)$ where $\overrightarrow{wv}$ is an arc. Since we have $\Delta^-(D)+\Delta^+(D)+1$ weights to play with, one can freely be assigned to $\overrightarrow{uv}$ without creating any sum conflict. Note also that the conflicts between the ends of ss-arcs in D' are fixed in D, as otherwise it would mean that D has ss-arcs. More precisely, for every ss-arc, at least one of its source and sink must be incident to lonely arcs. This concludes the proof. $\square$

The next result yields the answer in Row "$\sigma^+ \neq \sigma^-$", Column **Complexity** of Table 1.

Theorem 4.2. *Given a digraph D, deciding whether $\chi_{+,-}(D) \leq 2$ can be done in polynomial time.*

Proof. As observed in [1], finding a $(+,-)$-distinguishing k-arc-weighting of D is equivalent to finding a neighbour-sum-distinguishing k-edge-weighting of $B(D)$, the bipartite graph associated to D (see previous Section 2). Hence, $\chi_{+,-}(D) \leq 2$ if and only if $\chi_\Sigma(B(D)) \leq 2$. As proved by Thomassen, Wu and Zhang [15], a connected bipartite graph G verifies $\chi_\Sigma(G) \leq 2$ if and only if G is not an *odd multicactus*, a class of 2-degenerate 2-connected graphs that can be constructed and recognized easily.

Now, since constructing $B(D)$ from D can be done in polynomial time, and checking whether a bipartite graph is an odd multicactus can also be done in polynomial time (see [15] for more details), the claim follows. $\square$

We now prove the result in Row "$\sigma^+ \neq \sigma^+$", Column **Choosability** of Table 1. Assuming each arc $\overrightarrow{uv}$ of a digraph D is assigned a list $L(\overrightarrow{uv})$ of weights, by an *L-arc-weighting* of D we mean an arc-weighting where each arc is assigned a weight from its list.

Theorem 4.3. *Let D be a digraph, and $L : A(D) \to \mathbb{N}$ be any assignment of lists of size* 3 *to the arcs of D. Then D admits a $(+,+)$-distinguishing L-arc-weighting.*

Proof. The proof is by induction on the number of arcs in D. Since the claim can easily be verified by hand for digraphs with only a few arcs, we proceed with the general case. Choose v a vertex of D verifying $d^-(v) \leq d^+(v)$; such a vertex exists as $\sum_{v\in V(D)} d^-(v) = \sum_{v\in V(D)} d^+(v)$. Let D' be the digraph obtained from D by removing all arcs outgoing from v; by the induction hypothesis, there exists a $(+,+)$-distinguishing L-arc-weighting of D'. Our goal is to extend it to the arcs $\overrightarrow{vu_1}, ..., \overrightarrow{vu_{d^+(v)}}$ outgoing from v without creating sum conflicts. Note that assigning a weight to any such arc $\overrightarrow{vu_i}$ only affects $\sigma^+(v)$; thus, our goal is to assign weights (from their lists) to the $d^+(v)$ $\overrightarrow{vu_i}$'s in such a way that $\sigma(v)$ does not meet the value of σ^+ of any of the at most $2d^+(v)$ vertices neighbouring v in D.

The problem can be reformulated as follows: Given d 3-element lists $L_1, ..., L_d$, by picking elements $e_1, ..., e_d$ in $L_1, ..., L_d$, respectively, can we generate at least $2d+1$ values as $e_1 + ... + e_d$? We show below that this is the case, hence proving the claim.

Assume that the three elements in the L_i's are ordered in increasing order of their value. Let $p_1, ..., p_d$ be a pointer for the elements in $L_1, ..., L_d$, respectively (essentially, each pointer p_i is positioned on either the first, second or third element of L_i). By $v(p_i)$, we mean the value of the element p_i is currently pointing at in L_i. Assuming p_i is not pointing at the largest element of L_i, by *moving p_i to the right*, we mean making p_i point at the first element of L_i that is larger than the one it is currently pointing at.

Start from each p_i pointing at the smallest element of L_i. The first sum we consider is $v(p_1) + ... + v(p_d)$. To create the next $d-1$ sums, thus d sums so far in total, we consider $i = 2, ..., d$ successively, move p_i to the right, and consider $v(p_1) + ... + v(p_d)$ again. Since all L_i's are ordered from smallest to greatest, clearly each step we get a sum that is strictly bigger than the one obtained at the previous step. The next $d-1$ sums, thus $2d-1$ sums so far in total, are obtained by again considering $i = 2, ..., d$ successively, moving p_i to the right, and consider $v(p_1) + ... + v(p_d)$. The remaining two sums are obtained in two steps by moving p_1 to the right twice, and again looking at $v(p_1) + ... + v(p_d)$. $\square$

Last, we prove the results in the last row of Table 1.

Theorem 4.4. *Let D be a digraph, and $L : A(D) \to \mathbb{N}$ be any assignment of lists of size 2 to the arcs of D. Then D admits an L-arc-weighting such that, for every arc $\overrightarrow{uv}$, we have $(\sigma^-(u), \sigma^+(u)) \neq (\sigma^-(v), \sigma^+(v))$.*

Proof. Given an arc-weighting ω of D, for every vertex v we denote by $\Gamma(v)$ the pair $(\sigma^-(v), \sigma^+(v))$ resulting from ω. The crucial observation we are going to use is that the only situation where two vertices u, v cannot be distinguished by Γ is precisely when $\sigma^-(u) = \sigma^+(u) = \sigma^-(v) = \sigma^+(v)$.

The proof is by induction on the number of vertices of D. We focus on proving the general case. Let v be any vertex of D. According to the induction hypothesis, the digraph D' obtained by removing v from D has an L-arc-weighting which is as desired. We wish to extend it to all arcs (outgoing and incoming) incident to v. For every neighbour u of v in D, it is possible that, when assigning a weight w from $L(\overrightarrow{vu})$ (resp. $L(\overrightarrow{uv})$) to $\overrightarrow{vu}$ (resp. $\overrightarrow{uv}$), u is now involved in a conflict (regarding Γ) with one of its neighbours different from v. As explained earlier, this only occurs when $\sigma^-(u) + w = \sigma^+(u)$ (resp. $\sigma^+(u) + w = \sigma^-(u)$). From that perspective, we say that w is *unsafe* in $L(\overrightarrow{vu})$ (resp. $L(\overrightarrow{uv})$). A consequence is that, when assigning the second weight from the list to $\overrightarrow{vu}$ (resp. $\overrightarrow{uv}$), such a conflict involving u cannot occur. That second weight of the list is thus called *safe* (with respect to u). Thus, some of the neighbours of v in D are *fragile*, in the sense that the arc joining them to u has its list including an unsafe weight. This is because the weight to be assigned to the arc between v and a fragile vertex is somehow forced (the safe one must be assigned).

To every arc joining v and a fragile neighbour, let us thus assign the safe weight. Then:

- First assume that no arc incident to v remains to be weighted, i.e., all neighbours of v were fragile. By definition, every neighbour u of v, because it was fragile, now verifies $\sigma^-(u) \neq \sigma^+(u)$; since, under that condition, no vertex can be in conflict with u as stated earlier, in particular also v cannot; so the resulting arc-weighting of D is as desired.

- Finally assume that some arcs incident to v remain to be weighted, i.e., some neighbours of v were not fragile. By definition, whatever weight we assign to any such arc $\overrightarrow{vu}$ or $\overrightarrow{uv}$, it cannot be that u gets into a conflict, as we will eventually get $\sigma^-(u) \neq \sigma^+(u)$ in any case. All remaining such arcs we weight them arbitrarily in such a way that, eventually, also $\sigma^-(v) \neq \sigma^+(v)$ is satisfied. Such a condition can clearly be reached, as all lists have size 2, and weighting an arc incident to v alters only one of $\sigma^-(v), \sigma^+(v)$. Furthermore, that condition guarantees that v cannot be involved in a conflict. We are thus done here as well. $\square$

5. Conclusion and open questions

Our original intention in this work was to provide more details on the directed variant of the 1-2-3 Conjecture introduced in [7] by Horňák, Przybyło and Woźniak. We managed to fully prove Conjecture 1.2. We have also provided complexity results on that variant.

There have been quite a few works on the quest towards a directed variation of the 1-2-3 Conjecture, so we wanted to take this occasion to make a summary of all results we know on that very topic. This has resulted in Table 1. An interesting thing to emphasize is that most of these directed variants were proved through fairly easy proofs, sometimes even in much constrained settings (such as the list setting). Also, despite the fact that these proofs are rather easy, it is interesting that the employed arguments differ from a version to the others (sometimes easy inductive arguments apply, while equivalences to known cases of the 1-2-3 Conjecture had to be established other times); this shows that all these variants are easy for various reasons. We believe all this is an interesting phenomenon, as directed variants of problems generally tend to become harder than their undirected counterpart.

For these reasons, we however feel that the quest might be not over yet, and perhaps there are ways to define other variants whose behaviours would mimic those behind the original 1-2-3 Conjecture better.

About Table 1, a few results are still missing, and we would like to raise the following corresponding questions, whose answer would allow to complete the full picture.

Let us first consider the case of the cell in Row "$\sigma^+ \neq \sigma^-$", Column **Choosability**.

Question 5.1. *Let D be a nice digraph, and $L : A(D) \to \mathbb{N}$ be any assignment of lists of size 3 to the arcs of D. Does D always admit a $(+,-)$-distinguishing L-arc-weighting?*

As noted in [1], the $(+,-)$ variant of the 1-2-3 Conjecture is equivalent to the 1-2-3 Conjecture in bipartite graphs. Hence, Question 5.1 is equivalent to proving that all bipartite graphs satisfy the List 1-2-3 Conjecture of Bartnicki, Grytczuk and Niwczyk [2], which is wide open in general.

The same arguments apply to the $(-,+)$ variant, as Horňák, Przybyło and Woźniak proved that the notion of associated bipartite graph can also be employed in this context. If our Theorem 2.5 held in the list context, then this would be a first step towards the following two questions, which appear in Rows "$\sigma^+ \neq \sigma^-$ (lonely allowed)" and "$\sigma^+ \neq \sigma^-$ (nice digraphs)", Column **Choosability** of Table 1.

Question 5.2. *Let D be a digraph with no ss-arcs, and $L : A(D) \to \mathbb{N}$ be any assignment of lists of size k to the arcs of D. What is the least k such that D always admits a $(-,+)$-distinguishing L-arc-weighting?*

Question 5.3. *Let D be a nice digraph, and $L : A(D) \to \mathbb{N}$ be any assignment of lists of size 3 to the arcs of D. Does D always admit a $(-,+)$-distinguishing L-arc-weighting?*

An aspect we have not mentioned in Table 1, is the complexity of choosability for some of the variants (whose list index was proved to be at least 3). To the best of our knowledge, no such results are known here, and it might be interesting to establish new hardness reductions for that need.

We have also not mentioned the total versions of the variants in Table 1, or, in other words, directed variants of the 1-2 Conjecture raised by Przybyło and Woźniak [13]. The idea here is that not only the edges/arcs should be weighted but also the vertices, the sum of a vertex becoming the sum of weights on its incident edges/arcs plus its own weight. In the undirected case, it is expected that graphs become easier to weight in the total version, as the number 3 in the statement of the 1-2-3 Conjecture was dropped down to 2 in the statement of the 1-2 Conjecture.

All aspects mentioned in Table 1 could thus be investigated in the context of directed variants of the 1-2 Conjecture. Such results can actually already be found in the literature for some variants. For instance, in [3], the authors proved that the $(+,+)$ variant of the 1-2 Conjecture is false in a strong sense. In [1], the authors proved that the $(+,-)$ variant of the 1-2 Conjecture is false for a particular family of digraphs, but conjectured that it might be true for all other digraphs. To the best of our knowledge, this is pretty much everything that is known to date on this topic.

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